

DISCRETIZATION OF ELLIPTIC LINEAR PDE AND NEURAL NETWORK

1. PROBLEM SETUP

1.1. **HJB.** We want to solve a d-dimensions HJB given below:

- Domain

$$O = \{x \in \mathbb{R}^d : 0 < x_i < 1, i = 1, 2, \dots, d\}.$$

- Equation on O :

$$\left(\frac{1}{2}\Delta - \lambda\right)v(x) + \sum_{i=1}^d b_i(x) \frac{\partial v(x)}{\partial x_i} + \ell(x) = 0.$$

- Dirichlet data on ∂O :

$$v(x) = g(x).$$

1.2. Examples.

1.2.1. *Multidimensional PDE with quadratic function as its solution.* Consider a class of PDE with coefficients satisfying,

$$d - \lambda \|x - \frac{1}{2}\mathbf{1}\|_2^2 + b(x) \cdot (2x - \mathbf{1}) + \ell(x) = 0,$$

where $\mathbf{1}$ is an \mathbb{R}^d -vector with each element being 1. The exact solution is

$$v(x) = \|x - \frac{1}{2}\mathbf{1}\|_2^2 = \sum_{i=1}^d (x_i - \frac{1}{2})^2.$$

2. DISCRETIZATION

2.1. **FDM.** We introduce some notions of finite difference operators. Commonly used first order finite difference operators are FFD, BFD, and CFD. Forward Finite Difference (FFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{he_i} v(x) := \frac{v(x + he_i) - v(x)}{h}.$$

Backward Finite Difference (BFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{-he_i} v(x) := \frac{v(x - he_i) - v(x)}{-h}.$$

Central Finite Difference (CFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{\pm he_i} v(x) := \frac{1}{2}(\delta_{-he_i} + \delta_{he_i})v(x) = \frac{v(x + he_i) - v(x - he_i)}{2h}.$$

Second order finite difference operators are the followings:

$$\frac{\partial^2}{\partial x_i^2} v(x) \approx \delta_{-he_i} \delta_{he_i} v(x) = \frac{v(x + he_i) - 2v(x) + v(x - he_i)}{h^2}.$$

Although the next operator will not be used below, we will write it for its completeness. If $i \neq j$, we use

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} v(x) &\approx \delta_{\pm he_i} \delta_{\pm he_j} v(x) \\ &= \frac{v(x + he_i + he_j) - v(x + he_i - he_j) - v(x - he_i + he_j) + v(x - he_i - he_j)}{4h^2}. \end{aligned}$$

2.2. CFD on PDE. Approximations for PDE are

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{\pm he_i} v(x)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-he_i} \delta_{he_i} v(x).$$

For simplicity, if we set

$$\gamma = \frac{d}{d + h^2 \lambda}, \quad p^h(x \pm he_i | x) = \frac{1}{2d} (1 \pm 2hb_i(x)), \quad \ell^h(x) = \frac{h^2 \ell(x)}{d},$$

then it yields DPP

$$v(x) = \gamma \left\{ \ell^h(x) + \sum_{i=1}^d p^h(x + he_i | x) v(x + he_i) + p^h(x - he_i | x) v(x - he_i) \right\}.$$

2.3. UFD on PDE. Upwind finite difference(UFD) is the following:

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{he_i} v(x) \cdot I(b_i(x) \geq 0) + \delta_{-he_i} v(x) \cdot I(b_i(x) < 0)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-he_i} \delta_{he_i} v(x).$$

Then, with

$$c = d + h \sum_i |b_i(x)|, \quad \gamma = \frac{c}{c + h^2 \lambda}, \quad \ell^h = \frac{\ell(x) h^2}{c}, \quad p^h(x \pm he_i | x) = \frac{1 + 2hb_i^\pm(x)}{2c},$$

then it yields DPP

$$v(x) = \gamma \left\{ \ell^h(x) + \sum_{i=1}^d p^h(x + he_i | x) v(x + he_i) + p^h(x - he_i | x) v(x - he_i) \right\}.$$