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An Introduction to MERA

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www.heptnseminars.org

(online seminar series on tensor networks)

MPS	PEPS	MERA
Natural ansatz for 1d gapped systems	Higher dimensional extension of MPS	Natural ansatz for 1d critical systems
In practice, can be applied to 1d critical systems and also to 2d systems	Applied to both gapped and critical systems in 2d	Possible to extend to higher dimensions
Somewhat RG based	Not RG based	RG based

THIS TALK: MERA for 1d systems

Outline

PART 1: What is the MERA?

(1) Efficient representations of quantum many-body states from coarse-graining the lattice
(2) MERA as a numerical ansatz for 1d ground states

PART 2: MERA and critical systems

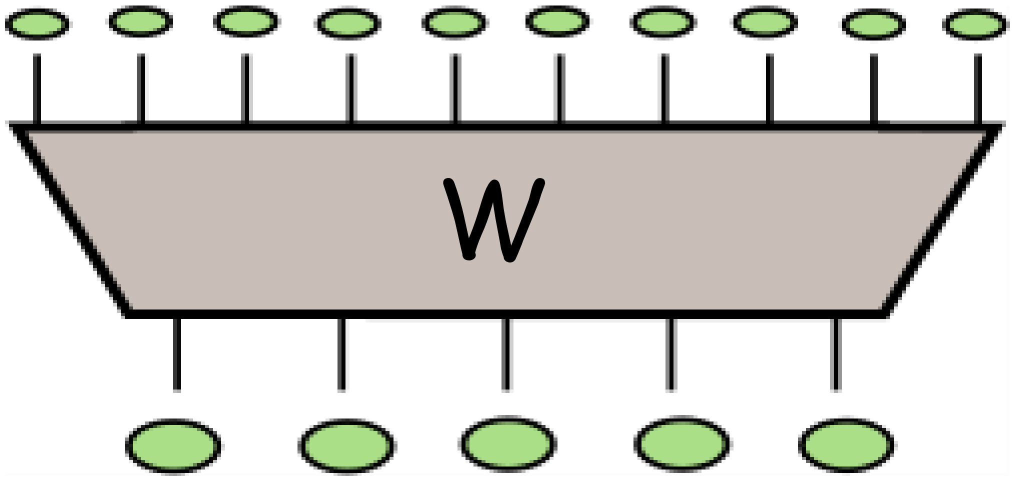
(3) Entanglement in the MERA
(4) infinite MERA as a lattice version of 2d CFTs
(5) MERA and holography

Efficient representations of many-body states

from coarse-graining the lattice

Definition of a coarse-graining or RG transformation

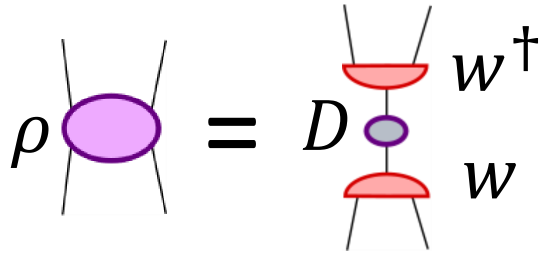
A linear operator (isometry) W that maps states/operators on a lattice to states/operators on a coarse-grained lattice



RG flow of the Hamiltonian: $H \to H_1 \to H_2 \to H_3 \to \ \dots \to H_F \to H_F \to \ \dots$

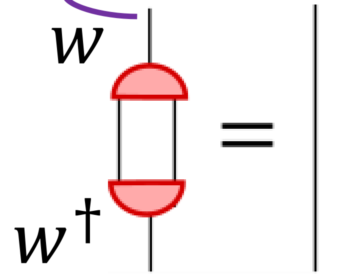
1) Must preserve the low energy subspace
2) Must reach a fixed-point (which captures universal low-energy properties)

The map W will be a tensor network

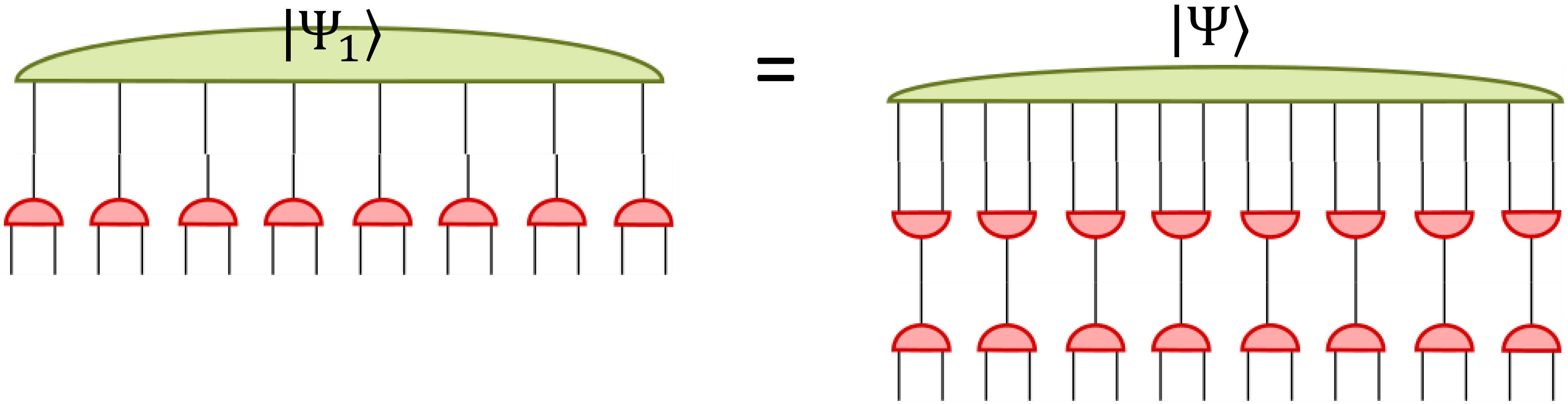


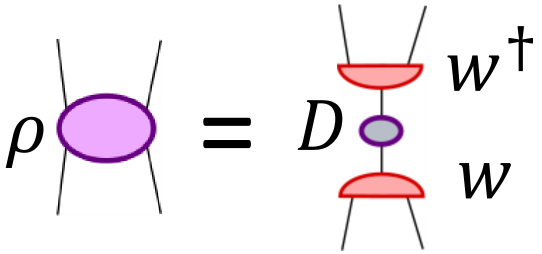
(eigenvalue decomposition)

Size of this index = rank of ρ



(isometry)





(eigenvalue decomposition)

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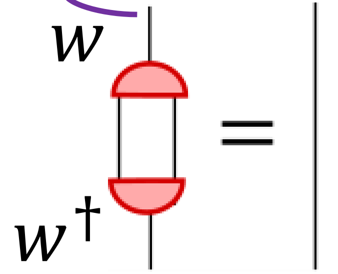
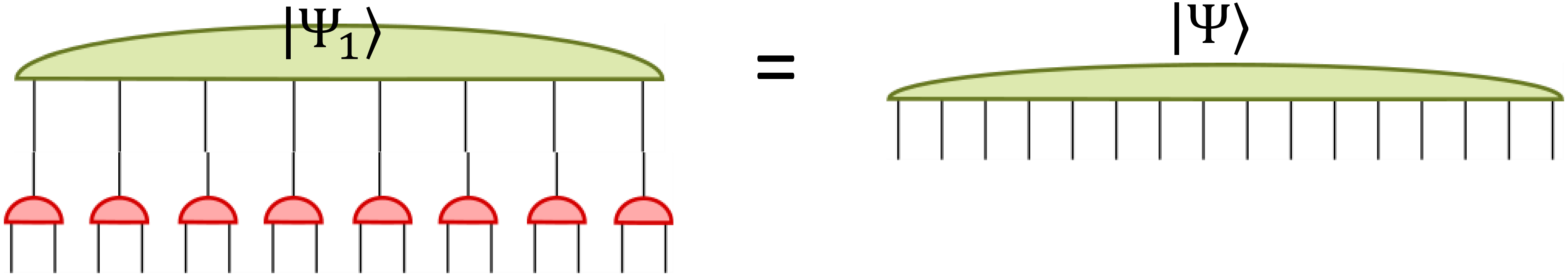
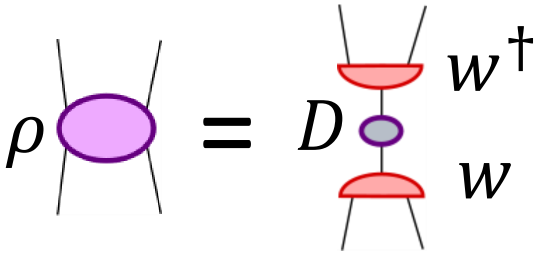


(isometry)



Alternative representation of $|\Psi\rangle$



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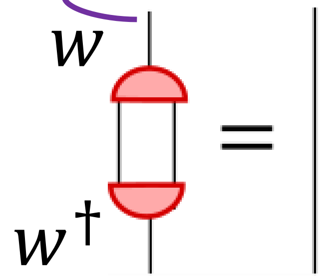
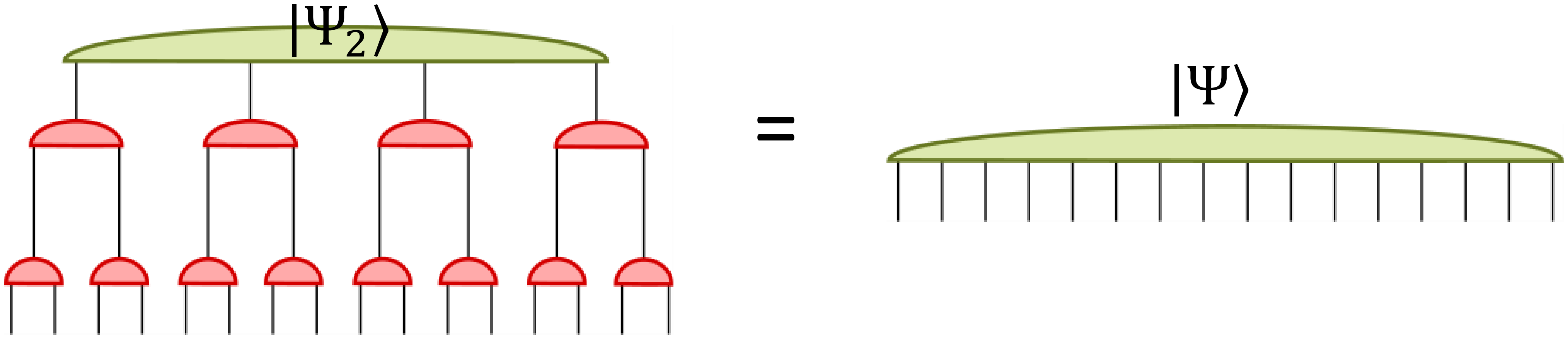


(isometry)



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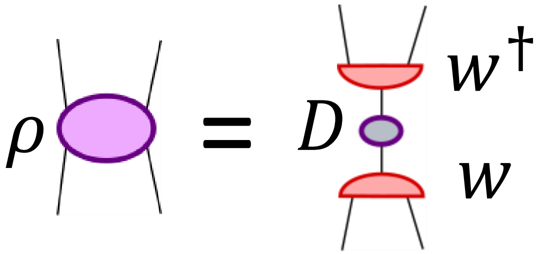
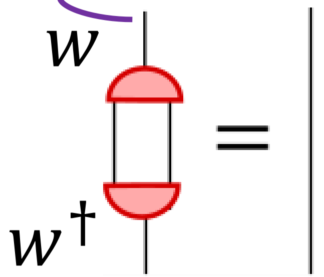


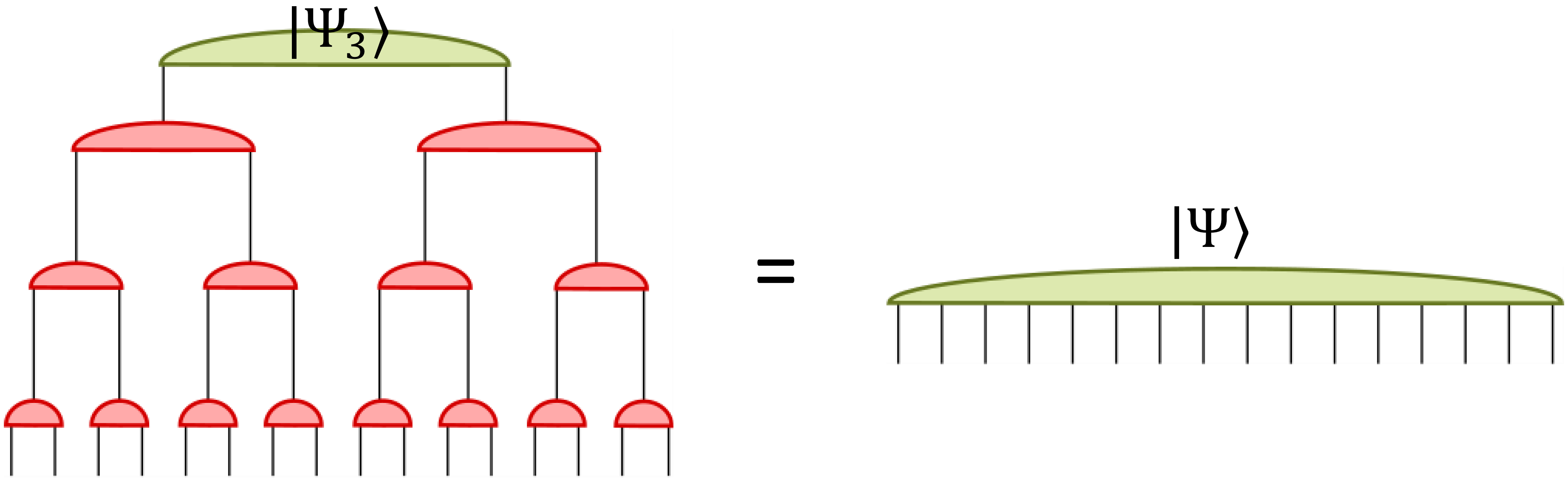
(eigenvalue decomposition)

Size of this index = rank of ρ

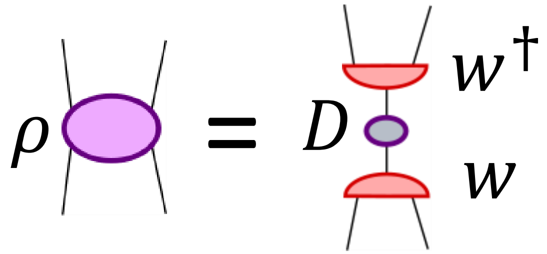


(isometry)

Tree tensor network



Alternative representation of $|\Psi\rangle$



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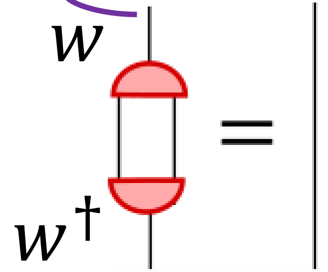
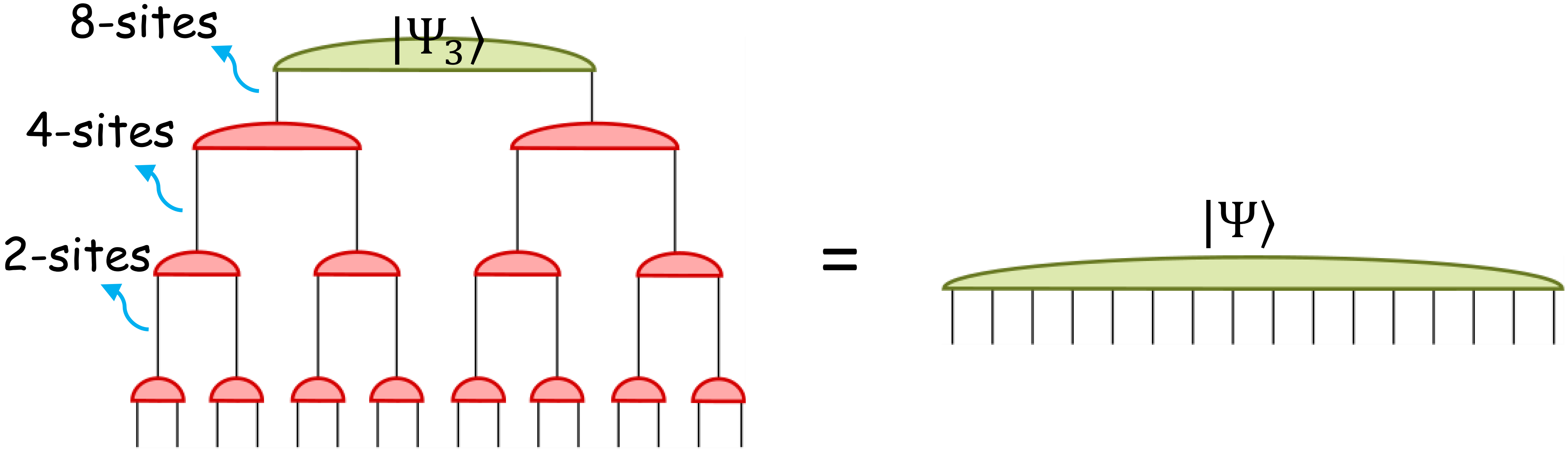


(isometry)

Tree tensor network



Alternative representation of $|\Psi\rangle$

So far we have not assumed anything about the state

The point is only that we have an alternative representation of the state as a tree tensor network

Let's look at gapped states first … We know that

$S(\rho_\ell)$ saturates $\Rightarrow$ $\boxed{\text{Rank } (\rho_\ell) \text{ saturates}}$

A bond index in the tree tensor network corresponds to a block of sites on the original lattice.

This means that the bond dimension saturates (say to χ) after a finite number of coarse-graining steps

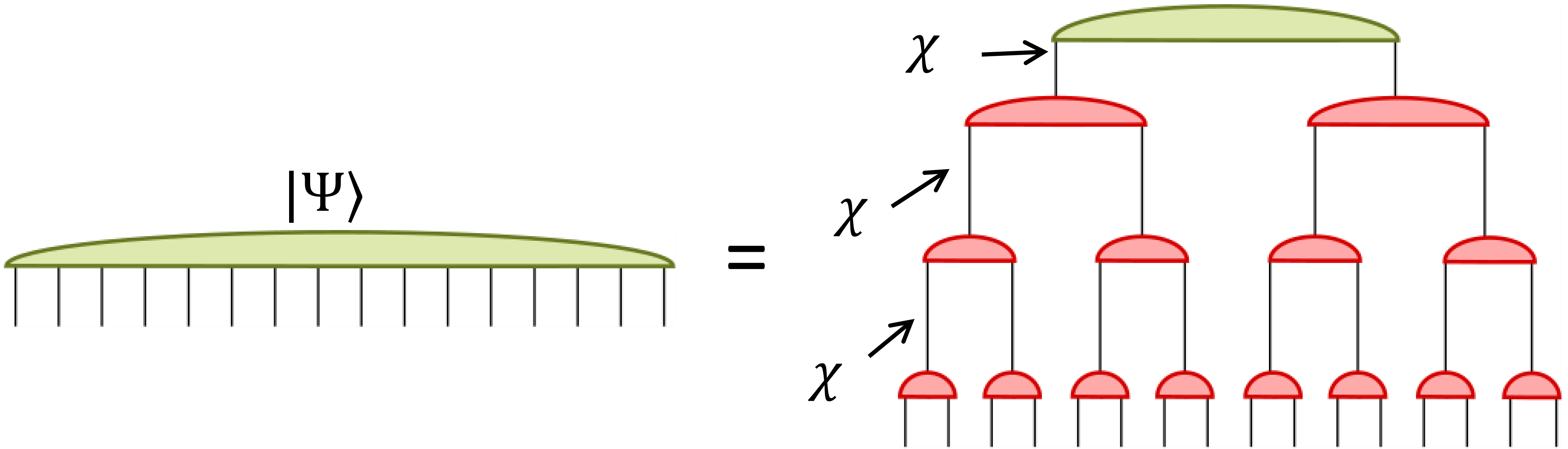
Quiz!

How many complex numbers in the tree representation of a 1d gapped ground state?

(i) $O\left(N^2\right)$

(ii) $O(\log N)$

(iii) $O\left(N\right)$



Quiz!

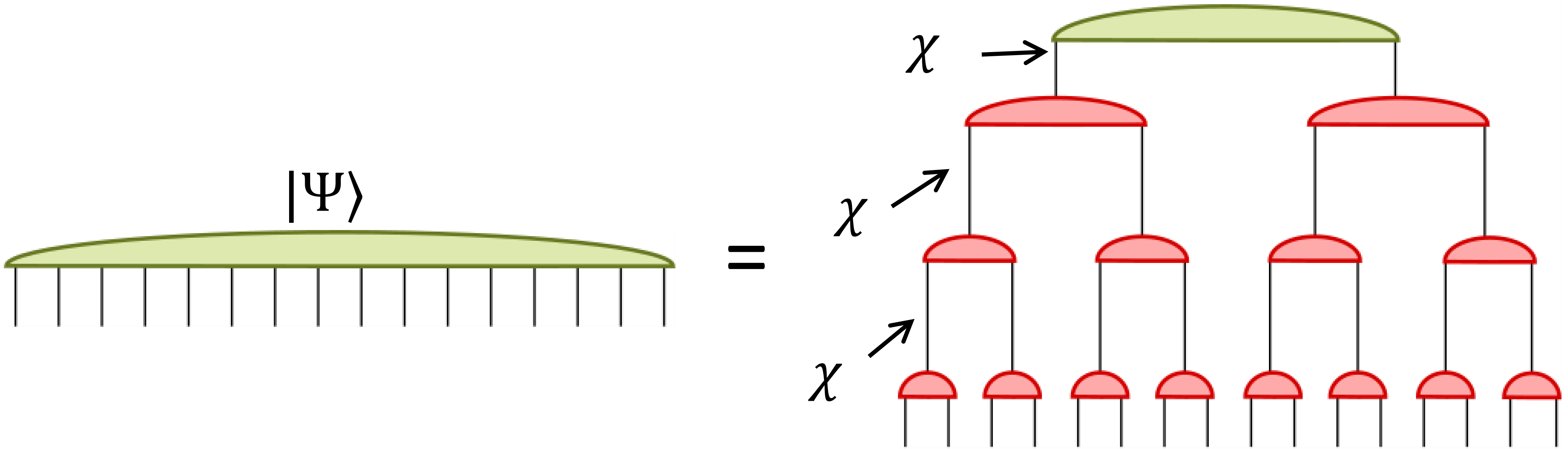
How many complex numbers in the tree representation of a 1d gapped ground state?

(i) $O\,(N^2)$

(ii) $O(\log N)$

(iii) $O\,(N)$

$O\,(N\,\chi^3)$



Tree tensor network can efficiently parameterize 1d gapped ground state

But being an efficient parameterization is not enough.

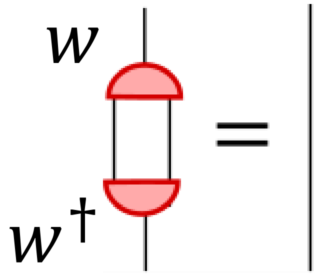
Can we also compute expectation values from the TTN efficiently (polynomial time)?

It turns out : yes.

