Long-time behavior of stochastic LQ control problem

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Abstract

This paper investigates the asymptotic behavior of the linear quadratic stochastic optimal control problems. By establishing a connection between the ergodic cost problem and the cell problem within weak KAM theory, we reveal turnpike properties from various perspectives.

1 Introduction

The research into the turnpike property originated in economics as a means of examining the stationary behavior during the transient time for long-horizon control problems. It was first proposed by von Neumann [18], and the terminology was introduced by Dorfman, Samuelson, and Solow [7]. The turnpike property refers to the situation where a solution to the optimization problem is concentrated in some static points which are evenly spaced along a specific path. Since then, the turnpike phenomenon has gained significant attention for finite and infinite dimensional problems in the context of deterministic discrete-time and continuous-time systems, see the book [5] as an excellent survey and a list of numerous references, including [16, 4, 14, 25, 19, 8, 26, 27, 28]. In particular, we mention the papers [19, 20] that discuss continuous-time linear quadratic (LQ) problems for ordinary differential equations (ODE) and the recent paper [22] by Sun, Wang, and Yong that addresses stochastic LQ optimal control problems. Moreover, Sun and Yong (see [23]) also established the exponential, integral, and mean-square turnpike properties for optimal pairs of mean-field linear stochastic differential equations, subject to the stabilizability condition for the state equation.

To be more specific, we consider a $\mathbb{R}^d$-valued standard Brownian motion $\{W(t)\}_{t\geq 0}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions. Let $T > 0$ be a given time horizon, $|\cdot|$ be the Euclidean norm of vectors in $\mathbb{R}^d$, and $\|\cdot\|_2$ be the spectral norm of matrices. We use $L^p_{\mathbb{F}}(\Omega \times [0, T])$ to denote the space of all $\mathbb{F}$-progressively measurable random processes $u = \{u(t)\}_{t\in[0,T]}$ satisfying $\mathbb{E}[\int_0^T |u(t)|^p dt] < \infty$. We also denote $\mathbf{O}_d$ to be the $d \times d$ matrix in which all the entries are 0 and $\mathbf{0}_d$ to be the d-dimensional column vector with all entries are 0. Consider the following diffusion process given by a linear stochastic differential equation (SDE)

$$dX(t) = (AX(t) + u(t) + b)\, dt + \sigma dW(t), \quad X(0) = x, \quad t \geq 0, \tag{1}$$

where $A \in \mathbb{S}^d$ is a $d \times d$ symmetric constant matrix, $b, x \in \mathbb{R}^d$ are constant vectors, and $\sigma \in \mathbb{R}^+$ is a positive constant.

The classical stochastic LQ control problem over the finite time horizon $[0, T]$ is to find an optimal control u^*_T from the space $\mathcal{U}_{[0,T]} := L^2_{\mathbb{F}}(\Omega \times [0, T])$ such that the quadratic cost functional

$$J_T(x; u_T) = \mathbb{E}\left[\int_0^T L(X(t), u_T(t))dt + g(X(T))\right] \tag{2}$$

is minimized for a given initial state $x \in \mathbb{R}^d$, i.e.,

$$V_T(x) := J_T\left(x; u_T^*\right) = \inf_{u_T \in \mathcal{U}_{[0,T]}} J_T(x; u_T), \tag{3}$$

where

$$L(x,u) = \frac{1}{2}\left(x^\top Q x + |u|^2\right) + q^\top x + r^\top u$$

and

$$g(x) = \frac{1}{2} x^\top N_1 x + x^\top N_2 + N_3$$

with Q, N_1 are a positive definite matrix in $\mathbb{S}^d$, $q, r, N_2 \in \mathbb{R}^d$, and $N_3 \in \mathbb{R}$. The corresponding optimal path is denoted by $X_T^*(t)$ for $t \in [0,T]$. Note that, we could set Q and N_1 be in $\mathbb{R}^{d\times d}$ in general.

The turnpike property of the above finite time control problem is associated with the following static optimization problem: Determine the point $(\hat{x}, \hat{u})$ to

$$\begin{cases} \text{minimize} \quad F(x,u) := \frac{1}{2}\left(x^\top Q x + |u|^2 + 2q^\top x + 2r^\top u\right) + \frac{1}{2}\sigma^2 \text{trace}(P), \\ \text{subject to} \quad Ax + u + b = \mathbf{0}_d, \end{cases} \tag{4}$$

where P is a positive definite solution to

$$P^2 - 2AP - Q = \mathbf{O}_d.$$

If the underlying control problem is deterministic, i.e., $\sigma = 0$, as it is shown in [19], the turnpike property refers to the following estimation: There exist some $\lambda > 0$ and $K > 0$ independent of t and T such that

$$|X_T^*(t) - \hat{x}| + |u_T^*(t) - \hat{u}| \le K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0,T].$$

Indeed, this turnpike property reveals that, for a sufficiently large T, one can achieve a good approximation of the optimal trajectory during the majority time period $[\delta T, (1-\delta)T]$ for some $0 < \delta << 1/2$ by simply staying at the stable point $\hat{x}$ of the static optimization problem in the sense

$$|X_T^*(t) - \hat{x}| + |u_T^*(t) - \hat{u}| \le 2Ke^{-\lambda\delta T}, \quad \forall t \in [\delta T, (1-\delta)T],\ \delta \in (0, 1/2).$$

However, extending this turnpike property to the stochastic control problem with $\sigma > 0$ poses challenges. This is because the presence of Brownian noise makes it impossible for any control to freeze the state unchanged at a fixed point. Recently, Sun and Yong (see Theorem 3.2 of [23]) proved the following (stochastic version) turnpike property

$$\mathbb{E}\left[|X_T^*(t) - X^*(t)|^2 + |u_T^*(t) - u^*(t)|^2\right] \le K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0,T] \tag{5}$$

by constructing two stochastic processes X^* and u^*, independent to T, satisfying $\mathbb{E}[X^*(t)] = \hat{x},\ \mathbb{E}[u^*(t)] = \hat{u}$.

In this study, we aim to reexamine the turnpike property, approaching it from a distinct perspective: the cell problem within the framework of weak Kolmogorov–Arnold–Moser (KAM) theory. Specifically, the objective of the cell problem is to seek the solution $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ to the following equation

$$c_0 = H\left(x, -\nabla v(x), -D^2 v(x)\right), \tag{6}$$

where the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ is given by

$$H(x, \bar{p}, \bar{q}) := \sup_{u\in\mathbb{R}^d} \left\{ (Ax + u + b)^\top \bar{p} + \frac{1}{2}\sigma^2 \mathrm{trace}(\bar{q}) - L(x, u) \right\}.$$

Note that uniqueness does not apply to the cell problem, as (v, c_0) is a solution if and only if $(v + k, c_0)$ is also a solution for any $k \in \mathbb{R}$. Therefore, when we refer to the uniqueness of the cell problem, we specifically mean uniqueness in the value of c_0.

Initially, the weak KAM Theory was developed by Fathi [11] and Mather [15], and is linked to the theory of homogenization for Hamilton-Jacobi (HJ) equations developed by Lions, Papanicolaou, and Varadhan in [13]. It provides the connection between a type of control problem and the cell problem, offering a representation of the optimal ergodic cost. In addition to its fundamental role in the theory of homogenization, the weak KAM theory has also been used to study the long-time behavior of dynamic control problems, including the ergodic behavior of the value function and the corresponding HJ equation in the deterministic case (see [17, 3, 9, 10, 24]), as well as the Hamilton-Jacobi-Bellman (HJB) equation in the stochastic case ([2, 12, 6]).

Compared to the above literature, this paper provides a distinct approach to show the turnpike properties in stochastic control theory by using the cell problem in PDE and contributions can be summarized as follows. Our first contribution lies in the formulation of a verification theorem connecting the cell problem to a specific class of infinite time horizon control problems, referred to as the probabilistic cell problem, see Lemma 4. Unlike the typical cell problem explored in the existing literature (e.g., [24]), the underlying cell problem in our context lacks uniqueness due to the non-compactness of the domain. It is the verification theorem, which establishes a tailored sufficient condition to distinguish the right solution to the probabilistic cell problem from multiple solutions of the cell problem. An immediate consequence of the verification theorem is the establishment of a link between the cell problem and the static optimization problem.

Our second contribution provides the connection between the probabilistic cell problem and the ergodic cost problem (refer to Remark 3.6.7 in [1]), which involves determining the constant

$$-c_* := \lim_{T\to\infty} \frac{1}{T} V_T(x) = \lim_{T\to\infty} \frac{1}{T} J_T(x; u_T^*). \tag{7}$$

The importance of this connection is that it unveils a new turnpike property in terms of the cost function in addition to the aforementioned turnpike property of (5) with respect to the control process and state process:

$$\lim_{T\to\infty} \frac{1}{T} J_T(x; u^*) = -c_*, \tag{8}$$

where u^* is a T-independent control process obtained from the probabilistic cell problem, see Theorem 2.

The five problems, namely, the Finite Time Stochastic Control Problem (3), the Static Optimization Problem (4), the Cell Problem (6), the Ergodic Cost Problem (7), and the Probabilistic Cell Problem introduced in this Section 1 will be interwoven throughout the remainder of this manuscript in the following manner: In Section 2, we prove the verification of the cell problem and further provide the consistency of the cell problem and the static optimization problem. Additionally, we identify the turnpike properties of (5) and (8) in Section 3. To further elucidate the results obtained in Section 3, we provide an illustrative example in Section 4. Proofs of certain lemmas are collected in Appendix 5.

Throughout the paper, we use $\mathbb{S}^d$ to denote the space of all $d \times d$ symmetric matrices, and $A \geq 0$ ($A > 0$) to denote the positive semidefinite (definite) matrix. If $A \geq 0$, then $\sqrt{A}$ denotes the unique $B \geq 0$ satisfying $B^2 = A$. We also use $\mathbf{O}_d$ denote the $d \times d$ matrix in which all the entries are 0 and $\mathbf{0}_d$ to be the d-dimensional column vector with all entries are 0. We also pose the following assumptions to coefficients:

(Cf) $A, Q, N_1 \in \mathbb{S}^d$. $b, x, q, r, N_2 \in \mathbb{R}^d$ are constant vectors, $\sigma \in \mathbb{R}^+$ is a positive constant, and $N_3 \in \mathbb{R}$ is a constant. Moreover,

$$Q > 0 \text{ and } A - \sqrt{A^2 + Q} < N_1 < A + \sqrt{A^2 + Q}.$$

In addition, we will use the following notations:

$$Z_1 = A + \sqrt{A^2 + Q},\ Z_2 = (Z_1 - A)^{-1}(Z_1 b - Z_1 r + q),$$

and

$$D_1 = Z_1 - A,\ D_2 = b - r - Z_2.$$

Notice that, from the above settings, it satisfies $D_1 > 0$ and $0 < Z_1 - N_1 < 2D_1$.

2 Cell problem and the verification theorem

We commence our exploration with the solvability of the cell problem (6) within the framework of weak KAM theory. In prevailing literature, the solution of the cell problem is unique up to the constant. Interestingly, this uniqueness fails in our framework attributed to the absence of compactness in the domain. This necessitates the formulation of a meticulously crafted verification theorem: How can we discern the appropriate solution from the multitude available to accurately characterize the associated optimal control problem?

