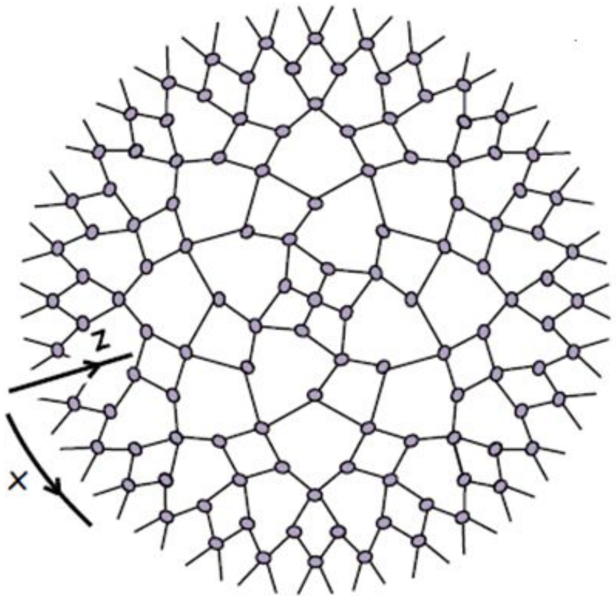
Tensor networks as toy models for holography?

Sukhi Singh

Institute for Quantum Optics and Quantum Information, Vienna



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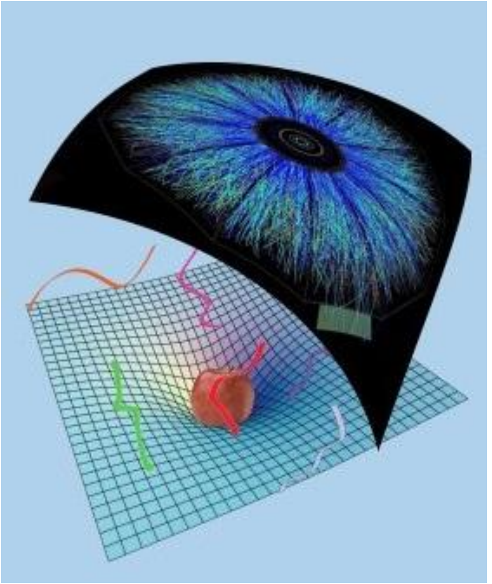
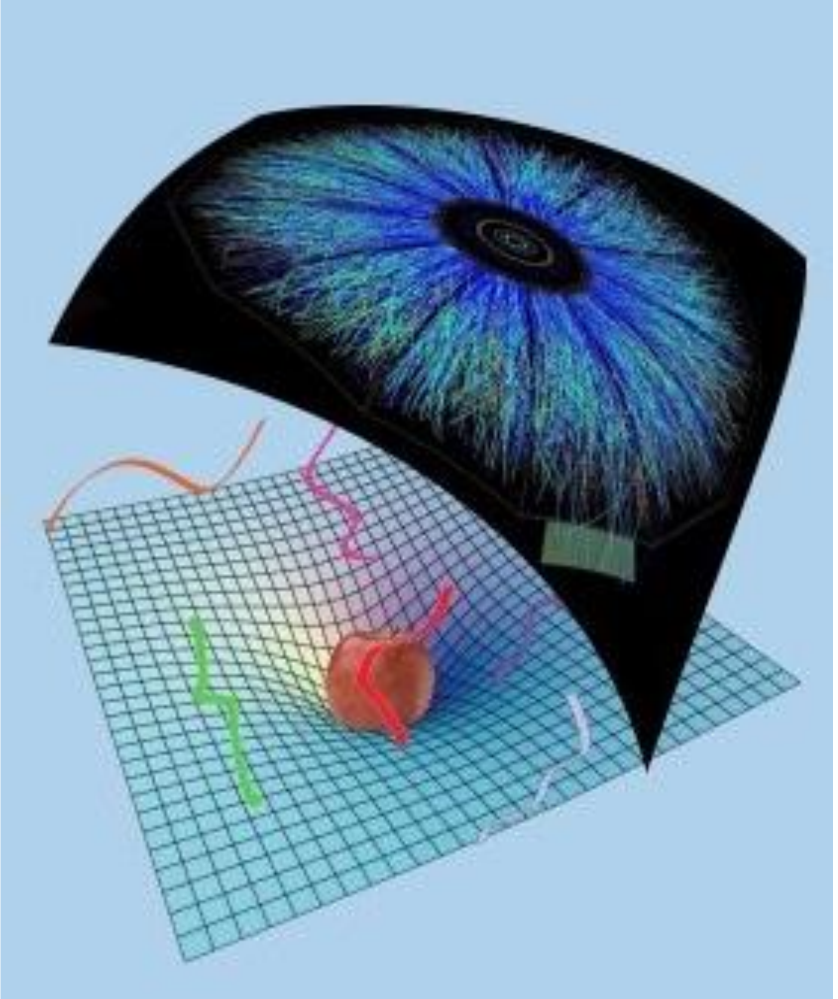


MERA

AdS/CFT

Contents

The holographic correspondence



Quantum gravity

- No universal agreement on what the problem is.

- The popular/conservative view is that gravity must be described by a quantum field theory.

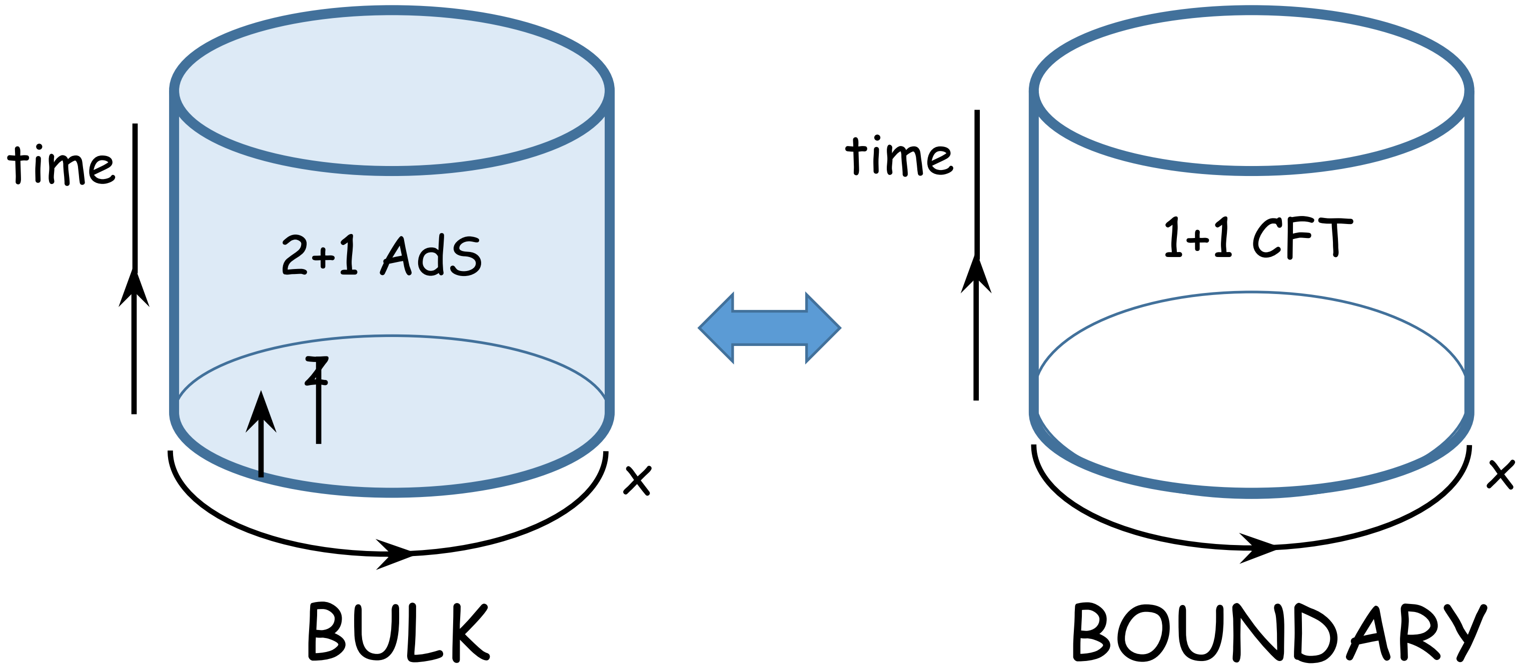
- **Challenges for a QFT description of gravity:**
  - A naïve QFT description is non-renormalizable and plagued with infinities.
  - Gravity is different from other forces since gravity corresponds to dynamical space time is dynamical while QFT is usually formulated on a fixed space time.
  - Gravity is not a fundamental force (entropic force)
  - …

The holographic principle

- Certain theories of gravity are equivalent to QFTs but in one less dimension!
- Conjecture from extrapolating insights from studying (quantum) black holes.
- QFT encodes the gravity system like a 2D hologram encodes a 3D image.
- This sounds like an intriguing idea but is there a concrete realization of this idea?
- The **AdS/CFT correspondence**

The AdS/CFT correspondence

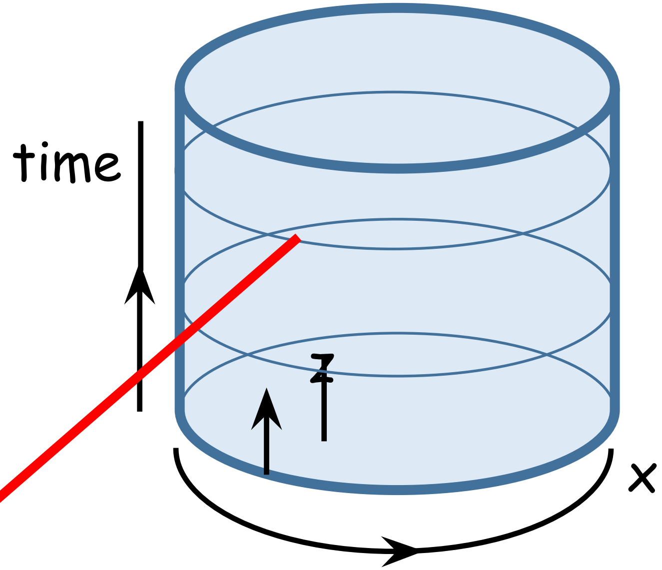
J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) *E. Witten, Adv.Theor. Math. Phys. 2, 253 (1998)*



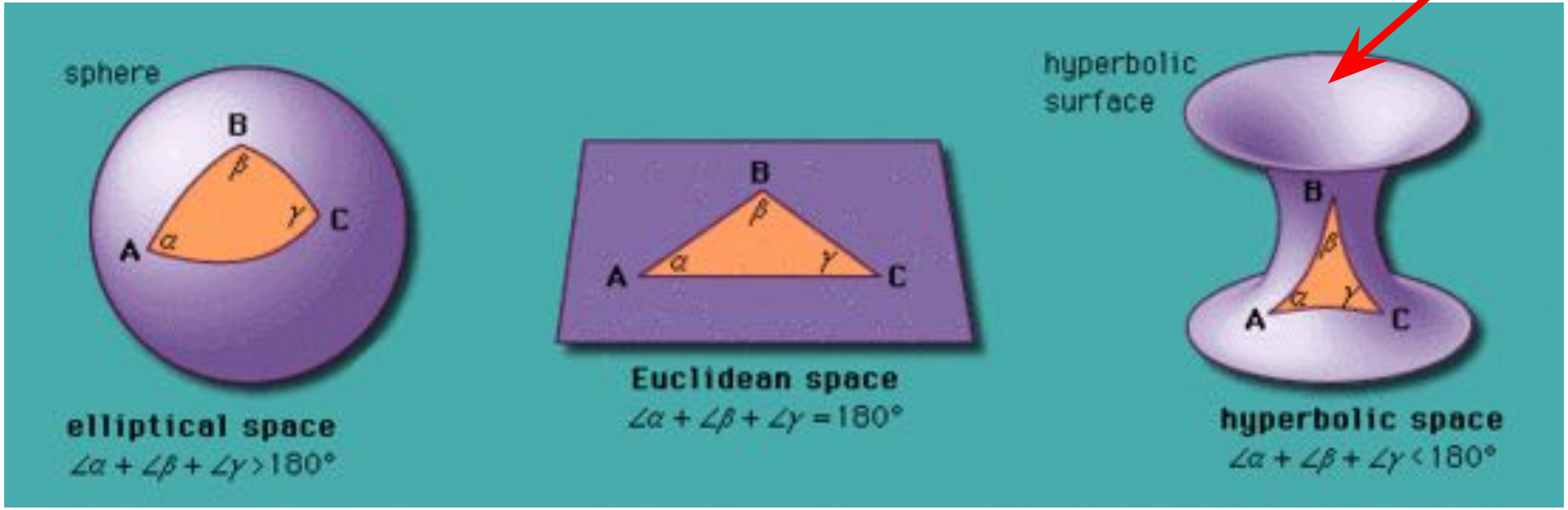
Gravity in **anti-deSitter (AdS) spacetime** ⟺ **Conformal field theory (CFT)**

Anti-deSitter spacetime

- Solution to the vacuum Einstein equations with negative cosmological constant.
- Space time with constant negative curvature.
- Can be foliated into hyperbolic time-like surfaces.



Penrose diagram of 2+1 AdS spacetime as a solid cylinder.



Conformal field theory

- Action is invariant under the action of the conformal group

- In 1+1 dims, the action is invariant under global change of scale.

$$x^\mu \to \lambda x^\mu$$

- In 1+1 dims, the symmetry is very constraining. A CFT is completely specified by the following list of data:

• A list of **primary fields**
• **Scaling dimension** of each field
• **Central charge**
• **OPE coefficients** (which determine the two point correlators of the primary fields)

The AdS/CFT correspondence

- The bulk and boundary theories are equivalent: Partition functions are equal.

- Concrete recipe to compute bulk correlation functions from the boundary theory and vice-versa.

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- AdS/CFT is a strong/weak type duality. Strongly coupled gravity (quantum gravity) is dual to a weakly coupled CFT, and vice-versa!

- A formulation of quantum gravity as a (weakly coupled) CFT.

- Also, useful in condensed matter physics since CFTs appear commonly as effective description of certain strongly correlated matter (more on this later)

Subir Sachdev et al

The AdS/CFT correspondence

- Examples of how boundary and bulk properties are related in the weak gravity regime:

The AdS/CFT correspondence

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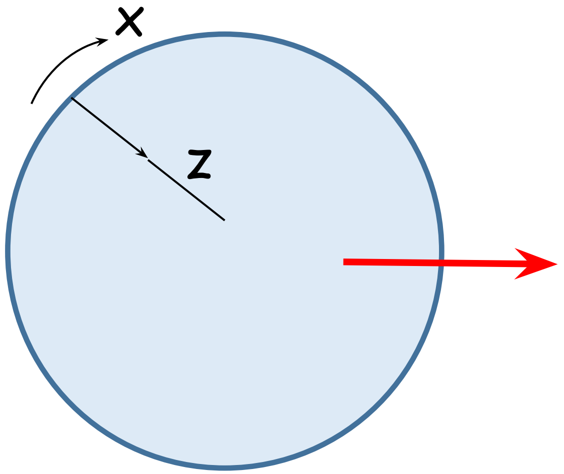

CFT ground state ⟺ Spatial slice of 2+1 AdS: 2D hyperbolic geometry

c: central charge ⟺ R: radius of curvature

J. Brown, M. Hennaux, Comm. Math. Phys., 1986

The AdS/CFT correspondence

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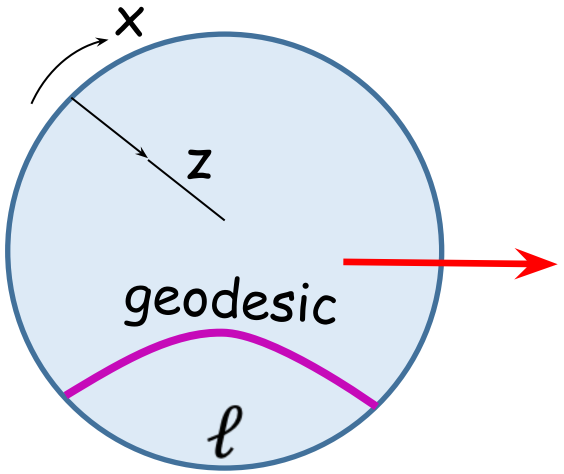

Spatial slice of 2+1 AdS:
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CFT ground state $\Leftrightarrow$

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Entanglement entropy ($\propto c \log \ell$) = Length of geodesic ($\propto R \log \ell$)

S. Ryu & T. Takayanagi, PRL 96, 181602 (2006)

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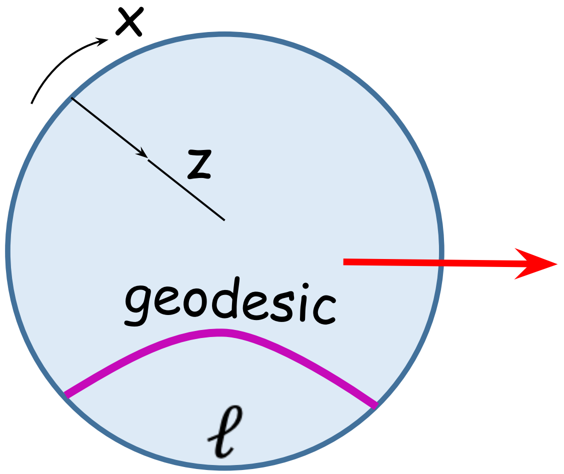

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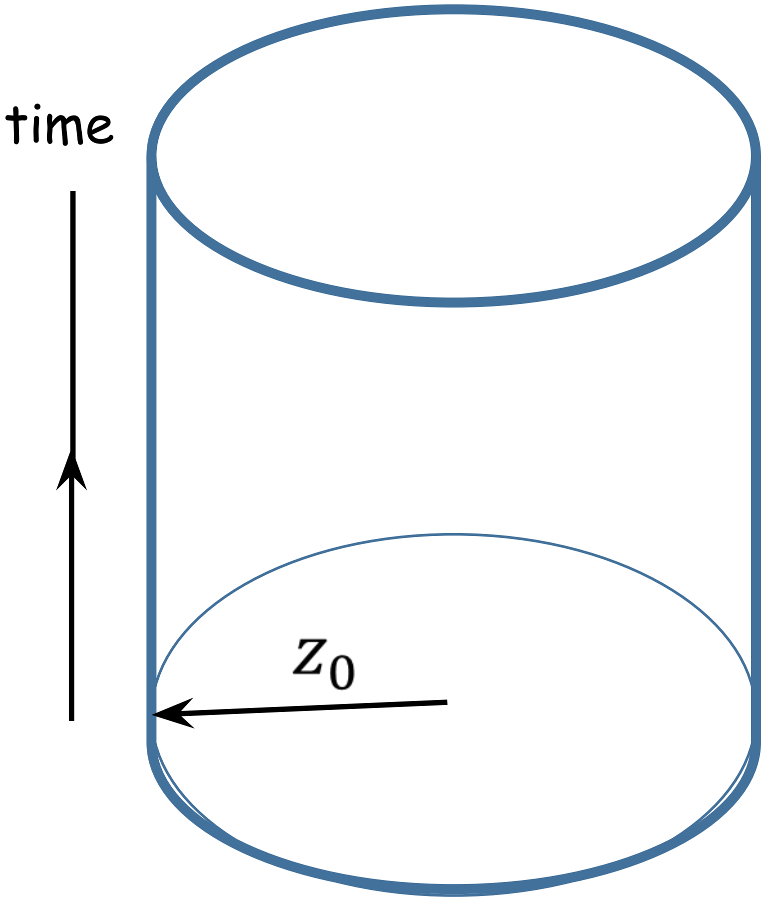
- A thermal state of the CFT corresponds to a AdS geometry with a black hole.

What is the extra dimension?

- The extra dimension on the gravity side corresponds to **length or energy scale** of the CFT.



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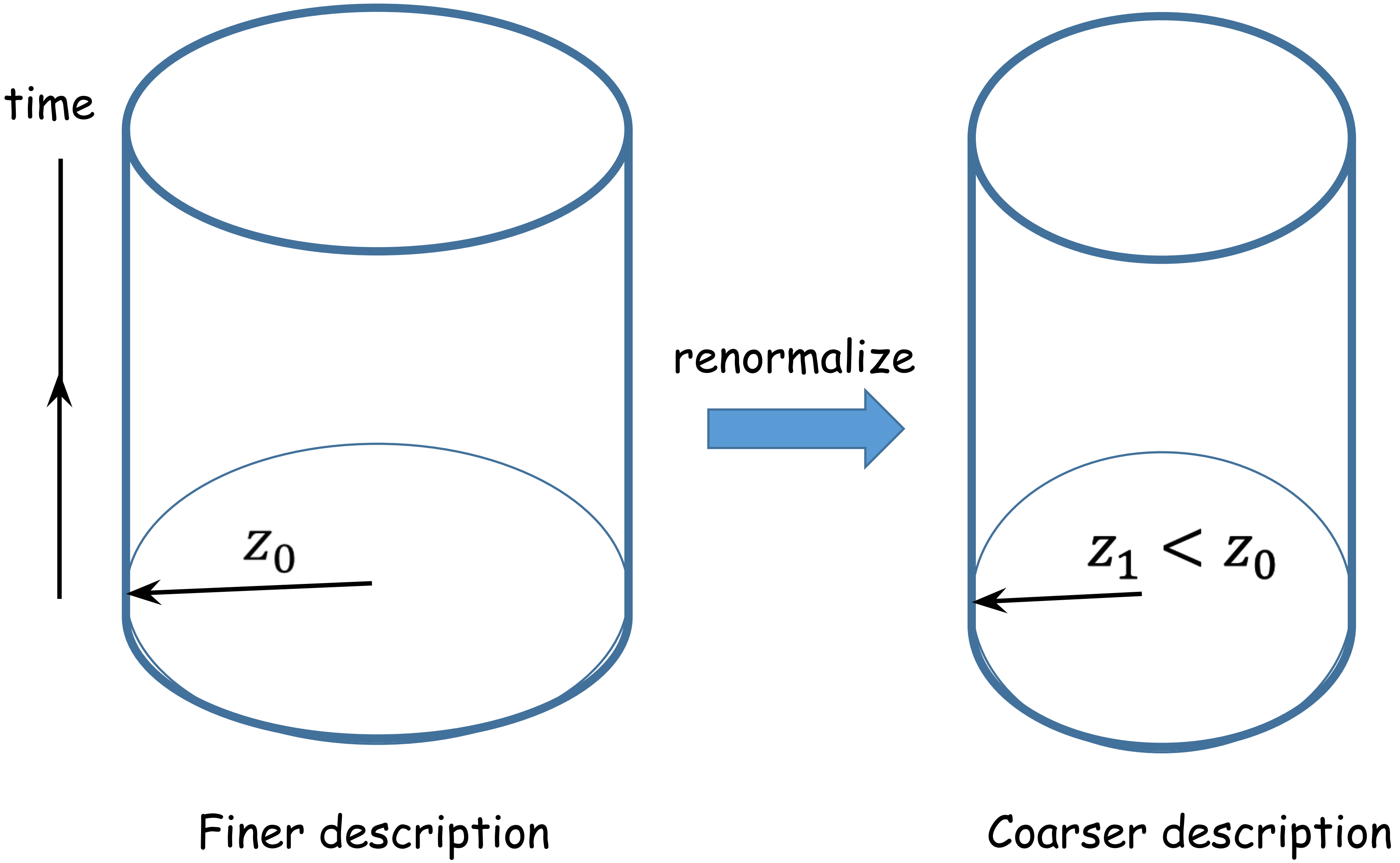

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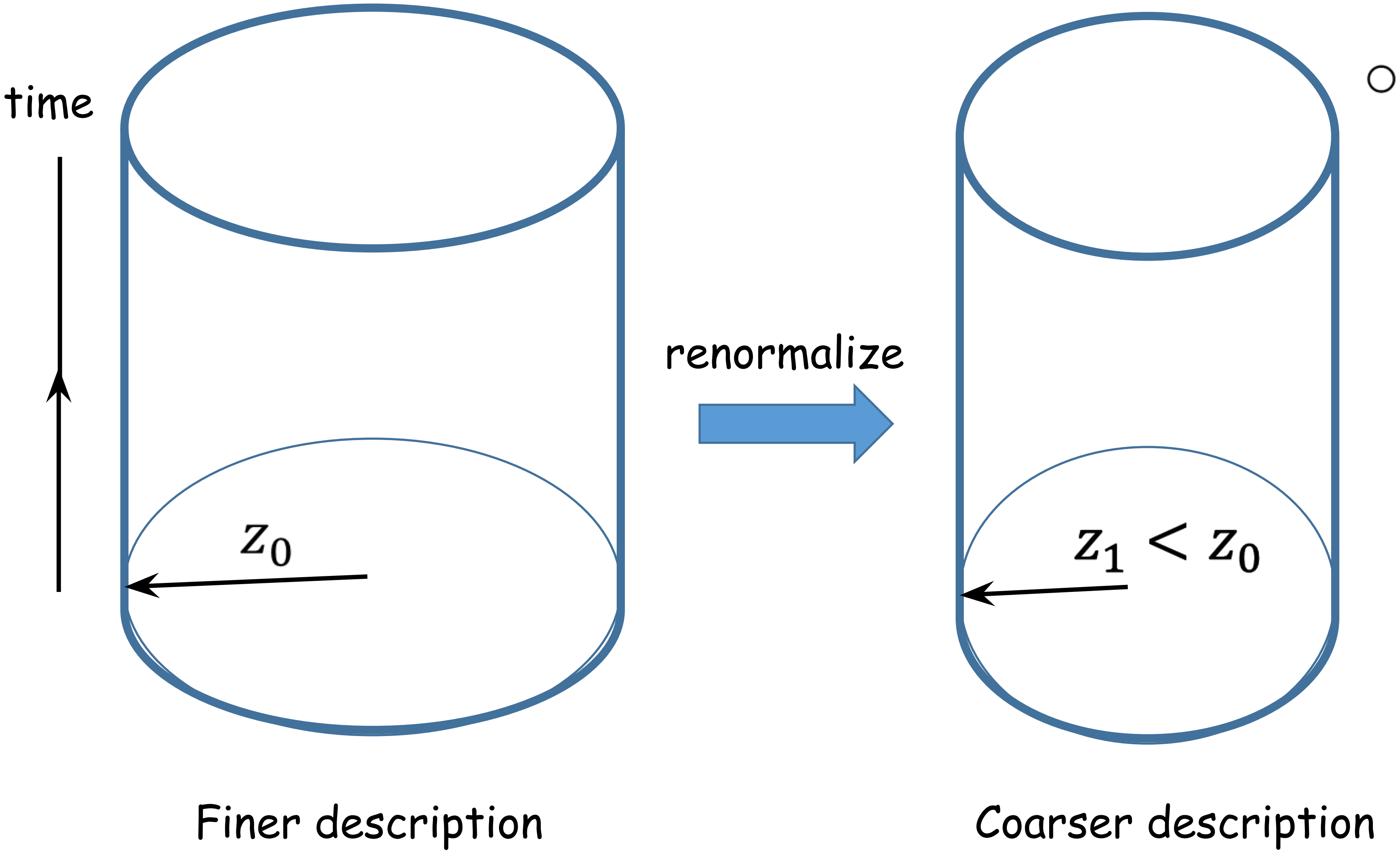

- 'Glue' effective descriptions of the CFT at different length scales $z_0 \to z_1 \to z_2 \to$ ... into a solid cylinder.

