Entanglement & non-locality in infinite 1D systems

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Based on

Z. Wang, SS, and M. Navascues, arXiv:1608.03485 (to appear soon in PRL)

- Entanglement and non-locality are hallmark features that signify clear departure from classical physics.

- Entanglement: A quantum many-body state is entangled if it is not multi-separable.

- (Bell) Non-locality: A conditional probability distribution is (Bell) non-local or non-classical if there is no local hidden variable model to reproduce it.

- **Goal:** Detection of entanglement and non-locality in 1D infinite <u>translation invariant</u> systems when you have access to <u>only</u> near-neighbors information.

Foundational motivation

- Detecting global properties of an infinite system from partial information (of a small number of neighbors).
- If some observers share a separable (classical) state and they have been promised that the state is part of an infinite TI state, they can detect if the total state is entangled (non-classical) in some cases.
- Reveals an interplay between TI and entanglement/non-locality.

Detection of entanglement/non-locality in condensed matter systems

- Can apply these results to detect entanglement and non-locality in condensed matter systems (typically, finite and non-TI!).

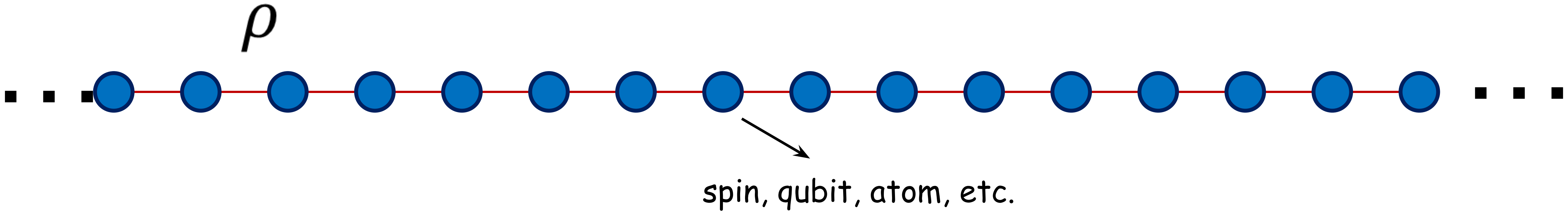
Previous work

- Global entanglement detection: Using bipartite entanglement between either a part of the chain with the rest of the chain or between two distant sites.
- W. K. Wootters, Contemporary Mathematics, 305, 299 (2002), arXiv:quant-ph/0001114.
- M. M. Wolf, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett., 92,087903 (2004).
- A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature, 416,608 (2002).
- J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys., 82,277 (2010).

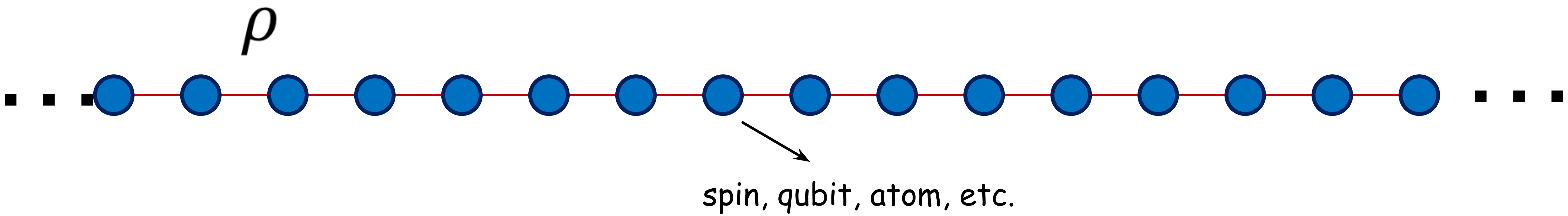
- Global non-locality detection: studied for finite number of parties.
- J. Tura, A. B. Sainz, T. Vértesi, A. Acín, M. Lewenstein, and R. Augusiak, Journal of Physics A: Mathematical and Theoretical, 47, 424024 (2014).
- J. Tura, G. De las Cuevas, R. Augusiak, M. Lewenstein, A. Acín, and J. I. Cirac, Phys. Rev. X, 7, 021005 (2017).

