MDP FROM THE DISCRETIZATION OF HJB

1. Problem setup

1.1. **HJB.** We want to solve a d-dimensions HJB given below:

- Domain
$$O = \{x \in \mathbb{R}^d : \|x\|_1 < 1\}.$$
- Equation on O:
$$(\frac{1}{2}\Delta - \lambda)v(x) + \inf_a \Big\{ \sum_{i=1}^d b_i(x,a)\frac{\partial v(x)}{\partial x_i} + \ell(x,a)\Big\} = 0.$$
- Dirichlet data on ∂O:
$$v(x) = g(x).$$

1.2. **Examples.**

1.2.1. *1-d HJB: Whittles Flypaper.* This example is taken from the reference *P97, Example 4 of [Roger and Williams 2000]* We want to solve HJB

$$\inf_a\{b(x,a)v'(x) + \frac{1}{2}\sigma^2 v''(x) - \lambda v(x) + \ell(x,a)\} = 0, \quad \text{on } O = (l,u)$$

with Dirichlet data

$$v(x) = g(x), \quad x = l, u.$$

Let parameters be given by

$$O = (0,z),\ \sigma = 1,\ b(x,a) = a,\ \lambda = 0,\ \ell(x,a) = \frac{1}{2}(a^2+1),\ g(x) = -\ln(c_1e^x + c_2e^{-x}).$$

The value function and the optimal policy are

$$v(x) = g(x),\ a^*(x) = -g'(x).$$

Ex. In the above Whittle's "flypaper", answer the following questions:

- show that v is concave.
- show that the optimal policy $|a^*(x)| \leq 1$.
- solve for the exact solution for terminal cost given by
$$g(x) = x^2.$$

1.2.2. *Multidimensional HJB with quadratic function as its solution.* Consider

$$\frac{1}{2}\Delta v + \inf_{a\in\mathbb{R}^d}\Big(a\cdot\nabla v + d + 2|x|^2 + \frac{1}{2}|a|^2\Big) = 0,\ x\in O.$$

with

$$v(x) = -|x|^2,\ x\in\partial O.$$

The exact solution is

$$v(x) = -|x|^2,\quad \text{with } a = 2x.$$

This means that the solution is invariant if $\inf_{a\in\mathbb{R}^d}$ is replaced by $\inf_{a\in 3O}$.

2. Discretization

2.1. **FDM.** We introduce some notions of finite difference operators. Commonly used first order finite difference operators are FFD, BFD, and CFD. Forward Finite Difference (FFD) is

$$\frac{\partial}{\partial x_i}v(x) \approx \delta_{he_i}v(x) := \frac{v(x+he_i)-v(x)}{h}.$$

Backward Finite Difference (BFD) is

$$\frac{\partial}{\partial x_i}v(x) \approx \delta_{-he_i}v(x) := \frac{v(x-he_i)-v(x)}{-h}.$$

Central Finite Difference (CFD) is

$$\frac{\partial}{\partial x_i}v(x) \approx \delta_{\pm he_i}v(x) := \frac{1}{2}(\delta_{-he_i}+\delta_{he_i})v(x) = \frac{v(x+he_i)-v(x-he_i)}{2h}.$$

Second order finite difference operators are the followings:

$$\frac{\partial^2}{\partial x_i^2}v(x) \approx \delta_{-he_i}\delta_{he_i}v(x) = \frac{v(x+he_i)-2v(x)+v(x-he_i)}{h^2}.$$

Although the next operator will not be used below, we will write it for its completeness. If $i\neq j$, we use

$$\begin{aligned}\tfrac{\partial^2}{\partial x_i\partial x_j}v(x) &\approx \delta_{\pm he_i}\delta_{\pm he_j}v(x)\\ &= \frac{v(x+he_i+he_j)-v(x+he_i-he_j)-v(x-he_i+he_j)+v(x-he_i-he_j)}{4h^2}.\end{aligned}$$

2.2. **CFD on HJB.** Approximations for HJB are

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{\pm he_i}v(x)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-he_i}\delta_{he_i}v(x).$$

For simplicity, if we set

$$\gamma = \frac{d}{d+h^2\lambda},\ p^h(x\pm he_i|x,a) = \frac{1}{2d}(1\pm hb_i(x,a)),\ \ell^h(x,a) = \frac{h^2\ell(x,a)}{d},$$

then it yields DPP

$$v(x) = \gamma \inf_a \Big\{ \ell^h(x,a) + \sum_{i=1}^d p^h(x+he_i|x)v(x+he_i) + p^h(x-he_i|x)v(x-he_i) \Big\}.$$