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On modified Euler methods for McKean–Vlasov stochastic differential equations with super-linear coefficients[☆]

Jiamin Jian[a,1], Qingshuo Song[b,1,*], Xiaojie Wang[c,1], Zhongqiang Zhang[b,1], Yuying Zhao[c,1]

[a] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
[b] Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA
[c] School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan, PR China

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ABSTRACT

We introduce a new class of numerical methods for solving McKean–Vlasov stochastic differential equations, which are relevant in the context of distribution-dependent or mean-field models, under super-linear growth conditions for both the drift and diffusion coefficients. Under certain non-globally Lipschitz conditions, the proposed numerical approaches have half-order convergence in the strong sense to the corresponding system of interacting particles associated with McKean–Vlasov SDEs. By leveraging a result on the propagation of chaos, we establish the full convergence rate of the modified Euler approximations to the solution of the McKean–Vlasov SDEs. Numerical experiments are included to validate the theoretical results.

1. Introduction

McKean–Vlasov stochastic differential equations (MV-SDEs), also known as distribution-dependent or mean-field SDEs, extend traditional SDEs by incorporating the collective behavior of multiple interacting particles. Initially proposed by McKean (1966), McKean (1967), this class of equations gained increasing attention following Dawson's foundational work (Dawson, 1983) and the development of the Lion's derivative with respect to the measure variables (Lions, 2007). Solving MV-SDEs is crucial in control theory as they model large-scale systems where individual components interact with the collective behavior of the group. They describe both the optimal path for mean-field controls (Bensoussan, Frehse, Yam, et al., 2013), Sznitman (1991) and the equilibrium trajectory for mean-field games (see Carmona & Delarue, 2018a, Carmona & Delarue, 2018b, Mishura & Veretennikov, 2020, Wu, Hu, Gao, & Yuan, 2022 and references therein). This type of SDEs are essential for simplifying control problems in distributed systems, such as robots, power grids, or financial markets, by focusing on the statistical distribution of agents rather than individual interactions. They also account for uncertainty in noisy environments, enabling robust control strategies. Applications include multi-agent systems (see Benachour, Roynette, Talay, & Vallois, 1998, Bossy & Talay, 1997) and other highly relevant fields like filtering (as highlighted in Crisan & Xiong, 2010). By reducing system complexity and providing feedback control for distributed systems, MV-SDEs make large-scale optimization problems more tractable.

McKean–Vlasov SDEs are also widely used to model random phenomena across various scientific domains, including physics, biology, engineering, and neural activities, such as Baladron, Fasoli, Faugeras, and Touboul (2012), Bolley, Cañizo, and Carrillo (2011), Bossy, Faugeras, and Talay (2015), Carmona and Delarue (2018a), Carmona and Delarue (2018b) Dreyer, Gaberšček, Guhlke, Huth, and Jamnik (2011), Guhlke, Gajewski, Maurelli, Friz, and Dreyer (2018), McKean (1966) and references therein. As a result, there has been a notable surge in interest in related research.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, where $\mathcal{F}_t$ is the augmented filtration of a standard m-dimensional Brownian motion $W = \{W(t)\}_{t\geq 0}$. For a fixed terminal time $T > 0$, we consider the following McKean–Vlasov SDEs

$$X_t = X_0 + \int_0^t b\,(s, X_s, \mathcal{L}(X_s))\,ds + \int_0^t \sigma\,(s, X_s, \mathcal{L}(X_s))\,dW(s),\ \ t \in [0,T],\ \ a.s., \tag{1}$$

where $\{\mathcal{L}(X_t)\}_{t\geq 0}$ is the flow of deterministic marginal distributions of $X = \{X_t\}_{t\geq 0}$, $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ denotes the drift function and $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$ is

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* Corresponding author.
E-mail addresses: jiaminj@umich.edu (J. Jian), qsong@wpi.edu (Q. Song), x.j.wang7@csu.edu.cn (X. Wang), zzhang7@wpi.edu (Z. Zhang), zhaoyuying78@gmail.com (Y. Zhao).

[1] All authors contributed equally.

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the diffusion function, expressed as $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$. In this notation, $\sigma_j : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ is the jth column of σ. Throughout this paper, the initial data X_0 is a $\mathcal{F}_0$-measurable random variable in $\mathbb{R}^d$ independent of W.

In general, such equations rarely have explicit solutions available and one usually falls back on their numerical solutions. If the measure flow $\{\mathcal{L}(X_t)\}_{t\geq 0}$ is known, then the coefficients b and σ are functions of time and space variables, and hence the MV-SDEs reduce to classical SDEs. It is widely acknowledged that in scenarios where the coefficients of SDEs lack globally Lipschitz continuity and exhibit super-linear growth, the commonly used Euler–Maruyama numerical solution fails to attain finite moments, leading to divergence in both strong and weak senses. This issue has been well-documented in the literature, as evidenced by, e.g., Higham, Mao, and Yuan (2007), Hutzenthaler, Jentzen, and Kloeden (2011), Mattingly, Stuart, and Higham (2002), Kumar and Sabanis (2017) and Milstein and Tretyakov (2005). A similar divergence phenomenon, referred to as particle corruption, was observed in the context of MV-SDEs (see Section 4.1 in dos Reis, Engelhardt, and Smith (2022) for more details). Therefore, special care must be taken to construct and analyze convergent numerical approximations in a non-globally Lipschitz setting and recent years have witnessed a proper growth of the literature on this interesting topic (Bao, Reisinger, Ren, & Stockinger, 2021; Chen & Dos Reis, 2022, 2024; Chen, Reis, & Stockinger, 2023; dos Reis et al., 2022; Gao, Guo, Hu, & Yuan, 2024; Kumar & Neelima, 2021; Kumar, Neelima, Reisinger, & Stockinger, 2022; Li, Mao, Song, Wu, & Yin, 2023; Liu, Shi, & Wu, 2023; Liu, Wu, & Wu, 2023; Neelima, Kumar, dos Reis, & Reisinger, 2020; Reisinger & Stockinger, 2022).

Under local Lipschitz and linear growth conditions, the Euler numerical method for approximating MV-SDEs was analyzed in Li et al. (2023). When the drift coefficients exhibit possible super-linear growth while the diffusion coefficients satisfy the linear growth condition, the moment boundedness and convergence rates of various numerical methods have been investigated in Bao et al. (2021), dos Reis et al. (2022), Reisinger and Stockinger (2022),Fang and Giles (2020), Kumar and Neelima (2021), and Liu, Wu, and Wu (2023). This analysis was further extended in Kumar et al. (2022) to MV-SDEs with common noise and in Neelima et al. (2020) to those with Lévy processes, allowing the diffusion coefficients to also exhibit super-linear growth. In addition, numerical methods have been proposed for solving a class of MV-SDEs with drift or diffusion components of convolution type. Specifically, Chen and Dos Reis (2022, 2024) addressed cases where both drifts and diffusions exhibit super-linear growth, while Chen et al. (2023) focused on drifts with super-linear growth and diffusion coefficients satisfying linear growth conditions. Finally, some works addressed numerical methods for McKean–Vlasov SDEs with Hölder continuous diffusion coefficient (see, e.g., Liu, Shi, & Wu, 2023). In terms of numerical methods, various approaches have been proposed in the literature:

- *Explicit tamed Euler methods*, were introduced and studied in dos Reis et al. (2022), Liu, Wu, and Wu (2023), Liu, Shi, and Wu (2023), Neelima et al. (2020), Bao et al. (2021), Kumar and Neelima (2021), and Kumar et al. (2022), which rely on certain taming modifications of coefficients of MV-SDEs in a form such as $\frac{b(t,x,\mu)}{1+h^{\beta}|b(t,x,\mu)|}$ $(0 < \beta \leq 1)$;
- *Adaptive numerical methods* may serve as a viable alternative to tamed numerical solutions, particularly in numerically solving super-linear drift and diffusion coefficients, as demonstrated in Reisinger and Stockinger (2022);
- *Truncated method* was proposed in Guo, He, and Li (2024) for the interacting particle system under a Khasminskii-type condition on the coefficients;
- *Projection-based particle method* was proposed in Belomestny and Schoenmakers (2018) to reduce the computational cost of solving MV-SDEs;
- *Implicit numerical methods*, such as the backward Euler method (dos Reis et al., 2022) and *split-step method* (see Chen & Dos Reis, 2022, 2024; Chen et al., 2023) were utilized to approximate MV-SDEs with superlinear coefficients.

It is worthwhile to note that, when numerically approximating stable (dissipativity) systems, stability (or dissipativity) preserving methods are particularly vital. Usually, implicit methods have excellent stability properties and can preserve the dissipativity (long time stability) of the system, but at a price of expensive costs. A cheap option is to rely on explicit methods. However, as pointed out by Chen and Dos Reis (2022), taming might destroy the strict dissipativity of the drift coefficients and the usual tamed methods would be confronted with long time stability issues. Our numerical experiments indicate different taming strategies would give different stability performances. Therefore, one should be careful with the choice of taming strategies for long-time simulations, which, albeit interesting and important, turns out to be non-trivial. Recently, several authors have made some attempts in this direction for usual SDEs. By truncating monotonic functions, a recent work (Johnston & Sabanis, 2024) proposed a polygonal (tamed) Euler method preserving the monotonicity of the drift coefficient. A new tamed Euler method was introduced by Neufeld, Ng, and Zhang (2025) to long-time approximate invariant measures of the Langevin SDEs.

In this work, we focus on strong convergence analysis over finite time horizons for numerically solving MV-SDEs and leave the study of long time approximations for future work. In the setting of the present article, the drift and diffusion coefficients are allowed to grow super-linearly in their spatial components. Rather than focusing on specific numerical solutions, we present a general framework to encompass a broader class of numerical methods. This allows us to establish moment boundedness and convergence rates within a general framework. A similar approach was adopted in Lionnet, dos Reis, and Szpruch (2018) for explicit numerical methods of forward–backward SDEs with drivers exhibiting polynomial growth.

The main contributions of this paper are summarized as follows:

- *New framework and new methods.* We establish a new framework to admit novel numerical methods for MV-SDEs with super-linear drift and diffusion, such as the sin Euler method and tanh Euler method. As demonstrated by numerical experiments, the tanh Euler method has better stability properties than the other explicit tamed methods and always produces reliable approximations.
- *Moment bound.* Lemma 13 establishes the boundedness of moments of the newly proposed methods (4) for MV-SDEs with super-linearly growing drift and diffusion coefficients.
- *Convergence rate.* We establish the strong convergence rates for a class of modified Euler methods in Theorem 19. It is shown that the proposed numerical methods have half-order convergence in the strong sense of the corresponding system of interacting particles associated with the MV-SDEs. Moreover, the full convergence rate of the modified Euler approximation to the solution of the McKean–Vlasov SDEs (1) is provided in Corollary 20 by leveraging a result on the propagation of chaos in Proposition 3.

The remainder of this article is structured as follows. In the forthcoming section, we present some necessary notations and our requirements on coefficients of MV-SDEs. A class of modified Euler methods and their uniform moment bounds are provided in Section 3. Section 4 derives the strong convergence rate of the modified Euler approximations to the system of interacting particles. Finally, some numerical results are demonstrated in Section 5.

2. Notations, assumptions, and preliminaries

In this section, we introduce notations and basic assumptions for the well-posedness of MV-SDEs. The system of interacting particles and the corresponding result of propagation of chaos are also presented.

