American option model and negative Fichera function on degenerate boundary

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Abstract We study American put option with stochastic volatility whose value function is associated with a 2-dimensional parabolic variational inequality with degenerate boundaries. Given the Fichera function on the boundary, we first analyze the existences of the strong solution and the properties of the 2-dimensional manifold for the free boundary. Thanks to the regularity result of the underlying PDE, we can also provide the uniqueness of the solution by the argument of the verification theorem together with the generalized Itos formula even though the solution may not be second order differentiable in the space variable across the free boundary.

1 Introduction

Option pricing is one of the most important topics in the quantitative finance research. Although the Black-Scholes model has been well studied, empirical evidence suggests that the Black-Scholes model is inadequate to describe asset returns and the behavior of the option markets. One possible remedy is to assume that the volatility of the asset price also follows a stochastic process, see [9] and [10].

In the standard Black-Scholes model, a standard logarithmic change of variables transforms the Black-Scholes equation into an equation with constant coefficients which can be studied by a general PDE theory directly. Different from the standard Black-Scholes PDE, the general PDE methods does not directly apply to the PDE

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associated with the option pricing underlying the stochastic volatility model in the following cases: 1) The pricing equation is degenerate at the boundary; 2) The drift and volatility coefficients may grows faster than linear, see [9].

In the related literatures, [9] derived a closed-form solution for the price of a European call option on some specific stochastic volatility models. [6] also studied the Black-Scholes equation of European option underlying stochastic volatility models, and showed that the value function was the unique classical solution to a PDE with a certain boundary behavior. Also, [2] showed a necessary and sufficient condition on the uniqueness of classical solution to the valuation PDE of European option in a general framework of stochastic volatility models. In contrast to the European option pricing on the stochastic volatility model, although there have been quite a few approximate solutions and numerical approaches, such as [1, 4], the study of the existence and uniqueness of strong solution for PDE related to American option price on the stochastic volatility model is rather limited. In particular, the unique solvability of PDE associated with American options of finite maturity with the presence of degenerate boundary and super-linear growth has not been studied in an appropriate Sobolev space.

Note that, American call options with no dividend is equivalent to the European call option. For this reason, we only consider a general framework of American put option model with stochastic volatility whose value function is associated by a 2-dimensional parabolic variational inequality. On the other hand, in the theory of linear PDE, boundary conditions along degenerate boundaries should not needed if the Fichera function is nonnegative, otherwise it should be imposed, see [16]. Therefore, we only consider the case when Fichera function is negative on the degenerate boundary $y = 0$ in this paper, and leave the other case in the future study.

To resolve solvability issue, we adopt similar methodology of [3] to work on a PDE of truncated version backward in time using appropriate penalty function and mollification. The main difference is that [3] studies constant drift and volatility, while the current paper considers functions of drift and volatility and the negative Fichera function plays crucial role in the proof.

Uniqueness issue is usually tackled by comparison result implied by Ishii's lemma with notions of viscosity solution, see [5]. However, this approach does not apply in this problem due to the fast growth of drift and volatility functions on unbounded domain. The approach to establish uniqueness in our paper is similar to the classical verification theorem conducted to classical PDE solution. In fact, a careful construction leads to a local regularity of the solutions in Sobolev space, and this enables us to apply generalized Ito's formula (see [12]) with weak derivatives. Note that this approach not only provides uniqueness of strong solution of PDE, but also provides that the value function of American option is exactly the unique strong solution.

In the next section, we first introduce the generalized stochastic volatility model. Section 3 shows the existence of strong solution to the truncated version of the variational inequality. We characterize the free boundary in section 4. Section 5 shows that the value function of the American option price is the unique strong solution to the variational inequality with appropriate boundary datum. Concluding remarks are given in section 6.

2 Stochastic volatility model

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a filtered probability space with a filtration $(\mathscr{F}_t)_{t\geq 0}$ that satisfies the usual conditions, W_t and B_t be two standard Brownian motions with correlation ρ. Suppose the stock price follows

$$(\text{Stk}) \quad dX_s = X_s(rds + \sigma(Y_s)dW_s), \quad X_t = x > 0,$$

and volatility follows

$$(\text{Vol}) \quad dY_s = \mu(Y_s)ds + b(Y_s)dB_s, \quad Y_t = y > 0.$$

Let $X^{x,t}$ and $Y^{y,t}$ be dynamics satisfying (Stk) and (Vol) with respective initial conditions on superscripts.

We consider an American put option underlying the asset X_s with strike $K > 0$ and maturity T, which has the payoff $(K - X_\tau)^+$ at the exercise time $\tau \in \mathscr{T}_{t,T}$. Here $\mathscr{T}_{t,T}$ denotes the set of all stopping times in $[t, T]$. Define value function of the optimal stopping by

$$V(x,y,t) = \sup_{\tau \in \mathscr{T}_{t,T}} \mathbb{E}_{x,y,t}[e^{-r(\tau - t)}(K - X_\tau)^+], \quad (x,y,t) \in \bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+ \times [0,T]. \tag{1}$$

Following assumptions will be imposed:

(A1) μ, σ^2, b^2 are locally Lipschitz continuous on $\mathbb{R}$ with $\mu(0) = \sigma(0) = b(0) = 0$ and $\sigma(y), b(y) > 0, \sigma'(y) \geq 0$ for all $y > 0$.
(A2) $|\mu| + b$ is at most linear growth, and $(\sigma^2)'$ is at most polynomial growth.

In the above, $\sigma'(y)$ and $(\sigma^2)'(y)$ stand for the first and second derivatives of $\sigma(y)$, respectively. Under the assumptions (A1)–(A2), we have unique non-negative, non-explosive strong solutions for both (Stk) and (Vol). Furthermore, $X_s^{x,t} > 0$ for all $s > t$. Such a stock price model includes Heston model.

Provided that the value function $V(x,y,t)$ is smooth enough, applying dynamic programming principle and Itô's formula, the value of the option $V(x,y,t)$ formally satisfies the variational inequality

$$\begin{cases} \min\Big\{ -\partial_t V - \mathscr{L}_x V,\ V - (K-x)^+ \Big\} = 0, & (x,y,t) \in Q := \mathbb{R}^+ \times \mathbb{R}^+ \times [0,T), \\ V(x,y,T) = (K-x)^+, & (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{cases} \tag{2}$$

where

$$\mathscr{L}_x V = \frac{1}{2}x^2\sigma^2(y)\partial_{xx}V + \rho x \sigma(y) b(y) \partial_{xy} V + \frac{1}{2} b^2(y) \partial_{yy} V + rx\partial_x V + \mu(y)\partial_y V - rV.$$

On the boundary $x = 0$, the Fichera condition on linear parabolic equation suggests us not to impose any boundary condition. On the boundary $y = 0$, the Fichera function is

$$F = \Big[\mu(y) - \frac{1}{2}\rho\sigma(y)b(y) - b(y)b'(y)\Big]\Big|_{y=0} = \mu(0) - \lim_{y\to 0} b(y)b'(y).$$

So

(F1) when $\mu(0) < \lim_{y\to 0} b(y)b'(y)$, one has to impose the boundary condition;

(F2) when $\mu(0) \geq \lim_{y\to 0} b(y)b'(y)$, one should not impose any boundary condition.

Although the problem (2) is a variational inequality instead of linear PDE, the Fichera condition in the linear PDE theory does not directly prove the existence of solution to the problem (2). Throughout this paper, we study the relation between value function (1) and PDE of (2) with an appropriate boundary data on $y = 0$ under the case (F1), while the case (F2) is left in the future study.