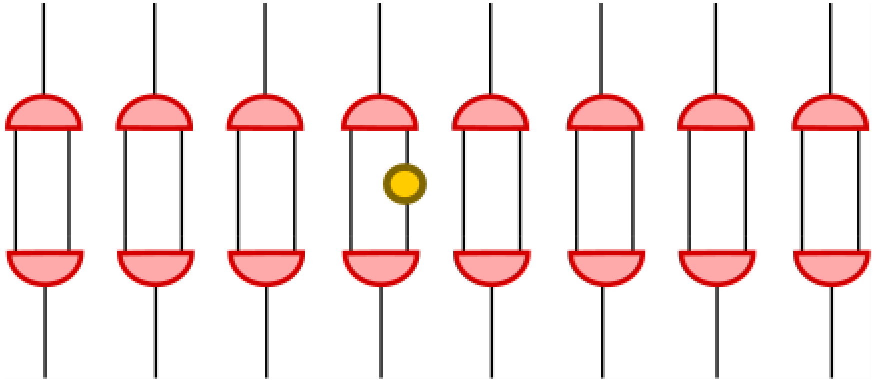
This is thanks to a particular feature of this coarse-graining transformation:

Local operators coarse-grain to local operators

1-site operators

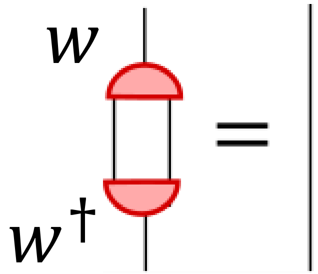


(isometry)



1-site operators

Cost : $O(\chi^4)$



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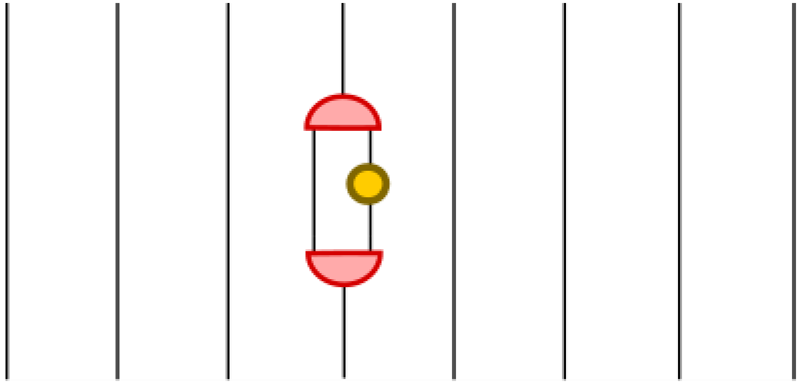
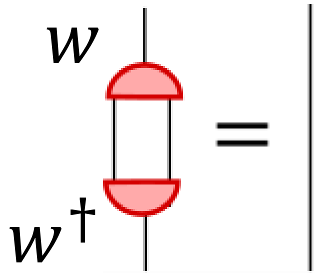


This also means that the cost of coarse-graining 1-site operators is independent of N

2-site operators



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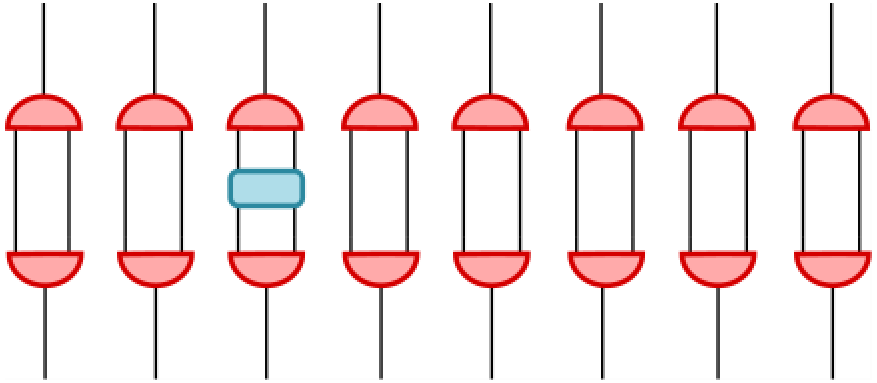
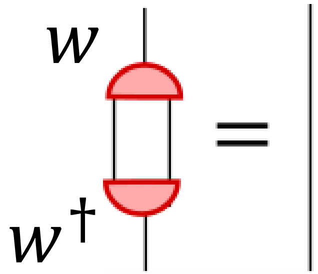
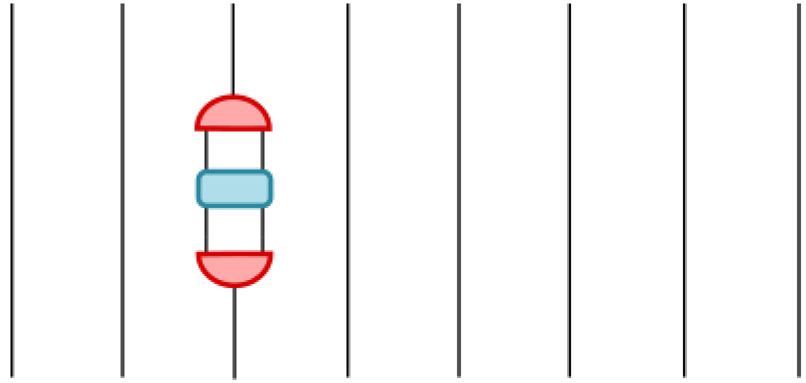


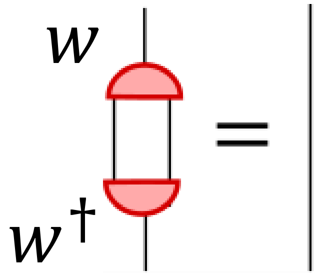
2-site operators



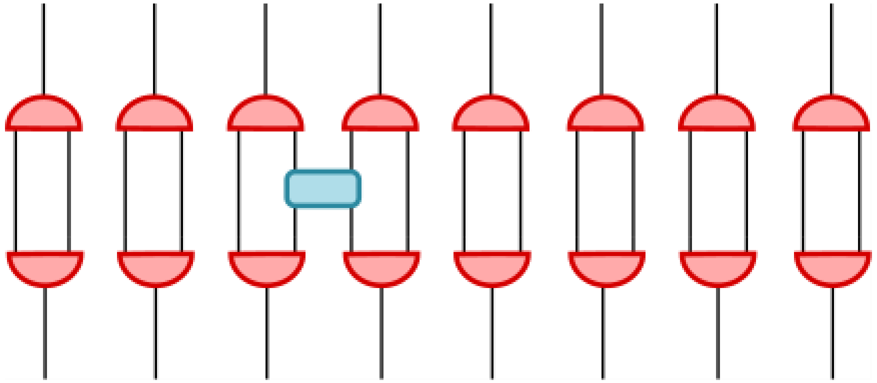
(isometry)



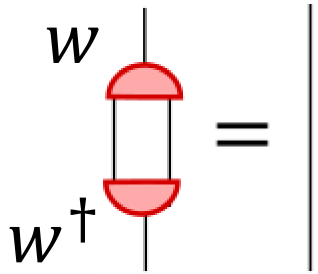
2-site operators



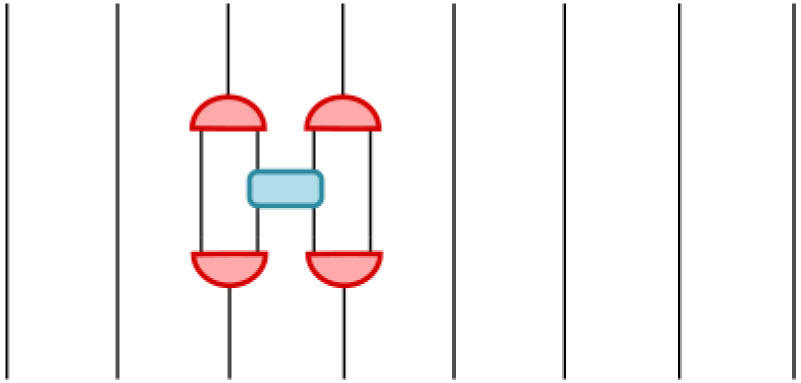
(isometry)



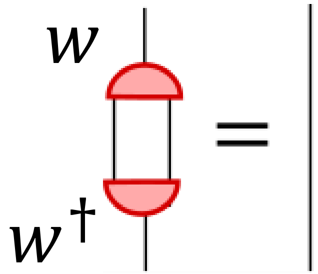
2-site operators



(isometry)

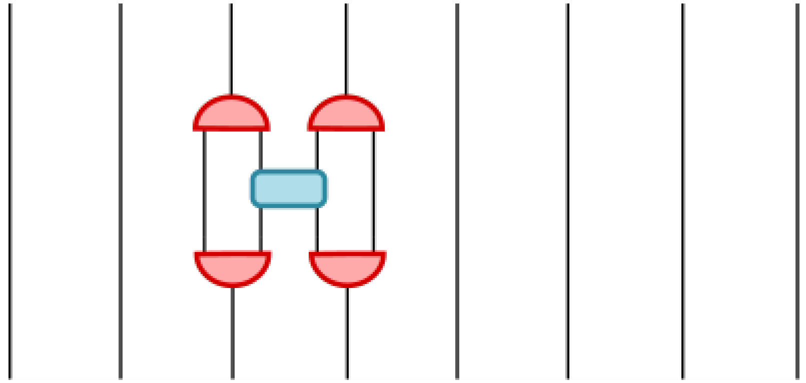


2-site operators



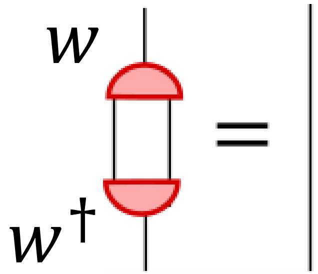
(isometry)

Cost : $O(\chi^6)$

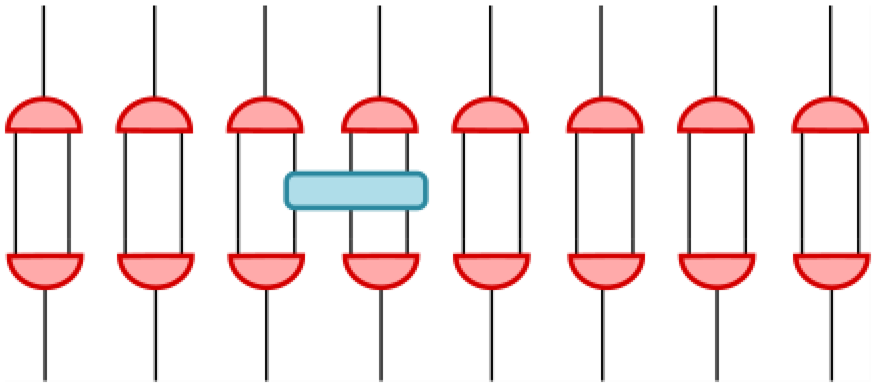


Cost of coarse-graining 2-site operators also independent of N

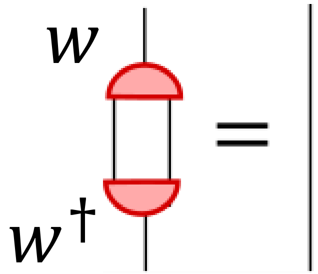
3-site operators



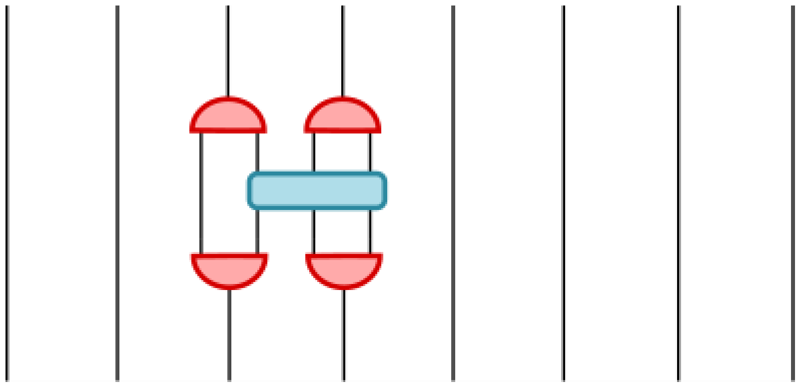
(isometry)



3-site operators

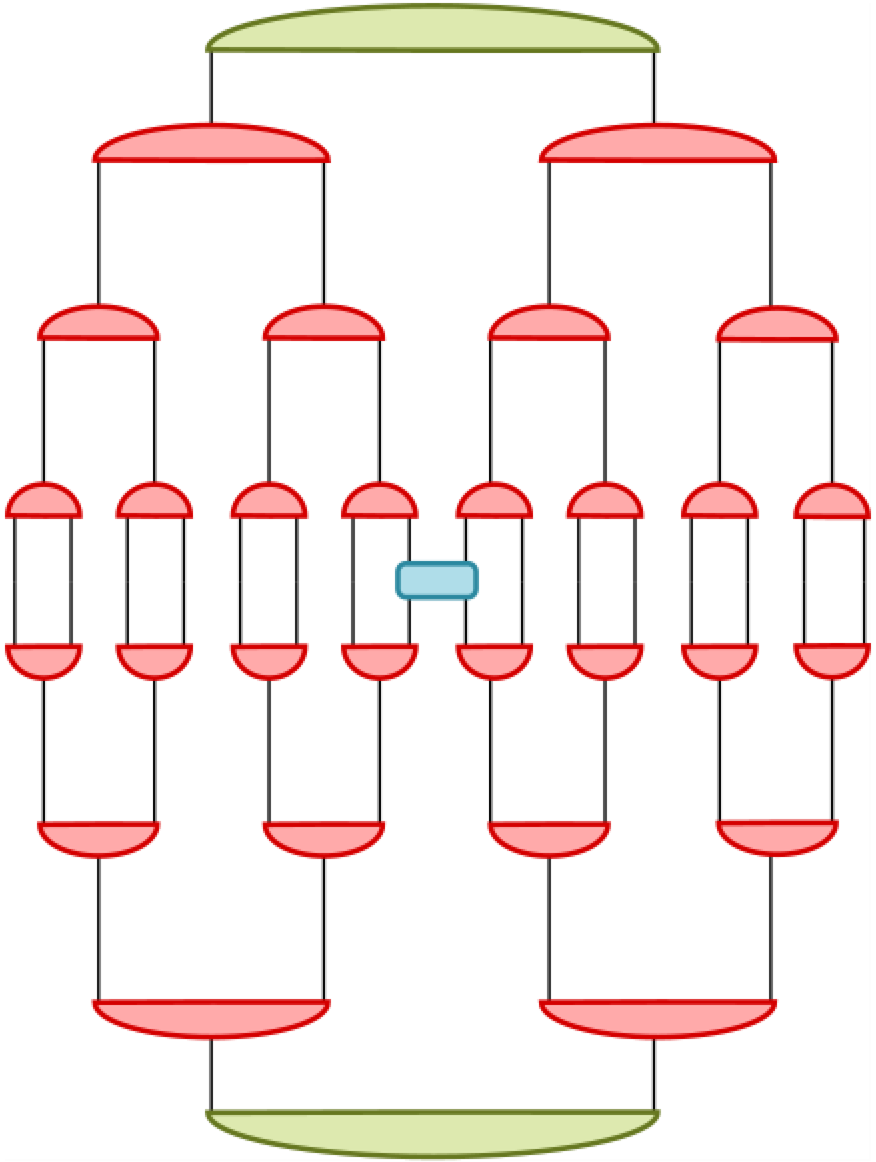


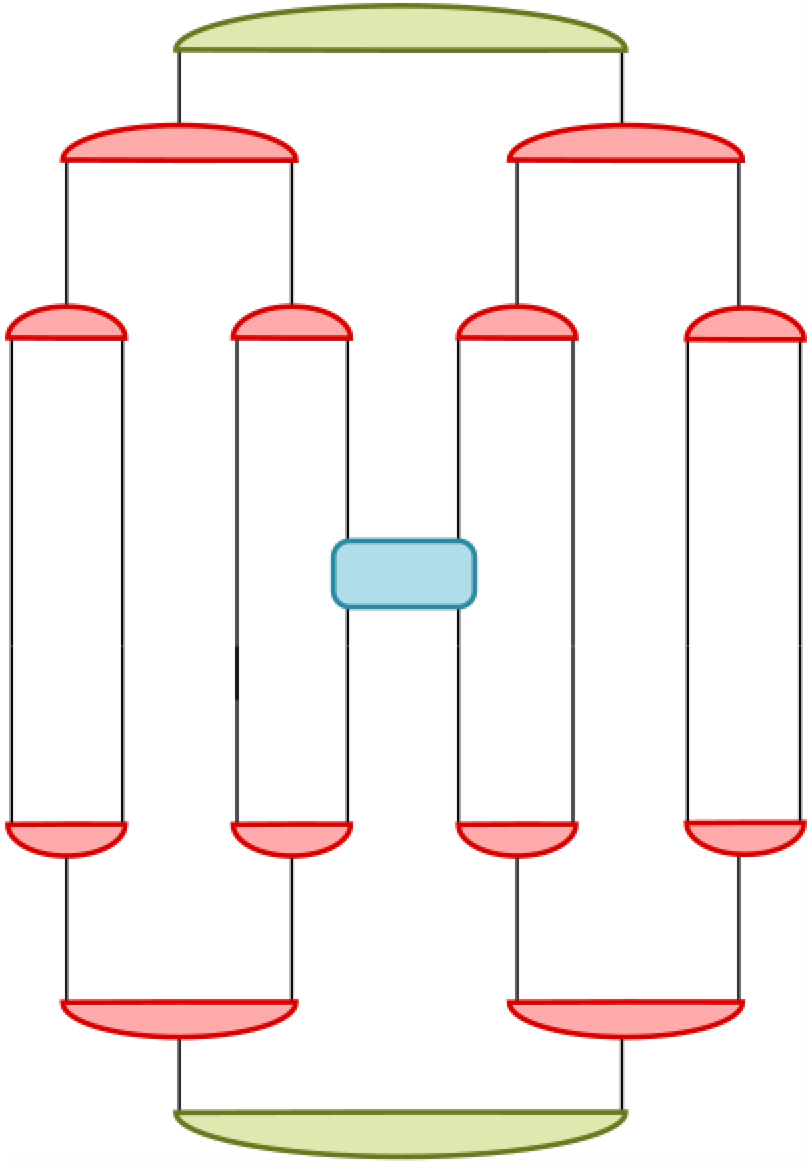
(isometry)

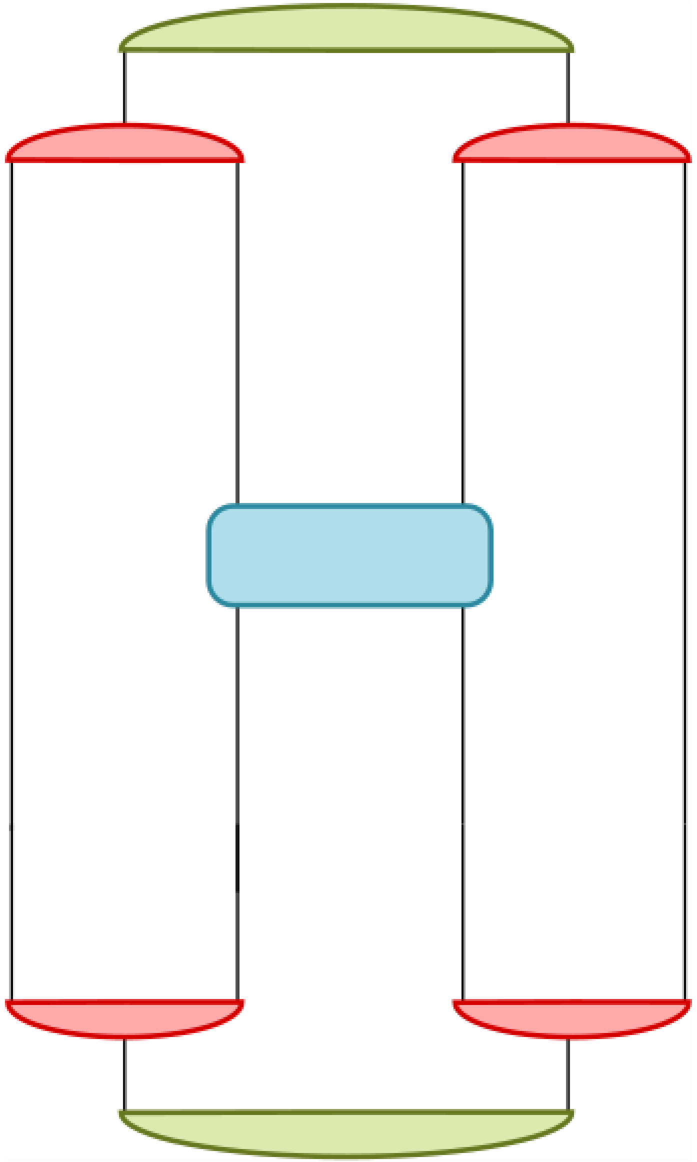


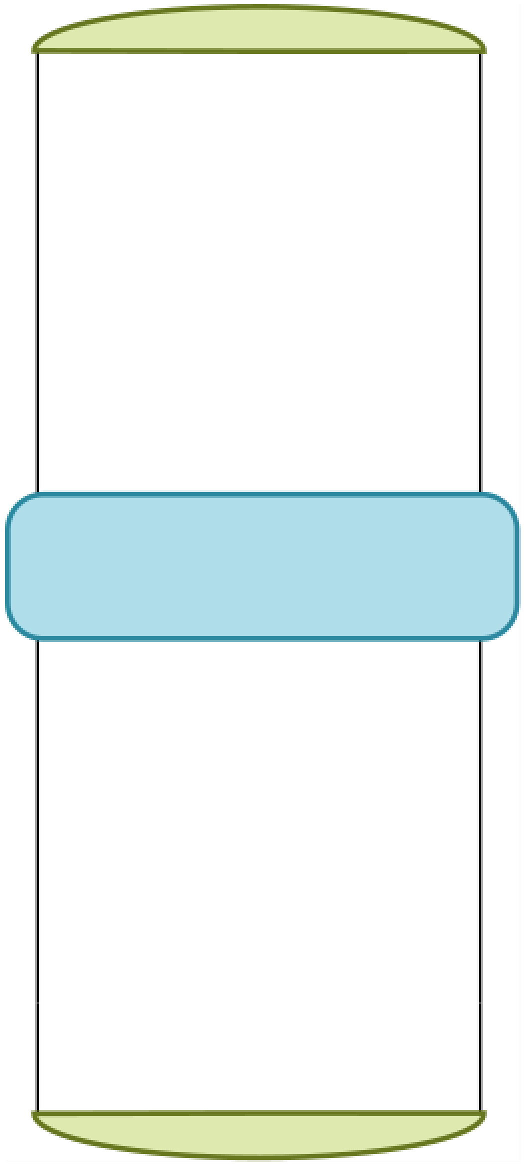
3 or more site operators shrink under coarse-graining (thus, also remain local)

Preservation of locality implies that expectation values of local operators can be computed efficiently

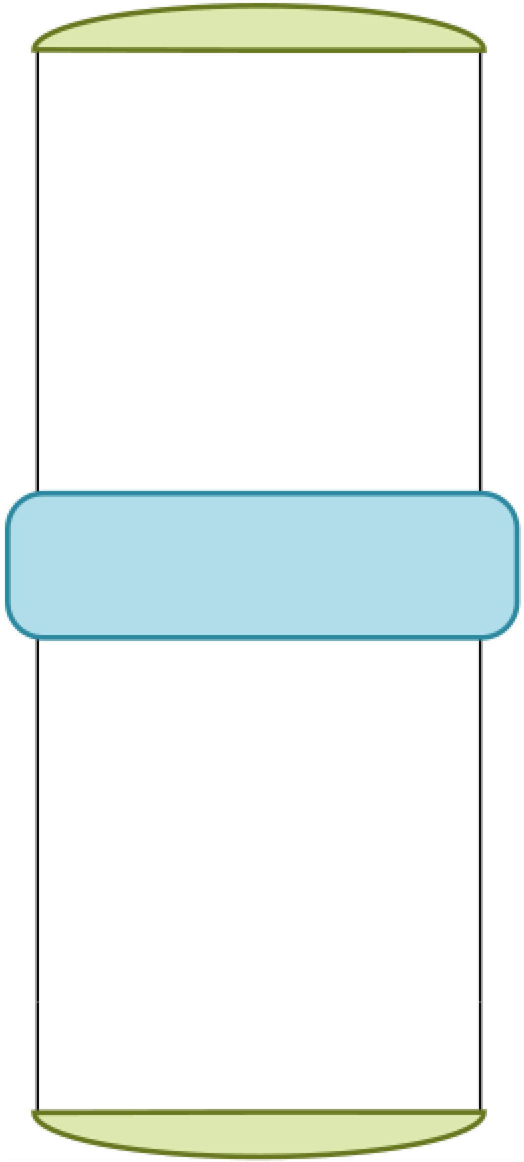








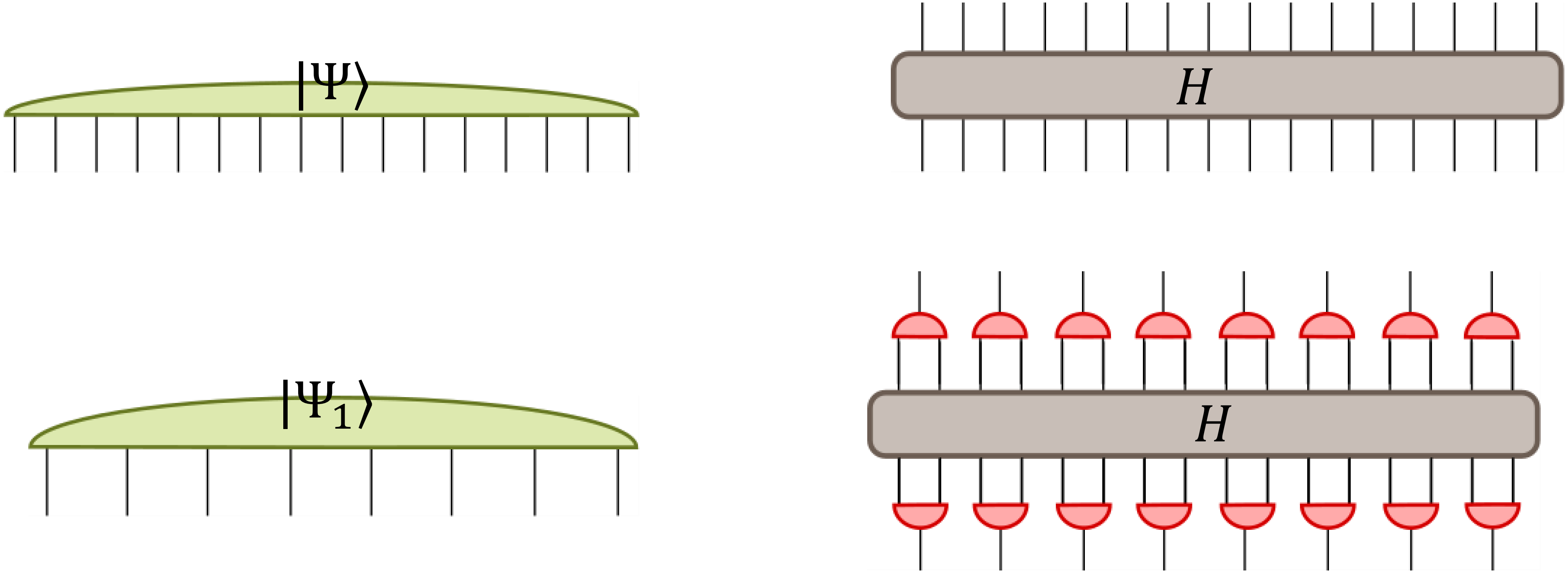
A TTN is an efficient ansatz for 1D gapped ground states.