2.1. Notations

Let $|\cdot|$ and $\langle\cdot,\cdot\rangle$ be the Euclidean norm and the inner product of vectors in $\mathbb{R}^d$, respectively. For a matrix A, we denote the Frobenius norm by $\|A\| = \sqrt{\mathrm{tr}(AA^\top)}$, where $A^\top$ is the transpose of A and $\mathrm{tr}(\cdot)$ is the trace function of matrices. Let δ_x be the Dirac measure at a point $x \in \mathbb{R}^d$.

To proceed, we denote $\mathcal{P}_2(\mathbb{R}^d)$ be the Wasserstein space of probability measures μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$ endowed with 2-Wasserstein metric $\mathcal{W}_2(\cdot,\cdot)$ defined by

$$\mathcal{W}_2(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^2 d\pi(x,y)\right)^{\frac{1}{2}},$$

where $\Pi(\mu,\nu)$ is the collection of all probability measures on $\mathbb{R}^d\times\mathbb{R}^d$ with its marginals agreeing with μ and ν.

2.2. Well-posedness of MV-SDEs

Next, we list the assumptions that are needed in this section. In the following, we use L and K to denote the generic constants which can be changed from line to line. Moreover, p_0 is denoted as a fixed positive constant that is sufficiently large and satisfies all the conditions specified in the inequalities presented in the theorem and lemmas of this paper.

Assumption 1.

(A1) $\mathbb{E}[|X_0|^{2p_0}] < \infty$ for a fixed constant $p_0 > 1$.

(A2) There exists a constant $L > 0$ such that

$$2\langle x, b(t,x,\mu)\rangle + (2p_0-1)\|\sigma(t,x,\mu)\|^2 \le L\left(1+|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right)$$

for all $t\in[0,T]$, $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

(A3) There exists a constant $L > 0$ such that

$$2\langle x-y, b(t,x,\mu)-b(t,y,\nu)\rangle + ||\sigma(t,x,\mu) - \sigma(t,y,\nu)||^2 \le L\left(|x-y|^2+\mathcal{W}_2^2(\mu,\nu)\right)$$

for all $t\in[0,T]$, $x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$.

(A4) For every $t \in [0,T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $b(t,\cdot,\mu)$ is a continuous function on $\mathbb{R}^d$ and for every $R > 0$ there exists $N_R \ge 0$ such that $\sup_{|x|\le R} |b(t,x,\delta_0)| \le N_R$ for all $t\in[0,T]$.

Assumption 1 plays a pivotal role in achieving existence, uniqueness, and moment boundedness concerning the McKean–Vlasov SDEs (1). Detailed proof of the following result could be found in (Kumar et al. (2022, Theorem 2.1) with common noise).

Proposition 2 *((Kumar et al., 2022, Theorem 2.1)). Let assumptions (A1), (A2), (A3) and (A4) in Assumption 1 be satisfied. Then, there exists a unique solution to (1) and the following boundedness of moments hold*

$$\sup_{0\le t\le T} \mathbb{E}\left[|X_t|^{2p_0}\right] \le K,$$

where p_0 is from (A1) and $K := K(L,\mathbb{E}[|X_0|^{2p_0}],d,m) > 0$ is a constant. Moreover,

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X_t|^{2q}\Big] \le K, \quad \forall q < p_0.$$

2.3. The interacting particle system and propagation of chaos

For a fixed $N\in\mathbb{N}$ and $i = 1,2,\ldots,N$, let (W^i, X_0^i) be N independent copies of (W, X_0). Note that, in the simulation of MV-SDEs, we need to approximate the measure $\mathcal{L}(X_t)$ for all $t \ge 0$, which is not required in the case of classical SDEs. We consider the N-dimensional system of interacting particles

$$X_t^{i,N} = X_0^i + \int_0^t b\left(s, X_s^{i,N}, \mu_s^{X,N}\right) ds + \int_0^t \sigma\left(s, X_s^{i,N}, \mu_s^{X,N}\right) dW^i(s), \quad a.s. \tag{2}$$

for all $t\in[0,T]$ and $i\in\{1,2,\ldots,N\}$, where $\mu_s^{X,N}$ is an empirical measure defined by

$$\mu_s^{X,N}(\cdot) = \frac{1}{N}\sum_{i=1}^N \delta_{X_s^{i,N}}(\cdot).$$

Note that, $X_t^{i,N}$ in the N-dimensional system of interacting particles (2) is a proper approximation to X_t in MV-SDEs (1) when N is large enough. This result is called the propagation of chaos. Due to distribution dependence in (1), we use the N-dimensional system of interacting particles (2) as a bridge to build the numerical approximations for MV-SDEs (1). In order to present the propagation of chaos, we consider the following system of non-interacting particles:

$$X_t^i = X_0^i + \int_0^t b\left(s, X_s^i, \mathcal{L}\left(X_s^i\right)\right) ds + \int_0^t \sigma\left(s, X_s^i, \mathcal{L}\left(X_s^i\right)\right) dW^i(s), \quad a.s. \tag{3}$$

for all $t\in[0,T]$ and $i\in\{1,2,\ldots,N\}$. Note that, if the MV-SDEs (1) have a unique solution, then

$$\mathcal{L}(X_t) = \mathcal{L}\left(X_t^i\right), \quad \forall i = 1,2,\ldots,N, \quad \forall t\in[0,T].$$

Under Assumption 1, Proposition 1 of Kumar et al. (2022) asserts the result of the propagation of chaos.

Proposition 3 (*Propagation of Chaos, Kumar et al. (2022, Proposition 1)*)*. Let assumptions (A1), (A2), (A3) and (A4) in Assumption 1 hold with $p_0 > 2$. Then,*

$$\sup_{i\in\{1,2,\ldots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] \le K \begin{cases} N^{-\frac{1}{2}}, & d < 4,\\ N^{-\frac{1}{2}}\ln(N), & d = 4,\\ N^{-\frac{2}{d}}, & d > 4,\end{cases}$$

where $K > 0$ is independent with N.

3. Modified Euler methods with moment bound

In this section, a class of modified Euler methods for the N-dimensional system of interacting particles (2) associated with the MV-SDEs (1) are proposed when the coefficients b and σ are allowed to grow super-linearly with respect to the state. Moreover, the boundedness of moments of the numerical approximation is also provided.

3.1. Modified Euler approximations

Let $n \in \mathbb{N}$ be given, we construct a uniform mesh on $[0, T]$ with $h = T/n \in (0, 1)$ being the stepsize and $t_k = kh$ for $k = 0, 1, \ldots, n$. For $i = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, n-1$, we consider modified Euler approximations

$$\begin{aligned} X^{i,N,n}_{t_{k+1}} &= X^{i,N,n}_{t_k} + \mathcal{T}_1\Big(b\big(t_k, X^{i,N,n}_{t_k}, \mu^{X,N,n}_{t_k}\big), h\Big)h \\ &\quad + \sum_{r=1}^{m} \mathcal{T}_2\Big(\sigma_r\big(t_k, X^{i,N,n}_{t_k}, \mu^{X,N,n}_{t_k}\big), h\Big)\Delta W^i_r(t_k) \end{aligned} \tag{4}$$

with $X^{i,N,n}_{t_0} = X^i_0$, where $\Delta W^i_r(t_k) = W^i_r(t_{k+1}) - W^i_r(t_k)$ and $\mathcal{T}_1, \mathcal{T}_2$ are operators satisfying

$$\mathcal{T}_1, \mathcal{T}_2 : \mathbb{R}^d \times (0, 1) \to \mathbb{R}^d.$$

For the simplicity of the notation, we denote $X^{i,N,n}_k := X^{i,N,n}_{t_k}$ for all $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, n$.

Next, we put some assumptions on the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ in (4), in order to achieve the moment boundedness of the modified Euler approximations.

Assumption 4.

(H1) There exists a constant $L > 0$ such that

$$|\mathcal{T}_1(x, h)| \le \min\left\{Lh^{-2}, |x|\right\}$$

$$|\mathcal{T}_2(x, h)| \le \min\left\{Lh^{-\frac{3}{2}}, |x|\right\}$$

for all $x \in \mathbb{R}^d$ and $h \in (0, 1)$.

(H2) There exist some constants $L, r_1, r_2 > 0$ such that

$$|\mathcal{T}_1(x, h) - x| \le Lh^{r_1}|x|^{r_2}$$

for all $x \in \mathbb{R}^d$ and $h \in (0, 1)$.

Remark 5. The condition *(H1)* in Assumption 4 ensures that the maps $\mathcal{T}_1, \mathcal{T}_2$ are controlled by the linear growth and their values are also bounded by the inverse of the step size h. This is essentially used to avoid moment explosion and maintain stability even when the drift or diffusion terms exhibit polynomial growth. Moreover, the condition *(H2)* serves as a consistency condition ensuring that the difference between $\mathcal{T}_1(x, h)$ and x is sufficiently close in the sense that $\mathcal{T}_1(x, h) \to x$ as $h \to 0$ for fixed $x \in \mathbb{R}^d$.

Also, we mention that Assumption 4 just provides sufficient conditions used to derive the moment bounds of the numerical approximations. We present in Section 3.2 some examples of modified Euler methods fulfilling Assumption 4. In the literature, there are numerical methods that do not satisfy the condition *(H1)* in Assumption 4, whose moment boundedness can be derived in a different way (see Example 10 quoted from Neelima et al., 2020).

3.2. Examples of modified Euler approximations

The following are some examples of modified Euler type methods (4), where $\mathcal{T}_1$ and $\mathcal{T}_2$ are explicitly given.

Example 6 (*Drift-Tamed Euler (DTE) dos Reis et al., 2022, Liu, Shi, & Wu, 2023*)**.** In dos Reis et al. (2022), Liu, Shi, and Wu (2023), the diffusion coefficient σ is assumed to be globally Lipschitz continuous (or Hölder continuous). In their setting, the diffusion coefficient σ does not need to be tamed, and a drift-tamed Euler method was introduced, where

$$\mathcal{T}_1(x, h) = \frac{x}{1 + h^{\lambda}|x|}, \ \mathcal{T}_2(x, h) = x, \ 0 < \lambda \le \tfrac{1}{2}. \tag{5}$$

In this work, we propose three new modified Euler methods as follows.

Example 7 (*Modified Euler Method (ME)*)**.** We propose a modified Euler method, where the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are given by

$$\mathcal{T}_1(x, h) = \mathcal{T}_2(x, h) = \frac{x}{1 + h|x|^2}. \tag{6}$$

It is straightforward to verify that the conditions *(H1)* and *(H2)* in Assumption 4 are satisfied with $r_1 \in (0, 1]$ and $r_2 = 3$.

Example 8 (*Tanh Euler Method (TE)*)**.** We introduce a tanh Euler method, where the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are as follows:

$$\mathcal{T}_1(x, h) = \mathcal{T}_2(x, h) = \frac{1}{h^{\alpha}}\tanh(h^{\alpha}x), \tag{7}$$

for some $\alpha \in (0, 3/2)$. The conditions *(H1)* and *(H2)* in Assumption 4 are satisfied when we choose $r_1 \in (0, \alpha]$ and $r_2 = 2$.

In a similar way, we propose the sin Euler method as follows.

Example 9 (*Sin Euler Method (SE)*)**.** In the sin Euler method, the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined by

$$\mathcal{T}_1(x, h) = \mathcal{T}_2(x, h) = \frac{1}{h^{\alpha}}\sin(h^{\alpha}x), \tag{8}$$

for some $\alpha \in (0, 3/2)$. The conditions *(H1)* and *(H2)* in Assumption 4 are fulfilled with $r_1 \in (0, \alpha)$ and $r_2 = 2$.

Next we provide tamed Euler type methods that do not satisfy Assumption 4 but the moment boundedness and strong convergence rates of the numerical solutions have be obtained in the literature (Neelima et al., 2020).