Then, what boundary condition should be imposed on the boundary $y = 0$? To proceed, we define $\nu := \inf\{s > t : Y_s = 0\}$ the first hitting time of the process Y_s to the boundary $y = 0$. Then for any stopping time $\tau > \nu$, we have

$$e^{-r(\tau-t)}(K - X_\tau)^+ \leq e^{-r(\nu-t)}(K - X_\nu)^+, \tag{3}$$

thus

$$V(x,0,t) = \sup_{\tau\in\mathscr{T}_{t,T}} \mathbb{E}_{x,0,t}[e^{-r(\tau-t)}(K - X_\tau)^+] \leq (K - x)^+.$$

On the other hand, by taking $\tau = t$,

$$V(x,0,t) = \sup_{\tau\in\mathscr{T}_{t,T}} \mathbb{E}_{x,0,t}[e^{-r(\tau-t)}(K - X_\tau)^+] \geq (K - x)^+.$$

Hence

$$V(x,0,t) = (K-x)^+ = \sup_{\tau\in\mathscr{T}_{t,T}} \mathbb{E}_{x,0,t}[e^{-r(\tau-t)}(K - X_\tau)^+], \quad (x,t) \in \mathbb{R}^+ \times [0,T). \tag{4}$$

Due to the non-linearity of the variational inequality (2), we may not expect the heuristic argument in the above assuming the enough regularity on V. In addition, Fichera condition on the boundary data is suitable only for linear second order PDE, see [16]. Our objective in this paper is to justify the regularity of the value function in the Sobolev space, so that the value function (1) can be characterized as the unique solution of the variational inequality (2) and an additional boundary data (4).

3 Solvability on the transformed problem

In order to obtain the existence of solution to the variational inequality (2) with boundary condition (4) in Sobolev space, we consider the existence of strong solution to the associated transformed problem of (2) with boundary condition (4) in this section.

To proceed, we take a simple logarithm transformation to the variational inequality. Let $s=\ln x,\ \theta=T-t,\ u(s,y,\theta)=V(x,y,t)$, then

$$\begin{aligned}\mathscr{L}_x V \;=\; &\frac{1}{2}\sigma^2(y)\partial_{ss}u+\rho\sigma(y)b(y)\partial_{sy}u\\ &\qquad\qquad+\frac{1}{2}b^2(y)\partial_{yy}u+\Big(r-\frac{1}{2}\sigma^2(y)\Big)\partial_s u+\mu(y)\partial_y u-ru\\ :=\; &\mathscr{L}_s u.\end{aligned}$$

Thus $u(s,y,\theta)$ satisfies

$$\begin{cases}\min\Big\{\partial_\theta u-\mathscr{L}_s u,\ u-(K-e^s)^+\Big\}=0, & (s,y,\theta)\in\mathscr{Q}:=\mathbb{R}\times\mathbb{R}^+\times(0,T],\\ u(s,y,0)=(K-e^s)^+, & s\in\mathbb{R},\ y\in\mathbb{R}^+,\\ u(s,0,\theta)=(K-e^s)^+, & s\in\mathbb{R},\ \theta\in(0,T].\end{cases}\tag{5}$$

Problem (5) is a variational inequality, we apply penalty approximation techniques to show the existence of strong solution to (5). Suppose $u_\varepsilon(s,y,\theta)$ satisfies

$$\begin{cases}\partial_\theta u_\varepsilon-\mathscr{L}_s^\varepsilon u_\varepsilon+\beta_\varepsilon(u_\varepsilon-\pi_\varepsilon(K-e^s))=0, & (s,y,\theta)\in\mathscr{Q},\\ u_\varepsilon(s,y,0)=\pi_\varepsilon(K-e^s), & s\in\mathbb{R},\ y\in\mathbb{R}^+,\\ u_\varepsilon(s,0,\theta)=\pi_\varepsilon(K-e^s), & s\in\mathbb{R},\ \theta\in(0,T].\end{cases}\tag{6}$$

where $\mathscr{L}_s^\varepsilon u=\frac{1}{2}(\sigma^2(y)+\varepsilon)\partial_{ss}u+\rho(\sigma(y)b(y)+\varepsilon)\partial_{sy}u+\frac{1}{2}(b^2(y)+\varepsilon)\partial_{yy}u+(r-\frac{1}{2}\sigma^2(y)-\frac{1}{2}\varepsilon)\partial_s u+\mu(y)\partial_y u-ru$, and $\beta_\varepsilon(\xi)$ (Fig. 1.), $\pi_\varepsilon(\xi)$ (Fig. 2.) satisfy

$$\beta_\varepsilon(\xi)\in C^2(-\infty,\ +\infty),\ \beta_\varepsilon(\xi)\le 0,\ \beta_\varepsilon(0)=-2r(K+1),\ \beta_\varepsilon'(\xi)\ge 0,\ \beta_\varepsilon''(\xi)\le 0.$$

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\xi)=\begin{cases}0, & \xi>0,\\ -\infty, & \xi<0,\end{cases}\qquad \pi_\varepsilon(\xi)=\begin{cases}\xi, & \xi\ge\varepsilon,\\ \nearrow, & |\xi|\le\varepsilon,\\ 0, & \xi\le-\varepsilon,\end{cases}$$

$$\pi_\varepsilon(\xi)\in C^\infty,\ 0\le\pi_\varepsilon'(\xi)\le 1,\ \pi_\varepsilon''(\xi)\ge 0,\ \lim_{\varepsilon\to 0}\pi_\varepsilon(\xi)=\xi^+.$$

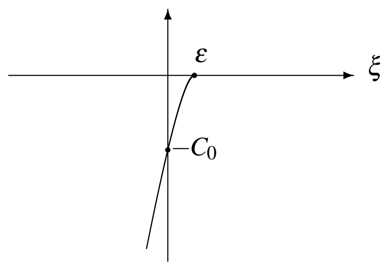

Fig. 1. $\beta_\varepsilon(\xi)$

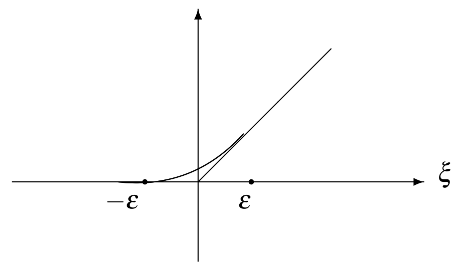

Fig. 2. $\pi_\varepsilon(\xi)$

Since $\mathscr{Q} = (-\infty,+\infty)\times(0,+\infty)\times(0,T]$ is infinitely, we consider the truncated version of (6), denote $\mathscr{Q}^N = (-N,N)\times(0,N)\times(0,T]$, let $u_\varepsilon^N(s,y,\theta)$ satisfy

$$\begin{cases} \partial_\theta u_\varepsilon^N - \mathscr{L}_s^\varepsilon u_\varepsilon^N + \beta_\varepsilon(u_\varepsilon^N - \pi_\varepsilon(K-e^s)) = 0, & (s,y,\theta)\in\mathscr{Q}^N, \\ u_\varepsilon^N(s,y,0) = \pi_\varepsilon(K-e^s), & (s,y)\in(-N,N)\times(0,N), \\ u_\varepsilon^N(s,0,\theta) = \pi_\varepsilon(K-e^s), & (s,\theta)\in(-N,N)\times(0,T], \\ \partial_y u_\varepsilon^N(s,N,\theta) = 0, & (s,\theta)\in(-N,N)\times(0,T], \\ u_\varepsilon^N(-N,y,\theta) = \pi_\varepsilon(K-e^{-N}), & (y,\theta)\in(0,N)\times(0,T], \\ \partial_s u_\varepsilon^N(N,y,\theta) = 0, & (y,\theta)\in(0,N)\times(0,T]. \end{cases} \tag{7}$$

Lemma 1. *For any fixed ε, $N>0$, there exists a unique solution $u_\varepsilon^N \in W_p^{2,1}(\mathscr{Q}^N)$ to the problem (7), and*

$$\pi_\varepsilon(K-e^s) \le u_\varepsilon^N(s,y,\theta) \le K+1, \tag{8}$$
$$\partial_\theta u_\varepsilon^N(s,y,\theta) \ge 0. \tag{9}$$

Proof. For any fixed ε, $N>0$, it is not hard to show by the fixed point theorem that problem (7) has a solution $u_\varepsilon^N \in W_p^{2,1}(\mathscr{Q}^N)$, and

$$|u_\varepsilon^N|_0 \le C(|\beta_\varepsilon(u_\varepsilon^N - \pi_\varepsilon(K-e^s))|_0 + |\pi_\varepsilon(K-e^s)|_0) \le C(|\beta_\varepsilon(-K-1)| + K+1) \le C.$$

The proof of uniqueness is a standard way as well.