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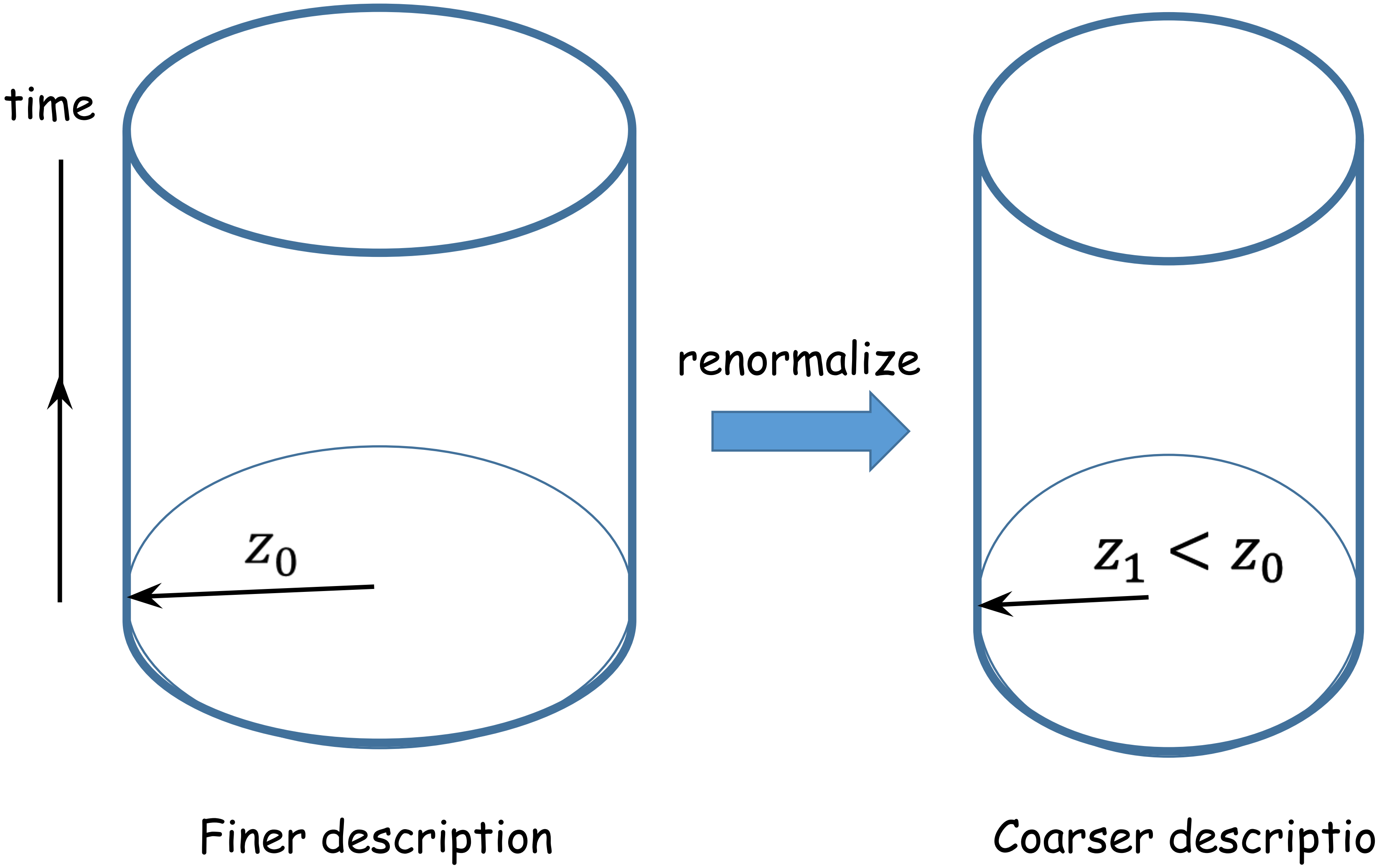

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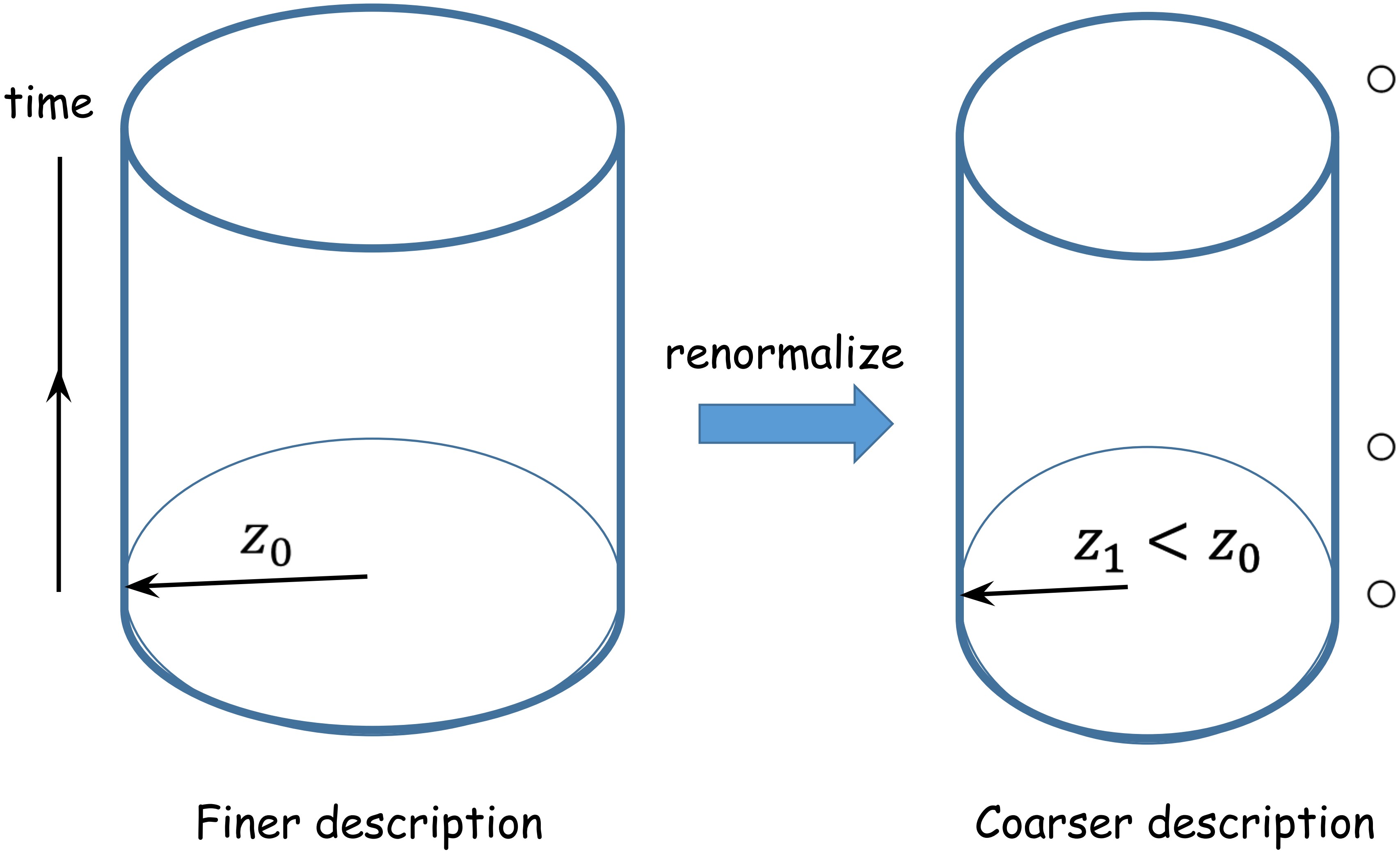
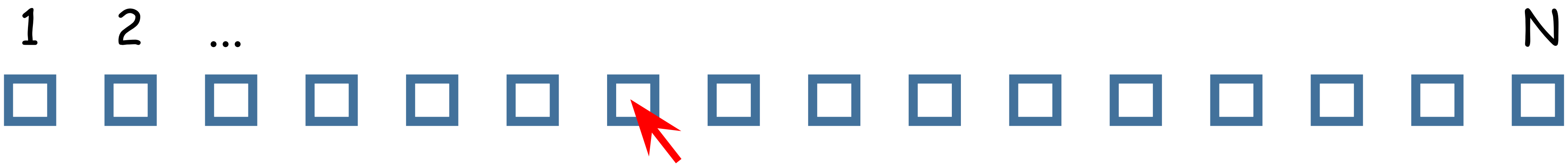


- 'Glue' effective descriptions of the CFT at different length scales $z_0 \to z_1 \to z_2 \to$ ... into a solid cylinder.
- Glued descriptions can be recast as gravity.
- AdS/CFT hints at a connection between renormalization & gravity.

Tensor network representations of ground states

Typical problem in condensed matter physics



qubit, atom, spin, … (Hilbert space of dimension d)

- Local and translational invariant Hamiltonian

$$H(m) = \sum_k X_k X_{k+1} + m\, Z_k$$

X, Z : Pauli matrices

1D quantum Ising model

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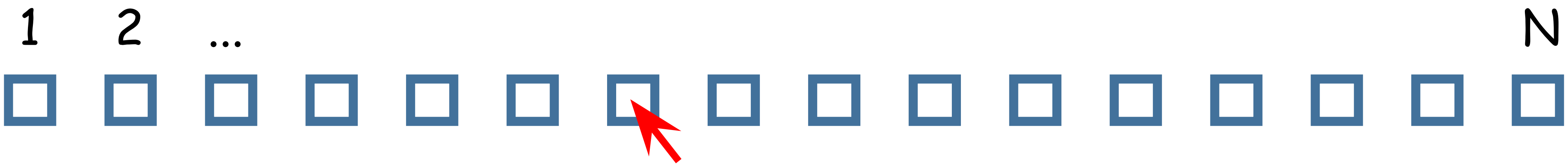

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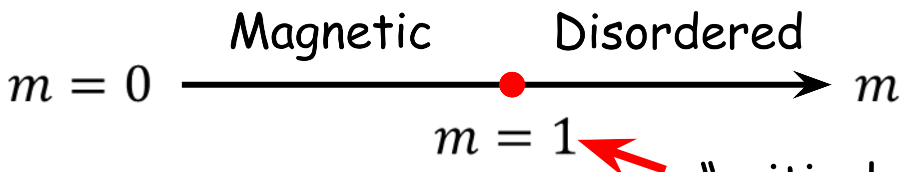
1D quantum Ising model

- Find the phase diagram of the system at zero temperature □

  Find the ground state for large *N*



"critical point" (gapless)

(described by a CFT in the continuum limit!)

Key challenge

- Exponentially large Hilbert space.

- Have to diagonalize an exponentially large Hamiltonian

- Ground state specified by d^N complex numbers

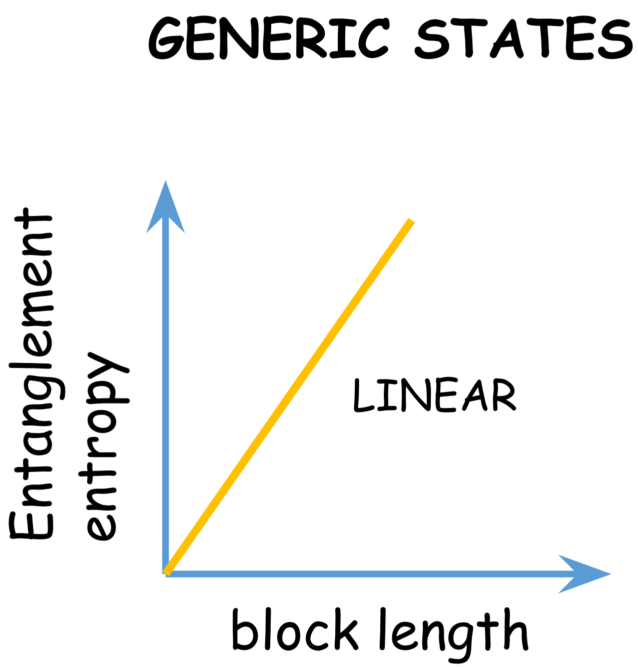
$$|\Psi\rangle = \sum \Psi_{i_1 i_2 \ldots i_N} \; |i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_N\rangle$$

$\{|i_k\rangle\}$: orthonormal basis on site k

(a) d=2, N=30 requires 17 GB memory
(b) d=3, N=20 requires 55 GB memory

But ground states of local Hamiltonians are atypical

- They have much less entanglement than generic states on the lattice.



Entanglement entropy $(\rho) = -Tr(\rho \log \rho)$

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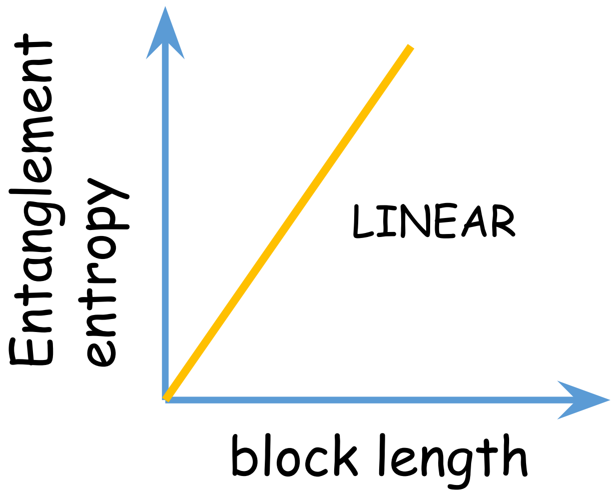
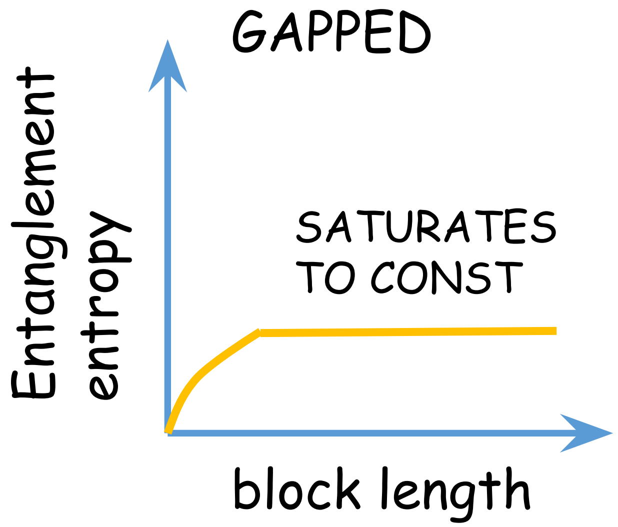
GENERIC STATES



GROUND STATES



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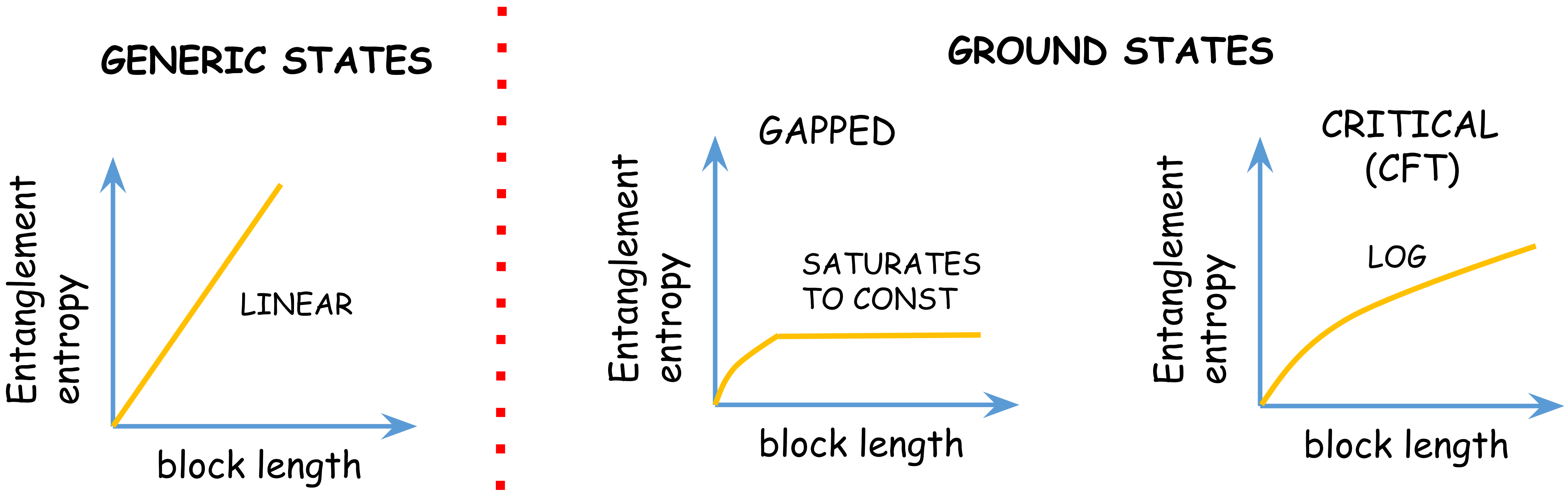

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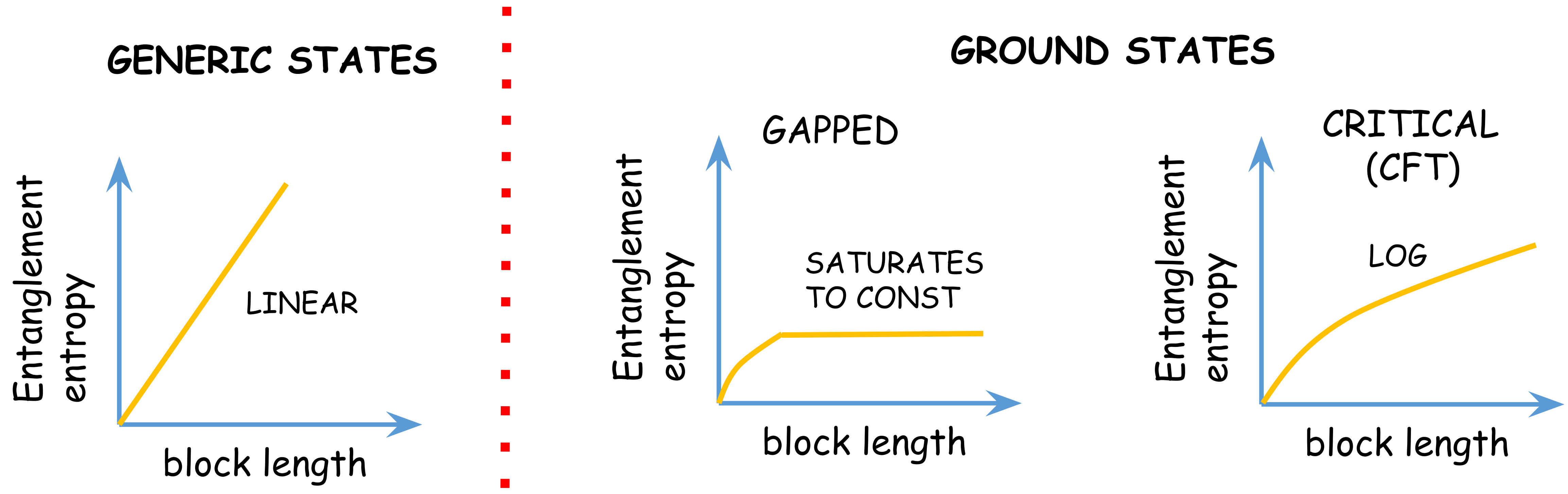


Entanglement entropy $(\rho) = -Tr(\rho \log \rho)$

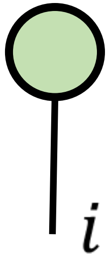
- Can we exploit this limited entanglement:

to build an efficient representation of the ground state?

Yes, as a **tensor network**.

