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STOCHASTIC MAXIMUM PRINCIPLE FOR A TIME-CHANGED MEAN FIELD GAME

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Abstract. This work studies a class of time-changed mean field game problems by the stochastic maximum principle, and establishes a duality principle for the derived forward backward stochastic differential equation. As an application, we provide an explicit solution for the mean of the subdiffusion under the framework of the linear-quadratic structure.

1. **Introduction.** In recent years, mean field game (MFG) problems have been developed in many directions since the pioneering work of Lasry-Lions model [8] and Huang-Caines-Malhame [5], see extensive details in Volume I and II of [2]. On the other hand, the subdiffusive model has been considered useful for several transport phenomena in e.g. physics and finance. It can describe particles being trapped and immobile during the constant periods (see [11]) or the asset price not changing in random periods of time (see [10]).

In this paper, we consider subdiffusive MFG models driven by a time-changed Brownian motion. Recently, [1, 15] characterized the subdiffusive MFG as the system of Hamilton-Jacobi-Bellman (HJB) and the Fokker-Planck equations (FPK). Our approach is different from [1, 15] in that we establish a time-changed McKean-Vlasov (MKV) forward backward stochastic differential equation (FBSDE) applying the stochastic maximum principle approach. In this regard, our approach is analogous to the well established one in [2] when the stochastic maximum principle applies to a traditional MFG problem without time-change, meanwhile also shares similarity with [12] for a time-changed stochastic control problem without the mean field term.

It is well known that the linear-quadratic structure is often used for the testbed of the control related theory due to its availability of the explicit form, see for instance [4, 13, 9]. As we test out our theory on time-changed MKV-FBSDE to the linear-quadratic structure, it does not immediately provide the desired Riccati ODE system due to the subdiffusive. To proceed, we further developed the dual FBSDE system by proving a duality principle on the time-changed MKV-FBSDE. The advantage of this idea is that, for some specific types of models such as the

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linear-quadric model, we can solve for the mean of the subdiffusion explicitly and hence obtain an explicit form of the MFG equilibrium.

2. **Main results.** Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual condition. Let B be a $(\mathcal{F}_t)$-adapted standard Brownian motion and $D = (D_t)_{t\geq 0}$ be a $(\mathcal{F}_t)$-adapted subordinator, which is independent of B, starting at 0 with Laplace exponent ψ with killing rate 0, drift 0, and Lévy measure ν; i.e. D is a one-dimensional nondecreasing Lévy process with càdlàg paths starting at 0 with Laplace transform

$$\mathbb{E}[e^{-sD_t}] = e^{-t\psi(s)}, \quad \text{where} \quad \psi(s) = \int_0^\infty (1 - e^{-sy})\, \nu(\mathrm{d}y), \quad s > 0,$$

with $\int_0^\infty (y \wedge 1)\, \nu(\mathrm{d}y) < \infty$. Let $E = (E_t)_{t\geq 0}$ be the inverse of D; i.e.

$$E_t := \inf\{u > 0; D_u > t\}, \quad t \geq 0.$$

We call E an *inverse subordinator*. If the subordinator D is stable with index $\beta \in (0, 1)$, then $\psi(s) = s^\beta$ and the corresponding time change E is called an inverse β-stable subordinator.

In this paper, we focus on the case when the Lévy measure ν is infinite, i.e. $\nu(0,\infty) = \infty$, which excludes compound Poisson subordinators from our discussion. This assumption implies that D has strictly increasing paths with infinitely many jumps, and therefore, E has continuous, nondecreasing paths starting at 0. Note that the jumps of D correspond to the (random) time intervals on which E is constant. During those constant periods, any time-changed process of the form $X \circ E = (X_{E_t})_{t\geq 0}$ also remains constant. If B is a standard Brownian motion independent of D, we can regard particles represented by the time-changed Brownian motion $B \circ E$ as being trapped and immobile during the constant periods.