- Here we focus on <u>multi-partite</u> entanglement/non-locality in 1D <u>infinite</u> systems

Entanglement detection



- Have a quantum state ρ of a 1D infinite lattice



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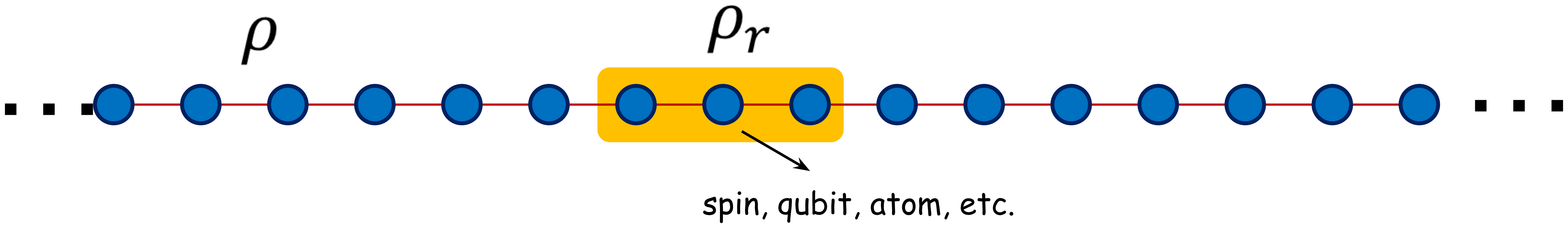
- Its <u>invariant under translations</u>: Reduced density matrix of any contiguous block of *r* sites is equal anywhere on the lattice (for all *r*).



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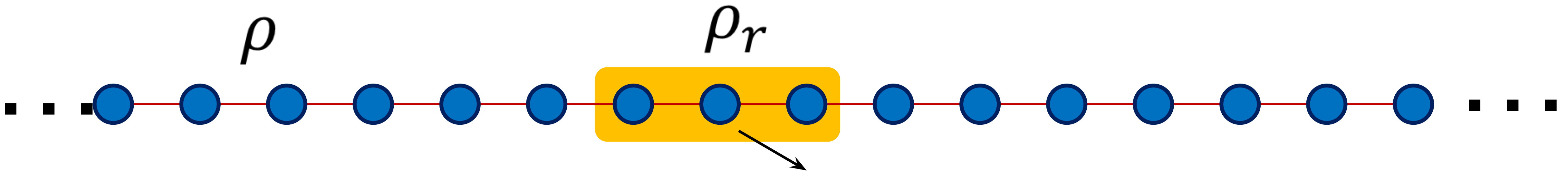

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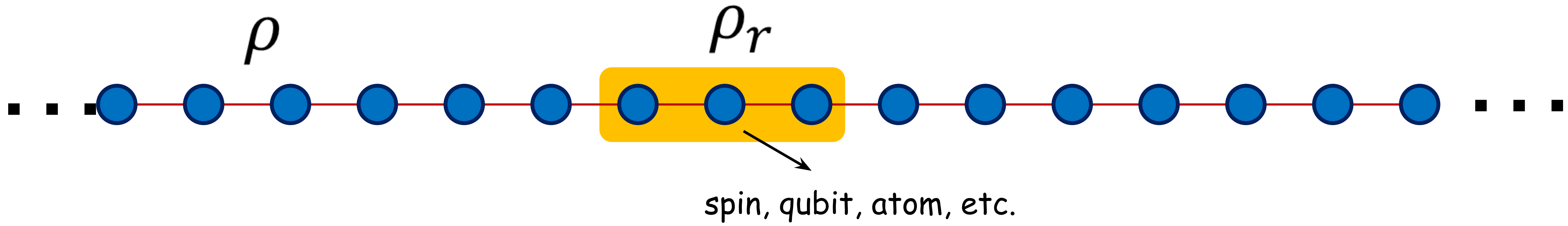
- Its invariant under translations: Reduced density matrix of any contiguous block of *r* sites is equal anywhere on the lattice (for all *r*).
- Say we can access only a part of the state ρ_r. Can we detect if the total state ρ is entangled or not?
- A quantum state is not entangled iff it is multi-separable:

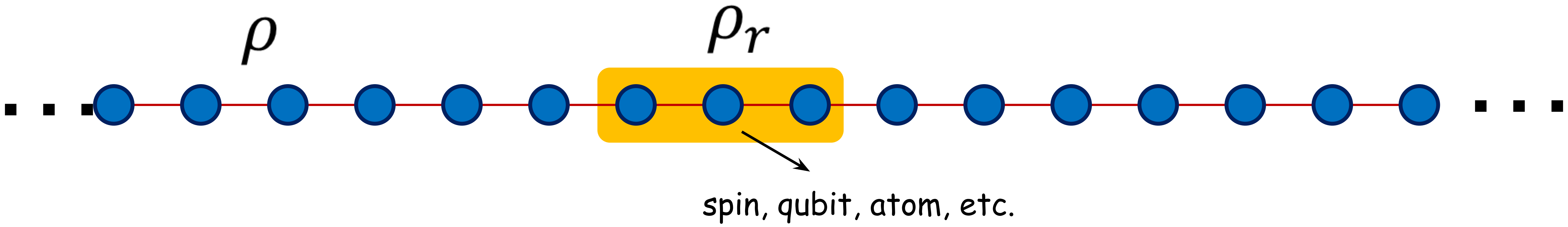
$$\rho = \int d\,\vec{\rho}\; \mathrm{P}(\rho_1, \rho_2, \dots)\; \rho_1 \otimes \rho_2 \otimes \cdots$$

some probability distribution

reduced density matrix on site 1



- Of course, if ρ_r is entangled e.g. ρ_r is not positive under partial transposition then the total state ρ is also entangled.



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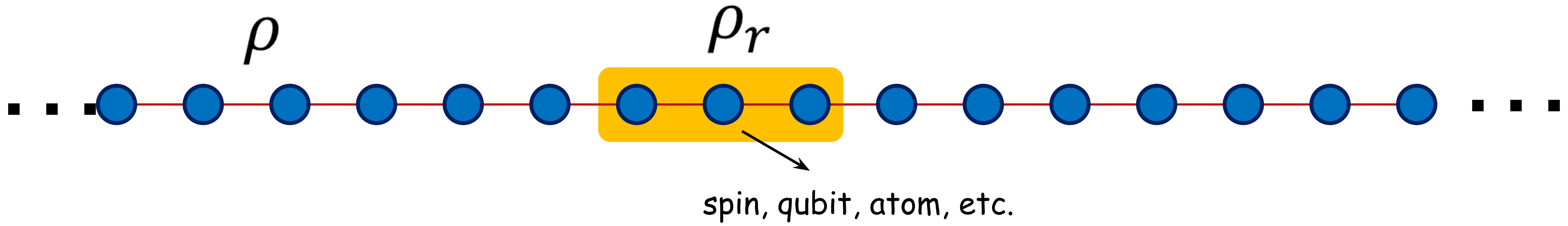

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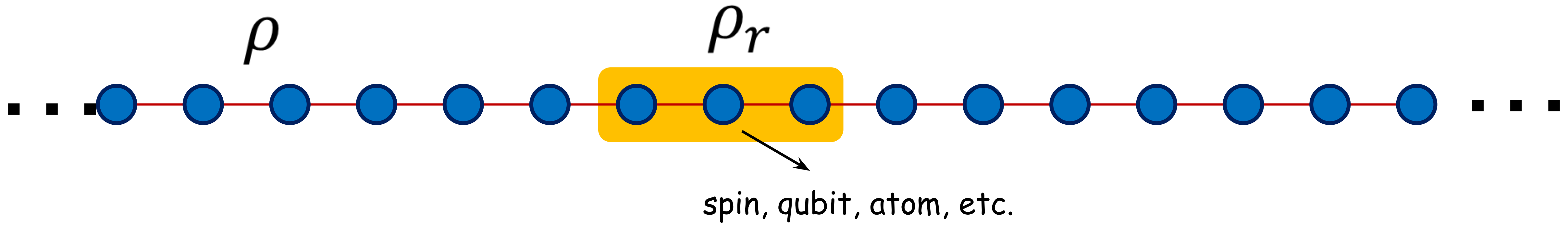

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- We are interested in the case where ρ_r is multi-separable.
- Will illustrate that the total state ρ can still be entangled in this case!
- **Approach:** Does ρ_r admit a <u>TI and separable extension</u>?

