Tensor networks as tools for classifying 1D quantum phases of matter

Sukhbinder Singh
Max-Planck Institute for Gravitational Physics, Potsdam
(Gravity, Quantum Fields, and Information Group)

Based on: SS, N McMahon, G Brennen, 1812.08500
PRB **99** 195139 (2019)

1D lattice + global on-site symmetry G

$$U_g U_h = U_{g.h}$$

Chen, Gu, and Wen, PRB (2011)
Schuch, Perez-Garcia, and Cirac, PRB (2011)
Pollmann, Berg, Turner, and Oshikawa, PRB

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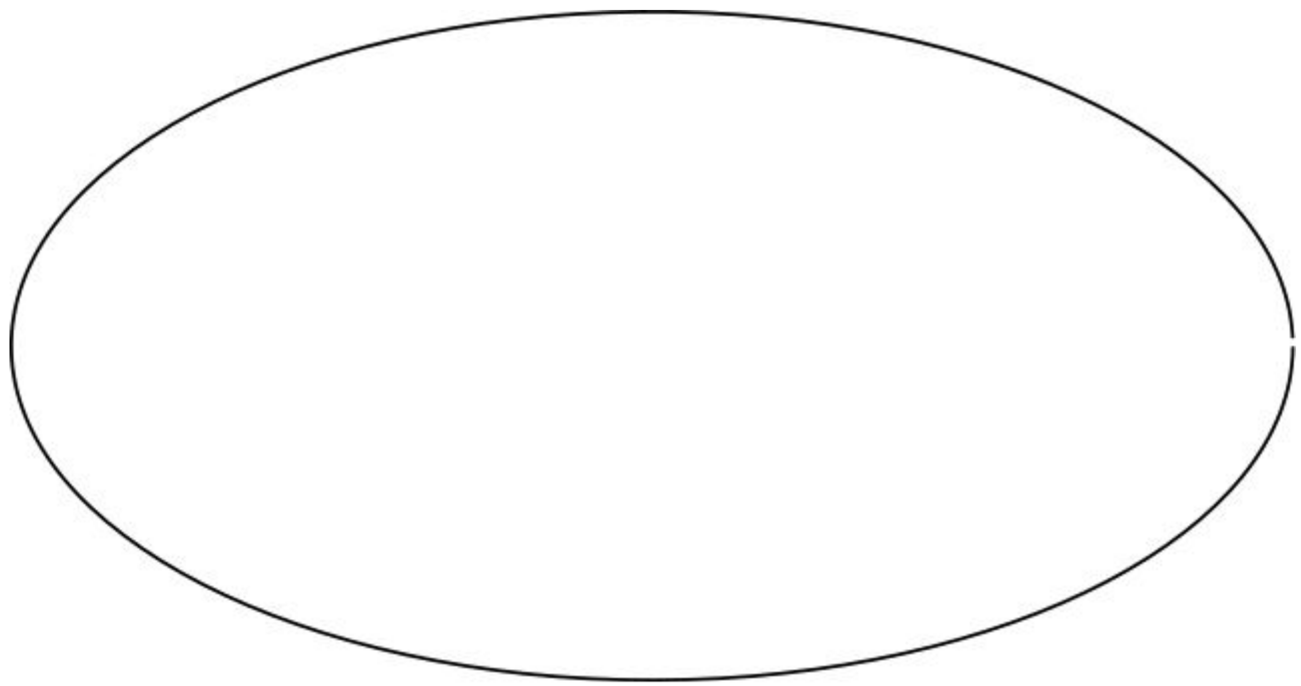

Space of local Hamiltonians with symmetry G

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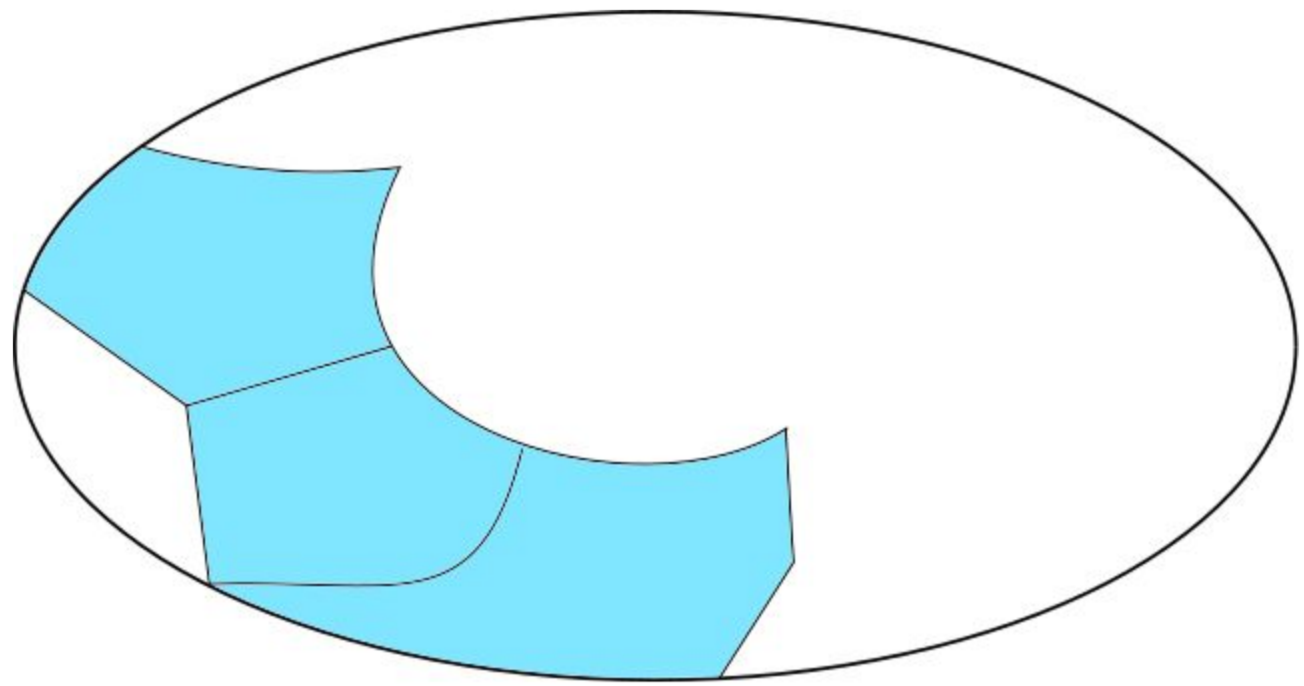
symmetry broken phase



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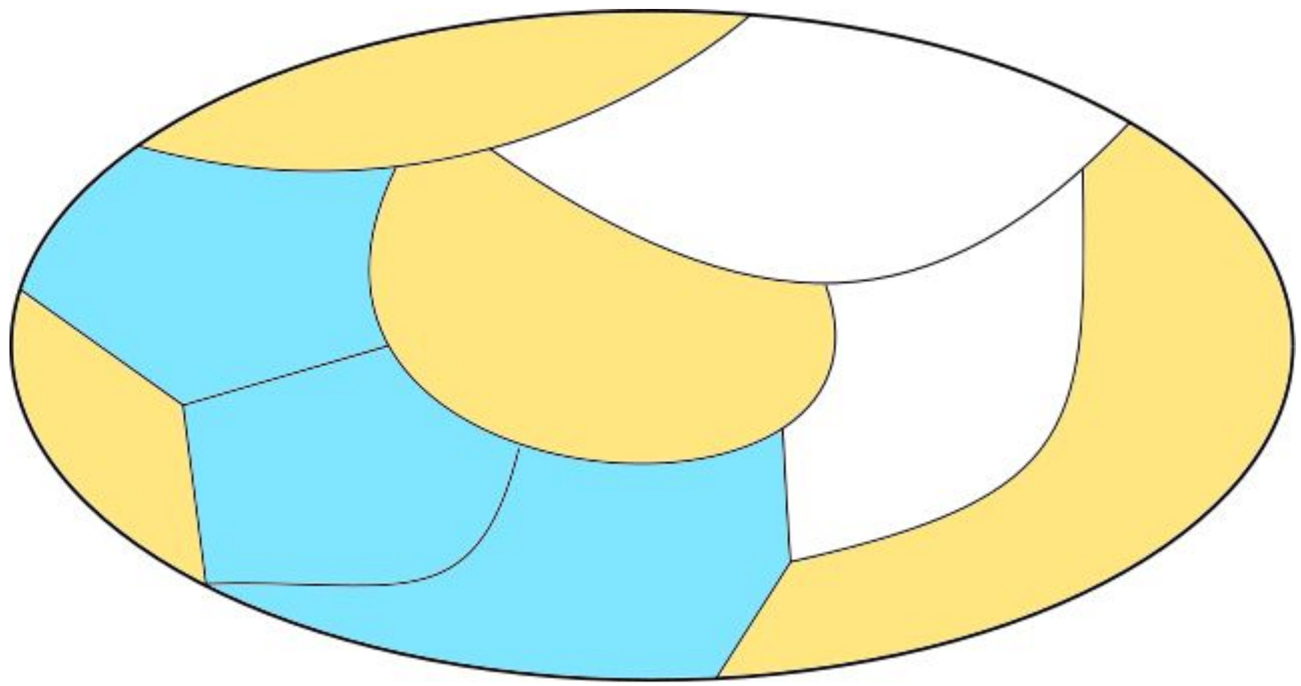

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Projective representations

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1D Symmetry protected phases

$\leftrightarrow$

Equivalence classes of projective representations

Chen, Gu, and Wen, PRB (2011)
Schuch, Perez-Garcia, and Cirac, PRB (2011)
Pollmann, Berg, Turner, and Oshikawa, PRB (2012)

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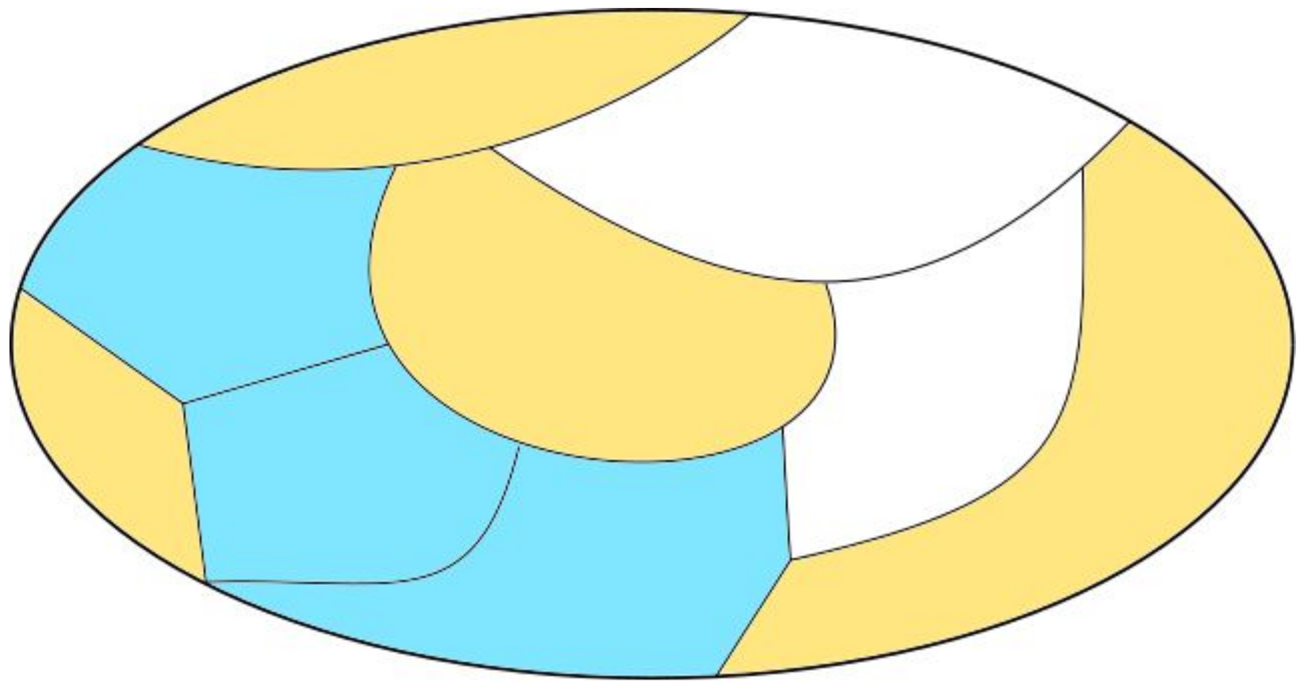

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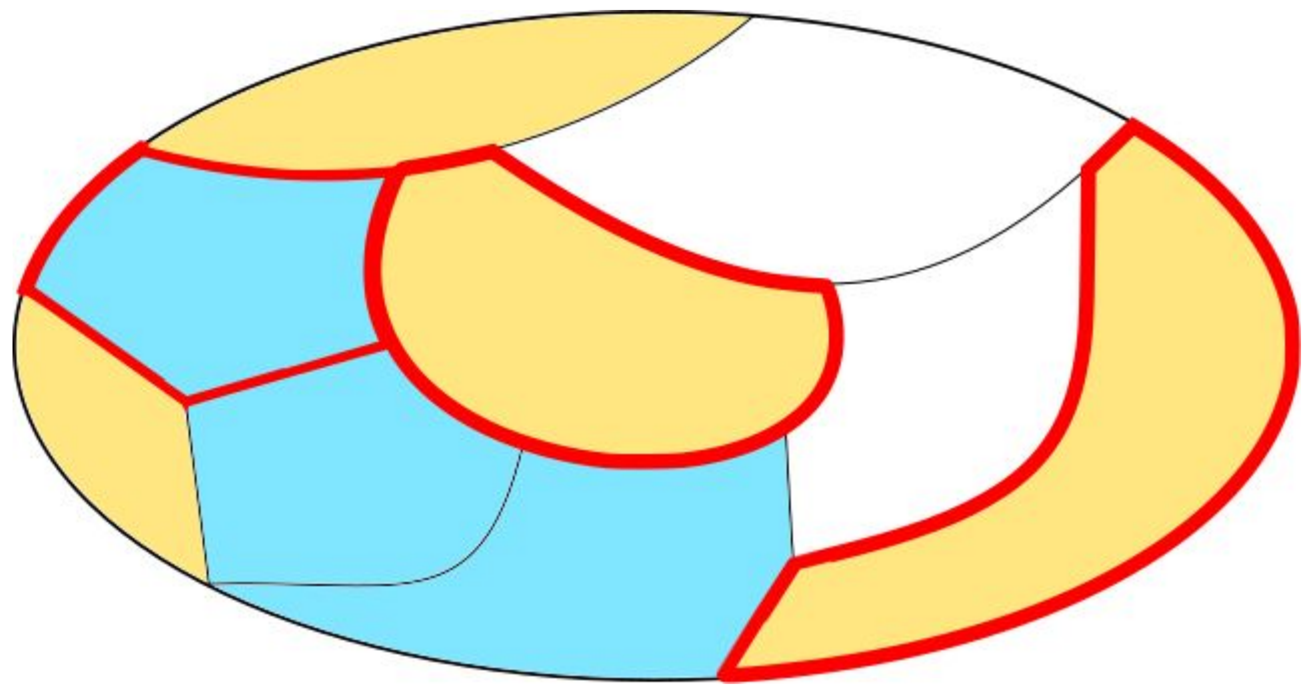

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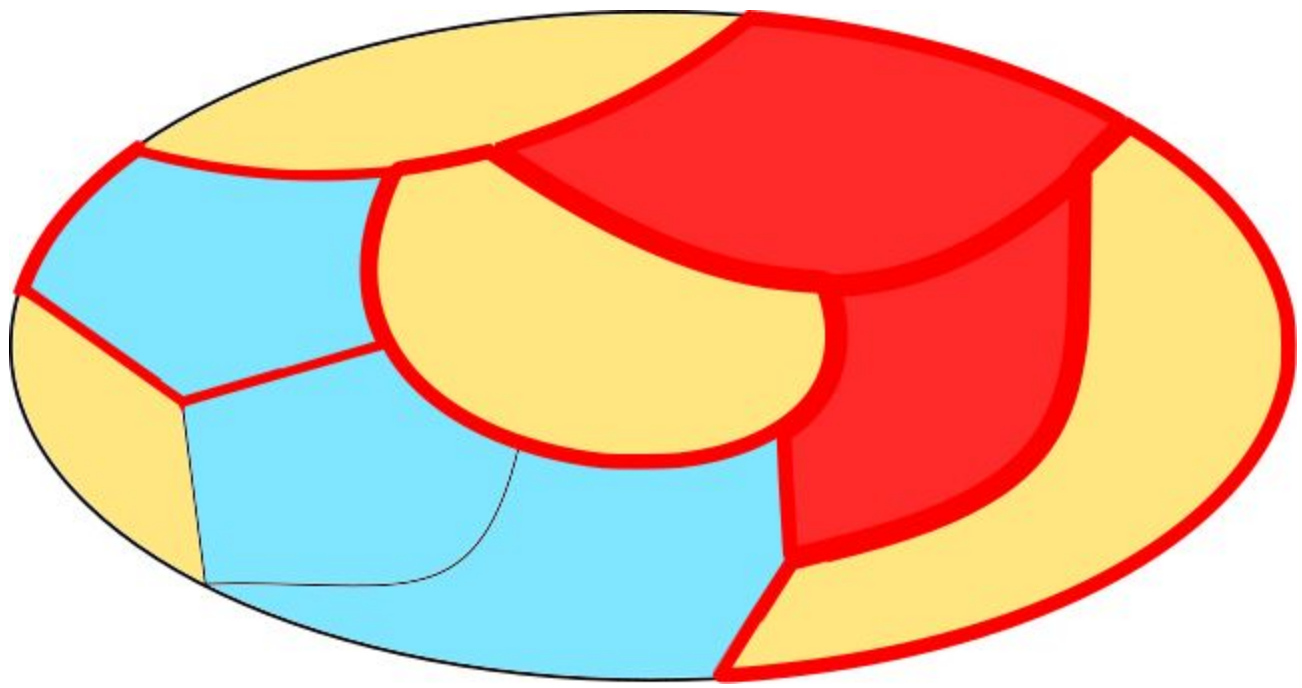

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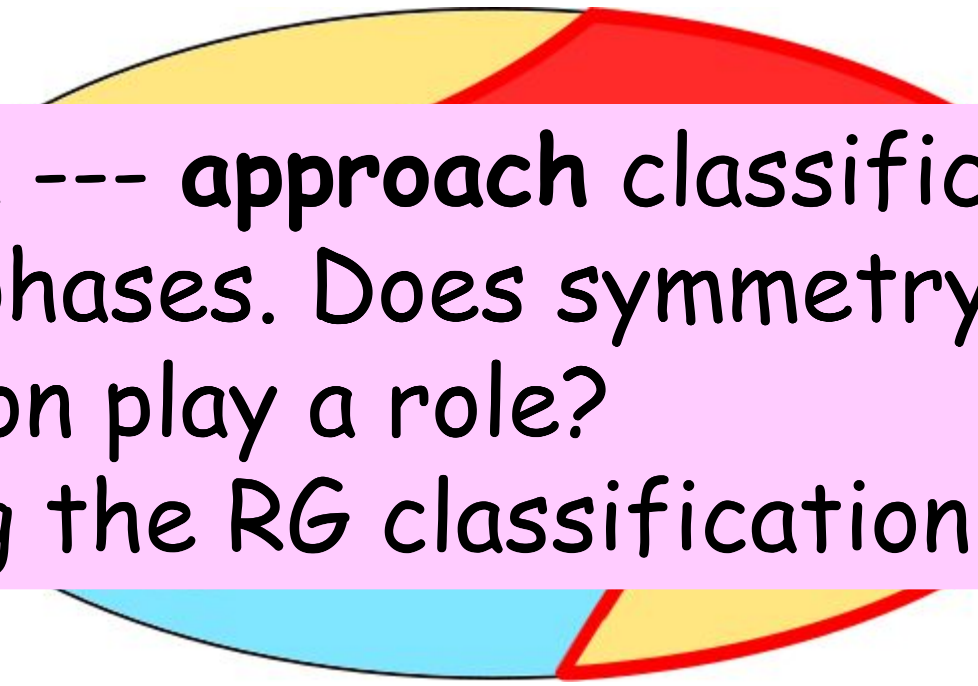

This talk --- **approach** classification of critical phases. Does symmetry protection play a role?
But using the RG classification of phases

Space of local Hamiltonians with symmetry G

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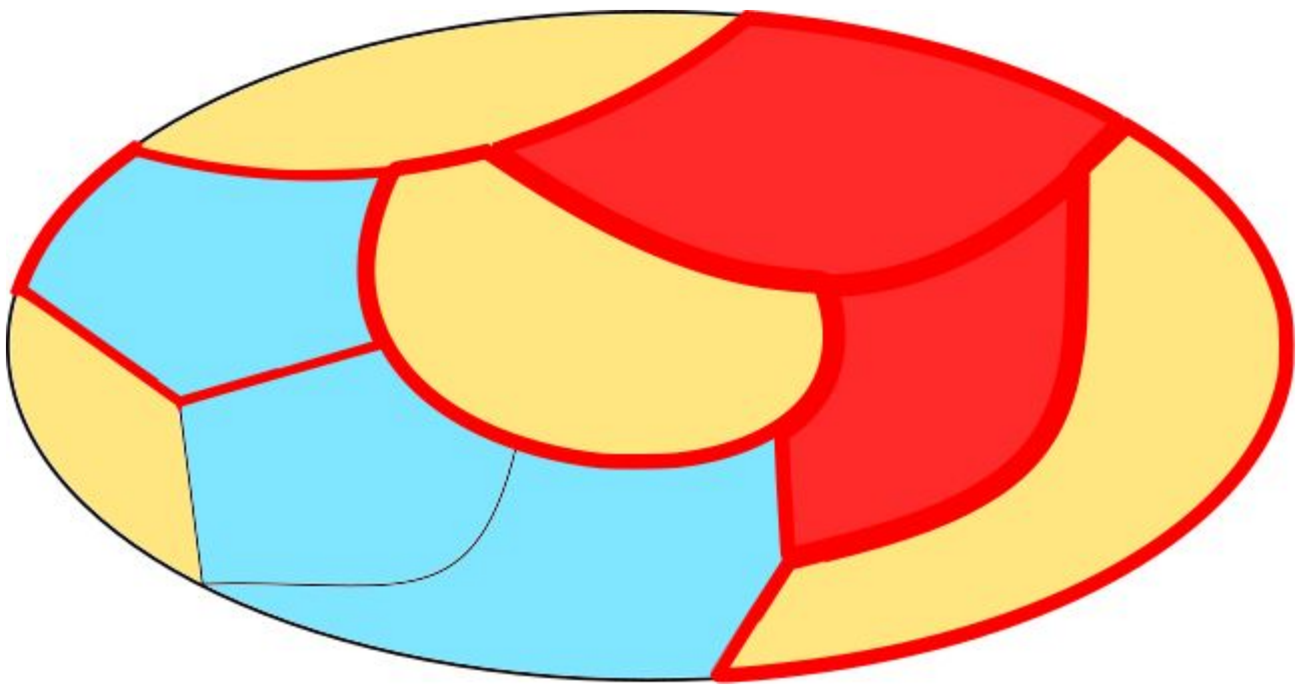
RG perspective on phases

Different phases essentially differ in large length scale properties



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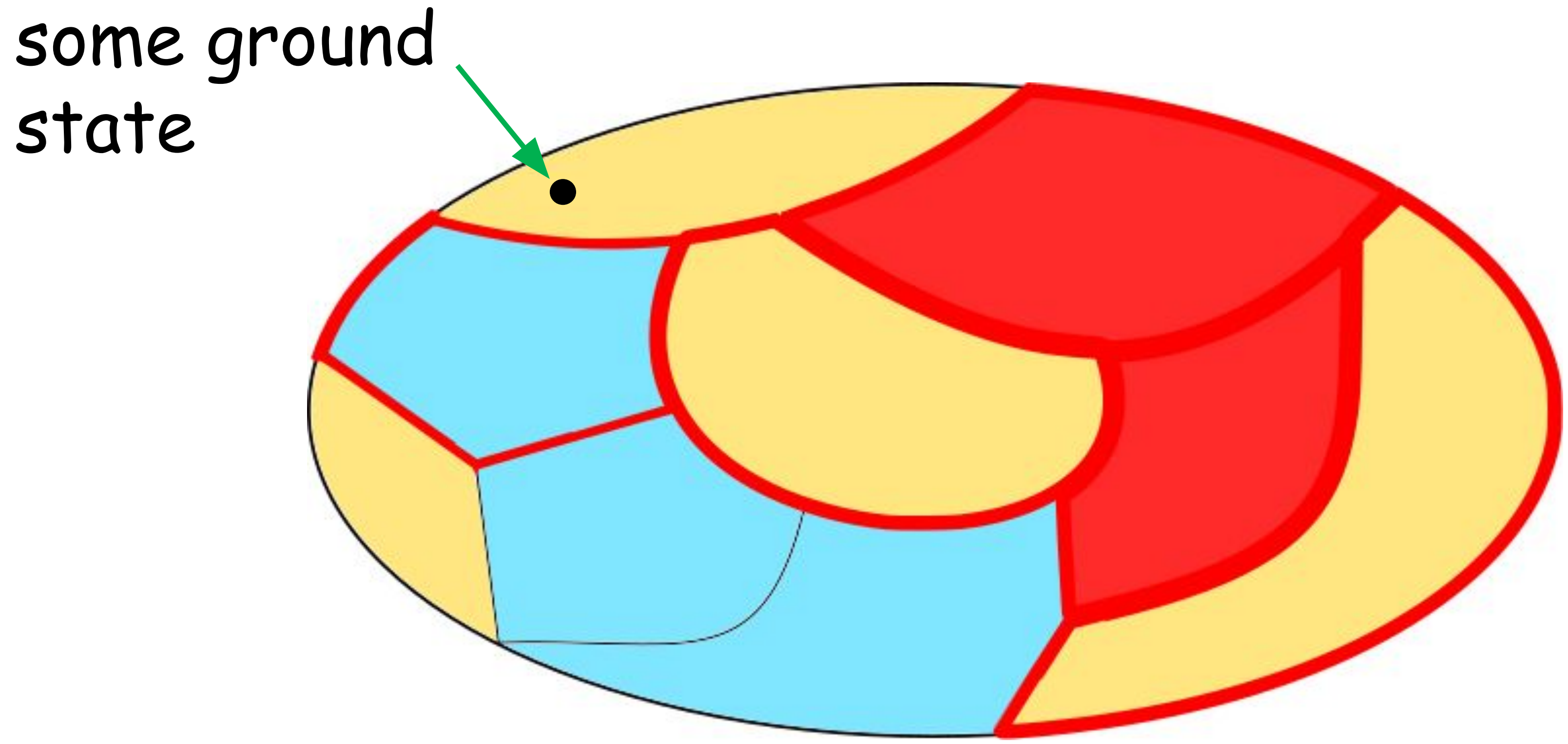

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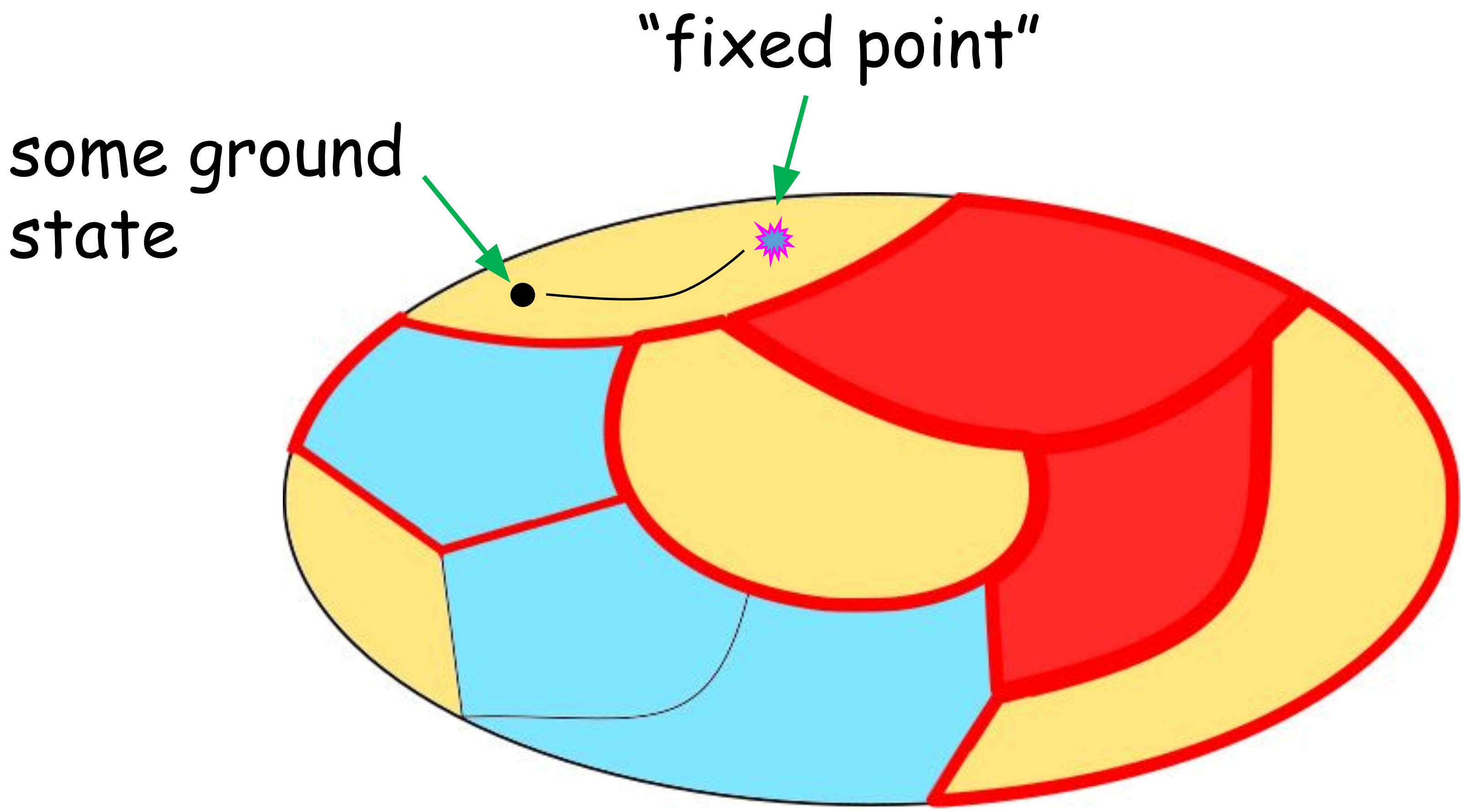

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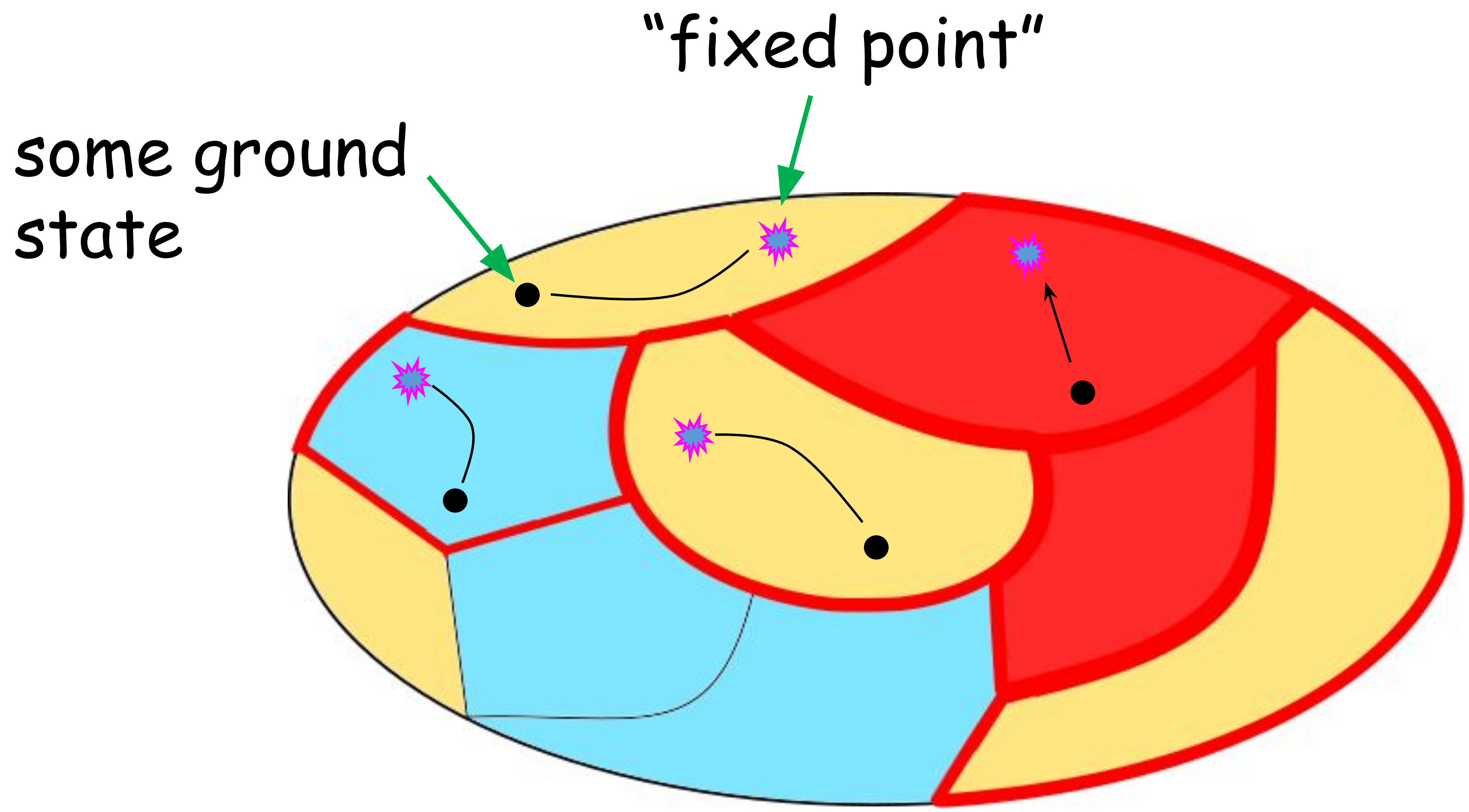