Example 10 (*Fully-Tamed Euler Method (FTE) Neelima et al., 2020*)**.** The author of Neelima et al. (2020) proposed a drift–diffusion fully tamed Euler method, where $\mathcal{T}_1, \mathcal{T}_2$ are given by

$$\begin{aligned} \mathcal{T}_1(b(t, x, \mu), h) &= \frac{b(t, x, \mu)}{1 + h^{\frac{1}{2}}|x|^{4\rho}}, \\ \mathcal{T}_2(\sigma_r(t, x, \mu), h) &= \frac{\sigma_r(t, x, \mu)}{1 + h^{\frac{1}{2}}|x|^{4\rho}}. \end{aligned} \tag{9}$$

Here ρ comes from the growth condition (A6) of the drift b below. It is not difficult to check that the mappings $\mathcal{T}_1, \mathcal{T}_2$ do not obey *(H1)* in Assumption 4, but satisfy

$$\begin{aligned} &|\mathcal{T}_1(b(t, x, \mu), h)| \\ \le\ &\min\left\{Lh^{-\frac{1}{4}}(1 + |x|) + \mathcal{W}_2(\mu, \delta_0), |b(t, x, \mu)|\right\}, \\ &|\mathcal{T}_2(\sigma_r(t, x, \mu), h)| \\ \le\ &\min\left\{Lh^{-\frac{1}{8}}(1 + |x|) + \mathcal{W}_2(\mu, \delta_0), |\sigma_r(t, x, \mu)|\right\}. \end{aligned}$$

3.3. Boundedness of moments of modified Euler approximations

As demonstrated in Higham et al. (2007), Hutzenthaler et al. (2011), Mattingly et al. (2002), Milstein and Tretyakov (2005), it has been established that the numerical approximation generated by the Euler–Maruyama method lacks finite moments, which is of paramount significance for achieving convergence toward the desired system of interacting particles. Subsequently, we establish that the modified Euler numerical approximations, as defined by (4), possess bounded high-order moments.

In the following, we give some assumptions on the regularity of the coefficients in MV-SDEs to establish the moment boundedness and the convergence rate of modified Euler approximations.

Assumption 11.

(A5) For some $p_1 > 1$, there exists a constant $L > 0$ such that

$$2\langle x-y, b(t,x,\mu)-b(t,y,\nu)\rangle + (2p_1-1)||\sigma(t,x,\mu) - \sigma(t,y,\nu)||^2 \leq L\left(|x-y|^2 + \mathcal{W}_2^2(\mu,\nu)\right)$$

for all $t \in [0,T], x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$.

(A6) There exist constants $L > 0$ and $\rho > 0$ such that

$$|b(t,x,\mu)-b(t,y,\nu)| \leq L\left(\left(1+|x|^{2\rho}+|y|^{2\rho}\right)|x-y|\right) + L\mathcal{W}_2(\mu,\nu)$$

for all $t \in [0,T], x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$.

(A7) There exist a constant $L > 0$ such that

$$|b(t,x,\mu)-b(s,x,\mu)| + \|\sigma(t,x,\mu)-\sigma(s,x,\mu)\| \leq L|t-s|^{\frac{1}{2}}$$

for all $t, s \in [0,T], x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Remark 12. We mention that, assumption (A5) is stronger than the assumption (A3) since $2p_1 - 1 > 1$. From assumptions (A5) and (A6), it follows that there exists a constant $K := K(L) > 0$ such that

$$\|\sigma(t,x,\mu)-\sigma(t,y,\upsilon)\| \leq K\left((1+|x|^{\rho}+|y|^{\rho})|x-y| + \mathcal{W}_2(\mu,\nu)\right)$$

for all $t \in [0,T], x, y \in \mathbb{R}^d$, and $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$.

Moreover, due to assumptions (A2), (A6) and (A7), there exists a constant $K := K(L,T) > 0$ such that

$$|b(t,x,\mu)| \leq K\left(1+|x|^{2\rho+1}+\mathcal{W}_2(\mu,\delta_0)\right), \tag{10}$$

$$\|\sigma(t,x,\mu)\| \leq K\left(1+|x|^{\rho+1}+\mathcal{W}_2(\mu,\delta_0)\right), \tag{11}$$

for all $t \in [0,T], x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Note that, (10) and (11) provide the growth condition for the coefficients b and σ, respectively.

Also, we mention that, in the above settings, the drift and diffusion coefficients of the MV-SDEs are assumed to be $\mathcal{W}_2$-Lipschitz with respect to the measure component. The following moment bound result, motivated by Zhao, Wang, and Zhang (2024), is essential for establishing the subsequent strong convergence result.

Lemma 13. *Suppose assumptions (A1), (A2), (A6), (A7), (H1), (H2) hold. Then, for all $k = 0, 1, \ldots, n$, there exists $\beta > 1$ and $K > 0$ independent of n and h such that*

$$\sup_{i\in\{1,2,\ldots,N\}} \mathbb{E}\left[|X_k^{i,N,n}|^{2p}\right] \leq K\left(1+\mathbb{E}\left[|X_0|^{2p\beta}\right]\right) \tag{12}$$

for all $p \in [1, \frac{2\bar{p}-\mathcal{G}}{2+4\mathcal{G}}]$, where

$$\mathcal{G} := \mathcal{G}(\rho, r_1, r_2) = \max\left\{6\rho, \frac{(2\rho+1)r_2-1}{r_1}\right\} \tag{13}$$

with $\rho > 0$ from (A6) and $r_1, r_2 > 0$ from (H2), and $\bar{p}$ satisfies $p_0 \geq \bar{p} \geq 1 + \frac{5}{2}\mathcal{G}$.

The proof of this lemma can be found in Appendix.

Remark 14. The boundedness of the moments of existing tamed numerical methods for MV-SDEs, such as those studied in Kumar et al. (2022), Neelima et al. (2020), was obtained based on an essential use of the following coercivity condition:

$$2\langle x, b_h(t,x,\mu)\rangle + (2p_0-1)\left\|\sigma_h(t,x,\mu)\right\|^2 \leq L\left(1+|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right) \tag{14}$$

for all $t \in [0,T], x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Here b_h, σ_h are certain taming modifications of the drift and diffusion coefficients b, σ. However, the modified Euler, tanh Euler and sin Euler methods proposed in Examples 7, 8, 9 fail to satisfy the condition (14). By formulating a different framework, here we employ new and different techniques to establish the desired moment bounds for these novel methods.

4. Strong convergence rate of modified Euler approximations

We prove the strong convergence rate of the modified Euler approximation (4) in this section. Firstly, we provide a continuous-time version of modified Euler approximations for (2). Let $\kappa_n(t) = \frac{\lfloor nt \rfloor}{n} = \sup\{s \in \{t_0, t_1, \ldots, t_n\}, s \leq t\}$ for all $t \in [0,T]$. The modified Euler approximation in continuous time is given by

$$X_t^{i,N,n} = X_0^i + \int_0^t \mathcal{T}_1\Big(b\big(\kappa_n(s), X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}\big), h\Big)ds + \sum_{r=1}^m \int_0^t \mathcal{T}_2\Big(\sigma_r\big(\kappa_n(s), X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}\big), h\Big)dW_r^i(s) \tag{15}$$

for all $t \in [0,T]$ and $i = 1, 2, \ldots, N$.

Before proceeding with the proof of the rate of convergence of the modified Euler approximation (15), we establish some lemmas in what follows.

The following lemma provides an estimation between $X_t^{i,N,n}$ and $X_{\kappa_n(t)}^{i,N,n}$.

Lemma 15. *Under the same conditions of Lemma 13, for all $i \in \{1, 2, \ldots, N\}, t \in [0,T]$ and $n, N \in \mathbb{N}$, we have the following inequality*

$$\mathbb{E}\left[\left|X_t^{i,N,n} - X_{\kappa_n(t)}^{i,N,n}\right|^{2p}\right] \leq Kh^p$$

for all $p \in [1, \frac{2\bar{p}-\mathcal{G}}{(2\rho+1)(2+4\mathcal{G})}]$, where $\mathcal{G}$ is given in (13) and $\bar{p}$ is a constant satisfying $p_0 \geq \bar{p} \geq (4\rho + \frac{5}{2})\mathcal{G} + 2\rho + 1$.

Proof. Applying Hölder's inequality and the Burkholder–Davis–Gundy inequality, we have that for all $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\left|X_t^{i,N,n} - X_{\kappa_n(t)}^{i,N,n}\right|^{2p}\right] \\
&\leq Kh^{2p-1} \mathbb{E}\left[\int_{\kappa_n(t)}^t \left|\mathcal{T}_1\Big(b(\kappa_n(s), X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}), h\Big)\right|^{2p} ds\right] \\
&\quad + Kh^{p-1} \mathbb{E}\left[\sum_{r=1}^m \int_{\kappa_n(t)}^t \left|\mathcal{T}_2(\sigma_r(\kappa_n(s), X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}), h)\right|^{2p} ds\right].
\end{aligned}$$

Under the assumption (*H1*), and the growth condition for the coefficients b and σ in (10) and (11) respectively, it can be shown that

$$\begin{aligned}
&\mathbb{E}\left[\left|X_t^{i,N,n} - X_{\kappa_n(t)}^{i,N,n}\right|^{2p}\right] \\
&\leq Kh^{2p-1}\mathbb{E}\Big[\int_{\kappa_n(t)}^t 1 + \left|X_{\kappa_n(s)}^{i,N,n}\right|^{2p(2\rho+1)} + \Big(\mathcal{W}_2^2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)\Big)^p ds\Big] \\
&\quad + Kh^{p-1}\mathbb{E}\Big[\int_{\kappa_n(t)}^t 1 + \left|X_{\kappa_n(s)}^{i,N,n}\right|^{2p(\rho+1)} + \Big(\mathcal{W}_2^2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)\Big)^p ds\Big].
\end{aligned}$$

From the identity (A.2), one can obtain

$$\mathbb{E}\Big[\big|X_t^{i,N,n}-X_{\kappa_n(t)}^{i,N,n}\big|^{2p}\Big]$$
$$\leq Kh^{2p}\Big(1+\sup_{s\in[\kappa_n(t),t]}\sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\big|X_{\kappa_n(s)}^{i,N,n}\big|^{2p(2\rho+1)}\Big]\Big)$$
$$+Kh^{p}\Big(1+\sup_{s\in[\kappa_n(t),t]}\sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\big|X_{\kappa_n(s)}^{i,N,n}\big|^{2p(\rho+1)}\Big]\Big).$$

By the result of Lemma 13, for all $i\in\{1,2,\ldots,N\}$, $t\in[0,T]$, n, $N\in\mathbb{N}$ and $p\in[1,\frac{2\bar{p}-\mathcal{G}}{(2\rho+1)(2+4\mathcal{G})}]$, we have

$$\mathbb{E}\Big[\big|X_t^{i,N,n}-X_{\kappa_n(t)}^{i,N,n}\big|^{2p}\Big]$$
$$\leq Kh^{p}\Big(1+\mathbb{E}\big[|X_0|^{2\beta p(2\rho+1)}\big]\Big)\leq Kh^p,$$

as required. $\square$

The following lemma gives the boundedness of moments for the modified Euler approximation (15).

Lemma 16. *Under the same conditions of Lemma 13, there exist $\beta>1$ and $K>0$ independent of n and h such that*

$$\sup_{i\in\{1,2,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\Big[\big|X_t^{i,N,n}\big|^{2p}\Big]\leq K\left(1+\mathbb{E}\left[|X_0|^{2p\beta}\right]\right)$$

for all $p\in[1,\frac{2\bar{p}-\mathcal{G}}{(2\rho+1)(2+4\mathcal{G})}]$, where $\mathcal{G}$ is given in (13) and $\bar{p}$ is a constant satisfying $p_0\geq\bar{p}\geq(4\rho+\frac{5}{2})\mathcal{G}+2\rho+1$.