Cost per step: $O(\chi^6)$

Total number of steps: $O(\log N)$

The tree representation of a gapped ground state also defines an RG flow



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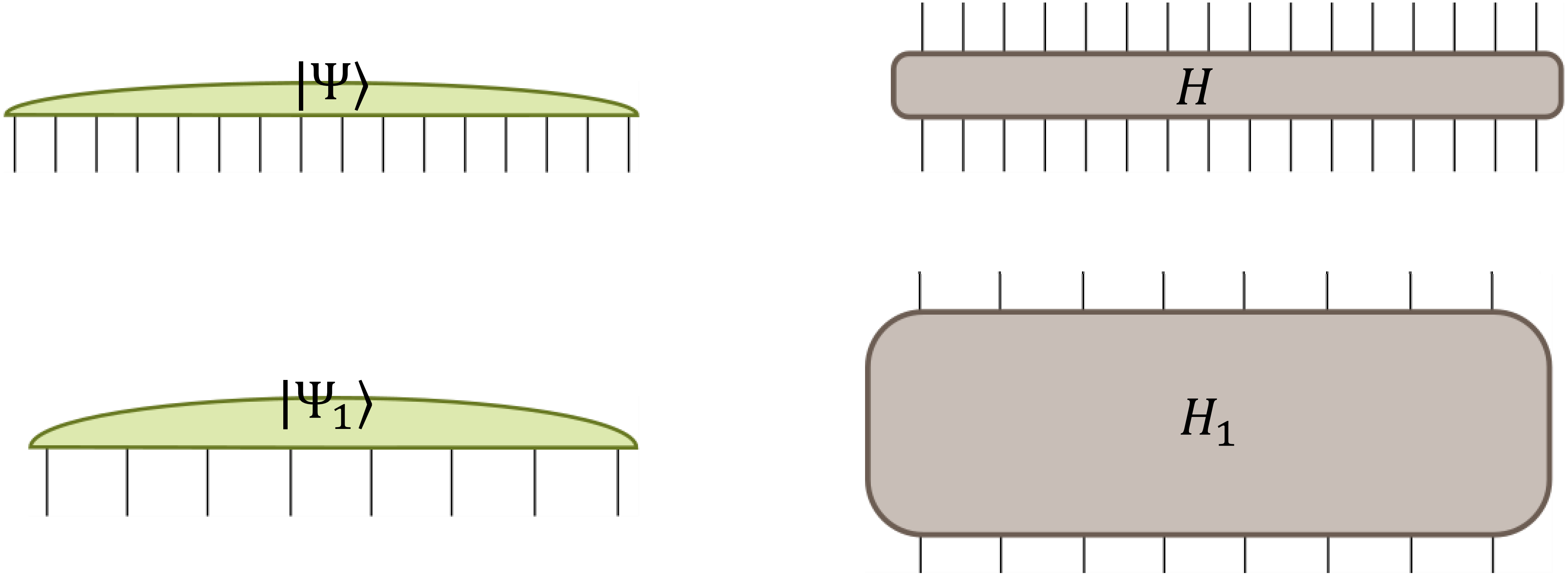

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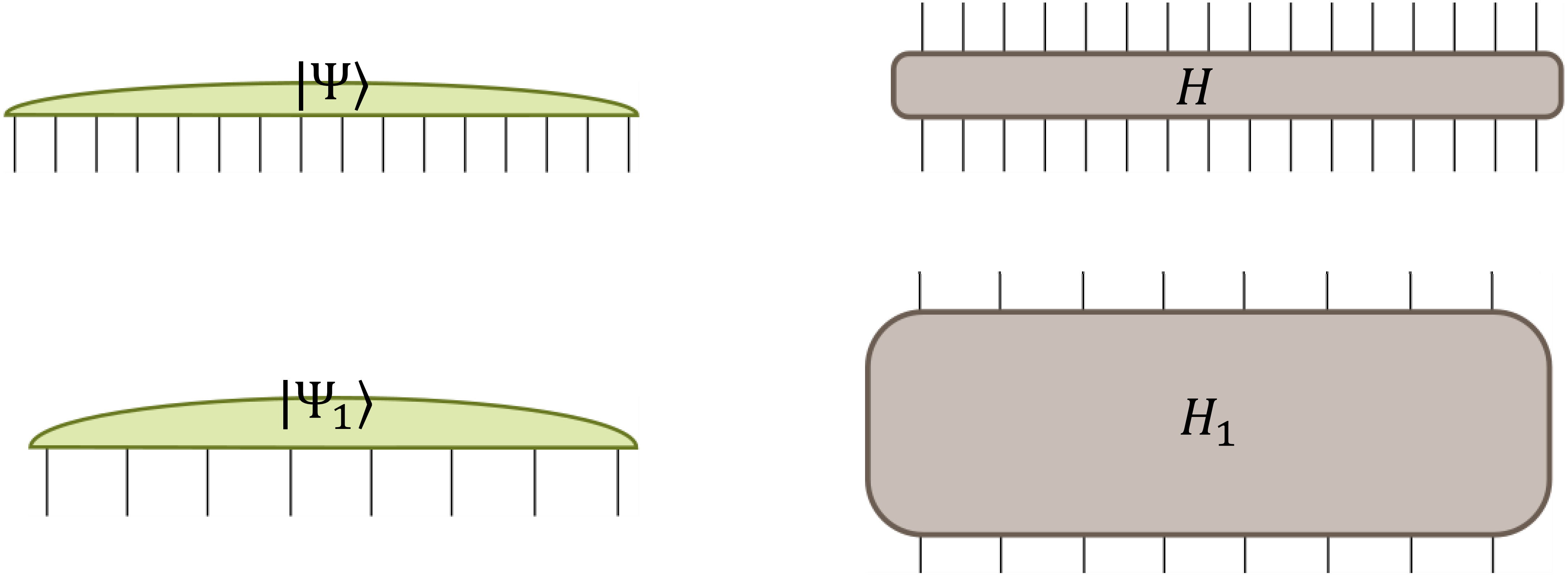
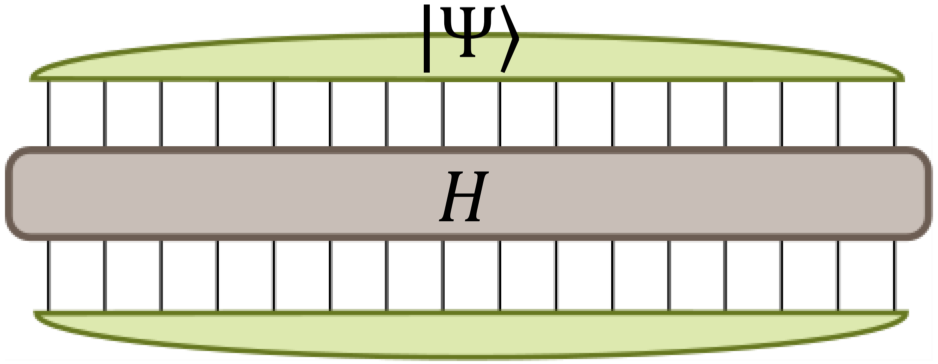


(i) $|\Psi_1\rangle$ is the ground state of H_1

(ii) $|\Psi_1\rangle$ has the same energy as $|\Psi\rangle$

ground state energy of H =



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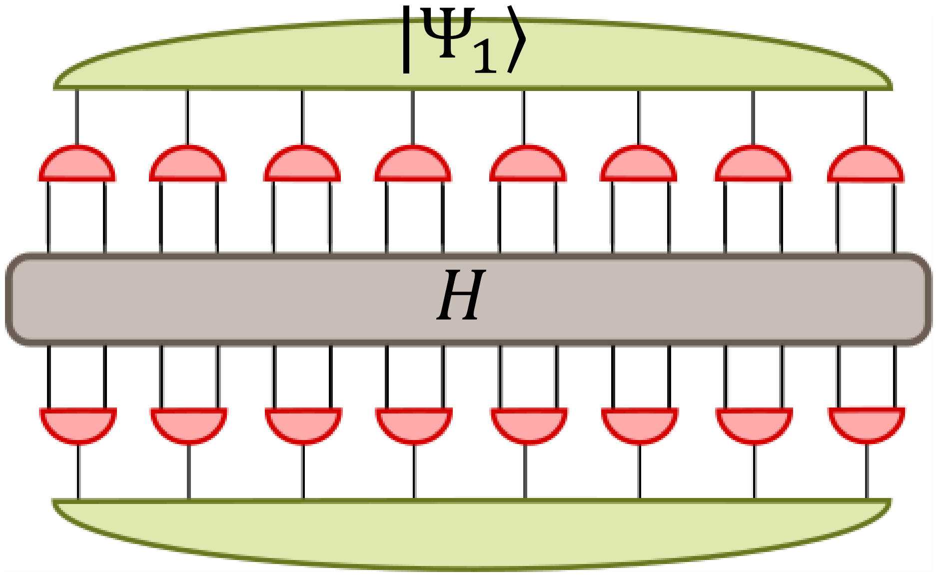
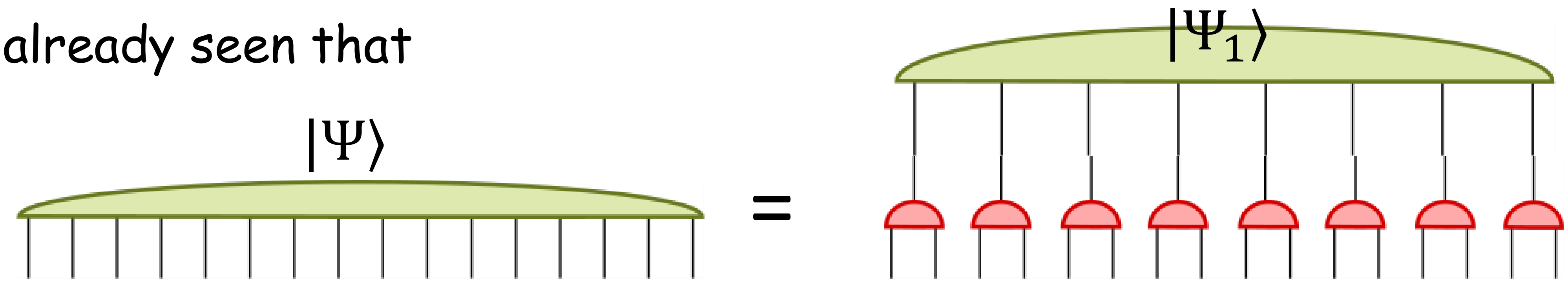


We have already seen that



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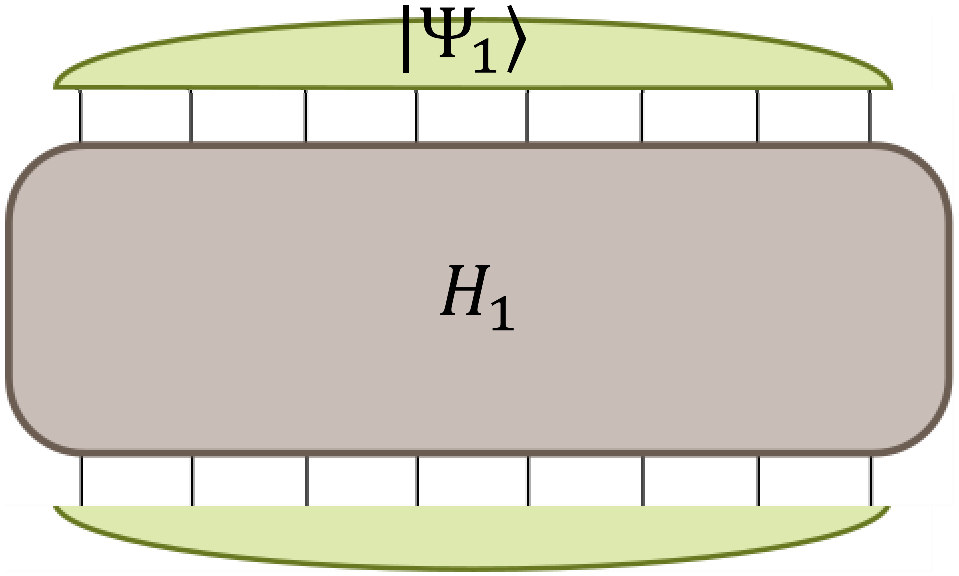
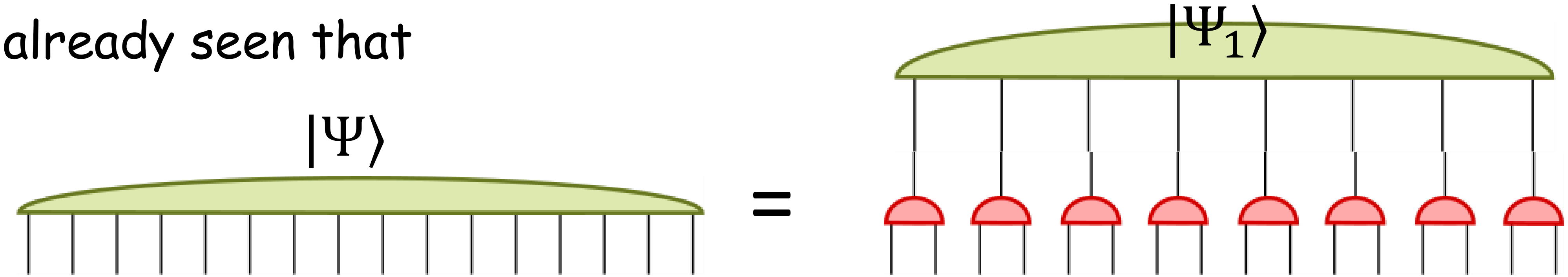


= Energy of $|\Psi_1\rangle$

Also, clearly $|\Psi_1\rangle$ is the ground state of H_1. Else this tensor network value can be minimized by replacing $|\Psi_1\rangle$ with the ground state.

But this cannot happen since $|\Psi\rangle$ is already the ground state of H.

We have already seen that



Not only energy match but also other expectation values are preserved

This means that the coarse-graining transformation preserves the ground state properties.

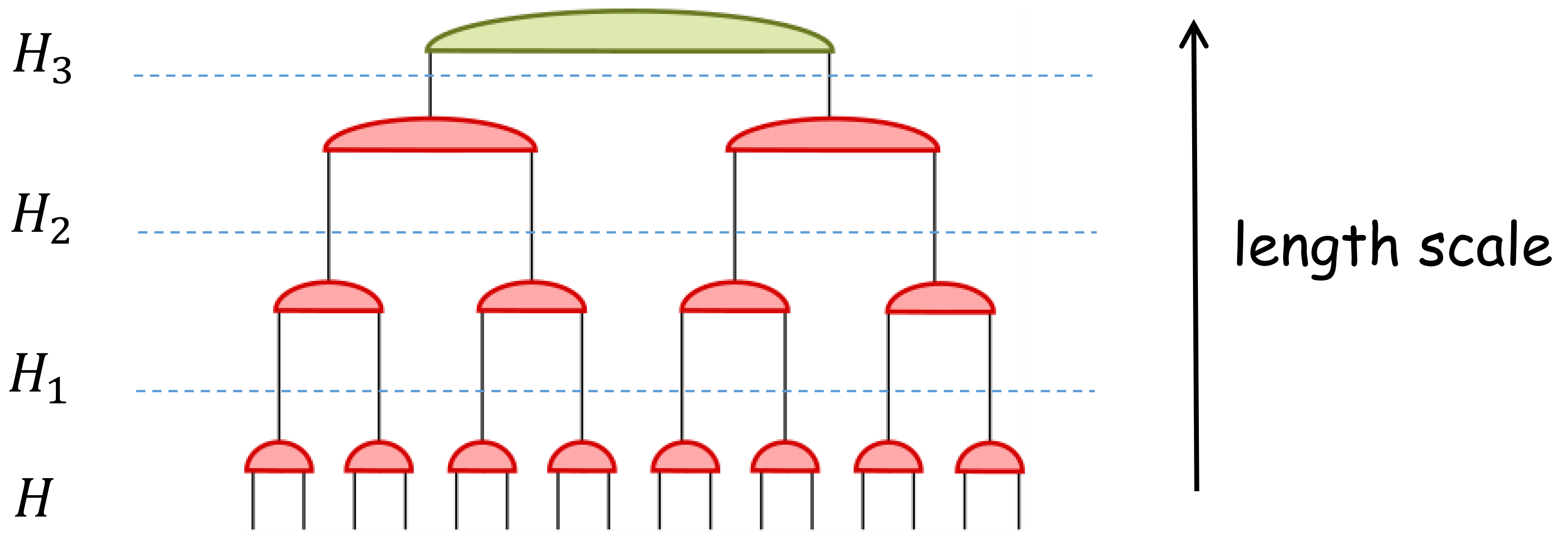
But H_1 is smaller than H.

So it must discard high energy states.

In fact, at the top we only have the ground state!

So the tree tensor network defines a RG flow in the space of Hamiltonians

The tree tensor network of a gapped ground state defines an RG flow



Summary: tree tensor network not only gives an efficient description of the gapped ground state but also defines an RG flow in the space of Hamiltonians.

What about critical ground states?

What about critical ground states?

$S(\rho_\ell)$ grows as $\log \ell$ $\Rightarrow$ Rank (ρ_ℓ) grows O(poly(ℓ))

Bond dimension grows polynomially with number of coarse-graining steps

Unsustainable RG flow. Worse: does not reach a fixed point

Not reproducing a fixed point at criticality not just a failure of aesthetics

Obtaining the correct fixed point will allow contact with **2d CFTs** that describe the critical system in the continuum

What about critical ground states?

$S(\rho_\ell)$ grows as $\log \ell$ $\Rightarrow$ Rank (ρ_ℓ) grows O(poly(ℓ))

Bond dimension grows polynomially with number of coarse-graining steps

Unsustainable RG flow. Worse: does not reach a fixed point

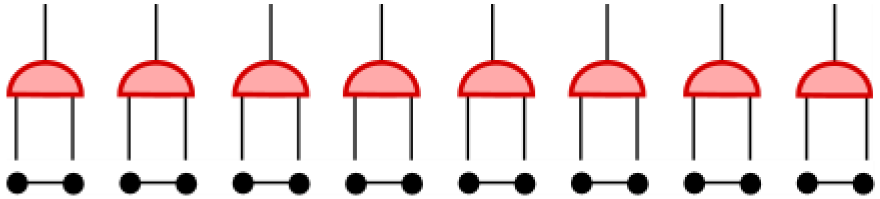
Crux of the problem:

(i) In a critical state entanglement appears at all length scales (as you block more and more sites, new short-range entanglement keeps appearing)

(ii) The coarse-graining transformation does not remove short-range correlations properly

RG flow in the tree can accumulate of short-range correlations

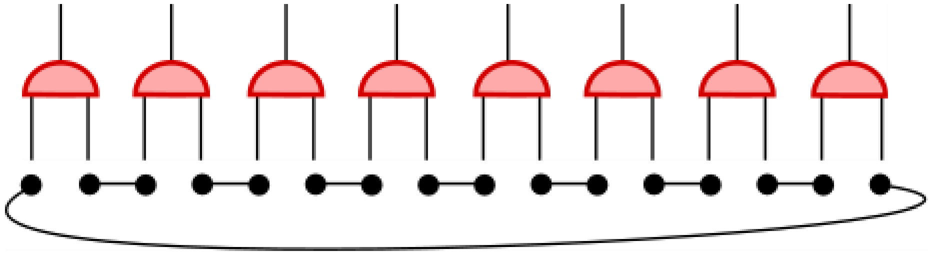
product state



maximally entangled
state

RG flow in the tree can accumulate of short-range correlations

entangled state



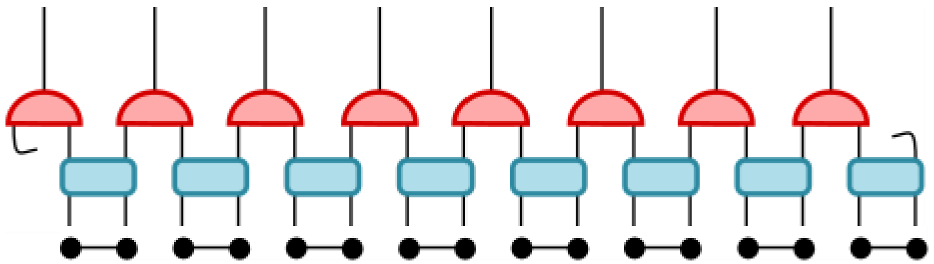
maximally entangled
state

In a critical state, where correlations appear at all length scales, some local correlations will not get removed properly

One fix: Introduce unitary gates that remove correlations locally

Disentanglers

product state



Disentanglers remove the entanglement

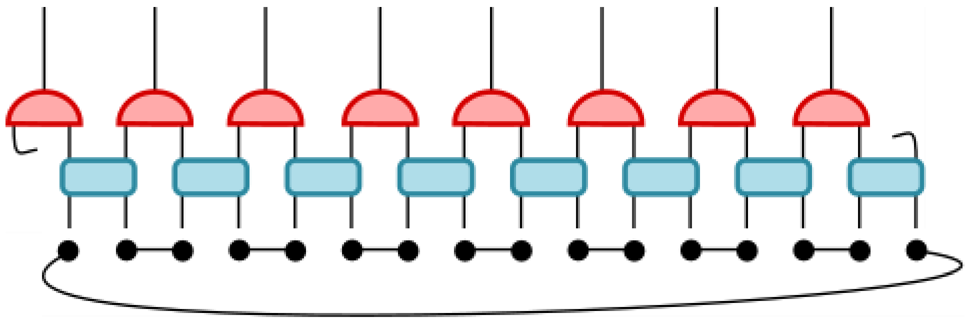
Isometries are trivial

One fix: Introduce unitary gates that remove correlations locally

Disentanglers

Entanglement renormalization

product state

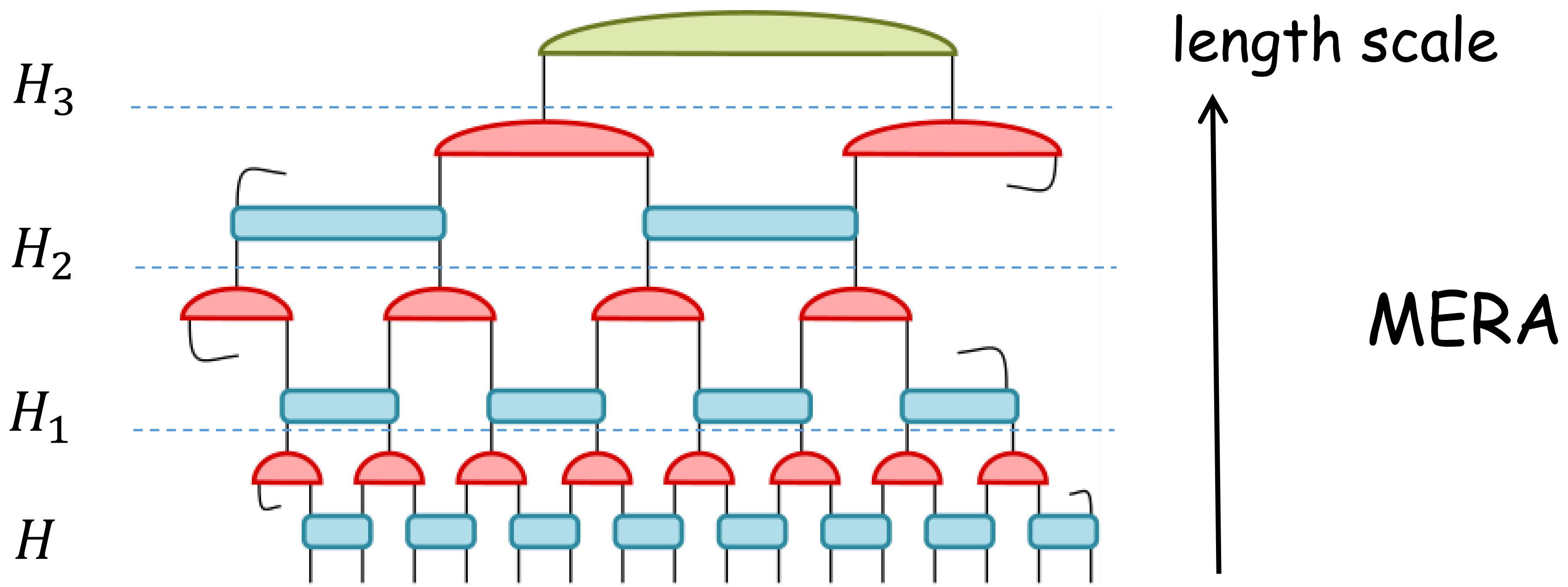


Disentanglers are trivial

But isometries remove the entanglement

Isometries + disentanglers are more effective at removing short range correlations

Tensor network generated by entanglement renormalization



Retains all the useful features of the tree: $O(N)$ parameters, preserves locality.

Also defines an RG flow, which can reach a fixed point at criticality.

MERA as a numerical ansatz for 1d ground states

So far we have discussed how to translate an exponentially large vector to a tree tensor network or MERA.

The isometries and disentanglers are determined from reduced density matrices in the state

In practice, we do not know the ground state to begin with.