Consider a subdiffusion X be the solution to a time-changed SDE

$$\mathrm{d}X_t = b(t, E_t, \mu_t, X_t, \alpha_t)\mathrm{d}E_t + \sigma \mathrm{d}B_{E_t}, \quad X_0 = \mu_0, \tag{1}$$

where σ is a constant, μ_t and $b(t_1, t_2, \mu, x, \alpha)$ are two deterministic functions satisfying some proper assumptions, and $(\alpha_t)_{t\geq 0} = (\alpha(t, E_t, \mu_t, X_t))_{t\geq 0} \in \mathcal{A}$ is an admissible control, i.e., a $\mathcal{F}_t$-adapted process for every t. We study the following problem:

(TCMFG):

(i). For each fixed deterministic continuous flow (μ_t), solve the standard stochastic control problem

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(t, E_t, \mu_t, X_t, \alpha_t)\mathrm{d}E_t + h(\mu_T, X_T)\right] \tag{2}$$

subject to (1), with $h(\cdot, x)$ being a concave function of variable x. Denote $\hat{\alpha}^\mu \in \mathcal{A}$ as its solution.

(ii). Find a particular flow μ such that $\mu_t = \mathbb{E}[\hat{X}^\mu_t]$ for ever $t > 0$, with $\hat{X}$ being the solution to (1) with μ_t and $\hat{\alpha}^\mu_t$.

Our approach has four steps. We first fix μ and set up a time-changed FBSDE for (X, Y) to solve (2), which has been studied in [12]. Then, using the idea of the *duality principle* introduced in [7], we establish a dual FBSDE for $(\bar{X}, \bar{Y})$ without time change such that $(X, Y) = (\bar{X}_E, \bar{Y}_E)$. By taking expectation on the dual FBSDE, we are able to construct a system of ODEs for the mean of $(\bar{X}, \bar{Y})$, denoted by $(\bar{\mu}, \bar{\nu})$, and solve it. Finally, by the fact that $\mu_t = \mathbb{E}[X_t] = \mathbb{E}[\bar{X}_{E_t}] = \mathbb{E}[\bar{\mu}_{E_t}]$, we recover the target μ_t such that $\mathbb{E}[X_t] = \mu_t$. We present the following main results for our approach.

Theorem 2.1. *For any fixed deterministic function μ_t starting from μ_0, let*

$$\hat{\alpha}(t_1, t_2, \mu, x, y) = argmin_{\alpha \in A} H(t_1, t_2, \mu, x, \alpha, y, q)$$

where the Hamiltonian $H(t_1, t_2, \mu, x, \alpha, y, q) \coloneqq b(t_1, t_2, \mu, x, \alpha)y + \sigma q + f(t_1, t_2, \mu, x, \alpha)$ and the function $h(\mu, x)$ are both differentiable and concave with respect to variable x. If (X, Y, q) solves the following FBSDE

$$\begin{cases} \mathrm{d}X_t = b(t, E_t, \mu_t, X_t, \hat{\alpha}(t, E_t, \mu_t, X_t, Y_t))\mathrm{d}E_t + \sigma \mathrm{d}B_{E_t} & X_0 = \mu_0 \\ \mathrm{d}Y_t = -H_x(t, E_t, \mu_t, X_t, \hat{\alpha}(t, E_t, \mu_t, X_t, Y_t))\mathrm{d}E_t + q_t \mathrm{d}B_{E_t} & Y_T = h_x(\mu_T, X_T) \end{cases} \tag{3}$$

with $\hat{\alpha}(t, E_t, \mu_t, X_t, Y_t) \in \mathcal{A}$ for every t. Then $(\hat{\alpha}(t, E_t, \mu_t, X_t, Y_t))_{t \geq 0}$ solves (2).