Graphical notation for tensors

$$|\mathrm{c}\rangle = \sum_{i=1}^{d} \mathrm{c}_i \; |i\rangle$$



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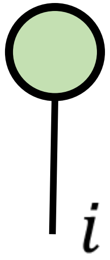
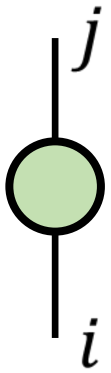


$$M = \sum M_{ij} \; |i\rangle\langle j|$$



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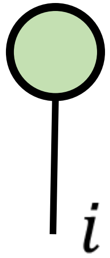

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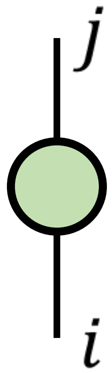

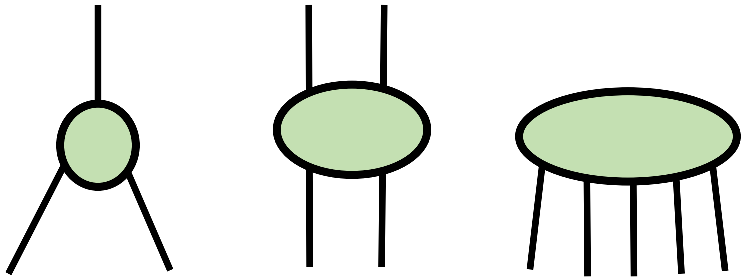

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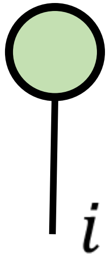

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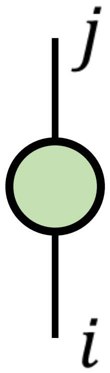
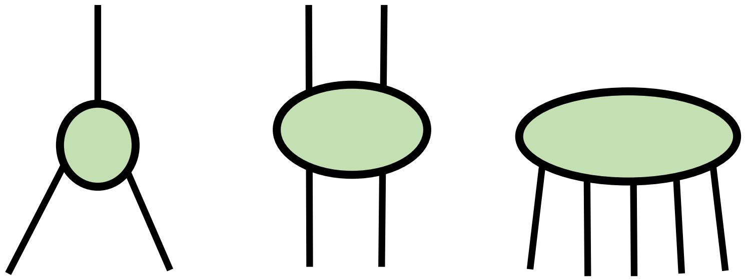
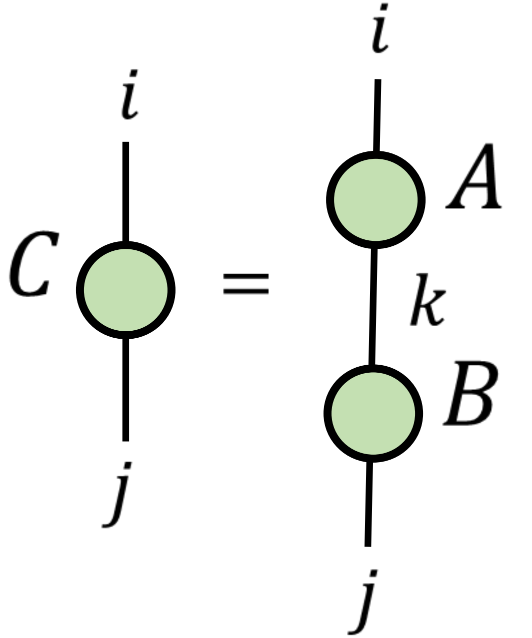




matrix product



$$C_{ij} = \sum_k A_{ik} B_{kj}$$

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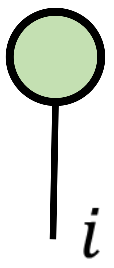

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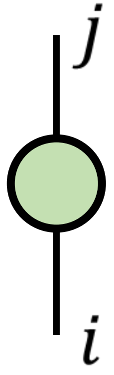

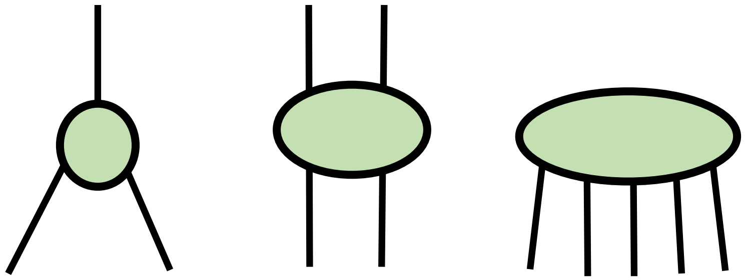

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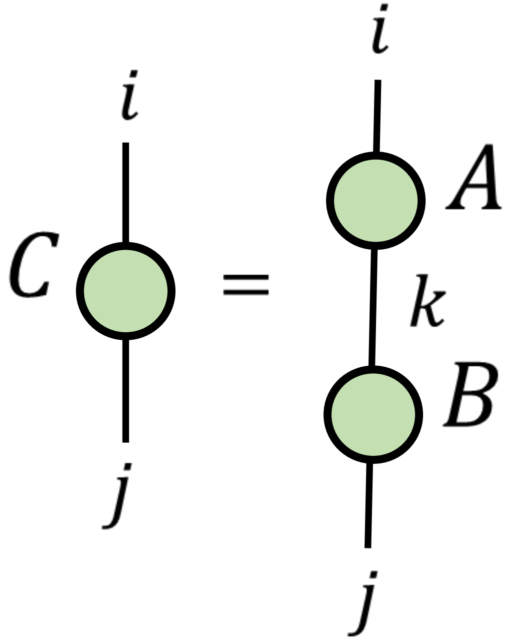

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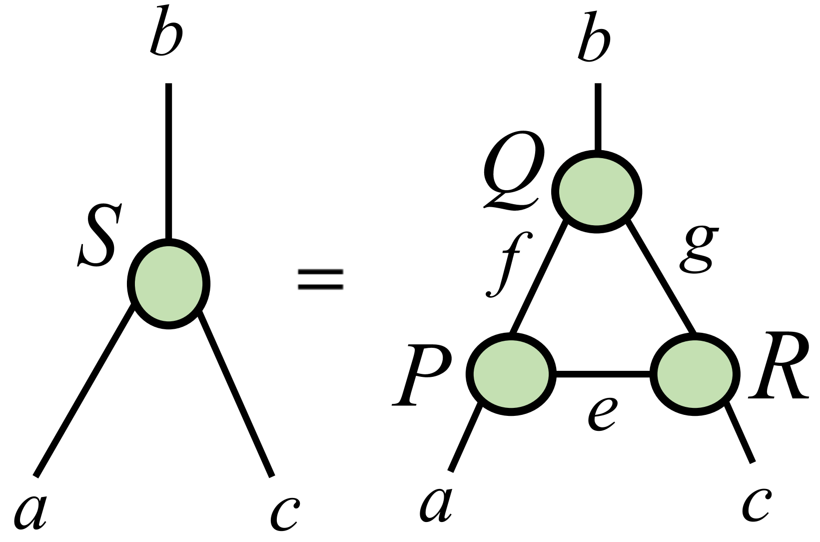
tensor contraction



$$S_{abc} = \sum_{efg} P_{afe} Q_{fbg} R_{egc}$$

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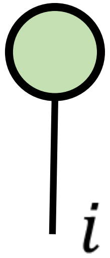

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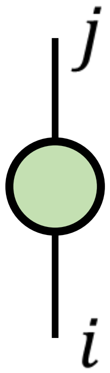
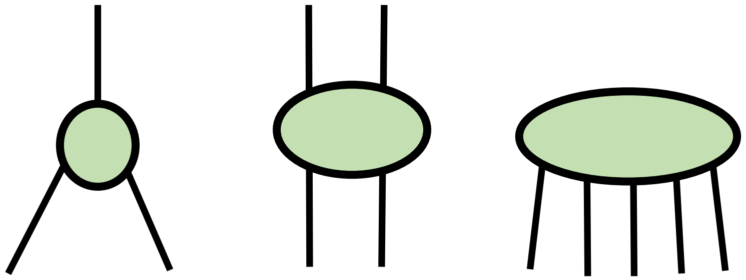
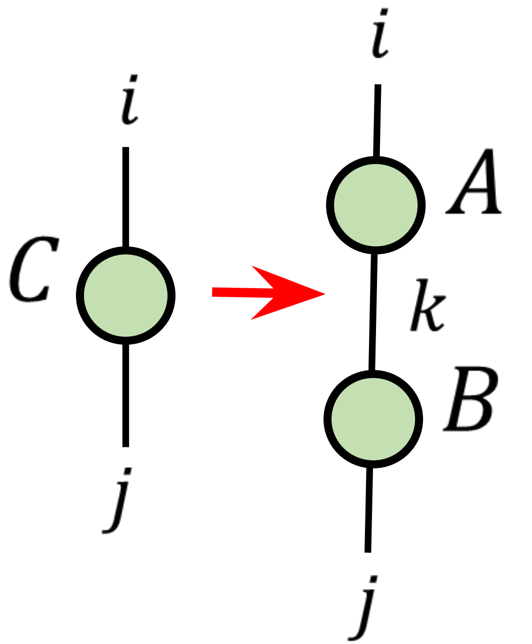




matrix decomposition



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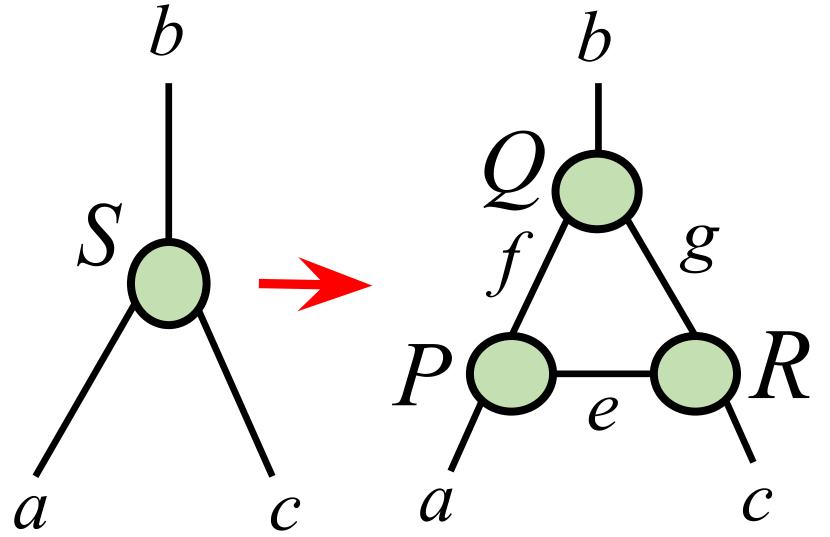

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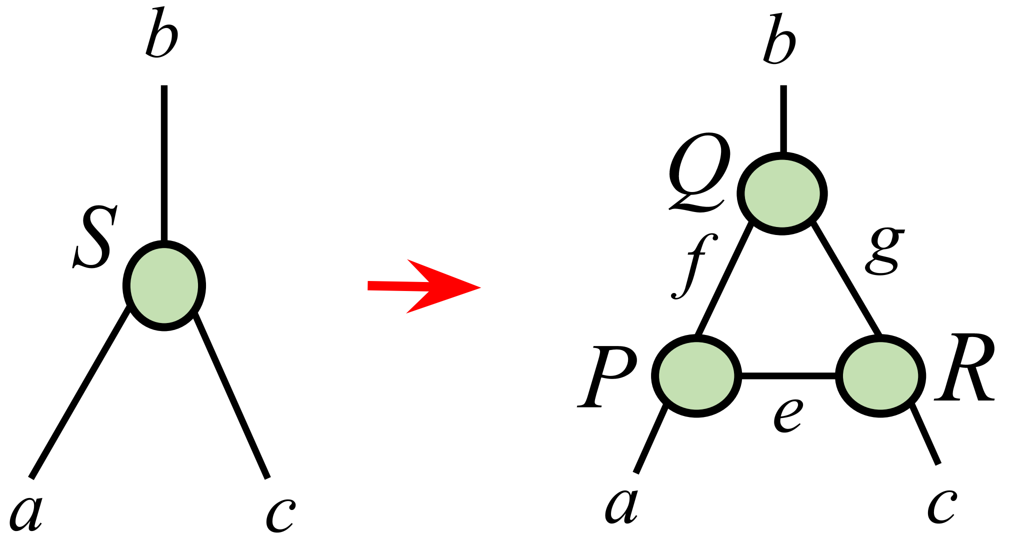
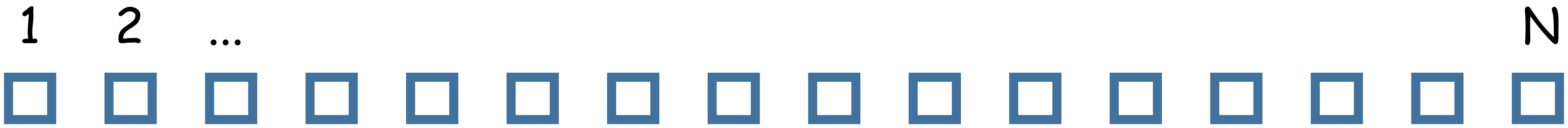
tensor

tensor network

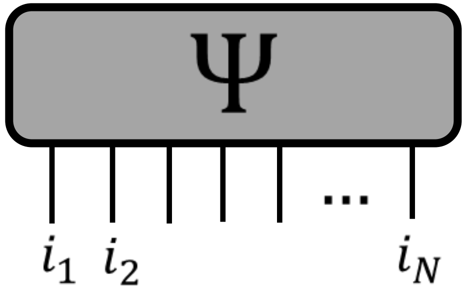


Ground states as tensor networks



$$|\Psi\rangle = \sum \Psi_{i_1 i_2 \ldots i_N} \, |i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_N\rangle$$

(tensor)



$=$

d^N coefficients

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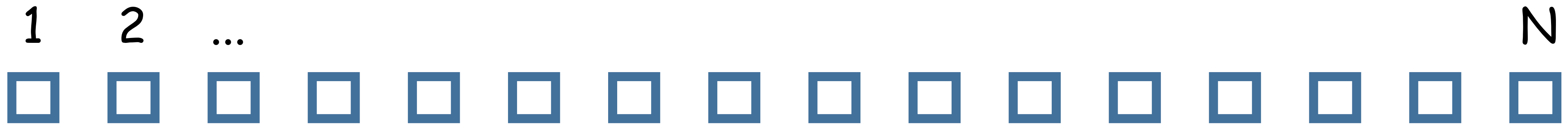

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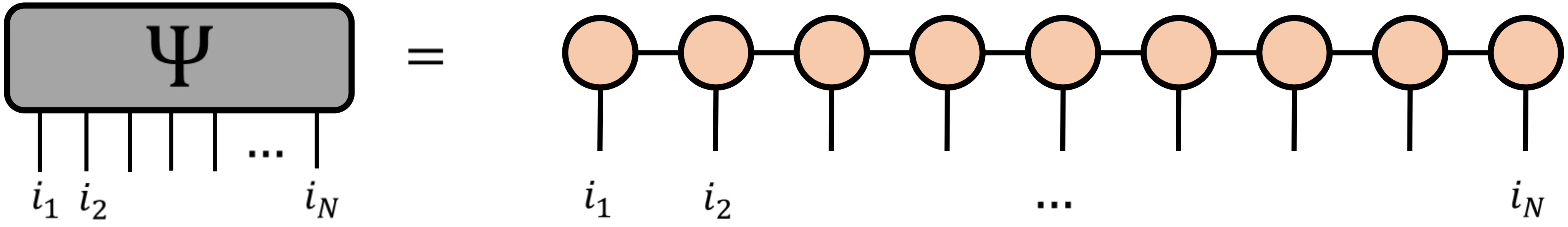

d^N coefficients

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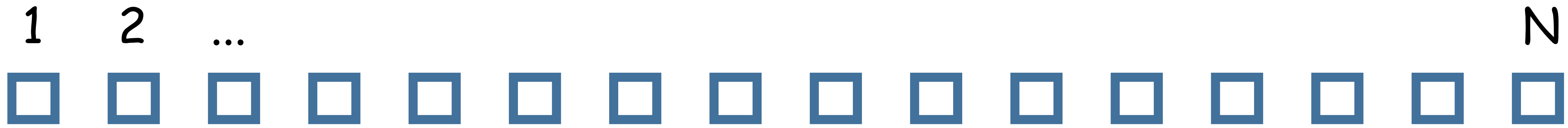

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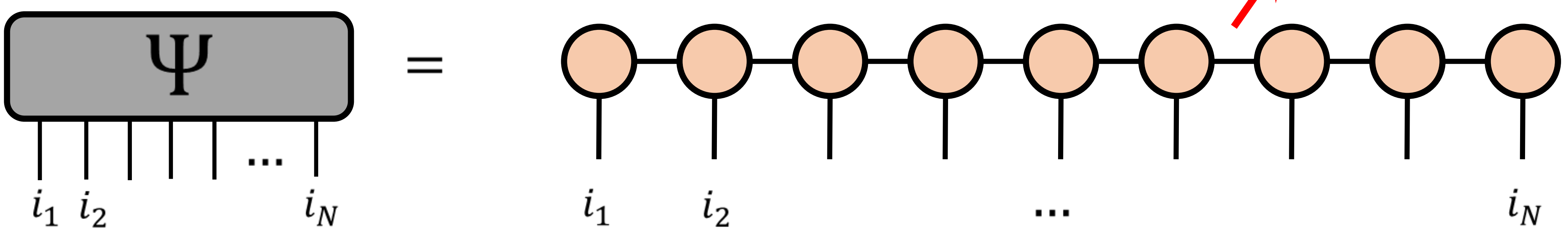
Bond index
$1, \ldots, \chi$



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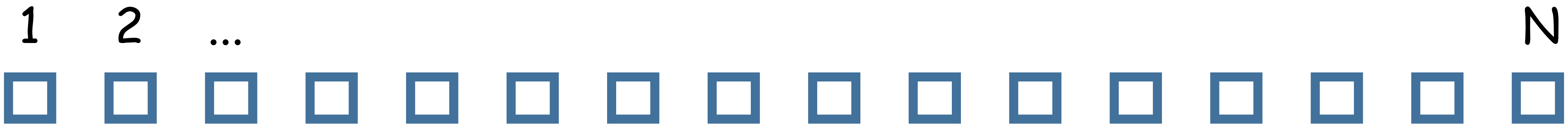

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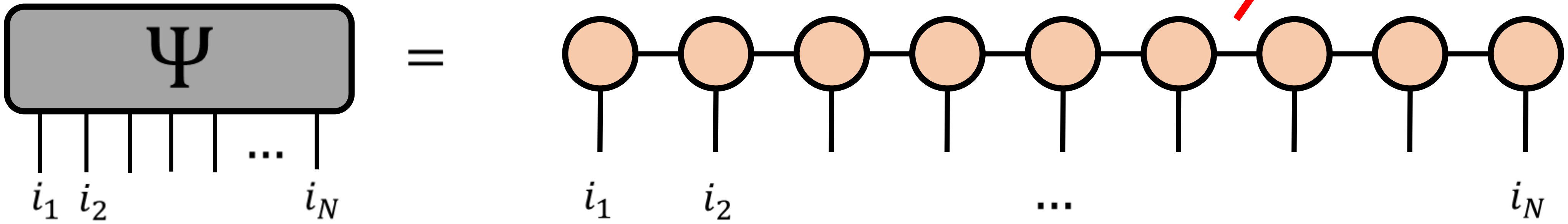
Bond index

$1, \ldots, \chi$

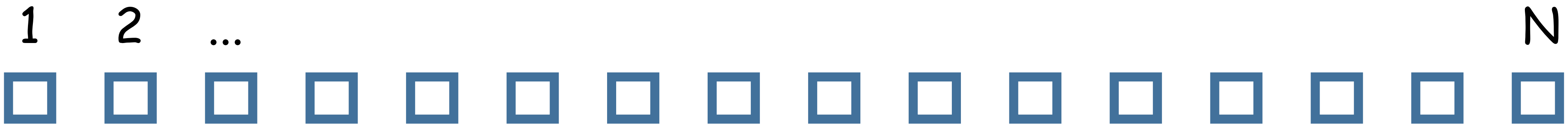


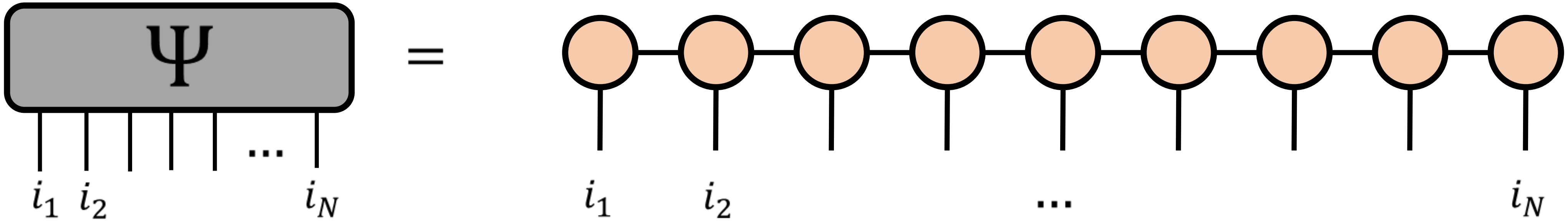
d^N coefficients

$O(N\chi^2 d)$ coefficients

$\chi = O(\exp N)$

Ground states as tensor networks



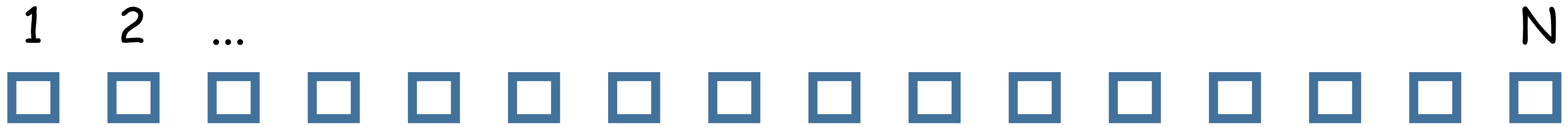


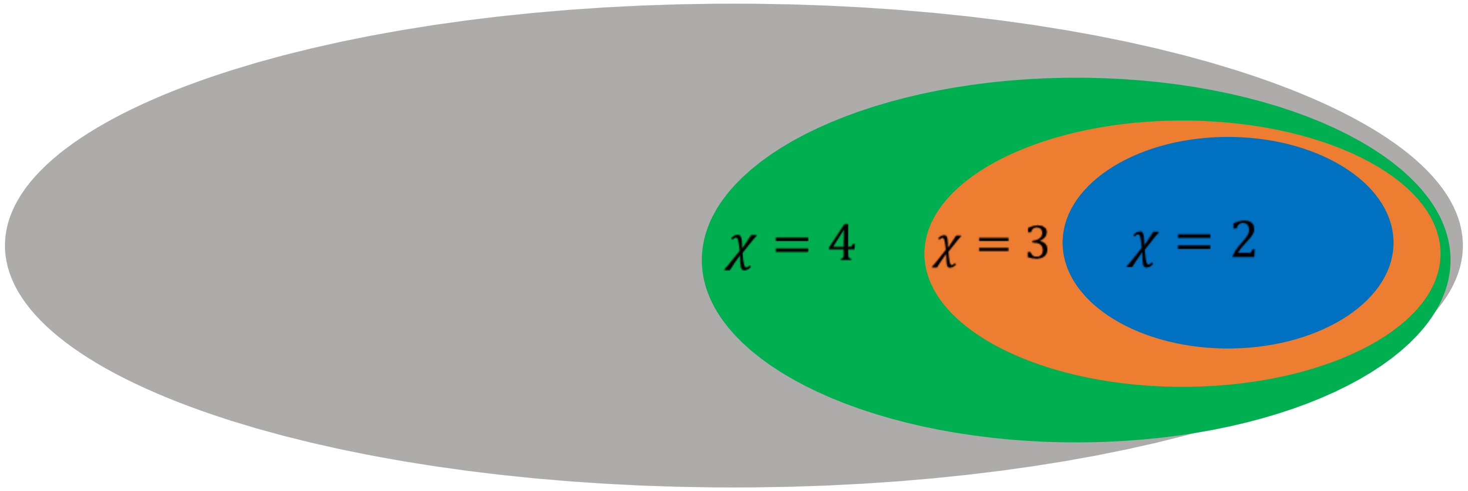
d^N coefficients

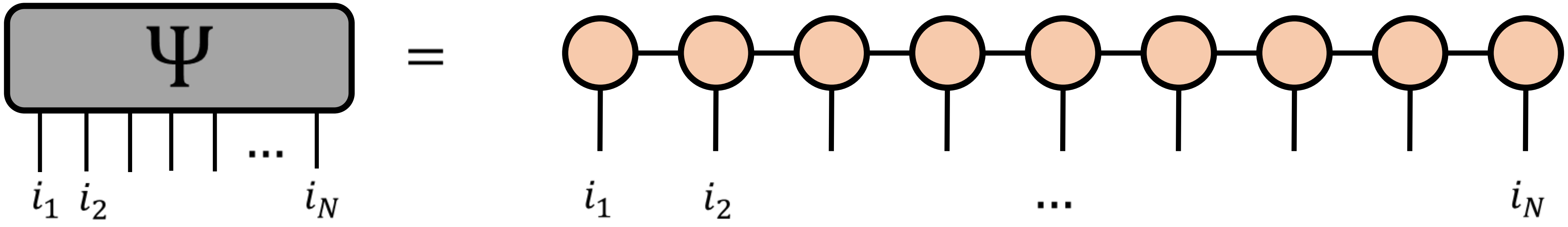
$O(N\chi^2 d)$ coefficients

$\chi = O(1)$

Ground states as tensor networks







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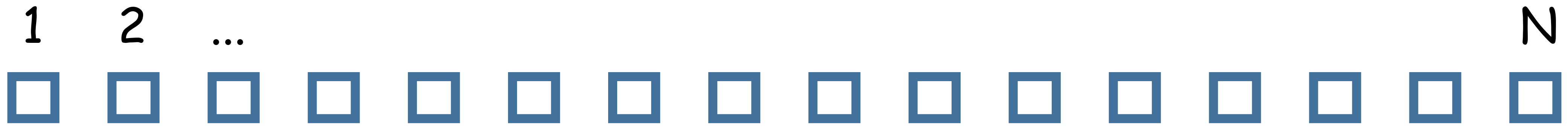
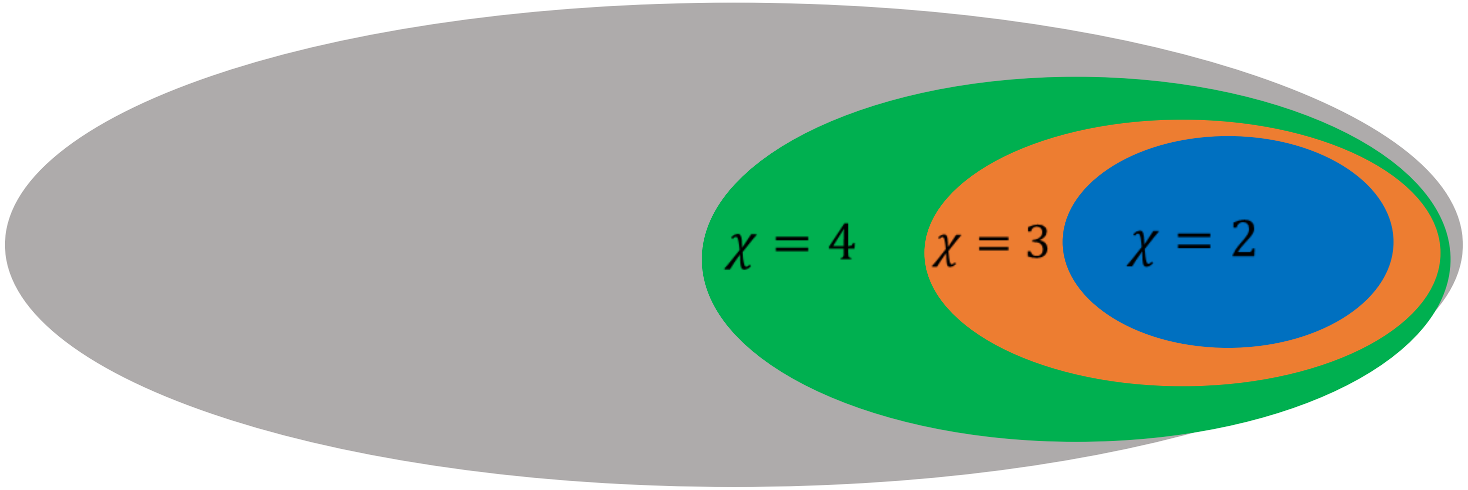
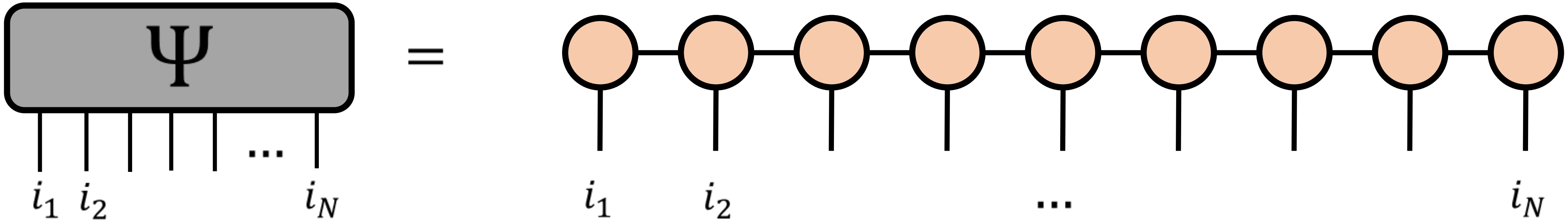