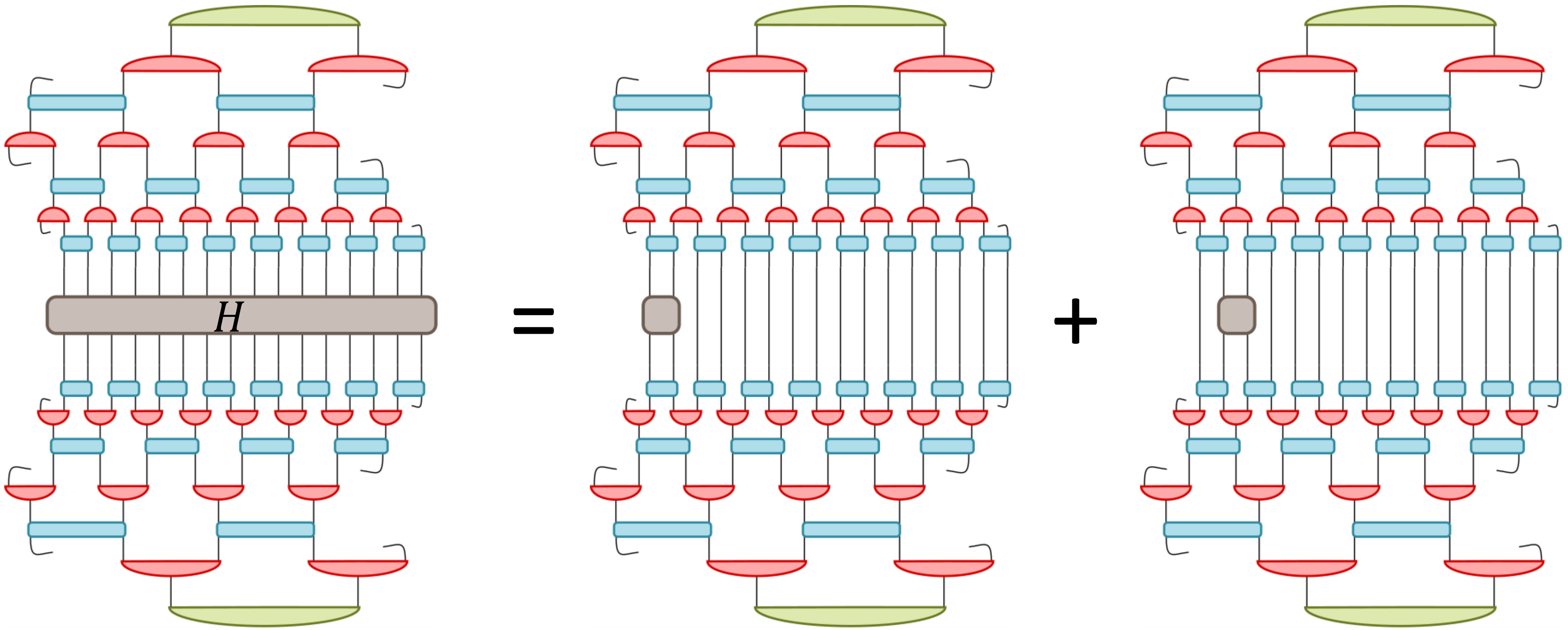
In this case, we can use the MERA as an ansatz for 1d ground states.

What we have argued so far is that it is capable of efficiently representing both gapped and critical ground states.

So how do we determine the ground state tensors in practice?

One method: variational energy minimization

Hamiltonian is local



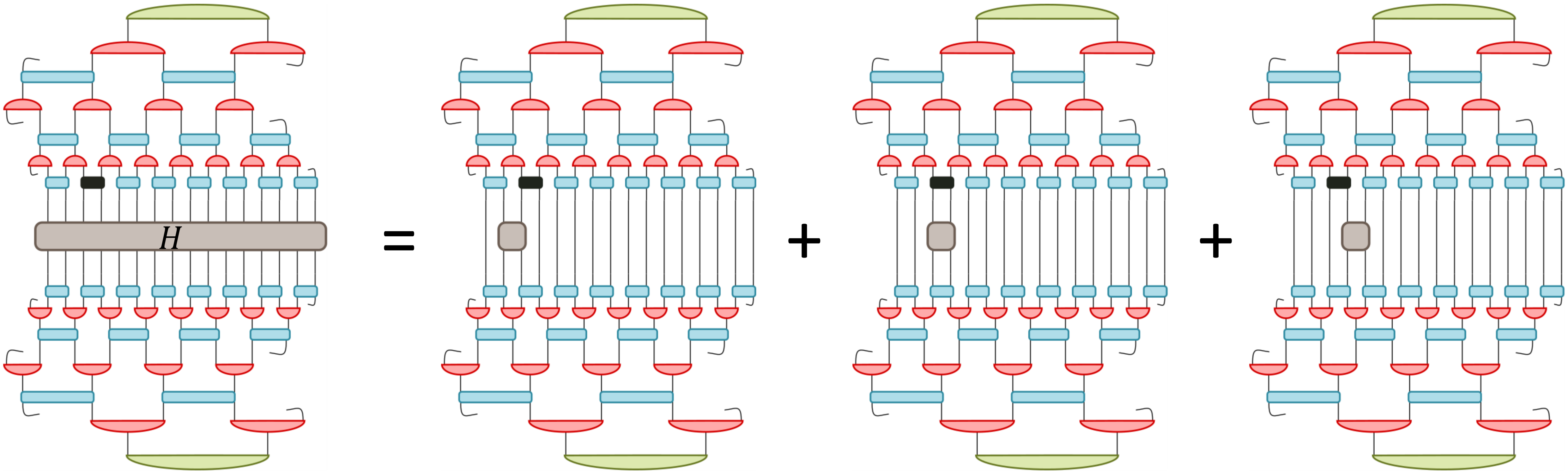
+ ... (N terms)

Minimize:

(over all tensors)

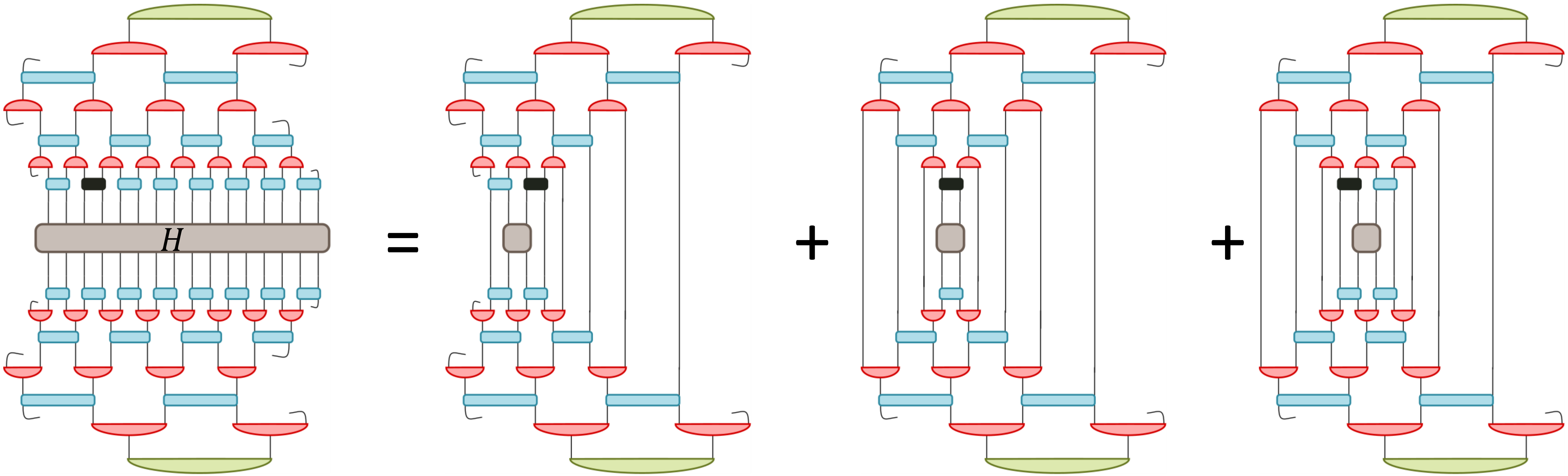
Optimize one tensor at time

Suppose we want to optimize the black disentangler



Only 3 terms contribute. (The black disentanglers cancels out in all other terms)

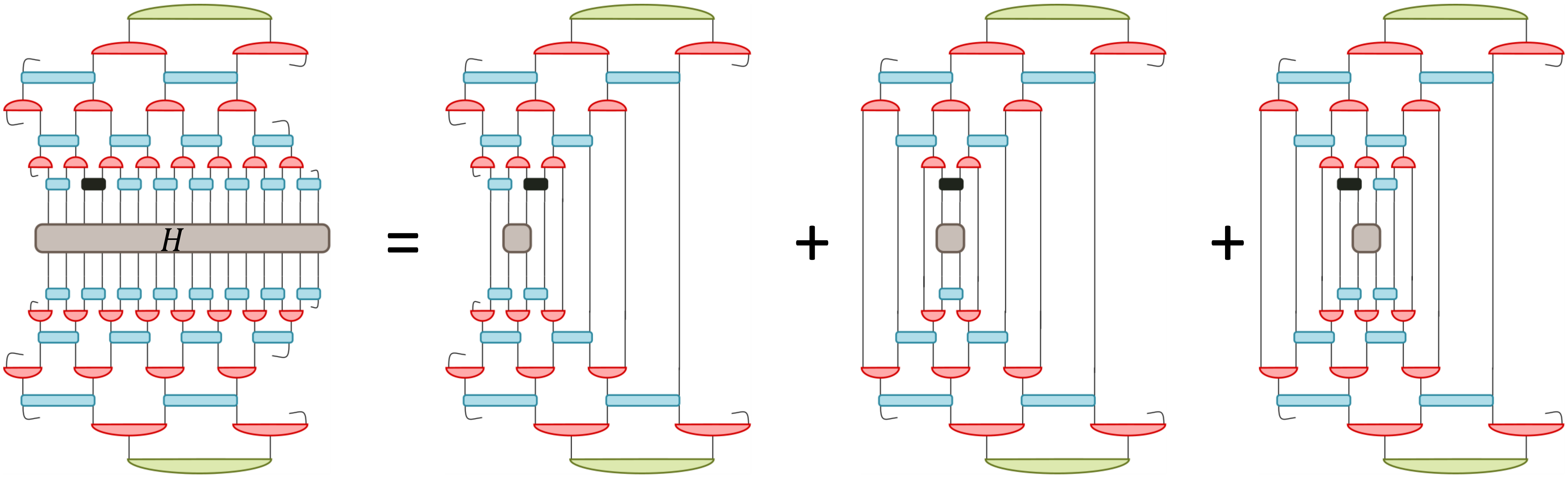
Suppose we want to optimize the black disentangler

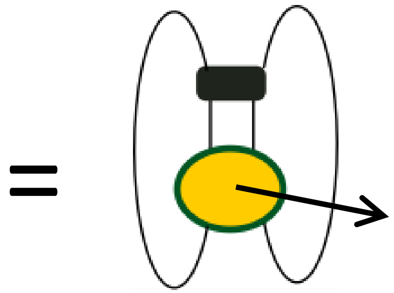


Only 3 terms contribute. (The black disentanglers cancels out in all other terms)

Furthermore, these terms also simplify significantly

Suppose we want to optimize the black disentangler





environment tensor

Optimal disentangler is obtained from the SVD of the environment tensor

In this way we can optimize one tensor at a time, say sweeping the tensor network from bottom to top

Repeat sweeps until e.g. energy converges

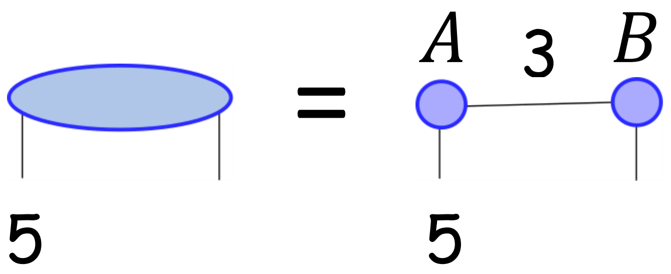
The numerically optimized MERA should describe an approximate RG flow as described earlier

PART 2

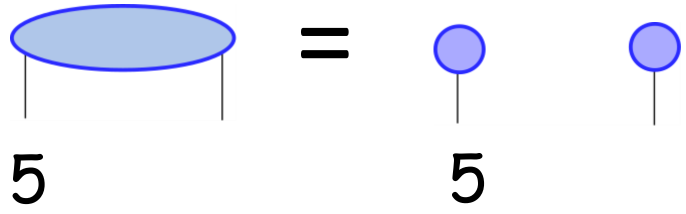
MERA and critical systems

Entanglement in the MERA

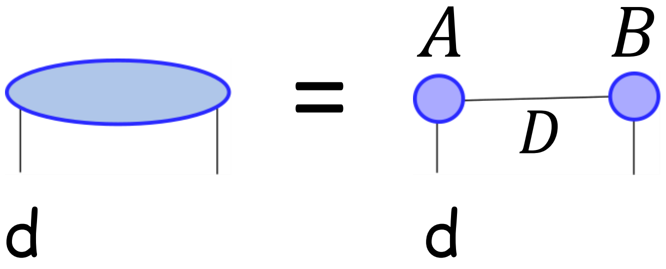
A simple observation …



Rank $(\rho_A) \leq 3$

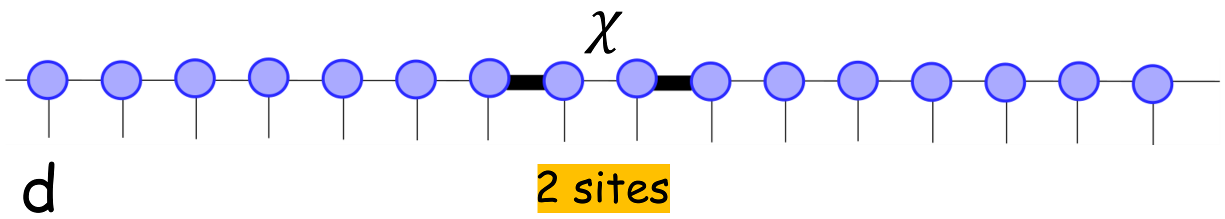


Rank $(\rho_A) = 1$



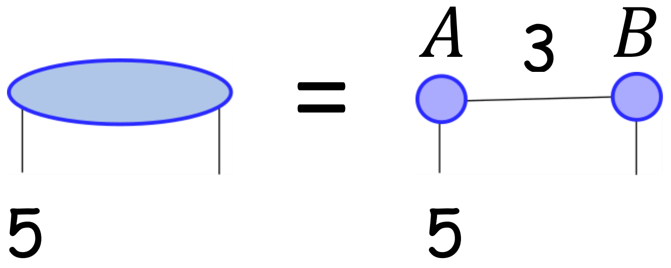
Rank $(\rho_A) \leq \min(\mathrm{d}, D)$

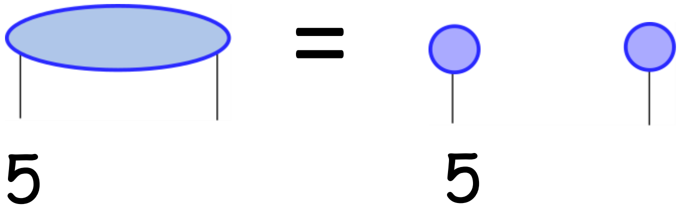
iMPS

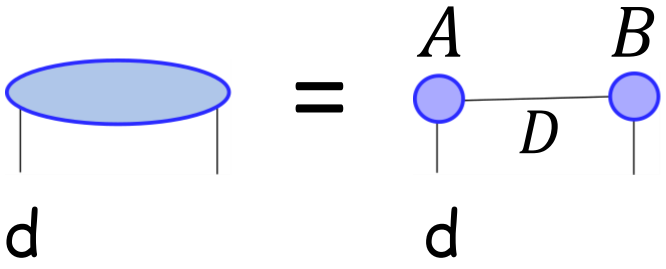


$$D = \chi^2$$

A simple observation …





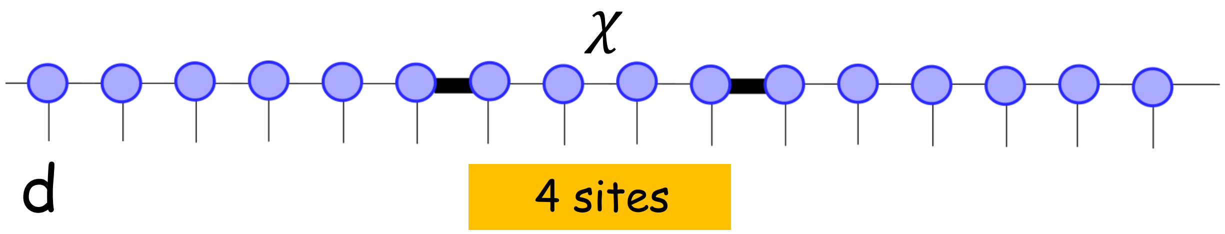


Rank $(\rho_A) \leq 3$

Rank $(\rho_A) = 1$

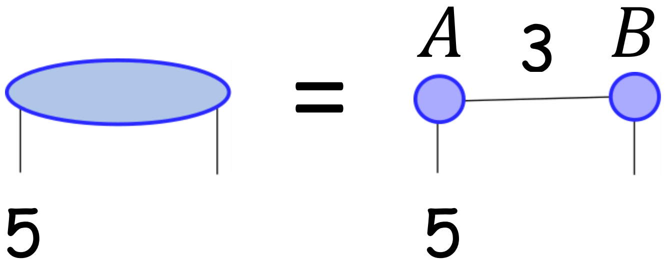
Rank $(\rho_A) \leq \min(\mathrm{d}, D)$

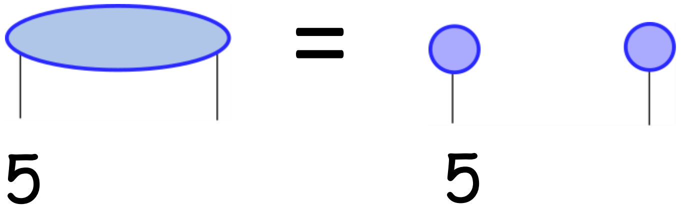
iMPS

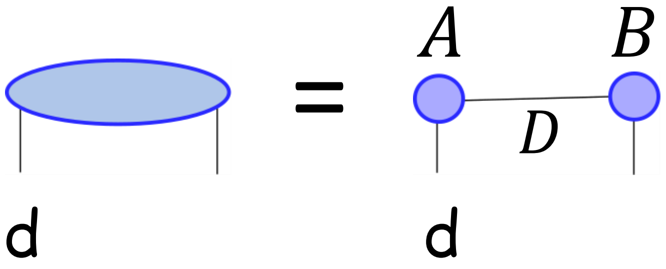


$$D = \chi^2$$

A simple observation …





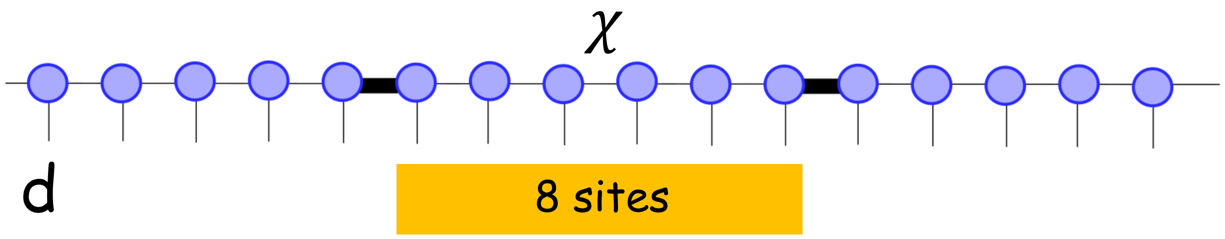


Rank $(\rho_A) \leq 3$

Rank $(\rho_A) = 1$

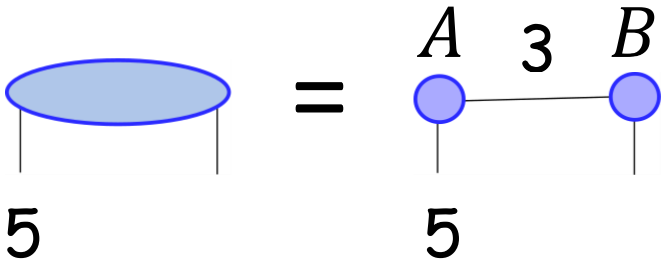
Rank $(\rho_A) \leq \min(\mathrm{d}, D)$

iMPS

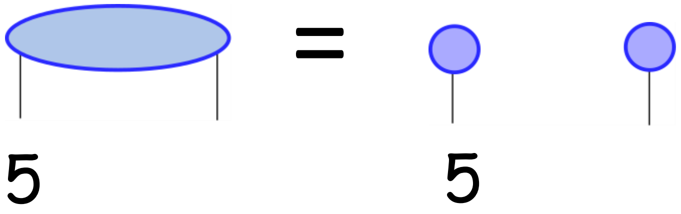


$$D = \chi^2$$

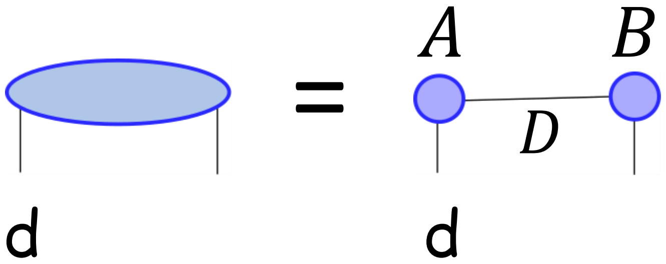
A simple observation ...