Proof. We follow the similar idea to the proof of Theorem 3.1 in [12]. Write

$$J(\alpha) := \mathbb{E}\left[\int_0^T f(t, E_t, \mu_t, X_t, \alpha_t)\mathrm{d}E_t + h(\mu_T, X_T)\right].$$

We want to prove, for any $\alpha \in \mathcal{A}$,

$$\begin{aligned} &J(\hat{\alpha}) - J(\alpha) \\ &= E\left[\int_0^T f(t, E_t, \mu_t, \hat{X}_t, \hat{\alpha}_t) - f(t, E_t, \mu_t, X_t, \alpha_t)\mathrm{d}E_t \right. \\ &\qquad \left. + h(\mu_T, \hat{X}_T) - h(\mu_T, X_T)\right] \geq 0. \end{aligned}$$

By the concavity of h and Itô's formula (see Lemma 2.2 in [12]), we have

$$\begin{aligned} &\mathbb{E}[h(\mu_T, \hat{X}_T) - h(\mu_T, X_T)] \geq \mathbb{E}[h_x(\mu_T, \hat{X}_T)(\hat{X}_T - X_T)] = \mathbb{E}[\hat{Y}_T(\hat{X}_T - X_T)] \\ &= \mathbb{E}\left[\int_0^T \hat{Y}_s \mathrm{d}(\hat{X}_s - X_s) + \int_0^T (\hat{X}_s - X_s)\mathrm{d}\hat{Y}_s + \int_0^T \mathrm{d}\hat{Y}_s \mathrm{d}(\hat{X}_s - X_s)\right] \\ &= \mathbb{E}\Big[\int_0^T \hat{Y}_s(b(s, E_s, \mu_s, \hat{X}_s, \hat{\alpha}_s) - b(s, E_s, \mu_s, X_s, \alpha_s))\mathrm{d}E_s \\ &\quad - \int_0^T (\hat{X}_s - X_s) H_x(s, E_s, \mu_s, \hat{X}_s, \hat{\alpha}_s)\mathrm{d}E_s\Big]. \end{aligned}$$

Therefore, by the concavity of the Hamiltonian $H(\cdot, x, \cdot)$ with respect to variable x,

$$\begin{aligned} J(\hat{\alpha}) - J(\alpha) &= \mathbb{E}\left[\int_0^T \hat{Y}_s(\hat{b}_s - b_s) + \hat{f}_s - f_s - (\hat{X}_s - X_s)\hat{H}_x(s)\mathrm{d}E_s\right] \\ &= \mathbb{E}\left[\int_0^T \hat{H}_s - H_s - (\hat{X}_s - X_s)\hat{H}_x(s)\mathrm{d}E_s\right] \geq 0. \end{aligned}$$

□

Remark 1. In the regular mean field theory, see examples in Section 4.7 of [2], after establishing the FBSDE, we take expectations on both equation to establish a system of ODEs by letting $\mu_t = \mathbb{E}[X_t]$ and $\nu_t = \mathbb{E}[Y_t]$, and solve for μ and ν. However, if we apply the same approach on (3), a drawback comes from the terms such as $\mathbb{E}[\int_0^T b(t, E_t, X_t, \mu_t, \hat{\alpha}_t)\mathrm{d}E_t]$, which can not be explicitly expressed. Therefore, we introduce the following the duality principle to tackle this problem.

Theorem 2.2 (Duality Principle)**.** *Suppose there exists a strong solution $(\bar{X}, \bar{Y}, \bar{q})$ to the following dual FBSDE*

$$\begin{cases} \mathrm{d}\bar{X}_t = b(D_t, t, \mu_{D_t}, \bar{X}_t, \hat{\alpha}(D_t, t, \mu_{D_t}, \bar{X}_t, \bar{Y}_t))\mathrm{d}t + \sigma \mathrm{d}B_t & \bar{X}_0 = \mu_0; \\ \mathrm{d}\bar{Y}_t = -H_x(D_t, t, \mu_{D_t}, \bar{X}_t, \hat{\alpha}(D_t, t, \mu_{D_t}, \bar{X}_t, \bar{Y}_t))\mathrm{d}t + \bar{q}_{D_t}\mathrm{d}B_t & \bar{Y}_{E_T} = h_x(\mu_T, \bar{X}_{E_T}). \end{cases} \tag{4}$$

Then $(X, Y, q) := (\bar{X} \circ E, \bar{Y} \circ E, q \circ E)$ solves (3) and $(\bar{X}, \bar{Y}, \bar{q}) = (X \circ D, Y \circ D, q \circ D)$.

