MDP FROM THE DISCRETIZATION OF HJB

1. Problem setup

1.1. HJB. We want to solve a d-dimensions HJB given below:

- Domain
$$O = \{x \in \mathbb{R}^d : \|x\|_2 < 1\}.$$
- Equation on O:
$$(\frac{1}{2}\Delta - \lambda)v(x) + \inf_a \Big\{ \sum_{i=1}^d b_i(x,a) \frac{\partial v(x)}{\partial x_i} + \ell(x,a) \Big\} = 0.$$
- Dirichlet data on ∂O:
$$v(x) = g(x).$$

1.2. An example. Consider
$$\frac{1}{2}\Delta v + \inf_{a\in\mathbb{R}^d} \Big(a \cdot \nabla v + d + 2|x|^2 + \frac{1}{2}|a|^2 \Big) = 0,\ x \in O.$$
with
$$v(x) = -|x|^2,\ x \in \partial O.$$
The exact solution is
$$v(x) = -|x|^2.$$

2. Discretization

2.1. FDM. We introduce some notions of finite difference operators. Commonly used first order finite difference operators are FFD, BFD, and CFD. Forward Finite Difference (FFD) is
$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{he_i} v(x) := \frac{v(x + he_i) - v(x)}{h}.$$
Backward Finite Difference (BFD) is
$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{-he_i} v(x) := \frac{v(x - he_i) - v(x)}{-h}.$$
Central Finite Difference (CFD) is
$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{\pm he_i} v(x) := \frac{1}{2}(\delta_{-he_i} + \delta_{he_i})v(x) = \frac{v(x + he_i) - v(x - he_i)}{2h}.$$
Second order finite difference operators are the followings:
$$\frac{\partial^2}{\partial x_i^2} v(x) \approx \delta_{-he_i}\delta_{he_i} v(x) = \frac{v(x + he_i) - 2v(x) + v(x - he_i)}{h^2}.$$

Although the operator below will not be used, we will write it for its completeness, if $i \neq j$,

$$\begin{aligned}\frac{\partial^2}{\partial x_i \partial x_j} v(x) &\approx \delta_{\pm h e_i} \delta_{\pm h e_j} v(x) \\ &= \frac{v(x + he_i + he_j) - v(x + he_i - he_j) - v(x - he_i + he_j) + v(x - he_i - he_j)}{4h^2}.\end{aligned}$$

2.2. **CFD on HJB.** Approximations for HJB are

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{\pm h e_i} v(x)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-h e_i} \delta_{h e_i} v(x).$$

For simplicity, if we set

$$\gamma = \frac{d}{d + h^2 \lambda}, \ p^h(x \pm he_i | x, a) = \frac{1}{2d}(1 \pm h b_i(x, a)), \ \ell^h(x, a) = \frac{h^2 \ell(x, a)}{d},$$

then it yields DPP

$$v(x) = \gamma \inf_a \Big\{ \ell^h(x, a) + \sum_{i=1}^d p^h(x + he_i | x) v(x + he_i) + p^h(x - he_i | x) v(x - he_i) \Big\}.$$