Matrix product states

Ostlund and Rommer, PRL 75, 3537 (1995)

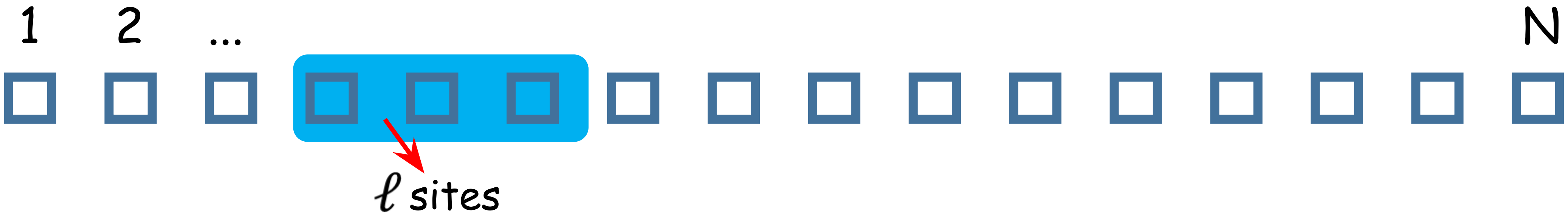


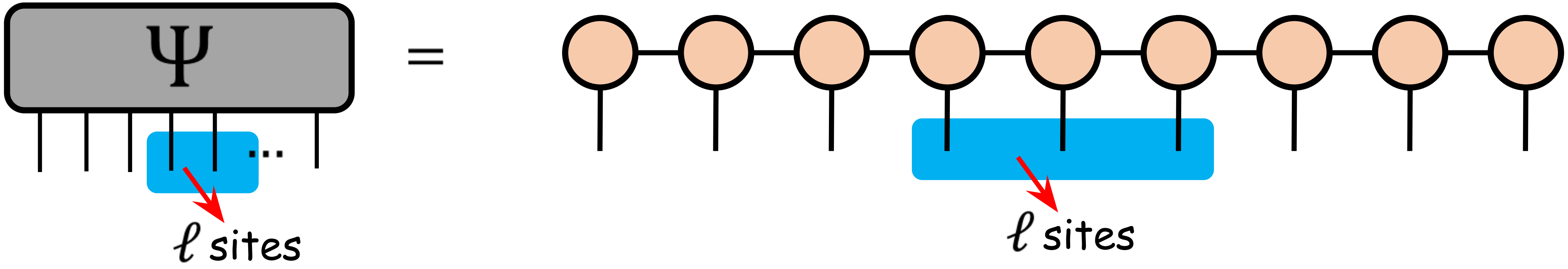
d^N coefficients

$O(N\chi^2 d)$ coefficients

$\chi = O(1)$

Ground states as tensor networks

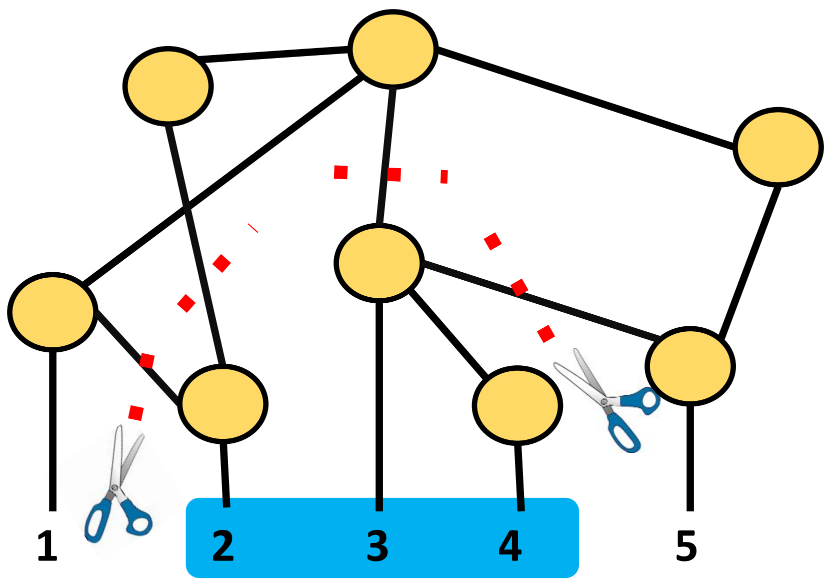


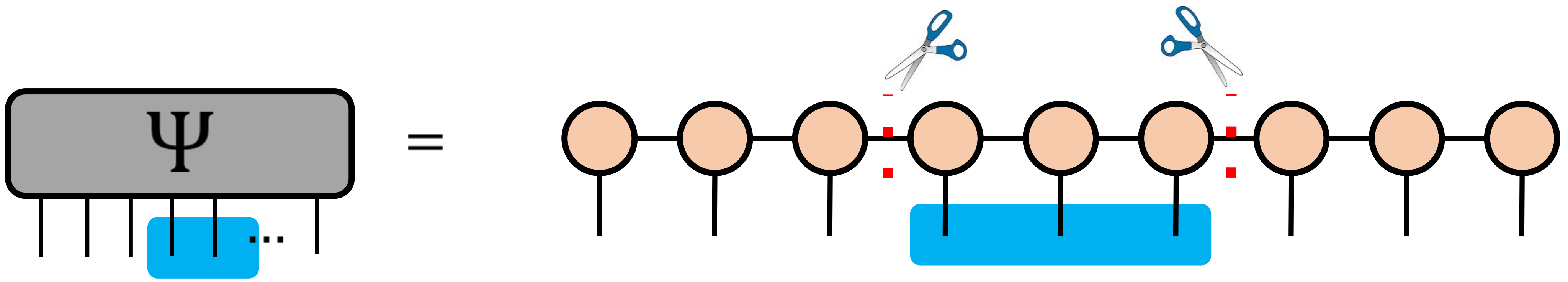


Rule for estimating upper bound of entanglement scaling from a TN

G. Evenbly and G. Vidal, J. Stat. Phys. 145, 891 (2011)

"Entanglement entropy $\propto$ Minimal number of bond cuts"

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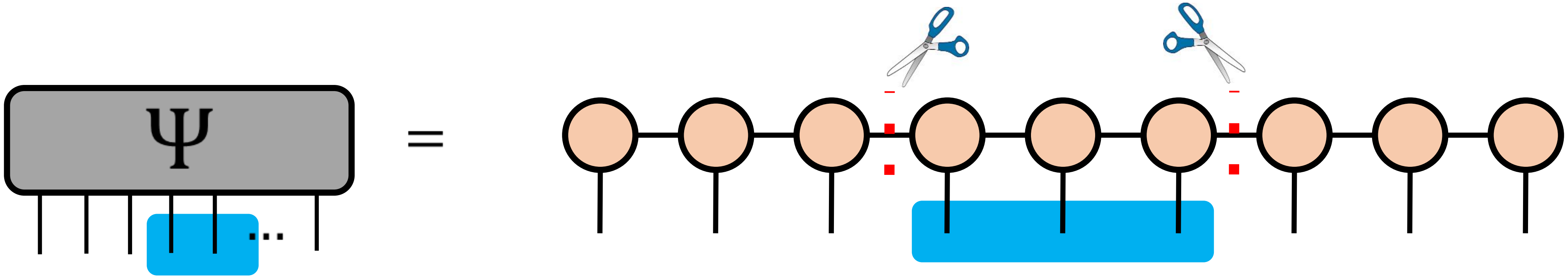
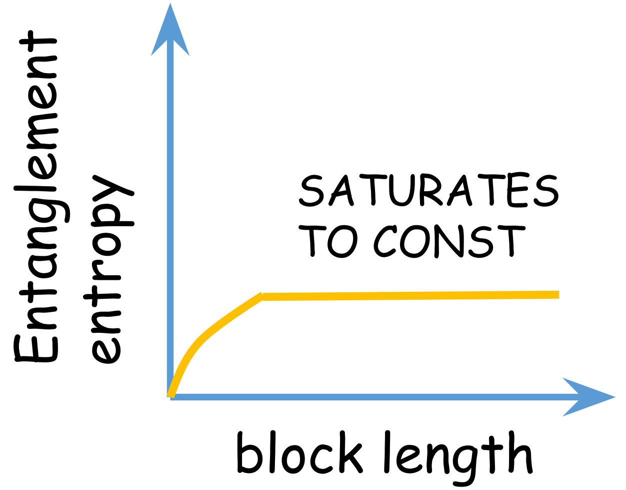
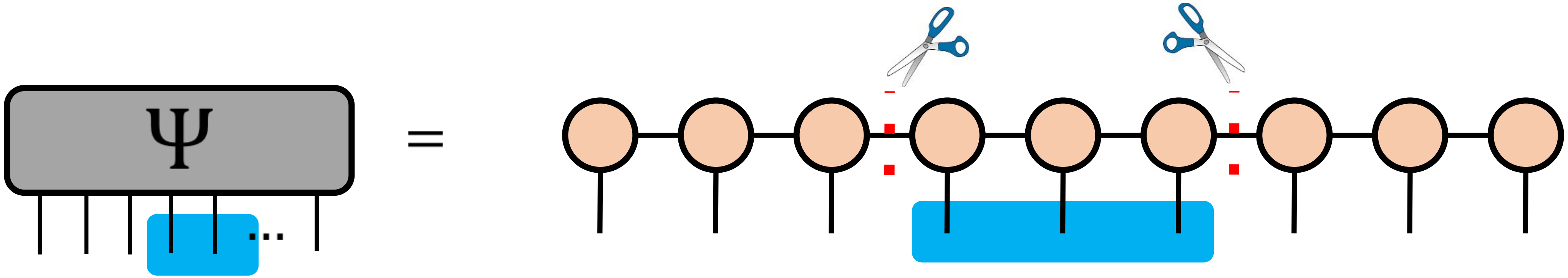
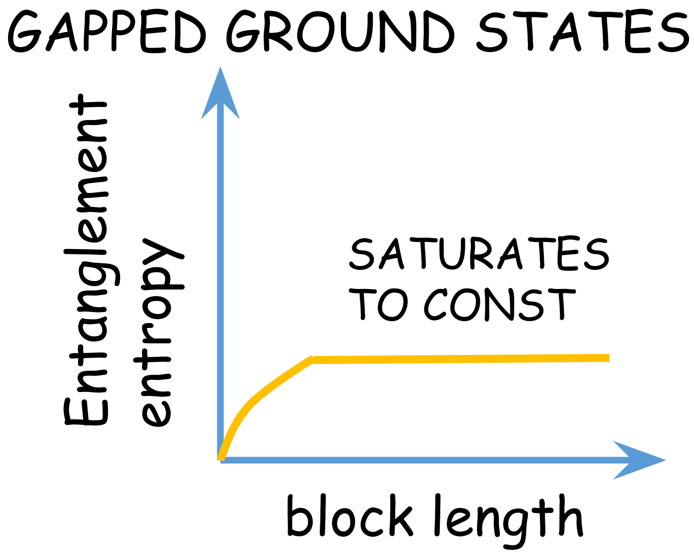


GAPPED GROUND STATES!





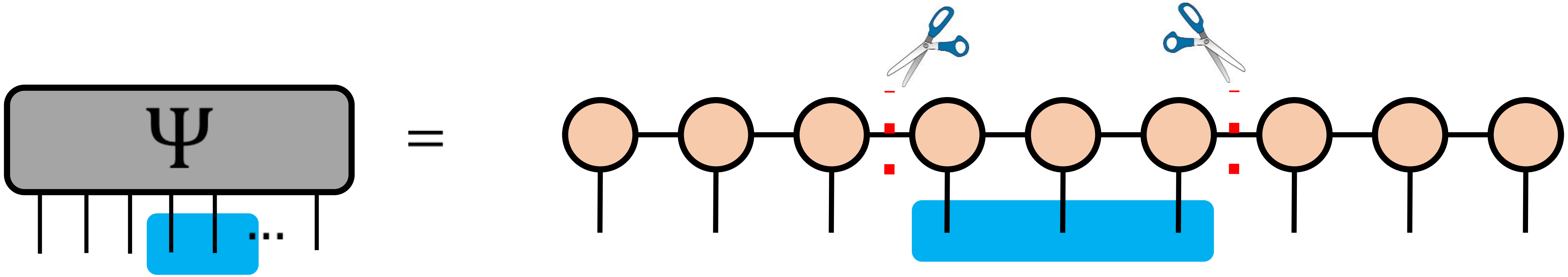


- MPS can accomodate the entanglement scaling of gapped ground states.
- Can use MPS as a variational ansatz for gapped ground states.
- Lots of supporting numerical evidence and theoretical results.

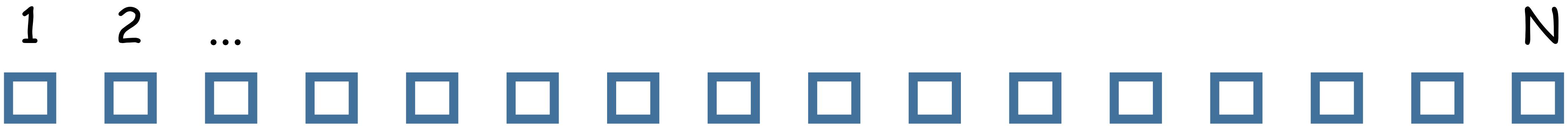
- We will be interested in **critical** ground states.

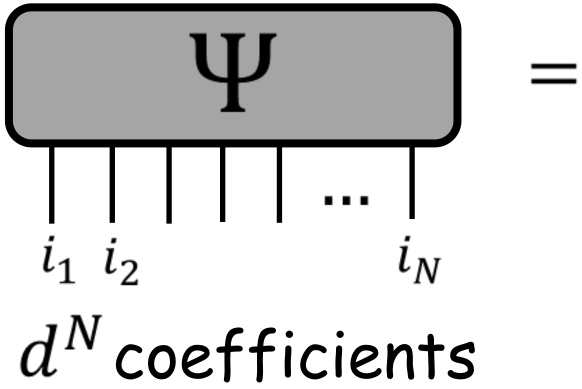
- Can we build a tensor network representation for critical ground states?

- **Lesson from the MPS:** entanglement in a tensor network state depends on the geometry of the tensor network.

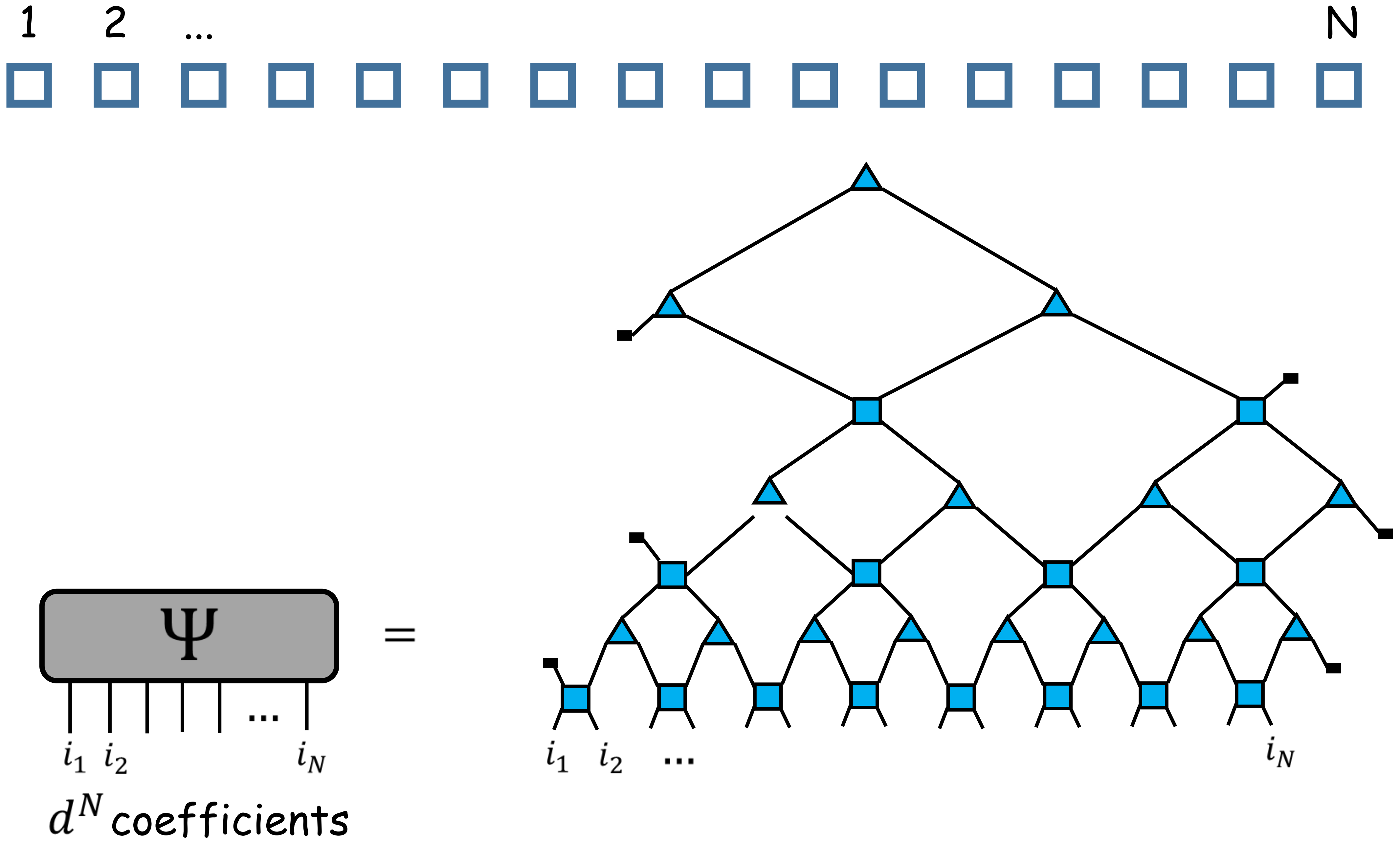


Critical ground states

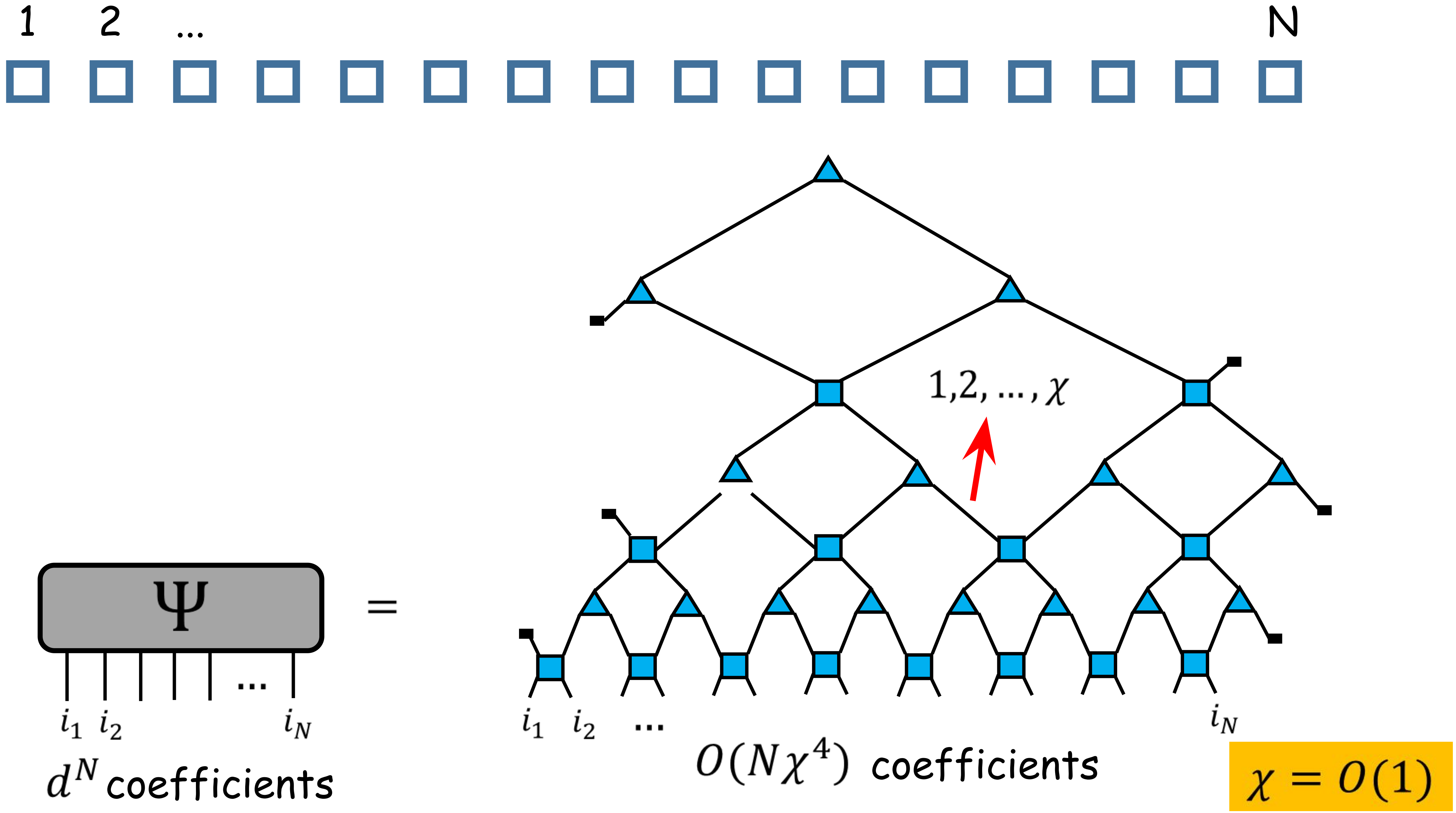




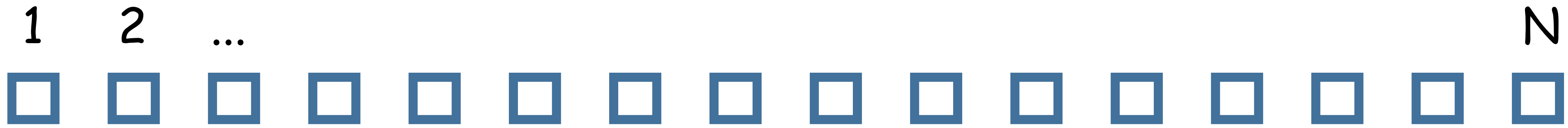
Critical ground states



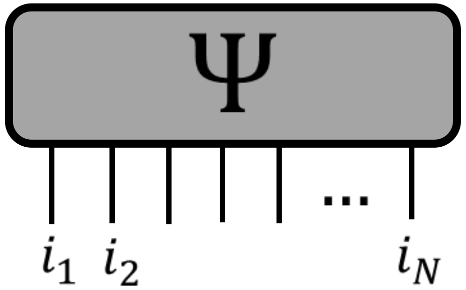
Critical ground states



Critical ground states



(1D state) $\rightarrow$ (2D TN)