Since

$$\begin{aligned} &\partial_\theta \pi_\varepsilon(K-e^s) - \mathscr{L}_s^\varepsilon(\pi_\varepsilon(K-e^s)) + \beta_\varepsilon(0) \\ &= -\frac{1}{2}(\sigma^2(y)+\varepsilon)(\pi_\varepsilon''(\cdot)e^{2s} - \pi_\varepsilon'(\cdot)e^s) + \left(r - \frac{1}{2}\sigma^2(y) - \frac{1}{2}\varepsilon\right)\pi_\varepsilon'(\cdot)e^s + r\pi_\varepsilon(K-e^s) + \beta_\varepsilon(0) \\ &\le r(K+\varepsilon) + r(K+\varepsilon) + \beta_\varepsilon(0) \le 0. \end{aligned}$$

Combining with the initial and boundary conditions, we know $\pi_\varepsilon(K-e^s)$ is a subsolution of (7). Similarly we know $K+1$ is a supersolution of (7).

Next we will prove (9). Set $u^\delta(s,y,\theta) := u_\varepsilon^N(s,y,\theta+\delta)$, then $u^\delta(s,y,\theta)$ satisfies

$$\begin{cases} \partial_\theta u^\delta - \mathscr{L}_s^\varepsilon u^\delta + \beta_\varepsilon(u^\delta - \pi_\varepsilon(K-e^s)) = 0, \\ u^\delta(s,y,0) = u_\varepsilon^N(s,y,\delta) \geq \pi_\varepsilon(K-e^s) = u_\varepsilon^N(s,y,0), \\ u^\delta(s,0,\theta) = \pi_\varepsilon(K-e^s), \\ \partial_y u^\delta(s,N,\theta) = 0, \\ u^\delta(-N,y,\theta) = \pi_\varepsilon(K-e^{-N}), \\ \partial_s u^\delta(N,y,\theta) = 0. \end{cases}$$

Applying the comparison principle, we have

$$u^\delta(s,y,\theta) \geq u_\varepsilon^N(s,y,\theta), \quad (s,y,\theta) \in (-N,N)\times(0,N)\times(0,T-\delta],$$

which yields $\partial_\theta u_\varepsilon^N \geq 0$.

Letting $N \to +\infty$, the existence of strong solution to the penalty problem (6) with some estimates is given in the following theorem.

Lemma 2. *For any fixed* $\varepsilon > 0$*, there exists a unique solution* $u_\varepsilon(s,y,\theta) \in W_{p,loc}^{2,1}(\mathscr{Q}) \cap C^1(\overline{\mathscr{Q}})$ *to the problem (6) for any* $1 < p < +\infty$*, and*

$$\pi_\varepsilon(K-e^s) \leq u_\varepsilon(s,y,\theta) \leq K+1, \tag{10}$$
$$\partial_\theta u_\varepsilon(s,y,\theta) \geq 0, \tag{11}$$
$$-e^s \leq \partial_s u_\varepsilon(s,y,\theta) \leq 0, \tag{12}$$
$$\partial_y u_\varepsilon(s,y,\theta) \geq 0. \tag{13}$$

Proof. For any fixed $\varepsilon > 0$, $R > 0$, applying $W_p^{2,1}$ interior estimate with part of boundary [13] to the problem (7) ($N > R$), then

$$|u_\varepsilon^N|_{W_p^{2,1}(\mathscr{Q}^R)} \leq C(|\beta_\varepsilon(u_\varepsilon^N - \pi_\varepsilon(K-e^s))|_{L^\infty(\mathscr{Q}^R)} + |\pi_\varepsilon(K-e^s)|_{W_p^{2,1}(\overline{\mathscr{Q}^R}\cap\{\theta=0\})}) \leq C,$$

where C depends on ε, R but is independent of N. Letting $N \to +\infty$, by the imbedding theorem, we know problem (6) has a solution $u_\varepsilon(s,y,\theta) \in W_{p,loc}^{2,1}(\mathscr{Q}) \cap C^1(\overline{\mathscr{Q}})$. (10)–(11) are consequences of (8)–(9).

Now we aim to prove (12). Differentiate the equation in (6) w.r.t. s and denote $w_1 = \partial_s u_\varepsilon$, then

$$\begin{cases} \partial_\theta w_1 - \mathscr{L}_s^\varepsilon w_1 + \beta_\varepsilon'(\cdot) w_1 = -\beta_\varepsilon'(\cdot)\pi_\varepsilon'(\cdot)e^s \leq 0, \quad (s,y,\theta) \in \mathscr{Q}, \\ w_1(s,y,0) = w_1(s,0,\theta) = -\pi_\varepsilon'(\cdot)e^s \leq 0. \end{cases} \tag{14}$$

Applying the maximum principle [17] we know $w_1 = \partial_s u_\varepsilon \leq 0$. In view of

$$(\partial_\theta - \mathscr{L}_s^\varepsilon)(-e^s) + \beta_\varepsilon'(\cdot)(-e^s) = -\beta_\varepsilon'(\cdot)e^s \leq -\beta_\varepsilon'(\cdot)\pi_\varepsilon'(\cdot)e^s.$$

Combining with the initial and boundary conditions, applying the comparison principle we have

$$-e^s \le w_1(s,y,\theta) = \partial_s u_\varepsilon(s,y,\theta) \le 0.$$

Finally we want to prove (13). We first differentiate (14) w.r.t. s, denote $w_2 = \partial_{ss} u_\varepsilon$, we obtain

$$\begin{cases} \partial_\theta w_2 - \mathscr{L}_s^\varepsilon w_2 + \beta_\varepsilon'(\cdot) w_2 = -\beta_\varepsilon'(\cdot)\pi_\varepsilon'(\cdot)e^s + \beta_\varepsilon'(\cdot)\pi_\varepsilon''(\cdot)e^{2s} - \beta_\varepsilon''(\cdot)[\pi_\varepsilon'(\cdot)e^s + w_1]^2, \\ w_2(s,y,0) = w_2(s,0,\theta) = -\pi_\varepsilon'(\cdot)e^s + \pi_\varepsilon''(\cdot)e^{2s}. \end{cases} \tag{15}$$

Set $w_3(s,y,\theta) := w_2(s,y,\theta) - w_1(s,y,\theta)$, in view of (14) and (15)

$$\begin{cases} \partial_\theta w_3 - \mathscr{L}_s^\varepsilon w_3 + \beta_\varepsilon'(\cdot) w_3 = \beta_\varepsilon'(\cdot)\pi_\varepsilon''(\cdot)e^{2s} - \beta_\varepsilon''(\cdot)[\pi_\varepsilon'(\cdot)e^s + w_1]^2 \ge 0, \\ w_3(s,y,0) = w_3(s,0,\theta) = \pi_\varepsilon''(\cdot)e^{2s} \ge 0. \end{cases}$$

Applying maximum principle we know $w_3(s,y,\theta) \ge 0$, i.e., $\partial_{ss}u_\varepsilon - \partial_s u_\varepsilon \ge 0$.

