

We are going to provide an explicit solution for the LQG problem in one dimensional setting. This will go over HJB and Ricatti equations.

1 Problem setting

We fix

$$\ell(a) = \frac{1}{2}a^2 \text{ and } g(x) = \frac{1}{2}x^2.$$

The controlled process is given by

$$dX_t = q(t)dt + dW(t),$$

where $q(\cdot)$ is an adapted control process and $W(\cdot)$ is a standard Brownian motion.

The value function is given by

$$v(t, x) = \sup_{q(\cdot)} \mathbb{E}^{x, t} \left[\int_t^T -\ell(q(s))ds - g(X(T)) \right].$$

Our goal is to solve for $(v, \hat{q}(\cdot))$, where $\hat{q}(\cdot)$ is the optimal control.

2 Analysis

The first attempt is to write down HJB equation and try to solve it due to the following verification theorem.

Proposition 1

If $u \in C^{2,1}$ solves HJB(u) equation

$$\begin{cases} \partial_t u + \frac{1}{2}\partial_{xx}u + H(\partial_x u) = 0, & \text{on } \mathbb{R} \times (0, T), \\ u(T, x) = -g(x) & \text{on } \mathbb{R}, \end{cases}$$

then $u = v$ and $\hat{q}(t) = \partial_x u(X_t, t)$. In the above, H is the Legendre transform of ℓ , i.e.

$$H(p) = \sup_q (q \cdot p - \ell(q)) = \frac{1}{2}p^2.$$

PROOF: later ?? \square

Next, we shall search for $\partial_x u$. To do that, we differentiate HJB(u) by ∂_x , and substitute $w = \partial_x u$ to obtain HJB(w),

$$\begin{cases} \partial_t w + \frac{1}{2}\partial_{xx}w + w\partial_x w = 0, & \text{on } \mathbb{R} \times (0, T), \\ w(T, x) = -g'(x) & \text{on } \mathbb{R}, \end{cases}$$

The above equation can lead to adjoint process in terms of Backward SDE, which is crucial in stochastic maximum principle. However, we shall not get into BSDE in this problem, and use it directly in this below.

From the form of $w(T, x) = -x$, we bravely guess

$$w(t, x) = \phi(t)x$$

for some negative $\phi(t)$, and plug into HJB(w)

$$\phi' + \phi^2 = 0, \text{ on } (0, T), \text{ and } \phi(T) = -1.$$

This is called Ricatti equation. We can explicitly write for ϕ and w as

$$\phi(t) = -\frac{1}{1+T-t}, \text{ and } w(x, t) = -\frac{x}{1+T-t} = \partial_x u. \quad (1)$$

This means ϕ is explicitly solvable, but we do not need the explicit form until the end. Starting from $w = \phi x$, we can write u by

$$u(x, t) = \psi(t) + \int_0^x w(y, t) dy = \psi(t) + \frac{1}{2} \phi(t) x^2.$$

Plugging this form into HJB, it yields

$$\psi'(t) + \frac{1}{2} \phi(t) = 0, \text{ and } \psi(T) = 0.$$

Now we invoke explicit ϕ to obtain ψ as

$$\psi(t) = -\frac{1}{2} \ln(1+T-t). \quad (2)$$

One can easily verify verification theorem, and reach the following answer.

Proposition 2

Let $\tau := T - t$. Then, the value and optimal control are given as

$$v(x, t) = \frac{-x^2}{2(1+\tau)} - \frac{1}{2} \ln(1+\tau), \quad \hat{q}(t) = \frac{-1}{1+\tau} \hat{X}_t.$$

In particular, the optimal trajectory \hat{X} follows O-U process, i.e.

$$d\hat{X}_t = \frac{-1}{1+\tau} \hat{X}_t dt + dW_t, \quad \hat{X}_0 = x.$$

Therefore, $\hat{X}_T^{x,0} \rightarrow 0$ in L^2 if $T \rightarrow \infty$, i.e.

$$\lim_{T \rightarrow \infty} \mathbb{E}^{x,0} (\hat{X}_T)^2 = 0.$$

See Example 1 of [\[here\]](#) about O-U process.

Remark 3

1. HJB(w) is Burger's equation.

Ex.

Consider controlled process

$$dX_t = 2\lambda q_t dt + \sigma dW_t$$

and value function

$$v(t, x) = \inf_q \mathbb{E}^{t,x} \left[\int_t^T q_t^2 dt + cX_T^2 \right].$$

Given that $\lambda, \sigma, c > 0$, prove that

$$v(t, x) = \frac{cx^2}{1 + 4\lambda^2 c(T - t)} + \sigma^2 c \ln(1 + 4\lambda^2 c(T - t))$$

and

$$q^*(t, x) = -\frac{2\lambda cx}{1 + 4c\lambda^2(T - t)}.$$