2.1 Existence and nonuniqueness of the cell problem

We recall the cell problem: Find the solution $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ to the following equation

$$c_0 = H\left(x, -\nabla v(x), -D^2 v(x)\right),$$

where the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ is given by

$$H(x, \bar{p}, \bar{q}) := \sup_{u\in\mathbb{R}^d} \left\{(Ax + u + b)^\top \bar{p} + \frac{1}{2}\sigma^2 \text{trace}(\bar{q}) - L(x, u)\right\} := \sup_{u\in\mathbb{R}^d} H^u(x, \bar{p}, \bar{q}). \tag{9}$$

Lemma 1. *The cell problem* (6) *can be solved by* (v, c_0) *in the form of*

$$v(x) = \frac{1}{2}x^\top Z_1 x + x^\top Z_2 + Z_3$$

and

$$-c_0 = \frac{1}{2}\sigma^2 \mathit{trace}(Z_1) + b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2, \tag{10}$$

where $(Z_1, Z_2) \in \mathbb{S}^d \times \mathbb{R}^d$ *can be any solution pair to the system of equations*

$$\begin{cases} Z_1^2 - 2AZ_1 - Q = \mathbf{O}_d, \\ (Z_1 - A)\, Z_2 - Z_1 b + Z_1 r - q = \mathbf{0}_d. \end{cases} \tag{11}$$

Proof. Assume (v, c_0) solves the cell problem (6) with v satisfying a quadratic form $v(x) = \frac{1}{2}x^\top Z_1 x + x^\top Z_2 + Z_3$. Note that Z_1 is symmetric, then $\nabla v(x) = Z_1 x + Z_2$ and $D^2 v(x) = Z_1$, and thus the cell problem (6) can be rewritten by $H(x, -(Z_1 x + Z_2), -Z_1) = c_0$. We first observe that

$$H(x, \bar{p}, \bar{q}) = (Ax + \bar{u}(x, \bar{p}) + b)^\top \bar{p} + \frac{1}{2}\sigma^2 \text{trace}(\bar{q}) - L(x, \bar{u}(x, \bar{p})),$$

where

$$\bar{u}(x, \bar{p}) = \bar{p} - r,$$

thus the cell problem can be reduced to

$$\begin{aligned} c_0 = &- (Ax - Z_1 x + b - r - Z_2)^\top (Z_1 x + Z_2) - \frac{1}{2}\sigma^2 \text{trace}(Z_1) - \frac{1}{2}x^\top Q x \\ &- \frac{1}{2}|Z_1 x + Z_2 + r|^2 - q^\top x + r^\top (Z_1 x + Z_2 + r) \end{aligned} \tag{12}$$

for all $x \in \mathbb{R}^d$. By setting the linear and quadratic terms with respect to x on the right hand side of (12) to 0, we obtain the system of equations (11). Combining like terms for the constants in (12) provides the expression c_0 of (10). □

The uniqueness of c_0 in the cell problem is established by Theorem 4.2 in [24] for the periodic domain. However, it is noteworthy that the uniqueness does not extend to our setting defined by equation (6). Recall that, by spectral theorem, if $A > 0$, it admits orthogonal decomposition $A = QDQ^\top$ for some orthogonal matrix Q and diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ with $\lambda_i > 0$. Moreover, any matrix in the form of

$$B = Q\text{diag}(\pm\sqrt{\lambda}_1, \pm\sqrt{\lambda}_2, \ldots, \pm\sqrt{\lambda}_d)Q^\top$$

satisfies $B^2 = A$. In this below, $\sqrt{A}$ is the unique choice of B satisfying $B > 0$ and $B^2 = A$. Also, $\sqrt{A}$ can be represented by $QD^{1/2}Q^\top$.

Lemma 2. *There exists multiple solution pairs (Z_1, Z_2) of (11) in $\mathbb{S}^d \times \mathbb{R}^d$, hence the c_0 of the cell problem (6) is not unique. Moreover, there exists unique solution pair (Z_1, Z_2) satisfying*

$$D_1 = Z_1 - A > 0.$$

Moreover, Z_1 and D_1 are commutative, hence they share the same eigenvector matrix.

Proof. Taking transpose to the first equation of (11), we have $Z_1^2 - 2Z_1 A - Q = \mathbf{O}_d$ as Z_1, A and Q are symmetric matrices, which implies that Z_1 and A are commutative, i.e., $Z_1 A = AZ_1$. Thus, $(Z_1 - A)^2 = A^2 + Q$ holds. By spectral theorem, since $A^2 + Q > 0$, there exists multiple solutions to Z_1 and the unique choice of Z_1 to have $D_1 = Z_1 - A > 0$ is

$$Z_1 = A + \sqrt{A^2 + Q}.$$

Accordingly, from the second equation of (11), Z_2 can be written in terms of Z_1 as

$$Z_2 = (Z_1 - A)^{-1}(Z_1 b - Z_1 r + q).$$

At the end, $D_1 Z_1 = Z_1 D_1 = Z_1^2 - AZ_1$ holds by commutativity of A and Z_1, thus they have the same eigenvector matrix by Page 305 of [21].

□

2.2 Verification theorem to probabilistic cell problem

The lack of uniqueness in the determination of c_0 as indicated by Lemma 2 introduces a compelling challenge when striving to identify the optimal value for c_0 in order to substantiate the verification theorem associated with its probabilistic counterparts. In the following discussion, our objective is to delineate the connection between the cell problem (6) and the probabilistic cell problem (13)-(14), which is referred to the verification procedure in the context of the control theory.

2.2.1 Probabilistic cell problem

We consider the following probabilistic cell problem.

Let $\mathcal{P}_2(\mathbb{R}^d)$ be the Wasserstein space of probability measures μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$ endowed with 2-Wasserstein metric $\mathcal{W}_2(\cdot, \cdot)$ defined by

$$\mathcal{W}_2(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_1, \mu_2)} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 d\pi(x, y) \right)^{\frac{1}{2}},$$

where $\Pi(\mu_1, \mu_2)$ is the collection of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with its marginals agreeing with μ_1 and μ_2. Moreover, we denote that $\langle \phi, \mu \rangle$ to be

$$\langle \phi, \mu \rangle = \int_{\mathbb{R}^d} \phi(x) \mu(dx)$$

for all function ϕ valued on $\mathbb{R}^d$ and all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. To proceed, we define $\mathcal{U}$ as the collection of all $\mathbb{F}$ progressively measurable processes such that

- its associated state process $X^u = X$ given by

  $$dX(t) = (AX(t) + u(t) + b)\, dt + \sigma dW(t), \quad X(0) = x, \quad t \geq 0,$$

  is well-defined;
- $\mathbb{E}[|X(t)|^2] < \infty$ for all $t > 0$ and $x \in \mathbb{R}^d$;
- the law of $(X(t), u(t))$ converges to some distribution $\mu_\infty \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ in 2-Wasserstein metric $\mathcal{W}_2$, i.e.,

  $$\lim_{t \to \infty} \mathcal{W}_2(Law(X(t), u(t)), \mu_\infty) = 0.$$

We define the probabilistic cell problem below: Determine (V, c) such that

$$-c = \inf_{u \in \mathcal{U}} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}\left[L(X(t), u(t))\right] dt, \tag{13}$$

and

$$V(x) = \inf_{u \in \mathcal{U}} \limsup_{T \to \infty} \int_0^T \mathbb{E}\left[L(X(t), u(t)) + c\right] dt. \tag{14}$$

Following the convention in stochastic control theory, we denote $V(x)$ as the value function of the probabilistic cell problem, provided it is well-defined. However, unlike the standard control problem, the objective of the probabilistic cell problem is not solely to identify a value function V; rather, it involves determining a pair (V, c) among many solutions of the cell problem (6).

2.2.2 Verification for a general setting

In this part, we will prove a verification for an infinite-time control problem under a general settings and all notations are independent to the rest of the paper.

The cell problem in the general setting is to find the pair $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ such that

$$c_0 = H\left(x, -\nabla v(x), -D^2 v(x)\right) := \sup_a H^a\left(x, -\nabla v(x), -D^2 v(x)\right), \tag{15}$$

where $H^a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ for any $a \in \mathbb{R}^d$ is given by

$$H^a(x, \bar{p}, \bar{q}) = \hat{b}(x, a) \cdot \bar{p} - \hat{L}(x, a) + \frac{1}{2}\hat{\sigma}^2 \text{trace}(\bar{q})$$

for some $\hat{b} : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\hat{\sigma} \in \mathbb{R}$, and $\hat{L} : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$.

Assumption 1. *$\hat{b}$ is Lipschitz continuous on $\mathbb{R}^d \times \mathbb{R}^d$, and $\hat{L}$ is locally Lipschitz continuous and satisfies quadratic growth, i.e., for all $(x, a) \in \mathbb{R}^d \times \mathbb{R}^d$,*

$$\left|\hat{L}(x, a)\right| \leq K\left(1 + |x|^2 + |a|^2\right) \text{ for some } K \in \mathbb{R}.$$

At the same time, we define its associated probabilistic cell problem: Consider $\mathbb{R}^d$-valued controlled process X, with a given $\mathbb{R}^d$-Brownian motion $W(t)$, given by

$$dX(t) = \hat{b}\left(X(t), u(t)\right) dt + \hat{\sigma} dW(t), \quad X(0) = x \in \mathbb{R}^d, \quad t \geq 0.$$

The objective is to determine (V, c) such that

$$-c = \inf_{u\in\mathcal{U}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\left[\hat{L}(X(t), u(t))\right] dt, \tag{16}$$

and

$$V(x) = \inf_{u\in\mathcal{U}} \limsup_{T\to\infty} \int_0^T \mathbb{E}\left[\hat{L}(X(t), u(t)) + c\right] dt. \tag{17}$$

Lemma 3. *Let $\mathcal{U}[\mu_\infty]$ be the collection of all control processes $u \in \mathcal{U}$ such that the law of $(X(t), u(t))$ converges to μ_∞. For an arbitrary control process $u \in \mathcal{U}[\mu_\infty]$ and any function $\phi : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ with a quadratic growth $|\phi(x, a)| \leq K(1 + |x|^2 + |a|^2)$ for all $x, a \in \mathbb{R}^d$, we have*

$$\lim_{t\to\infty} \mathbb{E}\left[\phi(X(t), u(t))\right] = \langle \phi, \mu_\infty \rangle,$$

and

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\left[\phi(X(t), u(t))\right] dt = \langle \phi, \mu_\infty \rangle.$$

Proof. Since $\mu_t := Law((X(t), u(t)))$ converges to some $\mu_\infty \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ in 2-Wasserstein distance, we have

$$\lim_{t\to\infty} \mathbb{E}[|X(t)|^2 + |u(t)|^2] = \int_{\mathbb{R}^d\times\mathbb{R}^d} (|x|^2 + |a|^2) d\mu_\infty(x, a).$$

By Skorohod representation theorem, one can find another stochastic process $(Y(t), v(t))$ in a different probability space, such that $(Y(t), v(t)) \to (Y_\infty, v_\infty)$ almost surely, as well as, $Law((Y(t), v(t))) = \mu_t$ and $Law((Y_\infty, v_\infty)) = \mu_\infty$. Hence, by the fact of

$$\phi(Y(t), v(t)) \leq K\left(1 + |Y(t)|^2 + |v(t)|^2\right)$$

and

$$\mathbb{E}\left[K\left(1+|Y(t)|^2+|v(t)|^2\right)\right] \to \mathbb{E}\left[K\left(1+|Y_\infty|^2+|v_\infty|^2\right)\right],$$

one can apply the dominated convergence theorem to $(Y(t), v(t))$ and obtain

$$\lim_{t\to\infty} \mathbb{E}\left[\phi(X(t), u(t))\right] = \lim_{t\to\infty} \mathbb{E}\left[\phi(Y(t), v(t))\right] = \mathbb{E}\left[\lim_{t\to\infty} \phi(Y(t), v(t))\right] = \langle \phi, \mu_\infty \rangle.$$

The second identity follows by applying the following fact to $f(t) = \mathbb{E}\left[\phi(X(t), u(t))\right]$,

- If $\lim_{t\to\infty} f(t) = f_\infty$, then $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt = f_\infty$.