Differentiate (6) w.r.t. y, denote $w_4(s,y,\theta) = \partial_y u_\varepsilon(s,y,\theta)$. Then we get

$$\begin{cases} \partial_\theta w_4 - \mathscr{L}_s^\varepsilon w_4 - \rho(\sigma'(y)b(y) + \sigma(y)b'(y))\partial_s w_4 - b(y)b'(y)\partial_y w_4 \\ \qquad\qquad -\mu'(y)w_4 + \beta_\varepsilon'(\cdot)w_4 = \sigma(y)\sigma'(y)(\partial_{ss}u_\varepsilon - \partial_s u_\varepsilon), \\ w_4(s,y,0) = 0, \\ w_4(s,0,\theta) \ge 0. \end{cases}$$

Since $\sigma'(y) \ge 0$, $u_\varepsilon \in C^{2,1}(\mathscr{Q})$ and $\partial_{ss}u_\varepsilon - \partial_s u_\varepsilon \ge 0$, by maximum principle [17], we have $\partial_y u_\varepsilon(s,y,\theta) \ge 0$.

Now we are able to show the solvability on the variational inequality (5) in the Sobolev space by the approximation of a subsequence of $\{u_\varepsilon\}$.

Lemma 3. *There exists a solution $u \in W_p^{2,1}(\mathscr{Q}_\delta^N \setminus B_h)$ to the problem (5), where $\mathscr{Q}_\delta^N = (-N,N) \times (\delta,N) \times (0,T]$, $B_h = (\ln K - h, \ln K + h) \times (0,+\infty) \times (0,T]$ for any N, δ, $h > 0$. Moreover,*

$$(K - e^s)^+ \le u(s,y,\theta) \le K + 1, \tag{16}$$

$$\partial_\theta u(s,y,\theta) \ge 0, \tag{17}$$

$$-e^s \le \partial_s u(s,y,\theta) \le 0, \tag{18}$$

$$\partial_y u(s,y,\theta) \ge 0. \tag{19}$$

Proof. Since $\sigma(y)$, $b(y)$ are continuous and $\sigma'(y) \ge 0$, in $\mathscr{Q}_{\frac{1}{2}}^N$, we have $\sigma^2(y) + \varepsilon \ge \sigma^2(\frac{1}{2}) > 0$, and $\lambda_1|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda_1|\xi|^2$, with Λ_1, λ_1 independent of ε. Applying $C^{\alpha,\alpha/2}$ estimate [14] and $W_p^{2,1}$ interior estimate with part of boundary [13], we have

$$|u_\varepsilon|_{C^{\alpha,\alpha/2}\left(\overline{\mathscr{Q}_{\frac{1}{2}}^N}\right)} \le C(|u_\varepsilon|_0 + |\beta_\varepsilon(u_\varepsilon - \pi_\varepsilon(K - e^s))|_0 + [\pi_\varepsilon(K - e^s)]_{C^\gamma(-N,N)}) \le C_1,$$

$$|u_\varepsilon|_{W_p^{2,1}(\mathscr{Q}_{\frac{1}{2}}^N \setminus B_h)} \le C(|u_\varepsilon|_{L^\infty} + |\beta_\varepsilon(u_\varepsilon - \pi_\varepsilon(K - e^s))|_{L^\infty} + |(K - e^s) \vee 0|_{W_p^{2,1}(\mathscr{Q}_{\frac{1}{2}}^N \setminus B_h)}) \le C_2,$$

where C_1, C_2 are independent of ε due to the estimate (10) and the definitions of $\beta_\varepsilon, \pi_\varepsilon$. Thus there exists a subsequence of $\{u_\varepsilon\}$, denote $\{u_\varepsilon^{(1)}\}$, and $u^{(1)} \in W_p^{2,1}(\mathscr{Q}^N_{\frac{1}{2}} \setminus B_h) \cap C\left(\overline{\mathscr{Q}^N_{\frac{1}{2}}}\right)$, such that

$$\begin{aligned} &u_\varepsilon^{(1)}(s,y,\theta) \rightharpoonup u^{(1)}(s,y,\theta) \quad in\, W_p^{2,1}(\mathscr{Q}^N_{\frac{1}{2}} \setminus B_h)\; weakly, \\ &u_\varepsilon^{(1)}(s,y,\theta) \to u^{(1)}(s,y,\theta) \quad in\, C\left(\overline{\mathscr{Q}^N_{\frac{1}{2}}}\right) uniformly. \end{aligned}$$

In a same way, in $\mathscr{Q}^N_{\frac{1}{3}}$, we have $\sigma^2(y)+\varepsilon \geq \sigma^2(\frac{1}{3})$, and $\lambda_2|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda_2|\xi|^2$, with $\Lambda_2,\ \lambda_2$ independent of ε. Thus there exists $\{u_\varepsilon^{(2)}\} \subseteq \{u_\varepsilon^{(1)}\},\ u^{(2)} \in W_p^{2,1}(\mathscr{Q}^N_{\frac{1}{3}} \setminus B_h) \cap C\left(\overline{\mathscr{Q}^N_{\frac{1}{3}}}\right)$, such that

$$\begin{aligned} &u_\varepsilon^{(2)}(s,y,\theta) \rightharpoonup u^{(2)}(s,y,\theta) \quad in\, W_p^{2,1}(\mathscr{Q}^N_{\frac{1}{3}} \setminus B_h)\; weakly, \\ &u_\varepsilon^{(2)}(s,y,\theta) \to u^{(2)}(s,y,\theta) \quad in\, C\left(\overline{\mathscr{Q}^N_{\frac{1}{3}}}\right) uniformly. \end{aligned}$$

Moreover,

$$u^{(2)}(s,y,\theta) = u^{(1)}(s,y,\theta), \quad (s,y,\theta) \in \mathscr{Q}^N_{\frac{1}{2}}.$$

Define $u(s,y,\theta) = u^{(k)}(s,y,\theta)$, if $(s,y,\theta) \in \mathscr{Q}^N_{\frac{1}{k+1}}$, abstracting diagram subsequence $\{u_{\varepsilon_k}^{(k)}\}$, for any $\delta,\ h,\ N > 0$, we have

$$\begin{aligned} &u_{\varepsilon_k}^{(k)}(s,y,\theta) \rightharpoonup u(s,y,\theta) \quad in\, W_p^{2,1}(\mathscr{Q}^N_\delta \setminus B_h)\; weakly, \\ &u_{\varepsilon_k}^{(k)}(s,y,\theta) \to u(s,y,\theta) \quad in\, C(\overline{\mathscr{Q}^N_\delta})\; uniformly, \end{aligned}$$

thus $u(s,y,\theta) \in W_p^{2,1}(\mathscr{Q}^N_\delta \setminus B_h) \cap C(\overline{\mathscr{Q}} \setminus \{y=0\})$ and $u(s,y,\theta)$ satisfies the variational inequality in (5) and the initial condition.