Rank $(\rho_A) \leq 3$

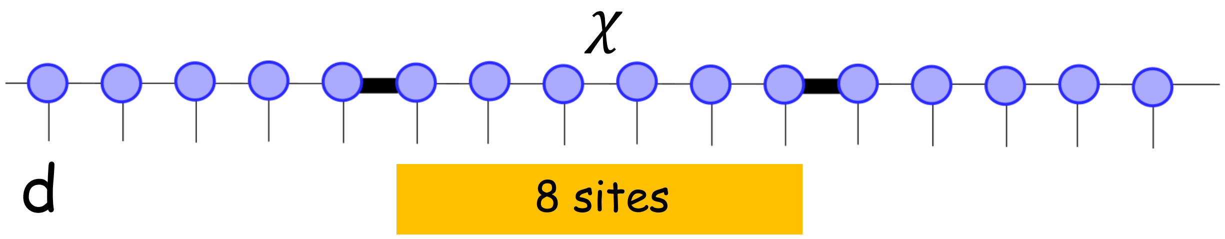


Rank $(\rho_A) = 1$



Rank $(\rho_A) \leq \min(\mathrm{d}, D)$

iMPS

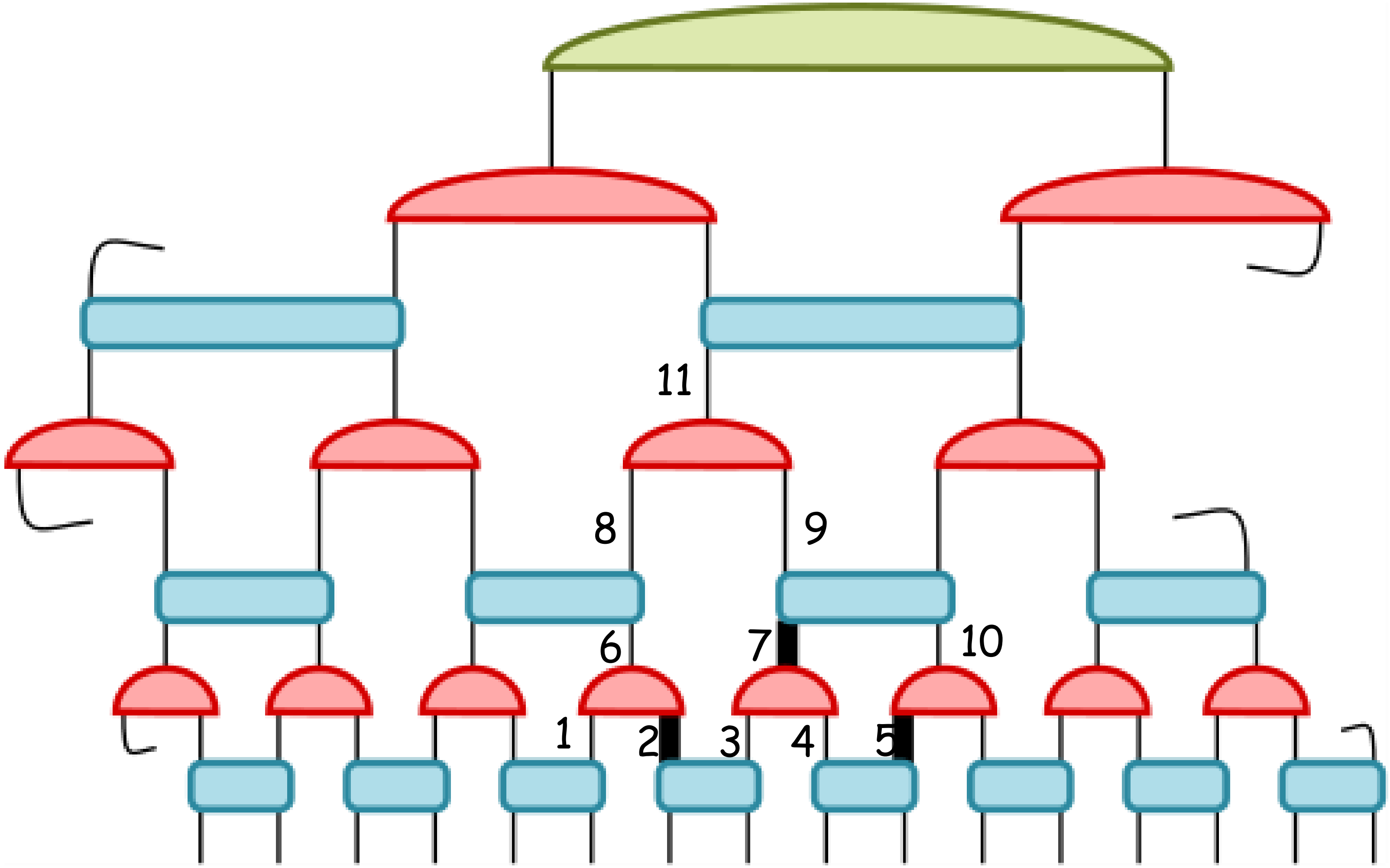


At some ℓ, $d^{\ell} > \chi^2$, thus Rank $(\rho_\ell) \leq \chi^2$

$$D = \chi^2$$

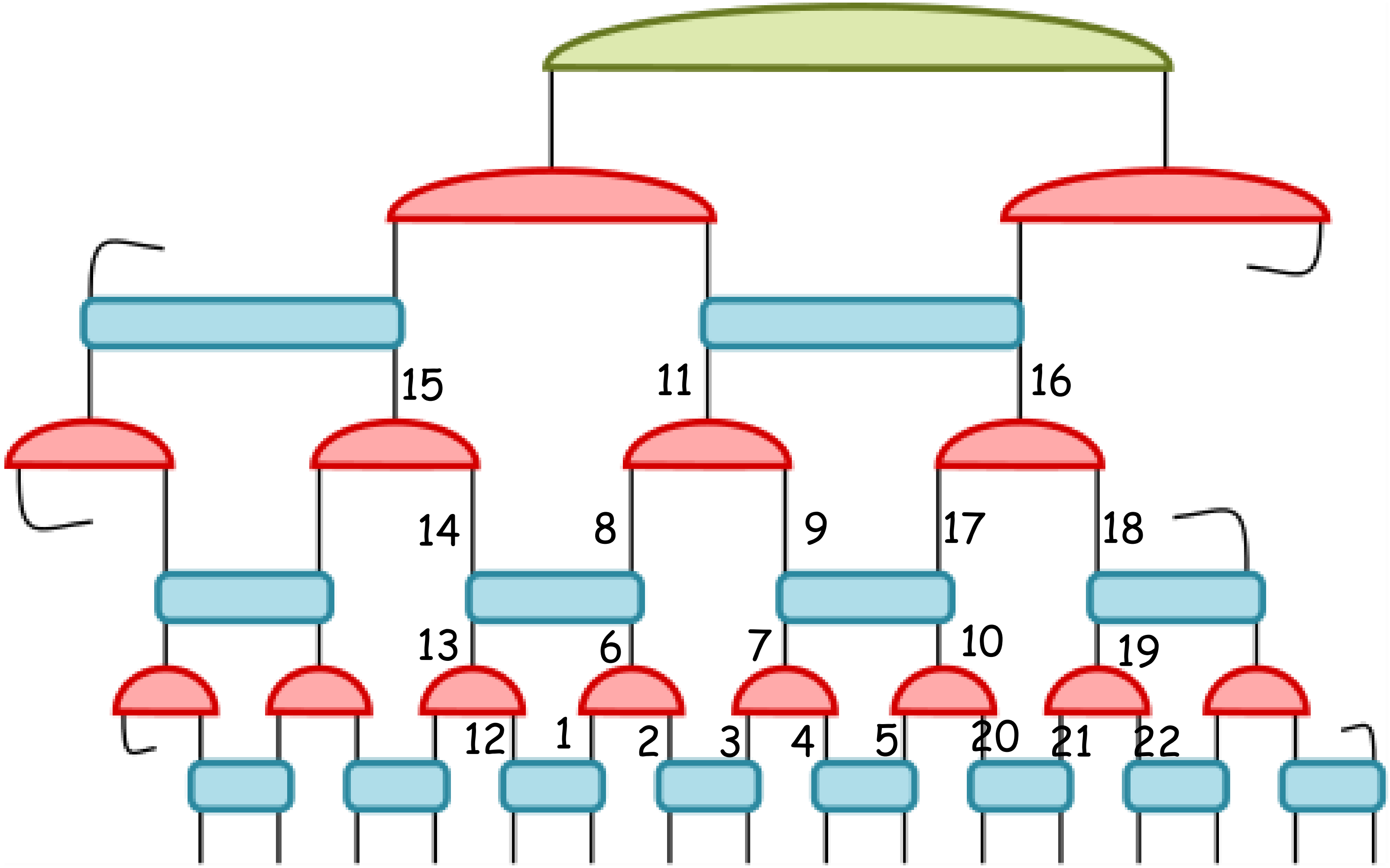
Matches the entanglement profile of gapped ground states

Quiz!

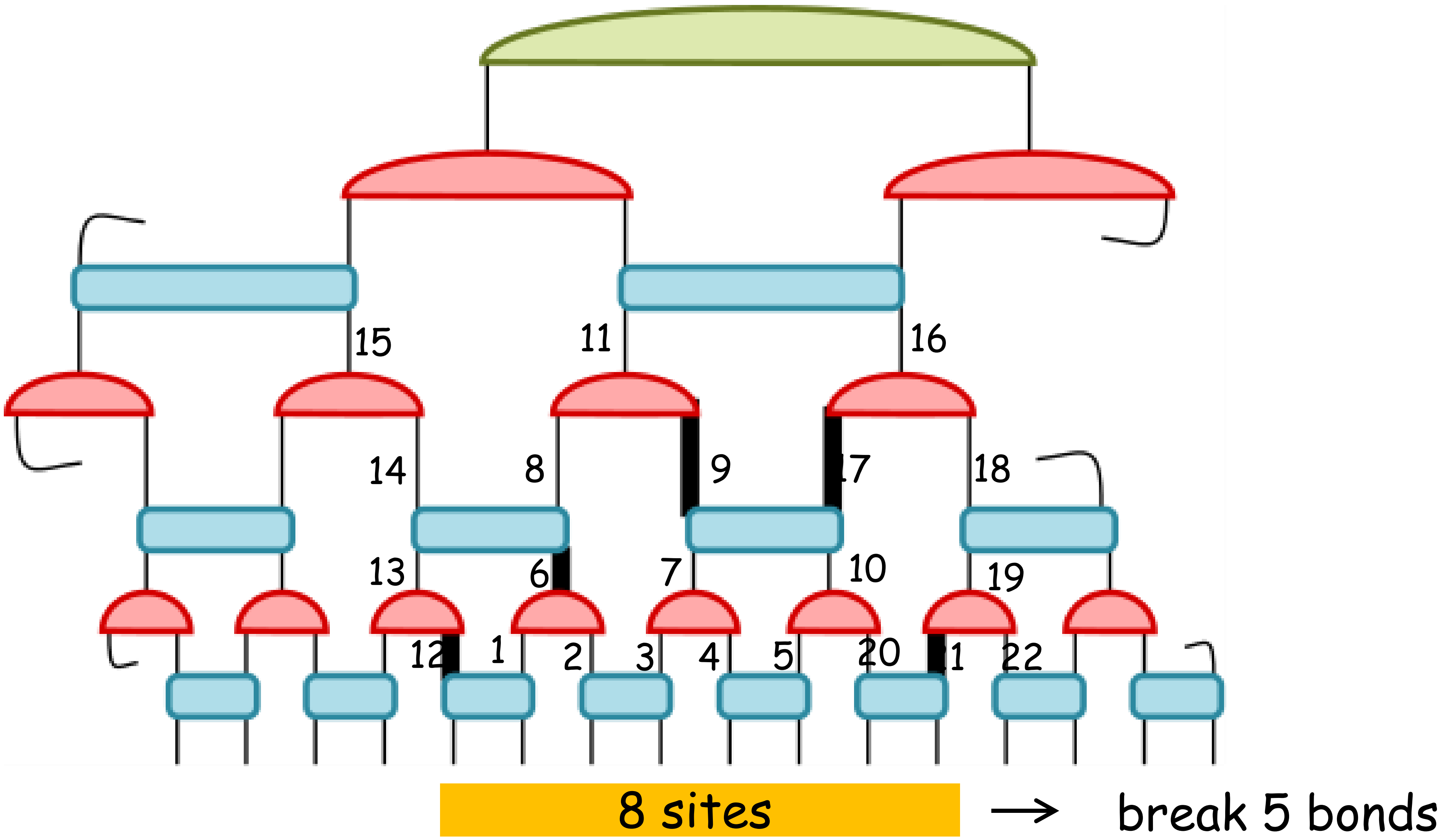


4 sites → break 3 bonds

Quiz!



8 sites



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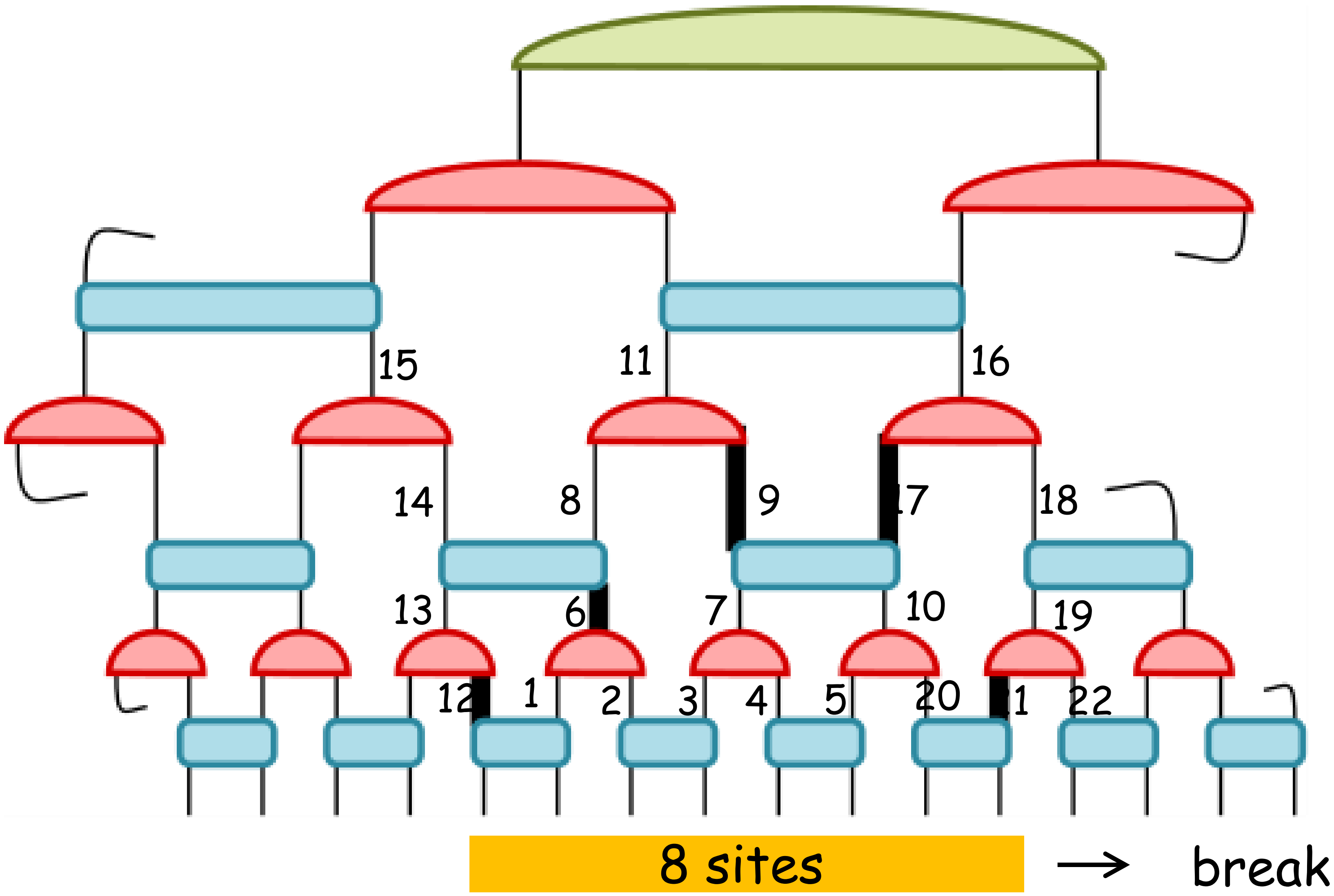
8 sites
→ break 5 bonds



→ break 5 bonds

ℓ sites → break $O(\log \ell)$ bonds $\Rightarrow$ Entropy $\sim O(\log \ell)$

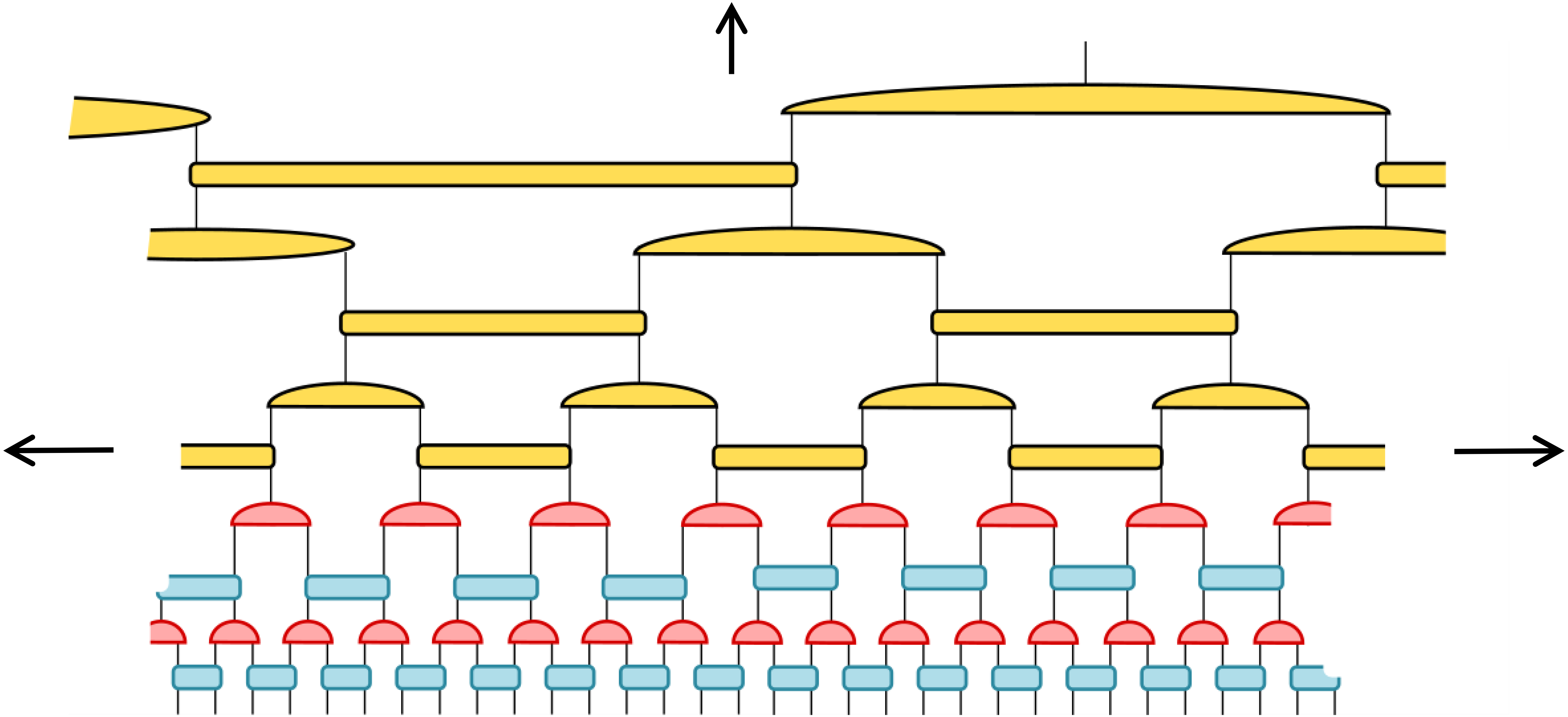
A random MERA state has logarithm scaling of entanglement entropy (similar to critical states)

One reason to hope that the MERA might be a **natural** ansatz for critical states

As another reason -- Next let us see how the MERA accurately captures the RG fixed point at criticality.

The infinite MERA:

a lattice approximation of a 2d CFT

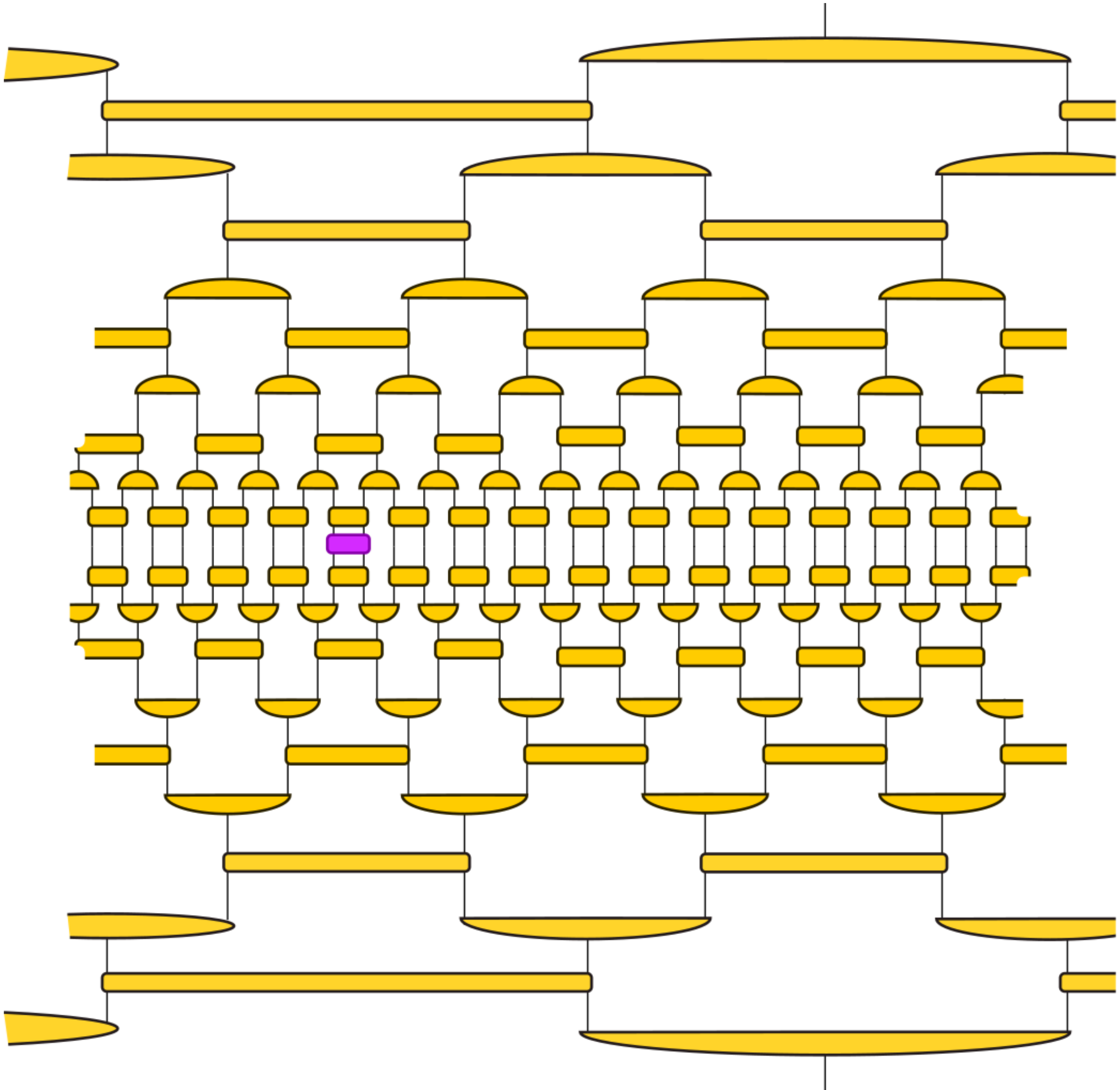


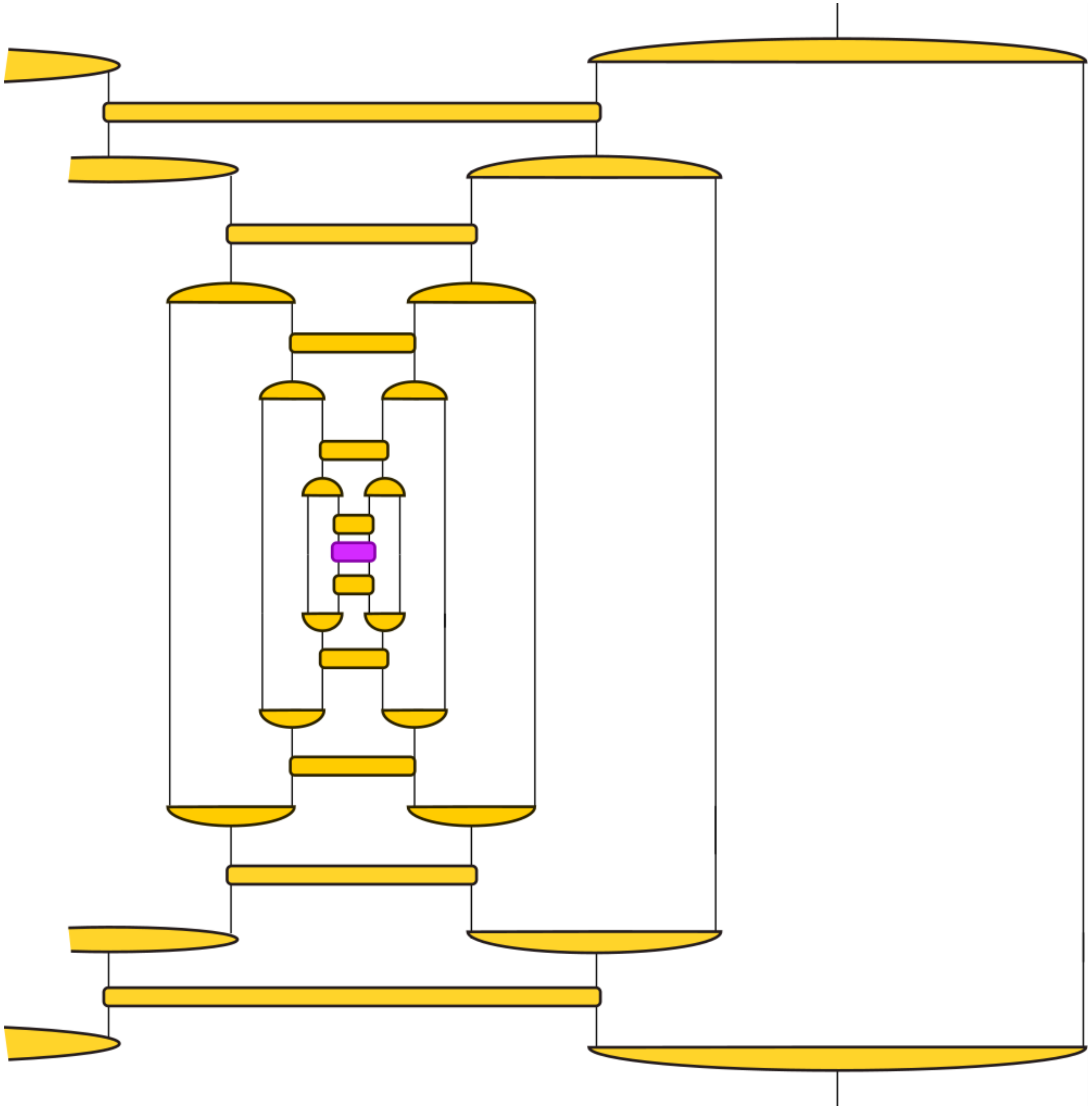
Choose identical tensors in each layer: approximate translation-invariance.