□

Lemma 4. *Suppose Assumption 1 holds. Let $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ solves the cell problem* (15)*. In addition, we assume*

1. *∇v is Lipschitz continuous;*
2. *There exists a unique maximizer of $H^a(x, \bar{p}, \bar{q})$ in the form of*

$$\bar{u}(x, \bar{p}) = \arg\max_a H^a(x, \bar{p}, \bar{q});$$

3. *The distribution of the process $(X^*(t), u^*(t))$ controlled by $u^*(t) = \bar{u}(X^*(t), -\nabla v(X^*(t)))$ converges to some $\mu^*_\infty \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ in 2-Wasserstein distance.*

Then, the pair

$$(V := v - \langle v, \pi_\# \mu^*_\infty \rangle, c_0)$$

solves the probabilistic cell problem (16)-(17)*, where $\pi_\#$ is the pushforward measure of the projection $\pi(x, a) = x$.*

Proof. Since $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ solves the cell problem (15), for a control $u \in \mathcal{U}$ and its associated state process $X^u = X$, by Itô's formula, we obtain

$$\begin{aligned} v(X(t)) = v(x) &+ \int_0^t \left(\hat{b}(X(s), u(s)) \cdot \nabla v(X(s)) + \frac{1}{2}\hat{\sigma}^2 \Delta v(X(s))\right) ds \\ &+ \int_0^t \hat{\sigma} \nabla v(X(s)) \cdot dW(s). \end{aligned}$$

Fixing $t > 0$ and taking expectation on both sides and note that

$$\mathbb{E}\left[\int_0^t \hat{\sigma}^2 \left|\nabla v(X(s))\right|^2 ds\right] \le \hat{\sigma}^2 K^2 \mathbb{E}\left[\int_0^t \left(1 + |X(s)|^2\right) ds\right] \le \hat{\sigma}^2 K^2 \left(t + \mathbb{E}\left[\int_0^t |X(s)|^2 ds\right]\right)$$

is finite, we have

$$\mathbb{E}\left[v(X(t))\right] = v(x) + \mathbb{E}\left[\int_0^t \left(\hat{b}(X(s), u(s)) \cdot \nabla v(X(s)) + \frac{1}{2}\hat{\sigma}^2 \Delta v(X(s))\right) ds\right].$$

The cell problem (15) implies that, for all $x, a \in \mathbb{R}^d$,

$$-\hat{b}(x, a) \cdot \nabla v(x) - \hat{L}(x, a) - \frac{1}{2}\hat{\sigma}^2 \Delta v(x) - c_0 \le 0,$$

hence

$$\hat{b}(X(s), u(s)) \cdot \nabla v(X(s)) + \frac{1}{2}\hat{\sigma}^2 \Delta v(X(s)) \geq -\hat{L}(X(s), u(s)) - c_0$$

for all $s \in [0, t]$. Thus

$$v(x) \leq \mathbb{E}\left[v(X(t))\right] + \mathbb{E}\left[\int_0^t \left(\hat{L}(X(s), u(s)) + c_0\right) ds\right]. \tag{18}$$

The inequality (18) holds for all $u \in \mathcal{U}$ and equality holds if $u = u^*$.

Moreover, due to Lipschitz continuity of ∇v, the value function v satisfies a quadratic growth condition, i.e., $|v(x)| \leq K(1 + |x|^2)$ for all $x \in \mathbb{R}^d$. By Lemma 3, we have

$$\lim_{t \to \infty} \mathbb{E}\left[v(X(t))\right] = \langle v, \pi_{\#}\mu_\infty \rangle,$$

where μ_∞ is the distribution limit of (X_t, u_t).

Next, by taking $\limsup_{T \to \infty} \frac{1}{T}$ on both sides of (18), we have

$$-c_0 \leq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}\left[\hat{L}(X(t), u(t))\right] dt. \tag{19}$$

The above inequality holds for all $u \in \mathcal{U}$ and equality holds if $u = u^*$. Thus, we conclude the first identity (16) of the probabilistic cell problem.

Moreover, by taking $\limsup_{t \to \infty}$ on both sides of (18), we have

$$v(x) \leq \langle v, \pi_{\#}\mu_\infty \rangle + \limsup_{t \to \infty} \int_0^t \mathbb{E}\left[\hat{L}(X(s), u(s)) + c_0\right] ds, \tag{20}$$

since $\mathbb{E}[|X(t)|^2] < \infty$ for all $t > 0$ and $\hat{L}(x, a)$ is quadratic growth with respect to x and a from Assumption 1. By Lemma 3 and (19), we have

$$\langle \hat{L}, \mu_\infty \rangle \geq \langle \hat{L}, \mu^*_\infty \rangle = -c_0,$$

and in particular, if $\langle \hat{L}, \mu_\infty \rangle \neq \langle \hat{L}, \mu^*_\infty \rangle$, then the right hand side of (20) is simply equal to ∞. Therefore, we can rewrite (20) as

$$v(x) \leq \langle v, \pi_{\#}\mu^*_\infty \rangle + \limsup_{t \to \infty} \int_0^t \mathbb{E}\left[\hat{L}(X(s), u(s)) + c_0\right] ds, \ \forall u \in \mathcal{U},$$

and an equality holds for u^*. Therefore, we conclude that

$$v(x) - \langle v, \pi_{\#}\mu^*_\infty \rangle = \inf_{u \in \mathcal{U}} \limsup_{t \to \infty} \int_0^t \mathbb{E}\left[\hat{L}(X(s), u(s)) + c_0\right] ds.$$

$\square$

2.2.3 Verification for LQG setting

In the following, we provide a complete characterization of the probabilistic cell problem (13)-(14) in the LQ setting by the help of cell problem given by Lemma 1.

Theorem 1. *Let $(v, c_0) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ be the solution to the cell problem* (6) *in the form of*

$$v(x) = \frac{1}{2} x^\top Z_1 x + x^\top Z_2$$

associated to the

$$Z_1 = A + \sqrt{A^2 + Q}, \; Z_2 = (Z_1 - A)^{-1}(Z_1 b - Z_1 r + q). \tag{21}$$

Then, the pair

$$(V := v - \langle v, \mu_\infty^* \rangle, c_0)$$

with

$$\mu_\infty^* = \mathcal{N}\left(D_1^{-1} D_2, \frac{1}{2}\sigma^2 D_1^{-1}\right), \; D_1 = Z_1 - A, \; D_2 = b - r - Z_2 \tag{22}$$

solves the probabilistic cell problem (13)-(14)*. Moreover, the optimal path X^* of the probabilistic cell problem* (13)-(14) *is an OU process given by*

$$dX^*(t) = (-D_1 X^*(t) + D_2)\, dt + \sigma dW(t), \quad X^*(0) = x, \tag{23}$$

whose distribution converges to μ_∞^* in 2-Wasserstein metric and the optimal control process u^* admits a feedback form of

$$u^*(t) = \bar{u}\left(X^*(t), -\nabla v\left(X^*(t)\right)\right) = -Z_1 X^*(t) - Z_2 - r, \tag{24}$$

where

$$\bar{u}(x, \bar{p}) = \bar{p} - r.$$

Proof. The cell problem (6) admits solutions in the form of

$$v(x) = \frac{1}{2} x^\top Z_1 x + x^\top Z_2 + Z_3$$

by Lemma 1. Note that $v - \langle v, \mu_\infty^* \rangle$ is independent to Z_3, hence it's enough to show the results with $Z_3 = 0$. By Lemma 2, there exists unique choice of (Z_1, Z_2) of (11) such that D_1 is positive definite.

Therefore, it's enough to check all of the assumptions in Lemma 4. First of all, v is a quadratic function and thus its first order derivative ∇v is Lipschitz continuous. Moreover, by the first order condition, the maximizer of H^a of (9) uniquely exists in the form of (24).

For the third assumption, we shall check the convergence of the process associated to the optimal control. The explicit solution to (23) can be written by

$$X^*(t) = \Phi_t x + \int_0^t \Phi_{t-s} ds D_2 + \sigma \int_0^t \Phi_{t-s} dW(s),$$

where Φ_t is a $d \times d$ fundamental matrix satisfying $\Phi_0 = I_d$ and the homogeneous matrix ODE

$$d\Phi_t = -D_1 \Phi_t dt.$$

It is clear that X^* is an Ornstein-Uhlenbeck (OU) process and

$$X^*(t) \sim \mathcal{N}(m_t, \nu_t),$$

where

$$m_t = \Phi_t x + \int_0^t \Phi_{t-s} ds D_2, \; \nu_t = \sigma^2 \int_0^t \Phi_{t-s}^2 ds$$

By Lemma 2, there exists unique choice of the positive definite D_1, and one can write its orthogonal diagonalization $D_1 = \tilde{Q}\Lambda\tilde{Q}^\top$, where $\tilde{Q}$ is an orthogonal matrix and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_d)$ is a diagonal matrix with all $\lambda_i > 0$. This implies that D_1 has its inverse in the form of $D_1^{-1} = \tilde{Q}\Lambda^{-1}\tilde{Q}^\top$ and Φ_t can be factored into $\Phi_t = e^{-D_1 t} = \tilde{Q}e^{-\Lambda t}\tilde{Q}^\top$. Moreover, we have

$$\int_0^t \Phi_{t-s} ds = (I_d - \Phi_t)D_1^{-1}, \int_0^t \Phi_{t-s}^2 ds = \frac{1}{2}(I_d - \Phi_t^2)D_1^{-1}.$$

Therefore, the mean and variance of X_t^* can be rewritten as

$$m_t = \Phi_t x + (I_d - \Phi_t)D_1^{-1}D_2,\ \nu_t = \frac{1}{2}\sigma^2(I_d - \Phi_t^2)D_1^{-1}.$$

In addition, due to the positive definiteness D_1, $\Phi_t \to 0$ as $t \to \infty$ and there exist

$$m_\infty = D_1^{-1}D_2 \in \mathbb{R}^d \text{ and } \nu_\infty = \frac{1}{2}\sigma^2 D_1^{-1} \in \mathbb{S}^d$$

such that m_t converges to m_∞ and ν_t converges to ν_∞. Hence, we obtain the desired result that $X^*(t) \xrightarrow{d} \bar{X}$ as $t \to \infty$, where

$$\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty) := \mu_\infty^* \tag{25}$$

is a normal random variable. The proof can be concluded from Lemma 4. □

2.3 Consistency with the static optimization problem

The static optimization problem (4) plays an important role in the study of the turnpike properties of the control problem in the deterministic type [19, 25] and the stochastic type [22]. In this subsection, we provide a result to show the equality between the constant $-c_0$ from the cell problem (6) and the value $F(\hat{x}, \hat{u})$ from the static optimization problem (4).