Next we will prove the continuity on the degenerate boundary $y=0$. For any $s_0 \in \mathbb{R} \setminus \{\ln K\}$, then there exists $\varepsilon_0 > 0$ such that $\pi_{\varepsilon_0}(K-e^{s_0}) = (K-e^{s_0})^+$, denote $w_0(s,y,\theta) = \pi_{\varepsilon_0}(K-e^s) + Ay^\alpha \geq 0$, with $0<\alpha<1$, and $A \geq 1$ to be determined, then for any $\varepsilon < \varepsilon_0$

$$\begin{aligned} &\partial_\theta w_0 - \mathscr{L}_s^\varepsilon w_0 + \beta_\varepsilon(w_0 - \pi_\varepsilon(K-e^s)) \\ = &-\frac{1}{2}(\sigma^2(y)+\varepsilon)\pi''_{\varepsilon_0}(\cdot)e^{2s} - \frac{1}{2}(b^2(y)+\varepsilon)A\alpha(\alpha-1)y^{\alpha-2} + r\pi'_{\varepsilon_0}(\cdot)e^s \\ &-\mu(y)\alpha Ay^{\alpha-1} + rw_0 + \beta_\varepsilon(\pi_{\varepsilon_0}(K-e^s) - \pi_\varepsilon(K-e^s) + Ay^\alpha) \\ \geq &-\frac{1}{2}(\sigma^2(y)+\varepsilon)\pi''_{\varepsilon_0}(\cdot)e^{2s} + \frac{1}{2}(b^2(y)+\varepsilon)A\alpha(1-\alpha)y^{\alpha-2} - \mu(y)\alpha Ay^{\alpha-1} + \beta_\varepsilon(0), \end{aligned}$$

since the negative Fichera function indicates $\mu(0) < \lim_{y\to 0} b(y)b'(y)$, in addition, $b^2(y) = O(y)$ or $b^2(y) = o(y)$ when $y \to 0$, hence there exists $\delta_0 > 0$ small enough and independent of ε such that

$$-\frac{1}{2}(\sigma^2(y)+\varepsilon)\pi''_{\varepsilon_0}(\cdot)e^{2s} + \frac{1}{2}(b^2(y)+\varepsilon)A\alpha(1-\alpha)y^{\alpha-2} - \mu(y)\alpha A y^{\alpha-1} + \beta_\varepsilon(0) \geq 0$$

for any $y \in (0,\delta_0)$. Moreover, we can choose A large enough such that $A\delta_0^\alpha \geq K+1$. Combining

$$\pi_{\varepsilon_0}(K-e^s) + Ay^\alpha \geq \pi_\varepsilon(K-e^s), \quad \varepsilon < \varepsilon_0,$$

applying comparison principle, we have

$$\pi_\varepsilon(K-e^s) \leq u_\varepsilon(s,y,\theta) \leq \pi_{\varepsilon_0}(K-e^s) + Ay^\alpha, \quad (s,y,\theta) \in \mathbb{R}\times(0,\delta_0)\times(0,T].$$

Letting $\varepsilon \to 0^+$ we have

$$(K-e^s)^+ \leq u(s,y,\theta) \leq \pi_{\varepsilon_0}(K-e^s) + Ay^\alpha, \quad (s,y,\theta) \in \mathbb{R}\times(0,\delta_0)\times(0,T].$$

In particular

$$(K-e^{s_0})^+ \leq u(s_0,y,\theta) \leq (K-e^{s_0})^+ + Ay^\alpha, \quad y \in (0,\delta_0).$$

Letting $y \to 0^+$, we obtain

$$u(s_0,0,\theta) = (K-e^{s_0})^+, \quad \theta \in (0,T],$$

since s_0 is arbitrary, then $u(s,0,\theta) = (K-e^s)^+$, $s \in \mathbb{R}\setminus\{\ln K\}$. Therefore $u(s,y,\theta)$ is a solution to the problem (5). (16)–(19) are consequences of (10)–(13).

4 Characterization of free boundary to the problem (5)

Variational inequality (5) is an obstacle problem, this section aims to characterize the free boundary arise from (5).

Lemma 4. *The solution to the problem (5) satisfies*

$$u(s,y,\theta) > 0, \quad (s,y,\theta) \in \mathbb{R}\times\mathbb{R}^+\times(0,T].$$

Proof. For any fixed $y_0 > 0$, we have

$$\begin{cases} \partial_\theta u - \mathscr{L}_s u \geq 0, & (s,y,\theta) \in \mathbb{R}\times(y_0,+\infty)\times(0,T], \\ u(s,y,0) = (K-e^s)^+ \geq 0, & (s,y) \in \mathbb{R}\times(y_0,+\infty), \\ u(s,y_0,\theta) \geq (K-e^s)^+ \geq 0, & (s,\theta) \in \mathbb{R}\times(0,T]. \end{cases}$$

Applying strong maximum principle, we obtain

$$u(s,y,\theta) > 0, \quad (s,y,\theta) \in \mathbb{R} \times (y_0, +\infty) \times (0,T].$$

Since y_0 is arbitrary, then we know

$$u(s,y,\theta) > 0, \quad (s,y,\theta) \in \mathbb{R} \times \mathbb{R}^+ \times (0,T].$$

In order to characterize the free boundary, we first define

$$\begin{aligned}
&\mathscr{C}[u] := \{(s,y,\theta) : u(s,y,\theta) = (K-e^s)^+\}(\text{Coincidence set}),\\
&\mathscr{N}[u] := \{(s,y,\theta) : u(s,y,\theta) > (K-e^s)^+\}(\text{Noncoincidence set}).
\end{aligned}$$

Thanks to the estimates (16)–(19) of the solution to (5), problem (5) gives rise to a free boundary that can be expressed as a function of (y,θ). The following three lemmas give the existence and properties of the free boundary.

Proposition 1. *There exists $h(y,\theta) : \mathbb{R}^+ \times (0,T] \to \mathbb{R}$, such that*

$$\mathscr{C}[u] = \{(s,y,\theta) \in \mathscr{Q} : s \le h(y,\theta),\ y \in \mathbb{R}^+,\ \theta \in (0,T]\}. \tag{20}$$

Moreover, for any fixed $y > 0$, $h(y,\theta)$ is monotonic decreasing w.r.t. θ; for any fixed $\theta \in (0,T]$, $h(y,\theta)$ is monotonic decreasing w.r.t. y.

Proof. Since $(K-e^s)^+ = 0$ when $s \ge \ln K$, in view of Lemma 4, we have

$$\{s \ge \ln K\} \subset \mathscr{N}[u], \quad \mathscr{C}[u] \subset \{s < \ln K\}.$$

Hence problem (5) is equivalent to the following problem

$$\begin{cases}
\min\Big\{\partial_\theta u - \mathscr{L}_s u,\ u-(K-e^s)\Big\} = 0, & (s,y,\theta) \in \mathscr{Q} := \mathbb{R} \times \mathbb{R}^+ \times (0,T],\\
u(s,y,0) = (K-e^s)^+, & s \in \mathbb{R},\ y \in \mathbb{R}^+,\\
u(s,0,\theta) = (K-e^s)^+, & s \in \mathbb{R},\ \theta \in (0,T].
\end{cases}$$

Together with (18), we can define

$$h(y,\theta) := \max\{s \in \mathbb{R} : u(s,y,\theta) = (K-e^s)\}, \quad (y,\theta) \in \mathbb{R}^+ \times (0,T],$$

by the definition of $h(y,\theta)$, we know (20) is true.

Suppose $h(y,\theta_1) = s_1$, notice that $\partial_\theta u(s,y,\theta) \ge 0$, then for any $\theta_2 \le \theta_1$,

$$0 \le u(s_1,y,\theta_2) - (K-e^{s_1}) \le u(s_1,y,\theta_1) - (K-e^{s_1}) = 0,$$

from which we infer that

$$u(s_1,y,\theta_2) = (K-e^{s_1}), \quad \theta_2 \le \theta_1.$$

By the definition of $h(y,\theta)$, we know $h(y,\theta_2) \geq s_1 = h(y,\theta_1)$, thus $h(y,\cdot)$ is monotonic decreasing w.r.t. θ.

Similarly, the monotonicity of $h(y,\theta)$ w.r.t. y can be deduced by virtue of $\partial_y u(s,y,\theta) \geq 0$ and the definition of $h(y,\theta)$.