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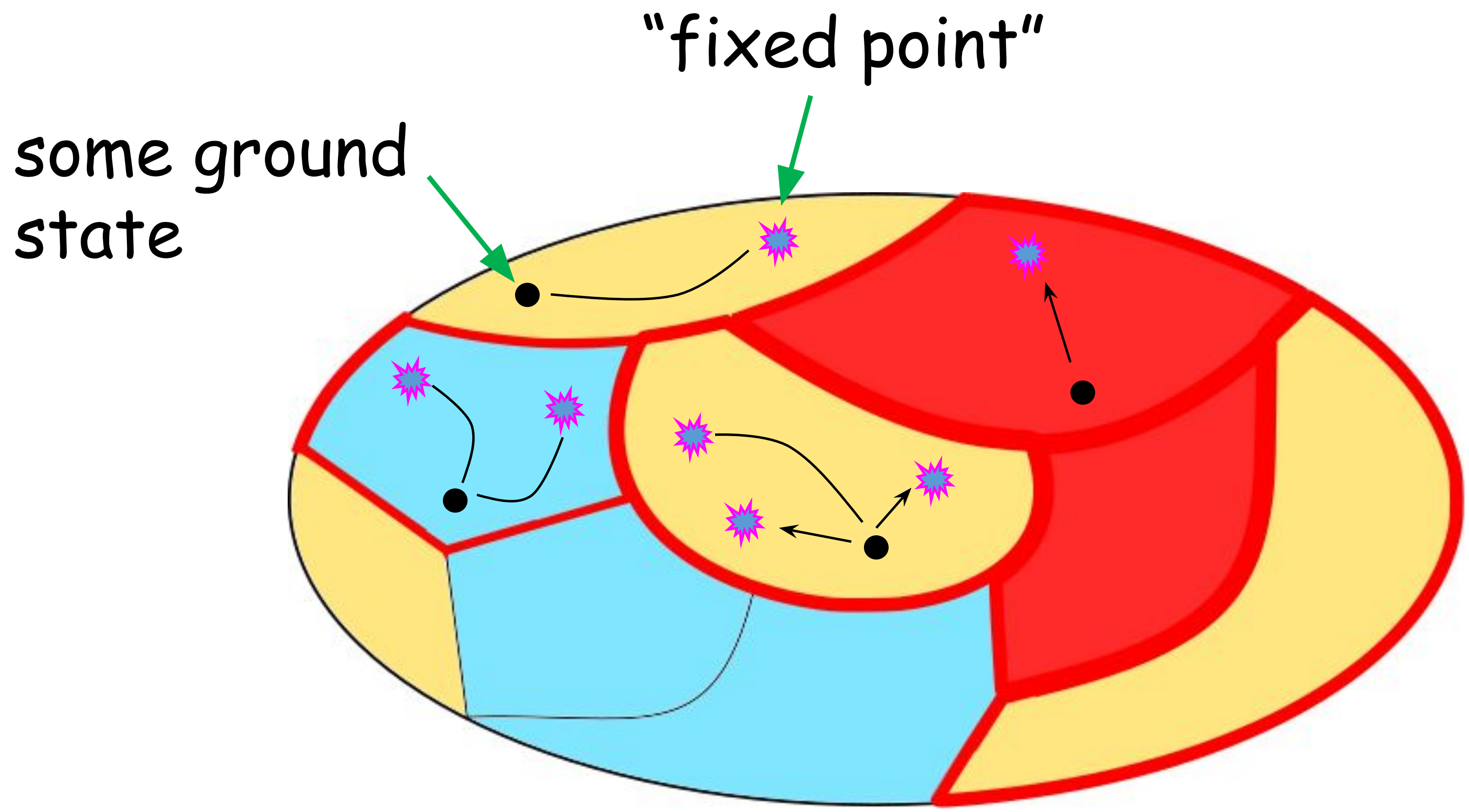

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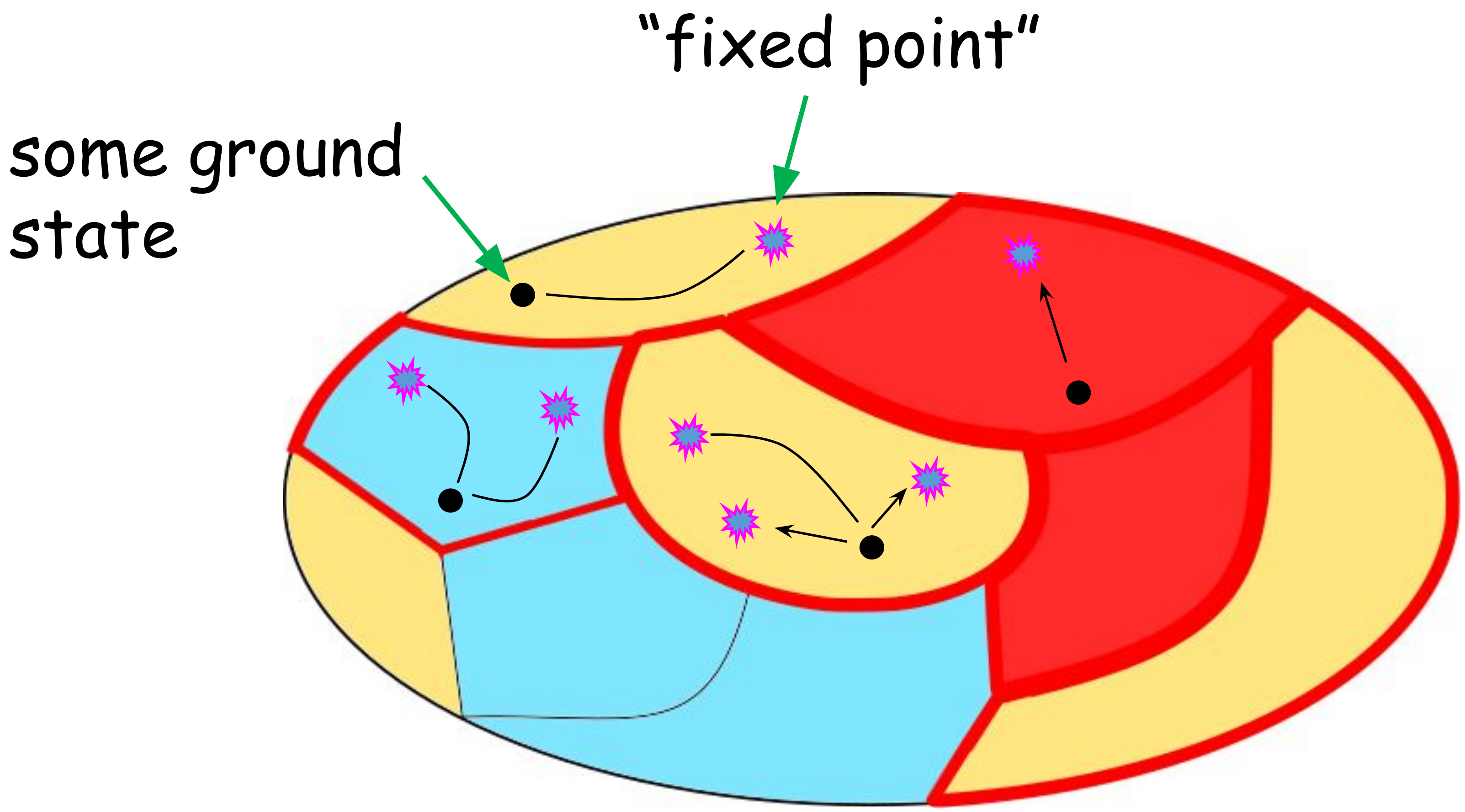

1. States in distinct phases must flow to distinct fixed points.

RG perspective on phases

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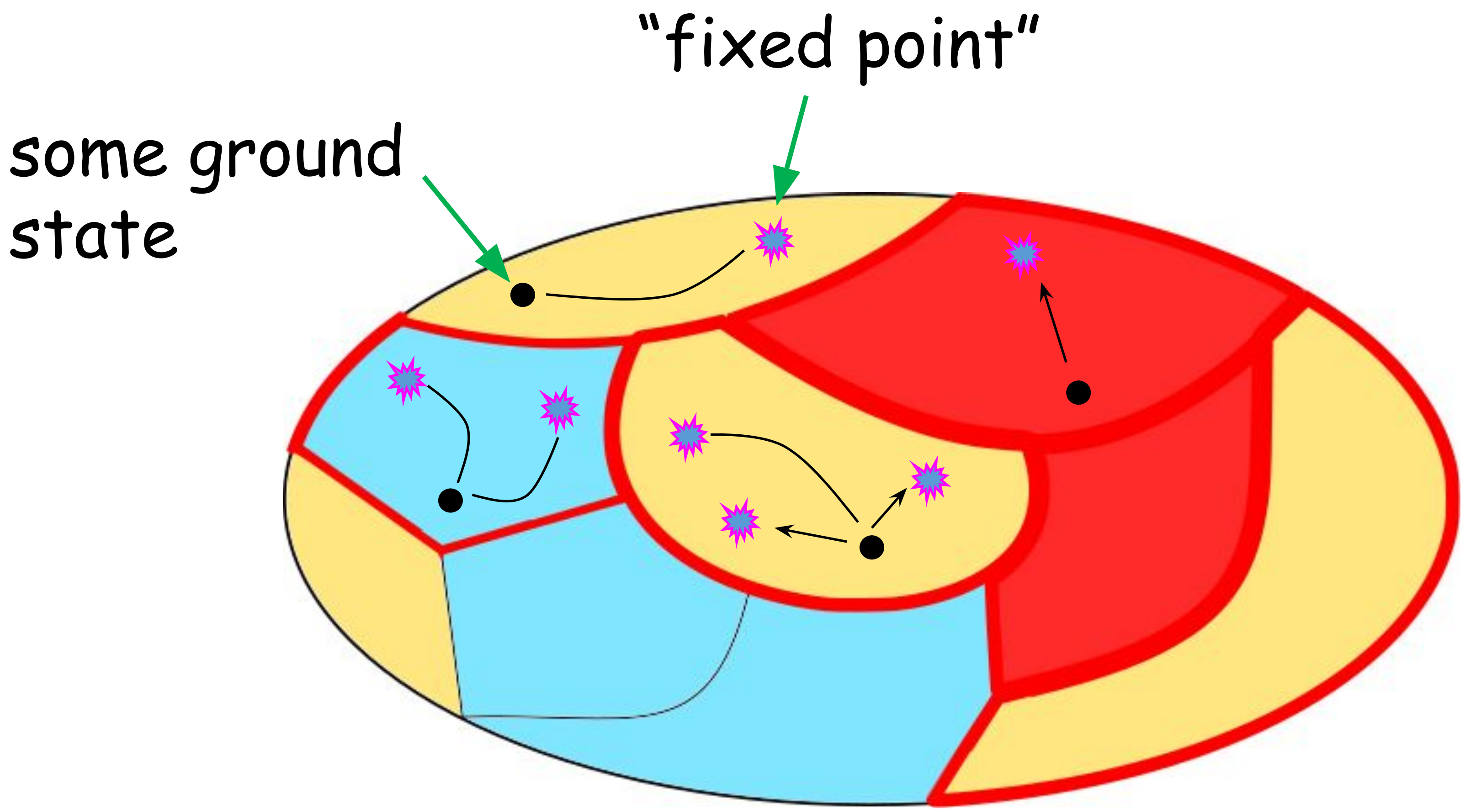

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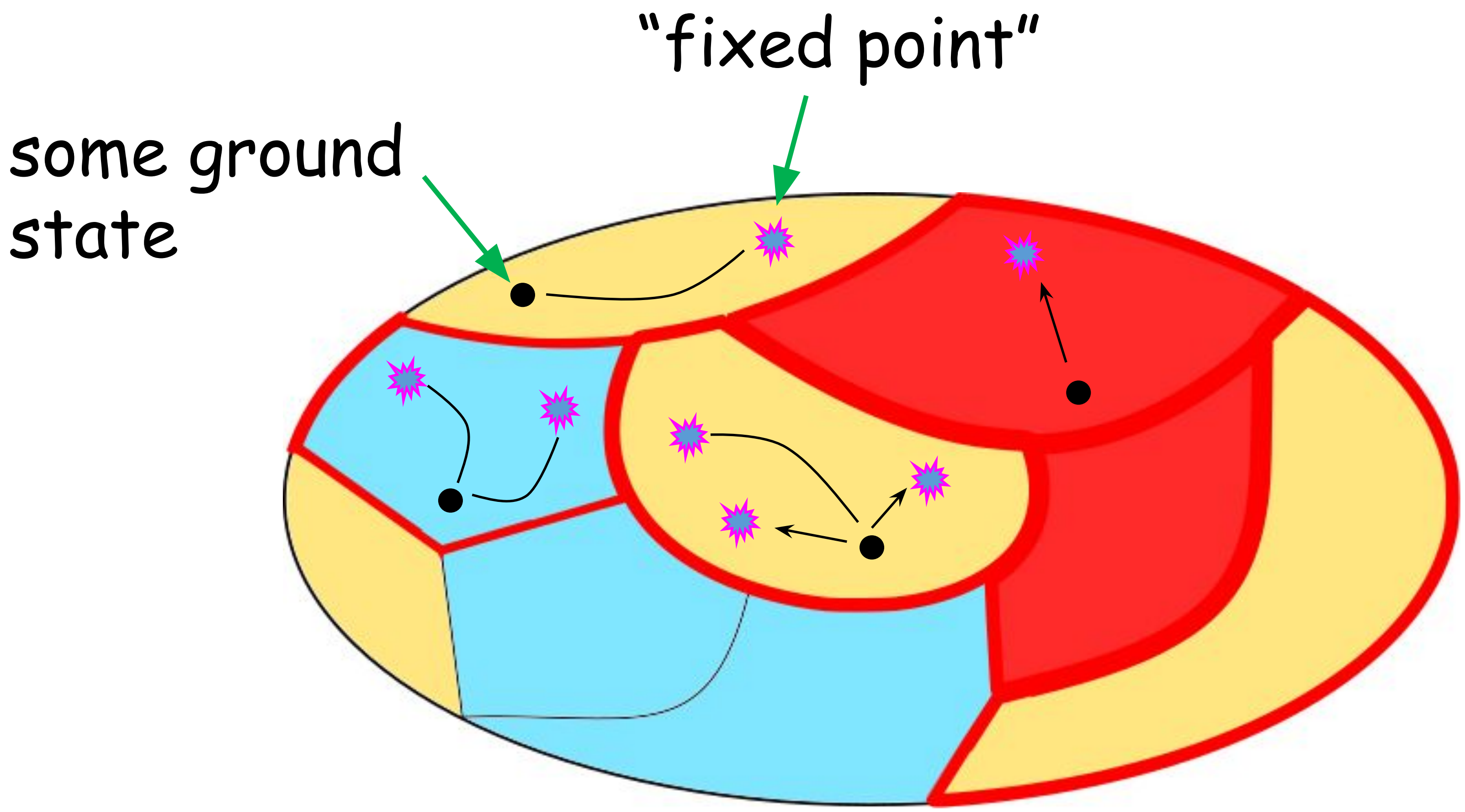

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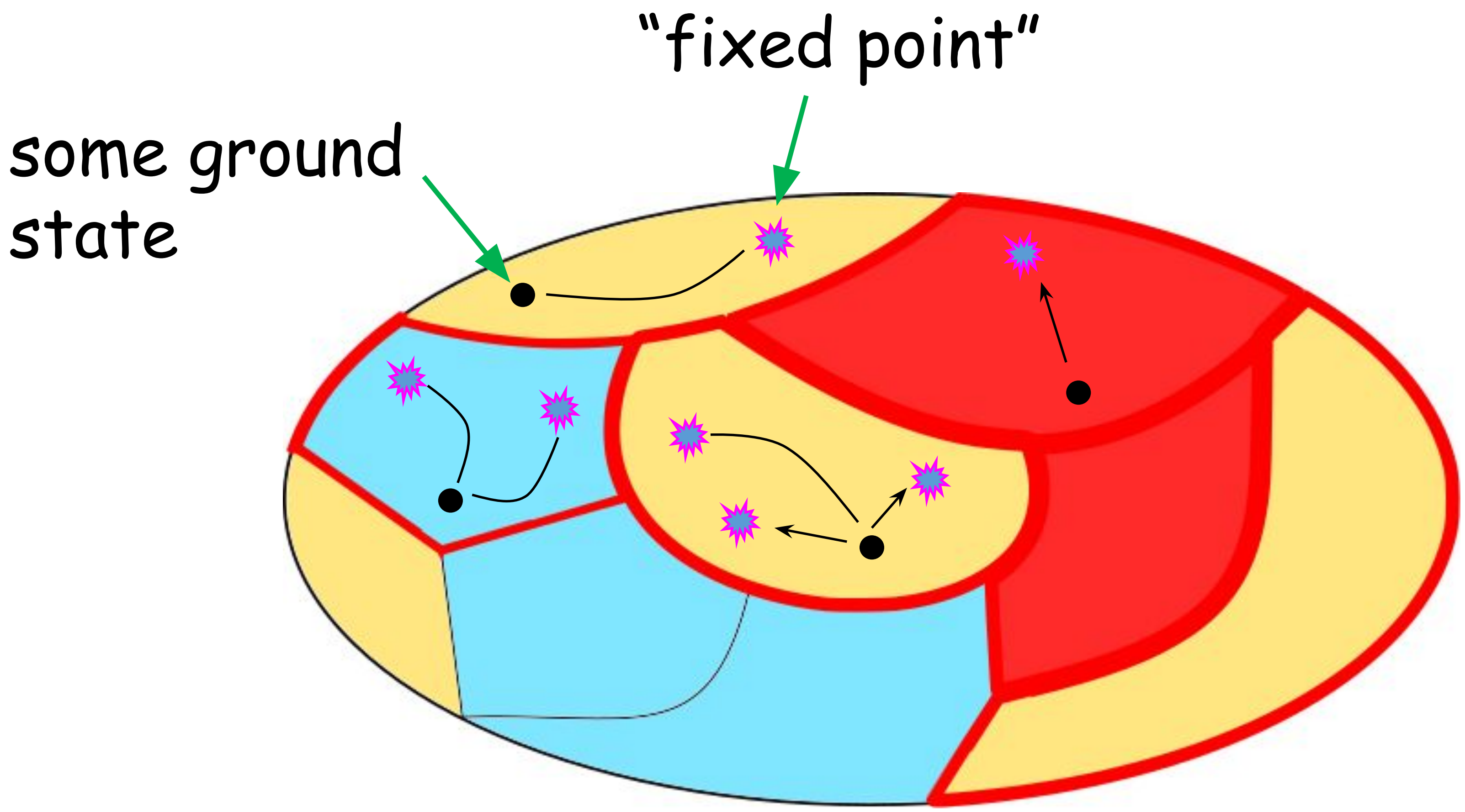
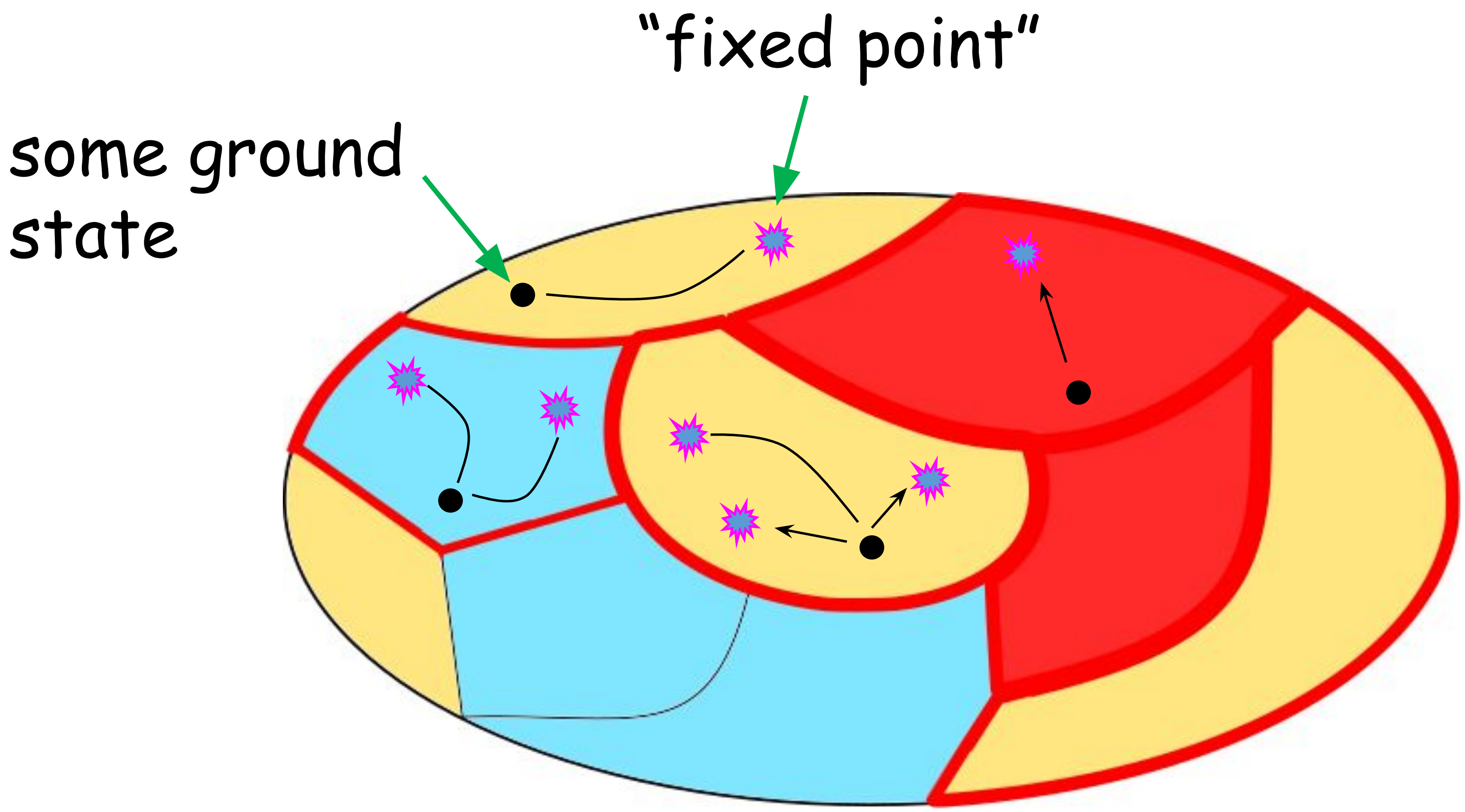


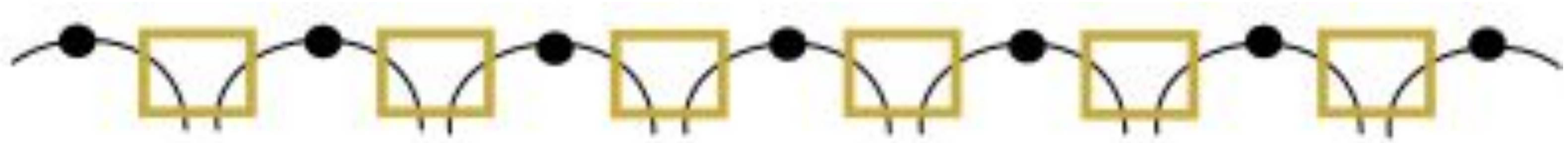
1. States in distinct phases must flow to distinct fixed points.
2. Gapped states must flow to short-range entangled fixed points.
3. Critical states must flow to long-range entangled fixed points.

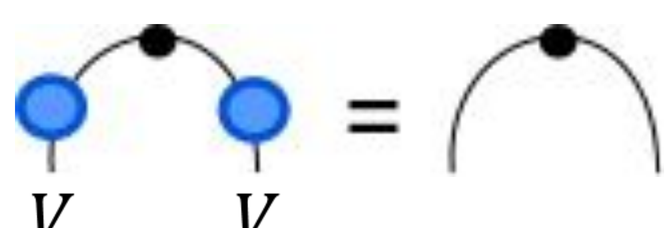
4. Symmetry protected phases flow to distinct fixed points only if the symmetry is protected along the RG.

RG perspective on phases

Different phases essentially differ in large length scale properties



$\left|\Psi_\phi^{\text{fixed}}\right\rangle \equiv$ 



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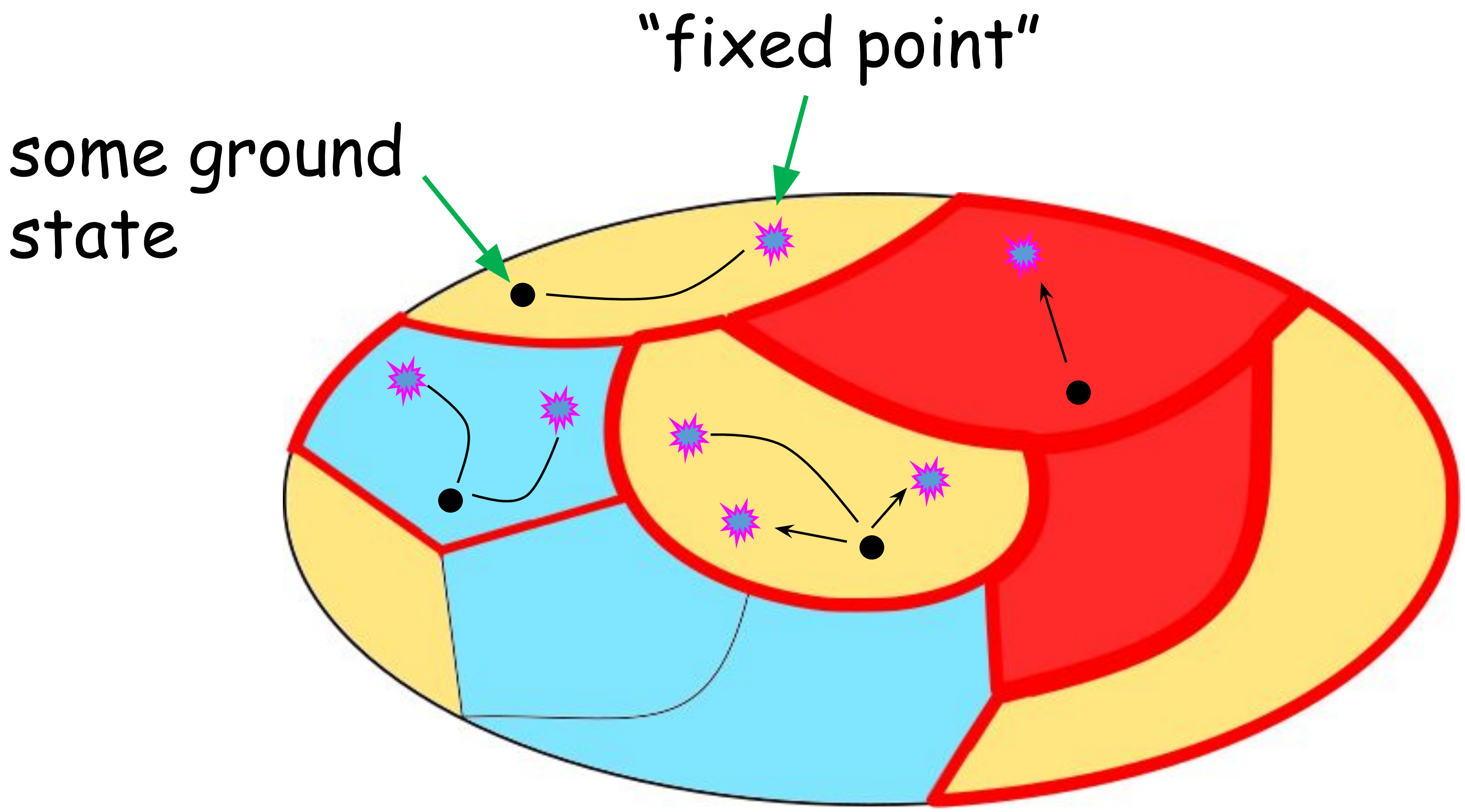

This talk --- approach classification of phases using specific symmetry protected RG flows, which are described by tensor networks

Outline

1. Tensor network representations of 1D ground states
   1.1 MPS
   1.2 MERA

2. MPS and 1D gapped symmetry protected phases

3. Symmetry protected MERA
   3.1 Gapped SP phases
   3.2 Critical SP phases

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Matrix product states (MPS)

1D gapped ground state

$$|\Psi\rangle = \sum \Psi_{i_1 i_{2 \ldots} i_N} \; |i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_N\rangle$$

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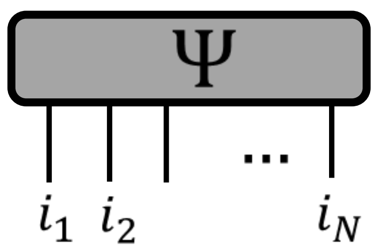

(tensor)

Limited correlations and entanglement

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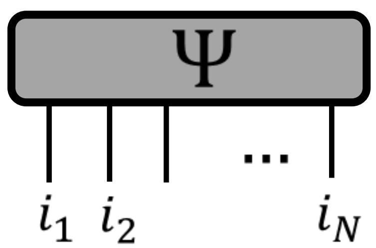

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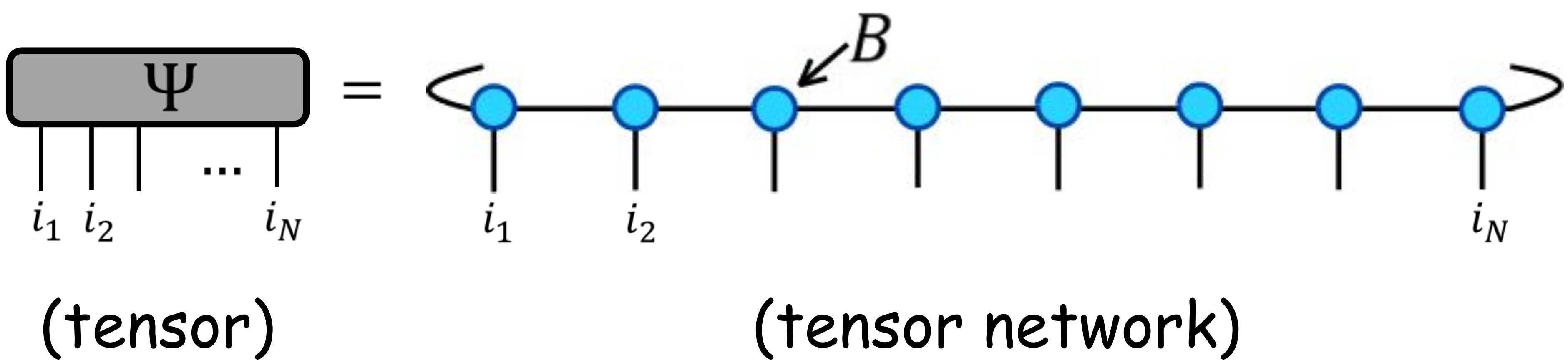

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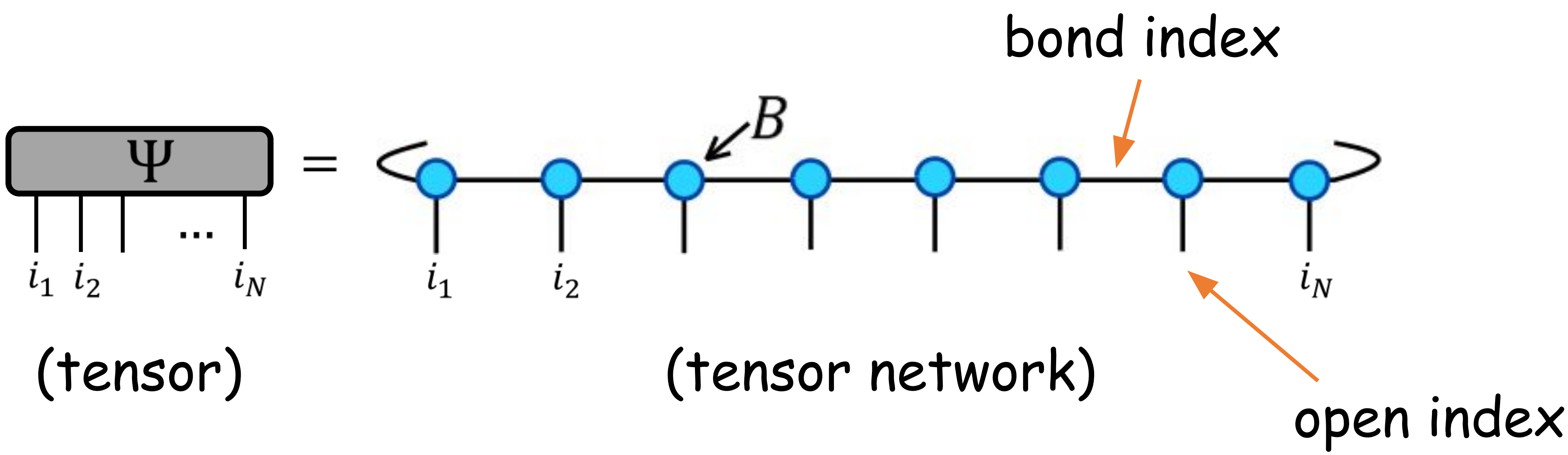