Recall that $L(x, u) = \frac{1}{2}(x^\top Qx + |u|^2 + 2q^\top x + 2r^\top u)$ and $F(x, u) = L(x, u) + \frac{1}{2}\sigma^2\mathrm{trace}(P)$, where P is a positive definite solution to $P^2 - 2AP - Q = \mathbf{O}_d$. The solution $(\hat{x}, \hat{u})$ to the static optimization problem (4) is the one that solves

$$\begin{cases} \text{Minimize} \quad L(x, u), \\ \text{subject to} \quad u = -(Ax + b), \end{cases}$$

since $\frac{1}{2}\sigma^2\mathrm{trace}(P)$ is a constant and it is independent with (x, u).

Lemma 5. *The optimal static value to the static optimization problem* (4) *is*

$$F(\hat{x}, \hat{u}) = -\frac{1}{2}(Ab + q - Ar)^\top \left(Q + A^2\right)^{-1}(Ab + q - Ar) + \frac{1}{2}b^\top b - r^\top b + \frac{1}{2}\sigma^2 \mathit{trace}(P), \tag{26}$$

where P is a positive solution to

$$P^2 - 2AP - Q = \mathbf{O}_d,$$

and the optimal solution $(\hat{x}, \hat{u})$ is given by

$$\begin{aligned} \hat{x} &= -\left(Q + A^2\right)^{-1}(Ab + q - Ar), \\ \hat{u} &= A\left(Q + A^2\right)^{-1}(Ab + q - Ar) - b. \end{aligned}$$

Proof. Plugging $u = -(Ax+b)$, we know that $L(x,u)$ is a quadratic function with respect to x and it is given by

$$L(x,u) = \frac{1}{2}x^\top \left(Q + A^2\right) x + (Ab + q - Ar)^\top x + \frac{1}{2}b^\top b - r^\top b.$$

It is straightforward to obtain the desired result by minimizing the quadratic function under the condition that $Q + A^2 > 0$. $\square$

Now, we are ready to show the consistency between $-c_0$ and $F(\hat{x}, \hat{u})$.

Lemma 6. *The constant $-c_0$ of* (6) *is identical to the value $F(\hat{x}, \hat{u})$ of the static optimization problem* (4) *.*

Proof. It's equivalent to verify the equality between the representation of in (10) and the optimal static value in (26). Choosing Z_1 be the solution to $Z_1^2 - 2AZ_1 - Q = \mathbf{O}_d$ such that $D_1 = Z_1 - A$ is positive definite, i.e., $Z_1 = A + \sqrt{A^2+Q}$, we obtain that $Z_1 = P$ and $Z_2 = (Z_1 - A)^{-1}(Z_1 b + q - Z_1 r)$. To verify that $-c_0 = F(\hat{x}, \hat{u})$, we only need to check that

$$b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2 = -\frac{1}{2}(Ab + q - Ar)^\top \left(Q + A^2\right)^{-1} (Ab + q - Ar) + \frac{1}{2}b^\top b - r^\top b.$$

By calculation and applying the representation of Z_2, we have

$$\begin{aligned}
& b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2 \\
= & -\frac{1}{2}\left((Z_1 - A)^{-1}(Z_1 b + q - Z_1 r) + r\right)^\top \left((Z_1 - A)^{-1}(Z_1 b + q - Z_1 r) + r\right) \\
& \quad + b^\top (Z_1 - A)^{-1}(Z_1 b + q - Z_1 r) \\
= & -\frac{1}{2}(Z_1 b + q - Z_1 r)^\top \left((Z_1 - A)^{-1}\right)^2 (Z_1 b + q - Z_1 r) - \frac{1}{2}r^\top r \\
& \quad + (Z_1 b + q - Z_1 r)^\top \left((Z_1 - A)^{-1}\right)^2 (Z_1 b - Ab - Z_1 r + Ar) \\
= & \frac{1}{2}(Z_1 b + q - Z_1 r)^\top \left((Z_1 - A)^{-1}\right)^2 (Z_1 b - Z_1 r - q - 2Ab + 2Ar) - \frac{1}{2}r^\top r.
\end{aligned}$$

Note that $Z_1 - A = \sqrt{Q + A^2}$, then $((Z_1 - A)^{-1})^2 = (Q + A^2)^{-1}$. Substituting Z_1 by $A + \sqrt{Q + A^2}$, we could obtain

$$\begin{aligned}
& b^\top Z_2 - \frac{1}{2}|Z_2 + r|^2 \\
= & \frac{1}{2}(Z_1 b + q - Z_1 r)^\top \left(Q + A^2\right)^{-1} (Z_1 b - Z_1 r - q - 2Ab + 2Ar) - \frac{1}{2}r^\top r \\
= & \frac{1}{2}\left(\left(A + \sqrt{Q + A^2}\right)(b - r) + q\right)^\top \left(Q + A^2\right)^{-1} \left(\left(A + \sqrt{Q + A^2}\right)(b - r) - q - 2Ab + 2Ar\right) - \frac{1}{2}r^\top r \\
= & -\frac{1}{2}(Ab + q - Ar)^\top \left(Q + A^2\right)^{-1} (Ab + q - Ar) + \frac{1}{2}(b - r)^\top (b - r) - \frac{1}{2}r^\top r \\
= & -\frac{1}{2}(Ab + q - Ar)^\top \left(Q + A^2\right)^{-1} (Ab + q - Ar) + \frac{1}{2}b^\top b - r^\top b,
\end{aligned}$$

which yields the desired result. $\square$

3 Turnpike property

In this section, we uncover the turnpike property applicable to the optimal trajectory X^* and optimal control u^*, both stemming from the probabilistic cell problem as outlined in (13)-(14). Moreover, distinct from the aforementioned turnpike property elucidated by (5), we also establish the turnpike behavior concerning the cost function:

$$\lim_{T\to\infty}\frac{1}{T}J_T(x;u^*) = -c_*,$$

where $-c_* := \lim_{T\to\infty}\frac{1}{T}J_T(x;u_T^*)$ is the ergodic cost defined in (7). In other words, achieving a near optimality in terms of the average cost over an extended period doesn't require calculating the optimal control u_T^* for every T. Instead, one only needs to compute the optimal control u^* for the probabilistic cell problem.

To accomplish this objective, it becomes imperative to establish a relation between the cost function $J_T(x;u^*)$ with the u^* from the probabilistic cell problem(13)-(14), and the corresponding function $V_T(x) = J_T(x,u_T^*)$ originating from the finite time stochastic control problem (3).

3.1 Main results on turnpike properties

To distinguish the finite time control problem (3) from the probabilistic cell problem (13)-(14), we denote by $\{X_T(t)\}_{0\le t\le T}$ as the underlying process controlled by finite time control $\{u_T(t)\}_{0\le t\le T}$ in the finite time stochastic control problem, while denote by $\{X(t)\}_{t\ge 0}$ as the underlying process controlled by infinite time control $\{u(t)\}_{t\ge 0}$ in the probabilistic cell problem. In other words, $\{X_T(t)\}_{0\le t\le T}$ follows

$$dX_T(t) = (AX_T(t) + u_T(t) + b)\,dt + \sigma dW(t), \quad t\in[0,T]$$

and $\{X(t)\}_{t\ge 0}$ follows

$$dX(t) = (AX(t) + u(t) + b)\,dt + \sigma dW(t), \quad t\ge 0.$$

We also recall that the value function of the finite time control problem (3) is

$$V_T(x) := J_T\left(x;u_T^*\right) = \inf_{u_T\in\mathcal{U}_{[0,T]}} J_T(x;u_T),$$

with its optimal control and optimal path denoted by u_T^* and X_T^*.

We recall that the value function $V(x)$ of the probabilistic cell problem is defined in (13)-(14) as

$$-c_0 = \inf_{u\in\mathcal{U}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\left[L(X(t),u(t))\right]dt$$

and

$$V(x) := \inf_{u\in\mathcal{U}}\limsup_{T\to\infty}\int_0^T \mathbb{E}\left[L\left(X_t,u_t\right)+c_0\right]dt.$$

The optimal control and optimal path of the probabilistic cell problem is denoted by u^* and X^* respectively, which are characterized by Theorem 1. Note that, the constant c in the above definition is replaced by c_0 of the cell problem corresponding to Z_1 such that D_1 is positive definite according to Theorem 1. Next, we present our main results.

Theorem 2. *Let $J_T(x;u^*)$ be the cost functional evaluated along the optimal control of the probabilistic cell problem u^* on the finite time horizon $[0,T]$, i.e.,*

$$J_T(x;u^*) := \mathbb{E}\left[\int_0^T L\left(X_T(s), u^*(s)\right) ds + g(X_T(T))\right].$$

For all $x \in \mathbb{R}^d$, the following estimation holds:

$$0 \leq J_T(x;u^*) - V_T(x) = O(1),$$

Moreover, the constant c_0 of the cell problem (6) is the ergodic cost defined via (7), i.e.,

$$c_0 = \lim_{T\to\infty} \frac{1}{T} V_T(x), \quad \forall x \in \mathbb{R}^d.$$

From the results of Theorem 2, we could establish the turnpike behavior in terms of the average cost function straightforwardly. Note that $V_T(x) \leq J_T(x;u^*) = V_T(x) + O(1)$, it follows that

$$\lim_{T\to\infty} \frac{1}{T} J_T(x;u^*) = \lim_{T\to\infty} \frac{1}{T} V_T(x) = -c_*. \tag{27}$$

Theorem 3. *For all $x \in \mathbb{R}^d$, there exist some $\lambda > 0$ and $K > 0$ independent of t and T such that*

$$\mathbb{E}\left[|X_T^*(t) - X^*(t)|^2 + |u_T^*(t) - u^*(t)|^2\right] \leq K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0,T].$$

3.2 Proofs

The proofs of these two theorems rely on the analytical expressions of the value functions for both the finite time control problem and probabilistic cell problem, and some comparison results between solutions to the Riccati system of ODEs. We first present some preliminary lemmas with their proof posted in Appendix 5. The first lemma gives an analytical expression for the value function of the finite time control problem.

Lemma 7. *The value function $V_T(x)$ of the finite time control problem has the following form*

$$V_T(x) = \frac{1}{2} x^\top \tilde{Z}_1(T) x + x^\top \tilde{Z}_2(T) + \tilde{Z}_3(T),$$

where $\{\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3\}$ solves the Riccati system of ODEs

$$\begin{cases} \dot{\tilde{Z}}_1(t) = 2A\tilde{Z}_1(t) - \tilde{Z}_1^2(t) + Q; \\ \dot{\tilde{Z}}_2(t) = A\tilde{Z}_2(t) - \tilde{Z}_1(t)\tilde{Z}_2(t) + \tilde{Z}_1(t)(b-r) + q; \\ \dot{\tilde{Z}}_3(t) = \frac{1}{2}\sigma^2 \mathit{trace}(\tilde{Z}_1(t)) + b^\top \tilde{Z}_2(t) - \frac{1}{2}|\tilde{Z}_2(t) + r|^2, \end{cases} \tag{28}$$

with initial conditions $\tilde{Z}_1(0) = N_1, \tilde{Z}_2(0) = N_2, \tilde{Z}_3(0) = N_3$. Moreover, the optimal feedback control of the finite time control problem is given by

$$u_T^*(t) = -\tilde{Z}_1(T-t)X_T^*(t) - \tilde{Z}_2(T-t) - r.$$

Next lemma gives a similar analytical structure for $J_T(x;u^*)$, which is the cost functional of the finite time control problem evaluated along the optimal control u^* from the probabilistic cell problem. Recall that (Z_1, Z_2) is the solution to the system of algebraic equations (11) in which Z_1 is uniquely chosen such that D_1 to be positive definite, see Lemma 2.