Proof. Applying Lemma 13 and Lemma 15, we obtain the desired result that

$$\sup_{i\in\{1,2,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\Big[\big|X_t^{i,N,n}\big|^{2p}\Big]$$
$$\leq\sup_{i\in\{1,2,\ldots,N\}}\sup_{t\in[0,T]}K\mathbb{E}\Big[\big|X_t^{i,N,n}-X_{\kappa_n(t)}^{i,N,n}\big|^{2p}\Big]$$
$$+\sup_{i\in\{1,2,\ldots,N\}}\sup_{t\in[0,T]}K\mathbb{E}\Big[\big|X_{\kappa_n(t)}^{i,N,n}\big|^{2p}\Big]$$
$$\leq K\Big(1+\mathbb{E}\big[|X_0|^{2p\beta}\big]\Big).$$

$\square$

Next, we provide the assumptions for the proof of the convergence rate of the time-continuous approximation (15).

Assumption 17.

(H3) There exist some constants $L>0, r_2>0$ and $r_1,r_3\geq\frac{1}{2}$ such that

$$|\mathcal{T}_1(x,h)-x|\leq Lh^{r_1}|x|^{r_2}$$
$$|\mathcal{T}_2(x,h)-x|\leq Lh^{r_3}|x|^{r_2}$$

for all $x\in\mathbb{R}^d$ and $h\in(0,1)$.

Note that, in the assumption (*H2*) of Assumption 4, we only need $r_1>0$. But in assumption (*H3*) of Assumption 17, we require that $r_1,r_3\geq\frac{1}{2}$. Thus, the assumption *(H3)* is slightly stronger than *(H2)* as $h\in(0,1)$.

Lemma 18. *Suppose the assumptions (A1), (A2), (A6), (A7), (H1), (H3) hold. Then, for all $p\in[1,\tilde{p}]$, there exists $K>0$ independent of n and h such that*

$$\mathbb{E}\Big[\Big|b\big(t,X_t^{i,N,n},\mu_t^{X,N,n}\big)$$
$$-\mathcal{T}_1\Big(b\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big),h\Big)\Big|^{2p}\Big]\leq Kh^p,$$
$$\sup_{r\in\{1,2,\ldots,m\}}\mathbb{E}\Big[\Big|\sigma_r\big(t,X_t^{i,N,n},\mu_t^{X,N,n}\big)$$
$$-\mathcal{T}_2\Big(\sigma_r\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big),h\Big)\Big|^{2p}\Big]\leq Kh^p,$$

where

$$\tilde{p}=\min\left\{\frac{2\bar{p}-\mathcal{G}}{r_2(2\rho+1)(2+4\mathcal{G})},\frac{2\bar{p}-\mathcal{G}}{4\rho(2\rho+1)(2+4\mathcal{G})},\frac{2\bar{p}-\mathcal{G}}{(2\rho+1)(2+4\mathcal{G})}\right\}\tag{16}$$

with $\mathcal{G}$ is given in (13) and $\bar{p}\leq p_0$ is a constant such that

$$\bar{p}\geq\max\Big\{\big(16\rho^2+8\rho+\frac{1}{2}\big)\mathcal{G}+8\rho^2+4\rho,$$
$$\big(2r_2(2\rho+1)+\frac{1}{2}\big)\mathcal{G}+r_2(2\rho+1),$$
$$\big(4\rho+\frac{5}{2}\big)\mathcal{G}+2\rho+1\Big\}.$$

Proof. By Lemma 15, Hölder's inequality, (*A6*), (*A7*), (*H3*), and the growth condition (10), we have

$$\mathbb{E}\Big[\Big|b\big(t,X_t^{i,N,n},\mu_t^{X,N,n}\big)$$
$$-\mathcal{T}_1\Big(b\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big),h\Big)\Big|^{2p}\Big]$$
$$\leq K\mathbb{E}\Big[\Big|b\big(t,X_t^{i,N,n},\mu_t^{X,N,n}\big)-b\big(t,X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big)\Big|^{2p}$$
$$+\Big|b\big(t,X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big)-b\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big)\Big|^{2p}$$
$$+\Big|b\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big)$$
$$-\mathcal{T}_1\Big(b\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big),h\Big)\Big|^{2p}\Big]$$
$$\leq K\mathbb{E}\Big[\Big(1+\big|X_t^{i,N,n}\big|^{2\rho}+\big|X_{\kappa_n(t)}^{i,N,n}\big|^{2\rho}\Big)^{2p}\big|X_t^{i,N,n}-X_{\kappa_n(t)}^{i,N,n}\big|^{2p}$$
$$+\mathcal{W}_2^{2p}\big(\mu_t^{X,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big)+\big|t-\kappa_n(t)\big|^p$$
$$+h^{2pr_1}\Big|b\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big)\Big|^{2pr_2}\Big]$$
$$\leq Kh^p\mathbb{E}\Big[\Big(1+\big|X_t^{i,N,n}\big|^{8p\rho}+\big|X_{\kappa_n(t)}^{i,N,n}\big|^{8p\rho}\Big)\Big]$$
$$+K\sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\big|X_t^{i,N,n}-X_{\kappa_n(t)}^{i,N,n}\big|^{2p}\Big]+Kh^p$$
$$+Kh^{2pr_1}\mathbb{E}\Big[1+\big|X_{\kappa_n(t)}^{i,N,n}\big|^{2pr_2(2\rho+1)}+\mathcal{W}_2^{2pr_2}\big(\mu_{\kappa_n(s)}^{X,N,n},\delta_0\big)\Big]$$

since

$$\mathcal{W}_2^2\big(\mu_t^{X,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big)\leq\sup_{i\in\{1,2,\ldots,N\}}\big|X_t^{i,N,n}-X_{\kappa_n(t)}^{i,N,n}\big|^2.$$

Then, Lemmas 13, 15 and 16 together show

$$\mathbb{E}\left[\ \Big|b\big(t,X_t^{i,N,n},\mu_t^{X,N,n}\big)\right.$$
$$\left.-\mathcal{T}_1\Big(b\big(\kappa_n(t),X_{\kappa_n(t)}^{i,N,n},\mu_{\kappa_n(t)}^{X,N,n}\big),h\Big)\Big|^{2p}\ \right]$$
$$\leq Kh^p\Big(1+\mathbb{E}\big[|X_0|^{8p\rho\beta}\big]\Big)+Kh^p$$
$$+Kh^{2pr_1}\Big(1+\mathbb{E}\big[|X_0|^{2p\beta r_2(2\rho+1)}\big]\Big)$$
$$+Kh^{2pr_1}\mathbb{E}\Big[\Big(\frac{1}{N}\sum_{i=1}^{N}\big|X_{\kappa_n(s)}^{i,N,n}\big|^2\Big)^{pr_2}\Big]$$
$$\leq Kh^p$$

since $r_1\geq\frac{1}{2}$. The proof is completed by performing a similar calculation for σ_r for all $r=1,2,\ldots,m$ with $r_2\geq\frac{1}{2}$. $\square$

Now, we are ready to prove the strong convergence rate of order $\frac{1}{2}$ for the modified Euler approximation (15) in the L^p sense.

Theorem 19. *Suppose assumptions (A1), (A2), (A5), (A6), (A7), (H1), (H3) are satisfied. Then, the modified Euler approximation (15) converges to the solution of the interacting particle system (2) in a strong sense with L^p convergence rate given by*

$$\sup_{i\in\{1,2,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\Big[\big|X_t^{i,N}-X_t^{i,N,n}\big|^{2p}\Big]\le Kh^p$$

for all $1\le p\le\min\{\frac{1}{2}p_1+\frac{1}{4},\tilde{p}\}$, where p_1 comes from assumption (A5), $\tilde{p}$ is given by (16) and $K>0$ does not depend on $n,N\in\mathbb{N}$.

Proof. From (2), (15), and Ito's formula, it follows that

$$\begin{aligned}
&\big|X_t^{i,N}-X_t^{i,N,n}\big|^{2p}\\
=\ &2p\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p-2}\langle X_s^{i,N}-X_s^{i,N,n},\\
&\quad b\big(s,X_s^{i,N},\mu_s^{X,N}\big)\\
&\quad-\mathcal{T}_1\Big(b\big(\kappa_n(s),X_{\kappa_n(s)}^{i,N,n},\mu_{\kappa_n(s)}^{X,N,n}\big),h\Big)\rangle ds\\
&+2p\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p-2}\sum_{r=1}^m\langle X_s^{i,N}-X_s^{i,N,n},\\
&\quad\sigma_r\big(s,X_s^{i,N},\mu_s^{X,N}\big)\\
&\quad-\mathcal{T}_2\Big(\sigma_r\big(\kappa_n(s),X_{\kappa_n(s)}^{i,N,n},\mu_{\kappa_n(s)}^{X,N,n}\big),h\Big)\rangle dW_r^i(s)\\
&+p(2p-1)\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p-2}\\
&\quad\sum_{r=1}^m\Big|\sigma_r\big(s,X_s^{i,N},\mu_s^{X,N}\big)\\
&\quad-\mathcal{T}_2\Big(\sigma_r\big(\kappa_n(s),X_{\kappa_n(s)}^{i,N,n},\mu_{\kappa_n(s)}^{X,N,n}\big),h\Big)\Big|^2ds.
\end{aligned}$$

Since $p\le\frac{1}{2}p_1+\frac{1}{4}$, taking expectation and applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{i,N}-X_t^{i,N,n}\big|^{2p}\Big]\\
\le\ &p\mathbb{E}\Big[\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p-2}\Big\{\langle 2\big(X_s^{i,N}-X_s^{i,N,n}\big),\\
&\quad b\big(s,X_s^{i,N},\mu_s^{X,N}\big)-b\big(s,X_s^{i,N,n},\mu_s^{X,N,n}\big)\rangle\\
&\quad+(2p_1-1)\sum_{r=1}^m\Big|\sigma_r\big(s,X_s^{i,N},\mu_s^{X,N}\big)\\
&\quad-\sigma_r\big(s,X_s^{i,N,n},\mu_s^{X,N,n}\big)\Big|^2\Big\}ds\Big]\\
&+K\mathbb{E}\Big[\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p-2}\langle\big(X_s^{i,N}-X_s^{i,N,n}\big),\\
&\quad b\big(s,X_s^{i,N,n},\mu_s^{X,N,n}\big)\\
&\quad-\mathcal{T}_1\Big(b\big(\kappa_n(s),X_{\kappa_n(s)}^{i,N,n},\mu_{\kappa_n(s)}^{X,N,n}\big),h\Big)\rangle ds\Big]\\
&+K\mathbb{E}\Big[\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p-2}\\
&\quad\sum_{r=1}^m\Big|\sigma_r\big(s,X_s^{i,N,n},\mu_s^{X,N,n}\big)\\
&\quad-\mathcal{T}_2\Big(\sigma_r\big(\kappa_n(s),X_{\kappa_n(s)}^{i,N,n},\mu_{\kappa_n(s)}^{X,N,n}\big),h\Big)\Big|^2ds\Big]
\end{aligned}$$

for all $t\in[0,T]$, $i\in\{1,2,\ldots,N\}$ and $n,N\in\mathbb{N}$. By using the Cauchy–Schwarz inequality, Young's inequality, and Assumption (A5), one can obtain