To prove the theorem, we follow the similar idea introduced in [7] Theorem 4.2 by applying the change-of-variable formula. A process Z is said to be in synchronization with a time change E if Z is constant on every interval $[E_{t-}, E_t]$ almost surely. Then, it is straightforward that any process is in synchronization with the continuous time change E. Denote $L(Z, \mathcal{F}_t)$ the class of $(\mathcal{F}_t)$-predictable processes H for which a stochastic integral driven by Z, denoted as $\int_0^t H_s \mathrm{d}Z_s$, can be constructed. The next lemma corresponds to Lemma 2.3 and Theorem 3.1 of [7].

Lemma 2.3 (Change-of-variable formula)**.** *Let Z be an $(\mathcal{F}_t)$-semimartingale which is in synchronization with the continuous finite time change E. If $H \in L(Z, \mathcal{F}_t)$, then $H_{E_t} \in L(Z \circ E, \mathcal{F}_{E_t})$. Moreover, for any $t \geq 0$,*

$$\int_0^{E_t} H_s \mathrm{d}Z_s = \int_0^t H_{E_s} \mathrm{d}Z_{E_s} \quad \textit{with probability 1.}$$

If $K \in L(Z \circ E, \mathcal{F}_{E_t})$, then $(K_{D_{t-}}) \in L(Z, \mathcal{F}_t)$. Moreover, for any $t \geq 0$

$$\int_0^t K_s \mathrm{d}Z_{E_s} = \int_0^{E_s} K_{D_{s-}} \mathrm{d}Z_s.$$

Proof of Theorem 2.2. On one hand, let $(\bar{X}, \bar{Y}, \bar{q})$ solves (4), we want to prove $(X, Y, q) := (\bar{X} \circ E, \bar{Y} \circ E, \bar{q} \circ E)$ solves (3). We first observe that E_s is a constant when $s \in [D_{E_s-}, D_{E_s}]$. This fact implies

$$\int_t^T g(D_{E_s}) - g(s)\mathrm{d}E_s = 0 \text{ and } \int_t^T g(D_{E_s}) - g(s)\mathrm{d}B_{E_s} = 0$$

for any function or functional g. By Lemma 2.3, we obtain

$$\begin{aligned} Y_t = \bar{Y}_{E_t} &= h_x(\mu_T, \bar{X}_{E_T}) - \int_{E_t}^{E_T} -H_x(D_s, s, \mu_{D_s}, \bar{X}_s, \bar{Y}_s)\mathrm{d}s - \int_{E_t}^{E_T} q_{D_s}\mathrm{d}B_s \\ &= h_x(\mu_T, X_T) - \int_t^T -H_x(D_{E_s}, E_s, \mu_{D_{E_s}}, \bar{X}_{E_s}, \bar{Y}_{E_s})\mathrm{d}E_s - \int_t^T q_{D_{E_s}}\mathrm{d}B_{E_s} \\ &= h_x(\mu_T, X_T) - \int_t^T -H_x(s, E_s, \mu_s, X_s, Y_s)\mathrm{d}E_s - \int_t^T q_s \mathrm{d}B_{E_s}, \end{aligned}$$

which solves (3). Similar approach works for X.

On the other hand, let (X, Y, q) solves (3), we want to prove $(\bar{X}, \bar{Y}) := (X \circ D, Y \circ D, q \circ D)$ solves (4). By Lemma 2.3, the continuity of the integrators and the fact $E_{D_s} = E_{D_{s-}} = s$, we obtain

$$\begin{aligned} Y_t &= h_x(\mu_T, X_T) - \int_t^T -H_x(s, E_s, \mu_s, X_s, Y_s)\mathrm{d}E_s - \int_t^T q_s \mathrm{d}B_{E_s} \\ &= h_x(\mu_T, X_T) - \int_{E_t}^{E_T} -H_x(D_{s-}, E_{D_{s-}}, \mu_{D_{s-}}, X_{D_{s-}}, Y_{D_{s-}})\mathrm{d}s - \int_{E_t}^{E_T} q_{D_{s-}}\mathrm{d}B_s \end{aligned}$$

$$= \quad h_x(\mu_T, X_T) - \int_{E_t}^{E_T} -H_x(D_s, s, \mu_{D_s}, X_{D_s}, Y_{D_s})\mathrm{d}s - \int_{E_t}^{E_T} q_{D_s}\mathrm{d}B_s.$$