Lemma 8. *The cost functional $J_T(x;u^*)$ of the finite time control problem evaluated along the optimal control u^* from the probabilistic cell problem has the form*

$$J_T(x;u^*) = \frac{1}{2}x^\top f_1(T)x + x^\top f_2(T) + f_3(T),$$

where $\{f_i : i = 1,2,3\}$ solves the Riccati system of ODEs

$$\begin{cases} \dot{f}_1(t) = -2(Z_1 - A)f_1(t) + Q + Z_1^2; \\ \dot{f}_2(t) = -(Z_1 - A)f_2(t) + f_1(t)(b - r - Z_2) + Z_1 Z_2 + q; \\ \dot{f}_3(t) = \frac{1}{2}\sigma^2 \mathit{trace}(f_1(t)) + (b - r - Z_2)^\top f_2(t) + \frac{1}{2}(|Z_2|^2 - |r|^2), \end{cases} \tag{29}$$

with initial conditions $f_1(0) = N_1, f_2(0) = N_2, f_3(0) = N_3$.

From the above lemmas, we can observe that

$$J_T(x;u^*) - V_T(x) = \frac{1}{2}x^\top \Gamma_1(T)x + x^\top \Gamma_2(T) + \Gamma_3(T), \tag{30}$$

where, for $i = 1,2,3$,

$$\Gamma_i(t) = f_i(t) - \tilde{Z}_i(t),\ i = 1,2,3.$$

If we introduce γ_i by

$$\gamma_i(t) = Z_i - \tilde{Z}_i(t),\ i = 1,2,$$

$\{\Gamma_i : i = 1,2,3\}$ satisfies the following system of ODE

$$\begin{cases} \dot{\Gamma}_1(t) = -2(Z_1 - A)\Gamma_1(t) + \gamma_1^2(t); \\ \dot{\Gamma}_2(t) = -(Z_1 - A)\Gamma_2(t) + \Gamma_1(t)(b - r - Z_2) + \gamma_1(t)\gamma_2(t); \\ \dot{\Gamma}_3(t) = \frac{1}{2}\sigma^2 \mathrm{trace}(\Gamma_1(t)) + (b - r - Z_2)^\top \Gamma_2(t) + \frac{1}{2}|\gamma_2(t)|^2, \end{cases} \tag{31}$$

with the initial conditions $\Gamma_1(0) = \mathbf{O}_d, \Gamma_2(0) = \mathbf{0}_d, \Gamma_3(0) = 0$. Moreover, $\{\gamma_1, \gamma_2\}$ is the solution to the system of ODEs

$$\begin{cases} \dot{\gamma}_1(t) = -2D_1\gamma_1(t) + \gamma_1^2(t); \\ \dot{\gamma}_2(t) = -D_1\gamma_2(t) + \gamma_1(t)\gamma_2(t) + \gamma_1(t)D_2, \end{cases} \tag{32}$$

with the initial conditions $\gamma_1(0) = Z_1 - N_1$ and $\gamma_2(0) = Z_2 - N_2$. In the above, we recall that

$$D_1 = Z_1 - A > 0, \quad D_2 = b - r - Z_2.$$

It is enough to give a proper estimations for the γ_1, γ_2 in (32) and $\Gamma_1, \Gamma_2, \Gamma_3$ in (31) to obtain the bound for the difference between $J_T(x;u^*)$ and $V_T(x)$.

Lemma 9. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$ be eigenvalues of D_1 in a descending order. Then, there exists a unique solution $\gamma_1, \gamma_2 \in C^1$ of the system of ODEs* (32) *satisfying, for some constant $k > 0$,*

$$\|\gamma_1(t)\|_2 \leq ke^{-2\lambda_d t},\ |\gamma_2(t)| \leq ke^{-\lambda_d t}, \quad \forall t > 0. \tag{33}$$

Moreover, $\gamma_1(t)$ is positive definite for all $t \in [0,T]$ and γ_1 is an decreasing function with respect to Loewner order.

Proof. ODE satisfied by γ_1 is

$$\dot{\gamma}_1(t) = -2D_1\gamma_1(t) + \gamma_1^2(t), \quad \gamma_1(0) = Z_1 - N_1.$$

Recall that $D_1 = \tilde{Q}\Lambda\tilde{Q}^\top$, where $\tilde{Q}$ is an orthogonal matrix and Λ is a diagonal matrix with descending diagonal entries λ_i. Since Z_1, N_1 and $\tilde{Z}_1(t)$ are symmetric matrices, $\gamma_1(t)$ is also symmetric as their difference. Taking transpose to the equation satisfied by γ_1, one can observe that D_1 and $\gamma_1(t)$ are commute, i.e., $D_1\gamma_1(t) = \gamma_1(t)D_1$. Thus, γ_1 and D_1 share the same eigenvector matrix $\tilde{Q}$ by the results in Page 305 of [21].

Hence, we can write $\gamma_1(t) = \tilde{Q}\Sigma(t)\tilde{Q}^\top$ for some $\Sigma(t) = \mathrm{diag}(\sigma_1(t), \sigma_2(t), \ldots, \sigma_d(t))$ for $t \geq 0$. It follows that

$$\dot{\Sigma}(t) = -2\Lambda\Sigma(t) + \Sigma^2(t), \quad \forall t > 0 \tag{34}$$

with initial condition $\Sigma(0) = \tilde{Q}^\top\gamma_1(0)\tilde{Q} = \tilde{Q}^\top(Z_1 - N_1)\tilde{Q}$.

It is equivalent to write

$$\dot{\sigma}_i(t) = -2\lambda_i\sigma_i(t) + \sigma_i^2(t), \quad t > 0,\ i = 1, 2, \ldots, d.$$

Recall that from (Cf),

$$0 < \gamma_1(0) = Z_1 - N_1 < 2D_1.$$

Hence, $2\Lambda > \Sigma(0) > 0$, i.e., $0 < \sigma_i(0) < 2\lambda_i$ for $i = 1, 2, \ldots, d$. Therefore, there exists unique solution $\{\sigma_i : i = 1, 2, \ldots, d\}$ in the form of

$$0 < \sigma_i(t) = \frac{2\lambda_i}{1 + \left(\frac{2\lambda_i}{\sigma_i(0)} - 1\right)e^{2\lambda_i t}} < \max_{i=1,2,\ldots,d}\left(\frac{2\lambda_i\sigma_i(0)}{2\lambda_i - \sigma_i(0)}\right)e^{-2\lambda_i t} := a_1 e^{-2\lambda_i t}, \quad \forall t \geq 0.$$

Clearly, σ_i is strictly decreasing on $[0, \infty)$, which implies that γ_1 is a strictly decreasing function on $[0, \infty)$ with respect to Loewner order. Moreover, since $\lambda_i > 0$ for all $i = 1, 2, \ldots, d$, $\gamma_1(t)$ is positive definite for all $t \geq 0$. Thus, we have

$$\|\gamma_1(t)\|_2 = \|\Sigma(t)\|_2 \leq a_1 e^{-2\lambda_d t}, \quad \forall t \geq 0. \tag{35}$$

By (32), γ_2 satisfies the ODE

$$\dot{\gamma}_2(t) = (\gamma_1(t) - D_1)\gamma_2(t) + \gamma_1(t)D_2, \quad \gamma_2(0) = Z_2 - N_2.$$

Denote that $A_1(t) = \gamma_1(t) - D_1$ for $t \geq 0$, the explicit form of γ_2 is given by

$$\gamma_2(t) = e^{\int_0^t A_1(s)ds}(Z_2 - N_2) + \int_0^t e^{\int_s^t A_1(r)dr}\gamma_1(s)D_2 ds \tag{36}$$

for all $t \geq 0$.

To proceed, we first observe from (35) that

$$\left\|\int_0^t \Sigma(s)ds\right\|_2 \leq \int_0^t \|\Sigma(s)\|_2\, ds \leq \frac{a_1}{2\lambda_d}(1 - e^{-2\lambda_d t}) \leq \frac{a_1}{2\lambda_d}.$$

From the fact that $\|e^D\|_2 = e^{\|D\|_2}$ for any $D \geq 0$, it implies that

$$\left\|e^{\int_0^t \Sigma(s)ds}\right\|_2 \leq a_2 := \exp\left\{\frac{a_1}{2\lambda_d}\right\}.$$

Hence, we have

$$\left\|e^{\int_0^t A_1(s)ds}\right\|_2 = \left\|e^{\int_0^t (\Sigma(s)-\Lambda)ds}\right\|_2 \leq \left\|e^{\int_0^t \Sigma(s)ds}\right\|_2 \left\|e^{-\Lambda t}\right\|_2 \leq a_2 e^{-\lambda_d t}. \tag{37}$$

Therefore, the estimate of (36) is

$$\begin{aligned}
|\gamma_2(t)| &\leq \left\|e^{\int_0^t A_1(s)ds}\right\|_2 \ |Z_2 - N_2| + \int_0^t \left\|e^{\int_s^t A_1(r)dr}\right\|_2 \|\gamma_1(s)\|_2 \ |D_2| ds \\
&\leq a_2 e^{-\lambda_d t}|Z_2 - N_2| + \int_0^t a_2 e^{-\lambda_d (t-s)} a_1 e^{-2\lambda_d s} ds |D_2| \\
&\leq \left(a_2|Z_2 - N_2| + \frac{a_1 a_2 |D_2|}{\lambda_d}\right) e^{-\lambda_d t} := a_3 e^{-\lambda_d t}
\end{aligned}$$

for all $t \geq 0$. □

Lemma 10. *With eigenvalues of D_1 denoted by Lemma 9, The system of ODEs (31) has a unique solution Γ_1, Γ_2 and $\Gamma_3 \in C^1$, and satisfies the following properties for some constant $k > 0$:*

$$\|\Gamma_1(t)\|_2 \leq k e^{-2\lambda_d t}, \quad |\Gamma_2(t)| \leq k e^{-\lambda_d t}, \quad |\Gamma_3(t)| \leq k, \ \forall t > 0. \tag{38}$$

Proof. We recall that, Γ_1 satisfies the ODE

$$\dot{\Gamma}_1(t) = -2D_1\Gamma_1(t) + \gamma_1^2(t), \quad \Gamma_1(0) = \mathbf{O}_d,$$

which can be written in terms of γ_1 in the form of

$$\Gamma_1(t) = \int_0^t e^{-2D_1(t-s)}\gamma_1^2(s)ds, \quad \forall t > 0.$$

By Lemma 9, the estimation of Γ_1 is

$$\|\Gamma_1(t)\|_2 \leq \int_0^t e^{-2\lambda_d(t-s)}\|\gamma_1(s)\|_2^2 ds \leq k e^{-2\lambda_d t}. \tag{39}$$

Moreover, from the explicit form of Γ_1, it is clear that $\Gamma_1(t)$ is positive semi-definite. Similarly, the ODE for Γ_2 is

$$\dot{\Gamma}_2(t) = -D_1\Gamma_2(t) + \Gamma_1(t)D_2 + \gamma_1(t)\gamma_2(t), \quad \Gamma_2(0) = \mathbf{0}_d,$$

which yields an expression in terms of γ_1, γ_2 and Γ_1:

$$\Gamma_2(t) = \int_0^t e^{-D_1(t-s)} \left(\Gamma_1(s)D_2 + \gamma_1(s)\gamma_2(s)\right) ds, \quad \forall t \in [0, T].$$