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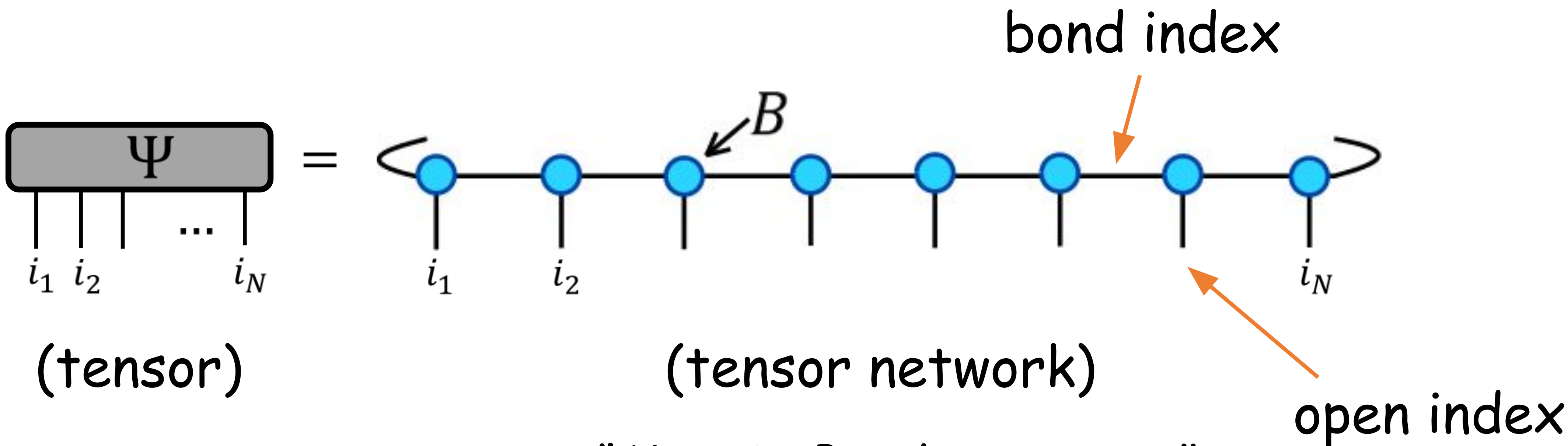

"Matrix Product State"

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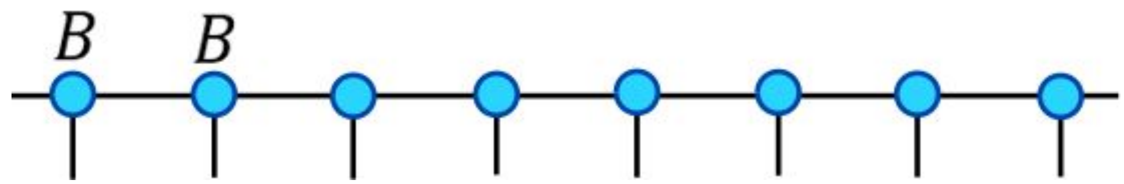
MERA

Based on a coarse-graining transformation

1D gapped ground state on an infinite lattice



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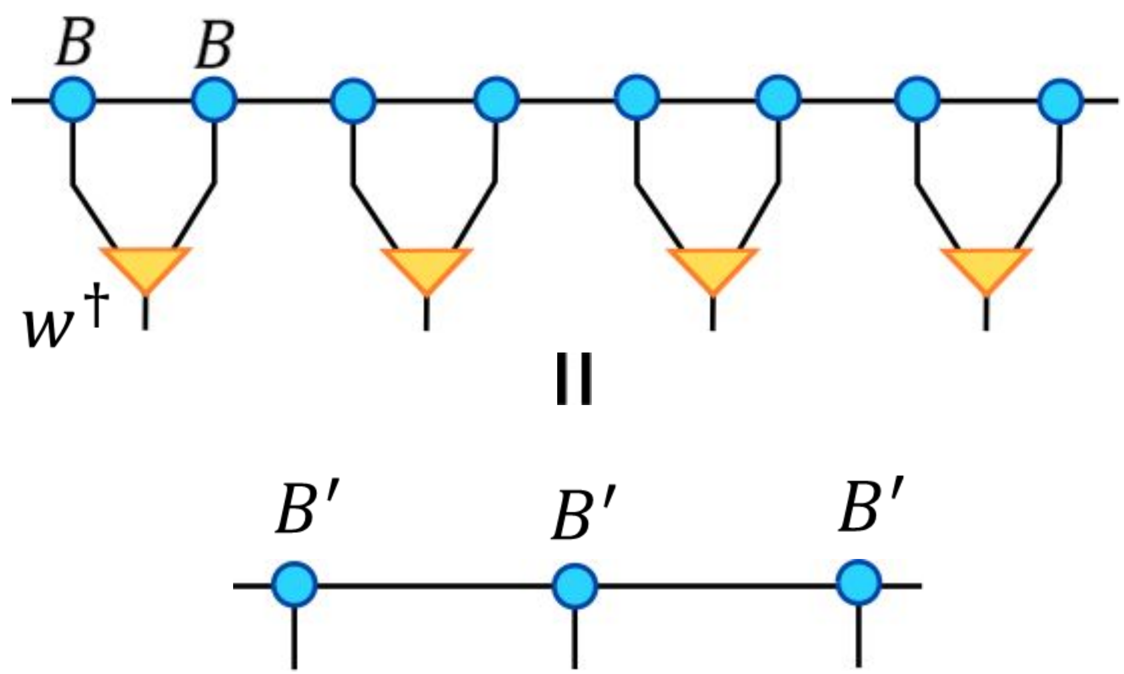

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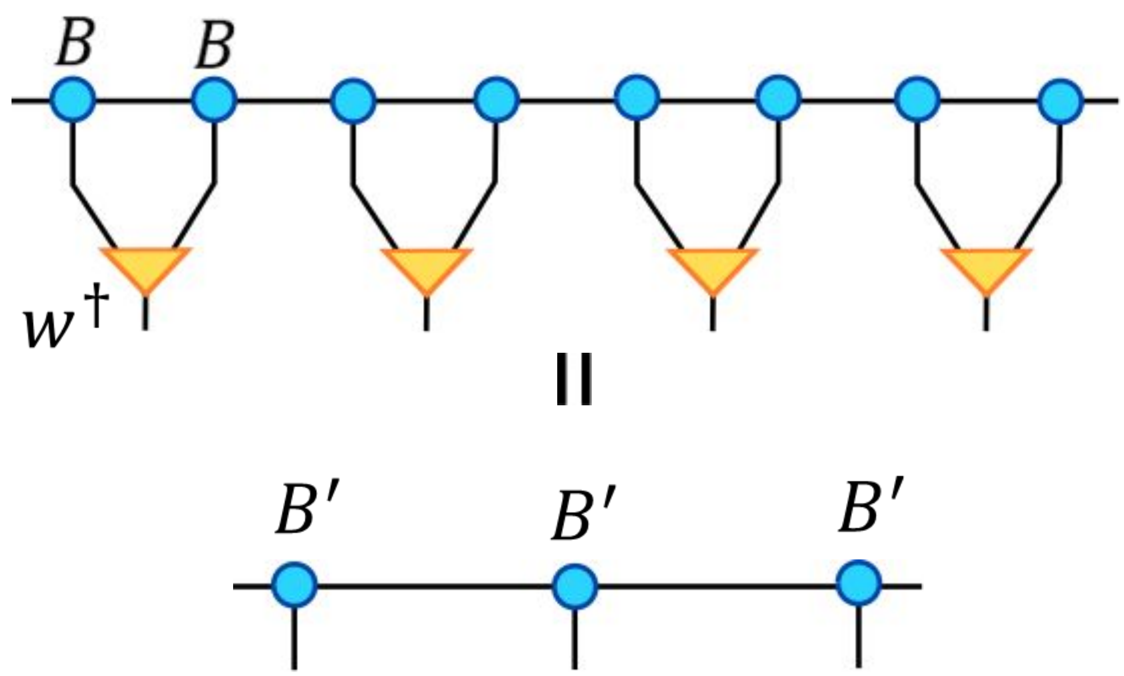

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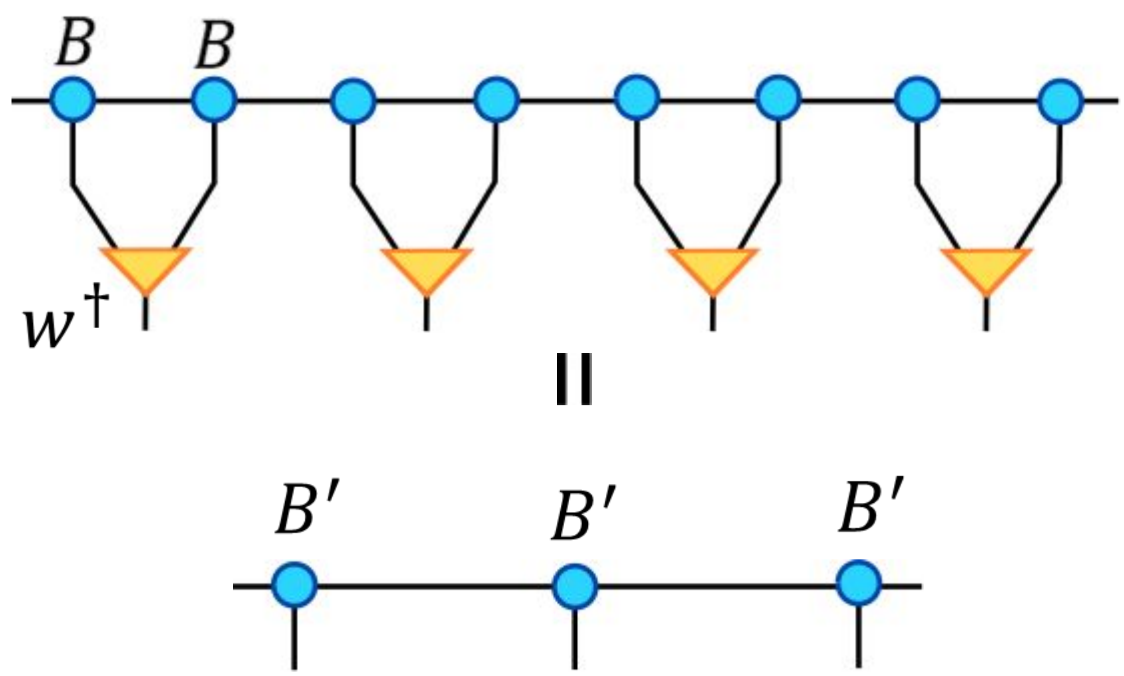
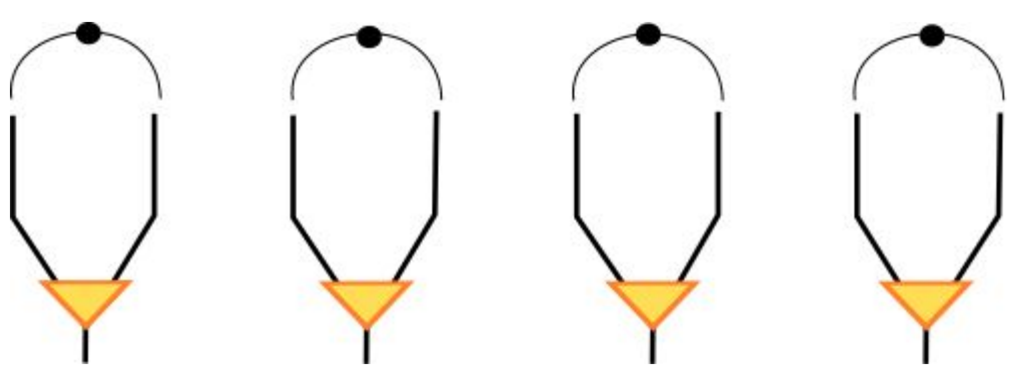
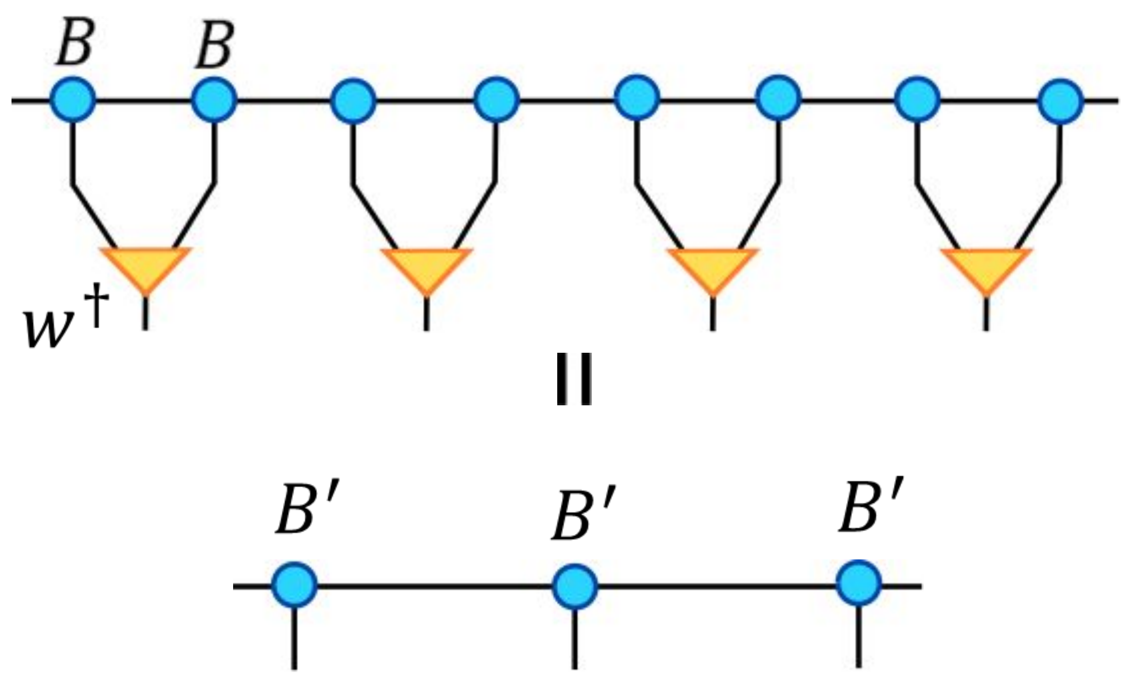
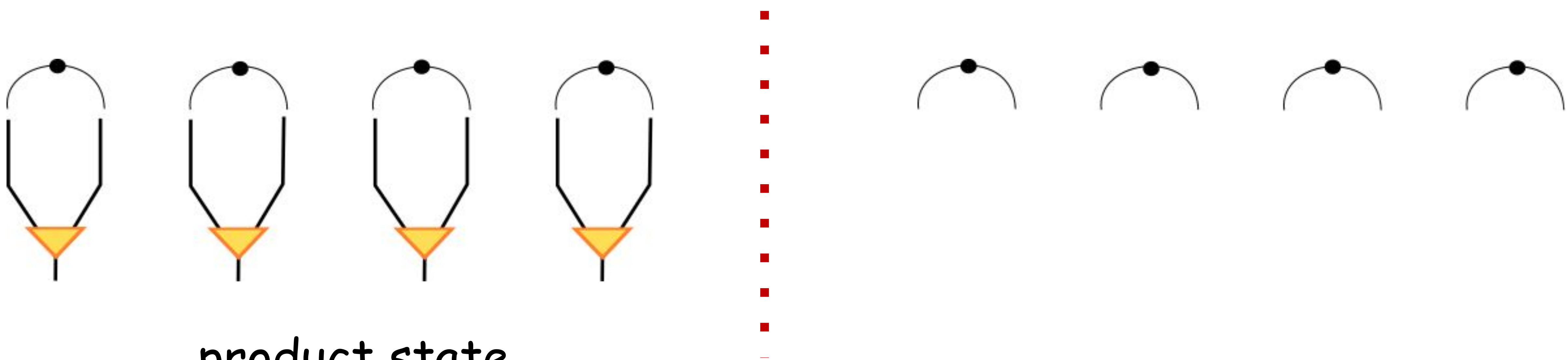




product state

1D gapped ground state on an infinite lattice





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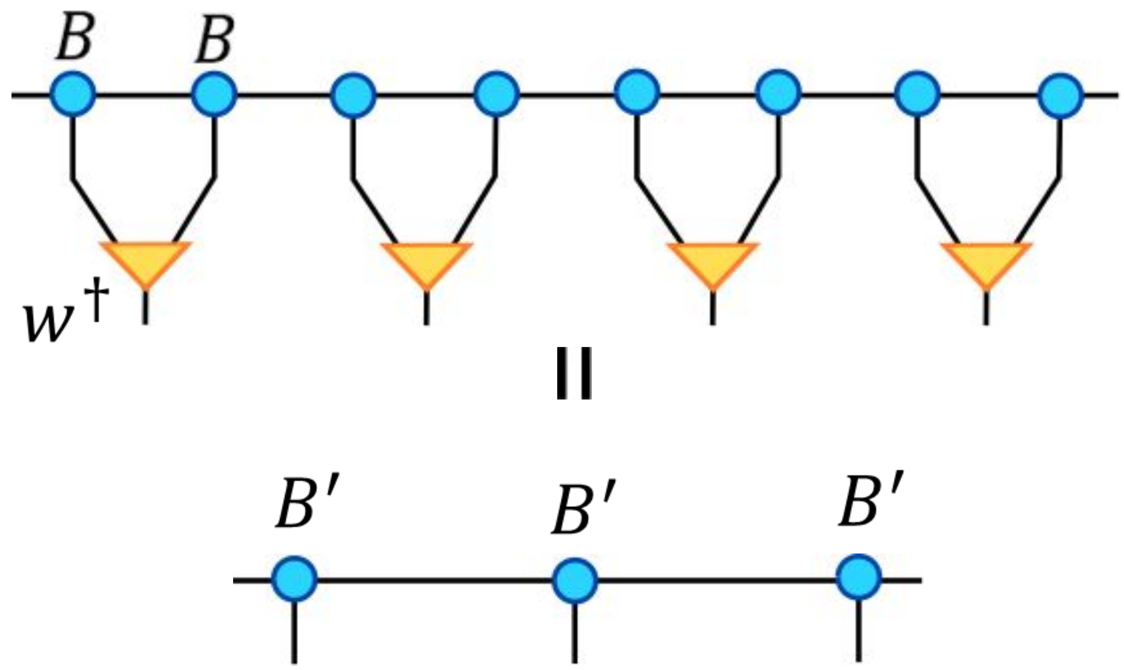
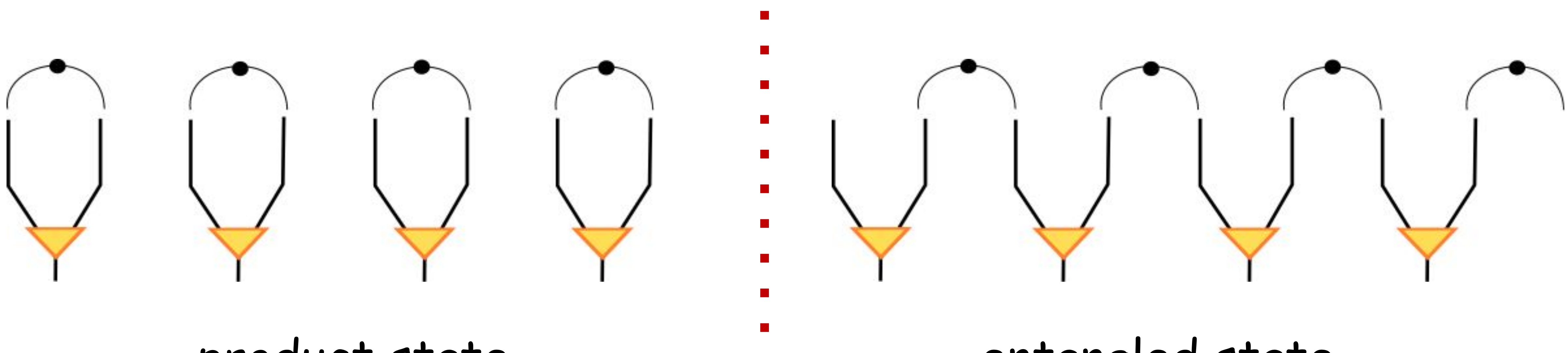
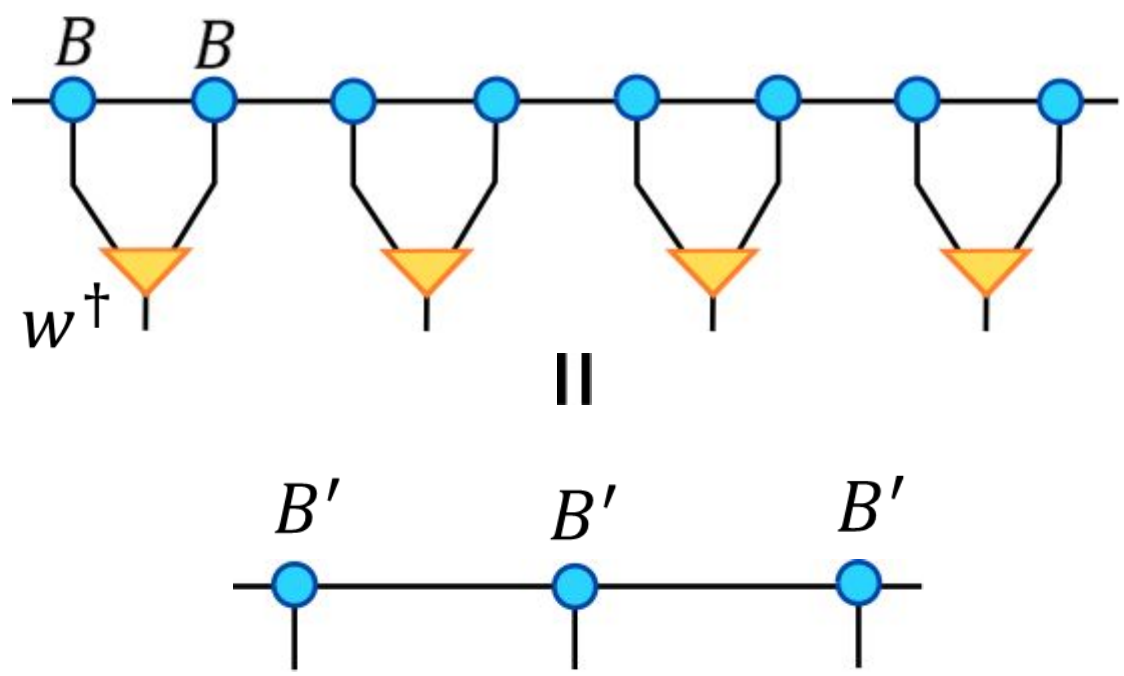




product state

entangled state



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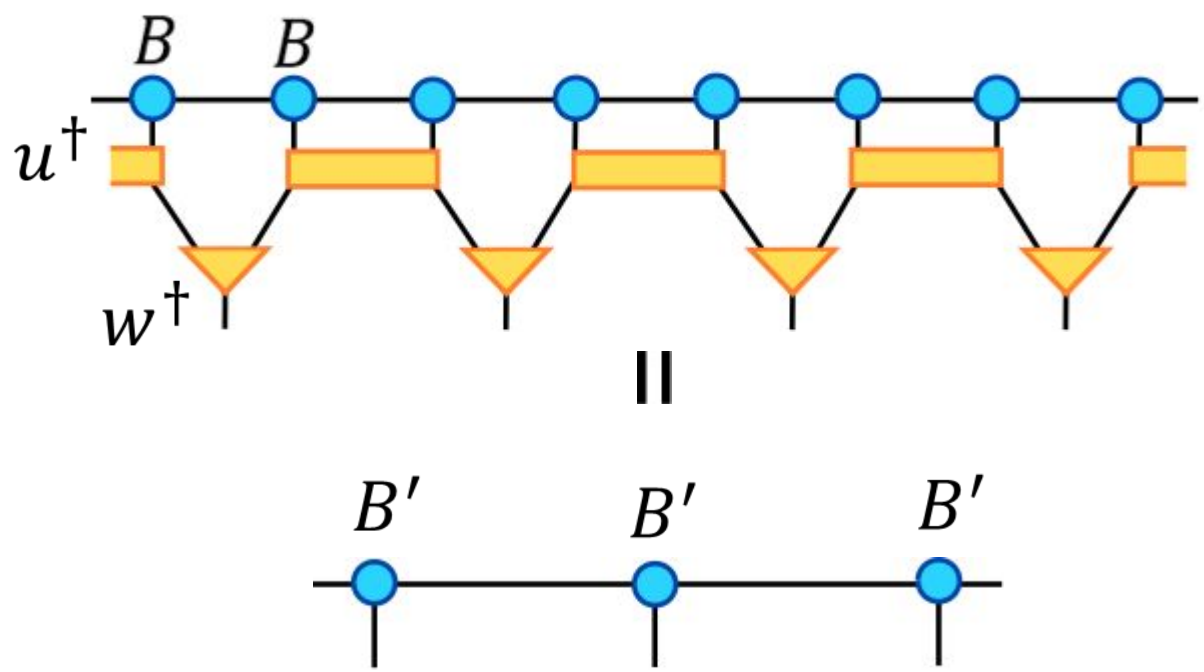
"Entanglement Renormalization" Vidal PRL (2007)



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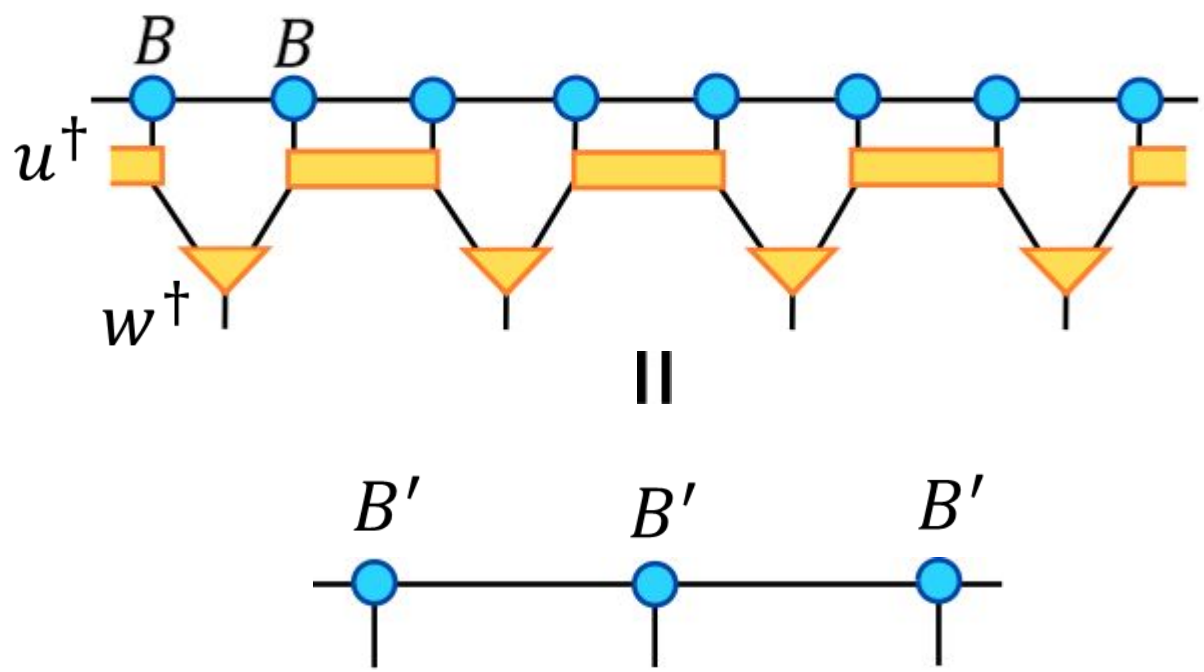

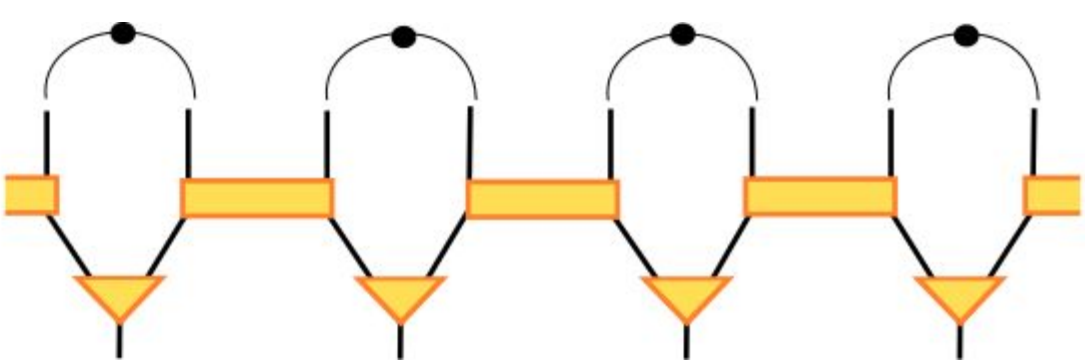

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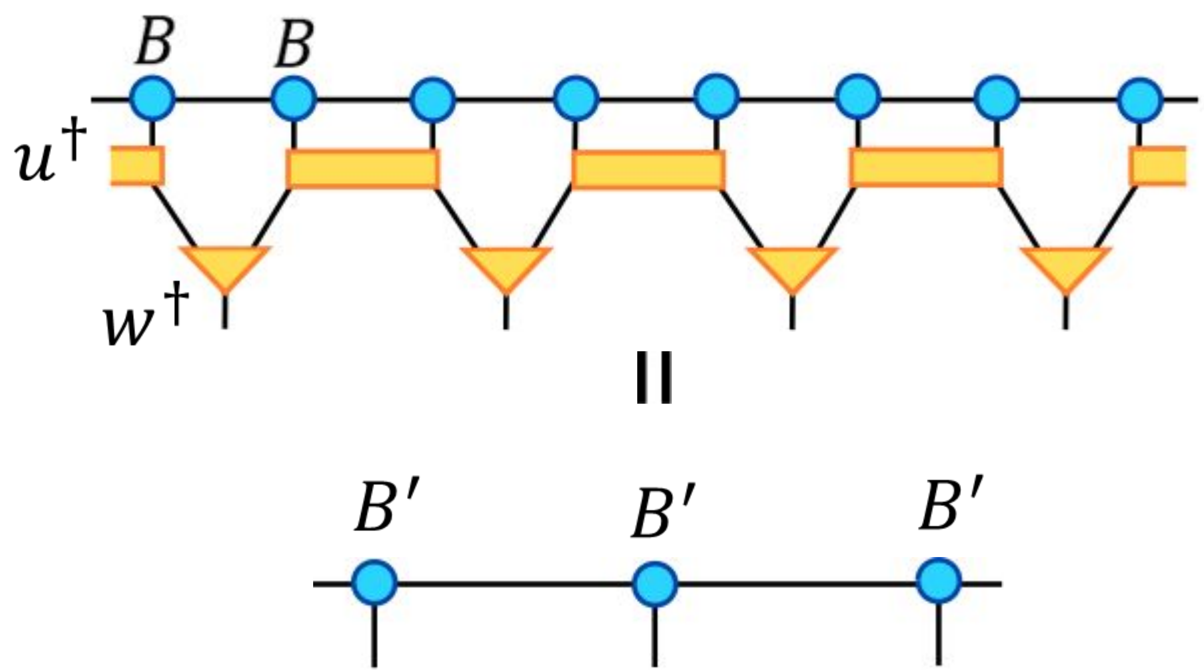