That is, can we find a 1D infinite TI state such that the reduced density matrix of any block of *r* sites is equal to ρ_r?

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can it realized as the marginal of an infinite TI distribution?

- Yes, if and only if

$$P_{1,\dots,r-1}(x_1, \dots, x_{r-1}) = P_{2,\dots,r}(x_2, \dots, x_r) \qquad \textbf{Lemma 1}$$

(Known for some time. Proof in paper.)

- Find all

$$P_{1,\dots,r}(x_1, \dots, x_r)$$

such that

$$P_{1,\dots,r-1}(x_1, \dots, x_{r-1}) = P_{2,\dots,r}(x_2, \dots, x_r)$$

- Can be solved using linear programming.

- Set of all such $P_{1,\dots,r}(x_1, \dots, x_r)$ form a polytope characterized by a finite number of vertices and facets.

- We have a multi-separable state ρ_r :

$$\rho_r = \int d\,\vec{\rho}\;\mathrm{P}(\rho_1, \ldots, \rho_r)\;\rho_1 \otimes \cdots \otimes \rho_r$$

- **Observation 1:** ρ_r has a TIS extension iff $\mathrm{P}(\rho_1, \ldots, \rho_r)$ satisfies Lemma 1.

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- **Hypothesis 1:** If ρ_r has a TIS extension then

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2) $\rho_{1,\dots,r-1} = \rho_{2,\dots,r}$

- These conditions are necessary. But are they also sufficient? No!

- All states TI states ρ must satisfy:

$$tr\left(\rho_{1,2}\ \sigma_y \otimes \sigma_x\right) \leq \frac{2}{\Pi} \qquad \textbf{(TI witness)}$$

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- Consider the state

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- ρ is separable and TI, $\rho_1 = \rho_2 = \frac{1}{2}I$ but $tr\left(\rho\ \sigma_y \otimes \sigma_x\right) = 1$
- Therefore, hypothesis 1 is false. Since these conditions cannot even guarantee the existence of a TI extension, separable or not.

- **Hypothesis 2:** If ρ_r has a TIS extension then

1) ρ_r must be separable
2) $\rho_{1,\dots,r-1} = \rho_{2,\dots,r}$
3) ρ_r has a TI extension (improvement from Hypothesis 1)

- These conditions are necessary. But are still <u>not</u> sufficient!

- Can show that all states $\rho_{1,2}$ with a TIS extension must satisfy:

$$tr\left(\rho_{1,2}\ \sigma_y \otimes \sigma_x\right) \leq \frac{1}{2} \qquad \textbf{(entanglement witness)}$$

- Got by applying **Observation 1** (quantum extension of **Lemma 1**)

- We identified a family of TI states ρ^{λ} of an infinite lattice of qubits such that $\rho^{\lambda}_{1,2}$ is separable. Some of these states $\rho^{\lambda}_{1,2}$ do not violate the **TI witness** (so they have TI extensions) but violate the **entanglement witness** inequality (so they <u>do not</u> have TIS extensions).

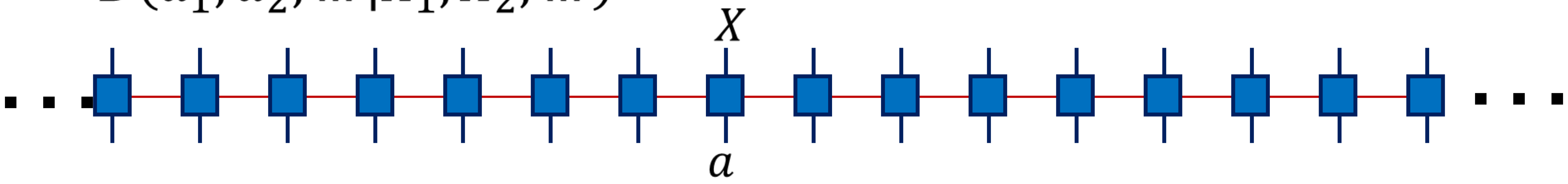
- Therefore even Hypothesis 2 is false.