By the estimation Lemma 9 and (39),

$$\begin{aligned}
|\Gamma_2(t)| &\leq \int_0^t \left\|e^{-D_1(t-s)}\right\|_2 \left(\|\Gamma_1(s)\|_2 |D_2| + \|\gamma_1(s)\|_2 |\gamma_2(s)|\right) ds \\
&\leq \int_0^t e^{-\lambda_d(t-s)} \left(k e^{-2\lambda_d s}|D_2| + k^2 e^{-3\lambda_d s}\right) ds \\
&\leq k e^{-\lambda_d t}.
\end{aligned} \tag{40}$$

Next, the term Γ_3 satisfies the ODE

$$\dot{\Gamma}_3(t) = \frac{1}{2}\sigma^2 \text{trace}(\Gamma_1(t)) + D_2^\top \Gamma_2(t) + \frac{1}{2}|\gamma_2(t)|^2, \quad \Gamma_3(0) = 0,$$

which can be rewritten with the above γ_2, Γ_1 and Γ_2:

$$\Gamma_3(t) = \int_0^t \left(\frac{1}{2}\sigma^2 \text{trace}(\Gamma_1(s)) + D_2^\top \Gamma_2(s) + \frac{1}{2}|\gamma_2(s)|^2 \right) ds, \quad \forall t \in [0, T].$$

Thus, we obtain the estimation for Γ_3 with the help of Lemma 9, (39), and (40)

$$|\Gamma_3(t)| \leq \int_t^T \left(\frac{1}{2} d\sigma^2 \|\Gamma_1(s)\|_2 + |D_2||\Gamma_2(s)| + \frac{1}{2}|\gamma_2(s)|^2 \right) ds \leq k. \tag{41}$$

□

Now, we are well prepared to prove Theorem 2.

Proof of Theorem 2. From (38) in Lemma 10, we have

$$\begin{aligned} |J_T(x; u^*) - V_T(x)| &= \left| \frac{1}{2} x^\top \Gamma_1(T) x + x^\top \Gamma_2(T) + \Gamma_3(T) \right| \\ &\leq \frac{1}{2} k e^{-2\lambda_d T} |x|^2 + k e^{-\lambda_d T} |x| + k. \end{aligned}$$

Hence, for all $x \in \mathbb{R}^d$, we obtain that $J_T(x; u^*) - V_T(x) = O(1)$. Since u_T^* is the optimal control of the finite time control problem, we have $0 \leq J_T(x; u^*) - V_T(x)$ for all $x \in \mathbb{R}^d$. Therefore, we obtain the desired result.

Next, the estimations of γ_1 and γ_2 in (33) can help us to verify that the identity $V_T(x) + c_0 T = o(T)$ holds, where c_0 is given by (10) and can also be obtained from the solution to the cell problem. Note that $V_T(x) = \frac{1}{2} x^\top \tilde{Z}_1(T) x + x^\top \tilde{Z}_2(T) + \tilde{Z}_3(T)$. Then, we only need to show that for all $x \in \mathbb{R}^d$

$$\lim_{T \to \infty} \frac{1}{T} \left| \frac{1}{2} x^\top \tilde{Z}_1(T) x + x^\top \tilde{Z}_2(T) + \tilde{Z}_3(T) + c_0 T \right| = 0.$$

From (33) in Lemma 9, we have the following inequalities

$$\left\| \tilde{Z}_1(T) \right\|_2 = \|Z_1 - \gamma_1(T)\|_2 \leq \|Z_1\|_2 + \|\gamma_1(T)\|_2 \leq \|Z_1\|_2 + k e^{-2\lambda_d T}$$

and

$$\left| \tilde{Z}_2(T) \right| = |Z_2 - \gamma_2(T)| \leq |Z_2| + |\gamma_2(T)| \leq |Z_2| + k e^{-\lambda_d T},$$

which implies that

$$\lim_{T \to \infty} \frac{1}{T} \left| \frac{1}{2} x^\top \tilde{Z}_1(T) x + x^\top \tilde{Z}_2(T) \right| = 0, \quad \forall x \in \mathbb{R}^d.$$

Moreover, from the ODE satisfied by $Z_3(t)$ in (28),

$$\begin{aligned} \tilde{Z}_3(T) &= N_3 + \int_0^T \left(\frac{1}{2}\sigma^2 \text{trace}\left(\tilde{Z}_1(s) \right) + b^\top \tilde{Z}_2(s) - \frac{1}{2} \left| \tilde{Z}_2(s) + r \right|^2 \right) ds \\ &= N_3 + \int_0^T \left(-\frac{1}{2}\sigma^2 \text{trace}\left(\gamma_1(s) \right) - D_2^\top \gamma_2(s) - \frac{1}{2} \left| \gamma_2(s) \right|^2 \right) ds - c_0 T, \end{aligned}$$

where $c_0 = -\frac{1}{2}\sigma^2 \mathrm{trace}(Z_1) - b^\top Z_2 + \frac{1}{2}|Z_2 + r|^2$. Then, it follows that

$$\lim_{T\to\infty} \frac{1}{T}\left|\tilde{Z}_3(T) + c_0 T\right| = \lim_{T\to\infty} \frac{1}{T}\left|N_3 + \int_0^T \left(-\frac{1}{2}\sigma^2 \mathrm{trace}\left(\gamma_1(s)\right) - D_2^\top \gamma_2(s) - \frac{1}{2}\left|\gamma_2(s)\right|^2\right) ds\right|.$$

Using the triangle inequality and the estimations (33), we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\left|N_3 + \int_0^T \left(-\frac{1}{2}\sigma^2 \mathrm{trace}\left(\gamma_1(s)\right) - D_2^\top \gamma_2(s) - \frac{1}{2}\left|\gamma_2(s)\right|^2\right) ds\right| \\
\le\ &\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{1}{2} d\sigma^2 \|\gamma_1(s)\|_2 + |D_2||\gamma_2(s)| + \frac{1}{2}|\gamma_2(s)|^2\right) ds \\
\le\ &\lim_{T\to\infty} \frac{1}{T}\frac{dk\sigma^2 + 4|D_2|k + k^2}{4\lambda_d} = 0,
\end{aligned}$$

which implies that $\lim_{T\to\infty} \frac{1}{T}|\tilde{Z}_3(T) + c_0 T| = 0$. Therefore, we obtain the desired result that

$$\lim_{T\to\infty} \frac{1}{T}|V_T(x) + c_0 T| = 0, \quad \forall x \in \mathbb{R}^d,$$

i.e., $V_T(x) + c_0 T = o(T)$ for all $x \in \mathbb{R}^d$ with c_0 given by (10). The identical result between c_0 and c_* yields from Definition (7). $\square$

Recall Theorem 1: From (24) and (23), the optimal control u^* of the probabilistic cell problem is

$$u^*(t) = -Z_1 X^*(t) - Z_2 - r,$$

and the optimal path X^* of the probabilistic cell problem is given by

$$dX^*(t) = -D_1 X^*(t)dt + D_2 dt + \sigma dW(t),$$

with $X^*(0) = x$, where D_1 and D_2 are given by (22). Moreover, the optimal path $X^*(t)$ in probabilistic cell problem (13)-(14)converges in distribution to a normal random variable $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ as $t \to \infty$, where $m_\infty = D_1^{-1} D_2$ and $\nu_\infty = \frac{1}{2}\sigma^2 D_1^{-1}$. Next, we prove the classical turnpike property that is descried in Theorem 3.

Proof of Theorem 3. Here we assume the optimal path X^* in (23) of the probabilistic cell problem has the initial point $X^*(0) = \bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$, a normal random variable independent to the Brownian motion W, instead of a real-valued vector $x \in \mathbb{R}^d$. Let $\tau(t) = T - t$. To calculate $\mathbb{E}[|X_T^*(t) - X^*(t)|^2]$, we first observe that the optimal control of the finite time control problem is given by

$$u_T^*(t) = -\tilde{Z}_1(\tau) X_T^*(t) - \tilde{Z}_2(\tau) - r$$

from the results in Lemma 7. Thus the optimal path for the finite time control problem satisfies

$$dX_T^*(t) = \left(A - \tilde{Z}_1(\tau)\right) X_T^*(t)dt + \left(b - r - \tilde{Z}_2(\tau)\right) dt + \sigma dW(t). \tag{42}$$

Denote that $\delta_T(t) = X_T^*(t) - X^*(t)$, by (23) and (42), we have

$$d\delta_T(t) = \left(A - \tilde{Z}_1(\tau)\right) \delta_T(t)dt + \gamma_1(\tau) X^*(t)dt + \gamma_2(\tau)dt$$

with the initial value $\delta_T(0) = X_T^*(0) - X^*(0) = x - \bar{X} \sim \mathcal{N}(x - m_\infty, \nu_\infty)$.

Let $\bar{A}_1(t) = A - \tilde{Z}_1(\tau) = \gamma_1(\tau) - D_1$ for all $t \in [0,T]$, then we have $\bar{A}_1(t) = A_1(\tau)$. By the similar method as the estimation in (37), for $0 \le s \le t \le T$, we have

$$\left\| e^{\int_0^t \bar{A}_1(r)dr} \right\|_2 \le a_2 e^{-\lambda_d t} \text{ and } \left\| e^{\int_s^t \bar{A}_1(r)dr} \right\|_2 \le a_2 e^{-\lambda_d (t-s)}.$$

Applying the integrating factor method, we obtain the explicit form of $\delta_T(t)$ as

$$\delta_T(t) = e^{\int_0^t \bar{A}_1(r)dr}\delta_T(0) + \int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_1(\tau(s))X^*(s)ds + \int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_2(\tau(s))ds.$$

Therefore, we have the following estimation:

$$\begin{aligned}\mathbb{E}\left[|\delta_T(t)|^2\right] \le 3\left\|e^{\int_0^t \bar{A}_1(r)dr}\right\|_2^2 \mathbb{E}\left[|\delta_T(0)|^2\right] + 3\mathbb{E}\left[\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_1(\tau(s))X^*(s)ds\right|^2\right] \\ + 3\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_2(\tau(s))ds\right|^2.\end{aligned}$$

Firstly, by calculation,

$$3\left\|e^{\int_0^t \bar{A}_1(r)dr}\right\|_2^2 \mathbb{E}\left[|\delta_T(0)|^2\right] \le 3a_2^2\mathbb{E}\left[\left|x - \bar{X}\right|^2\right]e^{-2\lambda_d t} \le K_1 e^{-2\lambda_d t}$$

for some constant $K_1 \ge 3a_2^2\mathbb{E}\left[|x - \bar{X}|^2\right]$. Next, using the estimation for γ_2 in (33) from Lemma 9 and by Hölder's inequality, we get

$$\begin{aligned}3\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_2(\tau(s))ds\right|^2 &\le 3\int_0^t \left\|e^{\int_s^t \bar{A}_1(r)dr}\right\|_2^2 ds \int_0^t |\gamma_2(\tau(s))|^2 ds \\ \le 3a_2^2k^2\int_0^t e^{-2\lambda_d(t-s)}ds\int_0^t e^{-2\lambda_d(T-s)}ds &\le K_2 e^{-2\lambda_d(T-t)}\end{aligned}$$

for some constant $K_2 \ge \frac{3a_2^2k^2}{4\lambda_d^2}$. Lastly, using the Hölder's inequality again, we have

$$3\mathbb{E}\left[\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_1(\tau(s))X^*(s)ds\right|^2\right] \le 3\int_0^t \left\|e^{\int_s^t \bar{A}_1(r)dr}\right\|_2^2 ds \; \mathbb{E}\left[\int_0^t \|\gamma_1(\tau(s))\|_2^2 \left|X^*(s)\right|^2 ds\right].$$

Similar with the calculation of expectation and variance of X^* in Subsection 2.2.3, we know the expectation and the variance of $X^*(s)$ in (23) with the initial $X^*(0) = \bar{X}$ is $\mathbb{E}\left[X^*(s)\right] = D_1^{-1}D_2$ and $\mathbb{V}ar\left(X^*(s)\right) = \frac{1}{2}\sigma^2 D_1^{-1}$ respectively, which implies $\mathbb{E}[|X^*(s)|^2] = a_6$ for all $s > 0$ for some positive constant a_6. Thus, from the estimation of γ_1 in Lemma 9, we obtain the estimation as follows

$$3\mathbb{E}\left[\left|\int_0^t e^{\int_s^t \bar{A}_1(r)dr}\gamma_1(\tau(s))X^*(s)ds\right|^2\right] \le 3k^2a_6a_2^2\int_0^t e^{-2\lambda_d(t-s)}ds\int_0^t e^{-4\lambda_d(T-s)}ds \le K_3 e^{-4\lambda_d(T-t)}$$

for some constant $K_3 \ge \frac{3k^2a_6a_2^2}{8\lambda_d^2}$.

