

# MDP FROM THE DISCRETIZATION OF HJB

## 1. PROBLEM SETUP

1.1. **HJB.** We want to solve a d-dimensions HJB given below:

- Domain

$$O = \{x \in \mathbb{R}^d : 0 < x_i < 1, i = 1, 2, \dots, d\}.$$

- Equation on  $O$ :

$$\left(\frac{1}{2}\Delta - \lambda\right)v(x) + \inf_a \left\{ \sum_{i=1}^d b_i(x, a) \frac{\partial v(x)}{\partial x_i} + \ell(x, a) \right\} = 0.$$

- Dirichlet data on  $\partial O$ :

$$v(x) = g(x).$$

## 1.2. Examples.

1.2.1. *1-d HJB: Whittles Flypaper.* This example is taken from the reference *P97, Example 4 of [Roger and Williams 2000]* We want to solve HJB

$$\inf_a \{b(x, a)v'(x) + \frac{1}{2}\sigma^2 v''(x) - \lambda v(x) + \ell(x, a)\} = 0, \quad \text{on } O = (l, u)$$

with Dirichlet data

$$v(x) = g(x), \quad x = l, u.$$

Let parameters be given by

$$O = (0, z), \quad \sigma = 1, \quad b(x, a) = a, \quad \lambda = 0, \quad \ell(x, a) = \frac{1}{2}(a^2 + 1), \quad g(x) = -\ln(c_1 e^x + c_2 e^{-x}).$$

The value function and the optimal policy are

$$v(x) = g(x), \quad a^*(x) = -g'(x).$$

**Ex.** In the above Whittle's "flypaper", answer the following questions:

- show that  $v$  is concave.
- show that the optimal policy  $|a^*(x)| \leq 1$ .
- solve for the exact solution for terminal cost given by

$$g(x) = x^2.$$

1.2.2. *Multidimensional HJB with quadratic function as its solution.* Consider

$$\frac{1}{2}\Delta v + \inf_{a \in \mathbb{R}^d} \left( a \cdot \nabla v + d + 2|x|^2 + \frac{1}{2}|a|^2 \right) = 0, \quad x \in O.$$

with

$$v(x) = -|x|^2, \quad x \in \partial O.$$

The exact solution is

$$v(x) = -|x|^2, \quad \text{with } a = 2x.$$

This means that the solution is invariant if  $\inf_{a \in \mathbb{R}^d}$  is replaced by  $\inf_{a \in 3O}$ .

## 2. DISCRETIZATION

2.1. **FDM.** We introduce some notions of finite difference operators. Commonly used first order finite difference operators are FFD, BFD, and CFD. Forward Finite Difference (FFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{he_i} v(x) := \frac{v(x + he_i) - v(x)}{h}.$$

Backward Finite Difference (BFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{-he_i} v(x) := \frac{v(x - he_i) - v(x)}{-h}.$$

Central Finite Difference (CFD) is

$$\frac{\partial}{\partial x_i} v(x) \approx \delta_{\pm he_i} v(x) := \frac{1}{2}(\delta_{-he_i} + \delta_{he_i})v(x) = \frac{v(x + he_i) - v(x - he_i)}{2h}.$$

Second order finite difference operators are the followings:

$$\frac{\partial^2}{\partial x_i^2} v(x) \approx \delta_{-he_i} \delta_{he_i} v(x) = \frac{v(x + he_i) - 2v(x) + v(x - he_i)}{h^2}.$$

Although the next operator will not be used below, we will write it for its completeness.

If  $i \neq j$ , we use

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} v(x) &\approx \delta_{\pm he_i} \delta_{\pm he_j} v(x) \\ &= \frac{v(x + he_i + he_j) - v(x + he_i - he_j) - v(x - he_i + he_j) + v(x - he_i - he_j)}{4h^2}. \end{aligned}$$

2.2. **CFD on HJB.** Approximations for HJB are

$$\frac{\partial v(x)}{\partial x_i} \leftarrow \delta_{\pm he_i} v(x)$$

and

$$\frac{\partial^2 v(x)}{\partial x_i^2} \leftarrow \delta_{-he_i} \delta_{he_i} v(x).$$

For simplicity, if we set

$$\gamma = \frac{d}{d + h^2 \lambda}, \quad p^h(x \pm he_i | x, a) = \frac{1}{2d}(1 \pm 2hb_i(x, a)), \quad \ell^h(x, a) = \frac{h^2 \ell(x, a)}{d},$$

then it yields DPP

$$v(x) = \gamma \inf_a \left\{ \ell^h(x, a) + \sum_{i=1}^d p^h(x + he_i | x, a) v(x + he_i) + p^h(x - he_i | x, a) v(x - he_i) \right\}.$$