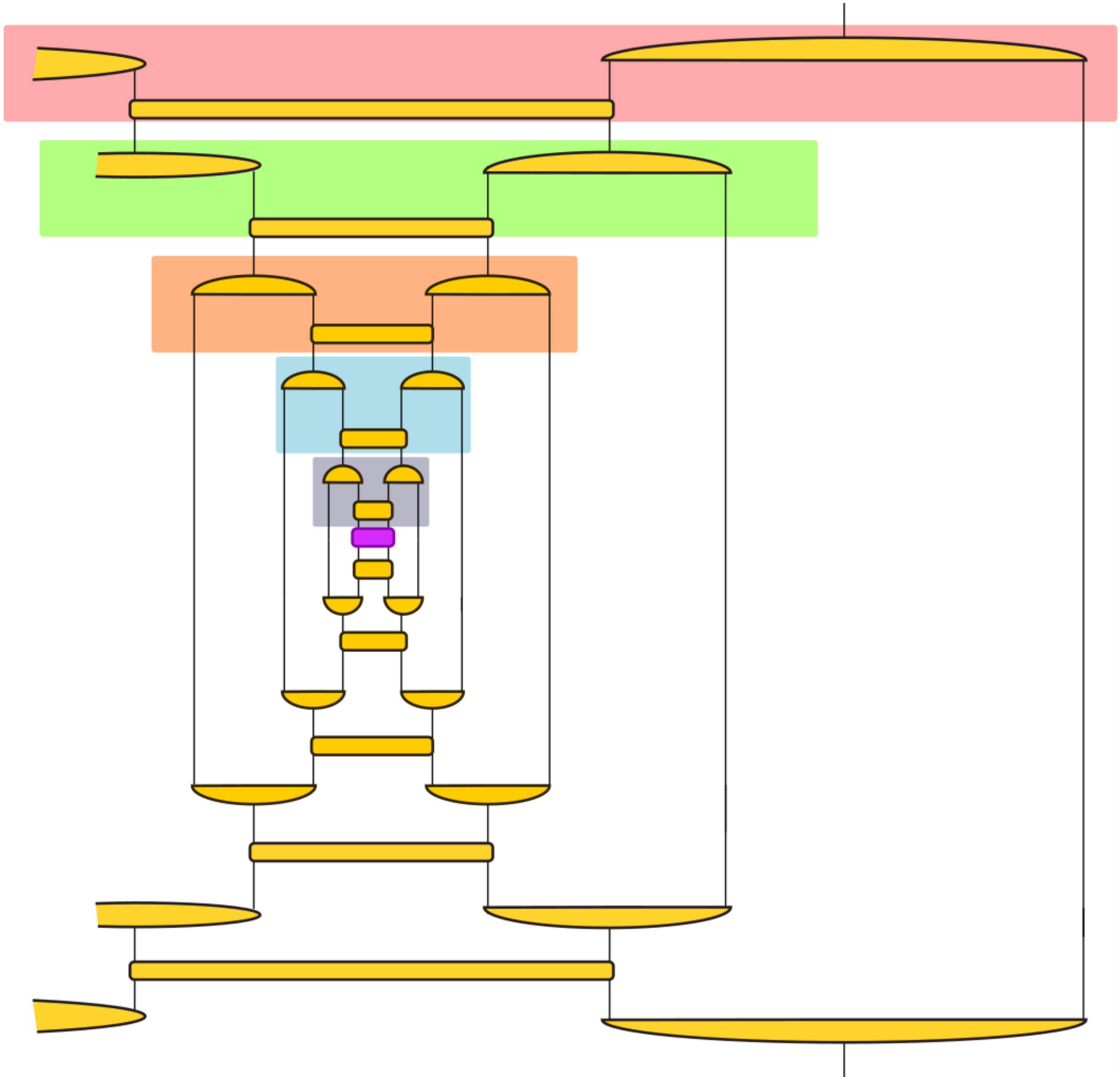
Choose identical tensors after some scale: exact scale-invariance.

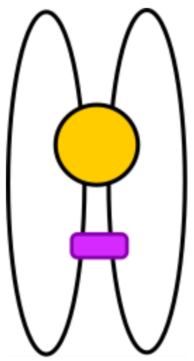
So the infinite MERA can be efficiently stored.

But how can we efficiently compute expectation values from such an infinite tensor network?

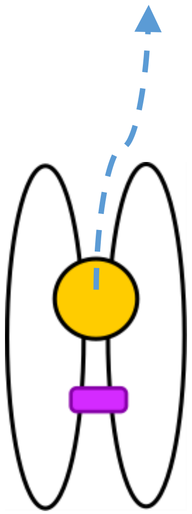


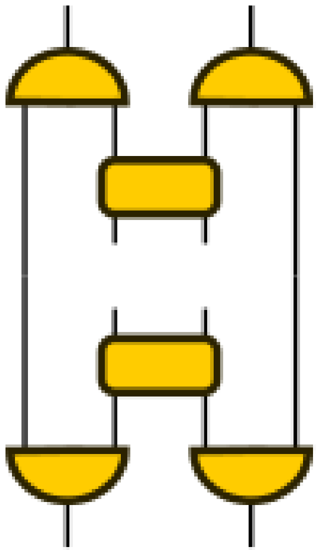






dominant eigenvector





“descending superoperator”

So the infinite MERA is also an efficient ansatz.

How do we use it in practice?

Number of transitional layers is an input parameter in numerics

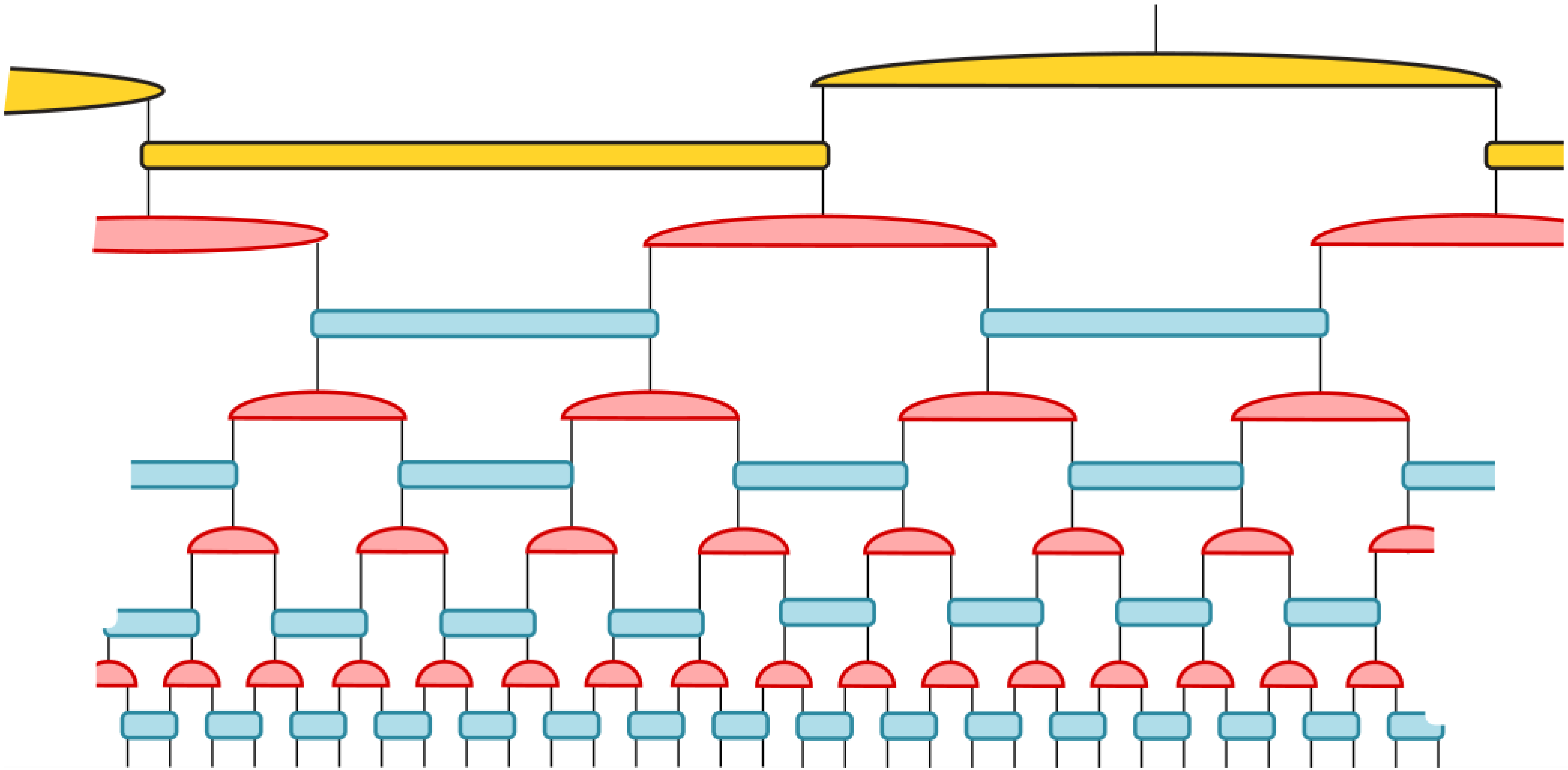
Converge energy for a small number of transitional layers

Increase the number of layers and check if energy lowers

Keep adding layers till the energy has converged (say L layers)

This numerically optimized MERA encodes a RG flow where the Hamiltonian reaches an approximate fixed point after L steps

$$H \to H_1 \to H_2 \to \cdots \to H_L \to H_L \to H_L \to \cdots$$



The yellow tensors encode a lot of important information about the 2d CFT that describes the critical system in the continuum limit

Conformal Field Theory in two dimensions

In 2d, conformal symmetry described by the infinite dimensional Virasoro algebra

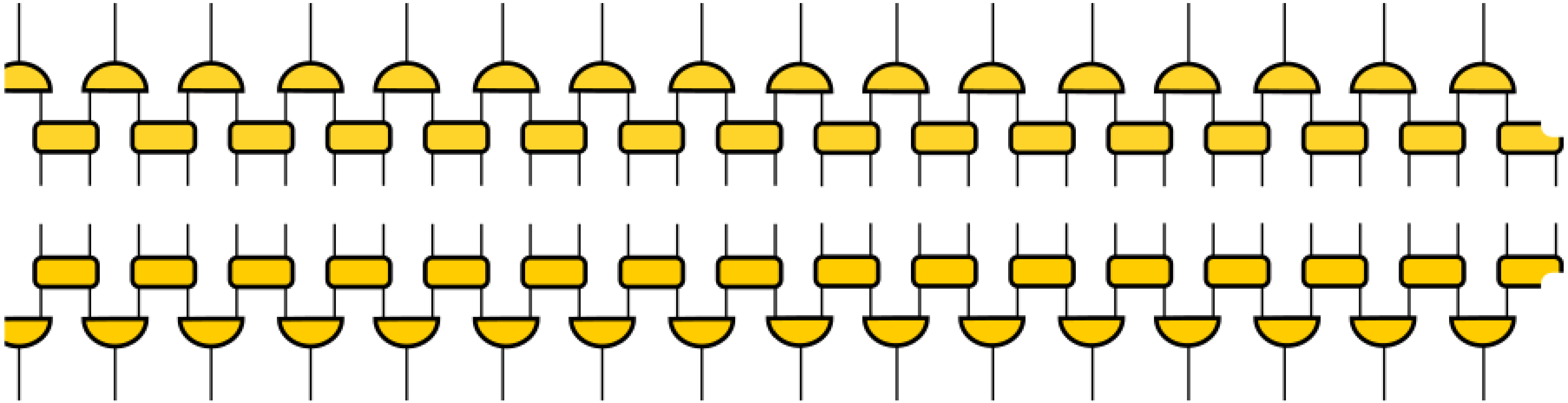
Scaling field operators $\phi_1, \phi_2, \phi_3, \ldots$ with scaling dimensions $\Delta_1, \Delta_2, \Delta_3, \ldots$

Scaling fields are eigenoperators of the dilation operator, and the scaling dimensions are the corresponding eigenvalues

2-point correlators $\langle \phi_\alpha(x) \phi_\beta(y) \rangle \sim \frac{C_{\alpha\beta}}{(x-y)^{\Delta_\alpha + \Delta_\beta}}$ (polynomial decay)

3-point correlators $\langle \phi_\alpha(x) \phi_\beta(y) \phi_\gamma(z) \rangle \sim \frac{C_{\alpha\beta\gamma}}{(x-y)^{\cdots}(y-z)^{\cdots}(x-z)^{\cdots}}$ (OPE coefficients)

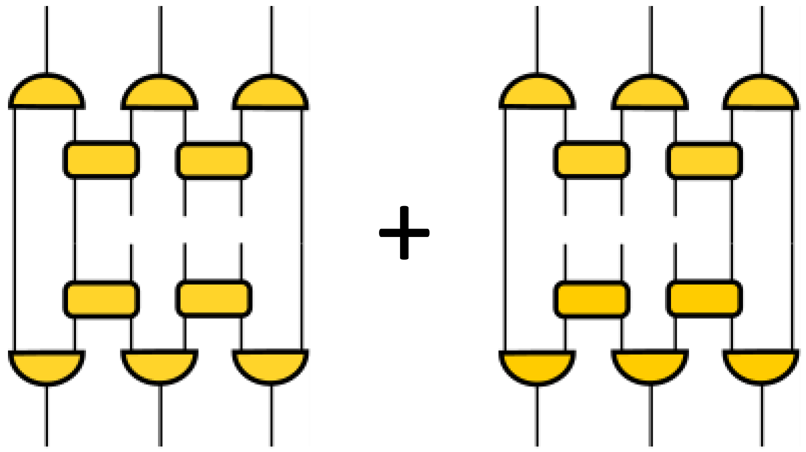
Dilation operator in the MERA (the scaling superoperator)



But we have seen that operators generally shrink under coarse-graining
(But eigenoperators cannot shrink!)

3-site operators remain 3-site, so we can find 3-site eigenoperators

3-site scaling superoperator

$D =$ 

Note: This data can also be extracted from the MPS, but its easier and more intuitive with the MERA

Numerical observations

- Eigenvalues of D match some of the scaling dimensions
- Eigenvectors are some lattice representation of the scaling fields
- Can compute the 3-point correlator of these scaling operators to determine the OPE coefficients
- Patches of the fixed point tensors also give lattice representations of conformal transformations
- Fixed point tensors encode essential CFT data and also know about the conformal symmetry

iMERA $\approx$ 2d CFT on the lattice

$\text{iMERA} \stackrel{?}{=} \text{2d CFT on the lattice}$

Open question

Does there exist an iMERA that encodes the CFT data exactly?

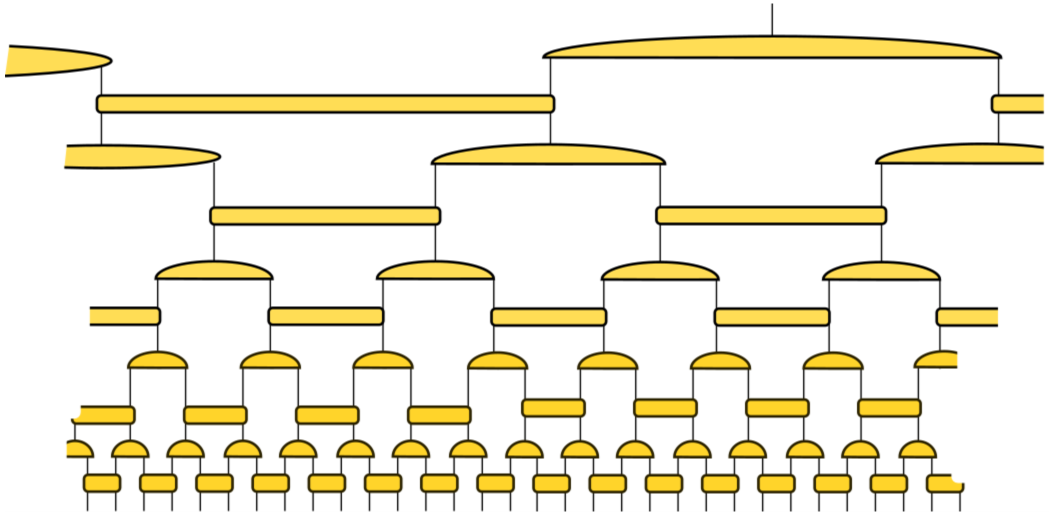
Of course, only a finite part of the infinite CFT data.

One requirement: incorporate the vacuum conformal symmetry in the MERA.

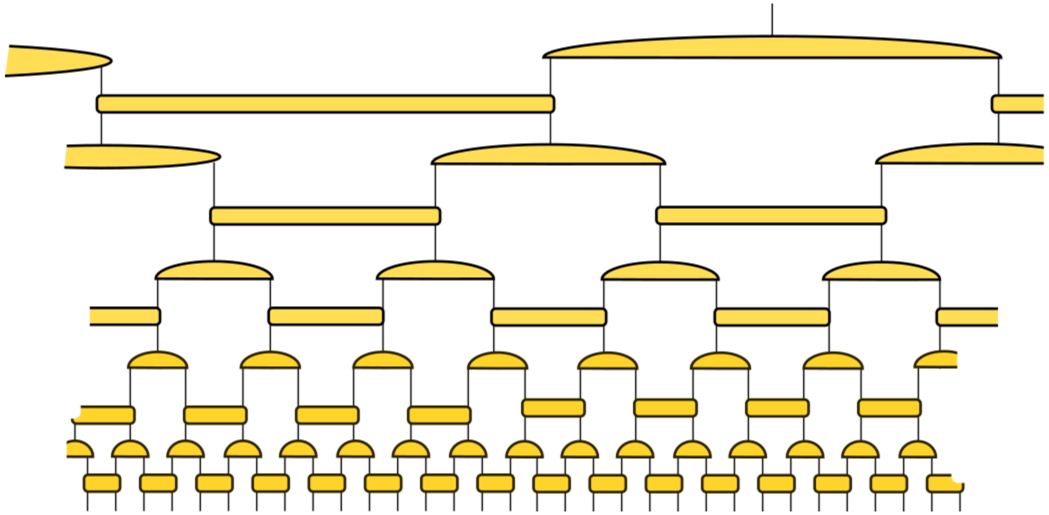
Vacuum conformal symmetry: scale invariance, translation invariance, special conformal transformations (those generated by the positive generators)

Fantasy/Folklore: If the MERA is scale + translation invariance and it is the ground state of a local Hamiltonian the it might possibly be an exact description of a 2d CFT on the lattice