Observe that

$$\bar{Y}_t = Y_{D_t} = h_x(\mu_T, \bar{X}_{E_t}) - \int_t^{E_T} -H_x(D_s, s, \mu_{D_s}, X_{D_s}, Y_{D_s})\mathrm{d}s - \int_t^{E_T} q_{D_s}\mathrm{d}B_s$$

solves (4). Similar approach works for X. $\square$

3. **An example from LQ time-changed MFG.** In this section, we solve LQ time-changed MFG applying the duality principle obtained above. To proceed, we take the expectations on both equations of (4) and obtain

$$\left\{ \begin{array}{ll} \bar{\mu}'_t = \mathbb{E}[b(D_t, t, \mu_{D_t}, \bar{X}_t, \hat{\alpha}(D_t, t, \mu_{D_t}, \bar{X}_t, \bar{Y}_t))] & \bar{\mu}_0 = \mu_0 \\ \bar{\nu}'_t = \mathbb{E}[-H_x(D_t, t, \mu_{D_t}, \bar{X}_t, \hat{\alpha}(D_t, t, \mu_{D_t}, \bar{X}_t, \bar{Y}_t))] & \bar{\nu}_T = E[h_x(\mu_{D_T}, \bar{X}_T)] \end{array} \right. , \quad (5)$$

where $\bar{\mu}_t = \mathbb{E}[\bar{X}_t]$ and $\bar{\nu}_t = \mathbb{E}[\bar{Y}_t]$.

Suppose $\bar{\mu}_t$ solves (5), which also satisfies $E[\bar{\mu}_{E_t}] < \infty$ for any t. Then we are able to recover μ_t using the duality again:

$$\mu_t = E[X_t] = E[\bar{X}_{E_t}] = E[\bar{\mu}_{E_t}].$$

The existence of $E[\bar{\mu}_{E_t}]$ can be verified using the criterion established on Theorem 1 in [6].

Let

$$\begin{aligned} b = b(x, \mu, \alpha) &= x + \mu + \alpha; \\ f = f(x, \mu, \alpha) &= \frac{1}{2}(x^2 + (x-\mu)^2 + \alpha^2); \\ h = h(x, \mu) &= \frac{1}{2}(x^2 + (x-\mu)^2). \end{aligned}$$

The Hamiltonian can be calculated as

$$H(x, \mu, \alpha, y, q) = (x + \mu + \alpha)y + \frac{1}{2}(x^2 + (x-\mu)^2 + \alpha^2) + \sigma q,$$

and is minimized at $\hat{\alpha} = -y$.

Then the time-changed FBSDE (3) reduces to

$$\left\{ \begin{array}{lcll} \mathrm{d}X_t &=& (X_t + \mu_t - Y_t)\mathrm{d}E_t + \sigma \mathrm{d}B_{E_t} & X_0 = \mu_0 \\ \mathrm{d}Y_t &=& -(2X_t + Y_t - \mu_t)\mathrm{d}E_t + q_t \mathrm{d}B_{E_t} & Y_T = X_T \end{array} \right. ,$$

and the dual FBSDE (4) reduces to

$$\left\{ \begin{array}{lcll} \mathrm{d}\bar{X}_t &=& (\bar{X}_t + \mu_{D_t} - \bar{Y}_t)\mathrm{d}t + \sigma \mathrm{d}B_t & \bar{X}_0 = \mu_0 \\ \mathrm{d}\bar{Y}_t &=& -(2\bar{X}_t + \bar{Y}_t - \mu_{D_t})\mathrm{d}t + q_{D_t} \mathrm{d}B_t & \bar{Y}_{E_T} = \bar{X}_{E_T} \end{array} \right. . \quad (6)$$