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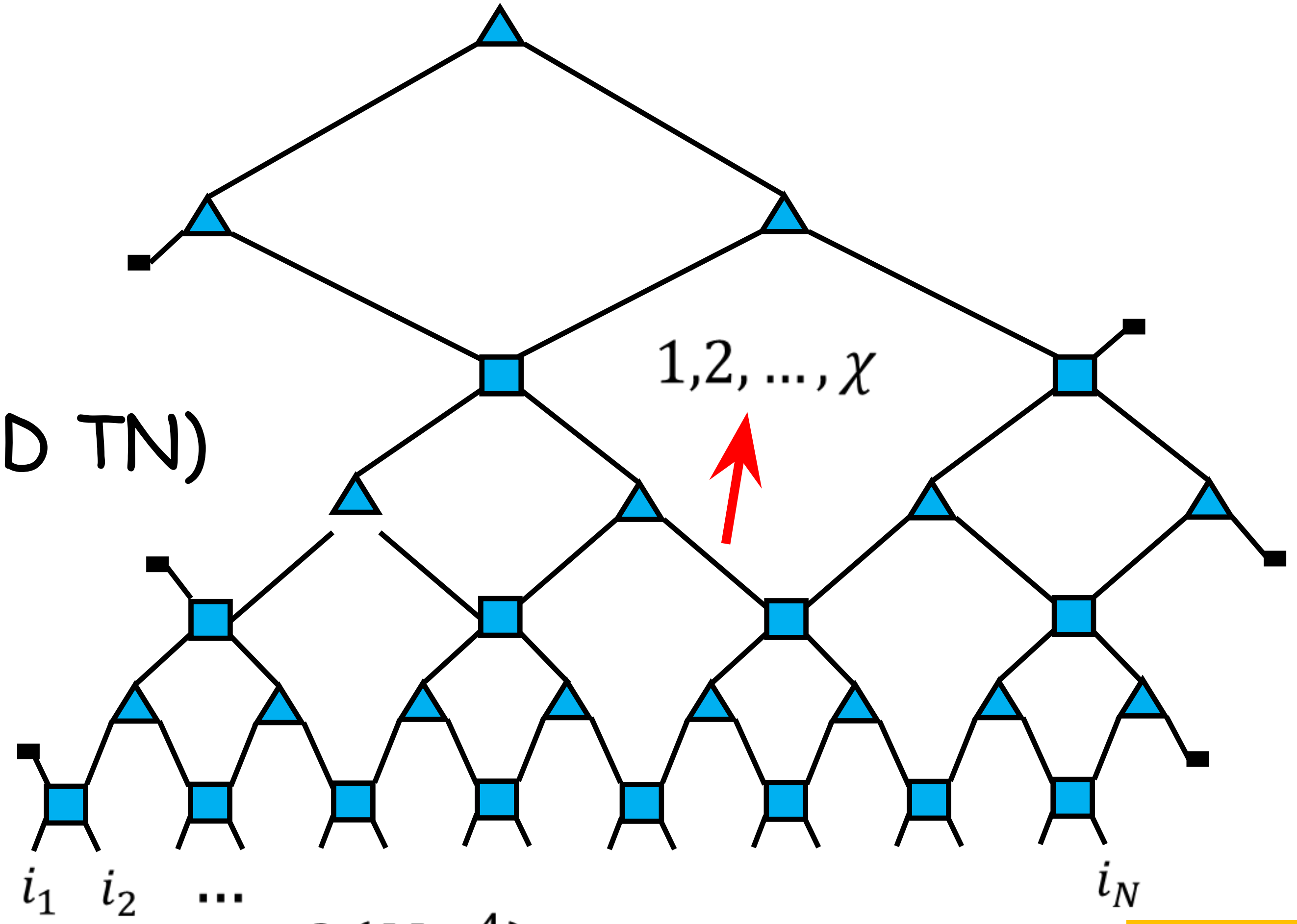
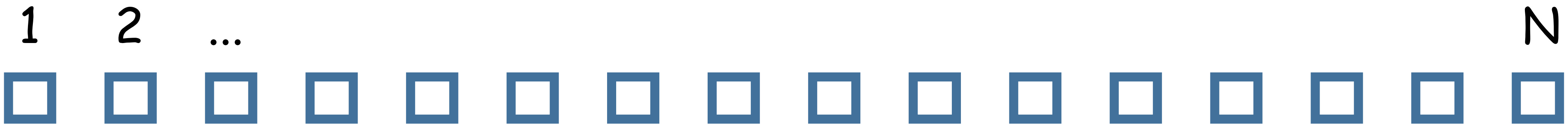


d^N coefficients

$O(N\chi^4)$ coefficients

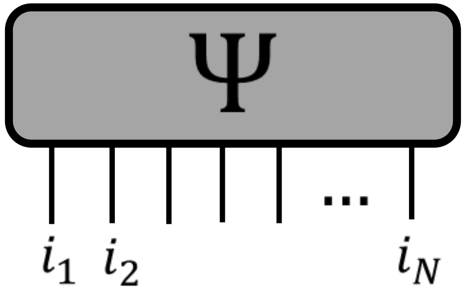
$\chi = O(1)$

Critical ground states



"Multi-scale entanglement renormalization ansatz"
(MERA) *G. Vidal, PRL 99 (2007); PRL 101 (2008)*

(1D state) $\rightarrow$ (2D TN)



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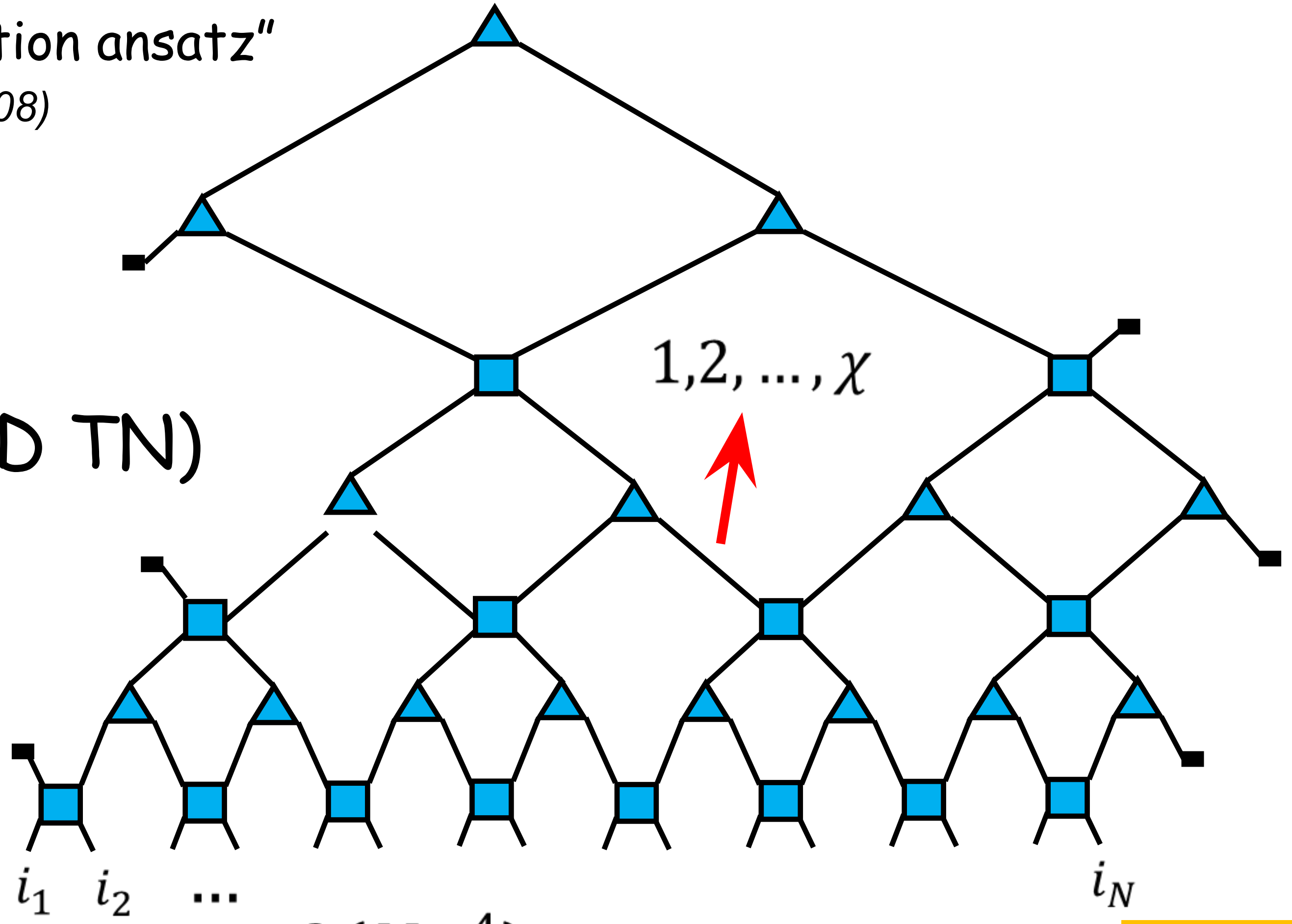

d^N coefficients

$O(N\chi^4)$ coefficients

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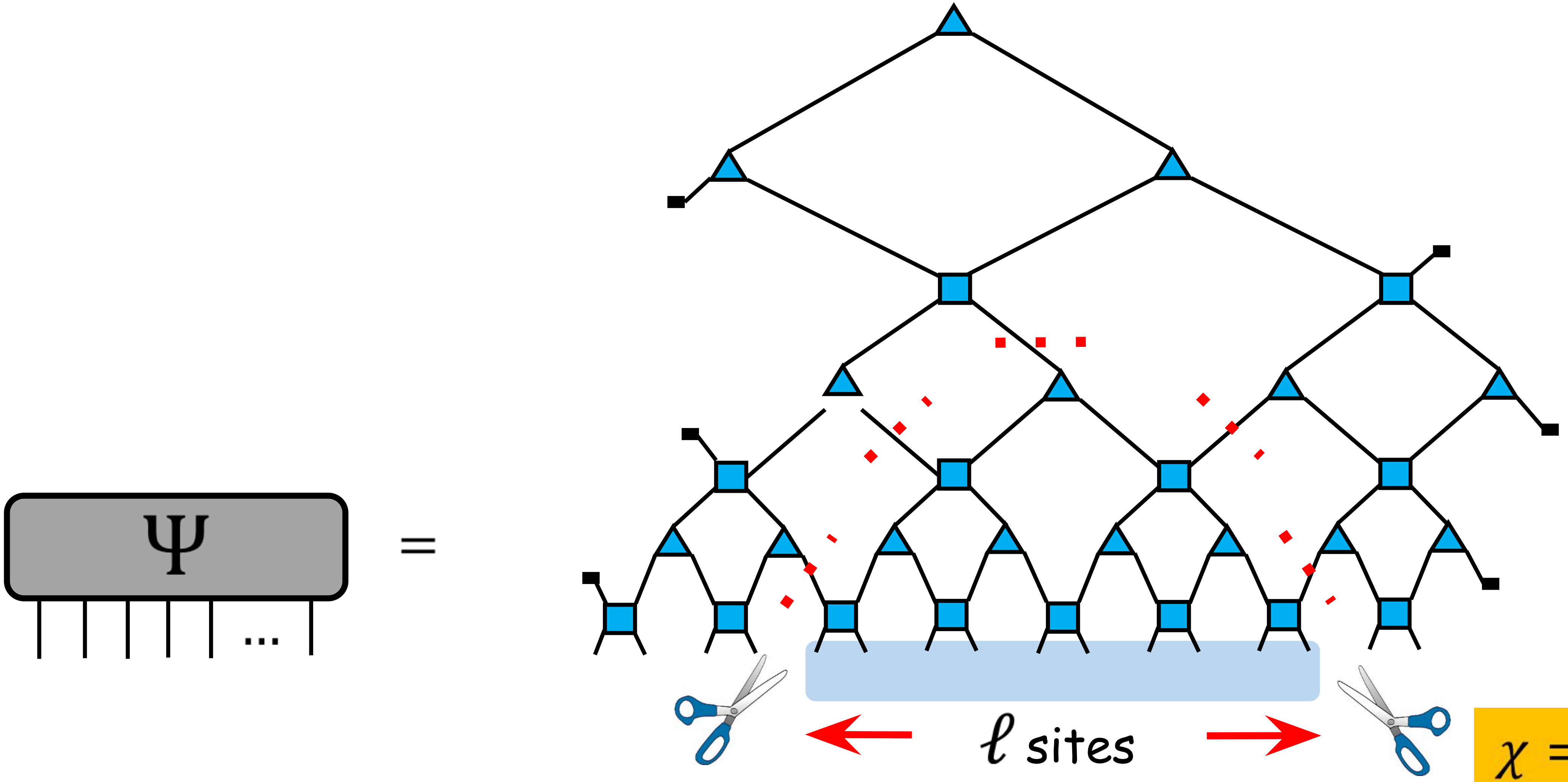
Entanglement scaling in the MERA

Minimal number of bond cuts $\propto \log \ell$

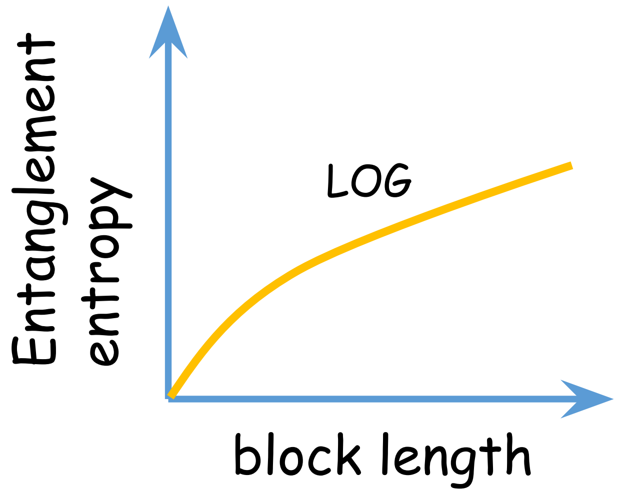


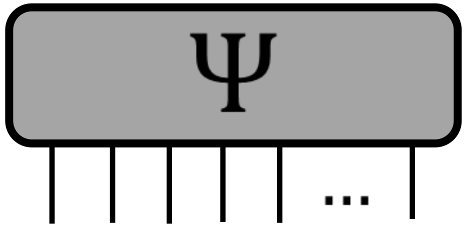
$\chi = O(1)$

Entanglement scaling in the MERA

Minimal number of bond cuts $\propto \log \ell$

CRITICAL GROUND STATES





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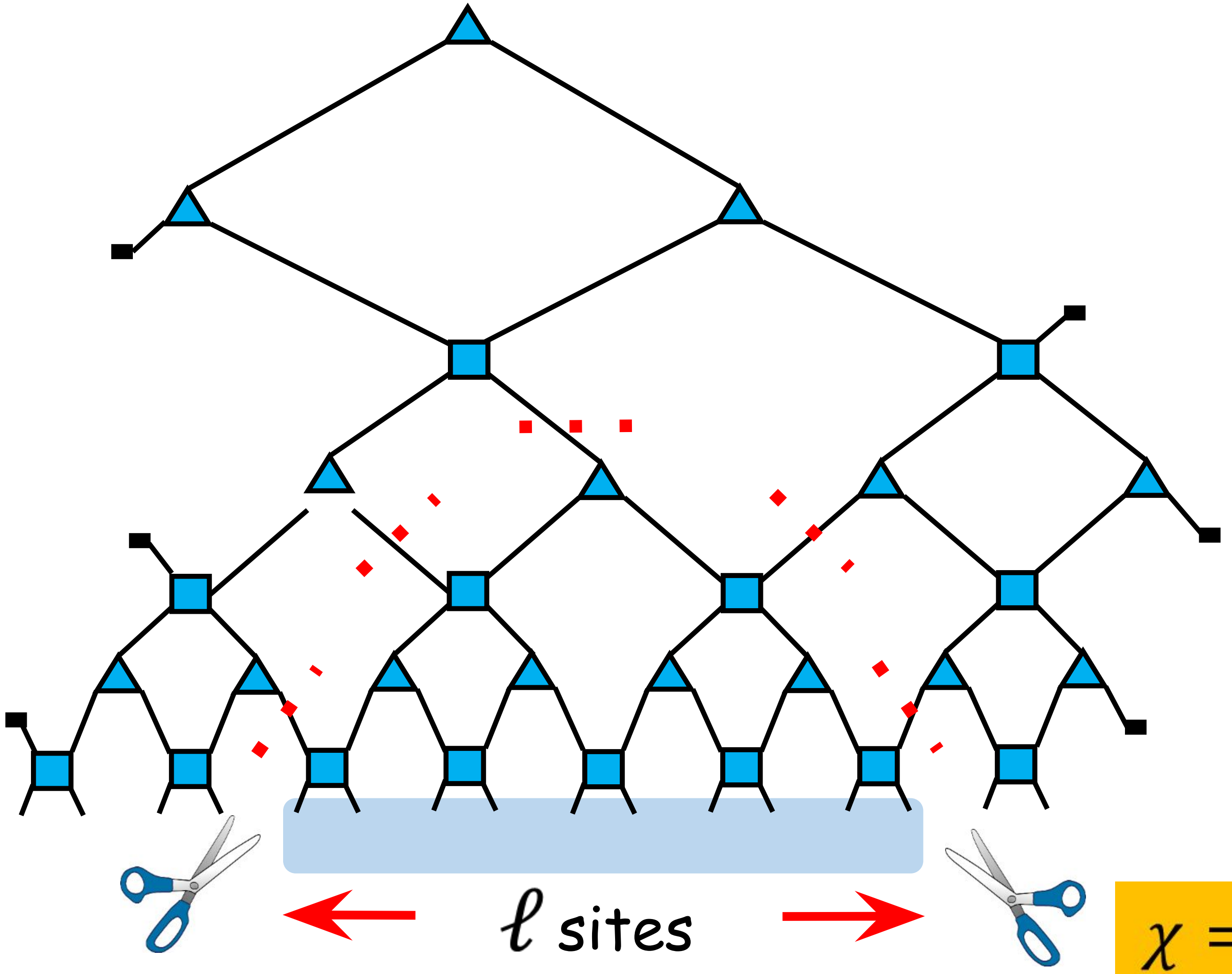
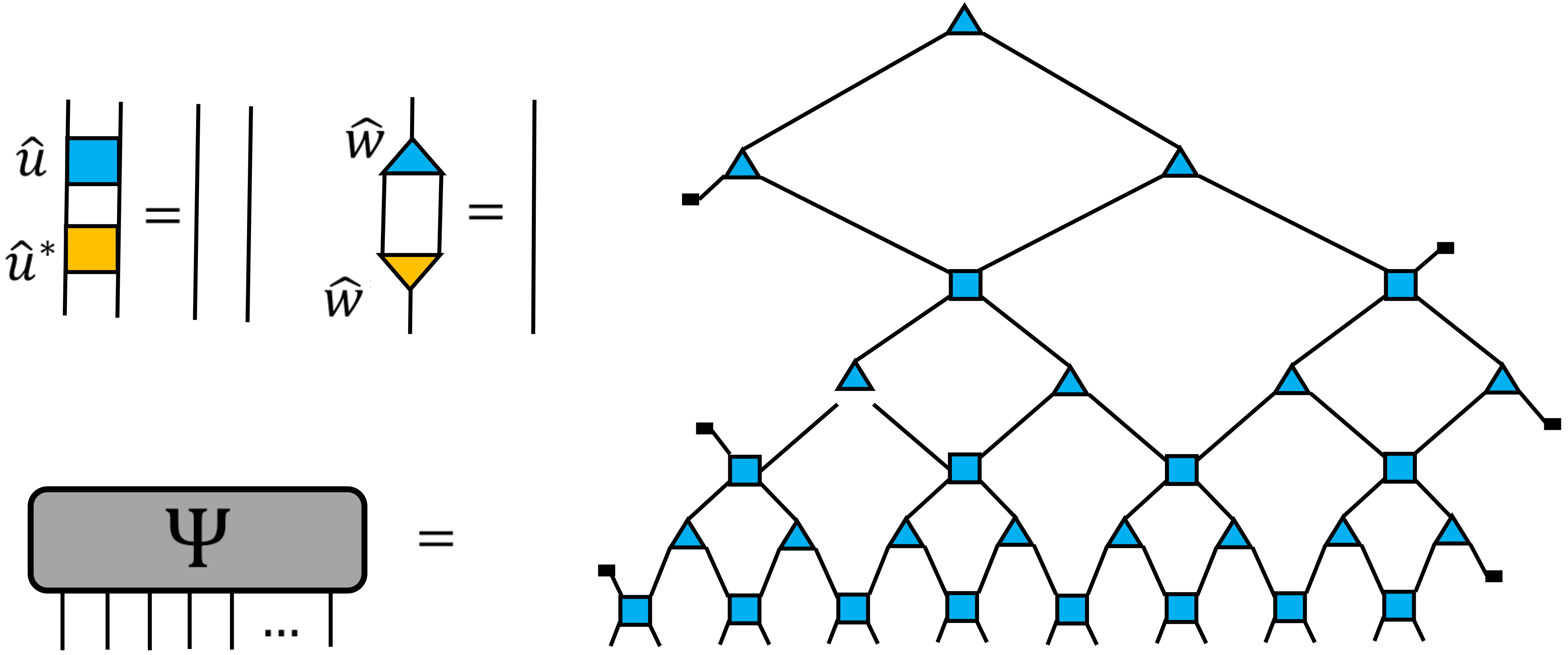


$\chi = O(1)$

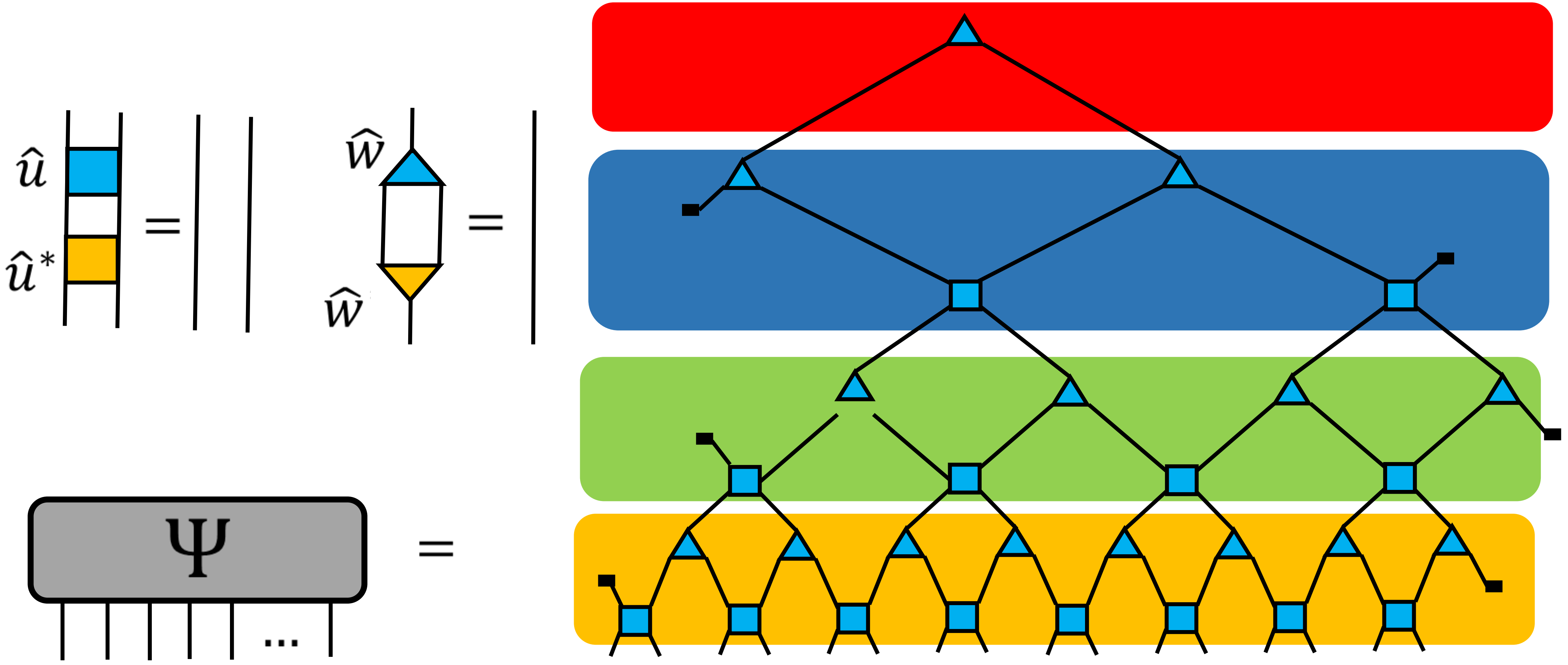
MERA as an efficient ansatz for critical ground states

- Algorithms known for: critical $\widehat{H}$ → ground state as MERA.
- Applied to several 1D quantum critical lattice models.
- Ground state properties can be efficiently computed.