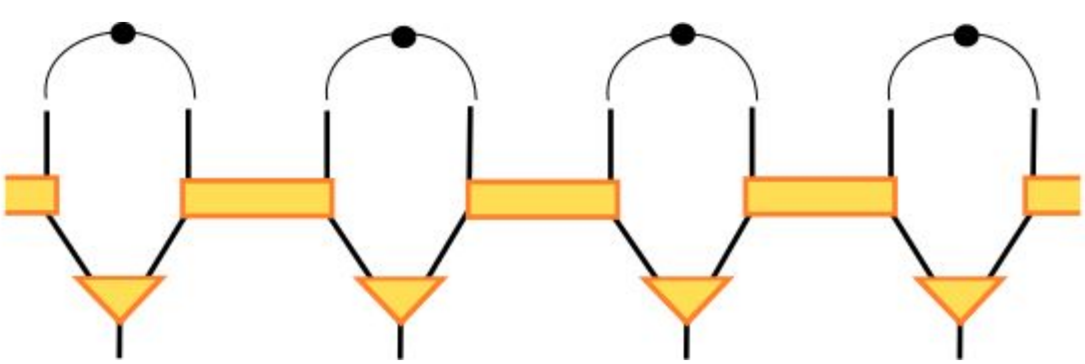

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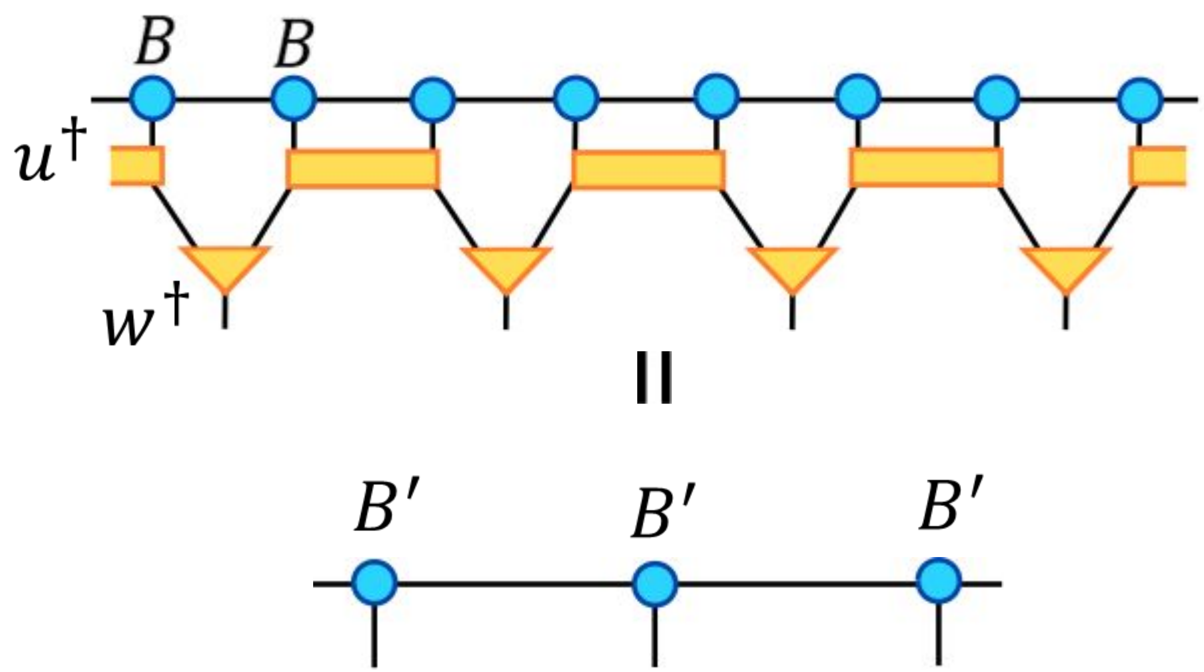

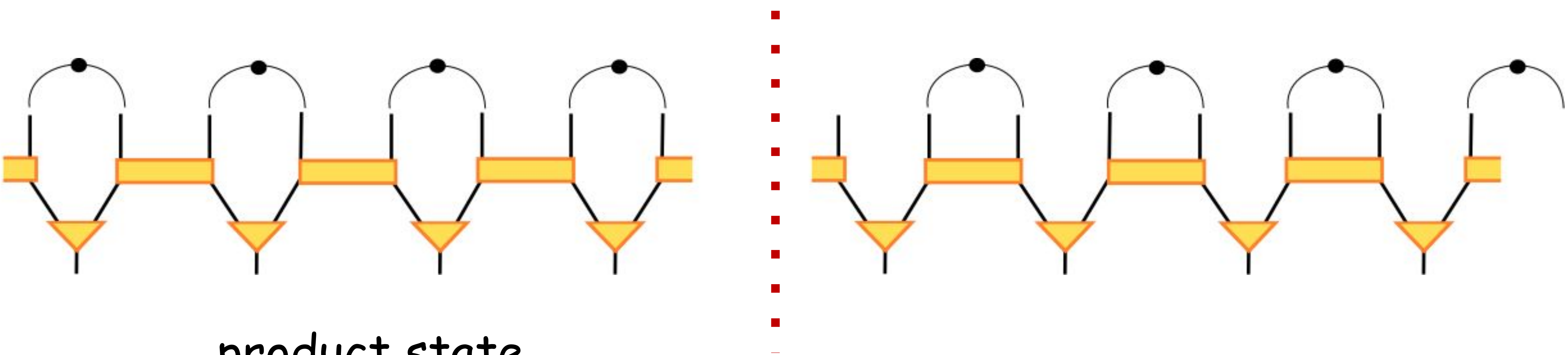

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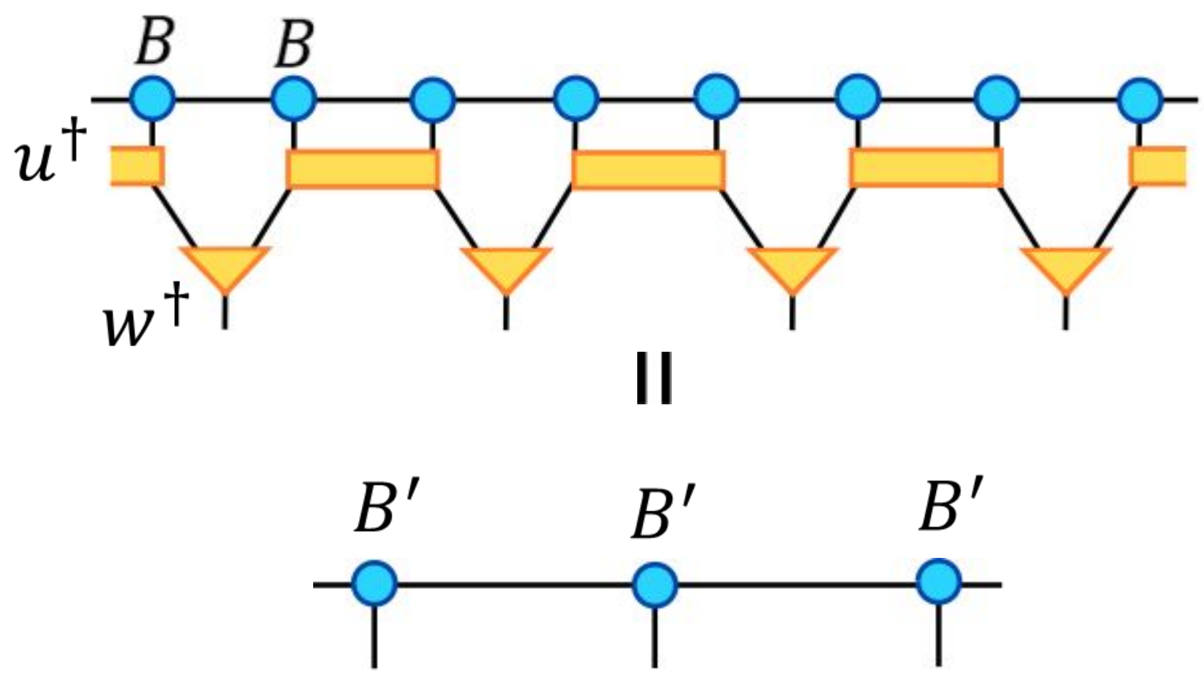

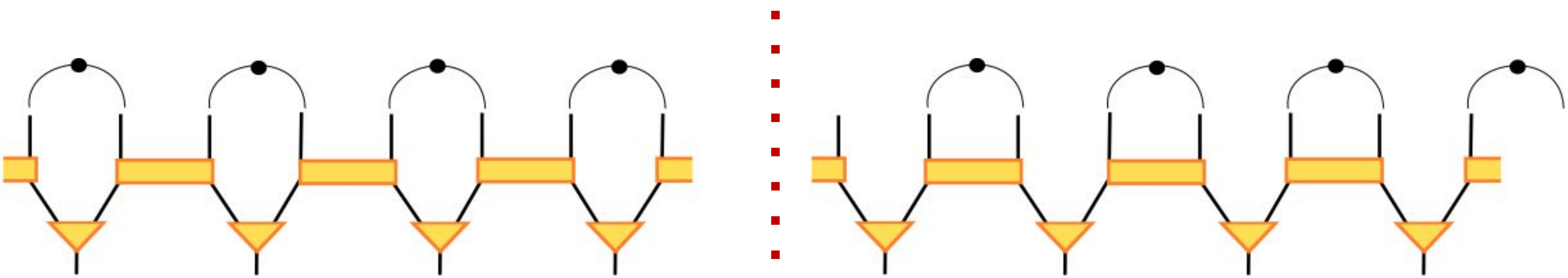

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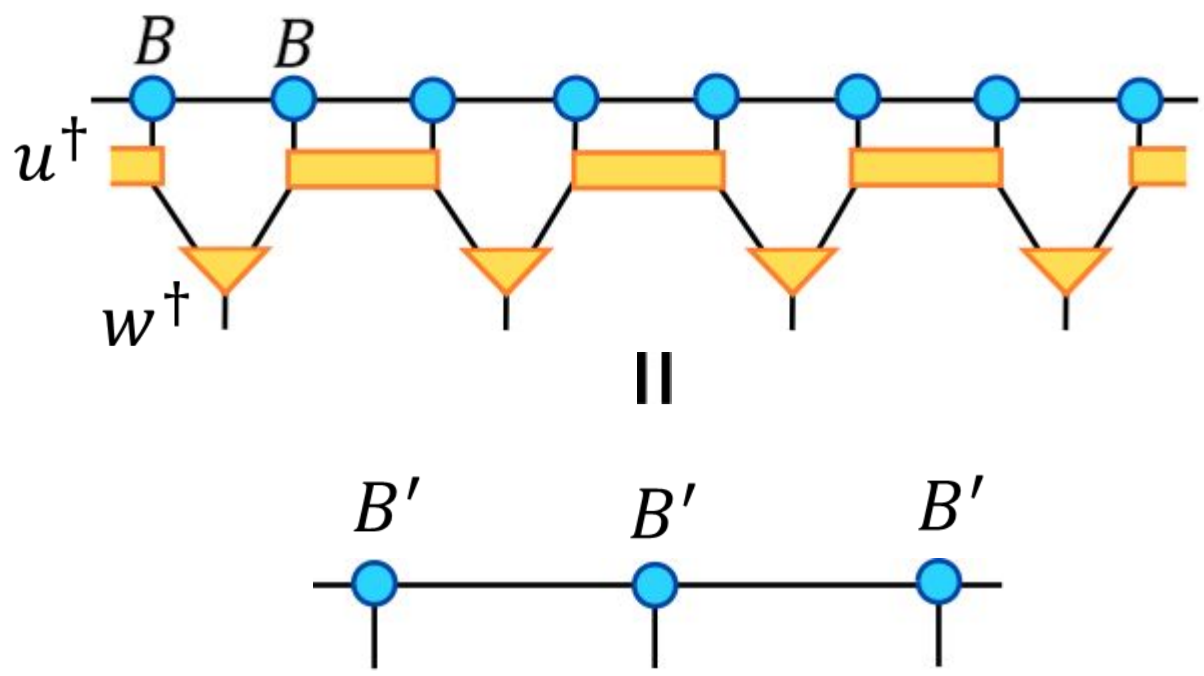

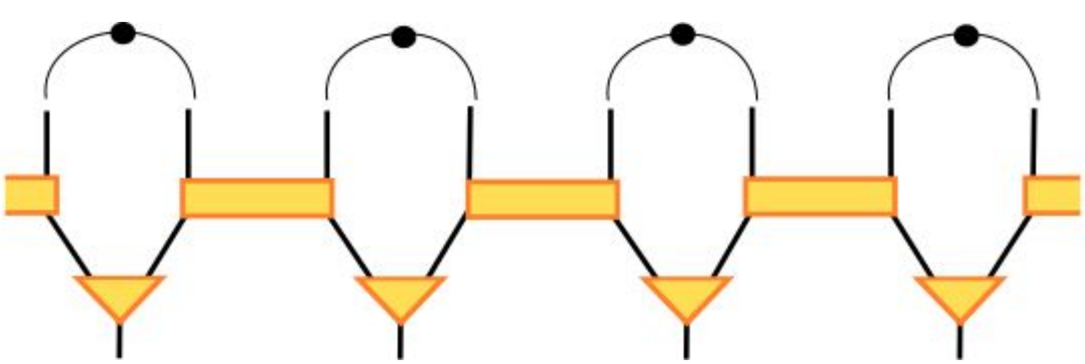

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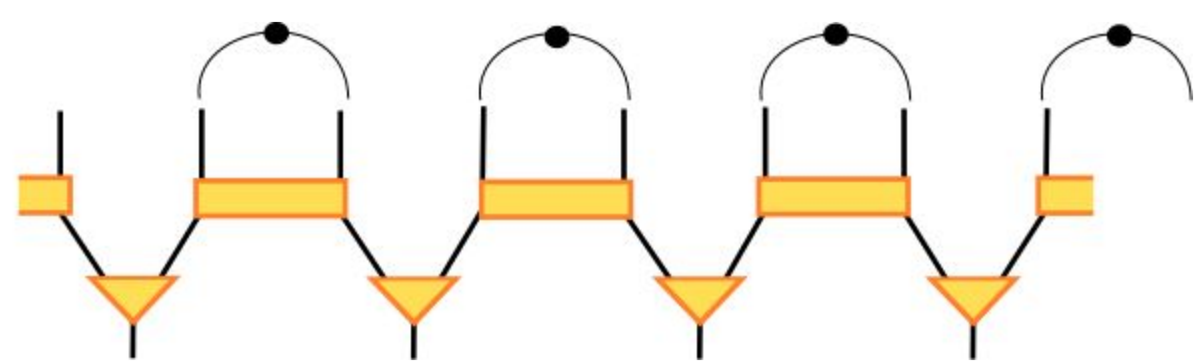
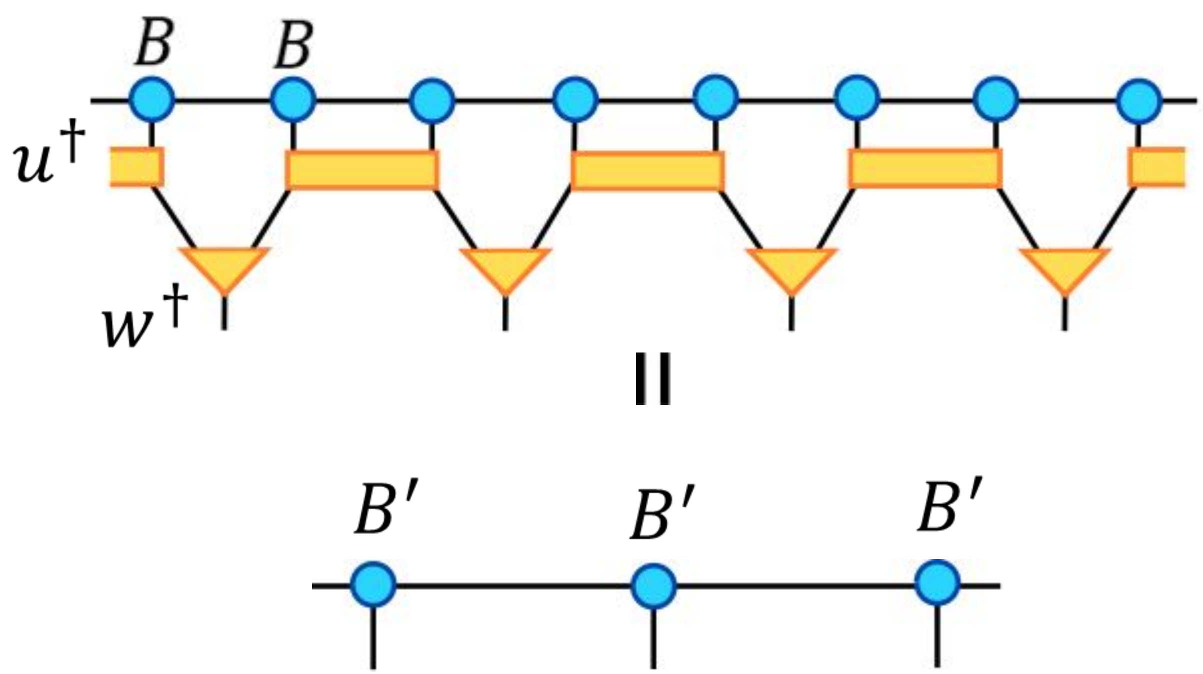


product state

More effective at removing short range details



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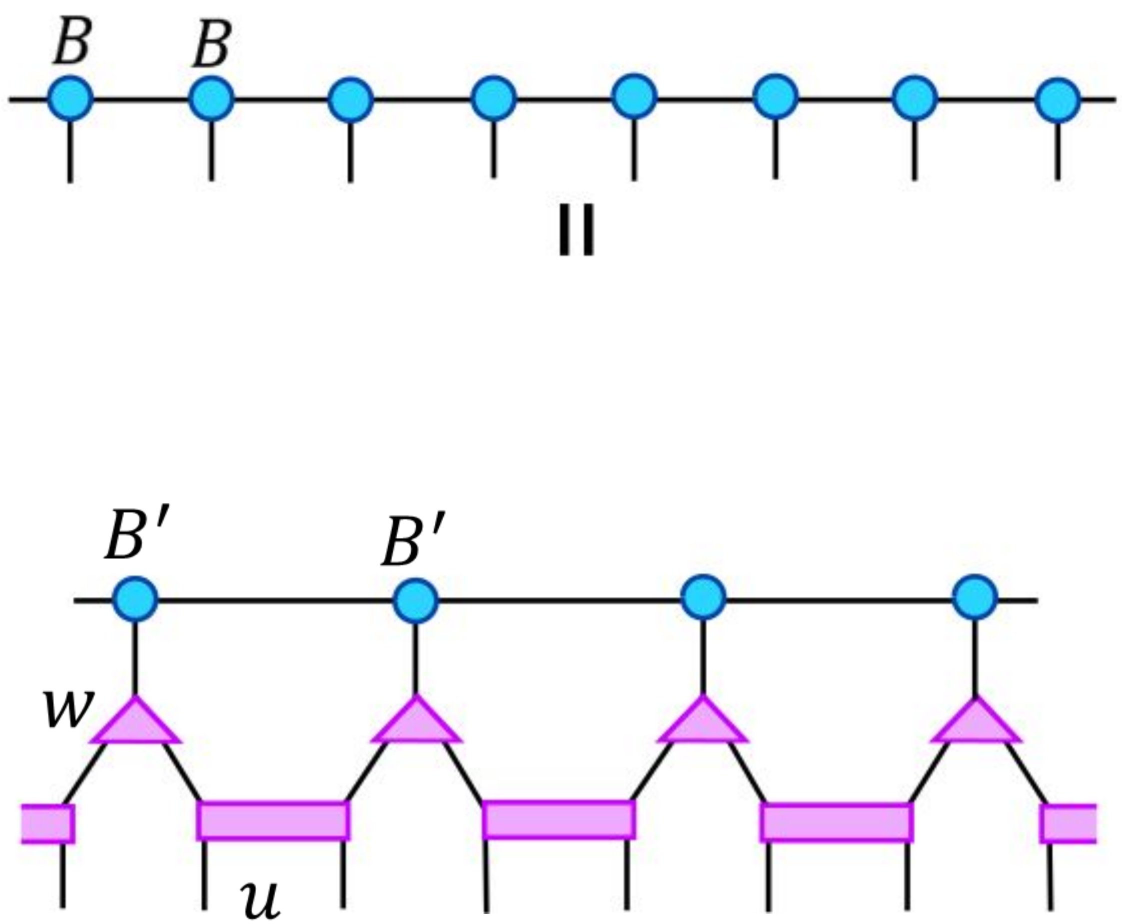

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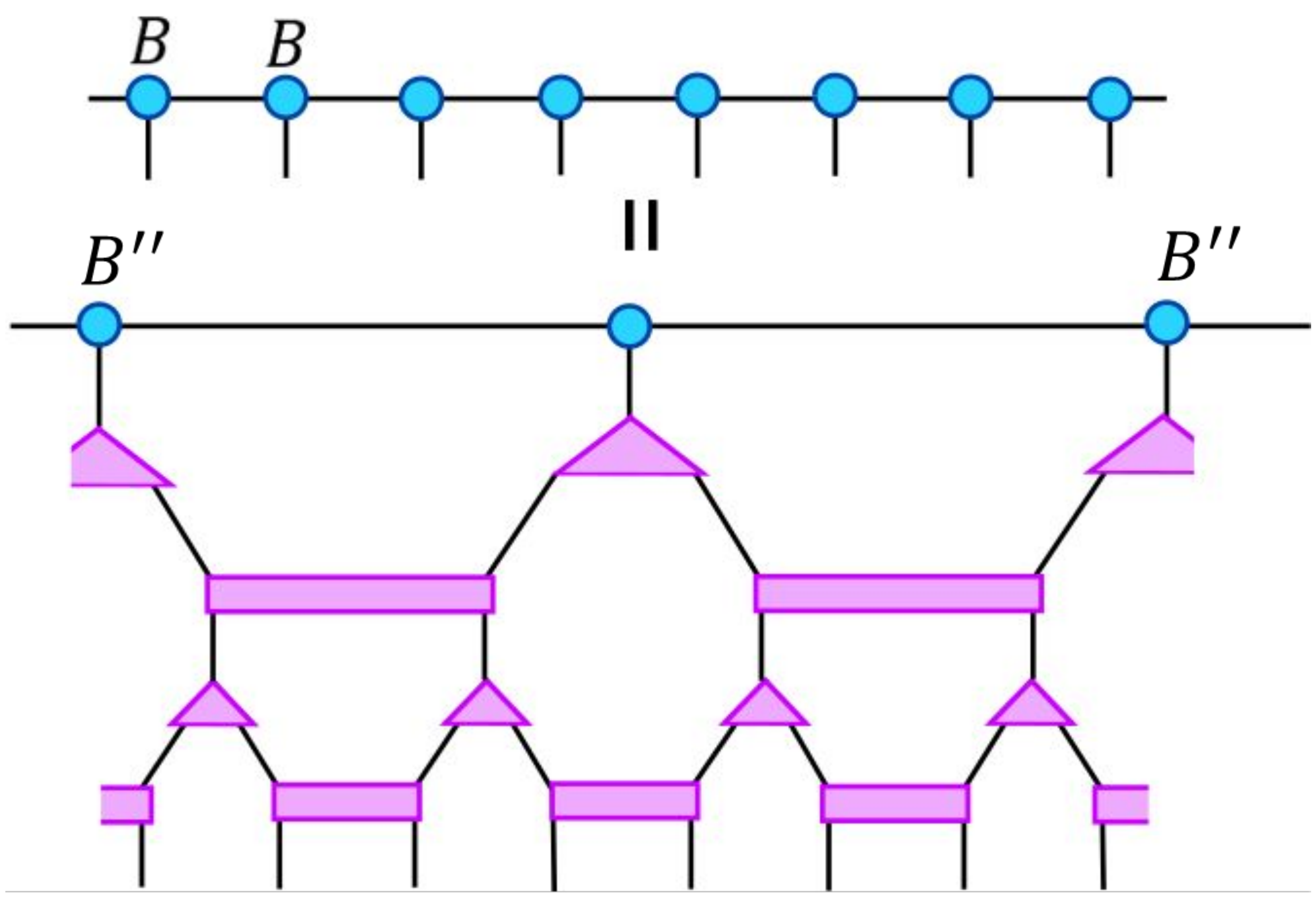

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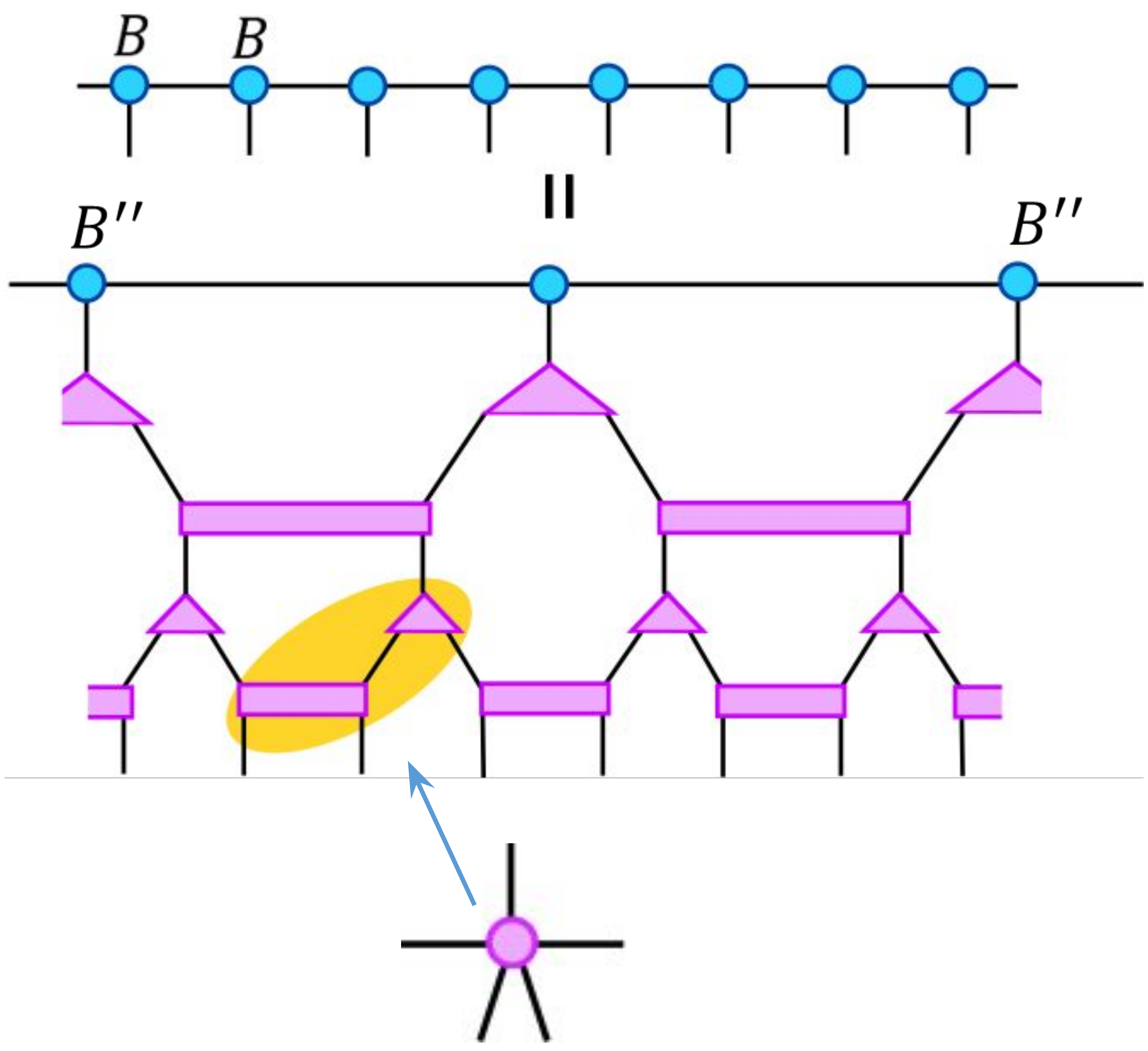

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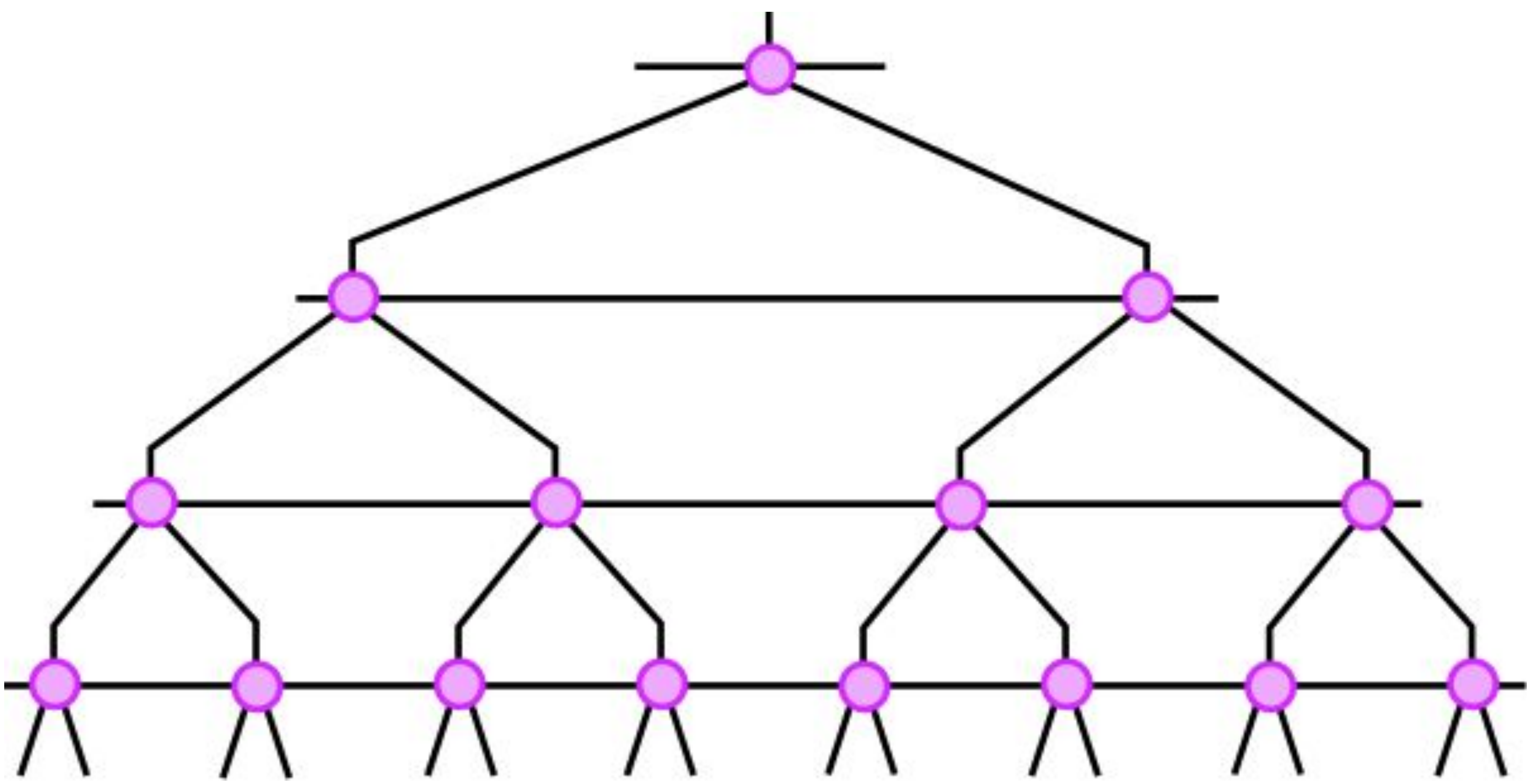
Multi-scale Entanglement Renormalization Ansatz (MERA)

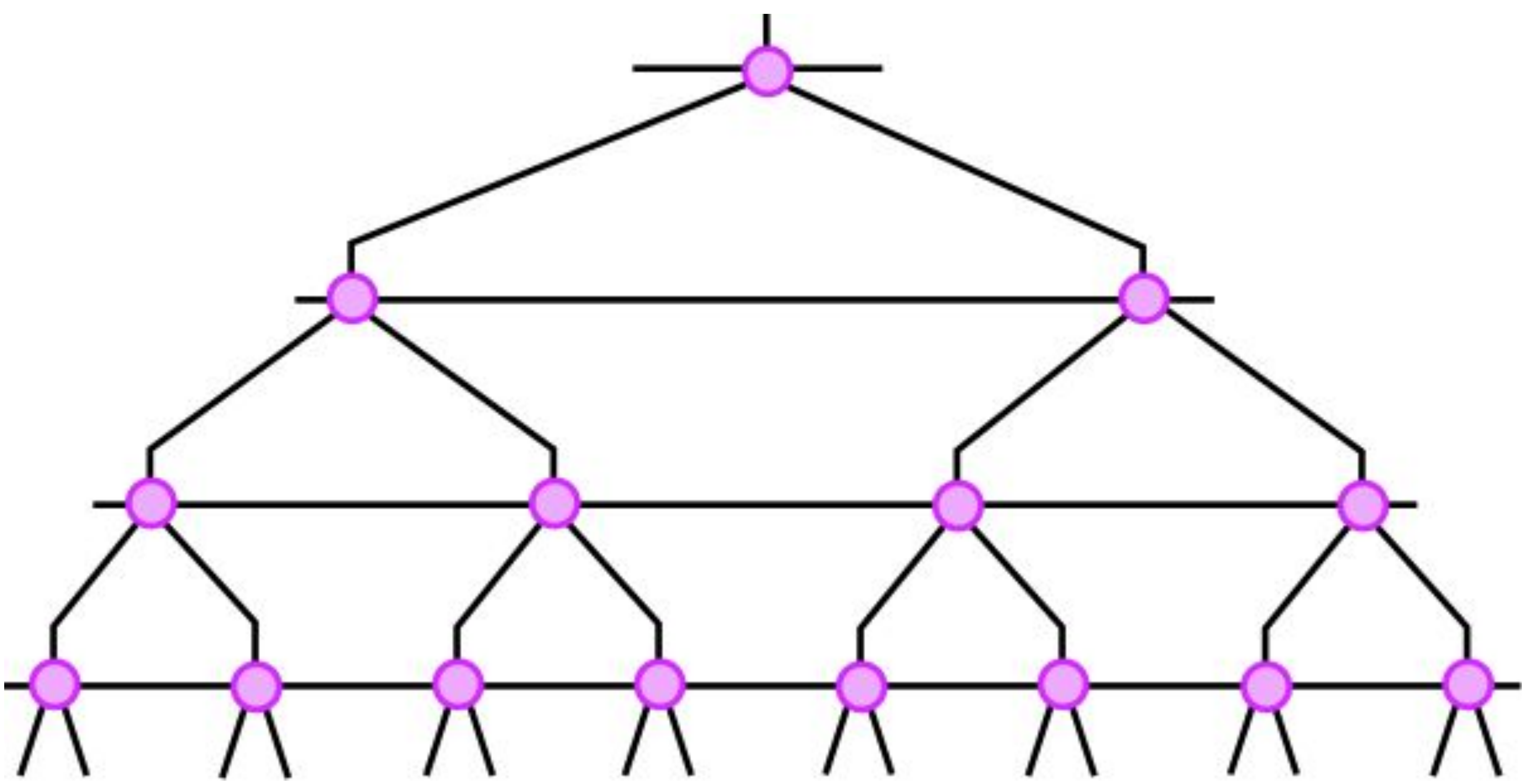
Vidal PRL (2008)