To summarize from the above estimation results, we have

$$\begin{aligned}\mathbb{E}\left[|\delta_T(t)|^2\right] &\le K_1 e^{-2\lambda_d t} + K_2 e^{-2\lambda_d(T-t)} + K_3 e^{-4\lambda_d(T-t)} \\ &\le K_4\left(e^{-2\lambda_d t} + e^{-2\lambda_d(T-t)}\right)\end{aligned} \tag{43}$$

for all $t \in [0, T]$ for some positive constant K_4.

Similarly, from the optimal control of the probabilistic cell problem and the optimal control of the finite time control problem, it is clear that

$$u_T^*(t) - u^*(t) = \gamma_1(\tau) X^*(t) - \tilde{Z}_1(\tau)\delta_T(t) + \gamma_2(\tau),$$

which yields the inequality

$$\mathbb{E}\left[|u_T^*(t) - u^*(t)|^2\right] \leq 3\left(\|\gamma_1(\tau)\|_2^2 \,\mathbb{E}\left[|X^*(t)|^2\right] + \left\|\tilde{Z}_1(\tau)\right\|_2^2 \mathbb{E}\left[|\delta_T(t)|^2\right] + |\gamma_2(\tau)|^2\right).$$

Due to the facts that $\gamma_1(\tau) = Z_1 - \tilde{Z}_1(\tau)$ and $\|\gamma_1(\tau)\|_2 \leq ke^{-2\lambda_d(T-t)}$, we have $\|\tilde{Z}_1(\tau)\|_2 \leq \|Z_1\|_2 + ke^{-2\lambda_d(T-t)} \leq \|Z_1\|_2 + k$ for all $t \in [0, T]$. Applying the estimations (33) in Lemma 9 and the inequality (43), we get the estimation

$$\mathbb{E}\left[|u_T^*(t) - u^*(t)|^2\right] \leq K_5\left(e^{-2\lambda_d t} + e^{-2\lambda_d(T-t)}\right)$$

for all $t \in [0, T]$ for some $K_5 > 0$ independent of t and T. Therefore, let $\lambda = 2\lambda_d$, the classical turnpike property

$$\mathbb{E}\left[|X_T^*(t) - X^*(t)|^2 + |u_T^*(t) - u^*(t)|^2\right] \leq K\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right), \quad \forall t \in [0, T]$$

is obtained for some $K \geq K_4 + K_5$. □

Remark 1. *In the above proof, we show the turnpike property between $X_T^*(t)$ and the optimal path $X^*(t)$ in (23) for the probabilistic cell problem taking an initial $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ instead of a real-valued vector $x \in \mathbb{R}^d$. The reason for taking this normal random variable as the initial is that the optimal path $X^*(t)$ starting with any constant initial $x \in \mathbb{R}^d$ converges in distribution to a normal random variable $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ as $t \to \infty$ from (25). We refer the asymptomatic $\bar{X} \sim \mathcal{N}(m_\infty, \nu_\infty)$ to be the equilibrium point of $X^*(t)$ in (23). The proof of Theorem 3 when X^* taking an initial value $x \in \mathbb{R}^d$ follows a similar approach.*

4 Example

In this section, we give an example to illustrate the results in Theorem 2 and Theorem 3. In this example, we consider the case when $d = 1$.

Let $A = 0,\ b = 0,\ Q = 1,\ q = 0,\ r = -1,\ N_1 = N_2 = N_3 = 0$. Then, the underlying process (1) is reduced to

$$dX(t) = u(t)dt + \sigma dW(t), \quad X_0 = x,$$

and the cost functional (2) can be simplified as following

$$J_T(0, x; u) = \mathbb{E}\left[\frac{1}{2}\int_0^T \left(X^2(t) + u^2(t) - 2u(t)\right) dt\right].$$

Firstly, we verify the result in Theorem 2. It is clear that $Z_1 = 1$ and $Z_2 = 1$ when we take $Z_1 > 0$ from the results in Lemma 1, and $\{\tilde{Z}_1(t), \tilde{Z}_2(t), \tilde{Z}_3(t) : t \in [0, T]\}$ is the solution to the system of ODEs

$$\begin{cases} \dot{\tilde{Z}}_1(t) = -\tilde{Z}_1^2(t) + 1; \\ \dot{\tilde{Z}}_2(t) = -\tilde{Z}_1(t)\tilde{Z}_2(t) + \tilde{Z}_1(t); \\ \dot{\tilde{Z}}_3(t) = \tilde{Z}_2(t) - \dfrac{1}{2}\tilde{Z}_2^2(t) + \dfrac{1}{2}\sigma^2\tilde{Z}_1(t) - \dfrac{1}{2}; \\ \tilde{Z}_1(0) = \tilde{Z}_2(0) = \tilde{Z}_3(0) = 0. \end{cases}$$

By calculation, we obtain the solution to the above system of ODEs as follows

$$\tilde{Z}_1(t) = \frac{1-e^{-2t}}{1+e^{-2t}}, \quad \tilde{Z}_2(t) = 1 - e^{-\int_0^t \tilde{Z}_1(s)ds} = 1 - \frac{2e^{-t}}{1+e^{-2t}},$$

and

$$\tilde{Z}_3(t) = \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 \ln\left(\frac{2}{1+e^{-2t}}\right) + \frac{1}{2} - \frac{1}{1+e^{-2t}}.$$

Similarly, we know that f_1, f_2, f_3 is the solution to

$$\begin{cases} \dot{f}_1(t) = -2f_1(t) + 2; \\ \dot{f}_2(t) = -f_2(t) + 1; \\ \dot{f}_3(t) = \frac{1}{2}\sigma^2 f_1(t); \\ f_1(0) = f_2(0) = f_3(0) = 0, \end{cases}$$

which gives that

$$f_1(t) = 1 - e^{-2t}, \; f_2(t) = 1 - e^{-t}, \; f_3(t) = \frac{1}{2}\sigma^2 t - \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 e^{-2t}.$$

Thus, we could obtain the difference between $f_i(t)$ and $\tilde{Z}_i(t)$ for $i = 1, 2, 3$ by

$$\Gamma_1(T) = e^{-2T}\frac{1-e^{-2T}}{1+e^{-2T}}, \quad \Gamma_2(T) = e^{-T}\frac{1-e^{-2T}}{1+e^{-2T}},$$

and

$$\Gamma_3(T) = \frac{1}{4}\sigma^2\left(e^{-2T} - 1 + 2\ln\left(\frac{2}{1+e^{-2T}}\right)\right) + \frac{1}{1+e^{-2T}} - \frac{1}{2}.$$

Note that for $x \in \mathbb{R}$ and T large enough,

$$J_T(x; u^*) - V_T(x) = \frac{1}{2}\Gamma_1(T)x^2 + \Gamma_2(T)x + \Gamma_3(T) \geq \Gamma_3(T) - \frac{(\Gamma_2(T))^2}{2\Gamma_1(T)} \geq 0.$$

On the other hand side, we have

$$\begin{aligned} &J_T(x; u^*) - V_T(x) \\ = &\frac{e^{-2T}(1-e^{-2T})}{2(1+e^{-2T})}x^2 + \frac{e^{-T}(1-e^{-2T})}{1+e^{-2T}}x + \frac{\sigma^2}{4}\left(e^{-2T} - 1 + 2\ln\left(\frac{2}{1+e^{-2T}}\right)\right) + \frac{1}{1+e^{-2T}} - \frac{1}{2}. \end{aligned}$$

Let $T \to \infty$, we find that

$$\lim_{T\to\infty}(J_T(x; u^*) - V_T(x)) = \frac{1}{4}\sigma^2(\ln 4 - 1) + \frac{1}{2},$$

which yields that $J_T(x; u^*) - V_T(x) = O(1)$ for all $x \in \mathbb{R}$. Thus, we obtain the desired result in Theorem 2 under this example.

Next, we check the result in Theorem 3. It is easy to get that $D_1 = 1$ and $D_2 = 0$ from (22). Denote $\delta_T(t) = X_T^*(t) - X^*(t)$ and $\tau(t) = T - t$, we have

$$d\delta_T(t) = -\tilde{Z}_1(\tau)\delta_T(t)dt + \left(1 - \tilde{Z}_2(\tau)\right)dt + \left(1 - \tilde{Z}_1(\tau)\right)X^*(t)dt$$

with the initial value $\delta_T(0) = x - \bar{X} \sim \mathcal{N}(x, \frac{\sigma^2}{2})$. Applying the integrating factor method, we have the explicit form of $\delta_T(t)$ as following

$$\delta_T(t) = \delta_T(0)e^{-\int_0^t \tilde{Z}_1(\tau(s))ds} + \int_0^t e^{-\int_s^t \tilde{Z}_1(\tau(r))dr}\left(1 - \tilde{Z}_2(\tau(s))\right)ds \\ + \int_0^t e^{-\int_s^t \tilde{Z}_1(\tau(r))dr}\left(1 - \tilde{Z}_1(\tau(s))\right)X^*(s)ds.$$

Then, we get the estimation

$$\mathbb{E}\left[|\delta_T(t)|^2\right] \leq 3e^{-\int_0^t 2\tilde{Z}_1(\tau(s))ds}\mathbb{E}\left[|\delta_T(0)|^2\right] + 3\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(\tau(r))dr}\left(1 - \tilde{Z}_2(\tau(s))\right)ds\right)^2 \\ + 3\mathbb{E}\left[\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(\tau(r))dr}\left(1 - \tilde{Z}_1(\tau(s))\right)X^*(s)ds\right)^2\right].$$

Firstly, since $\delta_T(0) = x - \bar{X} \sim \mathcal{N}(x, \frac{\sigma^2}{2})$, we know that

$$3e^{-\int_0^t 2\tilde{Z}_1(\tau(s))ds}\mathbb{E}\left[|\delta_T(0)|^2\right] = 3e^{-2t}\left(\frac{1+e^{2t-2T}}{1+e^{-2T}}\right)^2\left(x^2 + \frac{\sigma^2}{2}\right) \leq 12e^{-2t}\left(x^2 + \frac{\sigma^2}{2}\right).$$