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{i,N}-X_t^{i,N,n}\big|^{2p}\Big]\\
\le\ &K\mathbb{E}\Big[\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p}ds\Big]\\
&+K\mathbb{E}\Big[\int_0^t\mathcal{W}_2^{2p}\big(\mu_s^{X,N},\mu_s^{X,N,n}\big)ds\Big]\\
&+K\mathbb{E}\Big[\int_0^t\Big|b\big(s,X_s^{i,N,n},\mu_s^{X,N,n}\big)\\
&\quad-\mathcal{T}_1\Big(b\big(\kappa_n(s),X_{\kappa_n(s)}^{i,N,n},\mu_{\kappa_n(s)}^{X,N,n}\big),h\Big)\Big|^{2p}ds\Big]\\
&+K\mathbb{E}\Big[\int_0^t\sum_{r=1}^m\Big|\sigma_r\big(s,X_s^{i,N,n},\mu_s^{X,N,n}\big)\\
&\quad-\mathcal{T}_2\Big(\sigma_r\big(\kappa_n(s),X_{\kappa_n(s)}^{i,N,n},\mu_{\kappa_n(s)}^{X,N,n}\big),h\Big)\Big|^{2p}ds\Big].
\end{aligned}$$

Lemma 18 along with the following estimate

$$\mathcal{W}_2^2\big(\mu_s^{X,N},\mu_s^{X,N,n}\big)\le\frac{1}{N}\sum_{i=1}^N\big|X_s^{i,N}-X_s^{i,N,n}\big|^2$$

for all $s\in[0,t]$ yields

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_t^{i,N}-X_t^{i,N,n}\big|^{2p}\Big]\\
\le\ &K\mathbb{E}\Big[\int_0^t\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p}ds\Big]+K\mathbb{E}\Big[\int_0^t h^p ds\Big]\\
&+K\mathbb{E}\Big[\int_0^t\Big(\frac{1}{N}\sum_{i=1}^N\big|X_s^{i,N}-X_s^{i,N,n}\big|^2\Big)^p ds\Big]\\
\le\ &K\mathbb{E}\Big[\int_0^t\sup_{i\in\{1,2,\ldots,N\}}\big|X_s^{i,N}-X_s^{i,N,n}\big|^{2p}ds\Big]+Kh^p.
\end{aligned}$$

Therefore, the following inequality

$$\begin{aligned}
&\sup_{i\in\{1,2,\ldots,N\}}\sup_{r\in[0,t]}\mathbb{E}\Big[\big|X_r^{i,N}-X_r^{i,N,n}\big|^{2p}\Big]\\
\le\ &K\int_0^t\sup_{i\in\{1,2,\ldots,N\}}\sup_{r\in[0,s]}\mathbb{E}\Big[\big|X_r^{i,N}-X_r^{i,N,n}\big|^{2p}\Big]ds+Kh^p
\end{aligned}$$

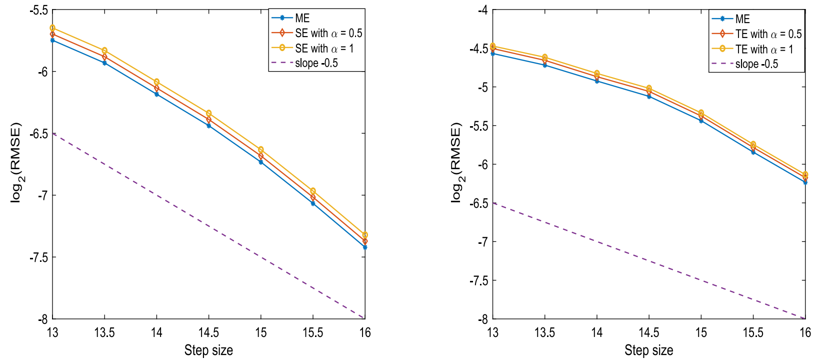
holds for all $t\in[0,T]$, $i\in\{1,2,\ldots,N\}$ and $n,N\in\mathbb{N}$. The application of Grönwall's inequality implies

$$\sup_{i\in\{1,2,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\Big[\big|X_t^{i,N}-X_t^{i,N,n}\big|^{2p}\Big]\le Kh^p.$$

The proof is thus finished. □

To ensure the completeness of our work, we now are ready to present the full numerical approximation errors of the modified Euler approximation (15) to the solution of McKean–Vlasov SDE (1). It is a straightforward result from the combination of Theorem 19 and Proposition 3.

Corollary 20. *Suppose assumptions (A1), (A2), (A3), (A4), (A5), (A6), (A7), (H1), (H3) are satisfied with $p_0>2$. Then, the modified Euler approach (15) converges to the solution of McKean–Vlasov SDEs (1)*



(a) Example 21 (b) Example 22

Fig. 1. Strong errors for Examples 21 and 22.

in a strong sense with L^2 convergence rate given by

$$\sup_{i\in\{1,2,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\Big[\big|X_t^i - X_t^{i,N,n}\big|^2\Big] \le K\begin{cases} h + N^{-\frac{1}{2}}, & d<4,\\ h + N^{-\frac{1}{2}}\ln(N), & d=4,\\ h+N^{-\frac{2}{d}}, & d>4,\end{cases}$$

where $K > 0$ is independent of $n, N \in \mathbb{N}$.

5. Numerical experiments

Some numerical experiments are conducted to illustrate the previous theoretical findings. Newton's method is used to solve implicit algebraic equations if necessary.

We begin by considering two examples to illustrate the mean square convergence rates of the modified Euler method (ME) (6), the tanh Euler method (TE) (7) and the sin Euler method (SE) (8). For these examples, the coefficients satisfy Assumption 11, where both the drift and diffusion terms are non-globally Lipschitz. To approximate the law $\mathcal{L}(X_{t_k})$ at each time step t_k for $k = 0, 1, \ldots, n$ by its empirical distribution, we apply the particle method with the number of particles $N = 100$.

As we do not know the exact solution of the considered examples, the strong convergence with respect to the number of time steps is assessed by comparing two solutions computed on a fine and coarse time grid, respectively, using the same samples of Brownian motion. The reference values of the models are computed based on a Monte Carlo method combined with the approximations and $h_{ref} = 2^{-17}$. We also apply Monte Carlo method to compute the root-mean-square error (RMSE) in step size $h = \{2^{-13}, 2^{-13.5}, 2^{-14}, 2^{-14.5}, 2^{-15}, 2^{-15.5}, 2^{-16}\}$ by $\text{RMSE} := \sqrt{\frac{1}{N}\sum_{i=1}^{N}\big(X_T^{i,N,n} - X_T^{i,N,n_h}\big)^2}$ at the terminal time $T = 1$, where i refers to the ith particle, and $n = T/h_{ref} = 2^{17}$ and $n_h = \lfloor T/h\rfloor$ are the corresponding time steps in the fine and coarse time grid, respectively.

Example 21. As the first test model, we consider the following McKean–Vlasov SDE

$$\begin{cases} dX_t = \big(X_t - X_t^3 + c\mathbb{E}[X_t]\big)\,dt \\ \qquad\qquad + \gamma\left(1 - X_t^2\right)dW(t), \quad t \in (0, T],\\ X_0 = x,\end{cases}$$

with the parameter values $\gamma = 0.5$, $c = 1$, and $x = 0$. The conditions in Assumption 11 are fulfilled with $\rho \ge 1$.

For this example, we test the RMSE of three different numerical methods ME (6), SE (8) with $\alpha = 1/2$ and SE (8) with $\alpha = 1$. As expected, the strong convergence rates of ME (6), SE (8) with $\alpha = 1/2$ and SE (8) with $\alpha = 1$ are close to $1/2$ from Fig. 1(a).

Example 22. For the second test model, we consider a McKean–Vlasov SDE in which the drift term preserves higher-order growth condition:

$$\begin{cases} dX_t = \big(1 - X_t^5 + X_t^3 + c\mathbb{E}[X_t]\big)\,dt \\ \qquad\qquad + \big(\gamma X_t^2 + 1\big)\,dW(t), \quad t \in (0, T],\\ X_0 = x,\end{cases}$$

with the parameter values $c = 1$, $\gamma = 0.01$ and $x = 0$. It is clear that Assumption 11 is satisfied if $\rho \ge 2$.

In Fig. 1(b), we reveal the RMSE of three different numerical methods ME (6), TE (7) with $\alpha = 1/2$ and TE (7) with $\alpha = 1$ against the same step sizes on the $\log_2(\cdot)$ scale. The strong error rate is $1/2$ as expected in Theorem 19.

The third example features a non-globally Lipschitz drift and a global Lipschitz diffusion with respect to the state. We test the drift-tamed Euler (DTE) (5), the modified Euler method (ME) (6), the tanh Euler method (TE) (7), the sin Euler method (SE) (8) and the fully-tamed Euler method (FTE) (9) facilitating a comparative analysis with an alternative numerical approach, the split-step method (SSM) proposed in Chen and Dos Reis (2022, 2024), Chen et al. (2023).

Example 23. We consider the double-well model considered in Chen and Dos Reis (2024), given by

$$\begin{cases} dX_t = \big(-\frac{5}{4}X_t^3 + 3X_t^2\mathbb{E}[X_t] - 3X_t\mathbb{E}[X_t^2] + \mathbb{E}[X_t^3]\big)\,dt \\ \qquad\qquad + X_t\,dW(t), \quad t \in (0, T],\\ X_0 \sim \mathcal{N}(\mu, \sigma^2),\end{cases}$$

where $\mathcal{N}(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2. There are three stable states $\{-2, 0, 2\}$ for this model (Tugaut, 2013).

Using the same setup as in Chen and Dos Reis (2024) with $N = 1000$, and computing the reference solution with a step size of $h = 10^{-4}$, Figs. 2–7 show density maps for DTE (5) with $\lambda = 1/2$, ME (6), TE (7) with $\alpha = 1$, SE (8) with $\alpha = 1$, FTE (9) and SSM. These results use a step size of $h = 10^{-2}$ and are shown at times $T = 1, 3, 10$ for two different initial distributions $\mathcal{N}(0, 1), \mathcal{N}(3, 9)$. Simulated paths of these methods with the initial distribution $\mathcal{N}(3, 9)$ are illustrated in Figs. 8–13.

As noted in Chen and Dos Reis (2024), for large initial values (i.e., $X_0 \sim \mathcal{N}(3, 9)$), the DTE (5) with $\lambda = 1/2$ produces unacceptable results (see Fig. 3). This happens because the method

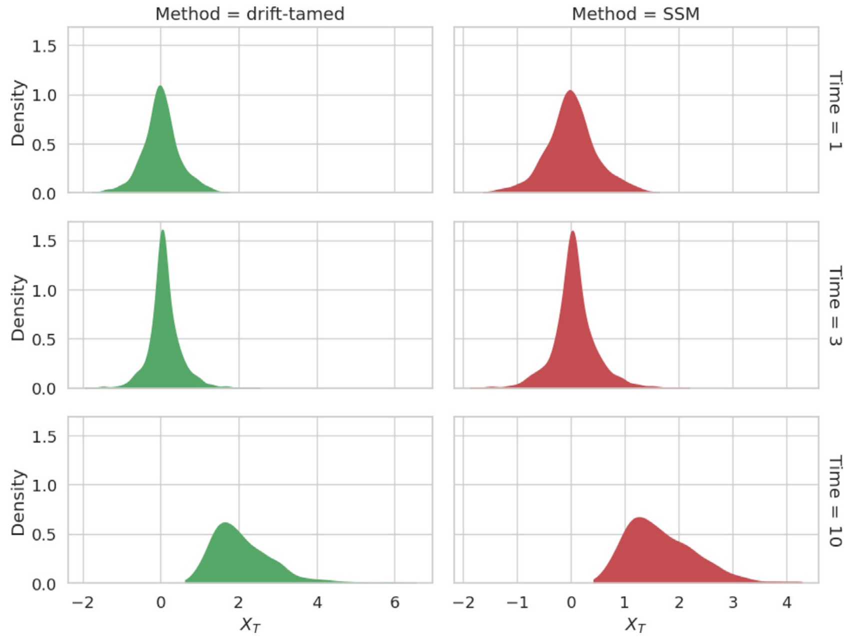


Fig. 2. Density with $X_0 \sim \mathcal{N}(0,\ 1)$.