Multi-scale Entanglement Renormalization Ansatz (MERA)

Vidal PRL (2008)

MERA representation of a state = RG flow + Fixed point

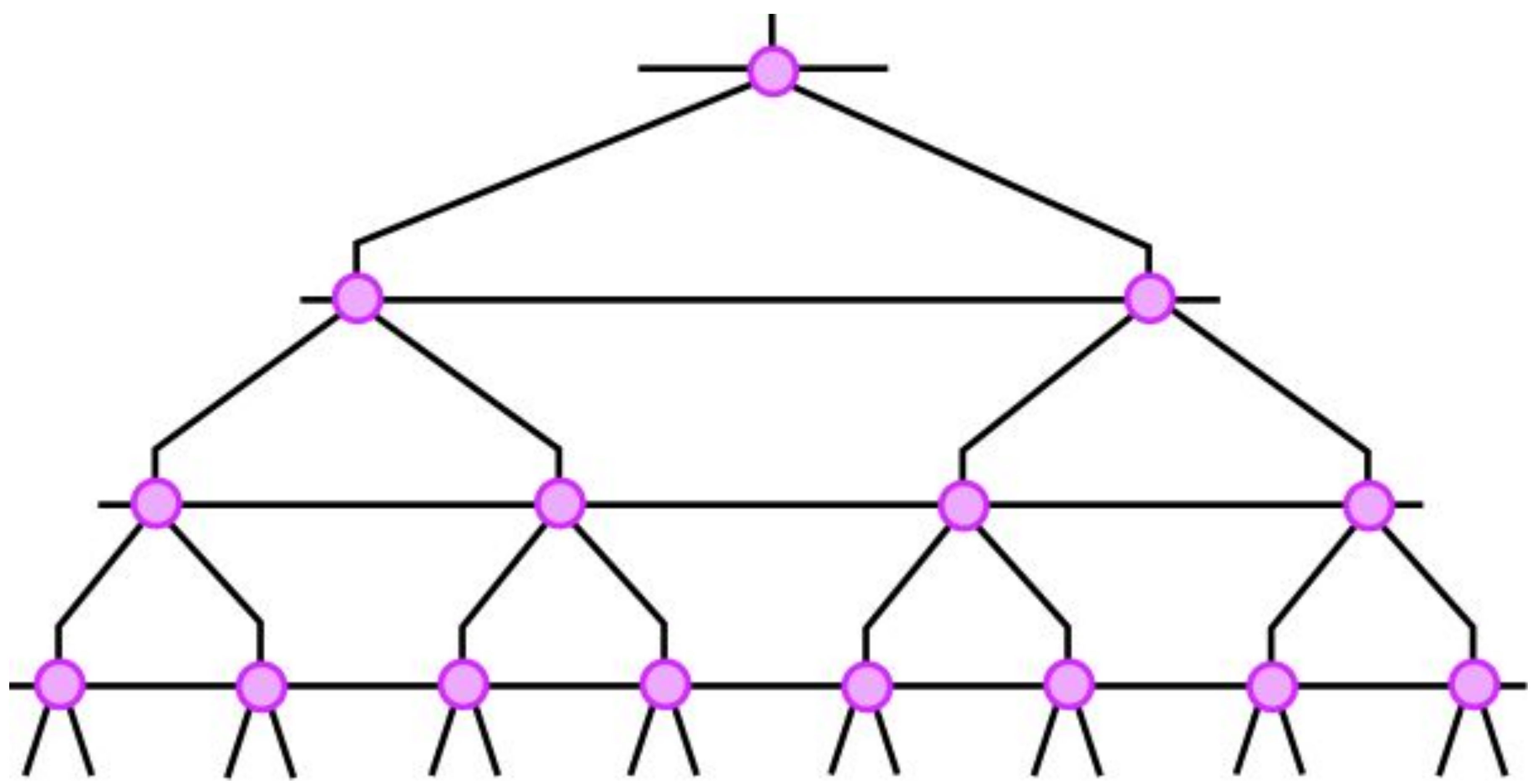


Multi-scale Entanglement Renormalization Ansatz (MERA)

Vidal PRL (2008)

MERA representation of a state = RG flow + Fixed point

MERA obtained from an MPS has finite correlation length.



Multi-scale Entanglement Renormalization Ansatz (MERA)

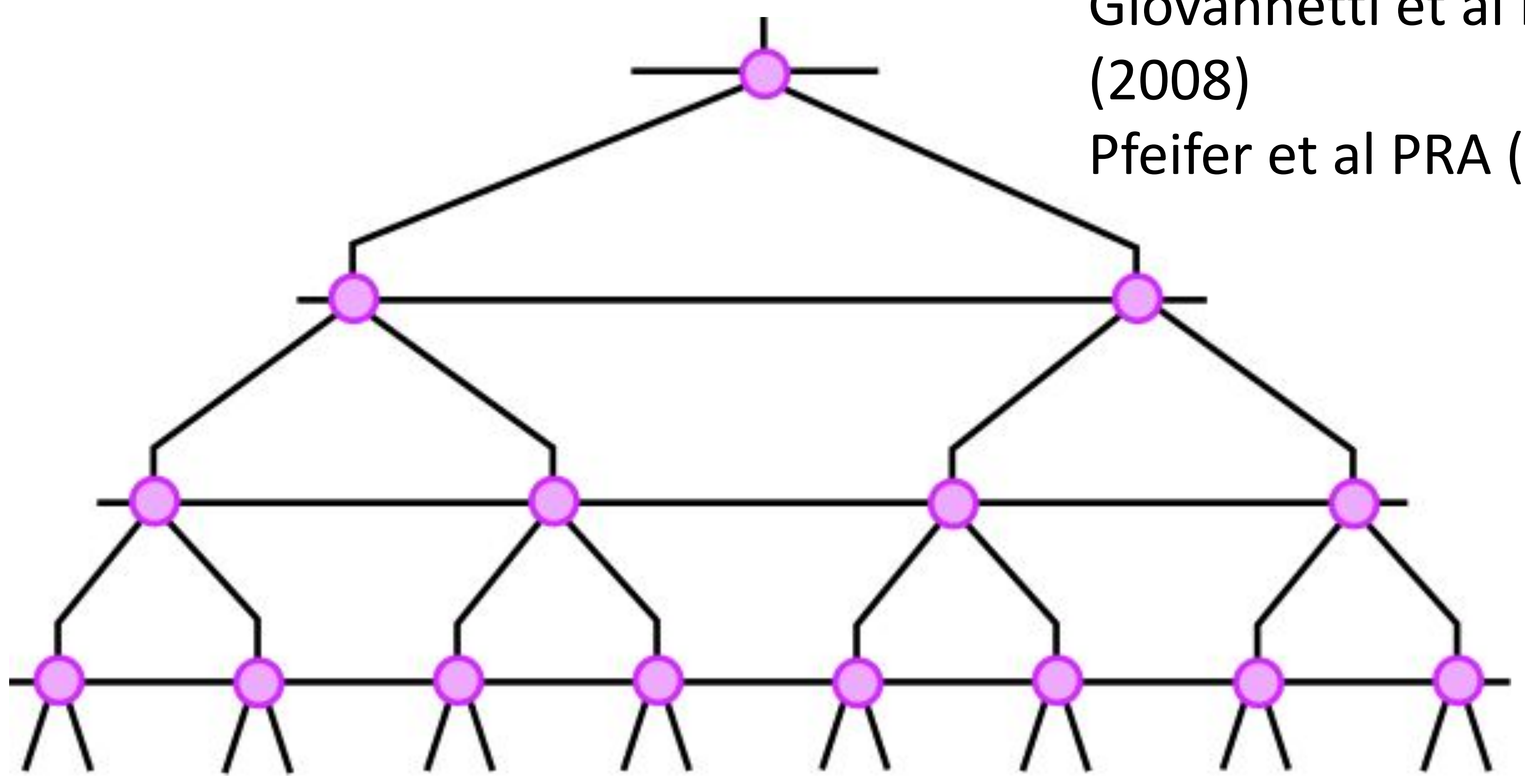
Vidal PRL (2008)

MERA representation of a state = RG flow + Fixed point

MERA obtained from an MPS has finite correlation length.

But generic choice of tensors describes a state with **polynomial decaying correlations** and **logarithmic growth** of entanglement entropy (properties of critical ground states).

Giovannetti et al PRL (2008)
Pfeifer et al PRA (2009)



MPS
&
1D gapped SP phases

Every gapped ground state can be faithfully represented as an MPS

Verstraete and Cirac, PRB (2006)

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Every MPS is the ground state of a local Hamiltonian

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Classifying 1D gapped SP phases

$\longleftrightarrow$

Classifying all symmetric MPSs

Chen, Gu, and Wen, PRB (2011)
Schuch, Perez-Garcia, and Cirac, PRB (2011)
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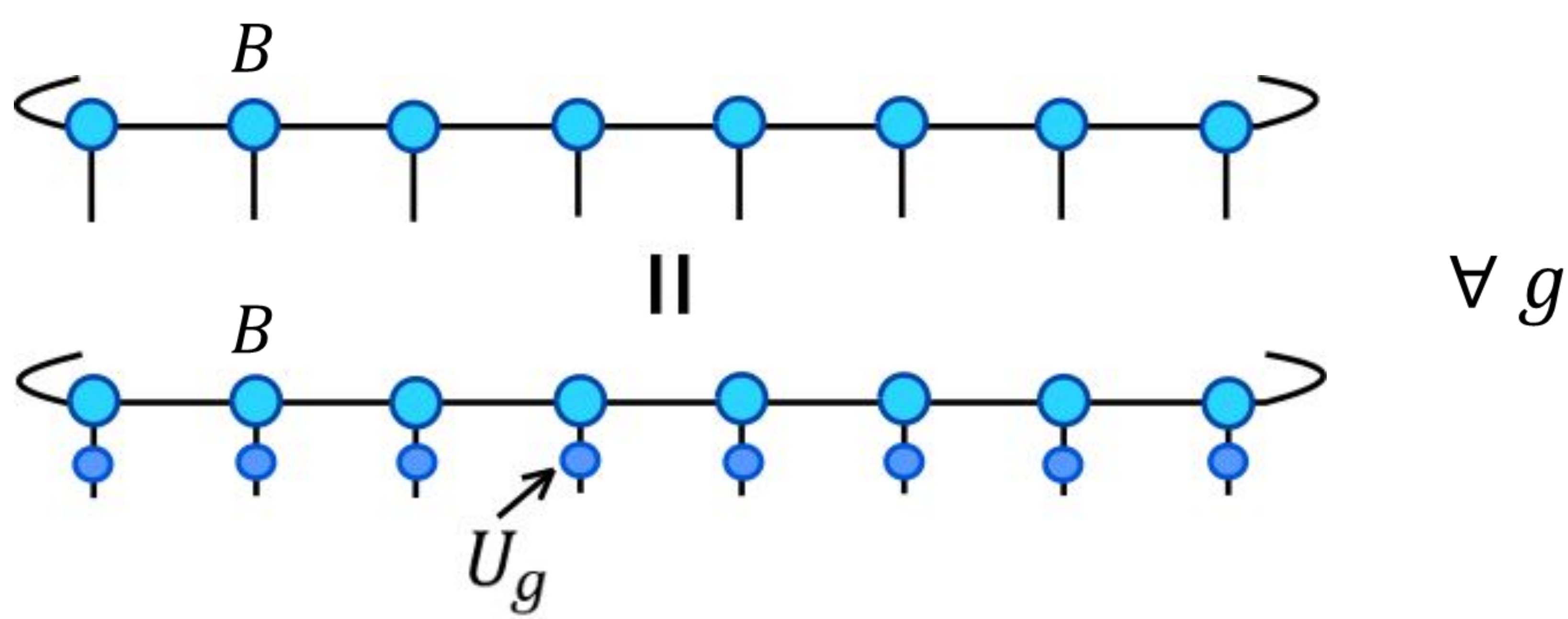
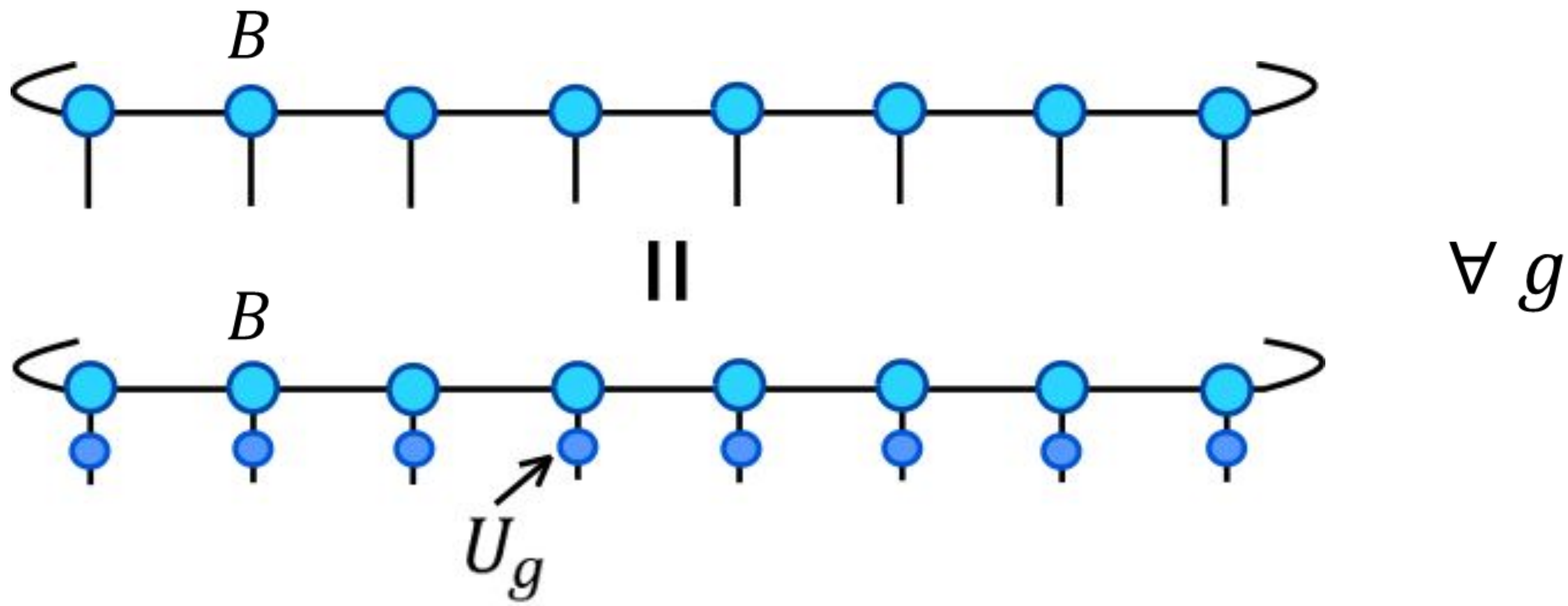
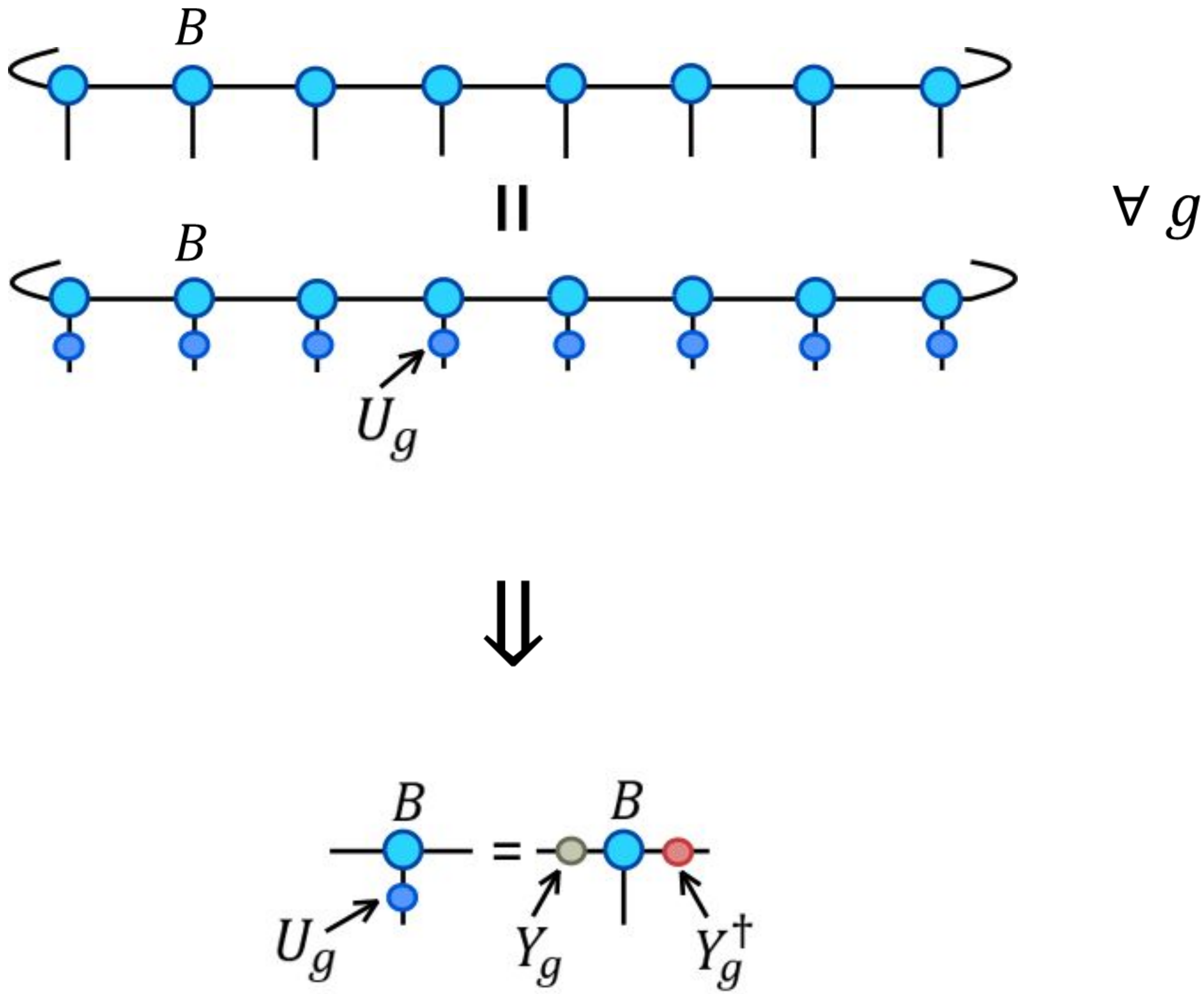
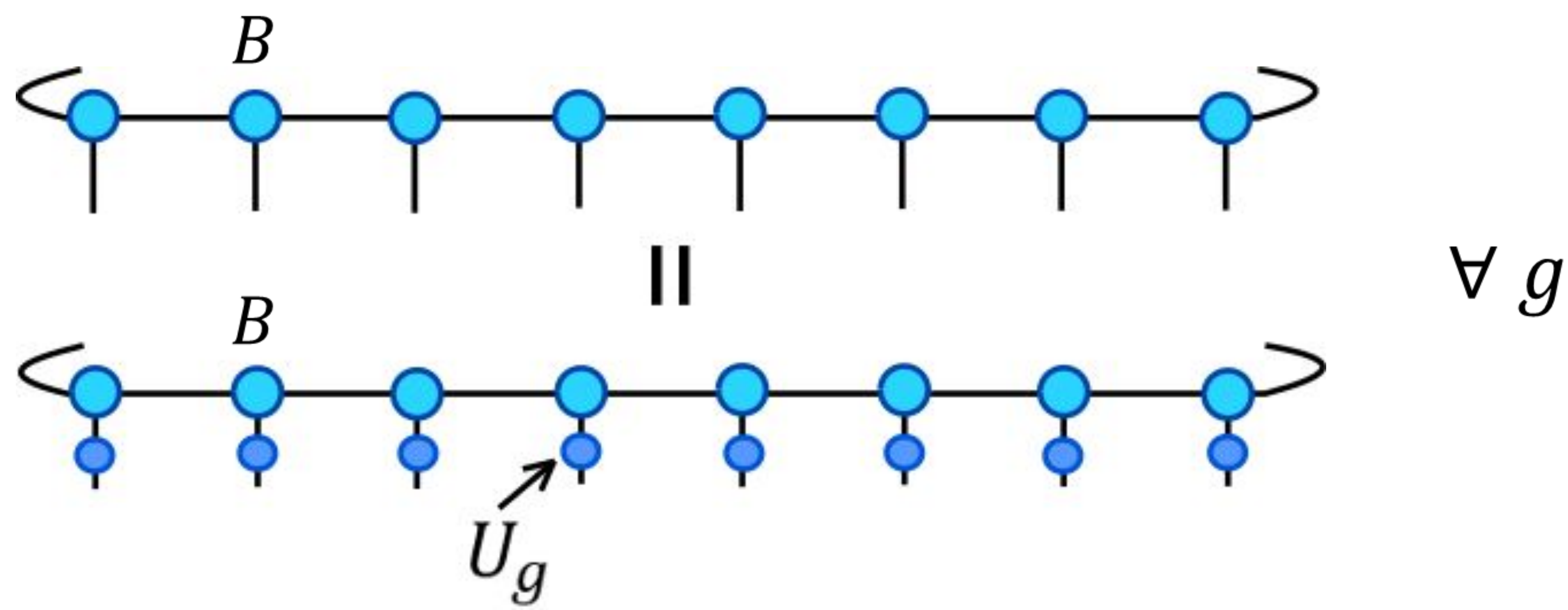




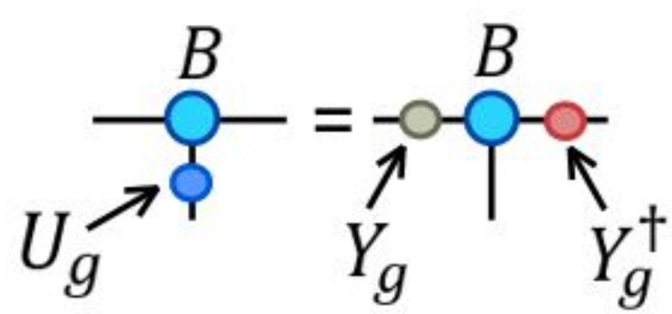
B
$=$
B
U_g
$\forall\ g$

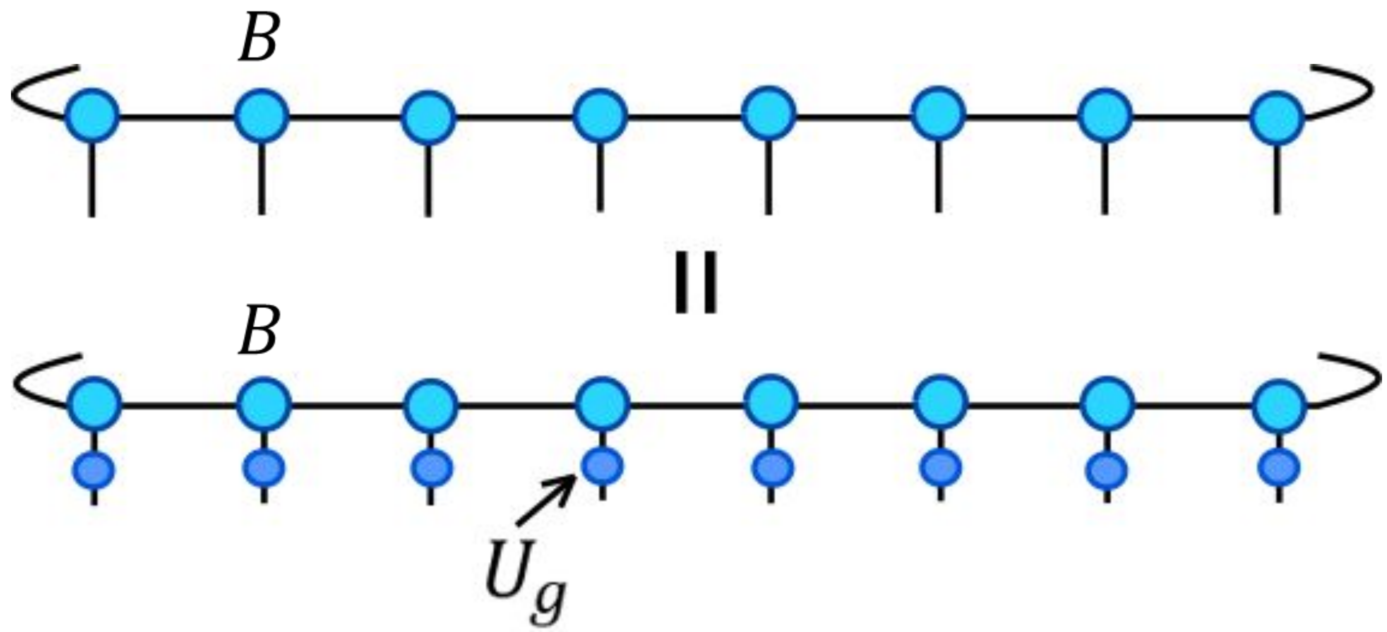


B
=
∀ g
B
U_g
⇓
B
=
B
U_g
Y_g
Y_g^†



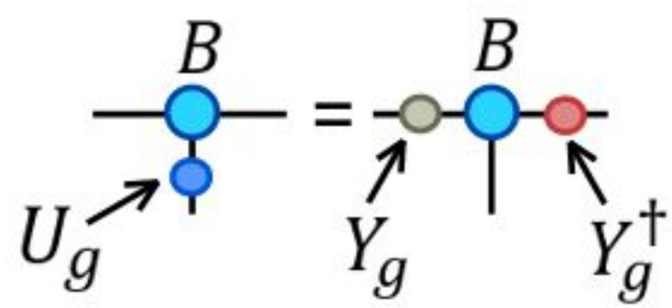
$\Downarrow$ If MPS is **normal** & in the **canonical form**





$\forall\, g$

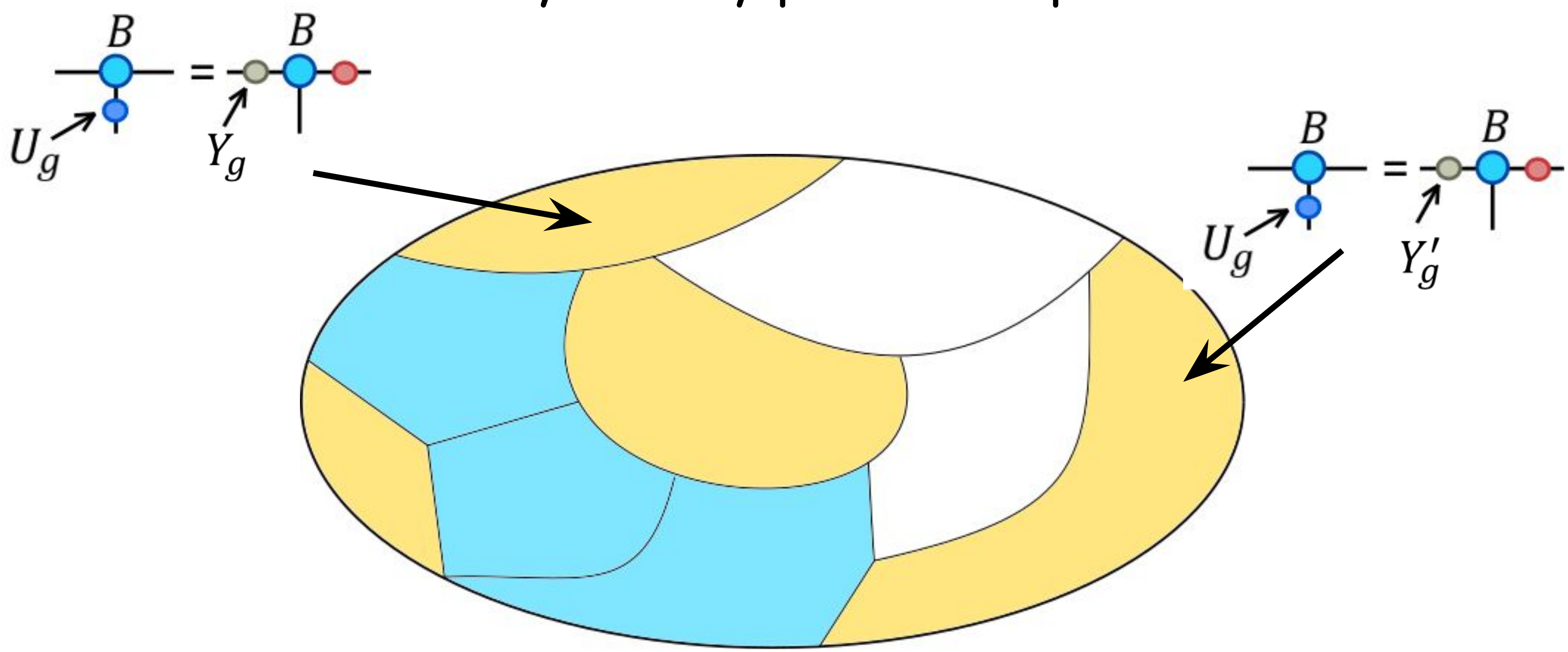
$\Downarrow$ If MPS is **normal** & in the **canonical form**



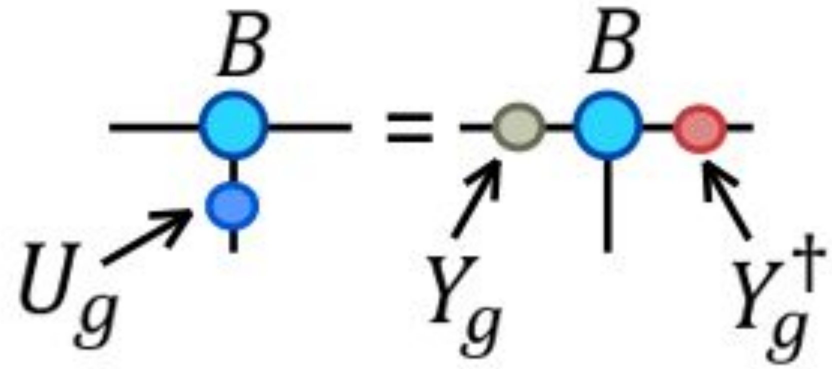
Y_g can be **projective** $\quad Y_g Y_h = e^{i\omega(g,h)} Y_{g.h}$

symmetry broken phase

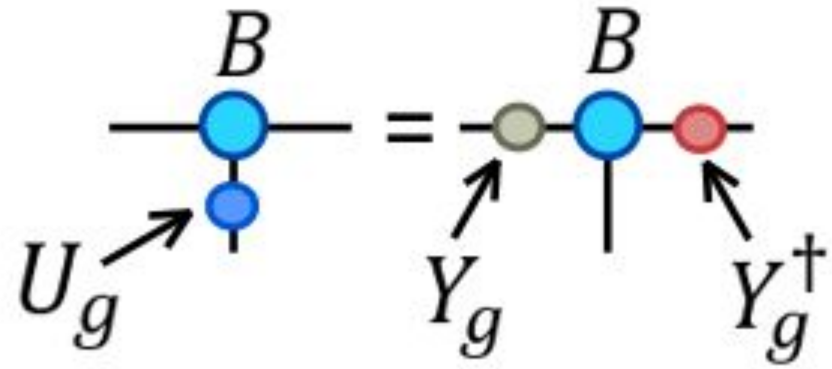
symmetry protected phase



Y_g and Y_g' are inequivalent projective representations



Star equation!



Star equation!