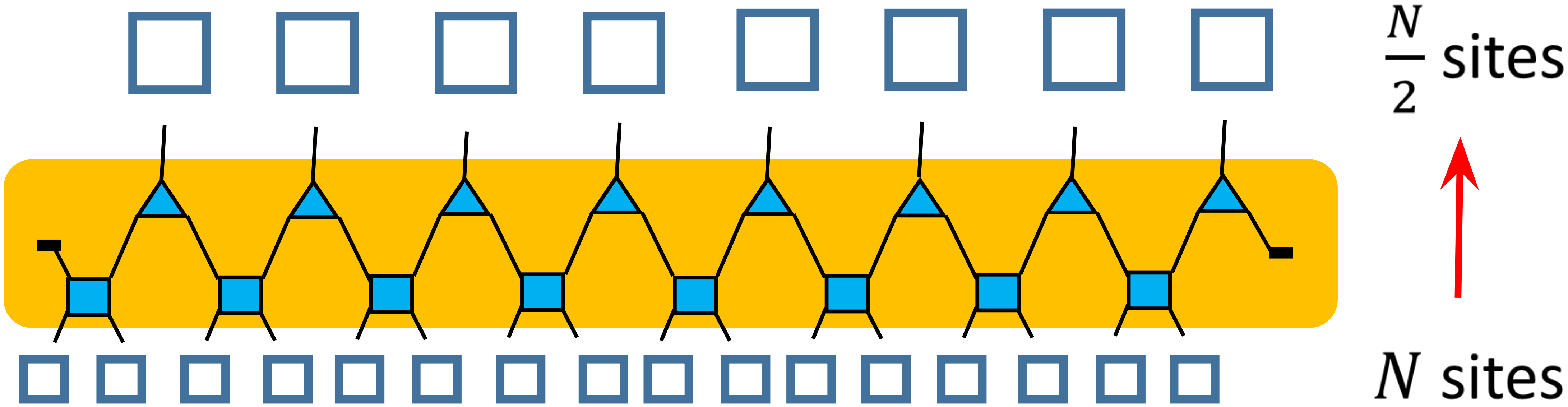
The MERA: more than just a representation



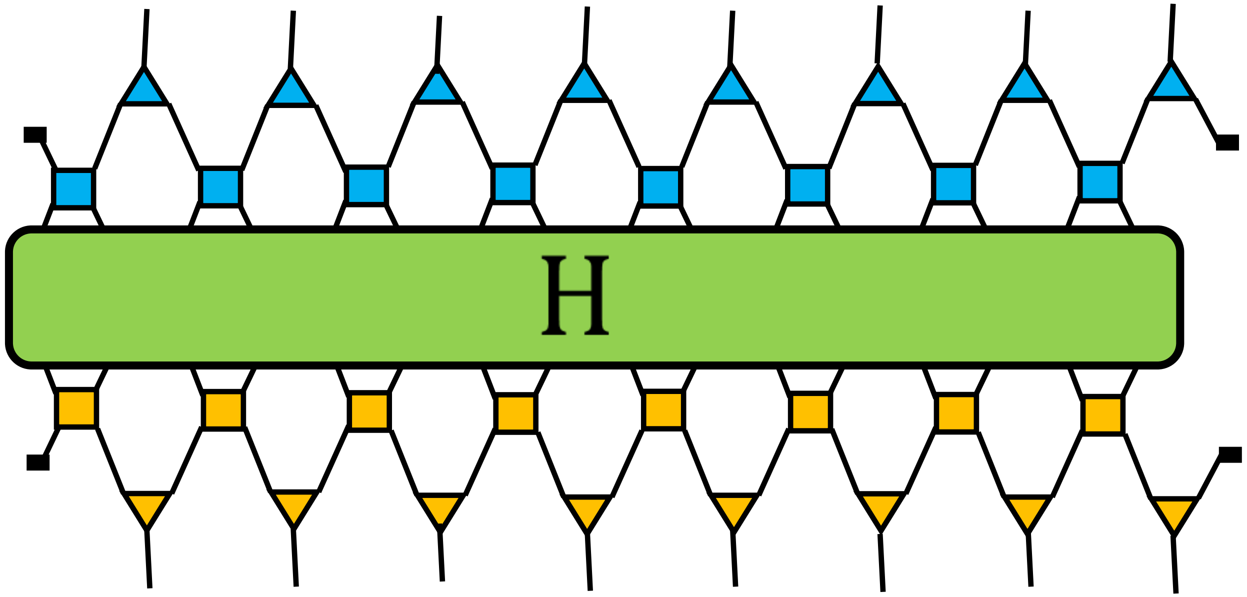
The MERA: more than just a representation



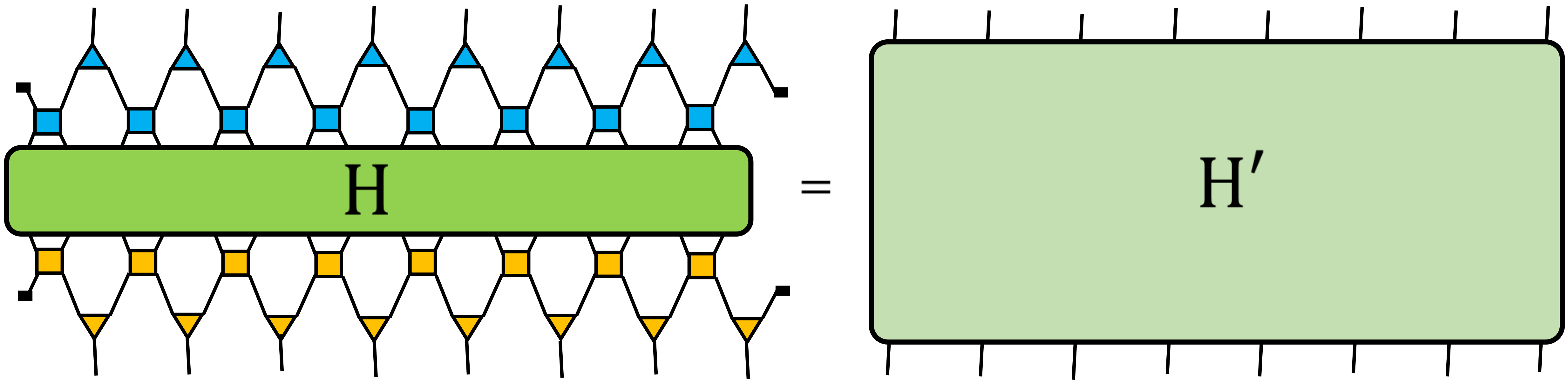
The MERA: more than just a representation



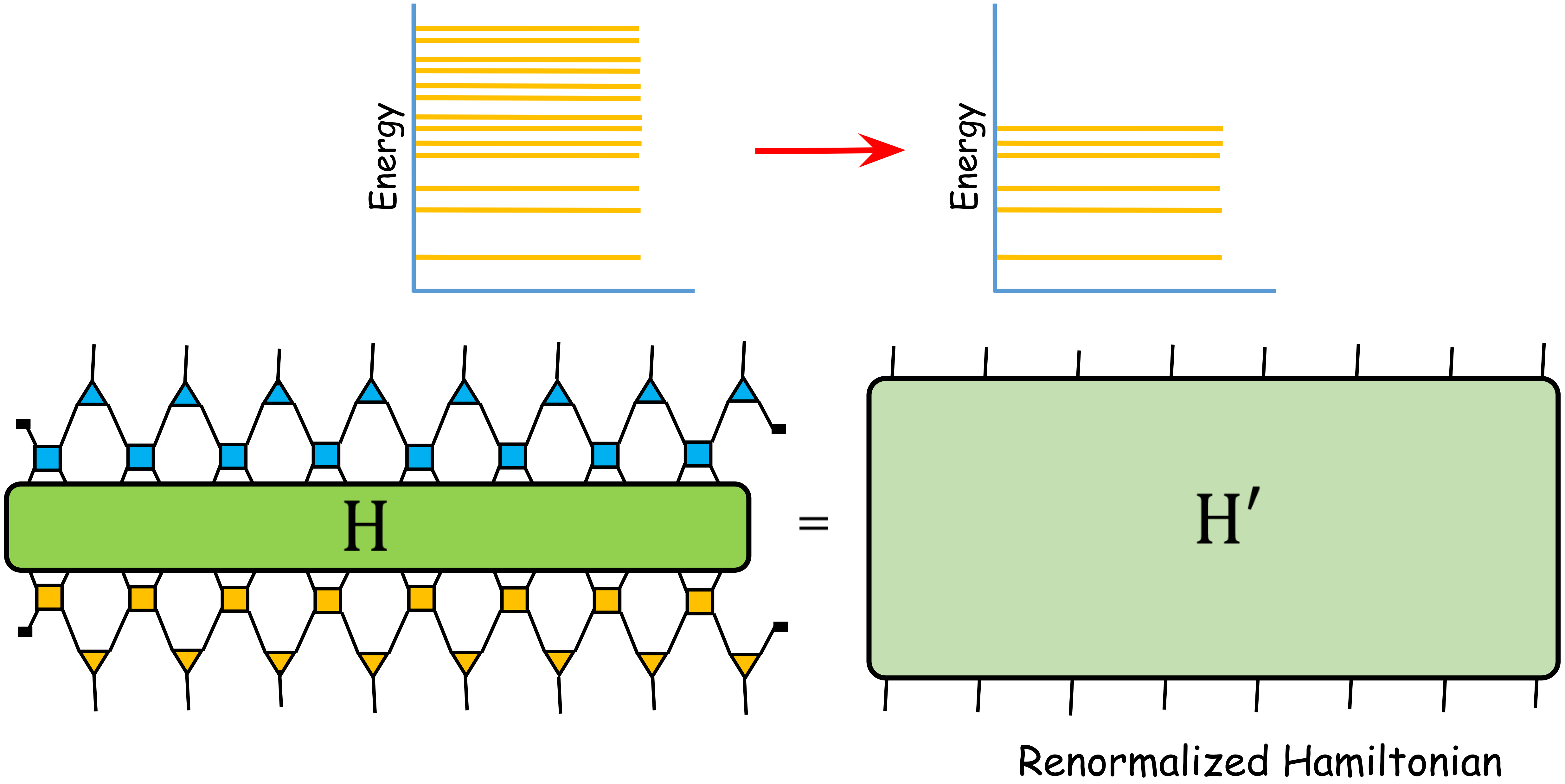
The MERA: more than just a representation



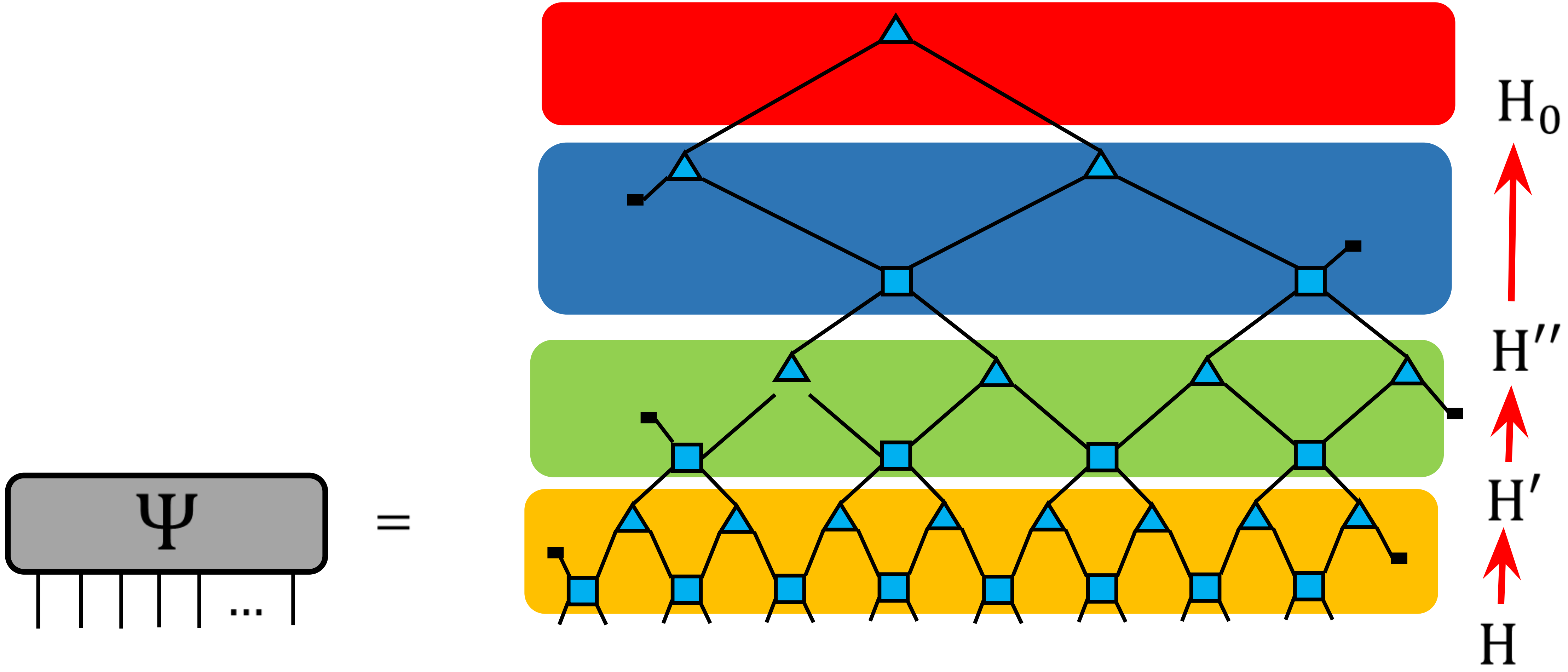
The MERA: more than just a representation



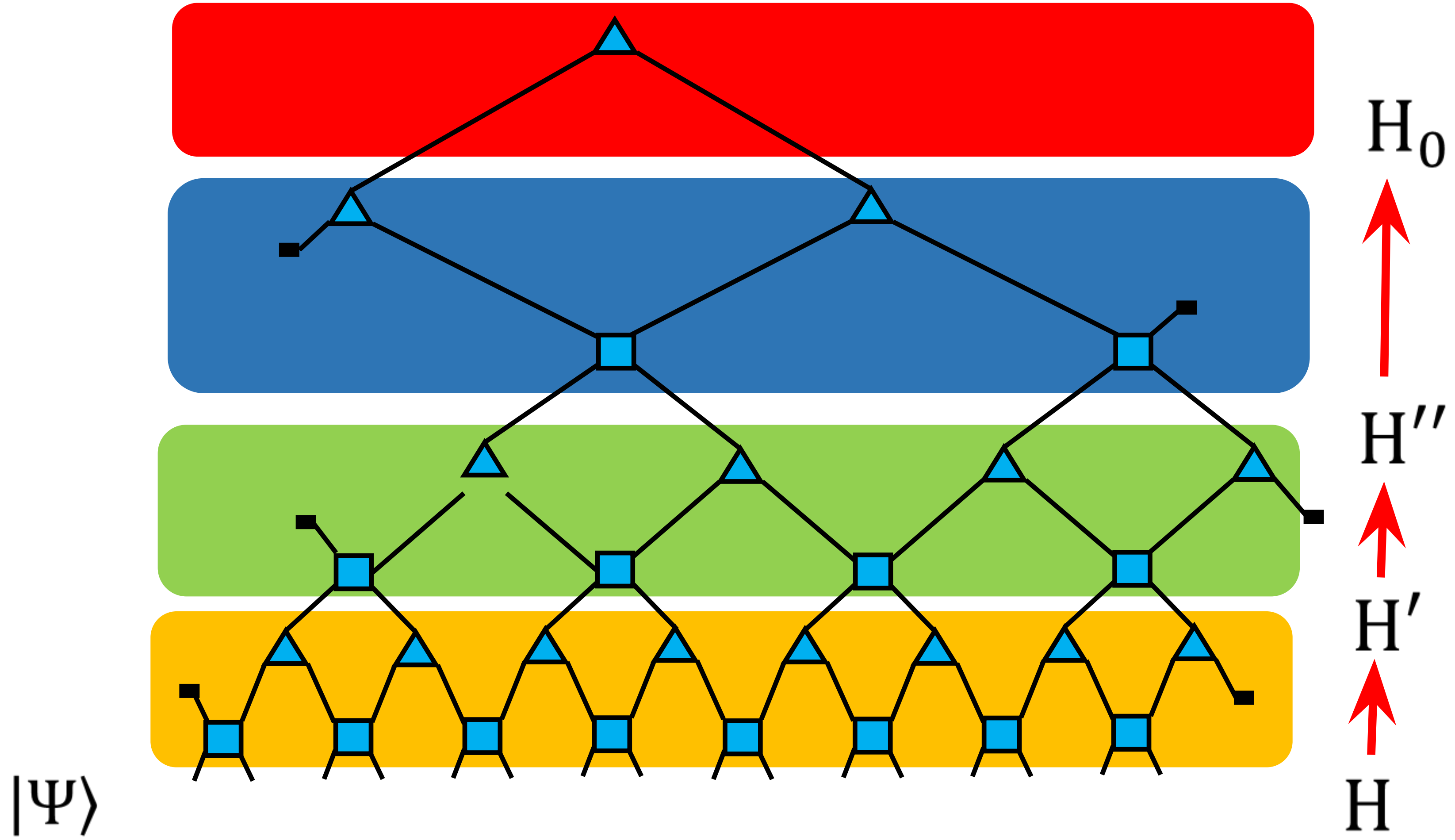
Single layer as a renormalization transformation