Let $\bar{\mu}_t = \mathbb{E}[\bar{X}_t]$ and $\bar{\nu}_t = \mathbb{E}[\bar{Y}_t]$, with the fact that $\mathbb{E}[\mu_{D_t}] = \mathbb{E}[X_{D_t}] = \mathbb{E}[\bar{X}_t] = \bar{\mu}_t$, we have the following ODEs

$$\left\{ \begin{array}{ll} \bar{\mu}'_t = 2\bar{\mu}_t - \bar{\nu}_t & \bar{\mu}_0 = \mu_0 \\ \bar{\nu}'_t = -\bar{\mu}_t - \bar{\nu}_t & \bar{\nu}_T = \bar{\mu}_T \end{array} \right. .$$

Then, we have $\bar{\mu}''_t - 3\bar{\mu}'_t - 3\bar{\mu}_t = 0$ with $\bar{\mu}_0 = \mu_0$. We can explicitly solve for $\bar{\mu}$ in the form of

$$\bar{\mu}_t = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \text{ where } C_1, C_2, r_2, r_2 \text{ are some constants.}$$

Therefore, with Theorem 1 in [6], for any $t > 0$,

$$\mu_t = \mathbb{E}\left[C_1 e^{r_1 E_t} + C_2 e^{r_2 E_t}\right] < \infty. \quad (7)$$

Remark 2. When D is a β-stable subordinator with $\beta \in (0,1)$, i.e., the Laplace exponent $\psi(s) = s^\beta$, the exponential moment of its inverse in (7) can be explicitly calculated by the Mittag-Leffler function: $\mathbb{E}[e^{rE_t}] = \mathbf{E}_\beta(rt^\beta) := \sum_{n=0}^\infty (rt^\beta)^n/\Gamma(n\beta + 1)$. See Lemma 3.4 in [6] for more details.

Remark 3. In order to establish the system of ODEs, we request the linearity on $\bar{X}$ and $\bar{Y}$ on the right hand side of (6). Thus, we are able to include more selections on functions b and f such that the calculations in the above example work. Assume

$$\begin{aligned} b(x,\mu,\alpha) &= C_1 x + C_2\mu + C_3 + b_2(\alpha); \\ f(x,\mu,\alpha) &= f_1(x,\mu) + C_4 b_2^2(\alpha), \quad C_4 > 0; \\ \frac{\partial f_1}{\partial x}(x,\mu) &= C_5 x + C_6 \mu + C_7, \end{aligned}$$

where C_1-C_7 are constants, $C_4 \neq 0$, b_1, b_2, f_1 and f_2 are deterministic functions, in particular, the support of the domain and range of b_2 is $\mathbb{R}$. Then, a similar system of ODE as (6) can be established as

$$\begin{cases} \bar{\mu}_t' = (C_1 + C_2)\bar{\mu}_t - \frac{1}{2C_4}\bar{\nu}_t + C_3 & \bar{\mu}_0 = \mu_0 \\ \bar{\nu}_t' = (-C_5 - C_6)\bar{\mu}_t - C_1\bar{\nu}_t - C_7 & \bar{\nu}_T = \bar{\mu}_T \end{cases}.$$

$\bar{\mu}$ is the solution to a homogeneous second order linear ODE with constant coefficients as $c_1\bar{\mu}_t'' + c_2\bar{\mu}_t' + c_3\bar{\mu}_t = 0$, where c_1-c_3 are constants depending on $C_1 - C_7$. Observe that the general shape of the solution depends on the roots the characteristic quadratic equation $c_1 r^2 + c_2 r + c_3 = 0$, named r_1 and r_2. Similar to (7), μ_t will take one of the following three forms, which are well defined by Theorem 1 in [6] and the Cauchy-Schwartz inequality: with some real constants C_8 and C_9

$$\begin{aligned} \mu_t &= \mathbb{E}\left[C_8 e^{r_1 E_t} + C_9 e^{r_2 E_t}\right] < \infty \quad \text{when} \quad r_1, r_2 \in \mathbb{R}, r_1 \neq r_2; \\ \mu_t &= \mathbb{E}\left[(C_8 E_t + C_9) e^{r_1 E_t}\right] < \infty \quad \text{when} \quad r_1 = r_2 \in \mathbb{R}; \\ \mu_t &= \mathbb{E}\left[e^{\alpha E_t}\left(C_8 \cos(\beta E_t) + C_9 \sin(\beta E_t)\right)\right] < \infty \\ &\qquad\qquad \text{when} \quad r_1 = \alpha + i\beta, r_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}. \end{aligned}$$

