$$dX_t = b(t, X_t)\,dt + \sqrt{\Sigma}\,dW_t$$

$$V(t,x) = \inf_m \mathbb{E}^{t,x,m}\left[\int_t^T l(s, X_s, m_s)\,ds + g(X_T)\right]$$

<u>DP</u>

$$V(t,x) = \inf_m \mathbb{E}^{t,x,m}\left[\int_t^{t+h} l\,ds + V(t+h, X_{t+h})\right]$$

<u>Q-value</u>

$$Q^h(t,x,m) = \mathbb{E}^{t,x,m}\left[\int_t^{t+h} l(s, X_s, m_s)\,ds + V(t+h, X_{t+h})\right]$$

<u>DP-Q</u>

$$\begin{cases} Q^h(t,x,m) = \mathbb{E}^{t,x,m}\left[\int_t^{t+h} l\,ds + \inf_m Q^h(t+h, X_{t+h}, m)\right] \\ V(t,x) = \inf_m Q^h(t,x,m) \end{cases}$$

HJB

$$
\begin{cases}
\inf_{a} \left\{ \partial_t v + L v + \ell \right\} = 0 \\
v(T, x) = g(x)
\end{cases}
$$

where

$$L \varphi = b \cdot D\varphi + \Delta V$$

Discretization (h, δ)

$$\partial_t^{\delta} \varphi = \frac{\varphi(t+\delta) - \varphi(t)}{\delta}$$

$$b \cdot D^h \varphi = \sum_{i=1}^{d} b_i^+ \frac{\varphi(x+he_i) - \varphi(x)}{h} - b_i^- \frac{\varphi(x) - \varphi(x-he_i)}{h}$$

$$\Delta^h \varphi = \sum_{i=1}^{d} \frac{\varphi(x+he_i) - 2\varphi(x) + \varphi(x-he_i)}{h^2}$$

$(HJB)^{h,\delta}$

$$\begin{cases} v(T,x) = g(x) \\ \inf_a \left(\partial_t^\delta v(t,x) + \mathcal{L}^h v(t+h,x) + l(t,x,a) \right) = 0 \end{cases}$$

$$v(t,x) = \inf_a \left\{ \sum_{x' \in N^h(x)} p^h(x' | t,x,a) v(t+h,x') + l(t,x,a) \right\} = 0$$

where

$$N_h(x) = \{ x' \,|\, x' = x \pm h e_i,\ i = 1 \cdots d \} \cup \{x\}$$

and $p^h(x' | t,x,a)$ is

$$\begin{cases} p(x \pm h e_i | t,x,a) = (b_i^{\pm} h + 1) \dfrac{\delta}{h^2} \\ p(x | t,x,a) = 1 - \sum_{i=1}^{d} \left(p(x+he_i) + p(x-he_i) \right) \end{cases}$$

Let

$$Q(t, x, a) = \sum_{x' \in N^h(x)} p^h(x' \mid t, x, a) V(t+h, x') + l(t, x, a).$$

Then

$$Q(t, x, a) = l(t, x, a) + \sum_{x' \in N^h(x)} p^h(x' \mid t, x, a) \inf_{a'} Q(t+h, x', a')$$