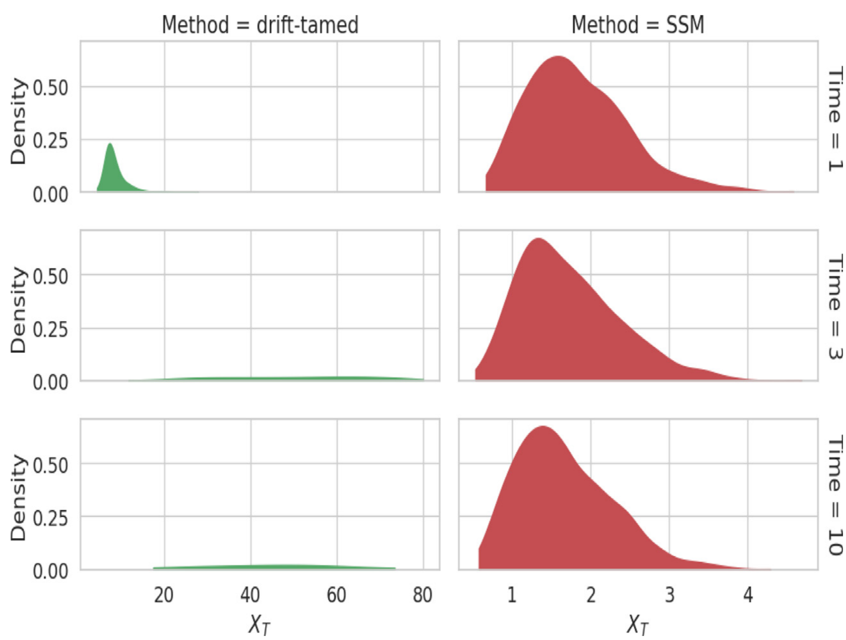


Fig. 3. Density with $X_0 \sim \mathcal{N}(3,\ 9)$.

(5) becomes unstable with the stepsize $h = 10^{-2}$, as clearly indicated by Fig. 8. Similar phenomenon can be also detected for the SE (8) (see Fig. 7, 13). In contrast, the SSM (Chen & Dos Reis, 2024, Chen & Dos Reis, 2022), ME (6), TE (7) with $\alpha = 1$ and FTE (9) produce acceptable results in the same setting. Interestingly, by decreasing the stepsize to e.g., $h = 0.004$, the DTE (5) with $\lambda = 1/2$ and SE (8) with $\alpha = 1$ can be then stable and give acceptable approximations.

In terms of the density maps depicted in Figs. 2–7, one can clearly observe that, the TE (7) performs most closely to the SSM and gives more reliable approximations than the other explicit methods.

To sum up, as an implicit method, the SSM method has better stability properties than the other explicit methods and can thus always produce reliable approximations even when treating MV-SDEs with relatively large initial values and relatively large stepsize h. However, some explicit methods, such as the DTE (5) and SE (8) with $\alpha = 1$, might be sensitive to the stepsize selection. Moreover, the above numerical results demonstrate that the TE (7) performs better than the other explicit methods.

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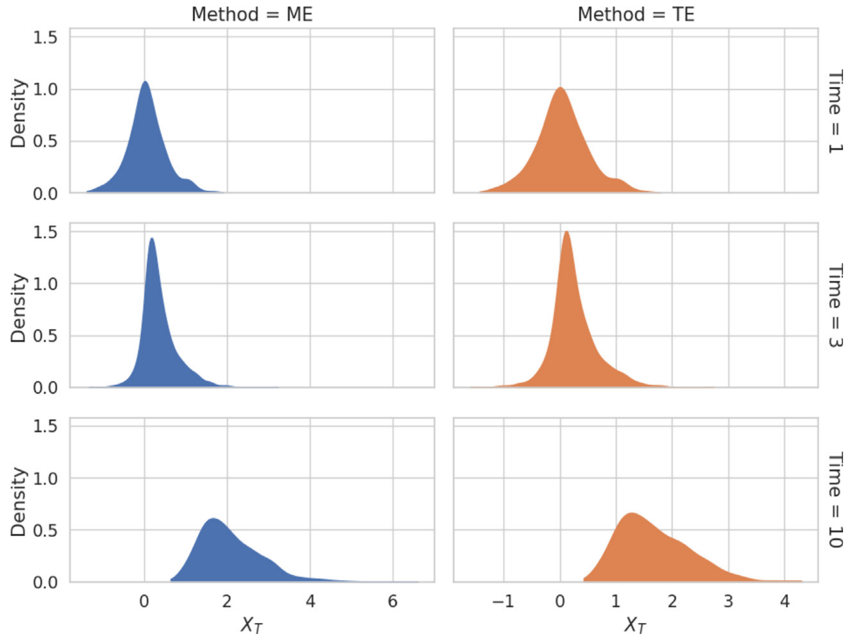


Fig. 4. Density with $X_0 \sim \mathcal{N}(0,\ 1)$.

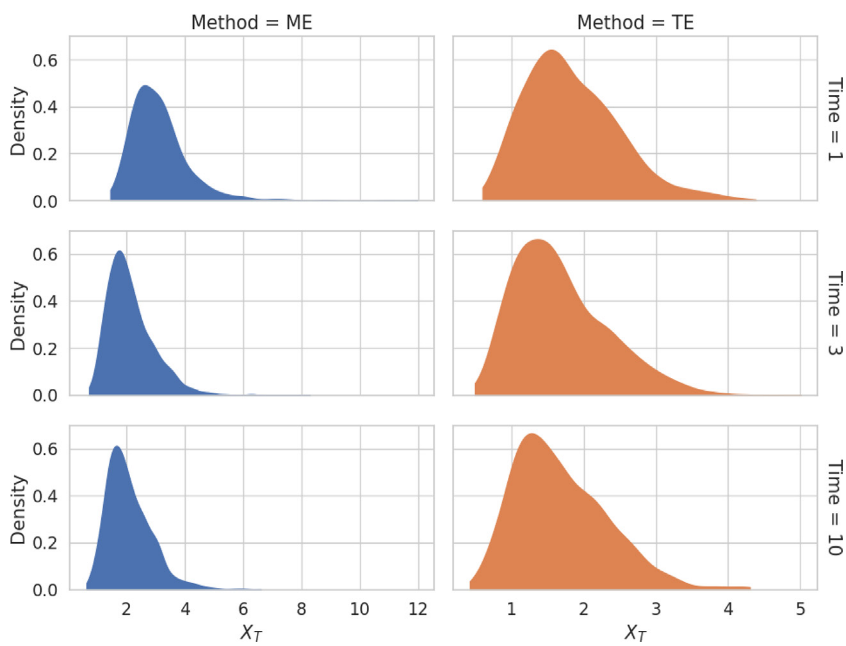


Fig. 5. Density with $X_0 \sim \mathcal{N}(3,\ 9)$.

Appendix. Proof of Lemma 13 in Section 3.3

Proof of Lemma 13. In the following, we use K to denote the generic constant which is independent of n and h. Let $\mathcal{R} > 0$ be sufficiently large and define a sequence of decreasing subevents

$$\Omega_{\mathcal{R},k} = \left\{\omega \in \Omega : \left|X_j^{i,N,n}(\omega)\right| \leq \mathcal{R},\ j = 0, 1, \ldots, k\right\}$$

for $k = 0, 1, \ldots, n$. We denote the complement of $\Omega_{\mathcal{R},k}$ by $\Omega^c_{\mathcal{R},k}$.

Firstly, we show that the boundedness of the moment is valid within a family of appropriate subevents $\left\{\Omega_{\mathcal{R},k}\right\}_{k\in\{0,1,\ldots,n\}}$. Note that

$$\mathbb{E}\left[\mathbb{1}_{\Omega_{\mathcal{R},k+1}}\left|X_{k+1}^{i,N,n}\right|^{2\bar{p}}\right] \leq \mathbb{E}\left[\mathbb{1}_{\Omega_{\mathcal{R},k}}\left|X_{k+1}^{i,N,n}\right|^{2\bar{p}}\right]$$

$$\leq \mathbb{E}\left[\mathbb{1}_{\Omega_{\mathcal{R},k}}\left|X_{k}^{i,N,n}\right|^{2\bar{p}}\right]$$

$$+\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\left|X_k^{i,N,n}\right|^{2\bar{p}-2}\Big(2\bar{p}\big\langle X_k^{i,N,n}, X_{k+1}^{i,N,n} - X_k^{i,N,n}\big\rangle$$

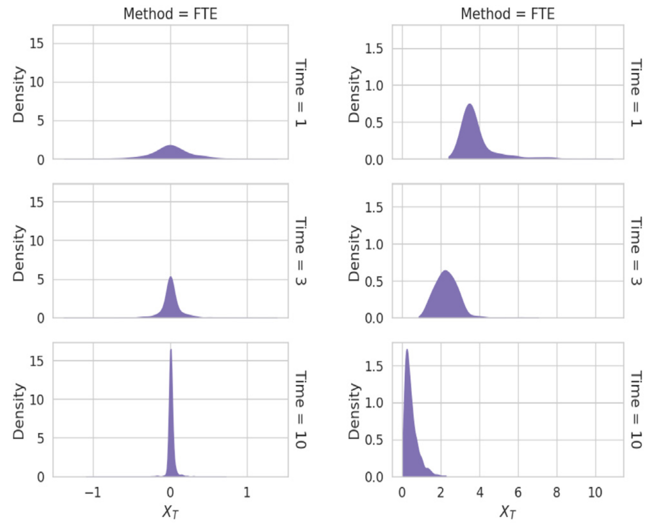
$$+\bar{p}(2\bar{p}-1)\left|X_{k+1}^{i,N,n} - X_k^{i,N,n}\right|^2\Big)\Big]$$

$$+K\sum_{l=3}^{2\bar{p}}\mathbb{E}\left[\mathbb{1}_{\Omega_{\mathcal{R},k}}\left|X_k^{i,N,n}\right|^{2\bar{p}-l}\left|X_{k+1}^{i,N,n} - X_k^{i,N,n}\right|^l\right]$$

$$:= \mathbb{E}\left[\mathbb{1}_{\Omega_{\mathcal{R},k}}\left|X_k^{i,N,n}\right|^{2\bar{p}}\right] + I_1 + I_2. \tag{A.1}$$

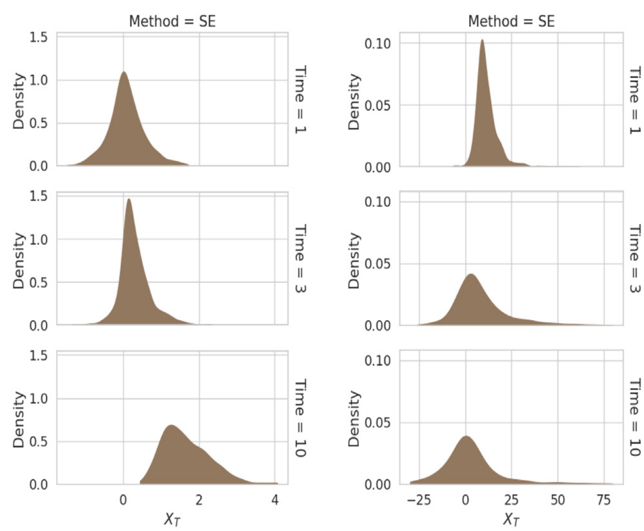
For I_1, according to (4) and $\Delta W_r^i(t_k)$ is independent with $X_k^{i,N,n}$ for all $r = 1, 2, \ldots, m$, one can derive

$$I_1 = 2\bar{p}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\left|X_k^{i,N,n}\right|^{2\bar{p}-2}$$



(a) (b)

Fig. 6. Density with $X_0 \sim \mathcal{N}(0, 1)$(a) and $X_0 \sim \mathcal{N}(3, 9)$(b).