4. **Further remarks.**

Our current presentation can be possibly further developed in various directions. For instance, the underlying system also can be generalized into an appropriate multidimensional state/control space, a common noise can be added to the underlying system, etc. However, we limit ourselves to being focused on the possibility of the extension into the subdiffusive MFG models by the stochastic maximum principle. Our conclusion is that, in parallel to the existing MFG theory, it is feasible to characterize the equilibrium by time-changed McKean-Vlasov FBSDE under appropriate conditions. Nevertheless, it is crucial to utilize the duality principle with an inverse time-change process for its application.

On the other hand, if the uniqueness holds both for the FBSDE system and the HJB-FPK system, we may expect their equivalence via the gradient of the value function. Similar arguments in the classical stochastic control theory are provided, see e.g. Section 6.4 of [14] for example with assumptions on regularities of the value function. We expect an equivalence between the solution to the FBSDE and the fractional HJB-FPK with proper conditions and it will be left to our future research direction.

In the end, we recall that duality principle Theorem 2.2 assumes the existence of the solution of dual FBSDE containing a process D independent of Brownian motion B given by

$$\begin{cases} \mathrm{d}\bar{X}_t = b(D_t, t, \mu_{D_t}, \bar{X}_t, \bar{Y}_t)\mathrm{d}t + \sigma \mathrm{d}B_t & \bar{X}_0 = \mu_0 \\ \mathrm{d}\bar{Y}_t = -H_x(D_t, t, \mu_{D_t}, \bar{X}_t, \bar{Y}_t)\mathrm{d}t + q_{D_t}\mathrm{d}B_t & \bar{Y}_{E_T} = h_x(\mu_T, \bar{X}_{E_T}) \end{cases}. \tag{8}$$

Next, we present a sufficient condition for this existence assumption.

($\mathcal{H}$): For any $x, y \in \mathbb{R}$, there exist two positive constants L and K, which do not depend on x and α, such that

$$|b(\cdot, x_1, y_1) - b(\cdot, x_2, y_2)| + |H_x(\cdot, x_1, y_1) - H_x(\cdot, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|);$$

$$|b(\cdot, x, y)| + |H_x \cdot, x, y)| \leq K(1 + |x| + |y|); \quad h_x \text{ is uniformly bounded.}$$

Proposition 1. *Assume* ($\mathcal{H}$) *is satisfied and*

$$\mathbb{E}_B \left[\int_0^T |\bar{X}_t|^2 + |\bar{Y}_t|^2 + |q_{D_t}|^2 \mathrm{d}t \right] < \infty,$$

then there exists a strong solution to (8).

Proof. Since the Brownian motion B and the subordinator D are assumed independent, it is possible to set up B and D on a product space $\Omega = \Omega_B \times \Omega_D$ with product measure $\mathbb{P} = \mathbb{P}_B \times \mathbb{P}_D$ with obvious notation. Let $\mathbb{E}_B$, $\mathbb{E}_D$ and $\mathbb{E}$ denote the expectations under the probability measures $\mathbb{P}_B$, $\mathbb{P}_D$ and $\mathbb{P}$, respectively. Then, the process $\bar{X}(\omega_1, \omega_2) : \Omega_B \times \Omega_D \to \mathbb{R}$ can be understood as stochastic process on the product space Ω, so to the process $\bar{Y}$ and q. Now, we consider the marginal solution of $(\bar{X}, \bar{Y}, q)$ on Ω_B, i.e., for each fixed $\omega_2 \in \Omega_D$, we write $(\bar{X}^{\omega_2}(\omega_1), \bar{Y}^{\omega_2}(\omega_1), q^{\omega_2}(\omega_1)) := (\bar{X}(\omega_1, \omega_2), \bar{Y}(\omega_1, \omega_2), q(\omega_1, \omega_2))$ for any $\omega_1 \in \Omega_B$ and rewrite the FBSDE as