1. Conceptual tool for classifying phases

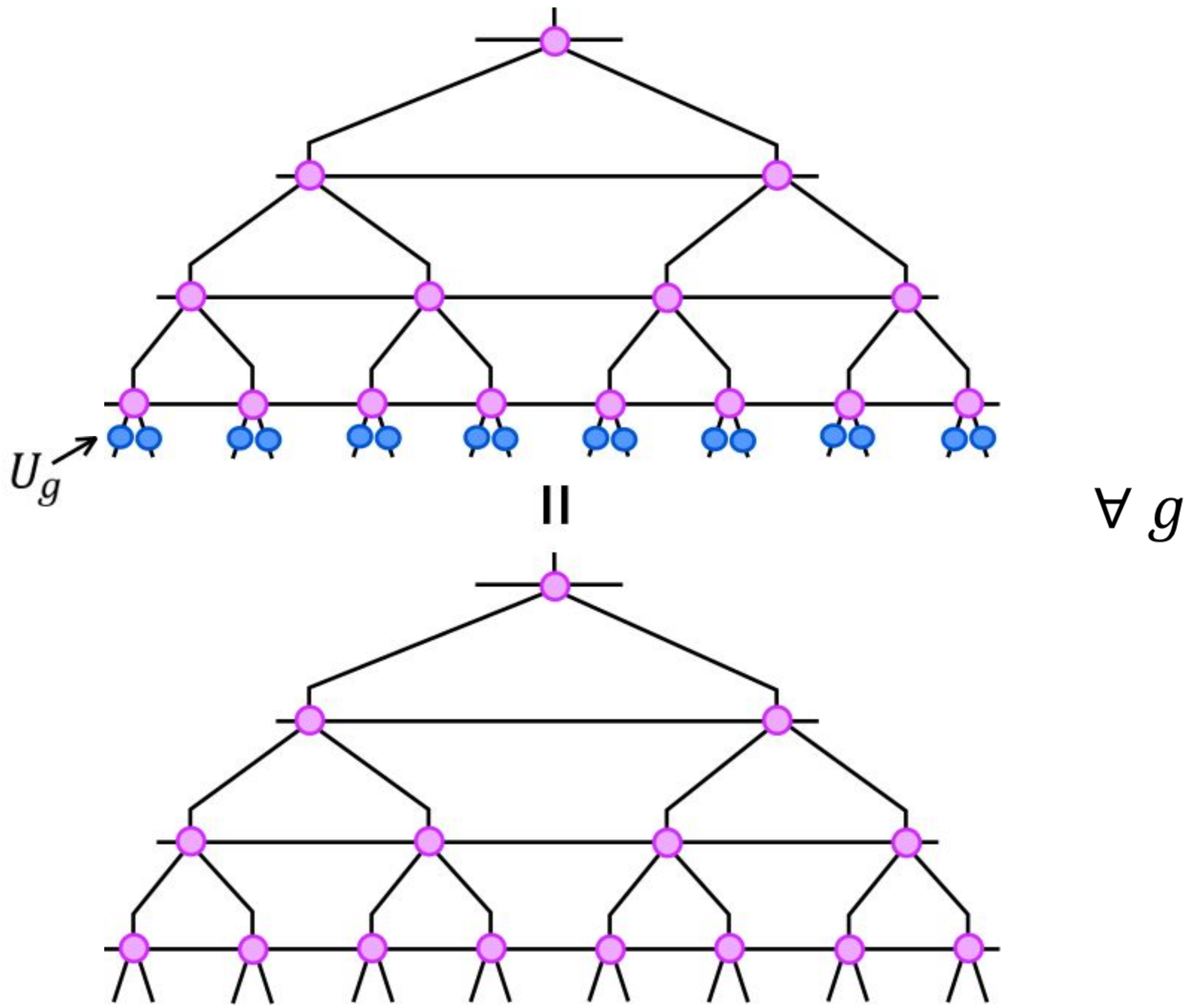
2. Detecting phases in numerical simulations

    Pollmann and Turner, PRB (2012)

    SS, PRB (2015)

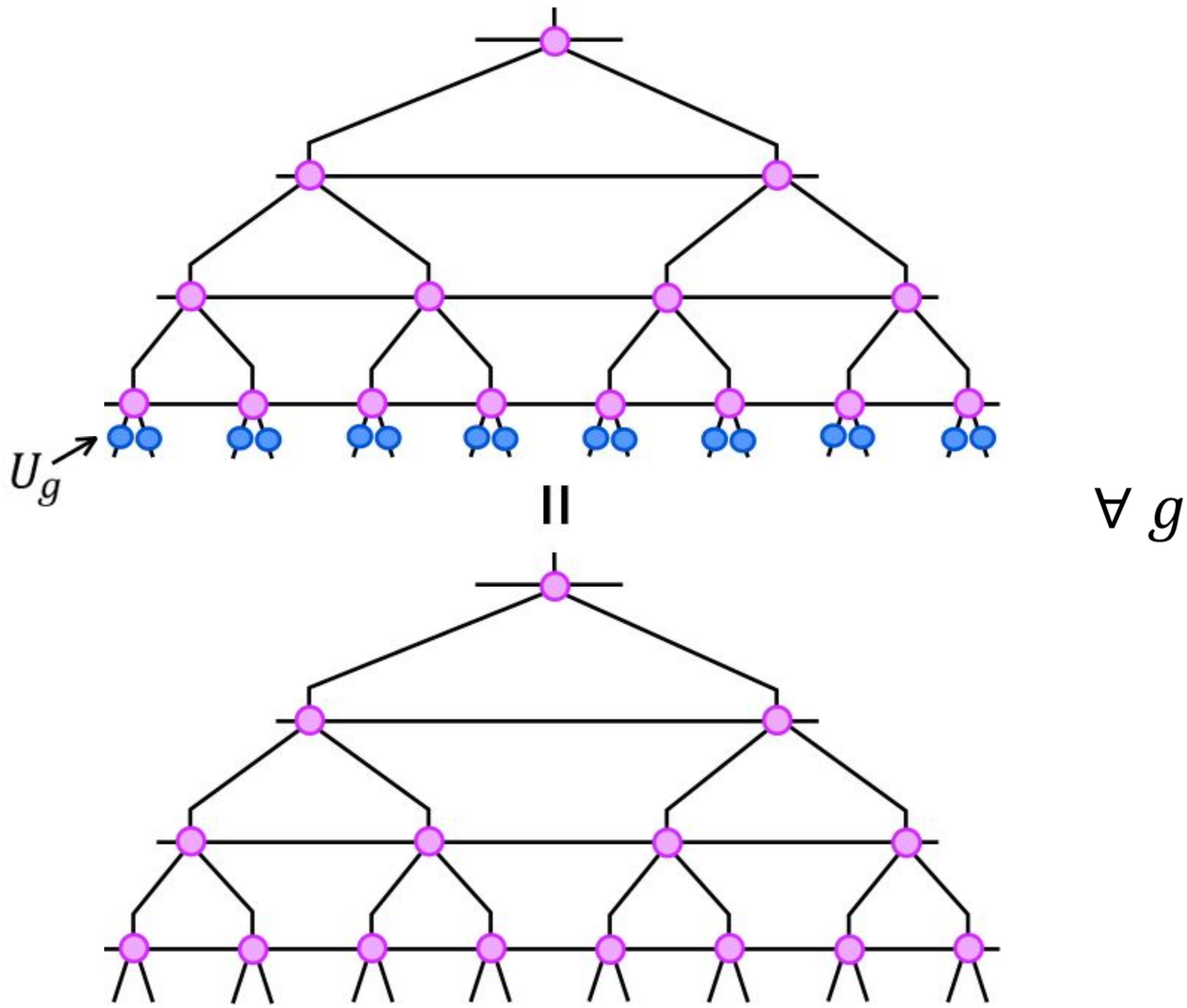
Symmetry protected MERA

Symmetry protected MERA

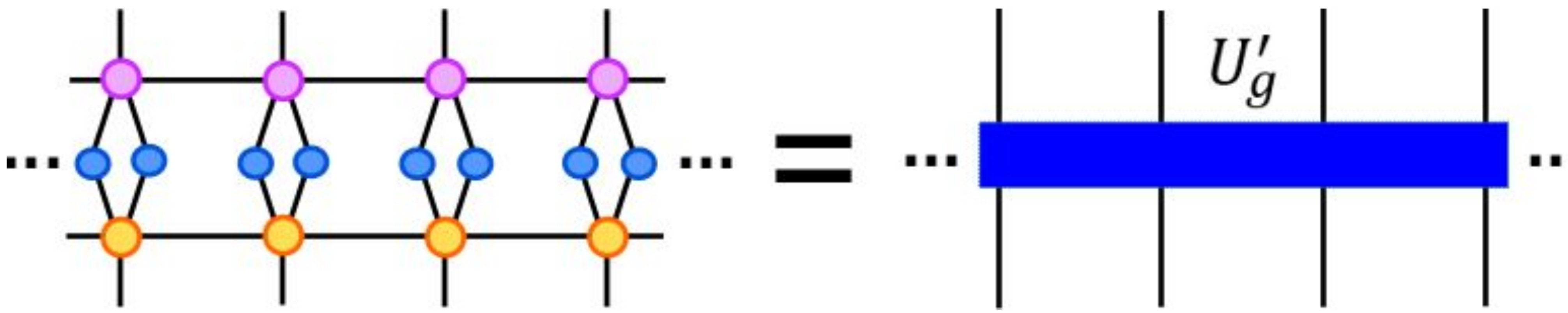


Symmetry protected MERA

This should include both gapped and critical states



Symmetry protected entanglement renormalization



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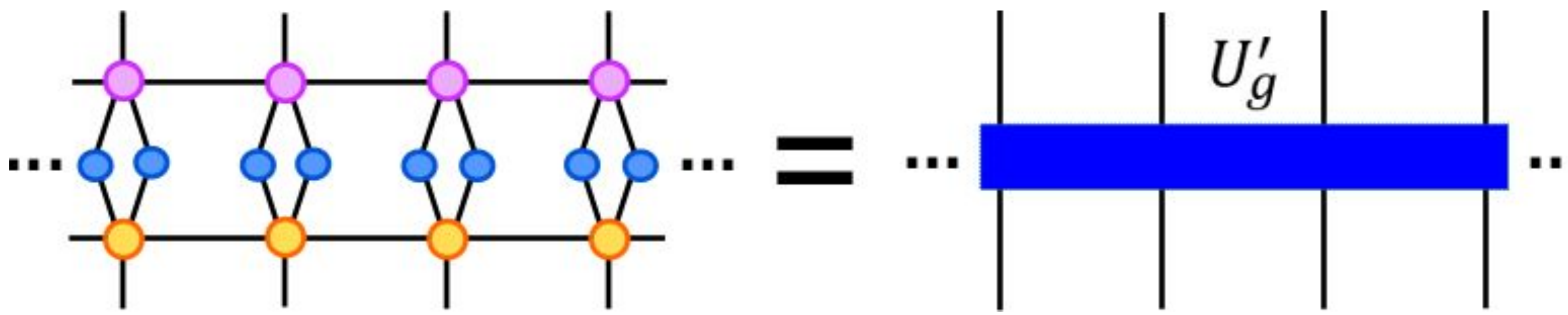


Assume U'_g is linear. But is it also on-site?

Argument for symmetry remaining on-site

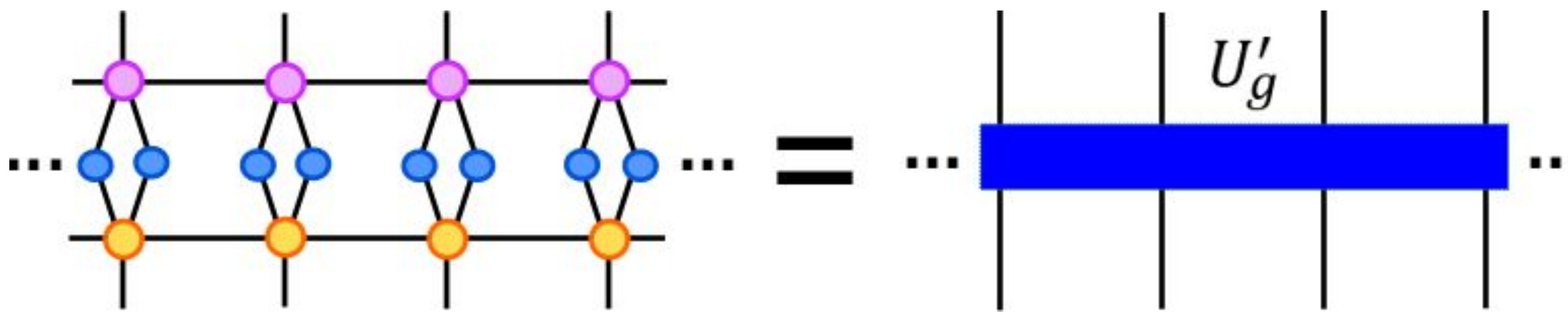
1. Global on-site symmetries can always be gauged

2. If a symmetry cannot be gauged it acts anomalously
   (acts in a non-onsite way)

3. At least in QFT legitimate RG flows must preserve any anomalies
   ('t Hooft anomaly matching condition)

4. A trivial anomaly (on-site) must remain trivial (on-site)
under a legitimate RG flow

Symmetry protected entanglement renormalization



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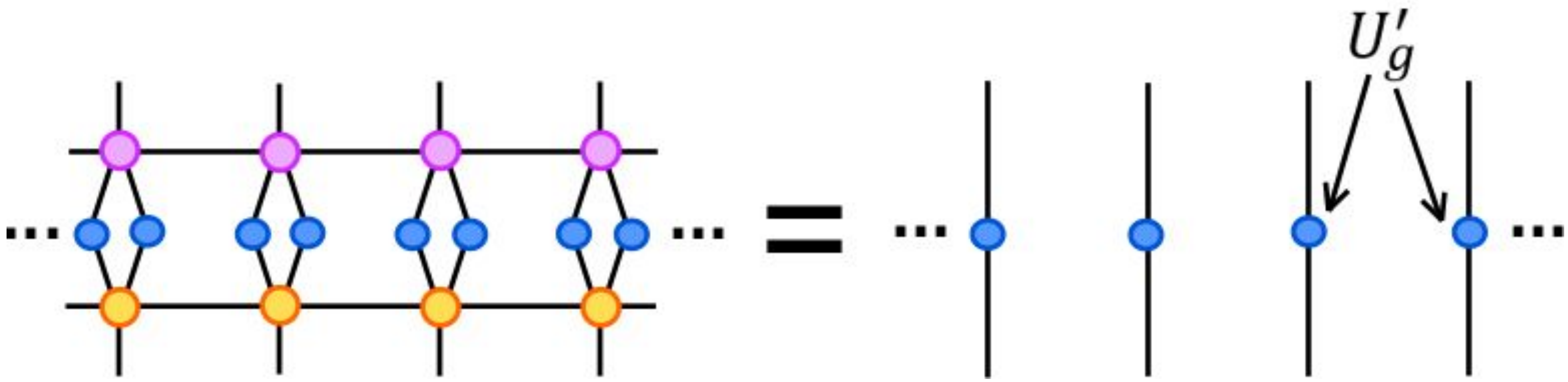
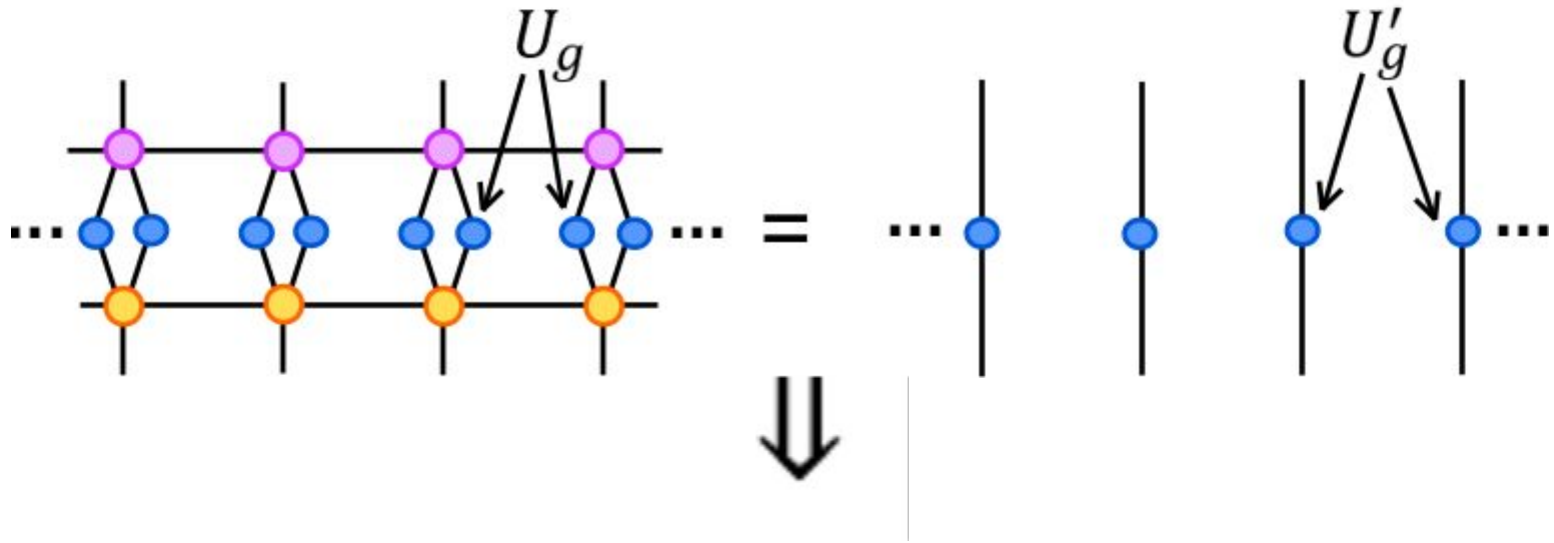
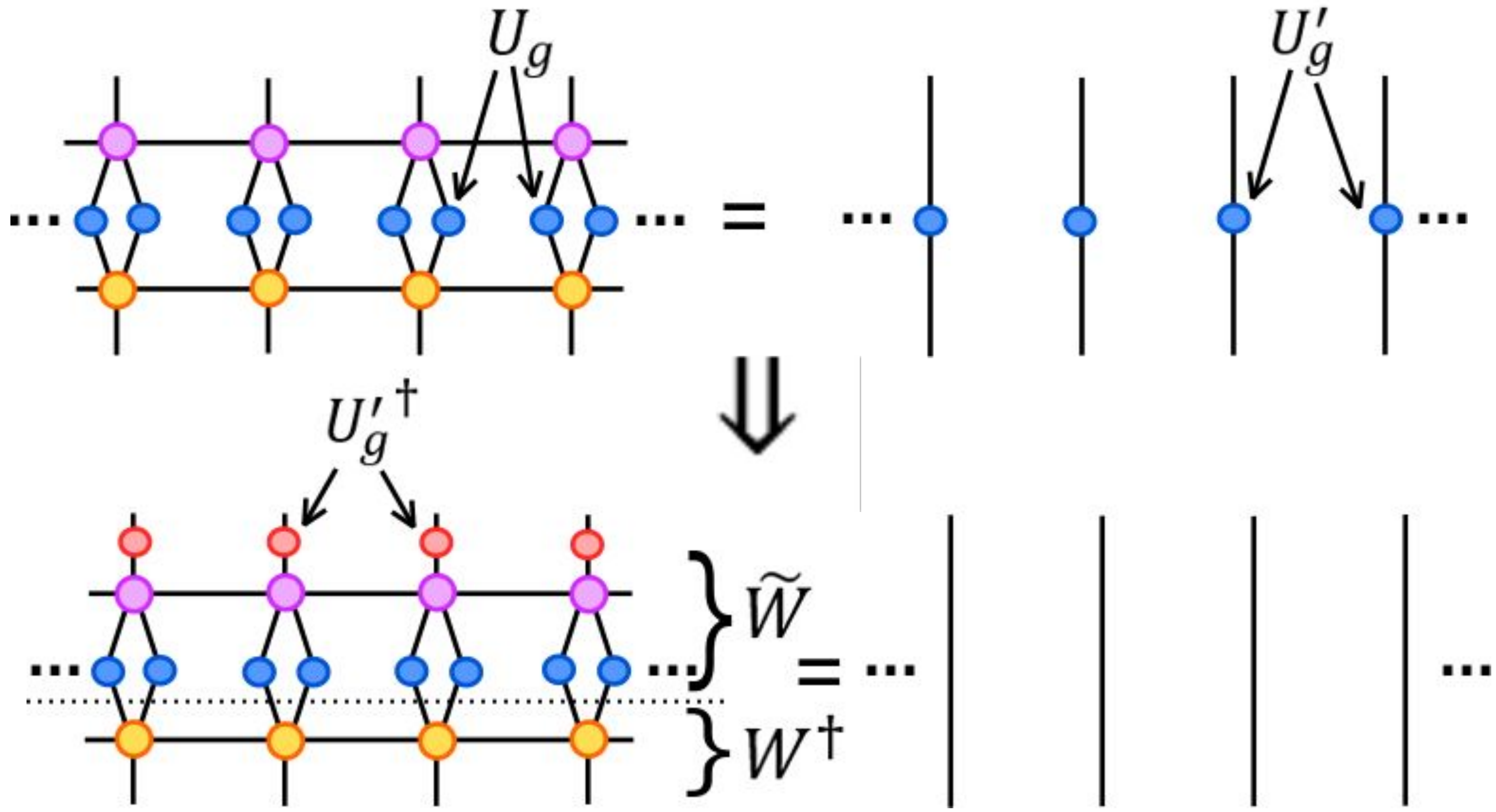


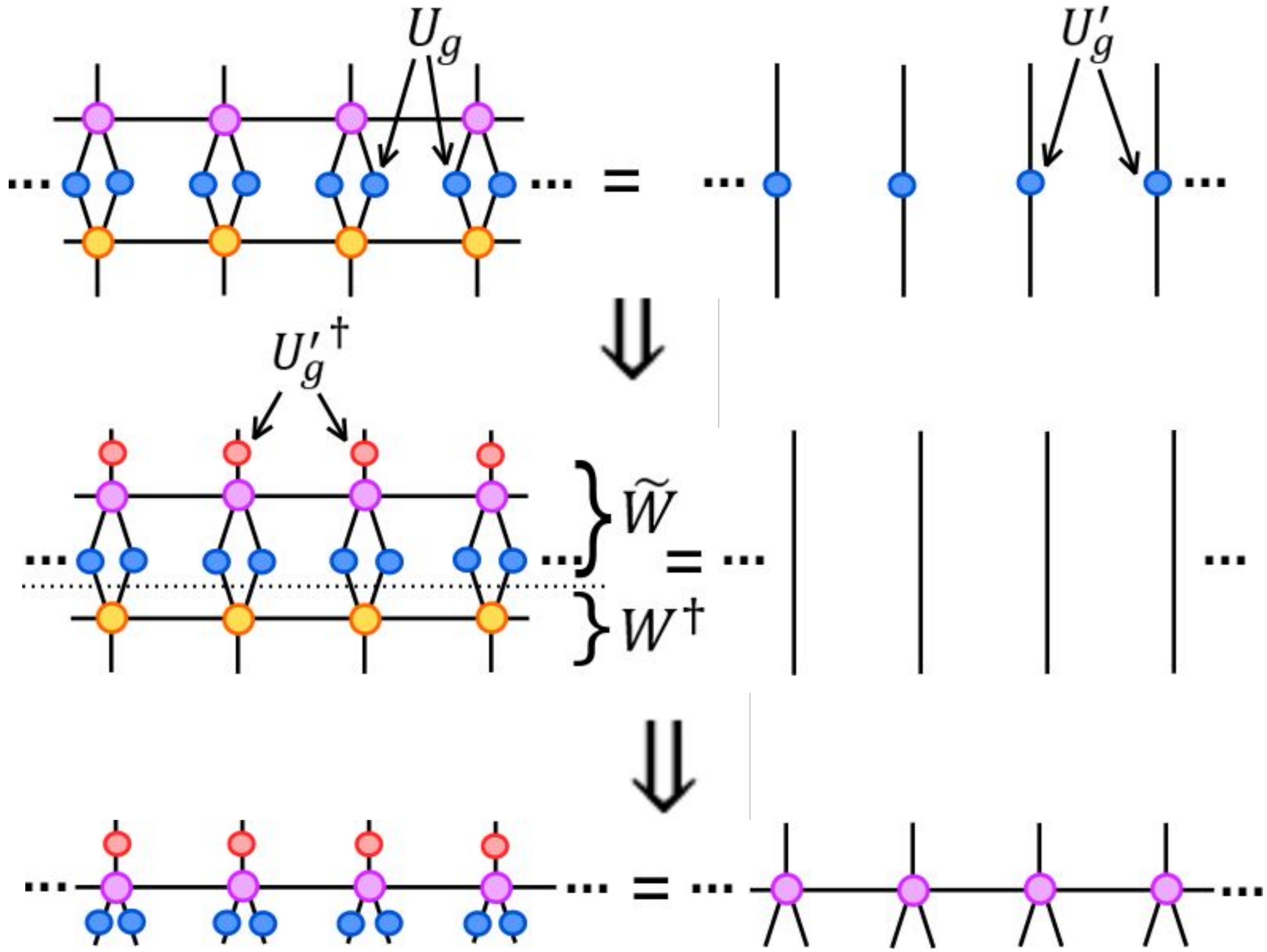
Assume U'_g is linear. But is it also on-site?

Yes (anomaly matching)

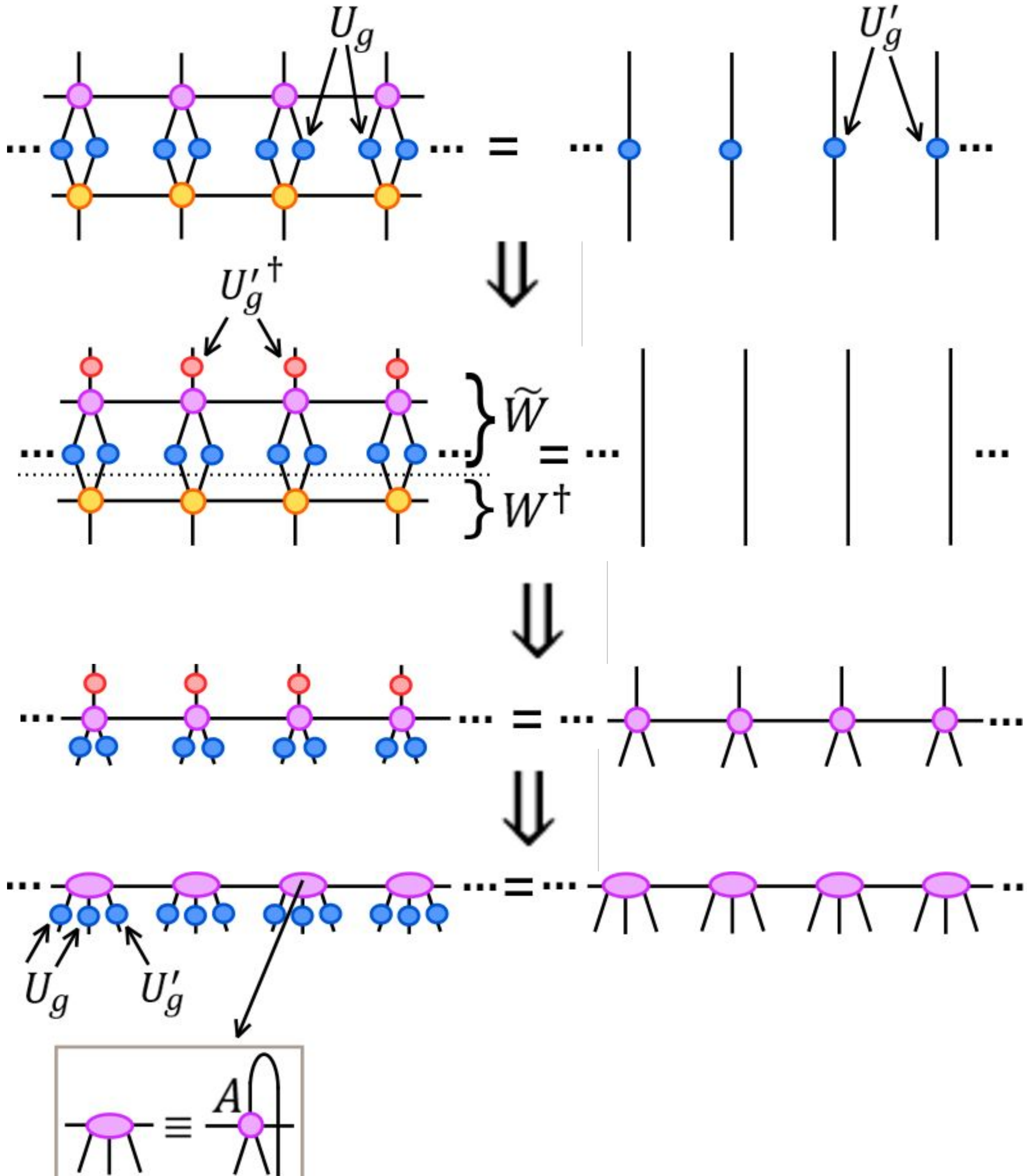


U_g
U_g'

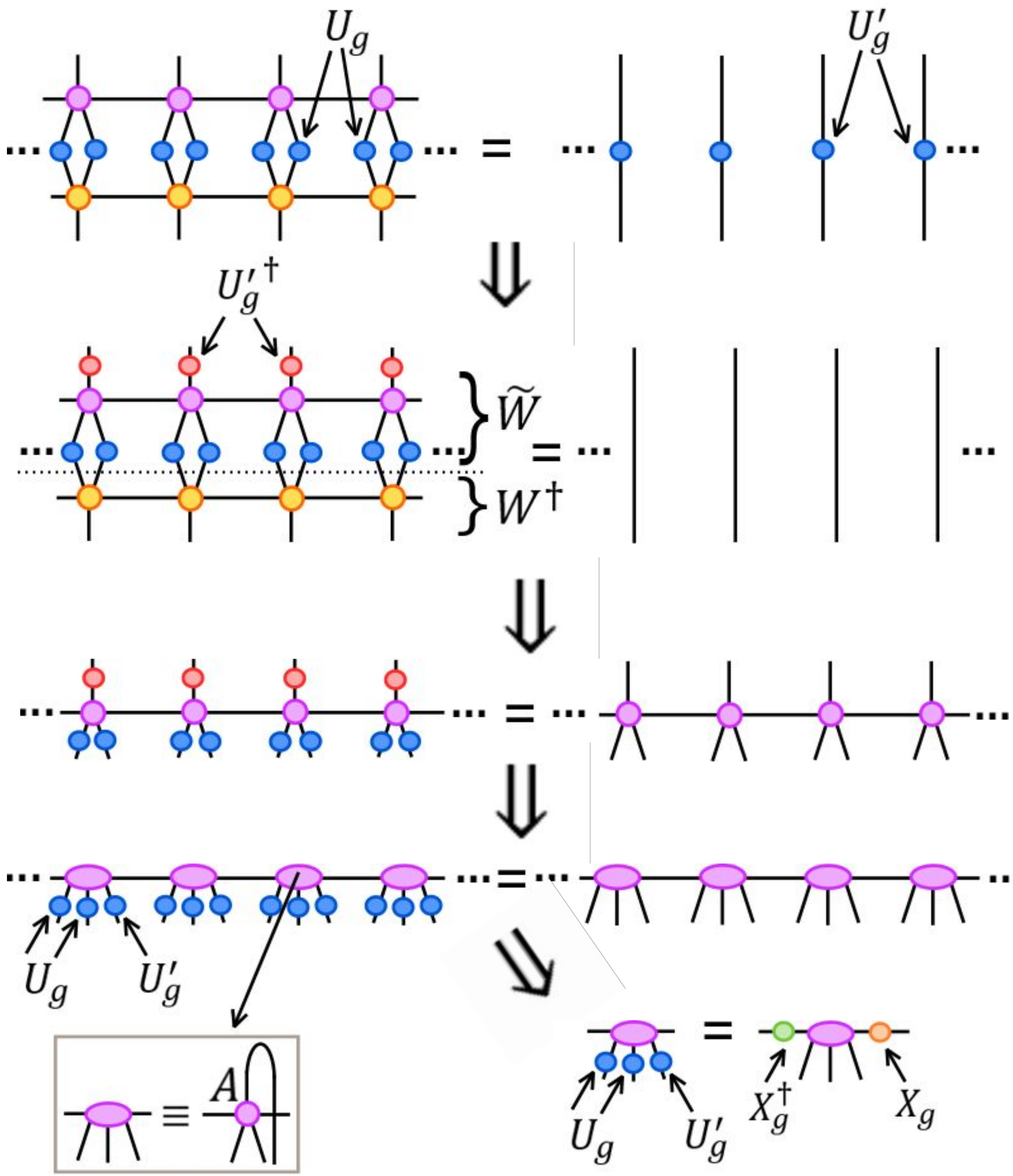

U_g
U'_g
$U'^{\dagger}_g$
$\widetilde{W}$
$W^{\dagger}$



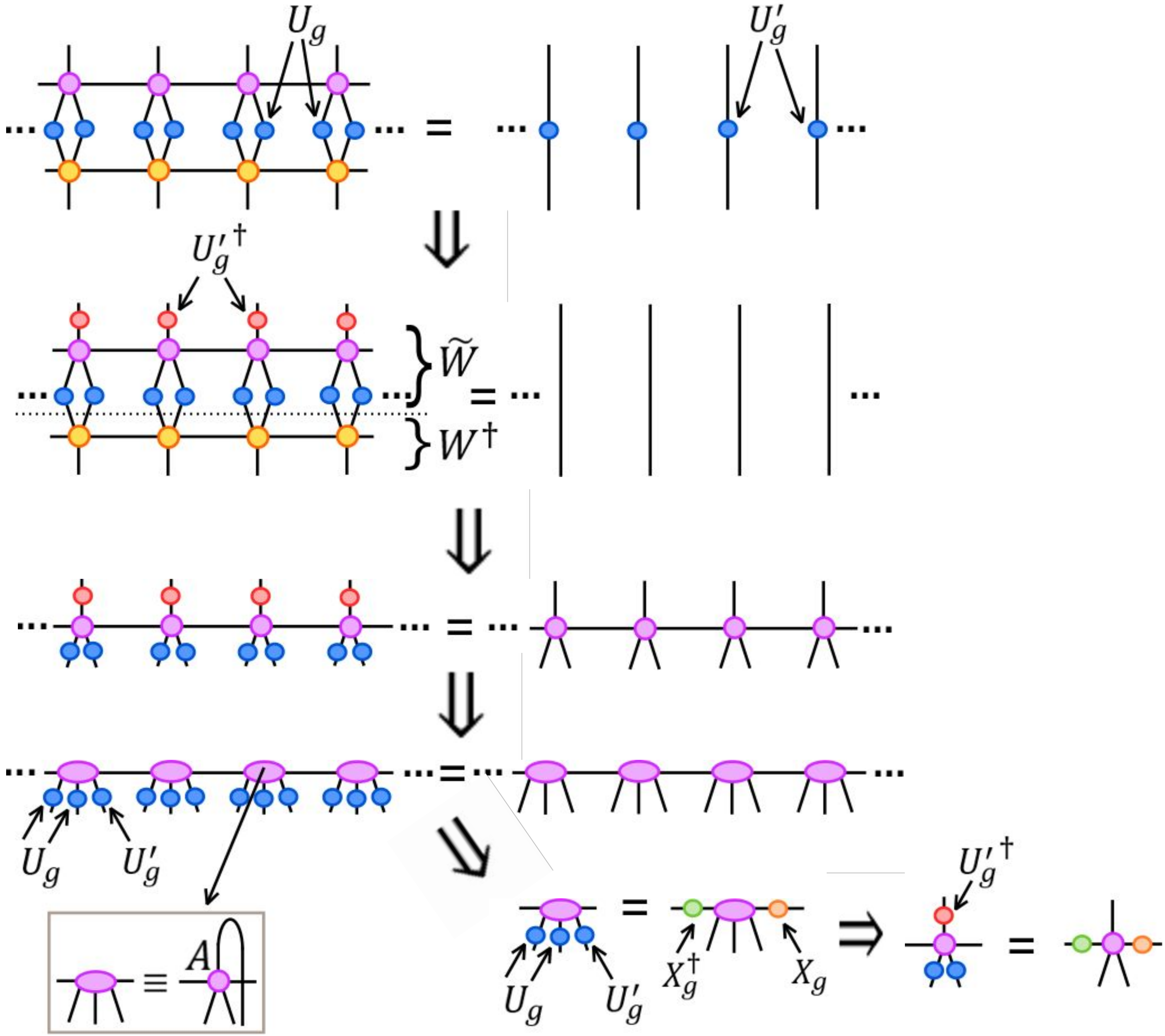
U_g
U_g'
$U_g'^\dagger$
$\widetilde{W}$
$W^\dagger$