Next, by the Hölder's inequality and some simplifications, we can estimate the second term as

$$3\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(\tau(r))dr}\left(1 - \tilde{Z}_2(\tau(s))\right)ds\right)^2 \leq 12e^{-2(T-t)}.$$

Lastly, using the Hölder's inequality again, we have

$$3\mathbb{E}\left[\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(\tau(r))dr}\left(1 - \tilde{Z}_1(\tau(s))\right)X^*(s)ds\right)^2\right] \leq 24e^{-4T}\int_0^t e^{4s}\mathbb{E}\left[(X^*(s))^2\right]ds.$$

Similar with the calculation of expectation and variance of X^* in Subsection 2.2.3, we know that

$$\mathbb{E}\left[X^*(s)\right] = \mathbb{E}\left[\bar{X}\right]e^{-s} = 0, \quad \mathbb{V}ar\left(X^*(s)\right) = \frac{\sigma^2}{2}\left(1 - e^{-2s}\right) + \mathbb{V}ar\left(\bar{X}\right)e^{-2s} = \frac{\sigma^2}{2},$$

as $\bar{X} \sim \mathcal{N}(0, \frac{\sigma^2}{2})$, which gives $\mathbb{E}[(X^*(s))^2] = \frac{\sigma^2}{2}$. Then

$$3\mathbb{E}\left[\left(\int_0^t e^{-\int_s^t \tilde{Z}_1(\tau(r))dr}\left(1 - \tilde{Z}_1(\tau(s))\right)X^*(s)ds\right)^2\right] \leq 3\sigma^2 e^{-4(T-t)}.$$

Thus, we obtain the desired inequality

$$\mathbb{E}\left[|\delta_T(t)|^2\right] \leq 12\left(x + \frac{\sigma^2}{2}\right)e^{-2t} + 12e^{-2(T-t)} + 3\sigma^2 e^{-4(T-t)} \\ \leq K_6\left(e^{-2t} + e^{-2(T-t)}\right)$$

for all $t \in [0, T]$, where $K_6 = \max\{12(x + \frac{\sigma^2}{2}), 12 + 3\sigma^2\}$. Similarly, since

$$u_T^*(t) - u^*(t) = 1 - \tilde{Z}_2(\tau) + \left(1 - \tilde{Z}_1(\tau)\right)X^*(t) - \tilde{Z}_1(\tau)\delta_T(t),$$

by some analytical calculations and simplifications, we have

$$\mathbb{E}\left[|u_T^*(t) - u^*(t)|^2\right] \leq K_7\left(e^{-2t} + e^{-2(T-t)}\right)$$

for all $t \in [0, T]$, where $K_7 \geq 12 + 6\sigma^2 + 3K_6$. Therefore, the desired turnpike property is obtained.

5 Appendix

Proof of Lemma 7. First, we can extend the value function from $V_T(x)$ to $V_T(t,x)$ with t for the initial time. From the standard dynamic programming principle, we obtain the HJB equation

$$\begin{cases} -\partial_t V_T(t,x) + H\left(x, -\nabla_x V_T(t,x), -D_x^2 V_T(t,x)\right) = 0, \\ V_T(T,x) = g(x), \end{cases} \tag{44}$$

where

$$\begin{aligned} L(x,u) &= \frac{1}{2}\left(x^\top Q x + |u|^2 + 2q^\top x + 2r^\top u\right), \\ H(x,\bar{p},\bar{q}) &= \sup_{u\in\mathbb{R}}\left\{(Ax+u+b)^\top \bar{p} + \frac{1}{2}\sigma^2 \text{trace}(\bar{q}) - L(x,u)\right\}. \end{aligned}$$

Taking derivative to the terms in the supermum with respect to u, and letting it be zero, we have $u + r + \nabla_x V_T(t,x) = 0$. Thus, the optimal feedback control is given by

$$u_T^*(t) = -\left(r + \nabla_x V_T\left(t, X_T^*(t)\right)\right),$$

and the value function $V_T(t,x)$ satisfies

$$\begin{aligned} 0 = \partial_t V_T(t,x) &+ x^\top A \nabla_x V_T(t,x) + (b-r)^\top \nabla_x V_T(t,x) - \frac{1}{2}|\nabla_x V_T(t,x)|^2 \\ &+ \frac{1}{2}\sigma^2 \Delta_x V_T(t,x) + \frac{1}{2}x^\top Q x + q^\top x - \frac{1}{2}|r|^2 \end{aligned} \tag{45}$$

with the terminal condition that $V_T(T,x) = g(x) = \frac{1}{2}x^\top N_1 x + x^\top N_2 + N_3$.

Next, we give the semi-explicit solution to the HJB equation (44). Suppose the solution V_T to the HJB equation (44) is in $C^{1,2}([0,T]\times\mathbb{R}^d)$, we assume the value function has the form

$$V_T(t,x) = \frac{1}{2}x^\top \tilde{Z}_1(t) x + x^\top \tilde{Z}_2(t) + \tilde{Z}_3(t),$$

where $\tilde{Z}_1 : [0,T] \mapsto \mathbb{S}^d$, $\tilde{Z}_2 : [0,T] \mapsto \mathbb{R}^d$ and $\tilde{Z}_3 : [0,T] \mapsto \mathbb{R}$ are real-valued functions in $C^1([0,T])$. Then, it is straightforward to get that

$$\begin{aligned} \partial_t V_T(t,x) &= \frac{1}{2}x^\top \dot{\tilde{Z}}_1(t) x + x^\top \dot{\tilde{Z}}_2(t) + \dot{\tilde{Z}}_3(t), \\ \nabla_x V_T(t,x) &= \tilde{Z}_1(t) x + \tilde{Z}_2(t), \\ D_x^2 V_T(t,x) &= \tilde{Z}_1(t). \end{aligned}$$

Plugging the above terms into equation (45), we have

$$\begin{aligned} 0 = \frac{1}{2}x^\top &\left(\dot{\tilde{Z}}_1(t) + 2A\tilde{Z}_1(t) - \tilde{Z}_1^2(t) + Q\right) x \\ &+ x^\top \left(\dot{\tilde{Z}}_2(t) + A\tilde{Z}_2(t) - \tilde{Z}_1(t)\tilde{Z}_2(t) + \tilde{Z}_1(t)(b-r) + q\right) \\ &+ \dot{\tilde{Z}}_3(t) + \frac{1}{2}\sigma^2 \text{trace}(\tilde{Z}_1(t)) + b^\top \tilde{Z}_2(t) - \frac{1}{2}|\tilde{Z}_2(t) + r|^2 \end{aligned}$$

for all $x \in \mathbb{R}^d$ with $\tilde{Z}_1(T) = N_1$, $\tilde{Z}_2(T) = N_2$, and $\tilde{Z}_3(T) = N_3$. Setting up the coefficients of linear and quadratic terms with respect to x and constant to 0, we obtain the Riccati system of ODEs (28) by reversing the time $\tau = T - t$ and using the same notations $\tilde{Z}_i$ for $i = 1,2,3$. Moreover, from the explicit form of $V_T(t,x)$, the optimal feedback control of the finite time control problem is given by $u_T^*(t) = -(\tilde{Z}_1(T-t)X_T^*(t) + \tilde{Z}_2(T-t) + r)$ for all $t \in [0,T]$, where $\tilde{Z}_1$ and $\tilde{Z}_2$ are from the Riccati system of ODEs (28). $\square$

Proof of Lemma 8. First, we extend the cost functional $J_T(x,u)$ to $J_T(t,x,u)$ with t being the initial time. From (24), the optimal control for the probabilistic cell problem is

$$u^*(t) = -\left(Z_1 X^*(t) + Z_2 + r\right),$$

which is a feedback control. If we take this control in the finite time control problem, then the underlying process becomes

$$\begin{aligned} dX_T(t) &= \left(AX_T(t) - (Z_1 X_T(t) + Z_2 + r) + b\right) dt + \sigma dW(t) \\ &= -D_1 X_T(t) dt + D_2 dt + \sigma dW(t) \end{aligned}$$

with $X_T(0) = x$, where D_1 and D_2 are constants given in (22). By inserting the optimal control of the probabilistic cell problem into the cost functional of the finite time control problem, we obtain $J_T(t,x;u^*)$ equals to

$$\mathbb{E}\left[\int_t^T \left(\frac{1}{2} X_T^\top(s)\left(Q + Z_1^2\right) X_T(s) + X_T^\top(s)(Z_1 Z_2 + q) + \frac{1}{2}\left(|Z_2|^2 - |r|^2\right)\right) ds\right].$$

From Feynman-Kac's formula, if $J_T(\cdot,\cdot;u^*) \in C^{1,2}([0,T]\times\mathbb{R}^d)$, it is the solution to the following PDE

$$\begin{cases} \partial_t J_T(t,x;u^*) + (-D_1 x + D_2)^\top \nabla_x J_T(t,x;u^*) + \dfrac{1}{2}\sigma^2 \Delta_x J_T(t,x;u^*) \\ \qquad + \dfrac{1}{2} x^\top \left(Q + Z_1^2\right) x + x^\top \left(Z_1 Z_2 + q\right) + \dfrac{1}{2}\left(|Z_2|^2 - |r|^2\right) = 0, \\ J_T(T,x;u^*) = g(x). \end{cases} \tag{46}$$

We consider that the solution to (46) has the form

$$J_T(t,x;u^*) = \frac{1}{2} x^\top f_1(t) x + x^\top f_2(t) + f_3(t),$$

where $f_1 : [0,T] \mapsto \mathbb{S}^d$, $f_2 : [0,T] \mapsto \mathbb{R}^d$ and $f_3 : [0,T] \mapsto \mathbb{R}$ are real-valued functions in $C^1([0,T])$. Then, it is clear that

$$\begin{aligned} \partial_t J_T(t,x;u^*) &= \frac{1}{2} x^\top \dot{f}_1(t) x + x^\top \dot{f}_2(t) + \dot{f}_3(t) \\ \nabla_x J_T(t,x;u^*) &= f_1(t) x + f_2(t) \\ D_x^2 J_T(t,x;u^*) &= f_1(t). \end{aligned}$$

Plugging the above terms into the PDE (46) satisfied by $J_T(t,x;u^*)$, we obtain

$$\begin{aligned} &\frac{1}{2} x^\top \dot{f}_1(t) x + x^\top \dot{f}_2(t) + \dot{f}_3(t) + (-D_1 x + D_2)^\top (f_1(t) x + f_2(t)) + \frac{1}{2}\sigma^2 \mathrm{trace}(f_1(t)) \\ &\qquad + \frac{1}{2} x^\top \left(Q + Z_1^2\right) x + x^\top \left(Z_1 Z_2 + q\right) + \frac{1}{2}\left(|Z_2|^2 - |r|^2\right) = 0 \end{aligned}$$

for all $x \in \mathbb{R}^d$ with $f_1(T) = N_1$, $f_2(T) = N_2$, and $f_3(T) = N_3$. Setting up the coefficients of linear and quadratic terms with respect to x and constant to 0, and plugging D_1, D_2 in (22) back, we obtain (29) by reversing the time $\tau = T - t$ and using the same notations f_i for $i = 1,2,3$. $\square$

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