$$\begin{cases} \mathrm{d}\bar{X}_t^{\omega_2} = b^{\omega_2}(t, \bar{X}_t^{\omega_2}, \bar{Y}_t^{\omega_2})\mathrm{d}t + \sigma \mathrm{d}B_t & \bar{X}_0 = \mu_0 \\ \mathrm{d}\bar{Y}_t^{\omega_2} = -H_x^{\omega_2}(t, \bar{X}_t^{\omega_2}, \bar{Y}_t^{\omega_2})\mathrm{d}t + q_t^{\omega_2}\mathrm{d}B_t & \bar{Y}_{E_T} = h_x(\mu_T, \bar{X}_{E_T}) \end{cases}, \tag{9}$$

where $b^{\omega_2}(t, x, y) := b(t, D_t, \mu_{D_t}, x, y)$, $H_x^{\omega_2}(t, x, y) := H_x(t, D_t, \mu_{D_t}, x, y)$ and $q_t^{\omega_2} := q_{D_t}$. Then, the assumption

$$\mathbb{E}_B \left[\int_0^T |\bar{X}_t^{\omega_2}|^2 + |\bar{Y}_t^{\omega_2}|^2 + |q_t^{\omega_2}|^2 \mathrm{d}t \right] < \infty,$$

for each ω_2, together with assumption $\mathcal{H}$, Theorem 1.1 of [3] yields that (9) has a unique strong solution in Ω_B. Then the existence and uniqueness of the a strong solution to FBSDE (8) follows straightforwardly. $\square$

REFERENCES

[1] F. Camilli and R. D. Maio, A time-fractional mean field game, *Adv. Differential Equations*, **24** (2019), 531-554.

[2] R. Carmona and F. Delarue, Probabilistic Theory of Mean Field Games with Applications, Springer, 2018.

[3] F. Delarue, On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case, *Stochastic Process. Appl.*, **99** (2002), 209-286.

[4] S. Gao and P. E. Caines, Grapbon linear quadratic regulation of large-scale networks of linear systems, *IEEE Trans. Automat. Control*, **65** (2020), 4090-4105.

[5] M. Huang, P. E. Caines and R. P. Malhame, Large population stochastic dynamic games: Closed-loop Mckean-Vlasov systems and the Nash certainty equivalence principle, *Commun Inf Syst*, **6** (2006), 221-251.

[6] S. Jin and K. Kobayashi, Strong approximation of stochastic differential equations driven by a time-changed Brownian motion with time-space-dependent coefficients, *J. Math. Anal. Appl.*, **476** (2019), 619-636.

[7] K. Kobayashi, Stochastic calculus for a time-changed semimartingale and the associated stochastic differential equations, *J. Theor. Prob.*, **24** (2011), 789-820.
[8] J. M. Lasry and P. L. Lions, Mean field games, *Jpn. J. Math.*, **2** (2007), 229-260.
[9] N. Li, X. Li and Z. Yu, Indefinite mean-field type linear-quadratic stochastic optimal control problems, *Automatica J. IFAC*, **122** (2020), 109267, 10 pp.
[10] M. Magdziarz, S. Orzeł and A. Weron, Option pricing in subdiffusive Bachelier model, *J. Stat. Phys.*, **145** (2011), 187-203.
[11] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1-77.
[12] E. Nane and Y. Ni, A time-changed stochastic control problem and its maximum principle, *Probab. Math. Statist.*, **41** (2021), 193-215.
[13] S. L. Nguyen and M. Huang, Linear-quadratic-Gaussian mixed games with continuum-parametrized minor players, *SIAM J. Control Optim.*, **50** (2012), 2907-2937.
[14] H. Pham, *Continuous-Time Stochastic Control and Optimization with Financial Applications*, Stochastic Modelling and Applied Probability, 61. Springer-Verlag, Berlin, 2009.
[15] Q. Tang and F. Camilli, Variational time-fractional mean field games, *Dyn. Games Appl.*, **10** (2020), 573-588.

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