Proposition 2. *$h(y,\theta)$ is continuous on $\mathbb{R}^+ \times [0,T]$ with*

$$h(y,0) := \lim_{\theta \to 0^+} h(y,\theta) = \ln K, \quad y > 0.$$

Proof. We first prove $h(y,\theta)$ is continuous w.r.t. θ. Suppose not. There exists $y_0 > 0,\ \theta_0 > 0$ such that $s_1 := h(y_0,\theta_0+) < h(y_0,\theta_0) := s_2$. Since $h(y_0,\theta_0+) = s_1$(see Fig. 3.), then

$$u(s,y_0,\theta) > K - e^s, \quad s > s_1,\ \theta > \theta_0.$$

In fact, $u \in W_p^{2,1}$ and the embedding theorem imply that u is uniformly continuous, thus there exists $\delta > 0$, take $\mathscr{S}_0 = (s_1,s_2) \times (y_0 - \delta, y_0)$ such that $U_0 := \mathscr{S}_0 \times (\theta_0,T] \subseteq \mathscr{N}[u]$, then

$$\partial_\theta u - \mathscr{L}_s u = 0, \quad (s,y,\theta) \in U_0.$$

Moreover, in view of $h(y_0,\theta_0) := s_2$, then

$$h(y,\theta_0) \geq s_2, \quad y_0 - \delta < y \leq y_0,$$

hence

$$u(s,y,\theta_0) = K - e^s, \quad s \leq s_2,\ y_0 - \delta < y \leq y_0.$$

In particular,

$$u(s,y,\theta_0) = K - e^s, \quad (s,y) \in \overline{U_0} \cap \{\theta = \theta_0\},$$

thus

$$\begin{aligned} \partial_\theta u\big|_{\theta=\theta_0} &= \mathscr{L}_s u\big|_{\theta=\theta_0} = \frac{1}{2}\sigma^2(y)(-e^s) + \Big(r - \frac{1}{2}\sigma^2(y)\Big)(-e^s) - r(K - e^s) \\ &= -rK < 0, \end{aligned}$$

which comes to a contradiction with the fact that $\partial_\theta u \geq 0$. Hence $h(y,\theta)$ is continuous w.r.t. θ.

Since $h(y,\theta)$ is monotonic decreasing w.r.t. θ, then we can define $h(y,0) := \lim_{\theta \to 0^+} h(y,\theta)$. In the same way we can prove $h(y,0) = \ln K$.

Now we aim to prove the continuity of $h(y,\theta)$ w.r.t. y. If this is not true, there exists $\theta_0,\ y_0 > 0$ such that $s_1 := h(y_0+,\theta_0) < h(y_0,\theta_0) := s_2$. Since $h(y,\theta)$ is continuous w.r.t. θ and $u(s,y,\theta) \in C^{1,1,1/2}(\mathscr{Q})$, take $U_0 := (\tilde{s},\bar{s}) \times (y_0,+\infty) \times (\tilde{\theta},\bar{\theta})$, where $(\tilde{s},\bar{s}) \times (\tilde{\theta},\bar{\theta}) \subseteq (h(y_0+,\theta), h(y_0,\theta))$(see Fig. 4.), then $u(s,y,\theta)$ satisfies

$$\partial_\theta u - \mathscr{L}_s u = 0, \quad (s,y,\theta) \in U_0,$$
$$u(s,y_0,\theta) = K - e^s, \quad (s,\theta) \in \overline{U_0} \cap \{y = y_0\},$$
$$\partial_y u(s,y_0,\theta) = 0, \quad (s,\theta) \in \overline{U_0} \cap \{y = y_0\}.$$

Thus

$$\partial_\theta u(s,y_0,\theta) = \partial_{y\theta} u(s,y_0,\theta) = 0, \quad (s,\theta) \in \overline{U_0} \cap \{y = y_0\}.$$

Since $\partial_\theta u \ge 0$ and $\partial_\theta(\partial_\theta u) - \mathscr{L}_s(\partial_\theta u) = 0$ in U_0, by Hopf lemma we know

$$\partial_{y\theta} u(s,y_0,\theta) > 0, \quad (s,\theta) \in \overline{U_0} \cap \{y = y_0\},$$

or

$$\partial_\theta u(s,y,\theta) \equiv 0, \quad (s,y,\theta) \in U_0,$$

but both come to contradictions.

Together with the monotonicity of $h(y,\theta)$ w.r.t. y and θ, we conclude that $h(y,\theta)$ is continuous on $\mathbb{R}^+ \times [0,T]$.

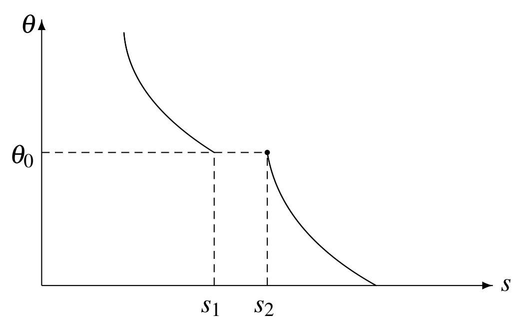

Fig. 3. Discontinuity of $h(y,\theta)$ w.r.t. θ

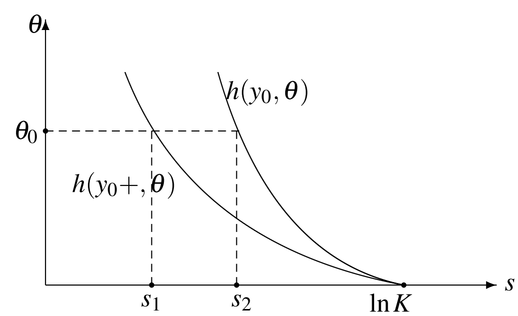

Fig. 4. Discontinuity of $h(y,\theta)$ w.r.t. y

Proposition 3. *The free boundary $h(y,\theta)$ satisfies*

$$h_0(y) \le h(y,\theta) < \ln K, \quad y > 0,\ \theta \in (0,T],$$

where $h_0(y)$ is the free boundary curve of

$$\min\{-\mathscr{L}_s u_\infty(s,y),\ u_\infty(s,y) - (K - e^s)^+\} = 0, \quad (s,y) \in \mathbb{R} \times \mathbb{R}^+. \tag{21}$$

Proof. Since $\partial_\theta u_\infty(s,y) = 0$, then we can rewrite (21) as

$$\begin{cases} \min\left\{\partial_\theta u_\infty - \mathscr{L}_s u_\infty,\ u_\infty - (K - e^s)^+\right\} = 0, \quad (s,y,\theta) \in \mathscr{Q}, \\ u_\infty(s,y)|_{\theta=0} = u_\infty(s,y) \ge (K - e^s)^+ = u(s,y,0). \end{cases}$$

Applying the monotonicity of solution of variational inequality w.r.t. initial value, we have

$$u_\infty(s,y) \geq u(s,y,\theta), \quad \theta \geq 0.$$

By the definitions of $h_0(y)$ and $h(y,\theta)$, we know

$$h_0(y) \leq h(y,\theta).$$

Now we will prove $h(y,\theta) < \ln K$. Suppose not. There exists $y_0 > 0$, $\theta_0 > 0$ such that $h(y_0,\theta_0) = \ln K$. Then by the monotonicity of $h(y,\theta)$ and the fact that $h(y,\theta) \leq \ln K$, we have

$$u(s,y,\theta) = K - e^s, \quad (s,y,\theta) \in [0,\ln K] \times (0,y_0] \times (0,\theta_0].$$

Thus

$$\partial_\theta u(\ln K, y, \theta) = \partial_{s\theta} u(\ln K, y, \theta) = 0, \quad (y,\theta) \in (0,y_0] \times (0,\theta_0].$$

Since $\partial_\theta u(s,y,\theta) \geq 0$ and $\partial_\theta(\partial_\theta u) - \mathscr{L}_s(\partial_\theta u) = 0, s > \ln K$. By Hopf lemma [7] we know

$$\partial_{s\theta} u(\ln K, y, \theta) > 0, \quad (y,\theta) \in (0,y_0] \times (0,\theta_0],$$

or

$$\partial_\theta u(s,y,\theta) = 0, \quad (s,y,\theta) \in (\ln K, +\infty) \times (0,y_0] \times (0,\theta_0],$$

but both come to contradictions.