Scale invariance

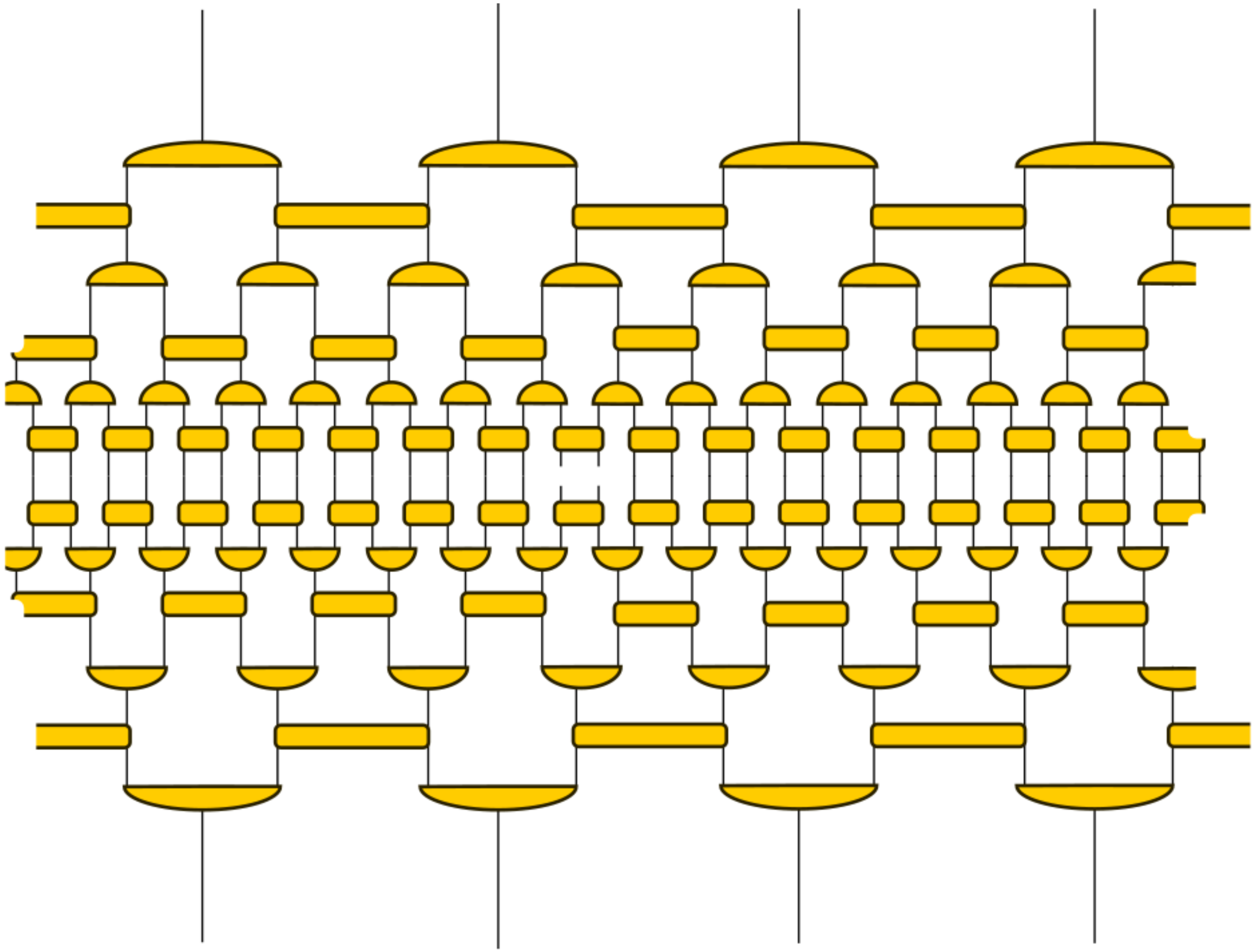


Scale invariance

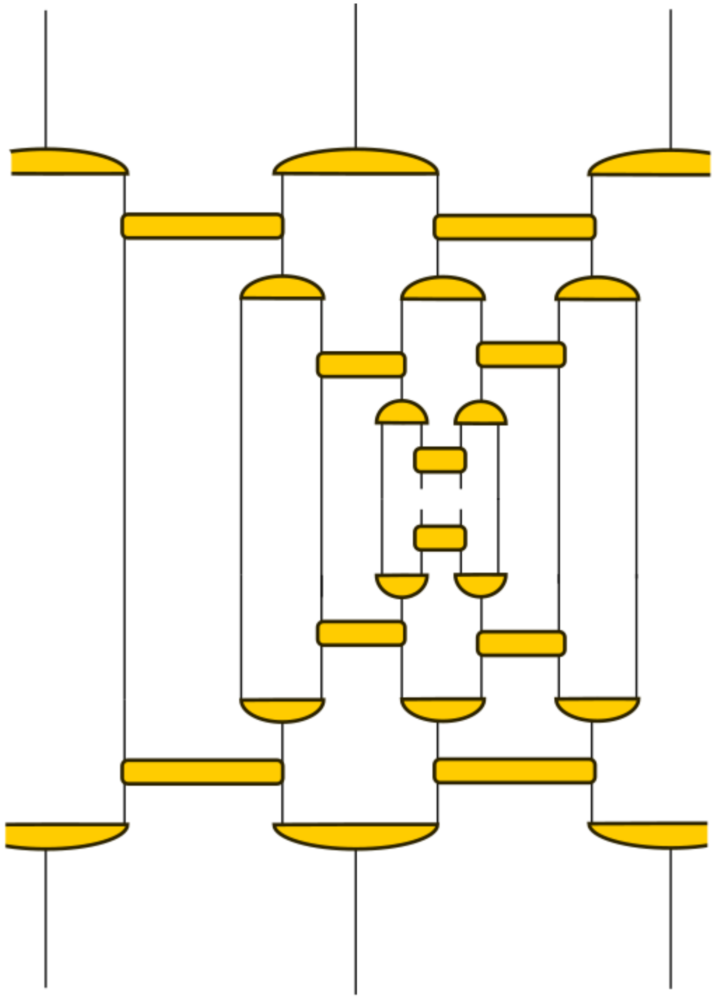


But most of these states are NOT translation invariant

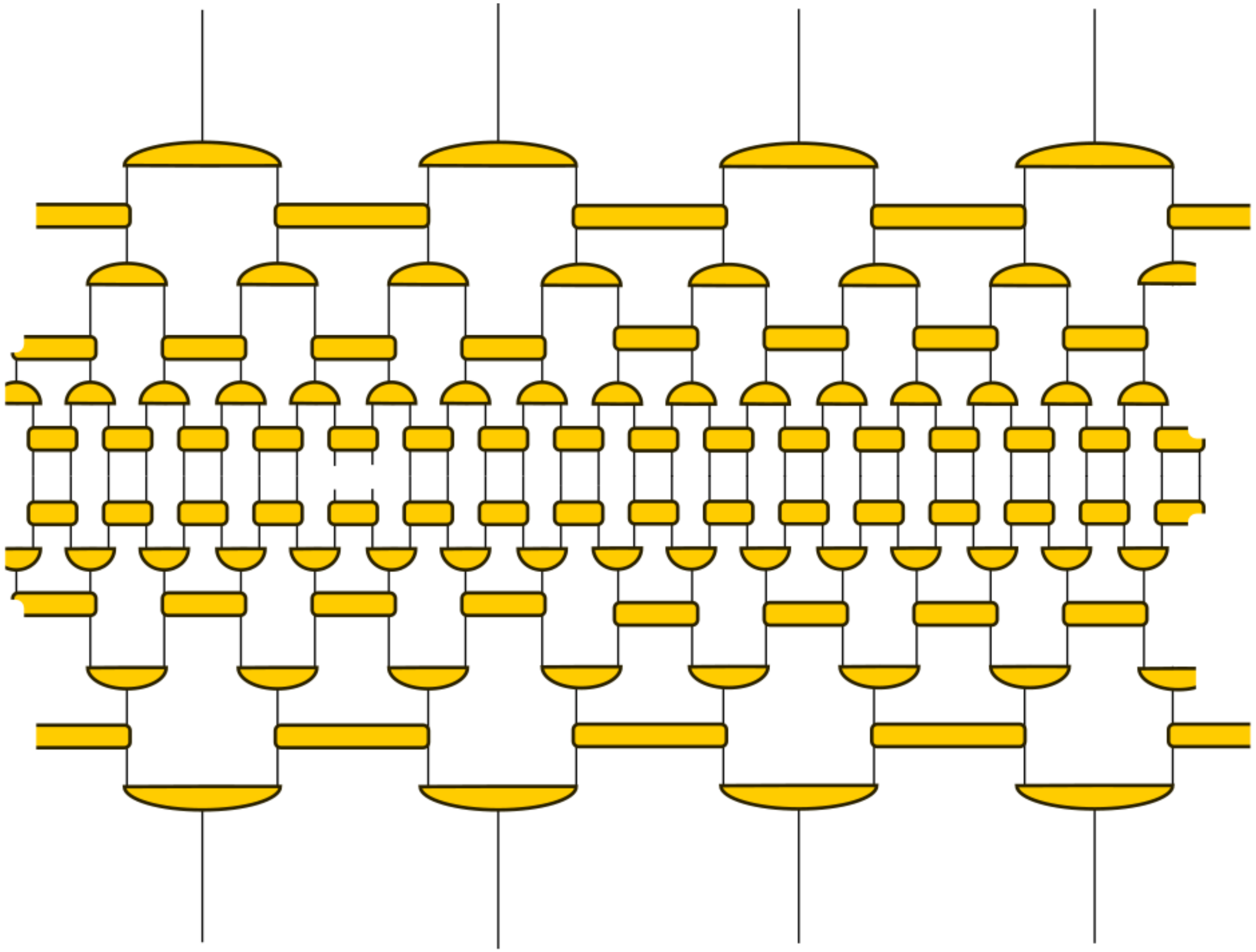
Translation invariance



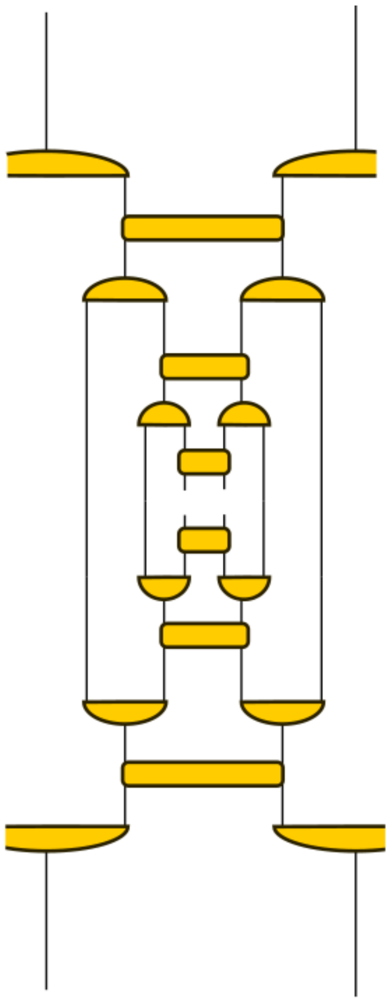
Translation invariance



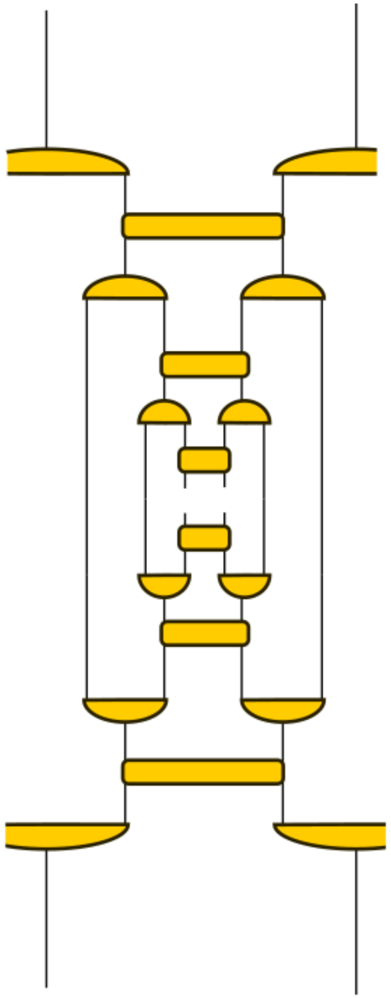
Translation invariance

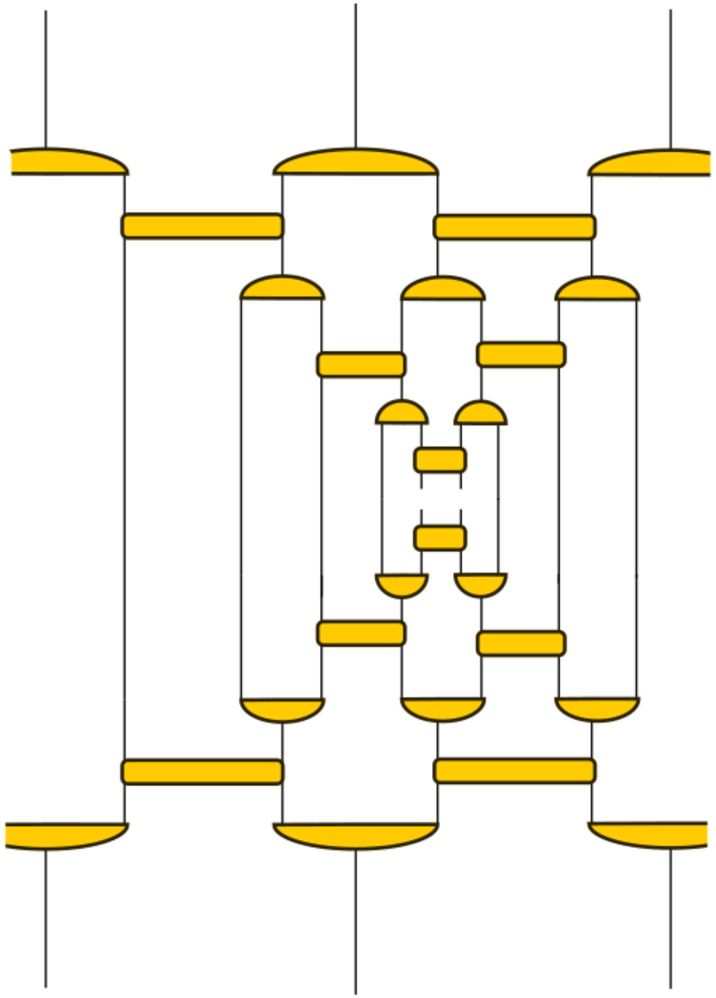


Translation invariance



Translation invariance





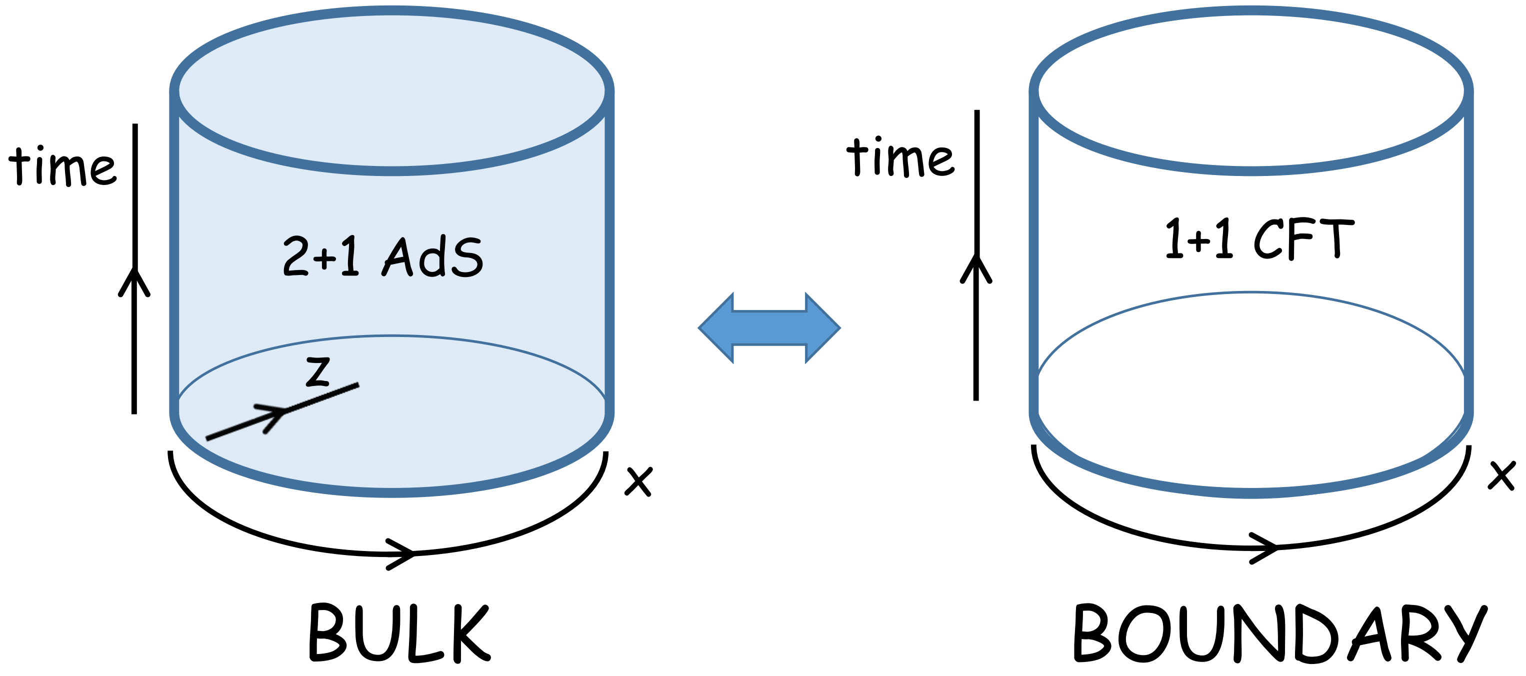
We can get approximate translation invariance in numerical simulations (by e.g. averaging environment tensors across the lattice)

Increasing bond dimension usually improves translation invariance and also the estimation of the CFT data

Can we make translation symmetry exact. (Open Problem)

MERA and holography: basic intro

The AdS/CFT correspondence



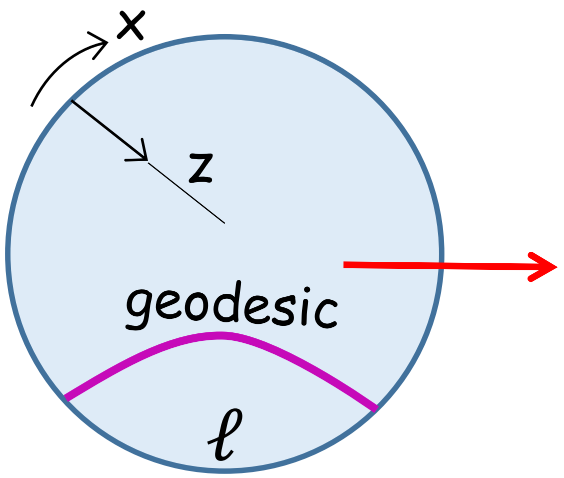
Gravity in 2+1 anti-deSitter spacetime ⟺ 1+1 CFT

Extra dimension on the gravity side ⟺ RG direction

The AdS/CFT correspondence

Certain 'Large' N QFTs correspond to (semi)-classical bulk gravity.

CFT ground state ⟺



Spatial slice of 2+1 AdS:
2D hyperbolic geometry

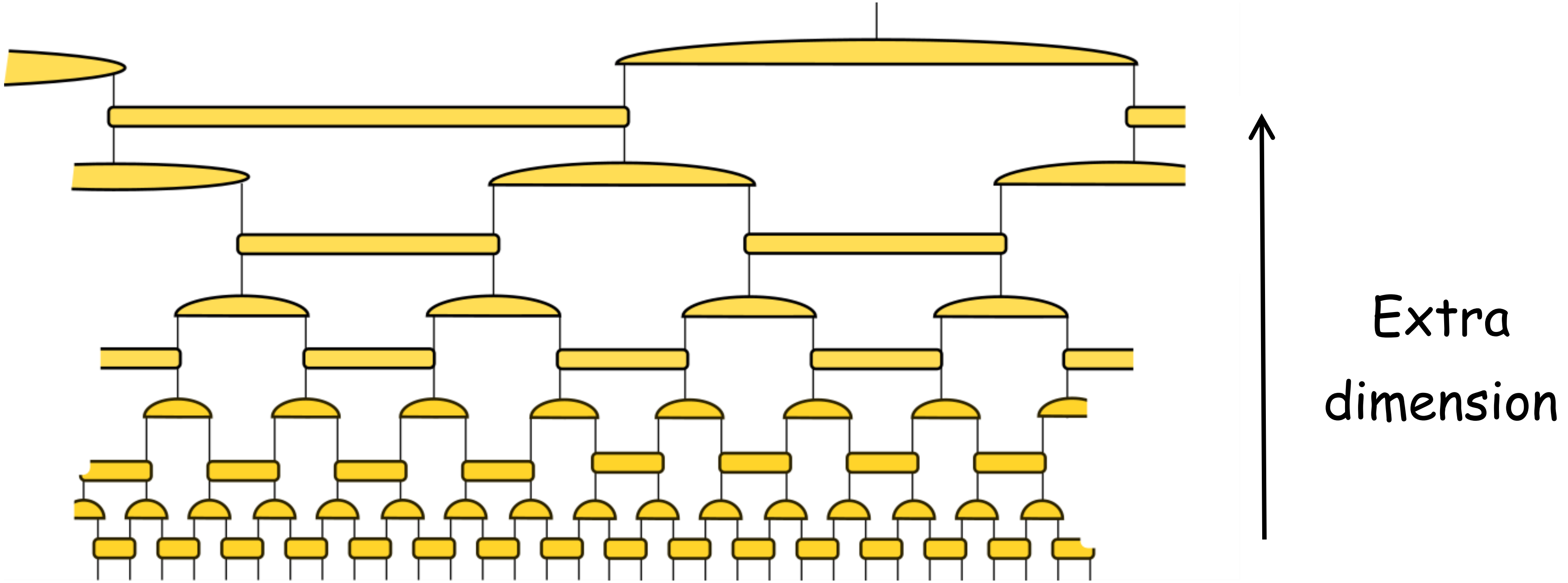
c: central charge ⟺ R: radius of curvature

J. Brown, M. Hennaux, Comm. Math. Phys., 1986

Entanglement entropy = Length of geodesic

$(\propto c \log \ell)$ $(\propto R \log \ell)$

S. Ryu & T. Takayanagi, PRL 96, 181602 (2006)



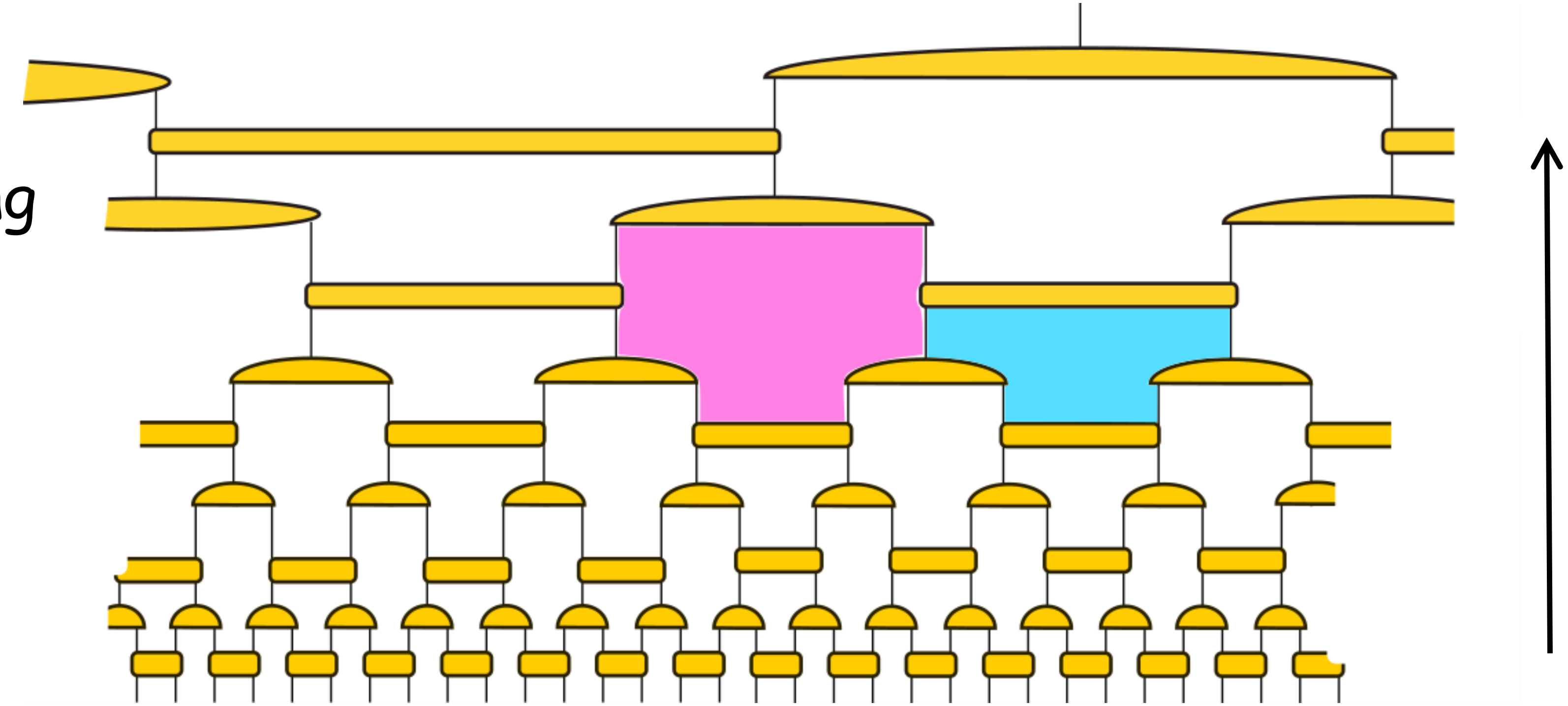
MERA is a good approximation of a 2d CFT

Has an extra dimension that corresponds to length scale

Is based on a negatively curved geometry

MERA as a tiling of the plane

(tiles are flat and regular)



Extra dimension

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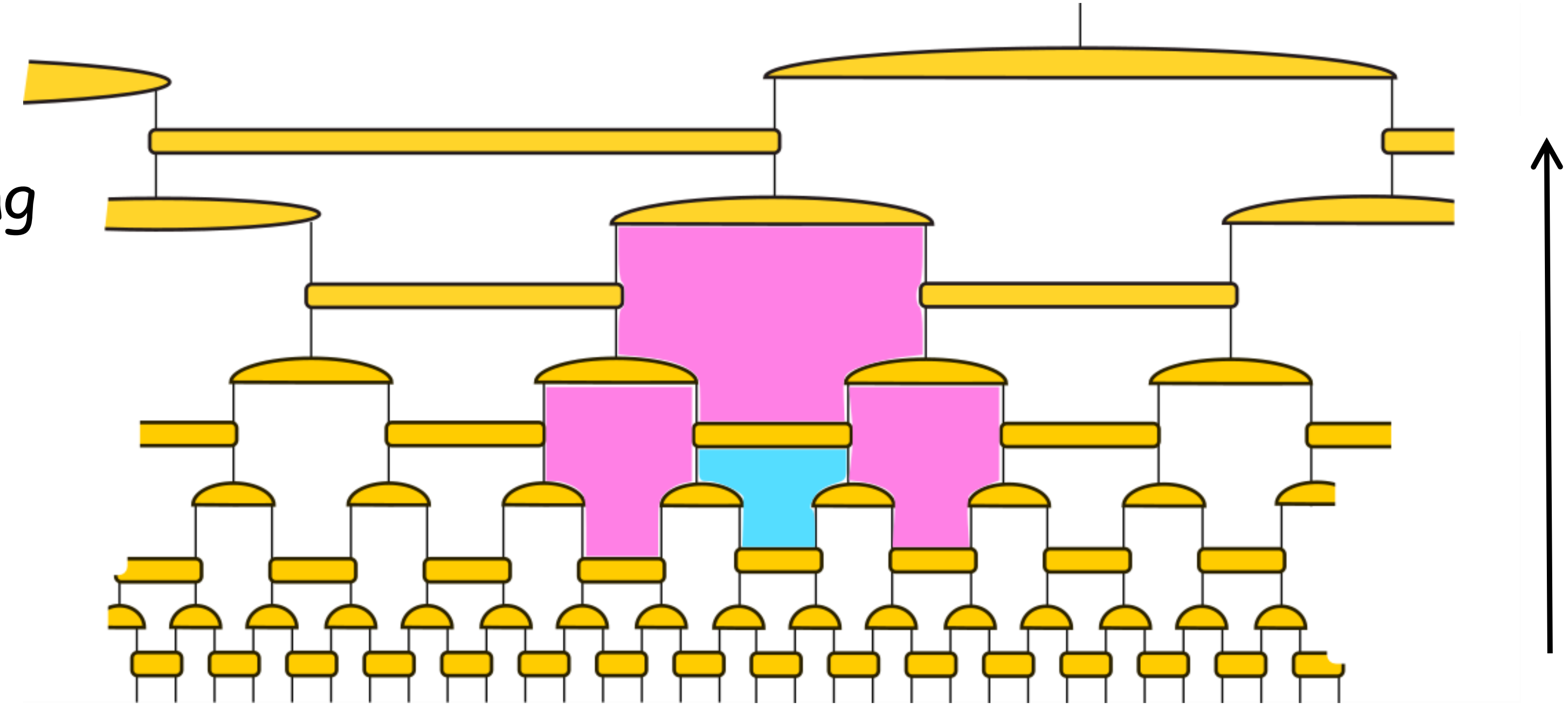
MERA as a tiling of the plane

(tiles are flat and regular)



Extra dimension

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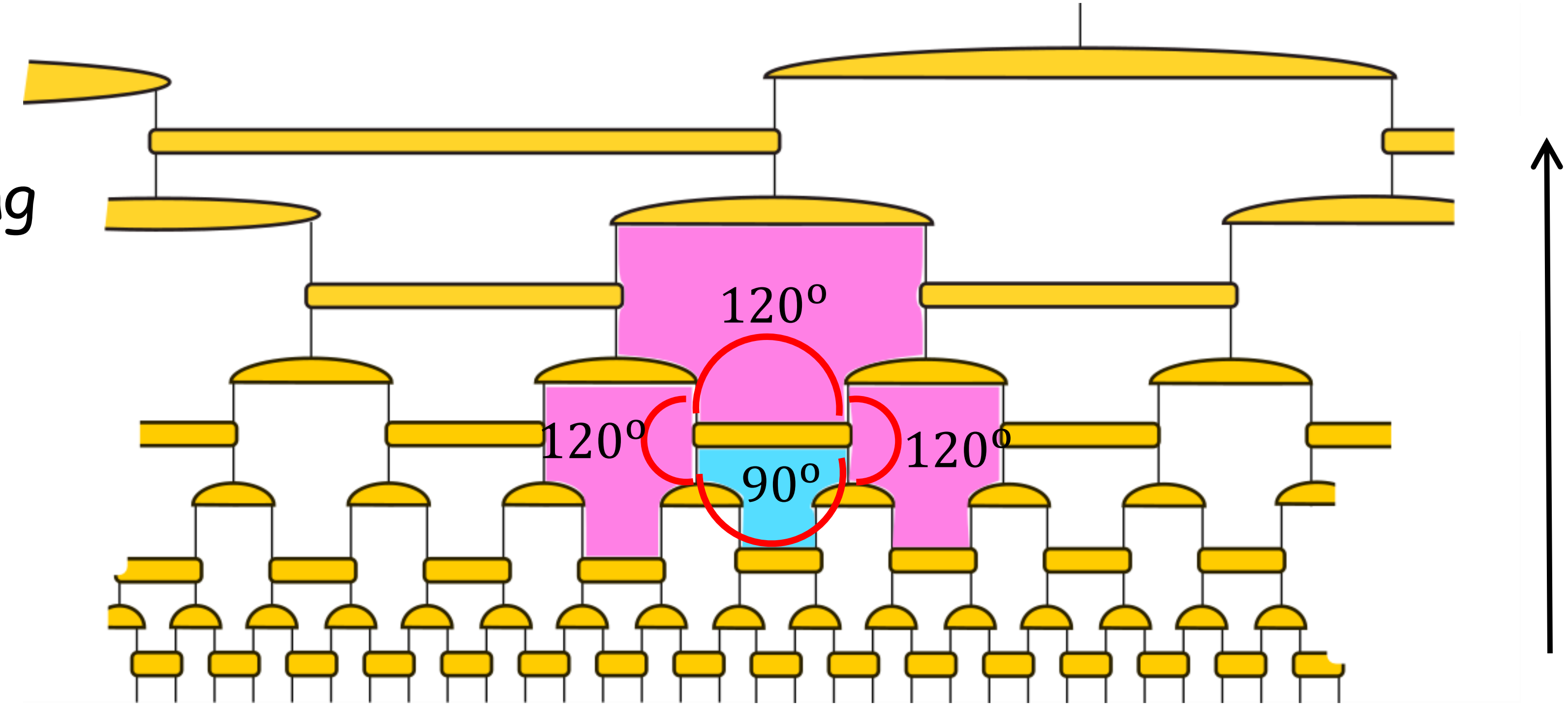