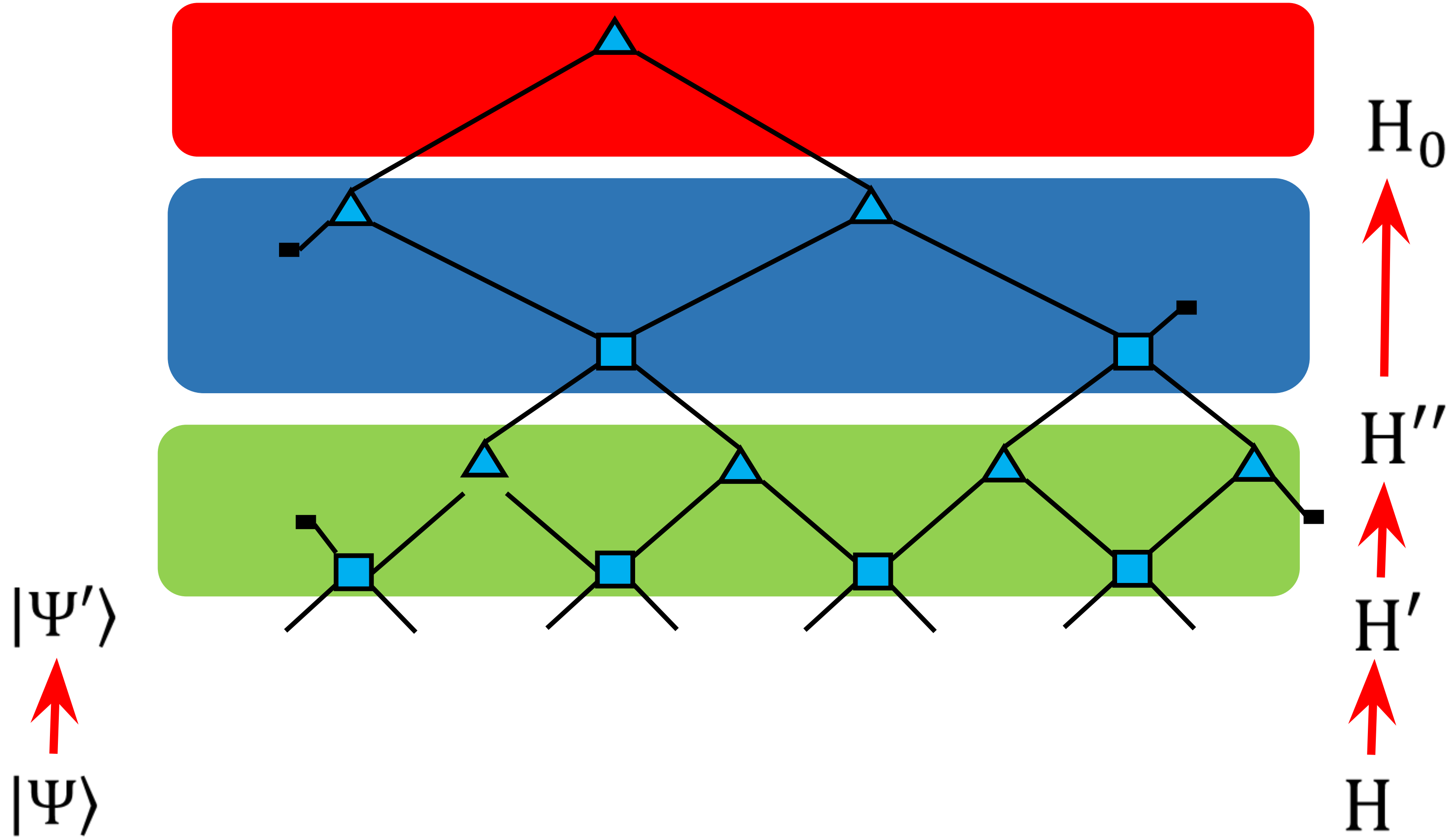
MERA as renormalization group (RG) flow



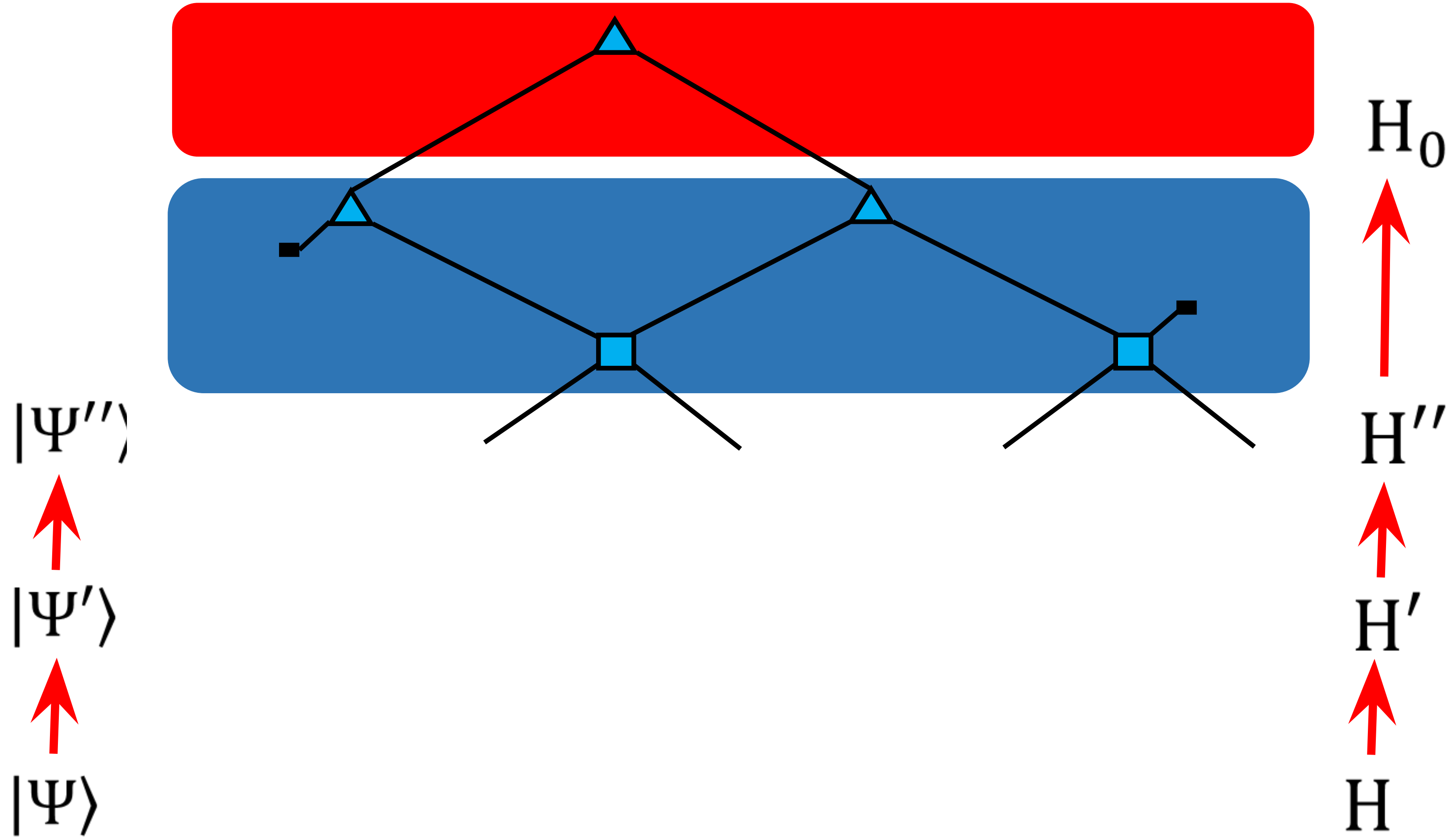
MERA as renormalization group (RG) flow



MERA as renormalization group (RG) flow



MERA as renormalization group (RG) flow



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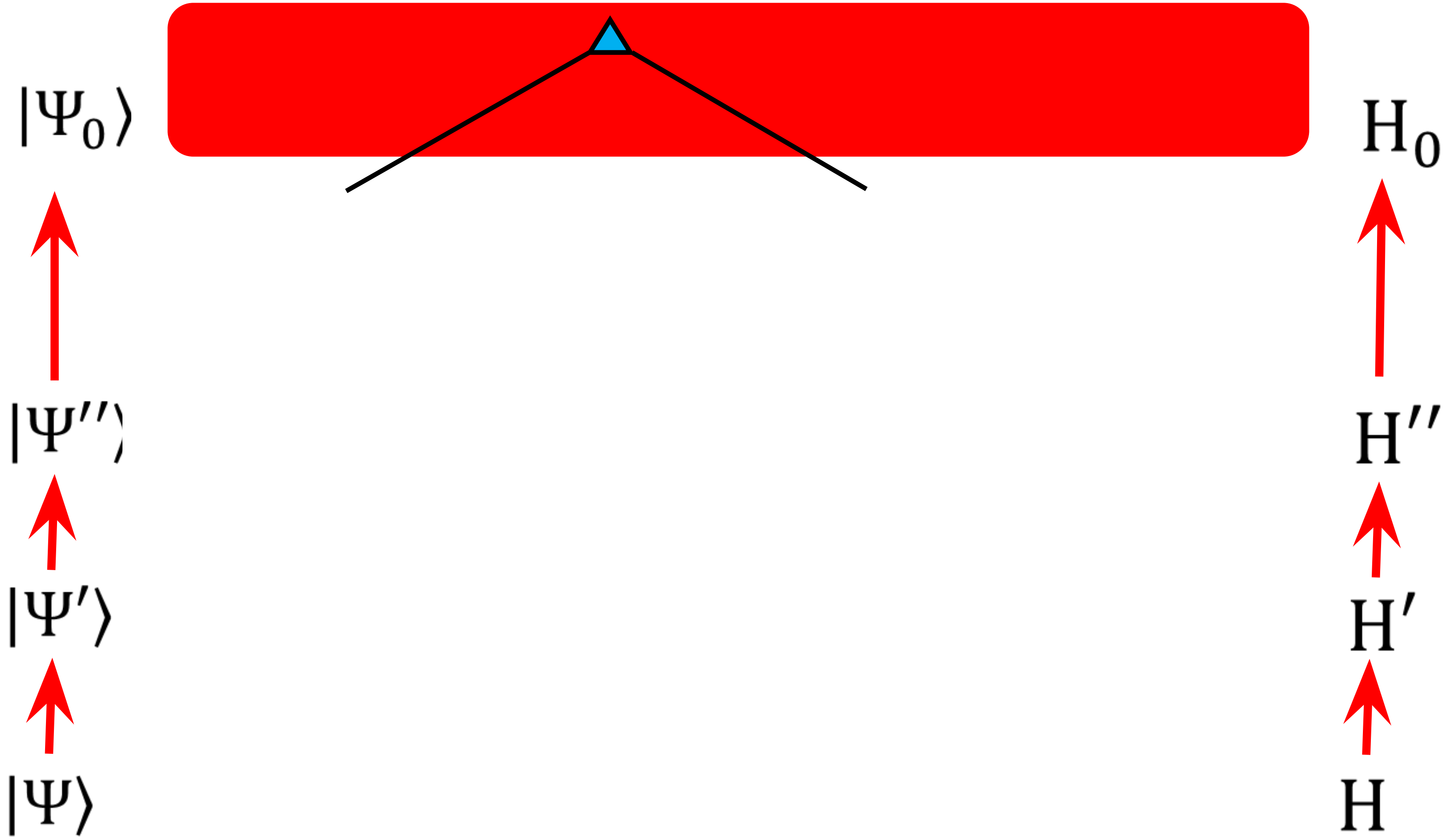

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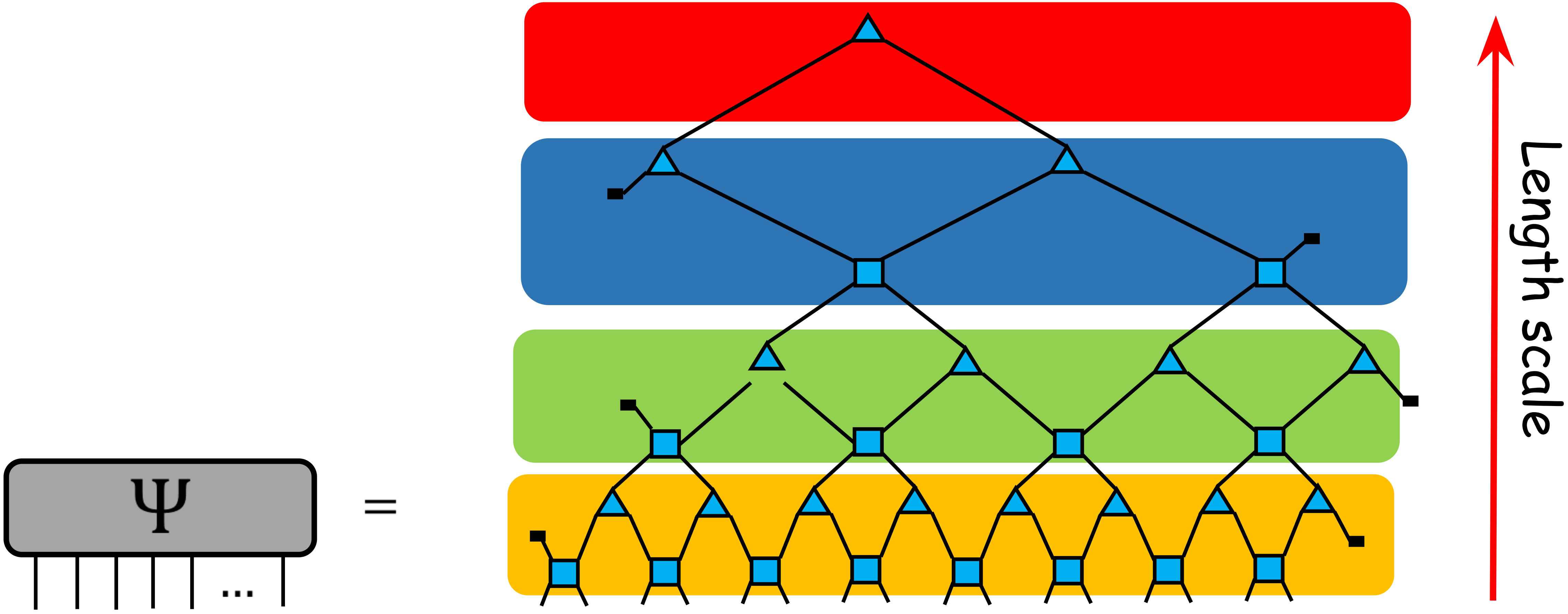
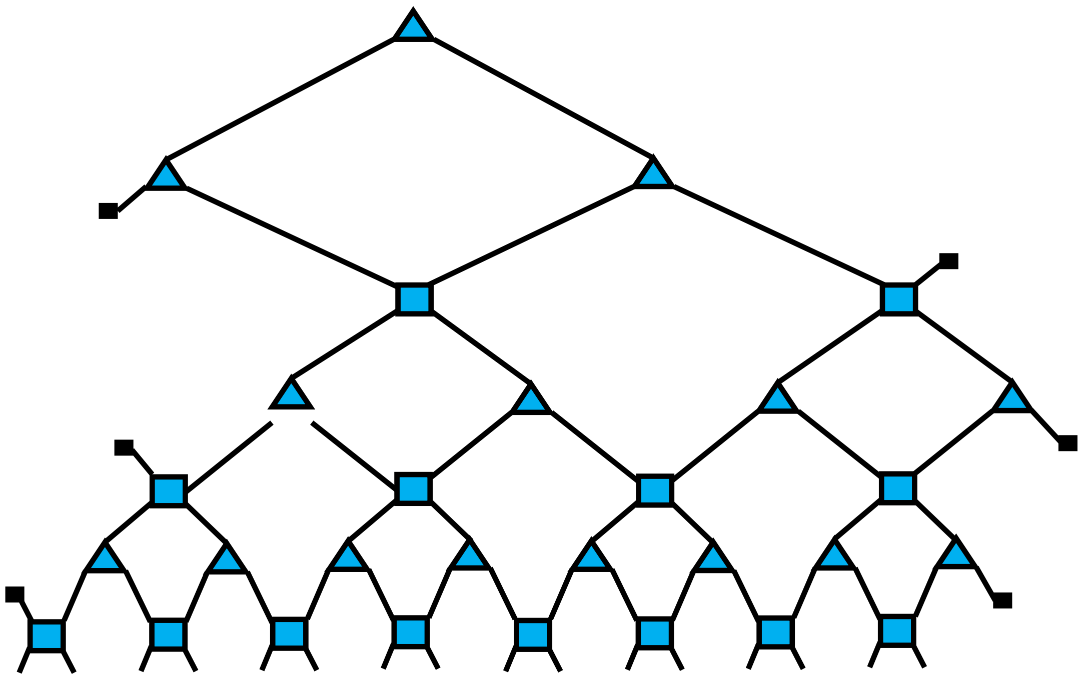
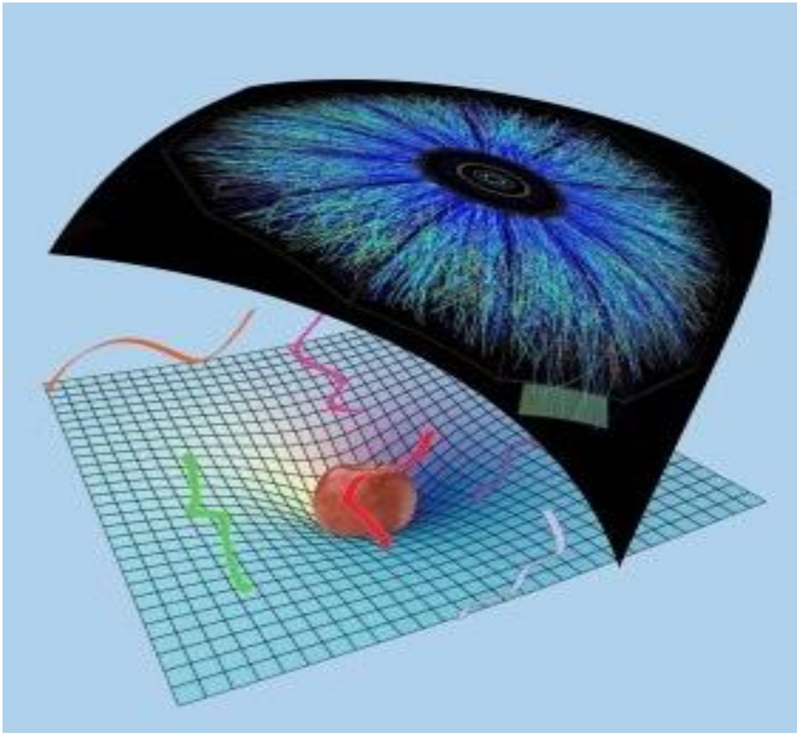


MERA and the AdS/CFT correspondence



MERA

?



Holographic correspondence

CFT side

- MERA is a “natural” ansatz for 1D critical ground states.

- Critical Hamiltonians are described by a CFT in the continuum At critical point, system becomes scale-invariant, critical ground state corresponds to the vacuum of the CFT.

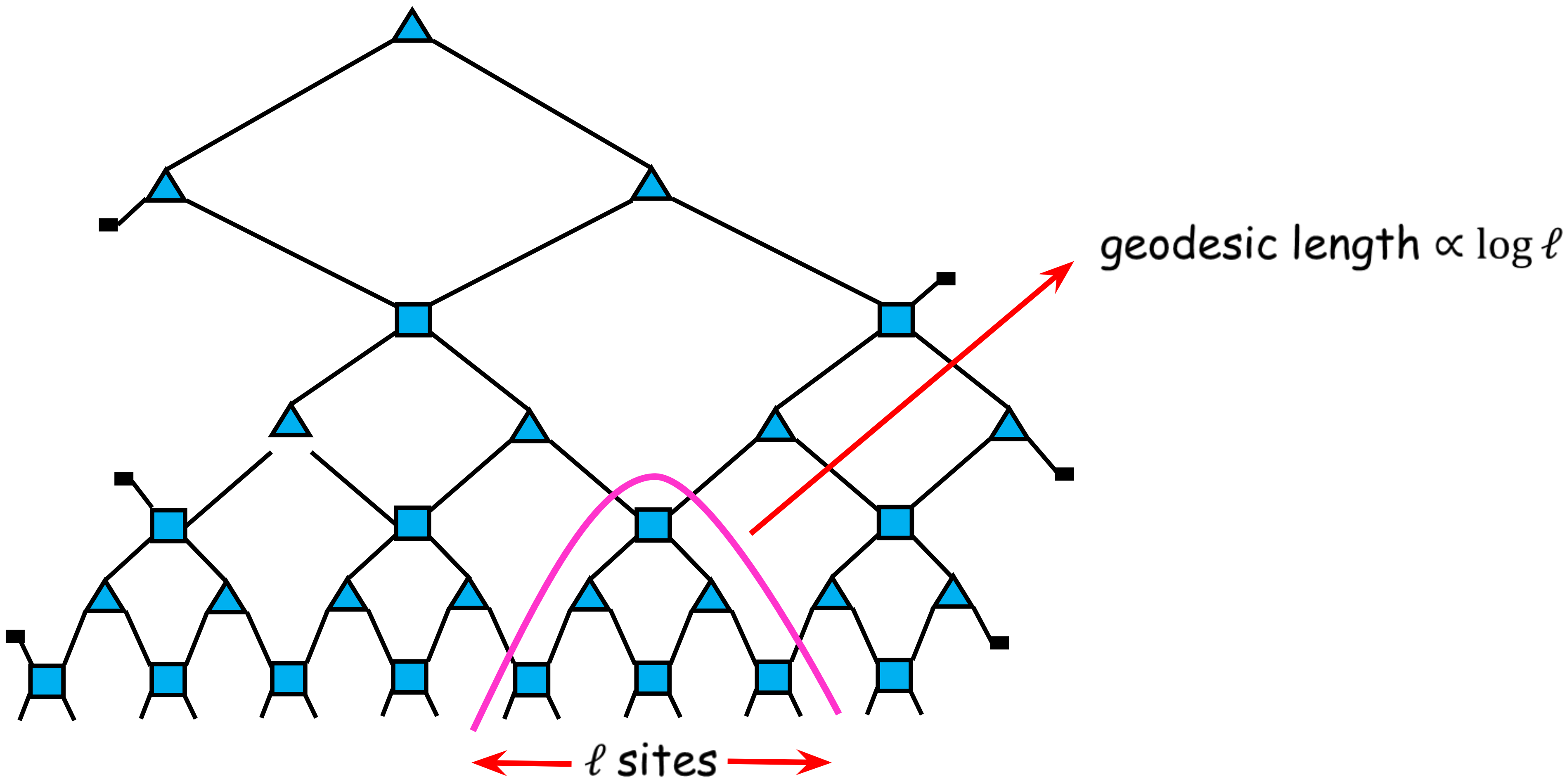
- The properties of the CFT (central charge, primary fields, scaling dimensions etc.) can be extracted easily from the MERA tensors.

- Can incorporate scale-invariance exactly into the MERA tensor network.

AdS side

MERA tensor network has a hyperbolic geometry!

B. Swingle, PRD 86, 065007 (2012)



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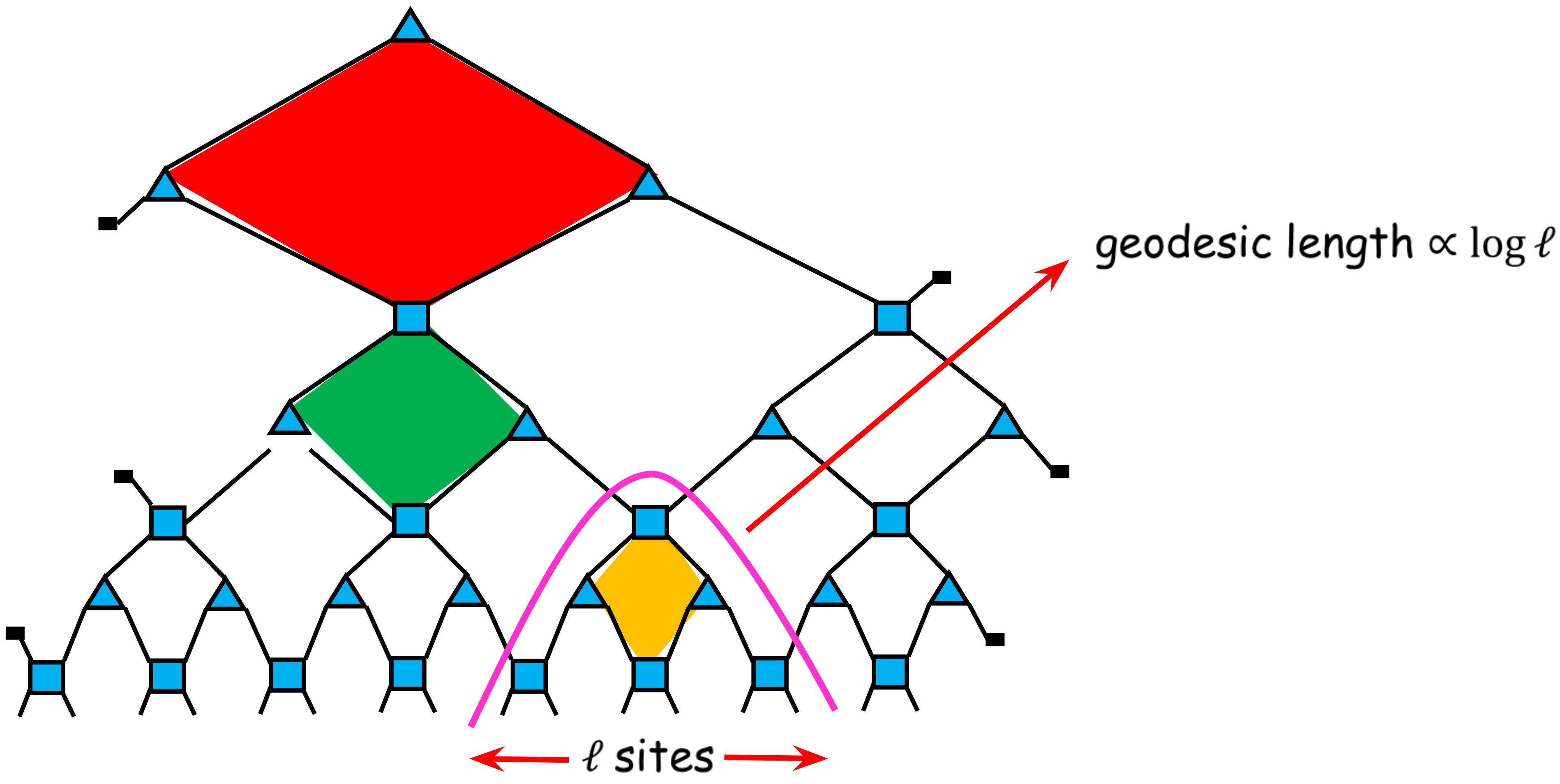