Remark 1. The numerical result of $h_0(y)$ under Heston model is given in [20], and by similar methods, under the assumptions (A1)–(A2), we can also obtain the existence of $h_0(y)$.

5 Characterization of the value function

Now, we are ready to present the characterization of the value function of (1) to the variational inequality (2) with boundary condition (4). To proceed, we present the solvability and regularity results on the variational inequality (2) with boundary condition (4) via the counterpart on the transformed problem obtained in the above.

Theorem 1. *Suppose a bounded function $v(x,y,t) \in W^{2,1}_{p,loc}(Q) \cap C(\bar{Q})$ satisfies variational inequality* (2) *with boundary condition* (4)*, the following assertions hold.*

1. $v(x,y,t)$ satisfies the following estimates

$$(K-x)^+ \leq v(x,y,t) \leq K+1, \tag{22}$$
$$-1 \leq \partial_x v(x,y,t) \leq 0, \tag{23}$$
$$\partial_y v(x,y,t) \geq 0. \tag{24}$$

2. *There exists a continuous function $g(y,t):\mathbb{R}^+\times[0,T)\to\mathbb{R}^+$, such that for any fixed $y>0$, $g(y,t)$ is monotonic increasing w.r.t. t; for any fixed $t\in[0,T)$, $g(y,t)$ is monotonic decreasing w.r.t. y with*

$$g(y,t)<g(y,T)=K,\quad y>0,\ t\in[0,T),$$

and

$$\begin{cases} -\partial_t v(x,y,t)-\mathscr{L}_x v(x,y,t)=0, & x>g(y,t),\\ v(x,y,t)=(K-x)^+, & 0\le x\le g(y,t). \end{cases}$$

3. *Especially, $v(x,y,t)\in C^{2,1}$ when $x>g(y,t)$.*

Proof. By the transformations $s=\ln x,\theta=T-t,u(s,y,\theta)=v(x,y,t)$, noting $x\partial_x v(x,y,t)=\partial_s u(s,y,\theta)$ and using estimates (16), (18) and (19), we can obtain (22)–(24).

Let $g(y,t)=\exp\{h(y,\theta)\}$, by proposition 1–2, we can conclude 2.

For any $x_0>g(y_0,t_0)$, then $v(x_0,y_0,t_0)>(K-x_0)^+$, since $v(x,y,t)$ is uniformly continuous, there exists a disk $B_\delta(x_0,y_0,t_0)$ with center (x_0,y_0,t_0) and radius δ such that

$$v(x,y,t)>(K-x)^+,\quad (x,y,t)\in B_\delta(x_0,y_0,t_0).$$

Applying $C^{2,1}$ interior estimate to

$$\partial_t v(x,y,t)+\mathscr{L}_x v(x,y,t)=0,\quad (x,y,t)\in B_\delta(x_0,y_0,t_0),$$

to obtain $v(x,y,t)\in C^{2,1}(B_\delta(x_0,y_0,t_0))$, hence $v(x,y,t)\in C^{2,1}$ when $x>g(y,t)$.

Finally, the uniqueness result is given in this below through the arguments of verification theorem.

Theorem 2. *Suppose there exists $v(x,y,t)\in W^{2,1}_{p,loc}(Q)$ to the problem* (2) *with boundary condition* (4)*, then $v(x,y,t)\ge V(x,y,t)$. If, in addition, there exists the region $\mathscr{N}[v]:=\{(x,y,t)\in Q, v(x,y,t)>(K-x)^+\}$ satisfies*

$$(\partial_t v+\mathscr{L}_x v)(X_s,Y_s,s)=0,\quad s\in[t,\tau^*],$$

for the stopping time $\tau^*:=\inf\{s>t:(X_s,Y_s,s)\notin\mathscr{N}[v]\}\wedge T$. Then the variational inequality (2) *with boundary condition* (4) *admits a unique solution in $W^{2,1}_{p,loc}(Q)$ and $v(x,y,t)=V(x,y,t)$.*

Proof. Let $\tau_x^\beta:=\inf\{s>t:X_s\le\frac{1}{\beta}\text{ or }\mathrm{X_s}\ge\beta\}\wedge\mathrm{T}$ be the first hitting time of the process X_s to the upper bound β or the lower bound $\frac{1}{\beta}$ or terminal time T, $\tau_y^\beta:=\inf\{s>t:Y_s\le\frac{1}{\beta}\text{ or }\mathrm{Y_s}\ge\beta\}\wedge\mathrm{T}$ be the first hitting time of the process Y_s to the upper bound β or the lower bound $\frac{1}{\beta}$ or terminal time T. Let $\tau\in\mathscr{T}_{t,\tau_x^\beta\wedge\tau_y^\beta}$, by the general Itô's formula [12],

$$e^{-r(\tau-t)}v(X_\tau,Y_\tau,\tau) = v(x,y,t) + \int_t^\tau e^{-r(s-t)}(\partial_t v + \mathscr{L}_x v)(X_s,Y_s,s)ds + \int_t^\tau e^{-r(s-t)}[\sigma(Y_s)X_s\partial_x v dW_s + b(Y_s)\partial_y v dB_s]. \quad (25)$$

Since $v(x,y,t)$ is bounded and the Itô integrals in (25) are local martingales, hence they are martingales. Moreover, by Theorem 1, we know $v(x,y,t)$ satisfies $\partial_t v + \mathscr{L}_x v \le 0$, $v(X_\tau,Y_\tau,\tau) \ge (K-X_\tau)^+$, hence

$$v(x,y,t) \ge \mathbb{E}_{x,y,t}[e^{-r(\tau-t)}(K-X_\tau)^+], \quad \tau \in \mathscr{T}_{t,\tau_x^\beta \wedge \tau_y^\beta}.$$

Since X_s is a positive, non-explosive local martingale, then

$$\lim_{\beta\to\infty} \tau_x^\beta = T, \quad a.s.-\mathbb{P}. \quad (26)$$

Since Y_s is non-negative, non-explosive local martingale, then

$$\lim_{\beta\to\infty} \tau_y^\beta = \nu \wedge T, \quad a.s.-\mathbb{P}, \quad (27)$$

where ν is the first hitting time of Y_s to the boundary $y=0$. Hence the arbitrariness of $\tau \in \mathscr{T}_{t,\tau_x^\beta \wedge \tau_y^\beta}$ and the above two limits imply that

$$v(x,y,t) \ge \sup_{\tau\in\mathscr{T}_{t,\nu\wedge T}} \mathbb{E}_{x,y,t}[e^{-r(\tau-t)}(K-X_\tau)^+]. \quad (28)$$

In view of (3), when $\nu < T$,

$$\mathbb{E}_{x,y,t}[e^{-r(\nu-t)}(K-X_\nu)^+] \ge \mathbb{E}_{x,y,t}[e^{-r(\tau-t)}(K-X_\tau)^+], \quad \tau \in \mathscr{T}_{\nu,T}.$$

Together with (28), we have

$$v(x,y,t) \ge \sup_{\tau\in\mathscr{T}_{t,T}} \mathbb{E}_{x,y,t}[e^{-r(\tau-t)}(K-X_\tau)^+] = V(x,y,t).$$