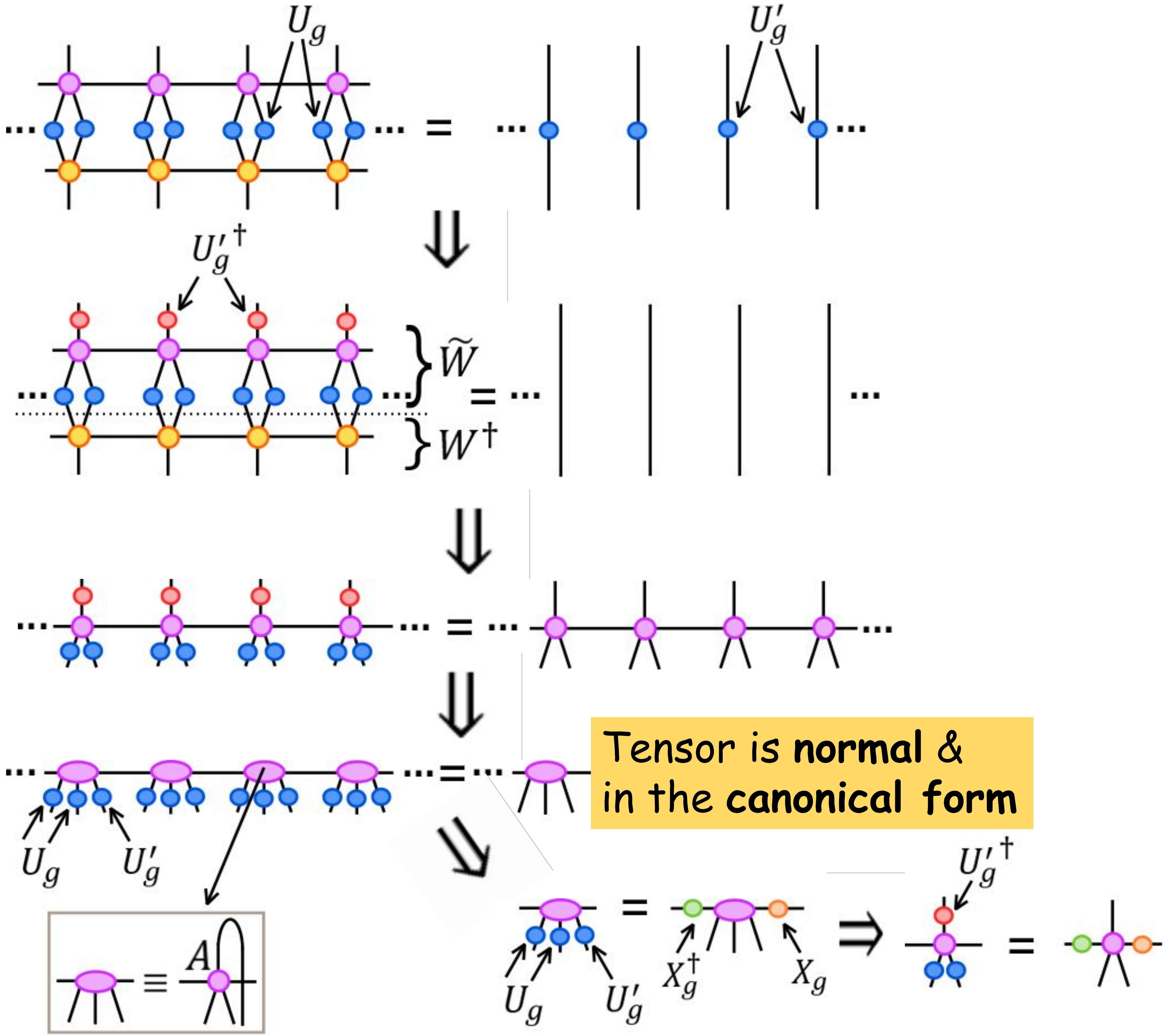
U_g
U'_g
$U'^{\dagger}_g$
$\widetilde{W}$
$W^{\dagger}$
U_g
U'_g
A



U_g
U'_g
${U'_g}^\dagger$
$\widetilde{W}$
$W^\dagger$
U_g
U'_g
A
$\equiv$
U_g
U'_g
$X_g^\dagger$
X_g



U_g
U'_g
$U'^{\dagger}_g$
$\widetilde{W}$
$W^{\dagger}$
U_g
U'_g
A
U_g
U'_g
$X^{\dagger}_g$
X_g
$U'^{\dagger}_g$



U_g
U'_g
$U'^{\dagger}_g$
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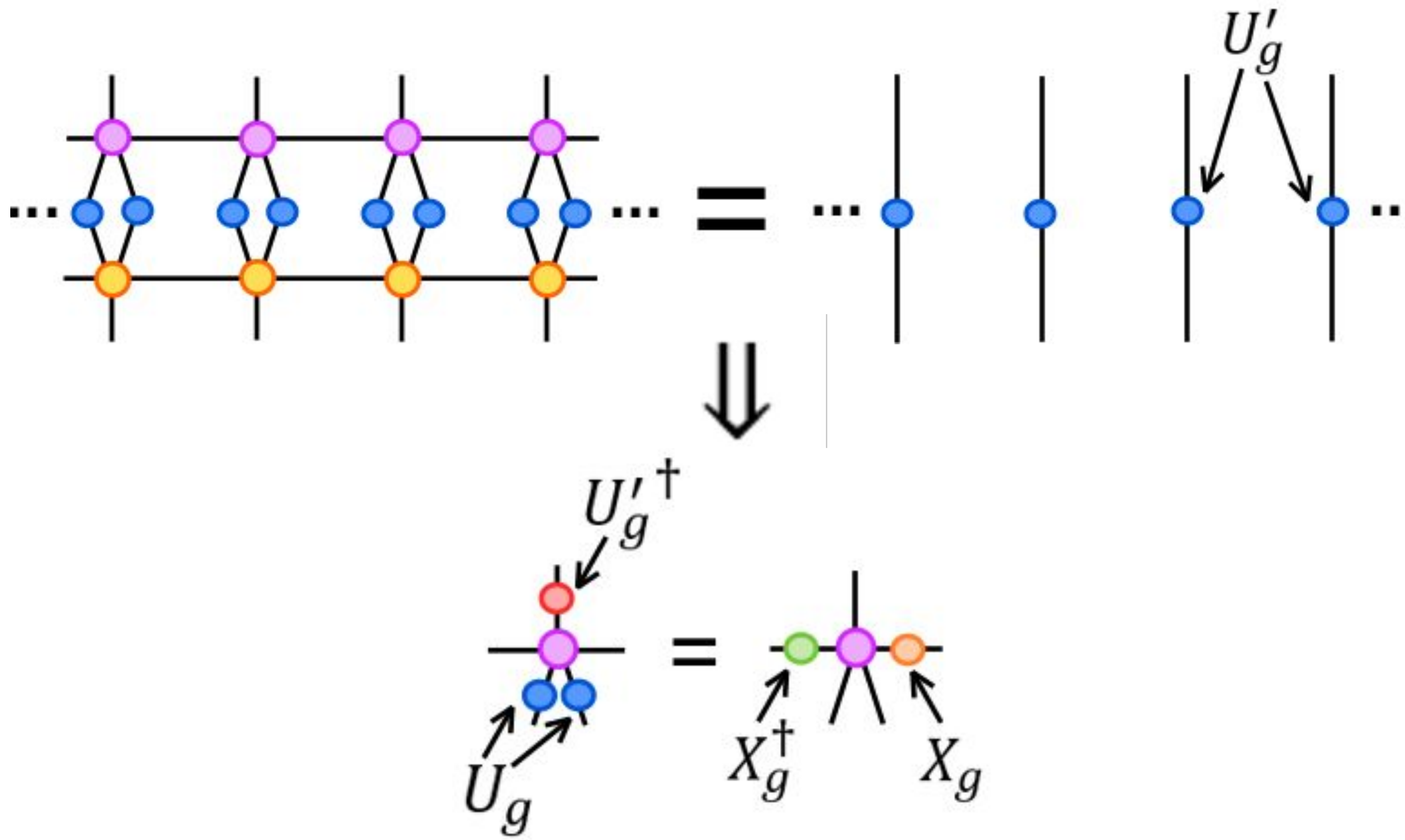
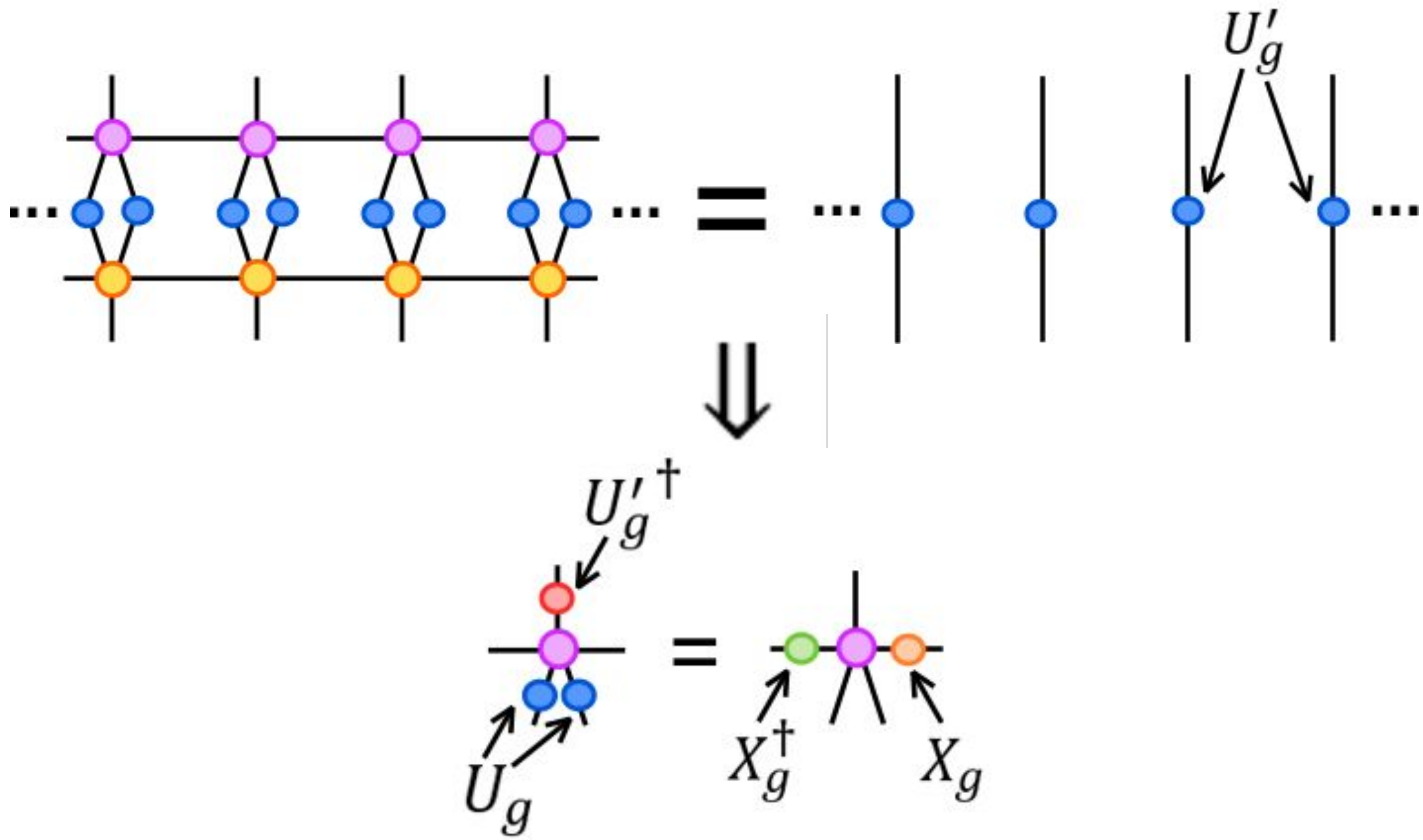
Tensor is **normal** &
in the **canonical form**
A
$X^{\dagger}_g$
X_g



U'_g
$U'^{\dagger}_g$
U_g
$X^{\dagger}_g$
X_g



X_g can be **projective**

In a gapped phase? (No)

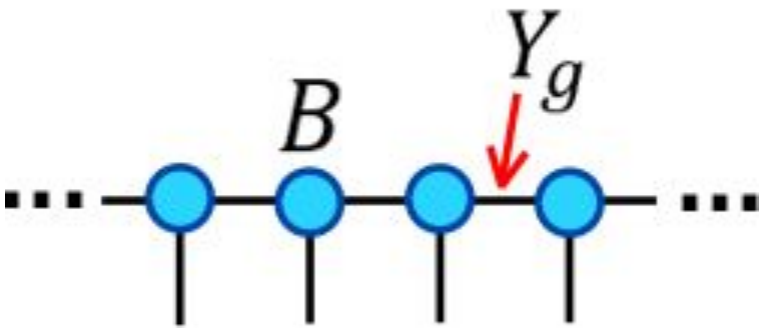
In a critical phase? (?)

MERA
and
1D gapped SP phases

Obtain MERA by coarse-graining MPS

MERA representation of a state = RG flow + Fixed point

Does MERA pick up a projective rep along the RG flow?



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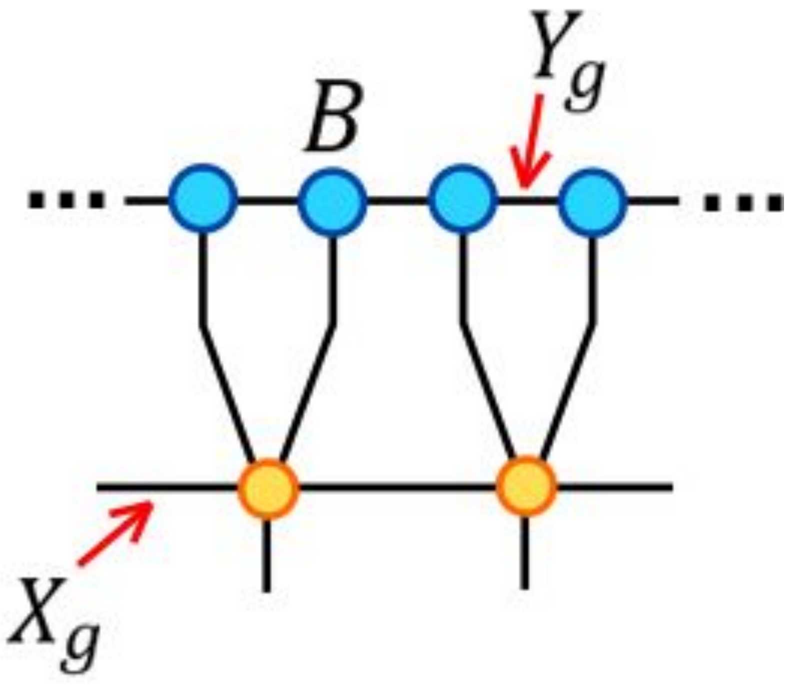

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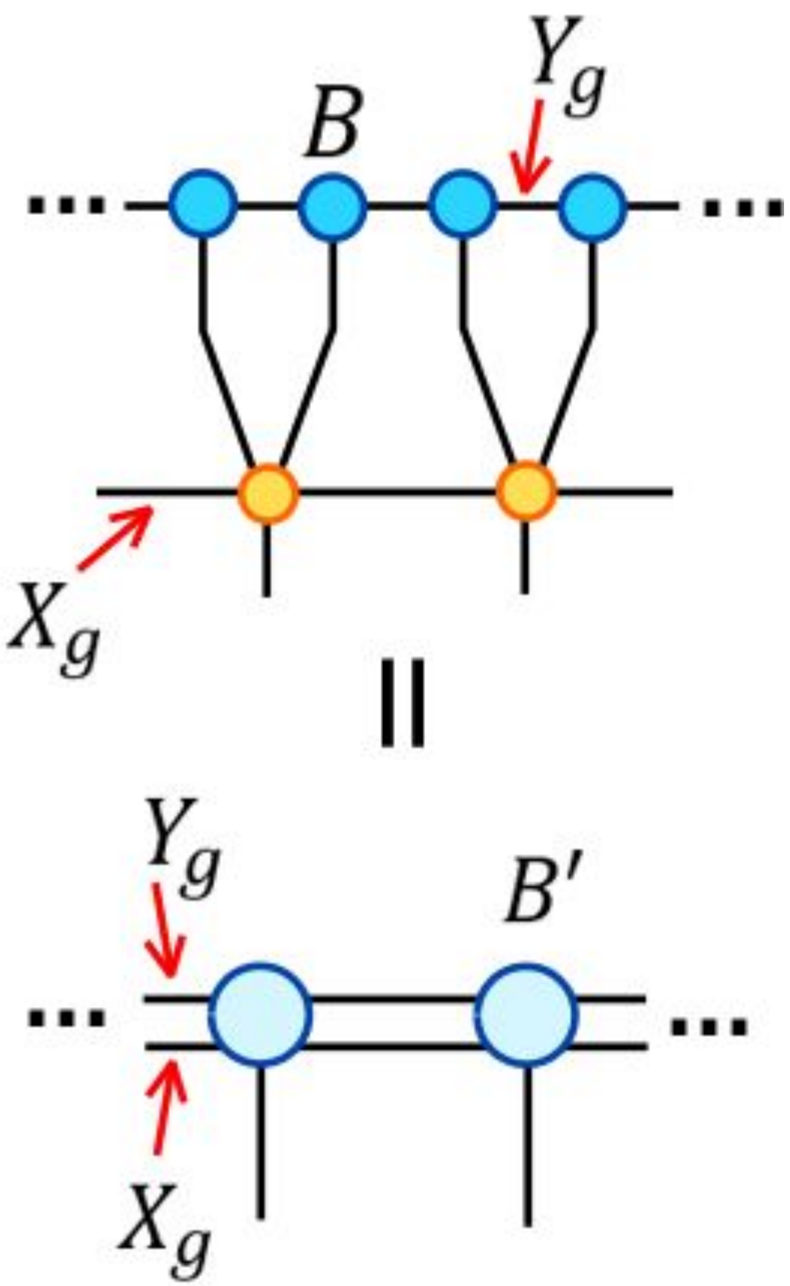

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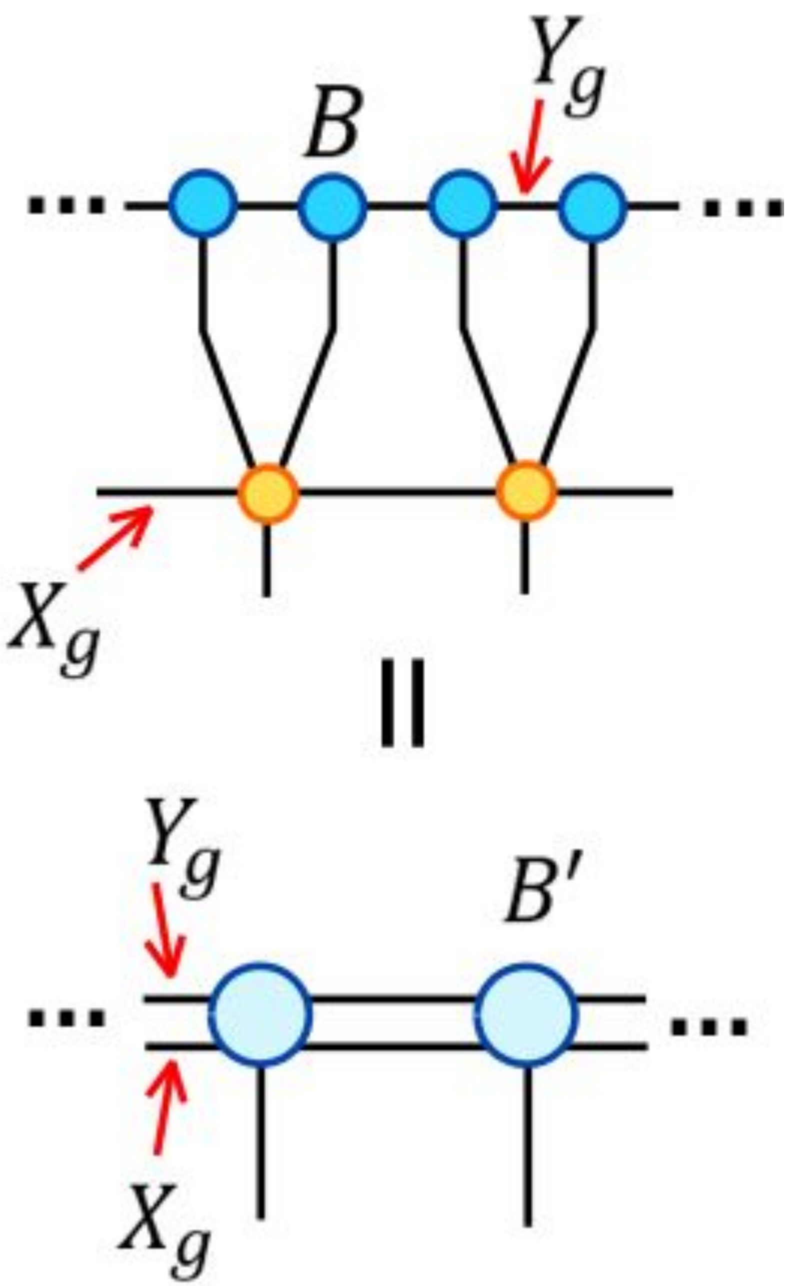

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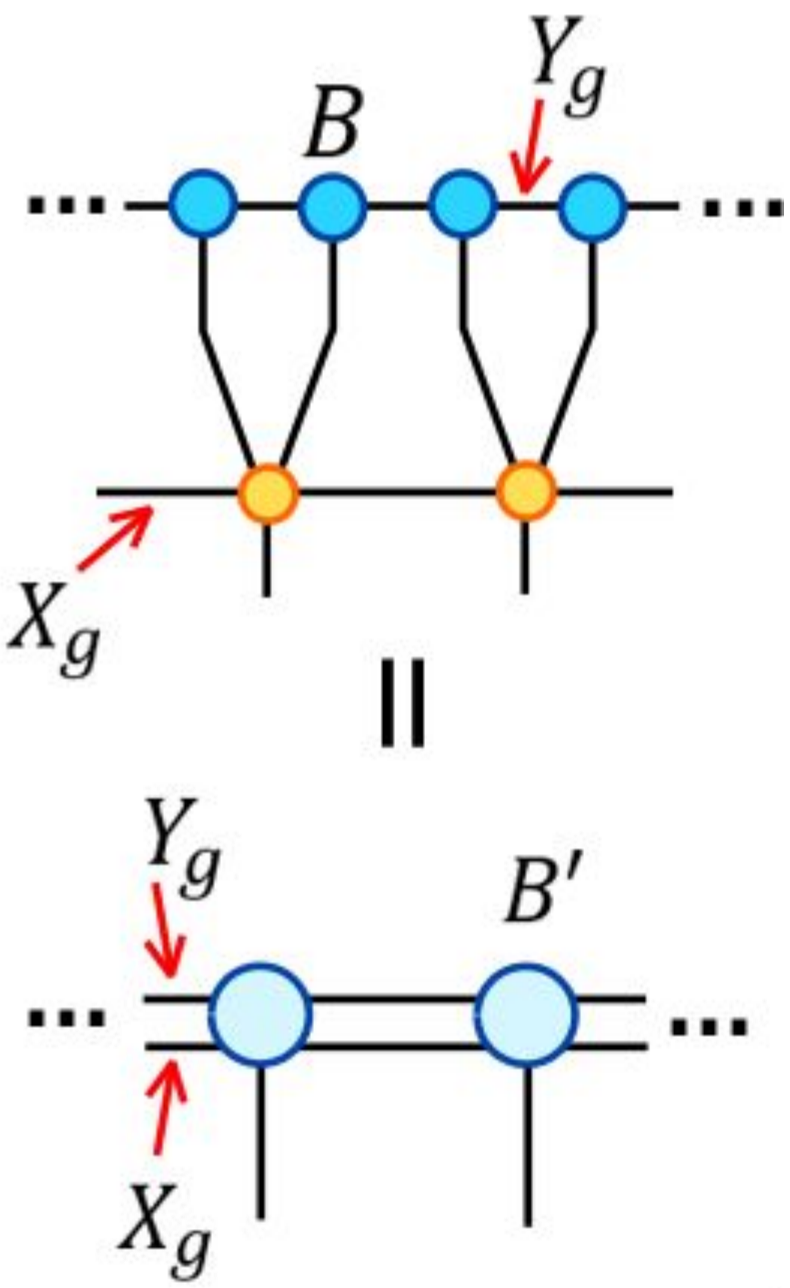

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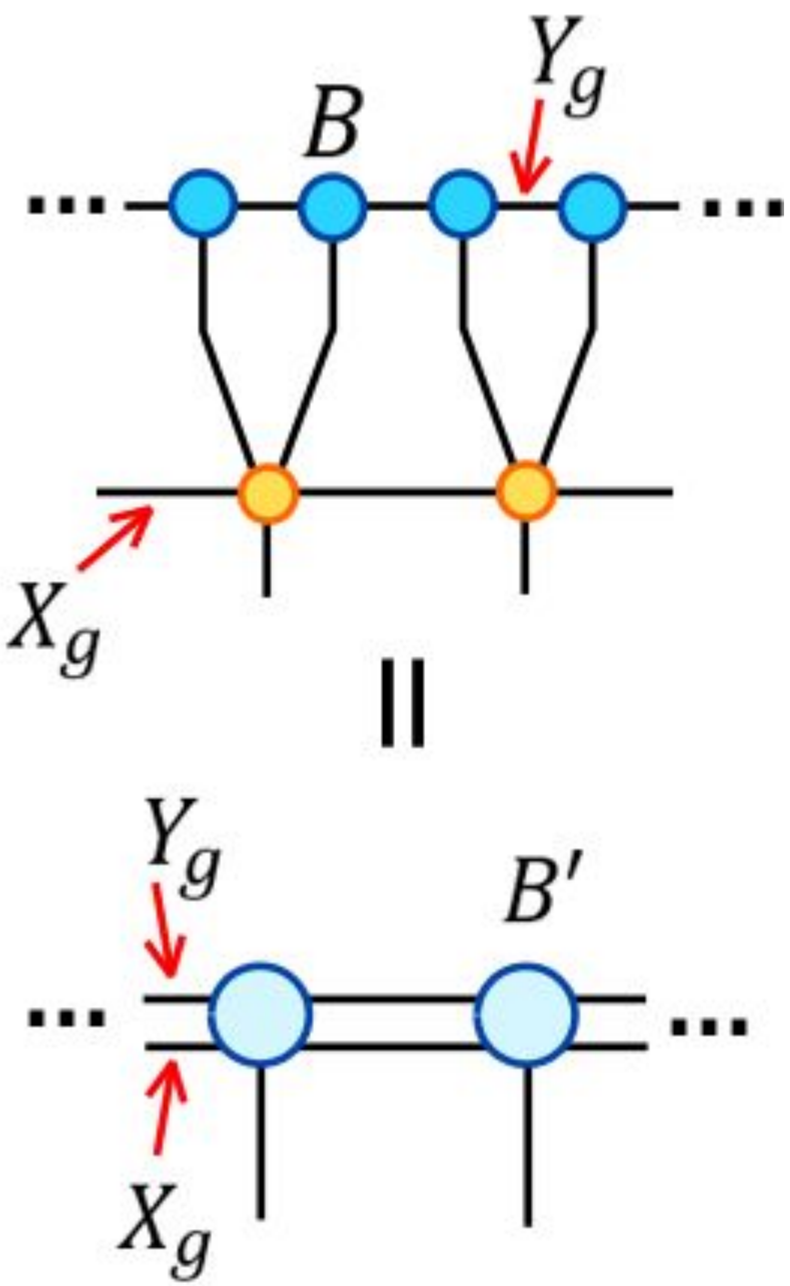

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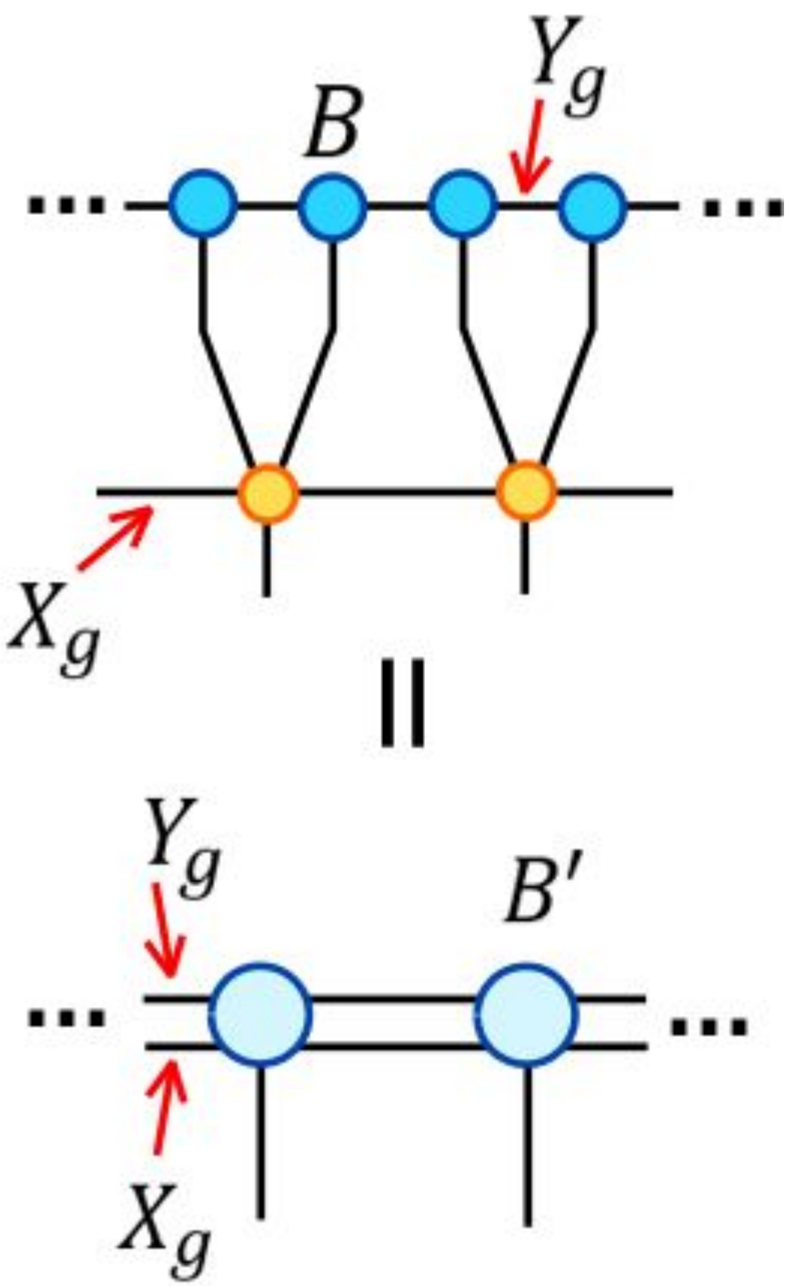

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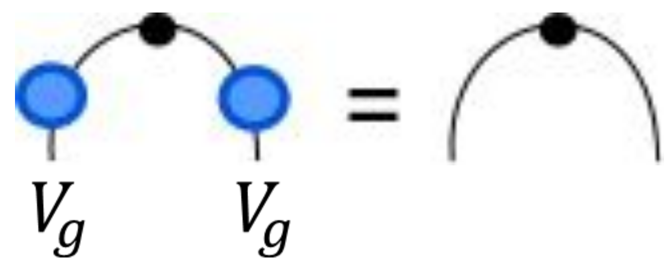
MERA does not pick up a projective rep at least along the RG flow.