Extra dimension

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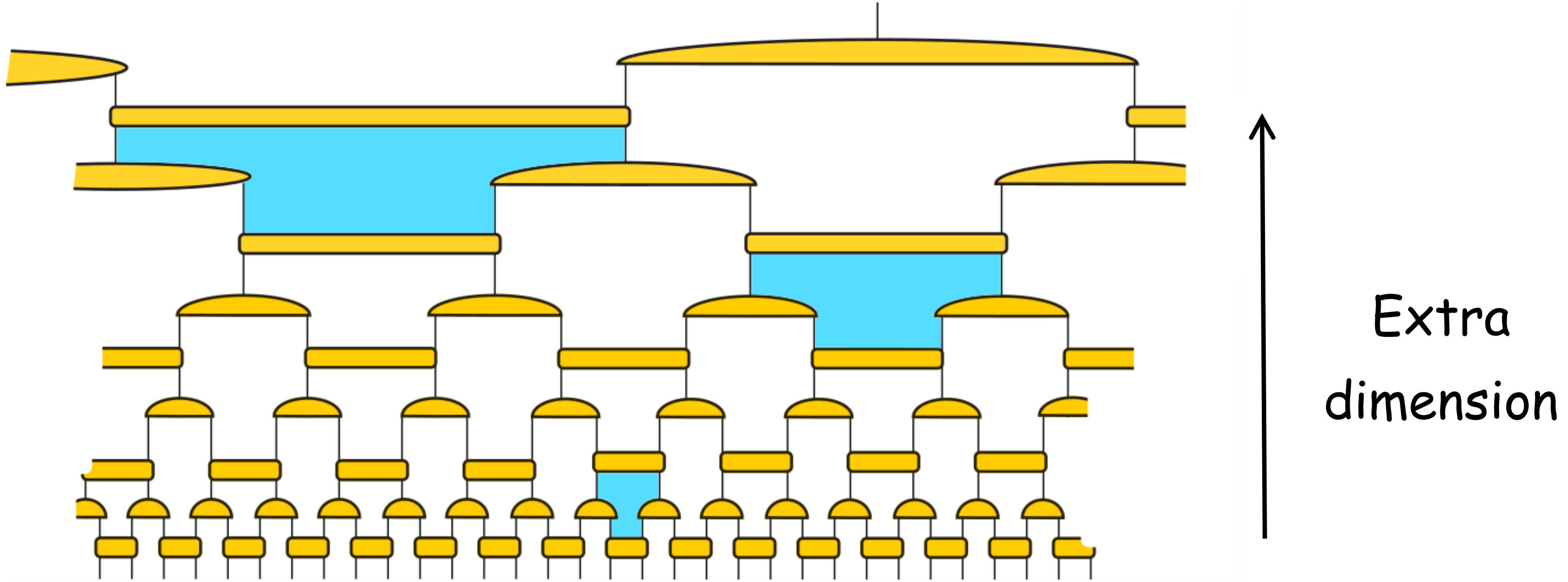

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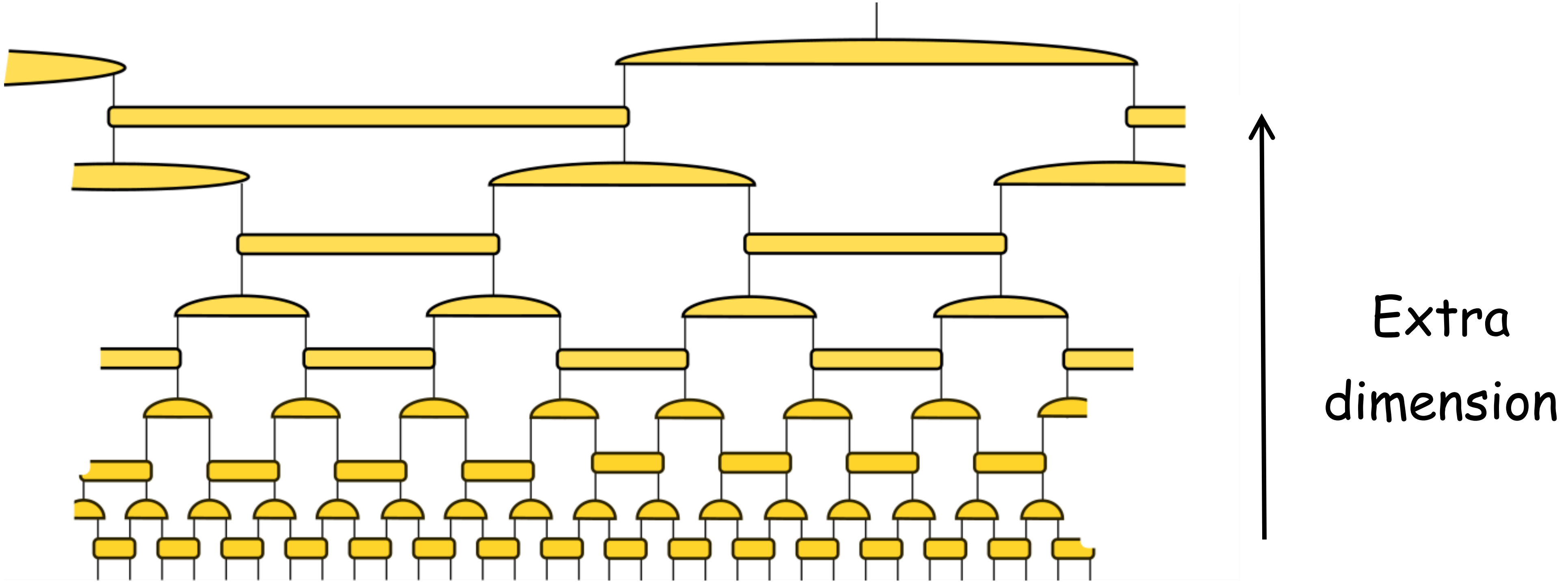
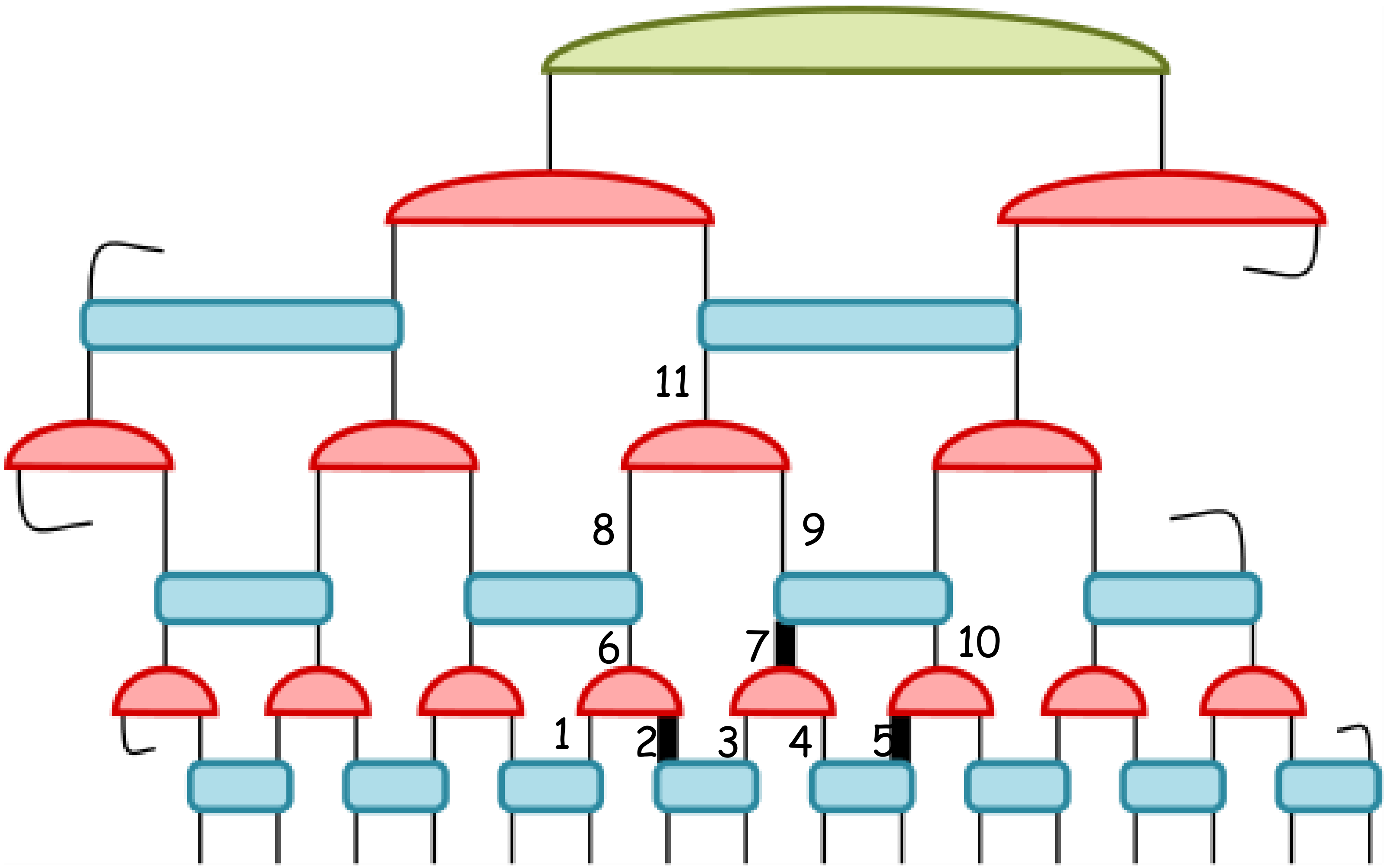


MERA is a good approximation of a 2d CFT

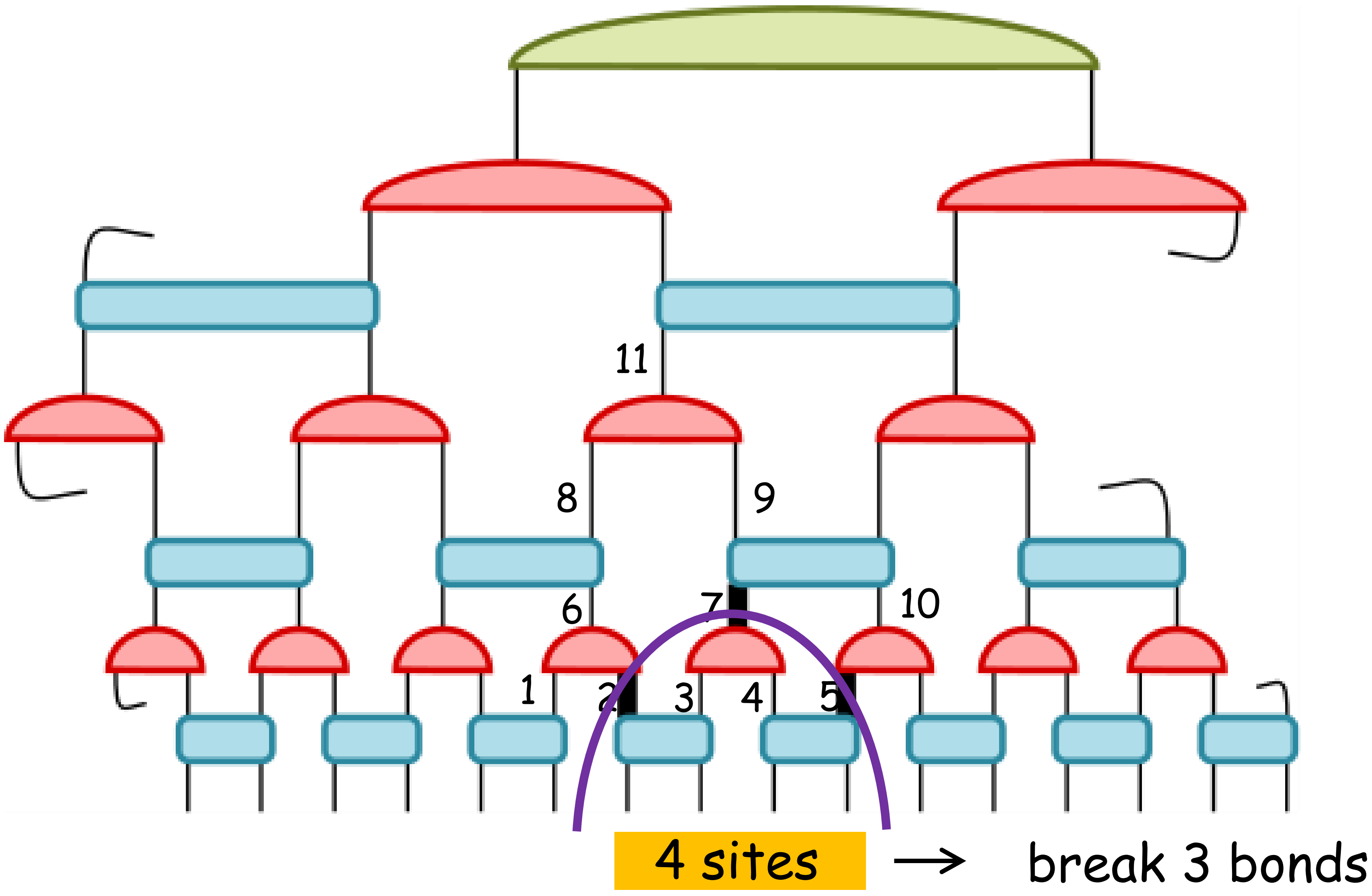
Has an extra dimension that corresponds to length scale

Is based on a negatively curved geometry

Bulk geodesics give bound on entanglement of intervals



4 sites → break 3 bonds



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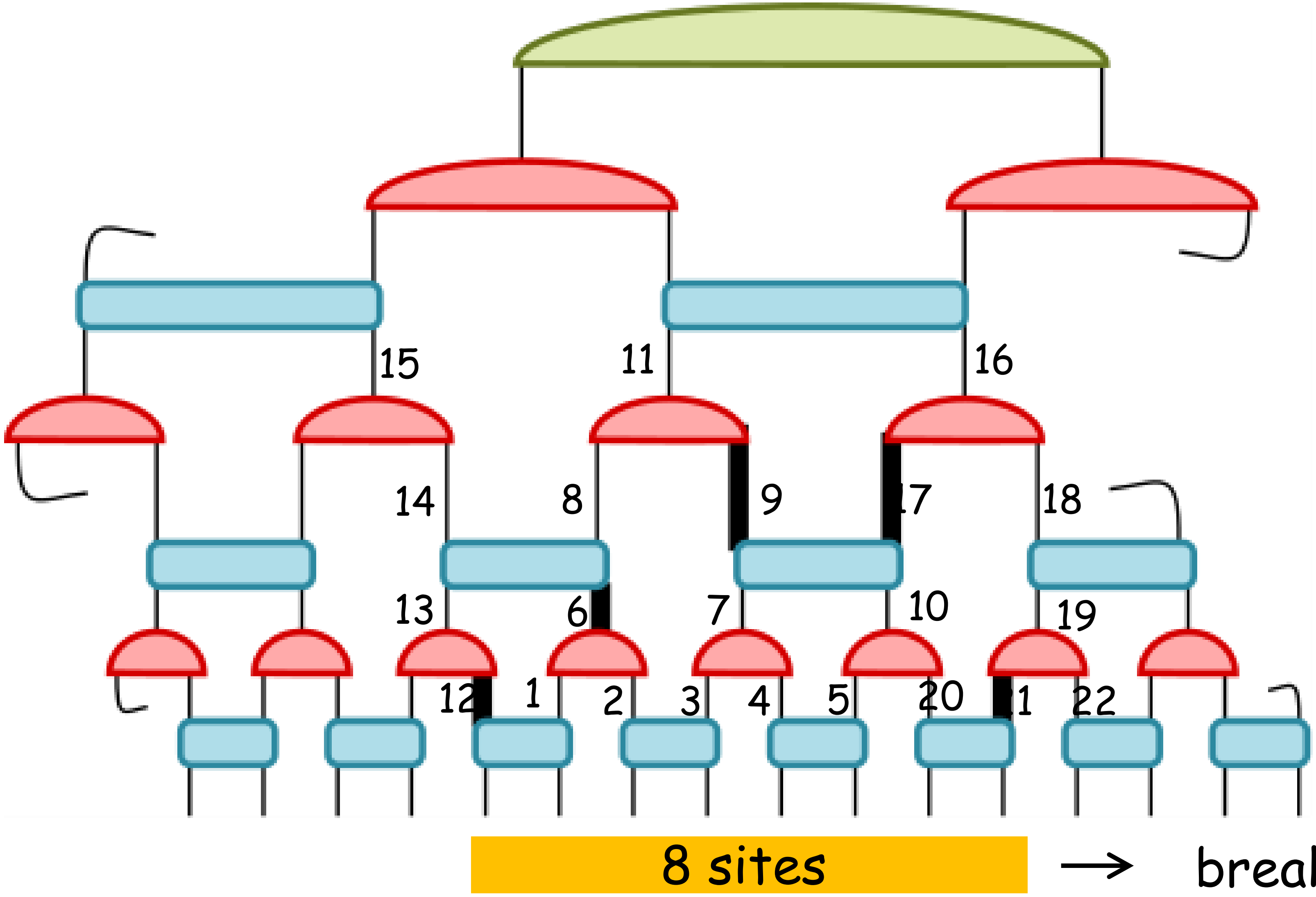
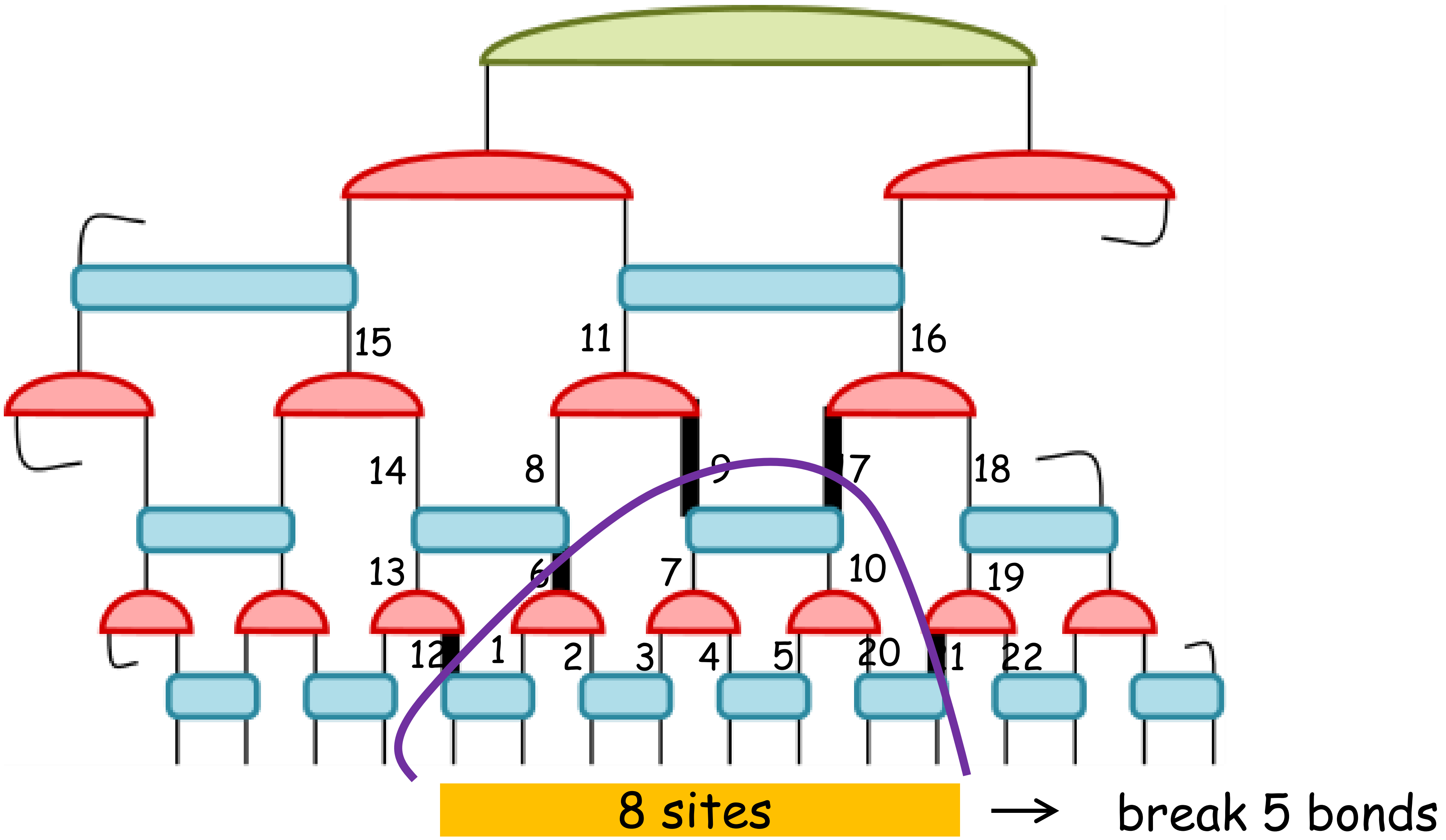
break 3 bonds



→ break 5 bonds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8 sites
break 5 bonds

But these are mostly qualitative comparisons

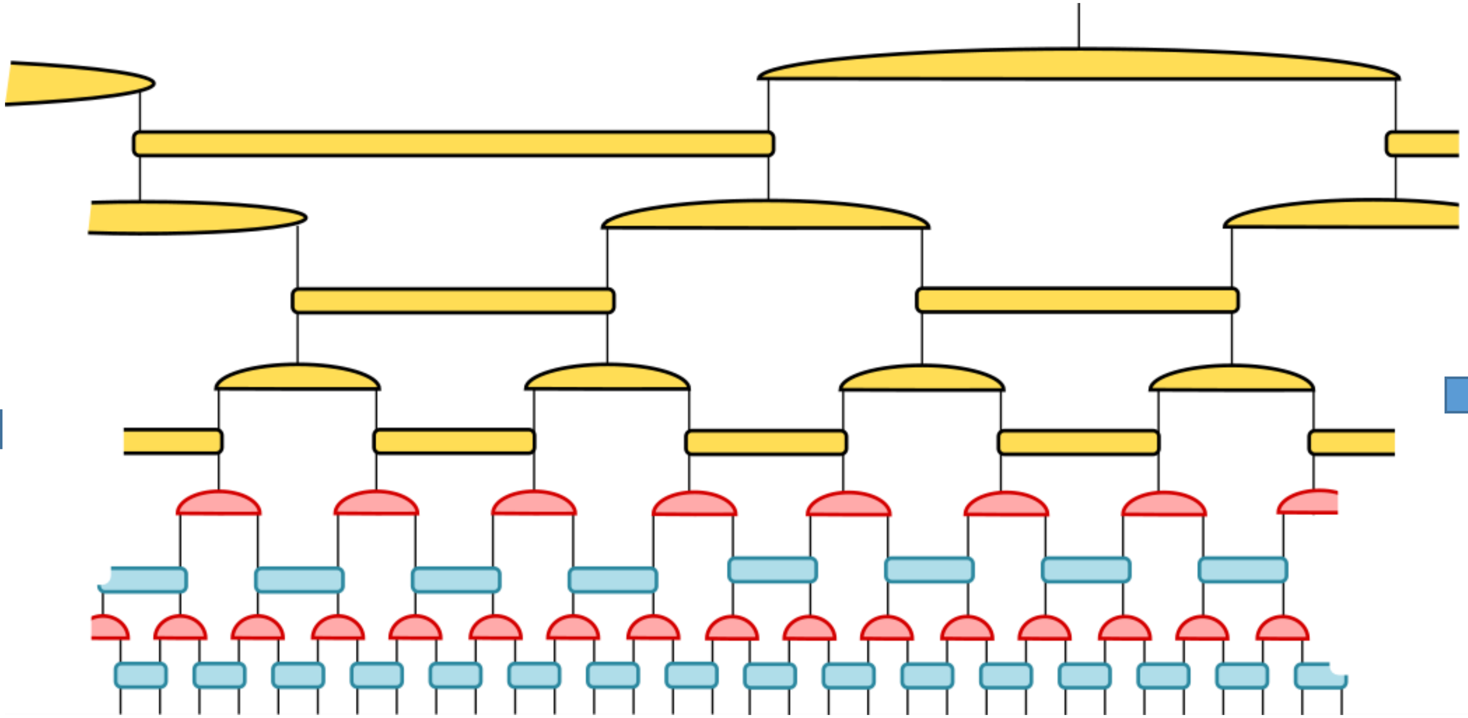
In particular, the emergent geometry should come from the tensors (state-dependent). (Recent proposal by Vidal in this direction)

Personal philosophy: Make the CFT side of the MERA exact by e.g. adding conformal symmetry. If there is any holography, it should pop out.

Adding translation symmetry may be a good first step
(in the tiling view, lack of translation invariance comes from the fact that MERA is an aperiodic tiling of the plane).

Summary

"Natural" ansatz for 1d critical ground states



RG flow

Lattice version of 2d CFTs

May have something to do with AdS/CFT

THANKS!

References

1) G. Vidal, PRL (2007), arXiv:cond-mat/0512165

2) G. Vidal, PRL (2008), arXiv:quant-ph/0610099

3) G. Evenbly and G. Vidal, PRB (2009), arXiv:0707.1454

4) R. N. C. Pfeifer, G. Evenbly, and G. Vidal, PRA (R) (2009), arXiv:0810.0580

5) Swingle, PRD (2012), arXiv:0905.1317