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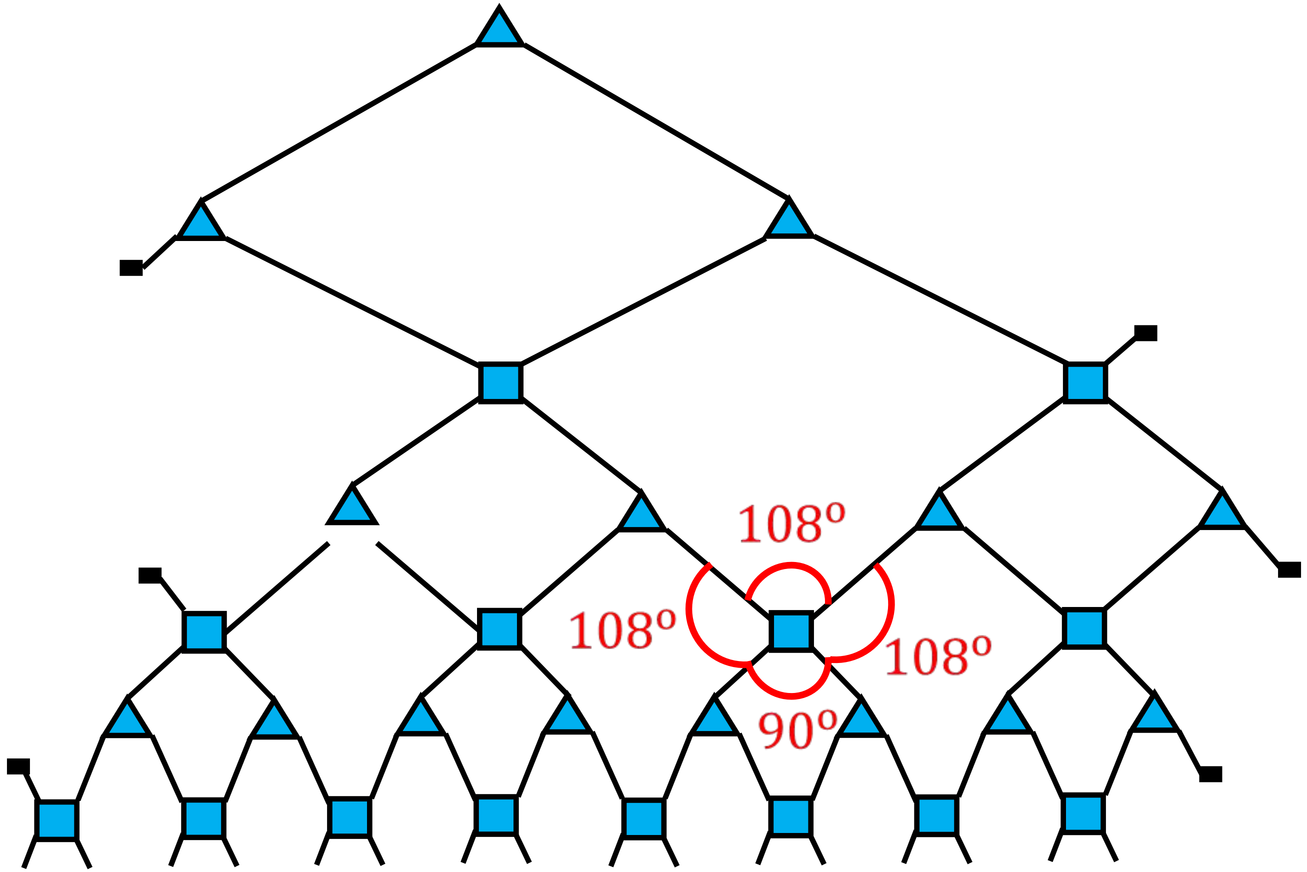
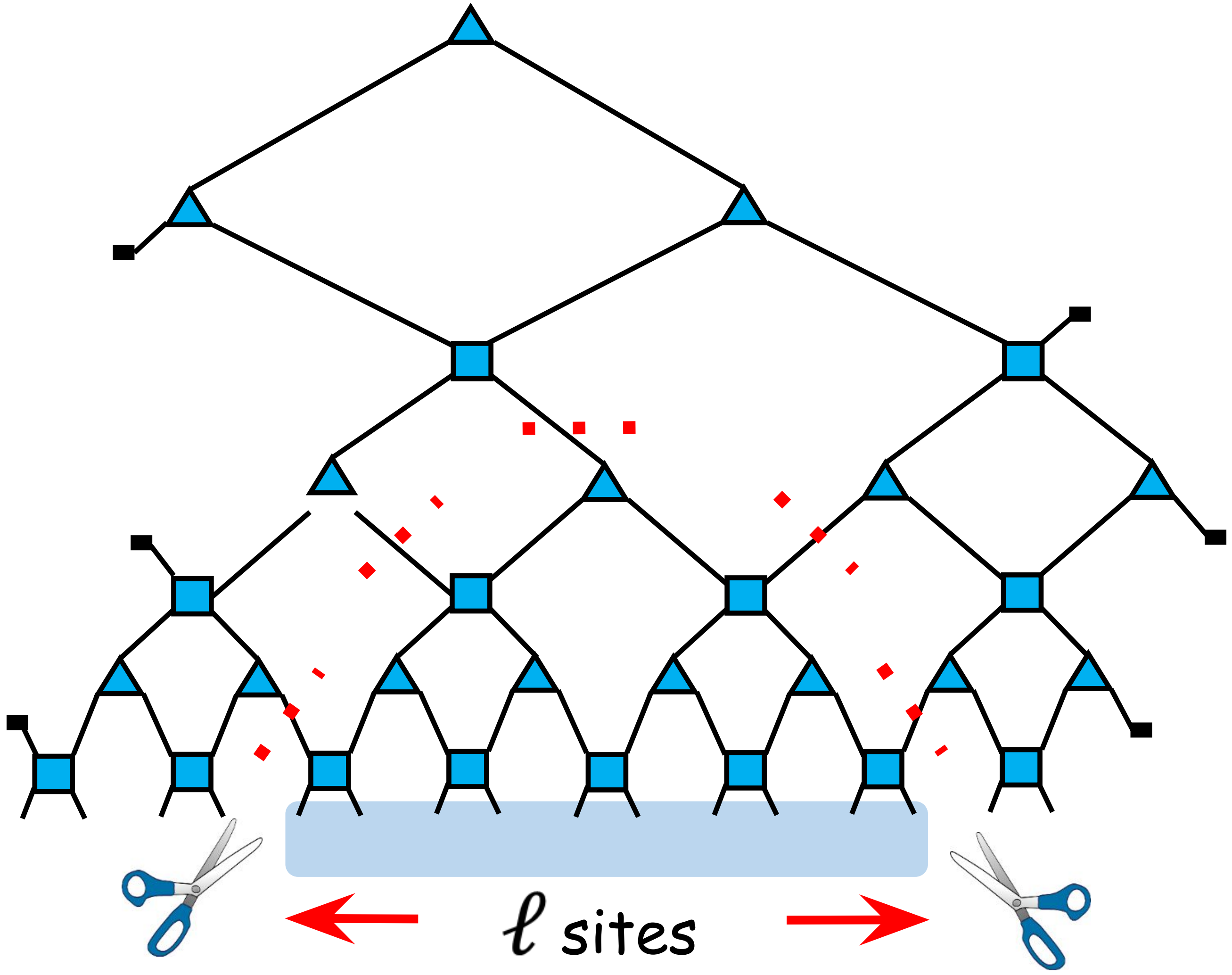


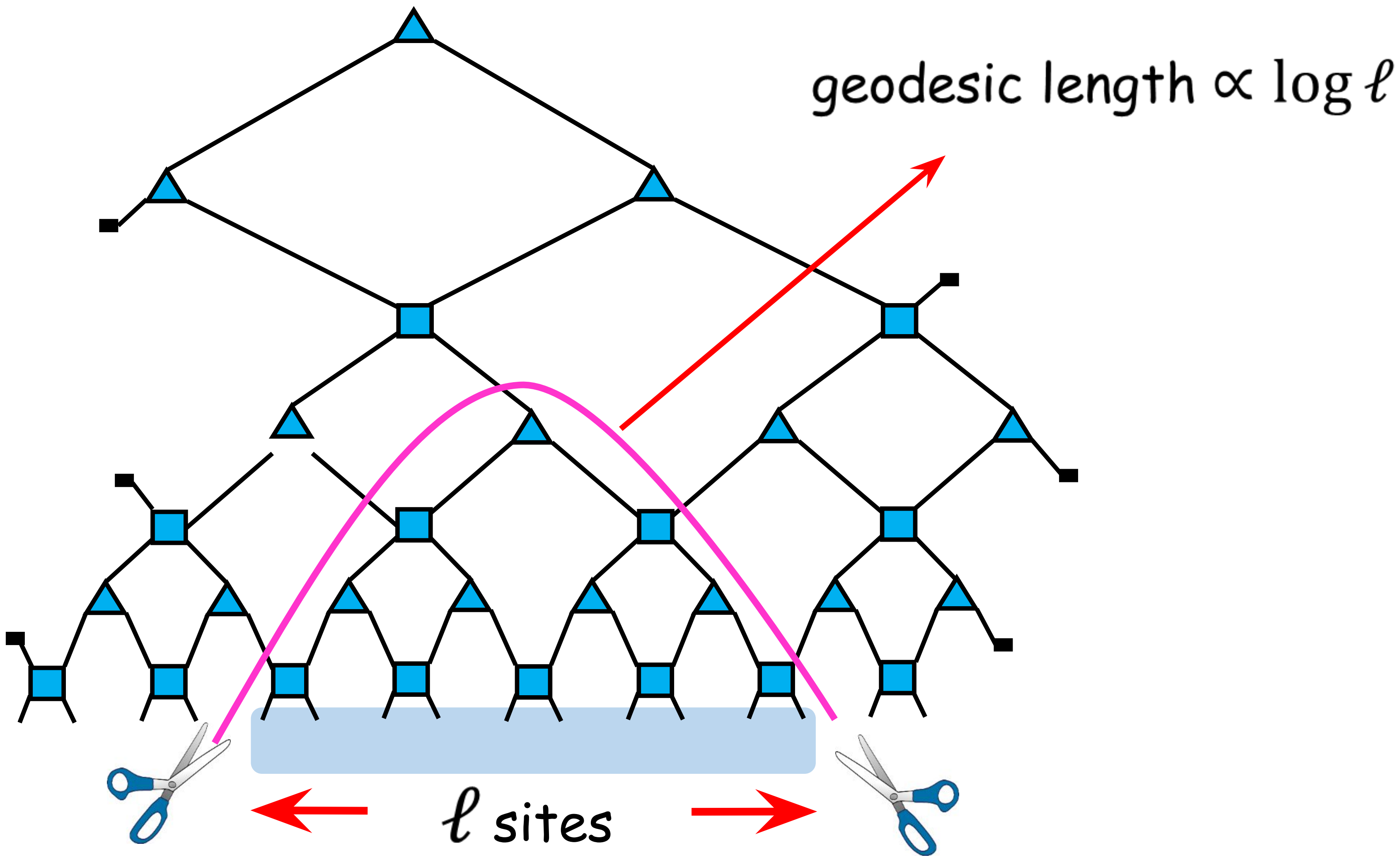
Entanglement scaling in the MERA

Minimal number of bond cuts $\propto \log \ell$



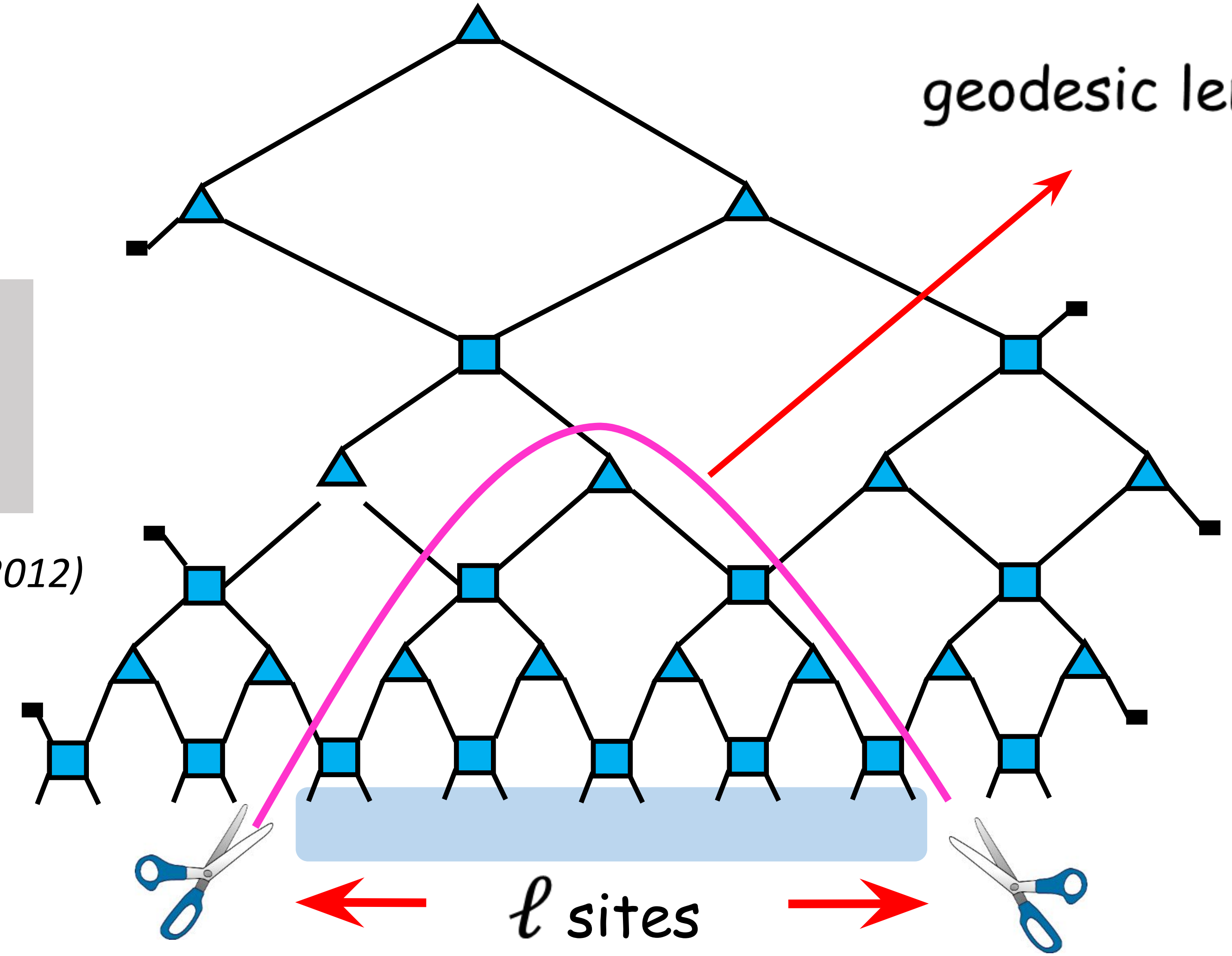
Entanglement scaling in the MERA

Block entropy identified with geodesics in the tensor network



Entanglement scaling in the MERA

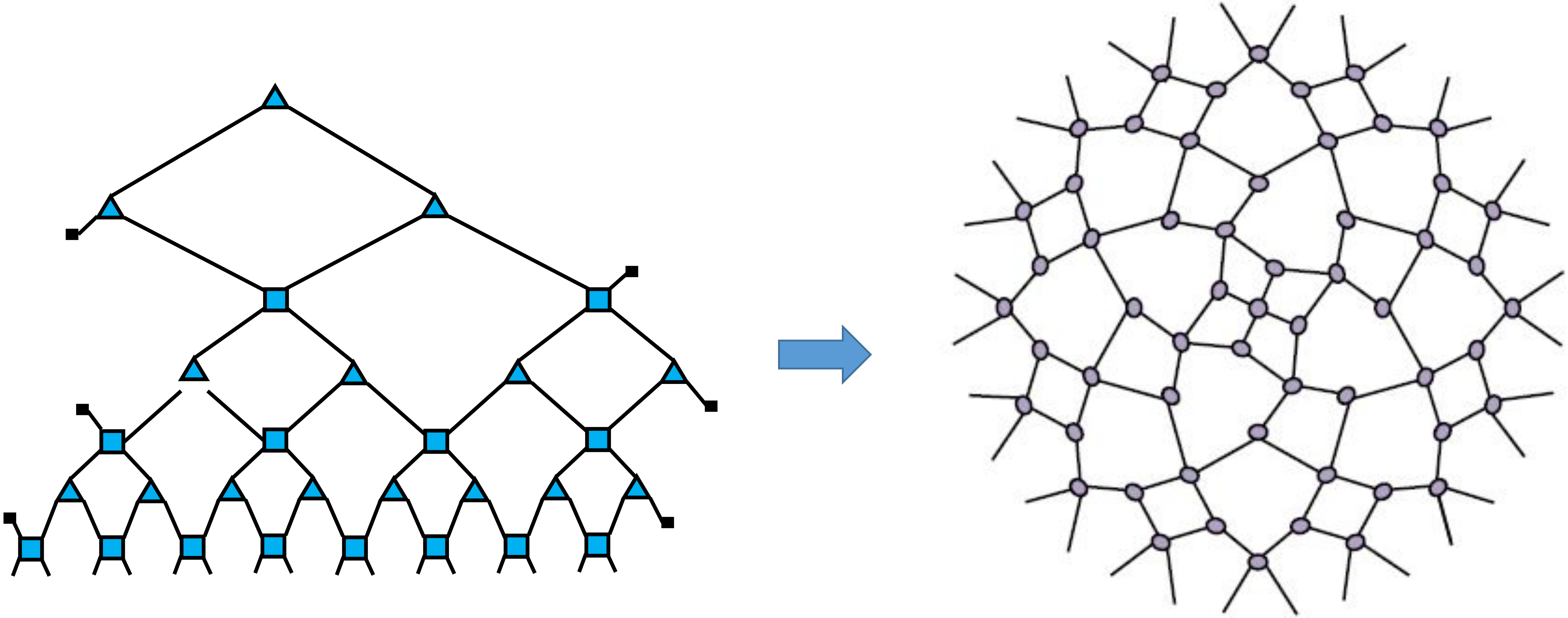
Block entropy identified with geodesics in the tensor network



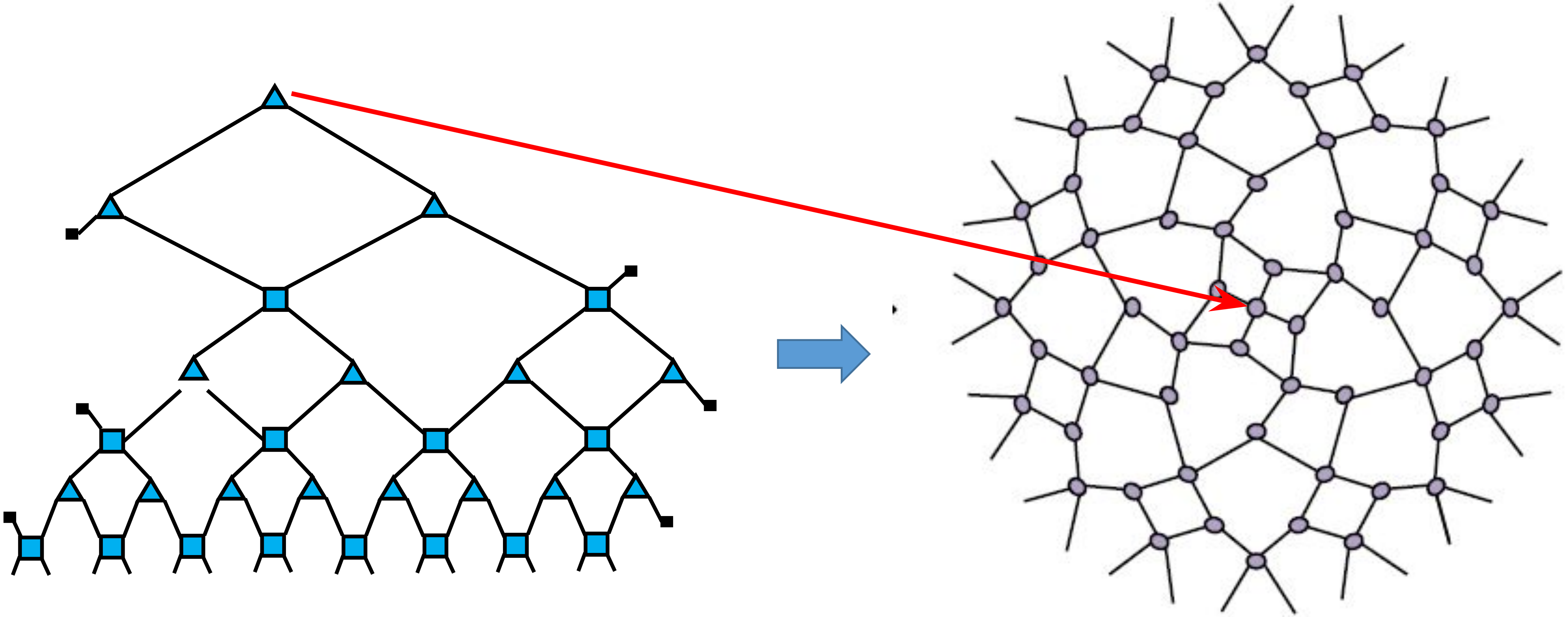
Reminds of the Ryu-Takayanagi formula

B. Swingle, PRD 86, 065007 (2012)

AdS side



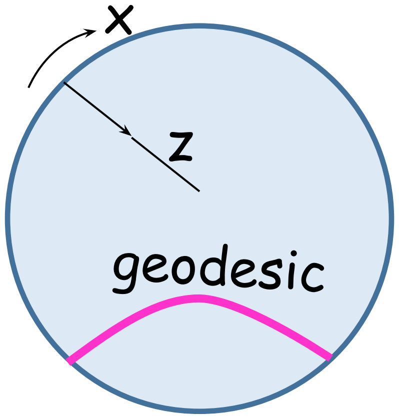
AdS side



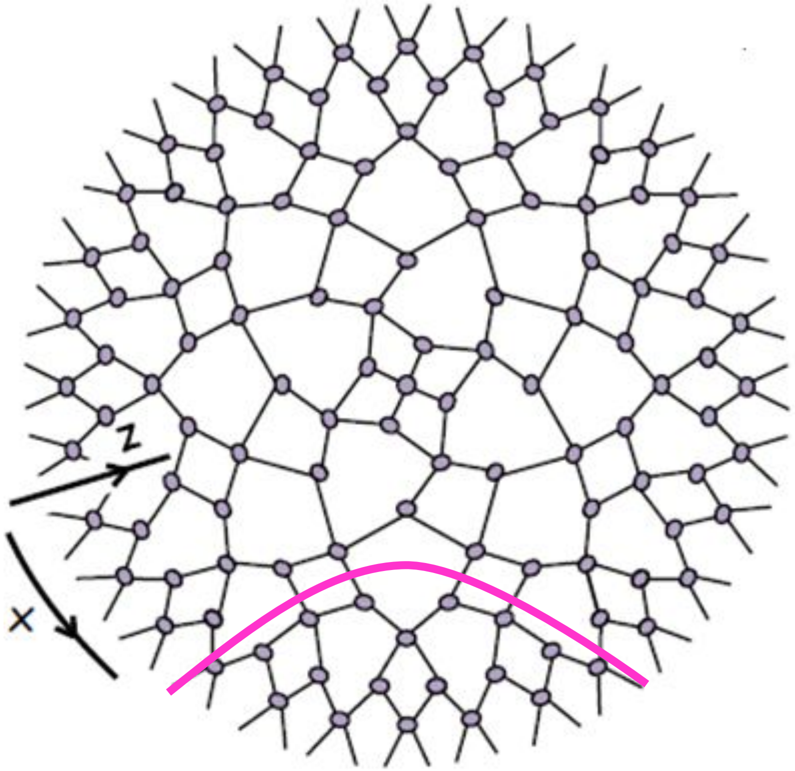
MERA/AdS conjecture

B. Swingle, PRD 86, 065007 (2012)

- MERA representation of a critical ground state can be regarded as a spatial slice of discretised 2+1 AdS spacetime.



Spatial slice of 2+1 AdS:
2D hyperbolic geometry



- Open indices correspond to sites of the lattice where the CFT lives.
- Extra dimension (z) corresponds to length scale.
- The MERA tensors that describe a critical ground state also encode a 2D gravity description.

MERA/AdS conjecture: what we have so far

- **Basic similarity:** MERA is a 2D representation of a 1D ground state and has a hyperbolic geometry, and the extra dimension in the tensor network corresponds to length scale of the boundary system.

- Some non-trivial features of the AdS/CFT correspondence are present in the MERA:
  - Analog of the Ryu-Takayanagi formula, *B. Swingle, PRD 86, 065007 (2012)*
  - Thermal state/black hole correspondence, *T. Hartman and J. Maldacena arXiv:1303.1080*
  - Bulk/boundary symmetries etc. *S. Singh and G. Vidal PRB (R) 88 121108 (2013)*

B. Swingle, arXiv:1209.3304.
J. Molina-Vilaplana and P. Sodano, JFEP 10, 11 (2011), arXiv:1108.1277.
H. Matsueda, M. Ishihara, and Y. Hashizume, Phys. Rev. D 87, 066002 (2013), arXiv:1208.0206.
M. Nozaki, S. Ryu, and T. Takayanagi, JHEP 10, 193 (2012), arXiv:1208.3469.
C. Beny, New J. Phys. 15, 023020 (2013), arXiv:1110.4872.
B. Czech, L. Lamprou, S. McCandlish, J. Sully, arXiv:1512.01548, SU-ITP-15/18, SLAC-PUB-16292.

Open questions

- Is the MERA/AdS conjecture true?

- Mostly qualitative observations. Can we make the conjecture more precise?

- We understand how MERA encodes the boundary critical ground state. How does the tensor network encode the dual bulk description?

  One potential approach

  SS, arXiv:1701.04778

  SS, N. McMahon, G. Brennen, arXiv:1702.00392

- Does the MERA encode a more general bulk/boundary correspondence, since in principle, one can represent the ground state of any CFT as a MERA (possibly including CFTs that are not known/expected to have a gravity dual).

Outlook

- If true, the MERA/AdS conjecture implies that a physical quantum gravity state is encoded in the MERA somehow, without even knowing the quantum gravity theory (but this in fact the promise of holography --- the bulk gravity is just a QFT in one lower dimension).

- Thus, MERA may be a toy model to explore the holographic correspondence.

- In reverse, insights from holography have already been applied to design better tensor networks for simulations of strongly correlated many-body systems e.g., the thermal state MERA, branching MERA etc.

THANKS!