

MDP FROM THE DISCRETIZATION OF HJB

1. PROBLEM SETUP

1.1. **HJB.** We want to solve a d-dimensions HJB given below:

- Domain

$$O = \{x \in \mathbb{R}^d : \|x\|_1 < 1\}.$$

- Equation on O :

$$\left(\frac{1}{2}\Delta - \lambda\right)v(x) + \inf_a \left\{ \sum_{i=1}^d b_i(x, a) \frac{\partial v(x)}{\partial x_i} + \ell(x, a) \right\} = 0.$$

- Dirichlet data on ∂O :

$$v(x) = g(x).$$

1.2. **An example.** Consider

$$\frac{1}{2}\Delta v + \inf_{a \in \mathbb{R}^d} \left(a \cdot \nabla v + d + 2|x|^2 + \frac{1}{2}|a|^2 \right) = 0, \quad x \in O.$$

with

$$v(x) = -|x|^2, \quad x \in \partial O.$$

The exact solution is

$$v(x) = -|x|^2, \quad \text{with } a = 2x.$$

This means that the solution is invariant if $\inf_{a \in \mathbb{R}^d}$ is replaced by $\inf_{a \in 3O}$.

2. DISCRETIZATION

2.1. **FDM.** We introduce some notions of finite difference operators. Commonly used first order finite difference operators are FFD, BFD, and CFD. Forward Finite Difference (FFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{he_i} v(x) := \frac{v(x + he_i) - v(x)}{h}.$$

Backward Finite Difference (BFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{-he_i} v(x) := \frac{v(x - he_i) - v(x)}{-h}.$$

Central Finite Difference (CFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{\pm he_i} v(x) := \frac{1}{2}(\delta_{-he_i} + \delta_{he_i})v(x) = \frac{v(x + he_i) - v(x - he_i)}{2h}.$$

Second order finite difference operators are the followings:

$$\frac{\partial^2}{\partial x_i^2} v(x) \approx \delta_{-he_i} \delta_{he_i} v(x) = \frac{v(x + he_i) - 2v(x) + v(x - he_i)}{h^2}.$$

Although the operator below will not be used, we will write it for its completeness, if $i \neq j$,

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} v(x) &\approx \delta_{\pm he_i} \delta_{\pm he_j} v(x) \\ &= \frac{v(x + he_i + he_j) - v(x + he_i - he_j) - v(x - he_i + he_j) + v(x - he_i - he_j)}{4h^2}. \end{aligned}$$

2.2. CFD on HJB. Approximations for HJB are

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{\pm he_i} v(x)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-he_i} \delta_{he_i} v(x).$$

For simplicity, if we set

$$\gamma = \frac{d}{d + h^2 \lambda}, \quad p^h(x \pm he_i | x, a) = \frac{1}{2d} (1 \pm hb_i(x, a)), \quad \ell^h(x, a) = \frac{h^2 \ell(x, a)}{d},$$

then it yields DPP

$$v(x) = \gamma \inf_a \left\{ \ell^h(x, a) + \sum_{i=1}^d p^h(x + he_i | x) v(x + he_i) + p^h(x - he_i | x) v(x - he_i) \right\}.$$