Does MERA pick up a projective rep at the fixed point?

$\left|\Psi_{\phi}^{\text{fixed}}\right\rangle \equiv$ 



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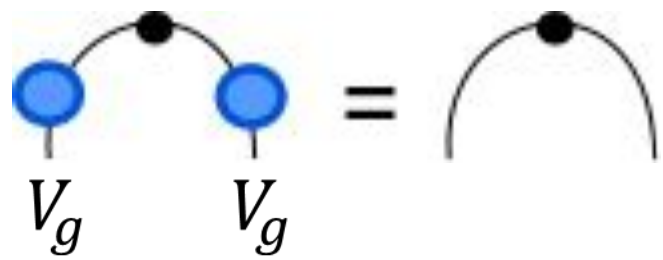
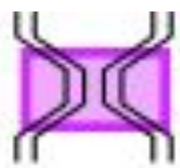


$u_\phi^{\text{fixed}} \equiv$ 
$w_\phi^{\text{fixed}} \equiv$ 

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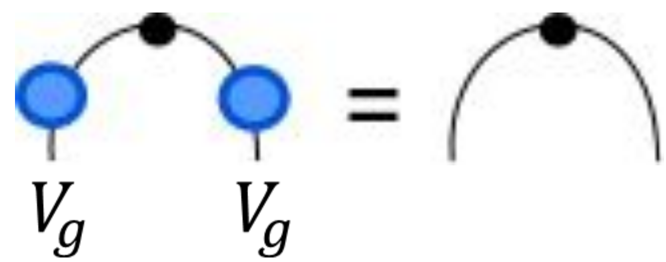
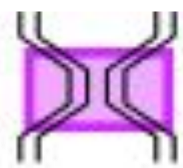
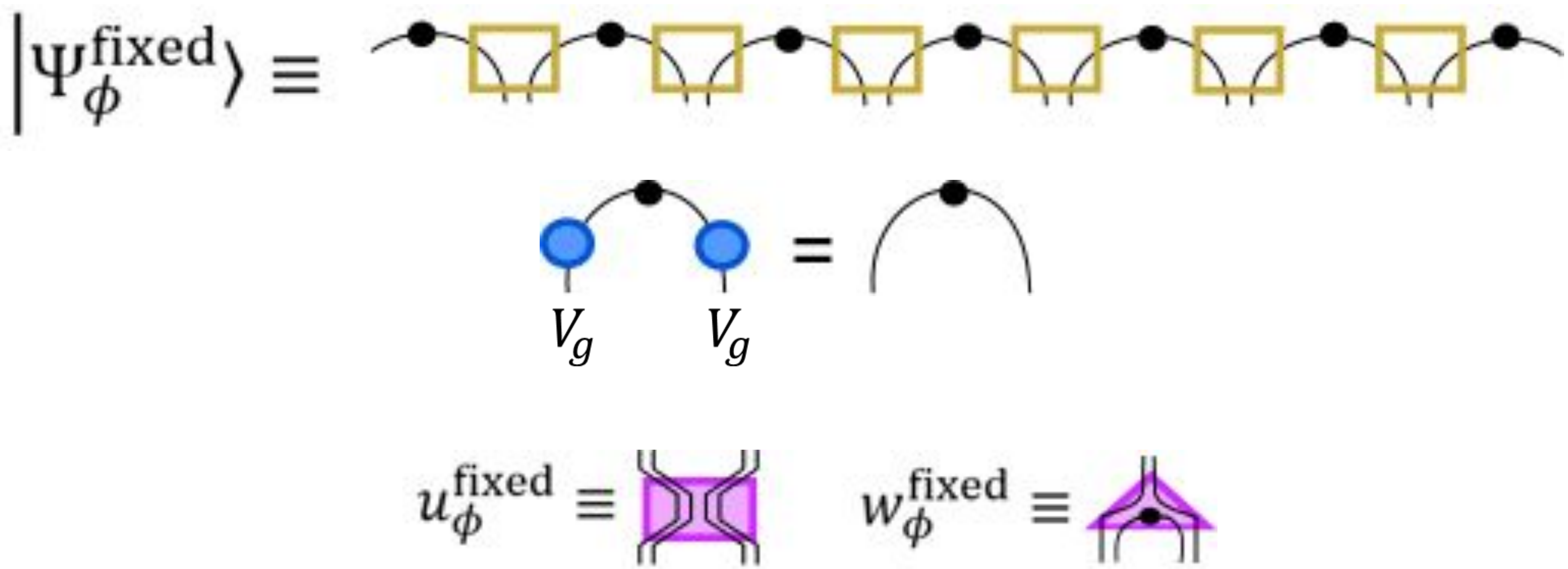


$u_\phi^{\text{fixed}} \equiv$  $w_\phi^{\text{fixed}} \equiv$ 

No projective rep also at the fixed point

Does MERA pick up a projective rep at the fixed point?



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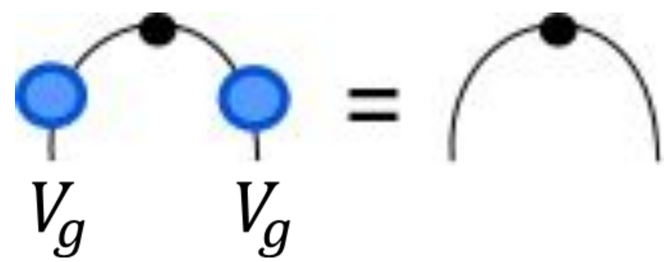
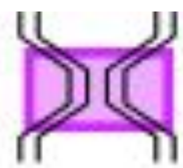


$u_\phi^{\mathrm{fixed}} \equiv$  $w_\phi^{\mathrm{fixed}} \equiv$ 

No projective rep also at the fixed point

MERA representation = RG flow + fixed point

So no projective reps in the MERA representation of a gapped symmetry protected ground state.

At a critical point?

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Our argument does not apply to critical states.

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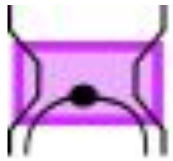

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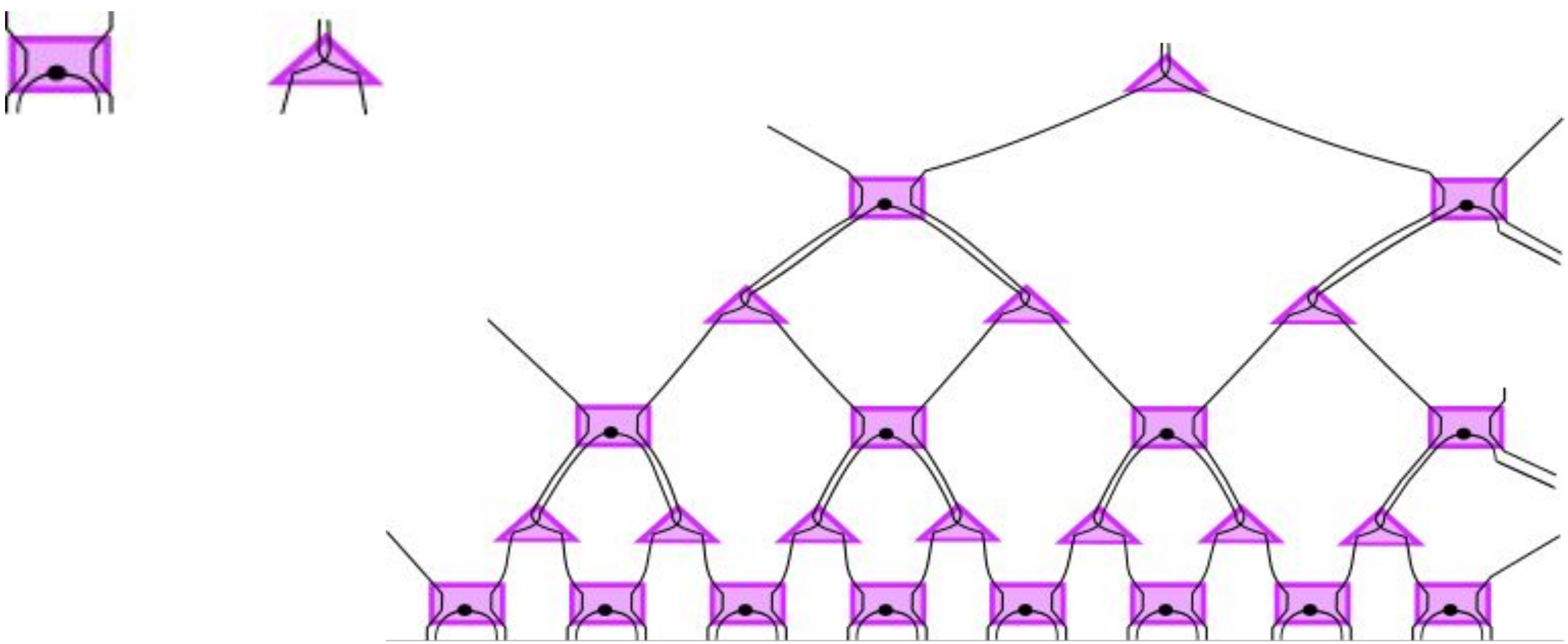

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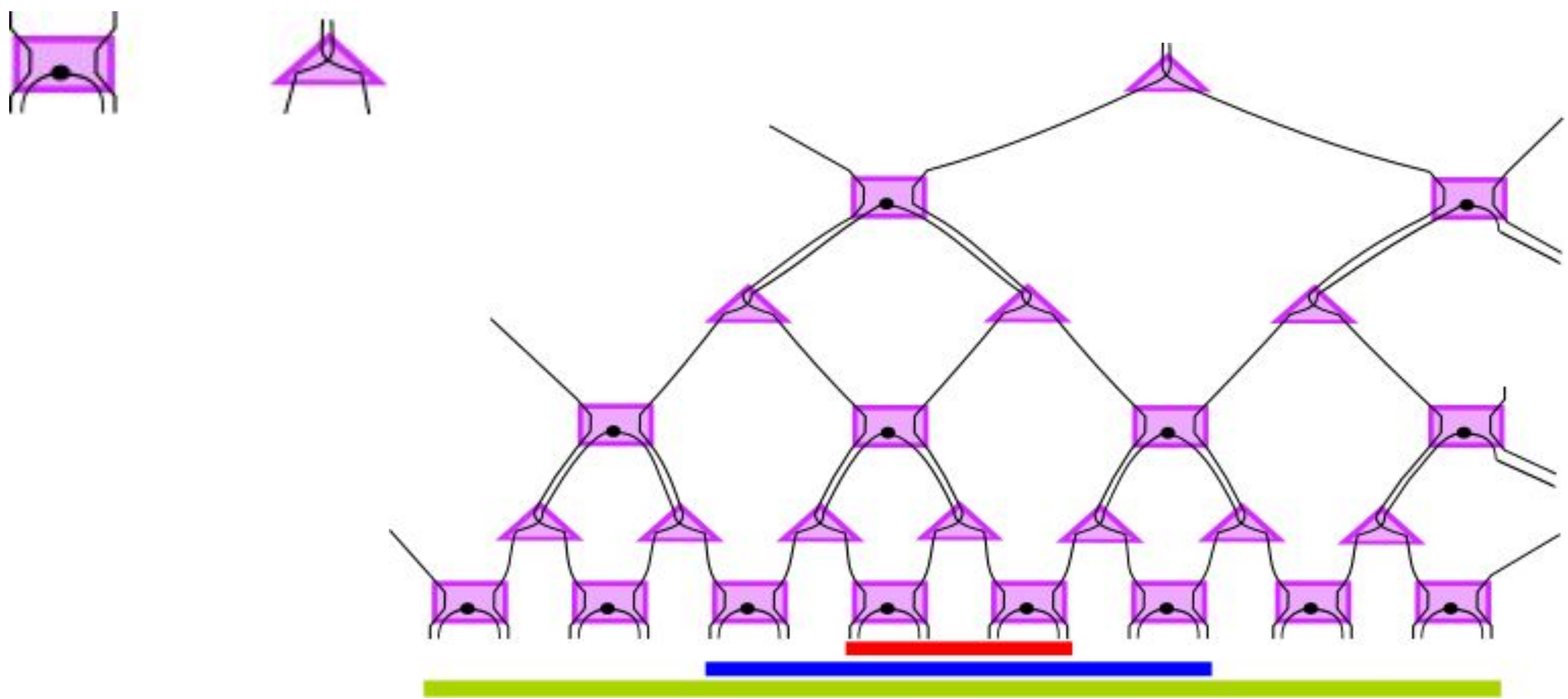

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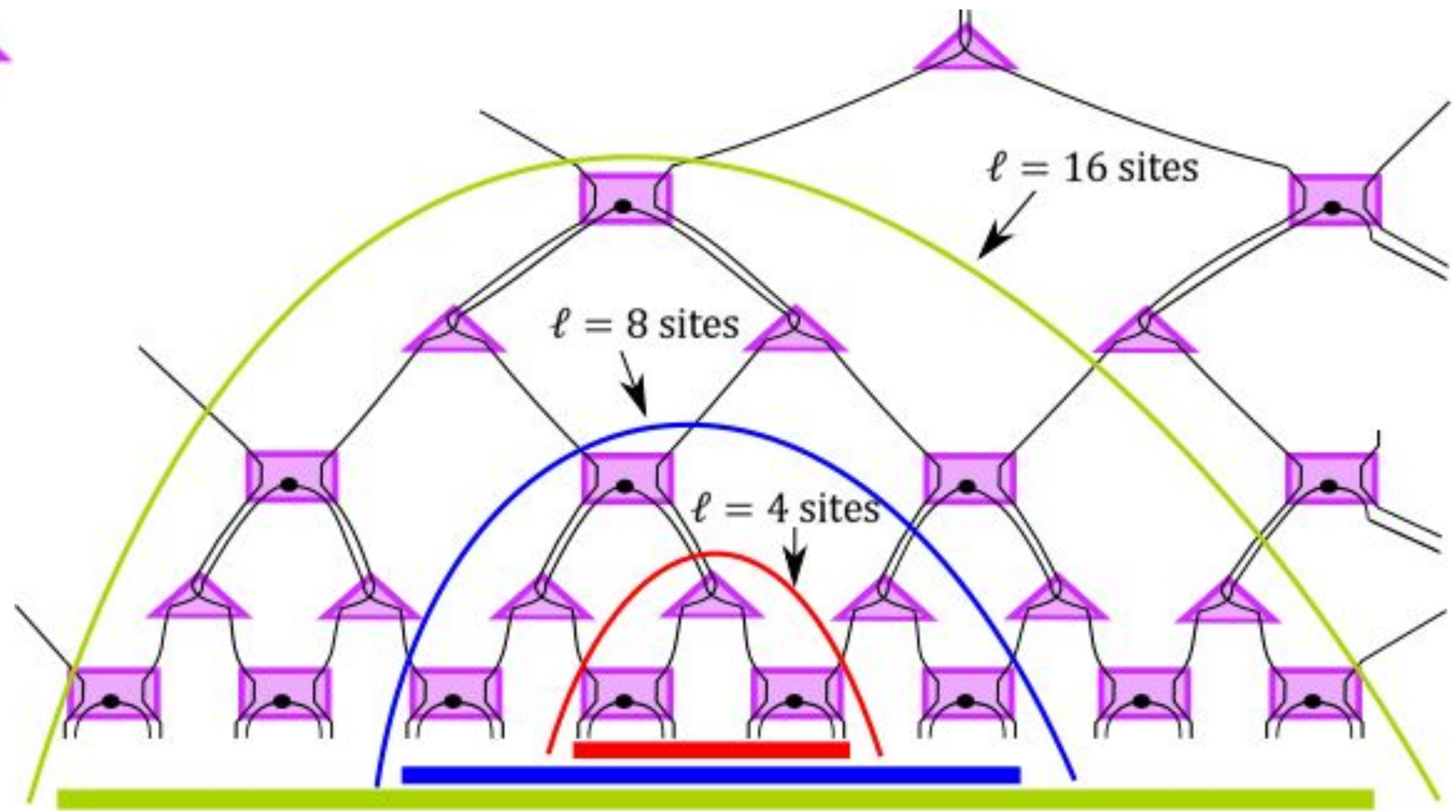

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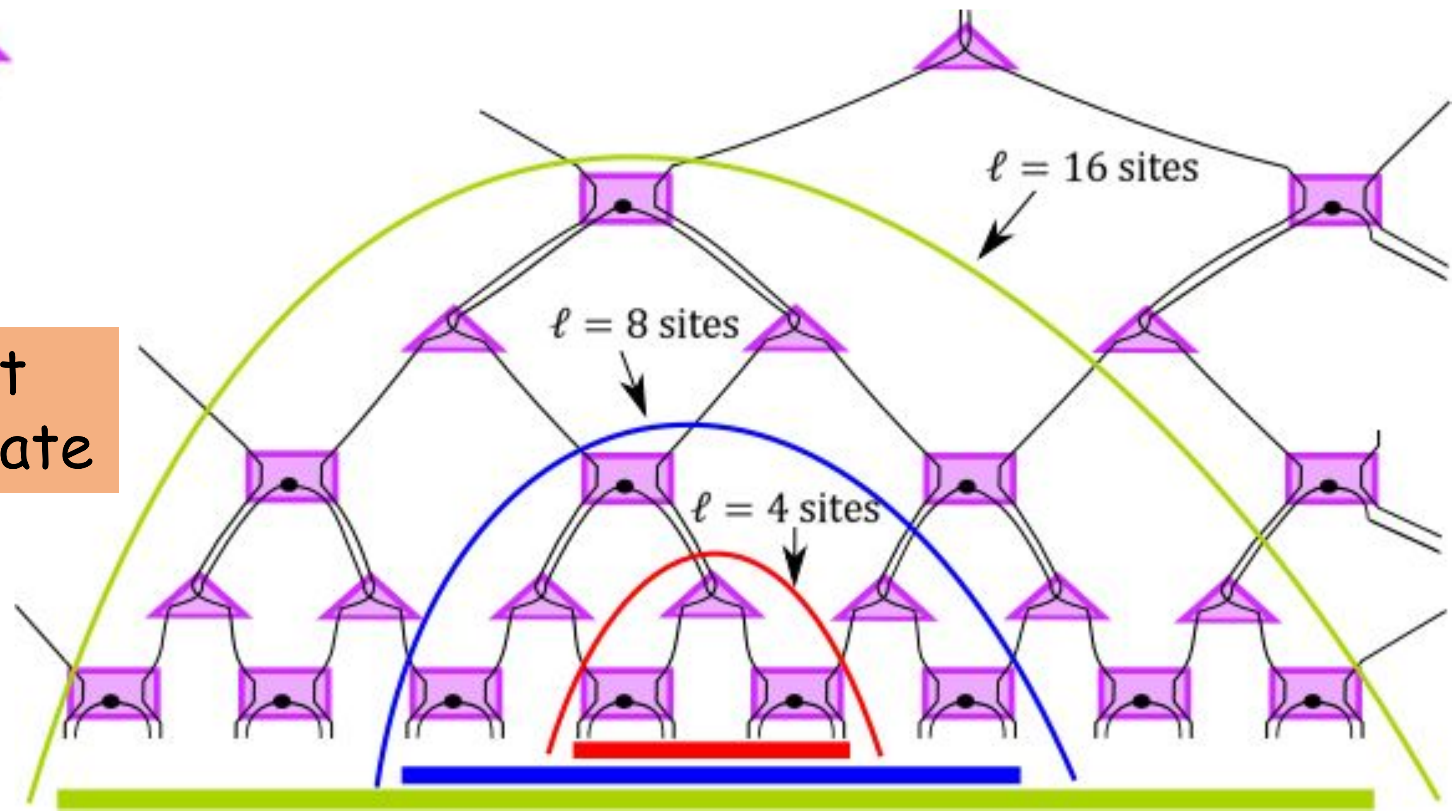
However, this is not a critical ground state



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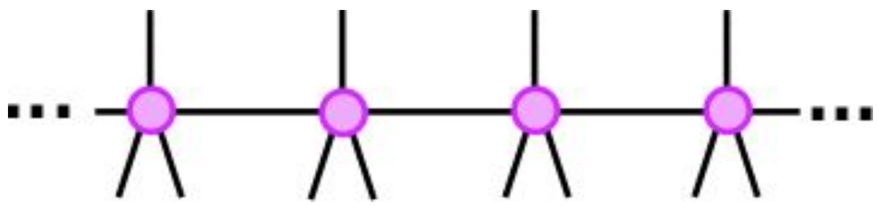

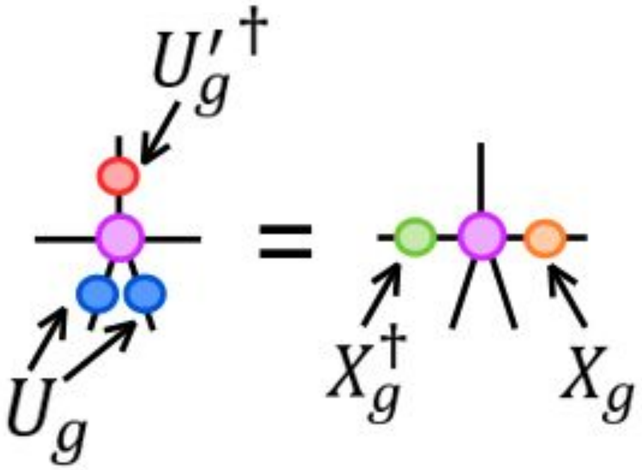

Summary

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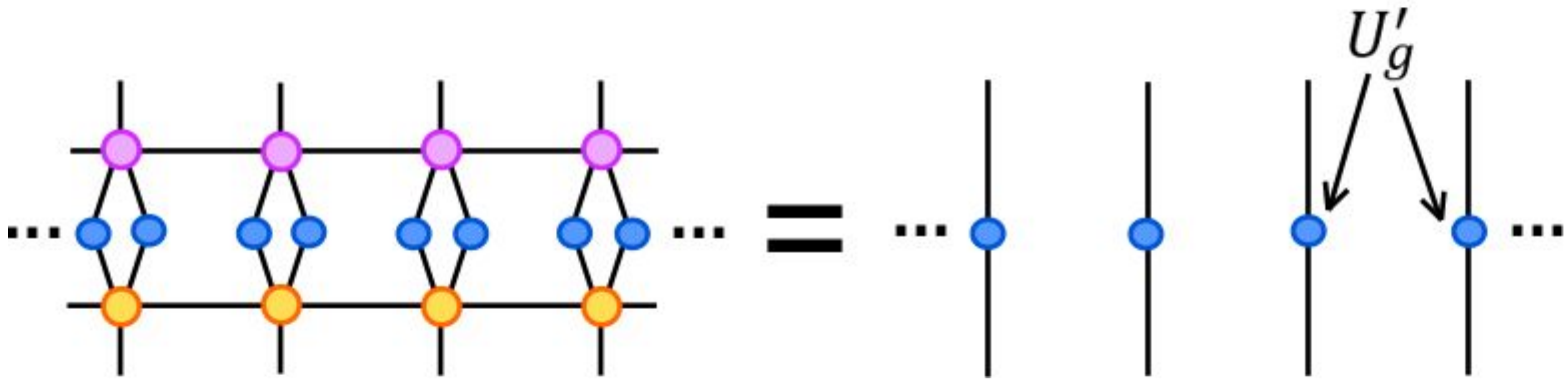
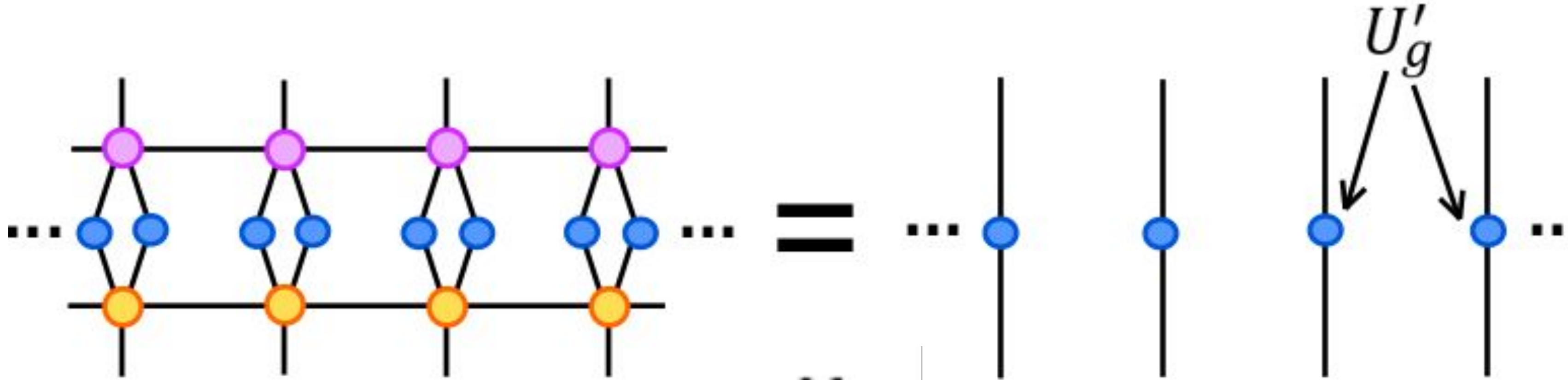
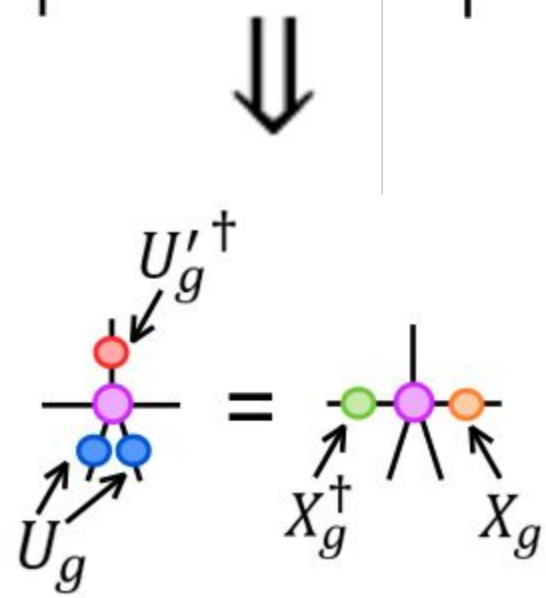


Symmetry protected
Entanglement renormalization

Summary

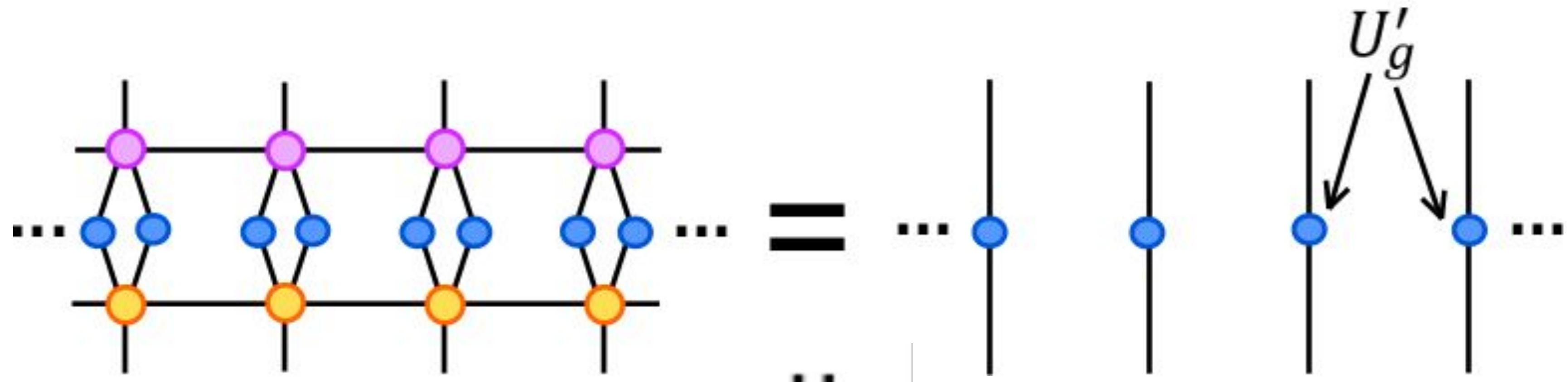




Symmetry protected
Entanglement renormalization

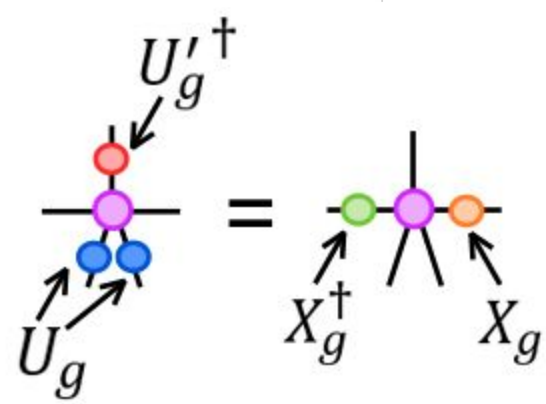
X_g can be **projective**

Summary



$\Downarrow$

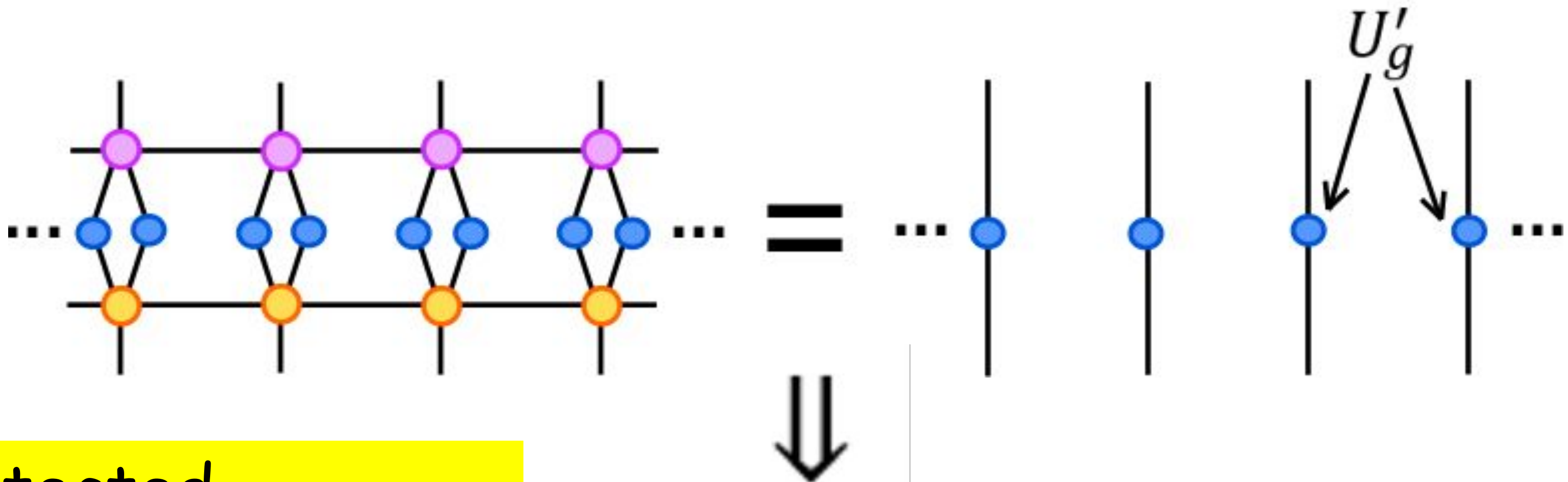
Symmetry protected
Entanglement renormalization



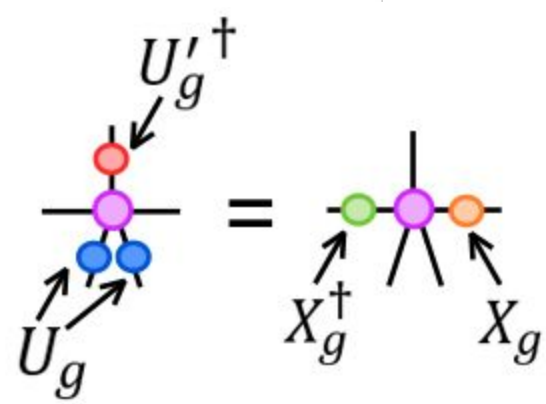
X_g can be **projective**

But X_g cannot be projective in a **gapped** phase.

Summary



Symmetry protected
Entanglement renormalization



X_g can be **projective**

But X_g cannot be projective in a **gapped** phase.

What's next?

Determine and classify critical fixed points <-> distinct critical SP phases?