(a) (b)

Fig. 7. Density with $X_0 \sim \mathcal{N}(0, 1)$(a) and $X_0 \sim \mathcal{N}(3, 9)$(b).

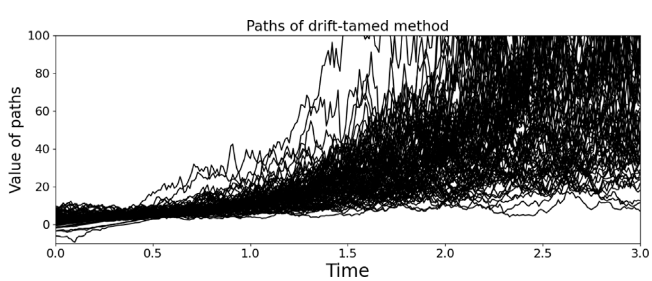


Fig. 8. Paths of drift-tamed method for Example 23.

$$\left\langle X_k^{i,N,n}, \mathcal{T}_1\left(b\left(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\right), h\right)h \right.$$
$$\left. -b\left(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\right)h\right\rangle\Big]$$

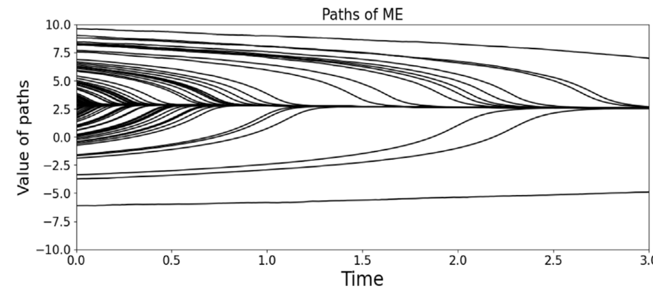


Fig. 9. Paths of modified Euler method for Example 23.

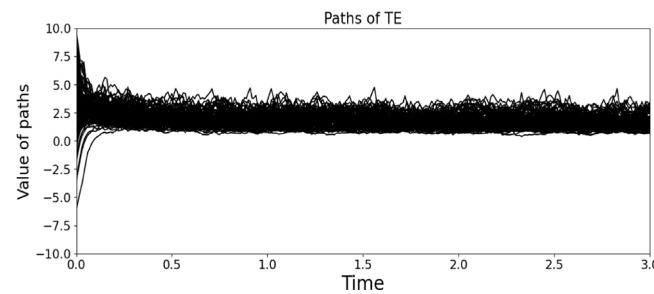


Fig. 10. Paths of tanh Euler method for Example 23.

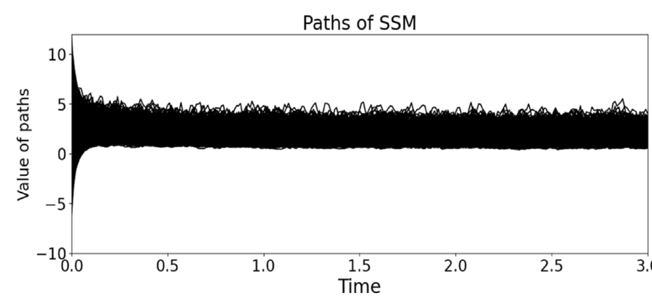


Fig. 11. Paths of split-step method for Example 23.

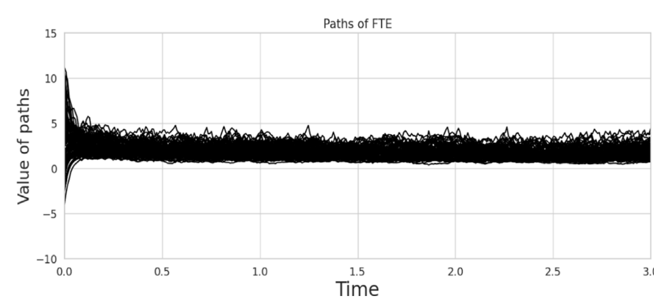


Fig. 12. Paths of fully-tamed Euler method for Example 23.

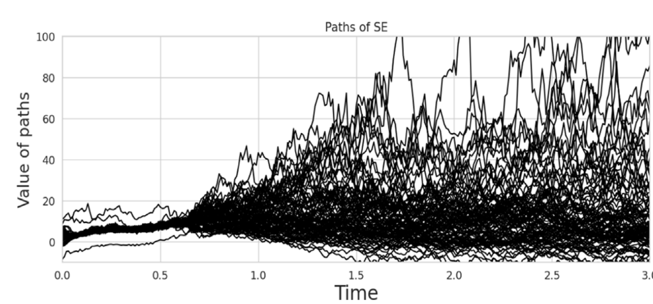


Fig. 13. Paths of sin Euler method for Example 23.

$$
\begin{aligned}
&+2\bar{p}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-2}\\
&\quad\big\langle X_k^{i,N,n}, b\big(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\big)h\big\rangle\Big]\\
&+\bar{p}\,(2\bar{p}-1)\,\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-2}\\
&\quad\big|\mathcal{T}_1\big(b\big(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\big), h\big)h\big|^2\Big]\\
&+\bar{p}(2\bar{p}-1)\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-2}\\
&\quad\big|\sum_{r=1}^{m}\mathcal{T}_2\big(\sigma_r\big(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\big), h\big)\Delta W_r^i(t_k)\big|^2\Big].
\end{aligned}
$$

Using the Schwartz inequality, assumptions (*A2*), (*H1*), (*H2*) and the growth condition for the coefficient b in (10), we have

$$
\begin{aligned}
I_1 \le\ & Kh^{1+r_1}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-1}\big(1+\big|X_k^{i,N,n}\big|^{2\rho+1}\\
&\quad+\mathcal{W}_2\big(\mu_{t_k}^{X,N,n}, \delta_0\big)\big)^{r_2}\Big]\\
&+Kh^2\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-2}\big(1+\big|X_k^{i,N,n}\big|^{2\rho+1}\\
&\quad+\mathcal{W}_2\big(\mu_{t_k}^{X,N,n}, \delta_0\big)\big)^{2}\Big]\\
&+Kh\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-2}\big(1+\big|X_k^{i,N,n}\big|^{2}\\
&\quad+\mathcal{W}_2^2\big(\mu_{t_k}^{X,N,n}, \delta_0\big)\big)\Big].
\end{aligned}
$$

Note that, by Lemma 2.3 of dos Reis, Salkeld, and Tugaut (2019),

$$
\mathcal{W}_2^2\big(\mu_{t_k}^{X,N,n}, \delta_0\big) = \frac{1}{N}\sum_{i=1}^{N}\big|X_k^{i,N,n}\big|^2, \tag{A.2}
$$

and some simplification, the estimation for I_1 is given as follows

$$
\begin{aligned}
I_1 \le\ & Kh + Kh\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}}\Big]\\
&+Kh^{1+r_1}\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-1+r_2(2\rho+1)}\Big]\\
&+Kh^2\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}+4\rho}\Big].
\end{aligned}
$$

Next, we focus on the estimation of I_2. By the assumption (*H1*), one can get

$$
\begin{aligned}
I_2 \le\ & K\sum_{l=3}^{2\bar{p}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-l}\\
&\Big(\big|\mathcal{T}_1\big(b\big(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\big), h\big)\big|^l h^l\\
&\quad+\big|\sum_{r=1}^{m}\mathcal{T}_2\big(\sigma_r\big(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\big), h\big)\Delta W_r^i(t_k)\big|^l\Big)\Big]\\
\le\ & K\sum_{l=3}^{2\bar{p}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-l}\Big(\big|b\big(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\big)\big|^l h^l\\
&\quad+h^{\frac{l}{2}}\sum_{r=1}^{m}\big|\sigma_r\big(t_k, X_k^{i,N,n}, \mu_{t_k}^{X,N,n}\big)\big|^l\Big)\Big].
\end{aligned}
$$

Then, the growth condition for the coefficients in (10) and (11) implies

$$
\begin{aligned}
I_2 \le\ & K\sum_{l=3}^{2\bar{p}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-l}\Big(1+\big|X_k^{i,N,n}\big|^{2\rho+1}\\
&\quad+\mathcal{W}_2\big(\mu_{t_k}^{X,N,n}, \delta_0\big)\Big)^l h^l\Big]\\
&+K\sum_{l=3}^{2\bar{p}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-l}\Big(1+\big|X_k^{i,N,n}\big|^{\rho+1}\\
&\quad+\mathcal{W}_2\big(\mu_{t_k}^{X,N,n}, \delta_0\big)\Big)^l h^{\frac{l}{2}}\Big]\\
\le\ & K\sum_{l=3}^{2\bar{p}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-l}\Big(h^l+h^{\frac{l}{2}}\\
&\quad+h^l\big|X_k^{i,N,n}\big|^{(2\rho+1)l}+h^l\mathcal{W}_2^l\big(\mu_{t_k}^{X,N,n}, \delta_0\big)\\
&\quad+h^{\frac{l}{2}}\big|X_k^{i,N,n}\big|^{(\rho+1)l}+h^{\frac{l}{2}}\mathcal{W}_2^l\big(\mu_{t_k}^{X,N,n}, \delta_0\big)\Big)\Big].
\end{aligned}
$$

It follows from (A.2) that

$$
\begin{aligned}
I_2 \le\ & Kh^{\frac{3}{2}}\\
&+K\sum_{l=3}^{2\bar{p}}\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}h^l\big|X_k^{i,N,n}\big|^{2\rho l+2\bar{p}}\Big]\\
&+K\sum_{l=3}^{2\bar{p}}\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}h^{\frac{l}{2}}\big|X_k^{i,N,n}\big|^{\rho l+2\bar{p}}\Big].
\end{aligned}
$$

Combining the above results, we have

$$
\begin{aligned}
&\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_{k+1}^{i,N,n}\big|^{2\bar{p}}\Big]\\
&\le Kh + Kh^2\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}+4\rho}\Big]\\
&\quad+Kh^{1+r_1}\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}-1+r_2(2\rho+1)}\Big]\\
&\quad+(1+Kh)\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}}\Big]\\
&\quad+K\sum_{l=3}^{2\bar{p}}\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}h^l\big|X_k^{i,N,n}\big|^{2\rho l+2\bar{p}}\Big]\\
&\quad+K\sum_{l=3}^{2\bar{p}}\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}h^{\frac{l}{2}}\big|X_k^{i,N,n}\big|^{\rho l+2\bar{p}}\Big].
\end{aligned}
$$