On the other hand, define $\tilde{\tau}_x^\beta := \inf\{s>t : X_s \ge \beta\} \wedge T$ be the first hitting time of the process X_s to the upper bound β or terminal time T, $\tilde{\tau}_y^\beta := \inf\{s>t : Y_s \ge \beta\} \wedge T$ be the first hitting time of the process Y_s to the upper bound β or terminal time T. By (26) and (27) we know

$$\lim_{\beta\to\infty} \tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta = T, \quad a.s.-\mathbb{P}.$$

Together with Monotone Convergence Theorem, we obtain

$$\lim_{\beta\to\infty}\mathbb{E}_{x,t}\left[X_T I_{\{\tilde{\tau}_x^\beta=T\}}\right]=\mathbb{E}_{x,t}\left[\lim_{\beta\to\infty}X_T I_{\{\tilde{\tau}_x^\beta=T\}}\right]=\mathbb{E}_{x,t}\left[X_T\right], \tag{29}$$

$$\lim_{\beta\to\infty}\mathbb{E}_{y,t}\left[Y_T I_{\{\tilde{\tau}_y^\beta=T\}}\right]=\mathbb{E}_{y,t}\left[\lim_{\beta\to\infty}Y_T I_{\{\tilde{\tau}_y^\beta=T\}}\right]=\mathbb{E}_{y,t}\left[Y_T\right]. \tag{30}$$

Moreover, by the definitions of $\tilde{\tau}_x^\beta, \tilde{\tau}_y^\beta$, we can have

$$\begin{aligned}\mathbb{E}_{x,t}\left[X_{\tilde{\tau}_x^\beta}\right] &= \mathbb{E}_{x,t}\left[X_{\tilde{\tau}_x^\beta} I_{\{\tilde{\tau}_x^\beta<T\}}\right]+\mathbb{E}_{x,t}\left[X_T I_{\{\tilde{\tau}_x^\beta=T\}}\right]\\ &= \beta\mathbb{P}\{\tilde{\tau}_x^\beta<T\}+\mathbb{E}_{x,t}\left[X_T I_{\{\tilde{\tau}_x^\beta=T\}}\right].\end{aligned}$$

Forcing the limit $\beta\to\infty$, due to (29),

$$\lim_{\beta\to\infty}\mathbb{E}_{x,t}[X_{\tilde{\tau}_x^\beta}]=\lim_{\beta\to\infty}\beta\mathbb{P}\{\tilde{\tau}_x^\beta<T\}+\mathbb{E}_{x,t}[X_T].$$

For all $\beta>x$, since $\{X_{\tilde{\tau}_x^\beta\wedge s}: s>t\}$ is a bounded local martingale, hence it is a martingale. So, $\mathbb{E}_{x,t}[X_{\tilde{\tau}_x^\beta}]=x$ for all $\beta>x$. Rearranging the above equality, we have

$$\lim_{\beta\to\infty}\beta\mathbb{P}\{\tilde{\tau}_x^\beta<T\}=x-\mathbb{E}_{x,t}[X_T]\le x. \tag{31}$$

Similarly,

$$\lim_{\beta\to\infty}\beta\mathbb{P}\{\tilde{\tau}_y^\beta<T\}=y-\mathbb{E}_{y,t}[Y_T]\le y. \tag{32}$$

By Theorem 1 we know $\mathscr{N}[v]=\{(x,y,t)\in Q: x>g(y,t)\}$, noting that $v(x,y,t)\in C^{2,1}$ and $\partial_t v+\mathscr{L}_x v=0$ in $\mathscr{N}[v]$, using the classical Itô's formula [15] in $[t,\tau^*\wedge\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta]$, we have

$$\begin{aligned}v(x,y,t) &= \mathbb{E}_{x,y,t}\left[e^{-r(\tau^*\wedge\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta-t)}v(X_{\tau^*\wedge\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},Y_{\tau^*\wedge\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},\tau^*\wedge\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta)\right]\\ &= \mathbb{E}_{x,y,t}\left[e^{-r(\tau^*-t)}v(X_{\tau^*},Y_{\tau^*},\tau^*)I_{\{\tau^*\le\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta\}}\right]\\ &\quad+\mathbb{E}_{x,y,t}\left[e^{-r(\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta-t)}v(X_{\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},Y_{\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta)I_{\{\tau^*>\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta\}}\right].\end{aligned}$$

Forcing $\beta\to+\infty$, since $\lim\limits_{\beta\to\infty}\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta=T$,

$$\begin{aligned}v(x,y,t) &= \mathbb{E}_{x,y,t}[e^{-r(\tau^*-t)}(K-X_{\tau^*})^+]\\ &\quad+\lim_{\beta\to\infty}\mathbb{E}_{x,y,t}\left[e^{-r(\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta-t)}v(X_{\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},Y_{\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta)I_{\{\tau^*>\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta\}}\right].\end{aligned}$$

In the following we show the limit $\lim\limits_{\beta\to\infty}\mathbb{E}_{x,y,t}\left[e^{-r(\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta-t)}v(X_{\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},Y_{\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta},\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta)I_{\{\tau^*>\tilde{\tau}_x^\beta\wedge\tilde{\tau}_y^\beta\}}\right]$ in the above equality is 0. Since $v(x,y,t)\le K+1$, then there exists

$g(\beta) = o(\beta)$, $\beta \to +\infty$ such that $v(X_{\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta}, Y_{\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta}, \tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta) \leq g(\beta) = o(\beta)$, together with (31) and (32), we can obtain

$$\begin{aligned}
0 &\leq \lim_{\beta \to \infty} \mathbb{E}_{x,y,t} \left[e^{-r(\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta - t)} v(X_{\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta}, Y_{\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta}, \tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta) I_{\{\tau^* > \tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta\}} \right] \\
&\leq \lim_{\beta \to \infty} g(\beta) \mathbb{E}_{x,y,t} \left[e^{-r(\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta - t)} I_{\{\tau^* > \tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta\}} \right] \\
&\leq \lim_{\beta \to \infty} \frac{g(\beta)}{\beta} \lim_{\beta \to \infty} \beta \mathbb{P}\{\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta < \tau^*\} \\
&\leq \lim_{\beta \to \infty} \frac{g(\beta)}{\beta} \lim_{\beta \to \infty} \beta \mathbb{P}\{\tilde{\tau}_x^\beta \wedge \tilde{\tau}_y^\beta < T\} = 0.
\end{aligned}$$

Hence $v(x,y,t) = \mathbb{E}_{x,y,t}[e^{-r(\tau^* - t)}(K - X_{\tau^*})^+]$, therefore $v(x,y,t) = V(x,y,t)$.

6 Conclusion

In this paper, we consider an American put option of stochastic volatility with negative Fichera function on the degenerate boundary $y = 0$, we impose a proper boundary condition from the definition of the option pricing to show that the solution to the associated variational inequality is unique, which is the value of the option, and the free boundary is the optimal exercise boundary of the option. Although the asset-price volatility coefficient may grow faster than linear growth and the domain is unbounded, we are able to show the uniqueness by verification theorem. In this paper we only consider the payoff function $(K - x)^+$, but the method in this paper will be useful for any nonnegative, continuous payoff function $f(x)$ which is of strictly sublinear growth, i.e., $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

The problem under study belongs to a general category of stochastic control problems, see for instance [18, 8]. Due to the nonlinearity, the numerical solution is always an issue in the general setup, see for instance [11] for Markov chain approximation method. In particular, regarding the current formulation of variational inequalities with Fichera functions on the boundary, it is unclear how its associated Markov chain behaves asymptotically as the step size goes to zero, see [19]. Appropriate scaling of step size at the boundary may be a key to make the obtained Markov chain consistent to the variational problem, and it will be pursued in our future work.

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