Choosing $\mathcal{R}=\mathcal{R}(h)=h^{-1/\mathcal{G}(\rho,r_1,r_2)}$ with $\mathcal{G}:=\mathcal{G}(\rho,r_1,r_2)$ given as (13), for all $l=3,4,\dots,2\bar{p}$, we have the following inequality

$$
\begin{aligned}
&\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{(2\rho+1)r_2+2\bar{p}-1}h^{r_1}\\
&=\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}}\Big(\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{\frac{(2\rho+1)r_2-1}{r_1}}h\Big)^{r_1}\\
&\le K\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}},
\end{aligned}
$$

where K is independent of h. Similarly, we derive that

$$
\begin{aligned}
&\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{4\rho+2\bar{p}}h \le K\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}},\\
&\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\rho l+2\bar{p}}h^{l-1} \le K\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}},\\
&\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{\rho l+2\bar{p}}h^{\frac{l}{2}-1} \le K\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_{k+1}^{i,N,n}\big|^{2\bar{p}}\Big] \le Kh\\
&\quad+(1+Kh)\sup_{i\in\{1,2,\dots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X_k^{i,N,n}\big|^{2\bar{p}}\Big],
\end{aligned}
\tag{A.3}
$$

which implies that

$$\sup_{i\in\{1,2,\ldots,N\}} \mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k+1}}\big|X^{i,N,n}_{k+1}\big|^{2\bar{p}}\Big] \le Kh + (1+Kh)\sup_{i\in\{1,2,\ldots,N\}} \mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2\bar{p}}\Big].$$

Therefore, by induction or Grönwall inequality in discrete time case, we have

$$\begin{aligned}&\sup_{i\in\{1,2,\ldots,N\}} \mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2\bar{p}}\Big]\\ &\le (1+Kh)^k\big(1+\mathbb{E}[|X_0|^{2\bar{p}}]\big) \\ &\le e^{Khk}\big(1+\mathbb{E}[|X_0|^{2\bar{p}}]\big) \le K\big(1+\mathbb{E}[|X_0|^{2\bar{p}}]\big),\end{aligned} \tag{A.4}$$

where K is a generic constant that is independent of h and n. It remains to estimate $\mathbb{E}[\mathbb{1}_{\Omega^c_{\mathcal{R},k}}|X^{i,N,n}_k|^{2p}]$. It follows from (4) and (*H1*) that

$$\begin{aligned}&\big|X^{i,N,n}_{k+1}\big|\\ &\le \big|X^{i,N,n}_{k}\big| + Lh^{-1} + \sum_{r=1}^{m} Lh^{-\frac{3}{2}}\big|W^i_r(t_{k+1})-W^i_r(t_k)\big|\\ &\le |X^i_0| + L(k+1)h^{-1}\\ &\quad + \sum_{j=0}^{k}\sum_{r=1}^{m} Lh^{-\frac{3}{2}}\big|W^i_r(t_{j+1})-W^i_r(t_j)\big|\end{aligned} \tag{A.5}$$

by induction. Note that

$$\begin{aligned}\mathbb{1}_{\Omega^c_{\mathcal{R},k}} &= 1-\mathbb{1}_{\Omega_{\mathcal{R},k}} = 1-\mathbb{1}_{\Omega_{\mathcal{R},k-1}}\mathbb{1}_{|X^{i,N,n}_k|\le\mathcal{R}}\\ &= \sum_{j=0}^{k}\mathbb{1}_{\Omega_{\mathcal{R},j-1}}\mathbb{1}_{|X^{i,N,n}_j|>\mathcal{R}},\end{aligned} \tag{A.6}$$

where we set $\mathbb{1}_{\Omega_{\mathcal{R},-1}}=1$. Then, applying (A.6), Hölder's inequality with $\frac{1}{p_1}+\frac{1}{q_1}=1$ for $q_1=\frac{2\bar{p}}{(4p+1)\mathcal{G}}>1$ due to $p\le\frac{2\bar{p}-\mathcal{G}}{2+4\mathcal{G}}$, and the Markov inequality, we derive that

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{\Omega^c_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2p}\Big]\\ &= \sum_{j=0}^{k}\mathbb{E}\Big[\big|X^{i,N,n}_{k}\big|^{2p}\mathbb{1}_{\Omega_{\mathcal{R},j-1}}\mathbb{1}_{|X^{i,N,n}_j|>\mathcal{R}}\Big]\\ &\le \sum_{j=0}^{k}\Big(\mathbb{E}\Big[\big|X^{i,N,n}_{k}\big|^{2pp_1}\Big]\Big)^{\frac{1}{p_1}}\Big(\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},j-1}}\mathbb{1}_{|X^{i,N,n}_j|>\mathcal{R}}\Big]\Big)^{\frac{1}{q_1}}\\ &= \Big(\mathbb{E}\Big[|X^{i,N,n}_{k}|^{2pp_1}\Big]\Big)^{\frac{1}{p_1}}\sum_{j=0}^{k}\Big(\mathbb{P}\Big(\mathbb{1}_{\Omega_{\mathcal{R},j-1}}|X^{i,N,n}_j|>\mathcal{R}\Big)\Big)^{\frac{1}{q_1}}\\ &\le \Big(\mathbb{E}\Big[|X^{i,N,n}_{k}|^{2pp_1}\Big]\Big)^{\frac{1}{p_1}}\sum_{j=0}^{k}\frac{\Big(\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},j-1}}|X^{i,N,n}_j|^{2\bar{p}}\Big]\Big)^{\frac{1}{q_1}}}{\mathcal{R}^{\frac{2\bar{p}}{q_1}}}.\end{aligned}$$

Note that, $p\le\frac{2\bar{p}-\mathcal{G}}{2+4\mathcal{G}}$ implies that $\bar{p}\ge pp_1$. Then, Hölder's inequality and (A.5) give

$$\begin{aligned}&\Big(\mathbb{E}\Big[|X^{i,N,n}_{k}|^{2pp_1}\Big]\Big)^{\frac{1}{p_1}} \le \Big(\mathbb{E}\Big[|X^{i,N,n}_{k}|^{2\bar{p}}\Big]\Big)^{\frac{p}{\bar{p}}}\\ &\le K\Big(\mathbb{E}\Big[|X^i_0|^{2\bar{p}}\Big]+(kh^{-1})^{2\bar{p}}\\ &\qquad +h^{-3\bar{p}}\mathbb{E}\Big[\Big(\sum_{j=0}^{k-1}\sum_{r=1}^{m}\big|W^i_r(t_{j+1})-W^i_r(t_j)\big|\Big)^{2\bar{p}}\Big]\Big)^{\frac{p}{\bar{p}}}\\ &\le Kh^{-4p}+K\Big(1+\mathbb{E}\Big[|X_0|^{2\bar{p}}\Big]\Big)^{\frac{p}{\bar{p}}}.\end{aligned}$$

According to the inequalities (A.3) and (A.4), we have

$$\begin{aligned}&\sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X^{i,N,n}_{k+1}\big|^{2\bar{p}}\Big]\\ &\le Kh+(1+Kh)K\Big(1+\mathbb{E}\Big[|X_0|^{2\bar{p}}\Big]\Big)\\ &\le K\Big(1+\mathbb{E}\Big[|X_0|^{2\bar{p}}\Big]\Big).\end{aligned}$$

Recall that $\mathcal{R}=h^{-\frac{1}{\mathcal{G}}}$, then $\mathcal{R}^{\frac{2\bar{p}}{q_1}}=h^{-\frac{2\bar{p}}{q_1\mathcal{G}}}$. Note that the inequality $\frac{2\bar{p}}{q_1\mathcal{G}}\ge 4p+1$ holds as $q_1=\frac{2\bar{p}}{(4p+1)\mathcal{G}}$. Thus,

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{\Omega^c_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2p}\Big]\\ &\le \Big(Kh^{-4p}+K\Big(1+\mathbb{E}\Big[|X_0|^{2\bar{p}}\Big]\Big)^{\frac{p}{\bar{p}}}\Big)\\ &\qquad (1+k)h^{\frac{2\bar{p}}{q_1\mathcal{G}}}K\Big(1+\mathbb{E}\Big[|X_0|^{2\bar{p}}\Big]\Big)^{\frac{1}{q_1}}\\ &\le K\Big(1+\mathbb{E}\Big[|X_0|^{2\bar{p}}\Big]\Big)^{\frac{1}{q_1}+\frac{p}{\bar{p}}}.\end{aligned} \tag{A.7}$$

Therefore, by using of Hölder's inequality, (A.4) and (A.7), we obtain the desired result as follows

$$\begin{aligned}&\sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\big|X^{i,N,n}_{k}\big|^{2p}\Big]\\ &= \sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2p}\Big]\\ &\qquad + \sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega^c_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2p}\Big]\\ &\le \Big(\sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2\bar{p}}\Big]\Big)^{\frac{p}{\bar{p}}}\\ &\qquad + \sup_{i\in\{1,2,\ldots,N\}}\mathbb{E}\Big[\mathbb{1}_{\Omega^c_{\mathcal{R},k}}\big|X^{i,N,n}_{k}\big|^{2p}\Big]\\ &\le K\left(1+\mathbb{E}\left[|X_0|^{2\bar{p}}\right]\right)^{\frac{p}{\bar{p}}}+K\left(1+\mathbb{E}\left[|X_0|^{2\bar{p}}\right]\right)^{\frac{1}{q_1}+\frac{p}{\bar{p}}}\\ &\le K\Big(1+\mathbb{E}\Big[|X_0|^{2\beta p}\Big]\Big),\end{aligned}$$

where $\beta=1+\frac{\bar{p}}{pq_1}>1$. $\square$

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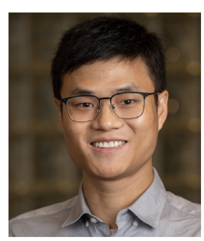
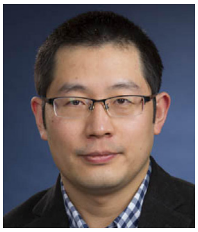


Jiamin Jian is a Postdoctoral Assistant Professor in the Department of Mathematics at the University of Michigan. His research focuses on stochastic control, mean field games, and financial mathematics. He earned his Ph.D. in Mathematical Sciences from Worcester Polytechnic Institute in May 2024. Prior to that, he obtained a master's degree in Mathematical Finance and Statistics from City University of Hong Kong (July 2019) and a bachelor's degree in Mathematics and Applied Mathematics from Nankai University (June 2018).

Qingshuo Song is a professor in the Department of Mathematical Sciences at WPI. His research interests include stochastic control theory and its applications to various engineering problems. He earned his B.Sc. from Nankai University and his MA and Ph.D. from Wayne State University in 2006. Before joining WPI in 2019, he held positions as an Associate Professor at City University of Hong Kong (2010–2018) and as a Postdoctoral Researcher at the University of Michigan (2009) and the University of Southern California (2006–2009).

Xiaojie Wang is a professor in the School of Mathematics and Statistic at Central South University (CSU). His research interests include numerical methods for stochastic (partial) differential equations, stochastic algorithms and their applications to sampling in generative AI and optimization. He earned his B.Sc. in 2007 and his Ph.D. in 2012, both from CSU.



Zhongqiang Zhang is an Associate Professor of Mathematics at Worcester Polytechnic Institute. His research interests include numerical methods for stochastic and integral differential equations, computational probability, and mathematics for machine learning. Before he joined Worcester Polytechnic Institute in 2014, he received Ph.D. degrees in mathematics at Shanghai University in 2011 and applied mathematics at Brown University in 2014. He has published a book titled Numerical Methods for Stochastic Partial Differential Equations with White Noise.

Yuying Zhao is a Postdoctoral Researcher in the School of Mathematics and Statistics at Central South University. Her research interests include stochastic differential equations, numerical analysis, and the mathematical foundations of generative modeling. She earned her Ph.D. in Mathematical Sciences from Central South University in May 2024 and was a visiting scholar at Worcester Polytechnic Institute from 2021 to 2023. Previously, she obtained her B.Sc. from Henan Normal University and her M.A. from Central South University in 2020.