- Could not find a simple (for practical detection), necessary and sufficient criterion for a TIS extension.

- But these examples illustrate that some separable states admit <u>only</u> entangled TI extensions. Thus, observers with access only to the partial state can detect the presence of global entanglement in these cases.

Non-locality detection

$B(a_1, a_2, \dots | X_1, X_2, \dots)$



- Each site is a black box with $X = 1,2, \dots, n_x$ inputs (measurement settings) and a$= 1,2, \dots, n_a$ outputs (measurement outcomes)

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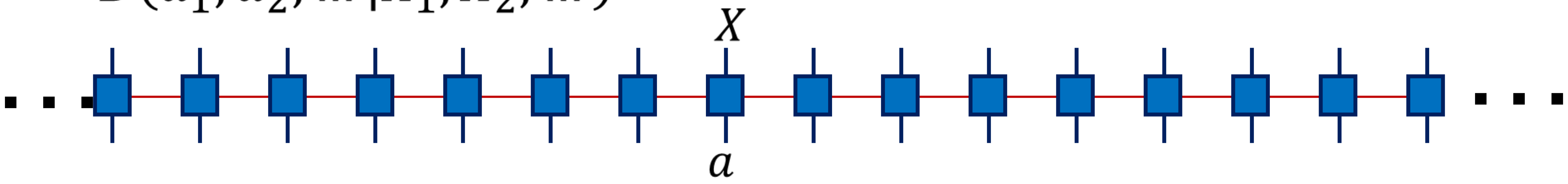

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- Have a conditional probability distribution of a 1D infinite system

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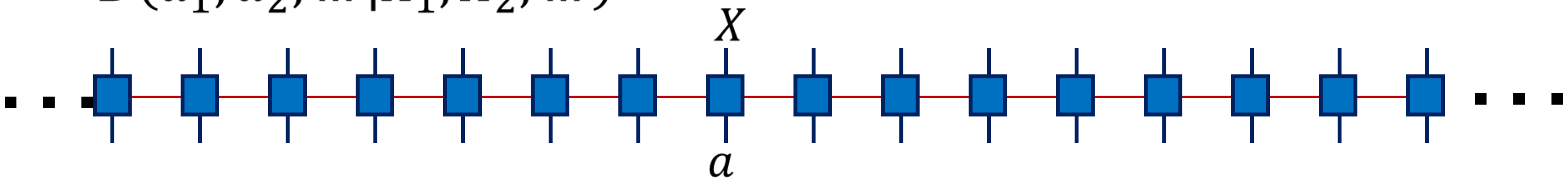

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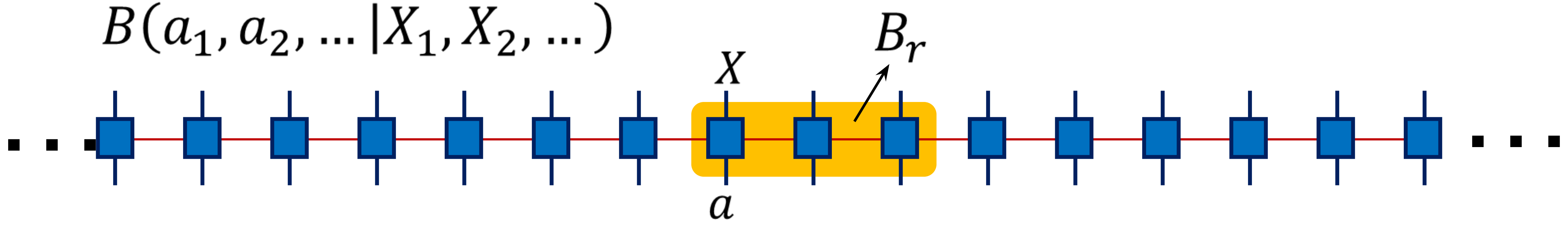
- Box B is <u>translation invariant</u>: the marginal box B_r of any r sites is the same anywhere along the chain (for all r).



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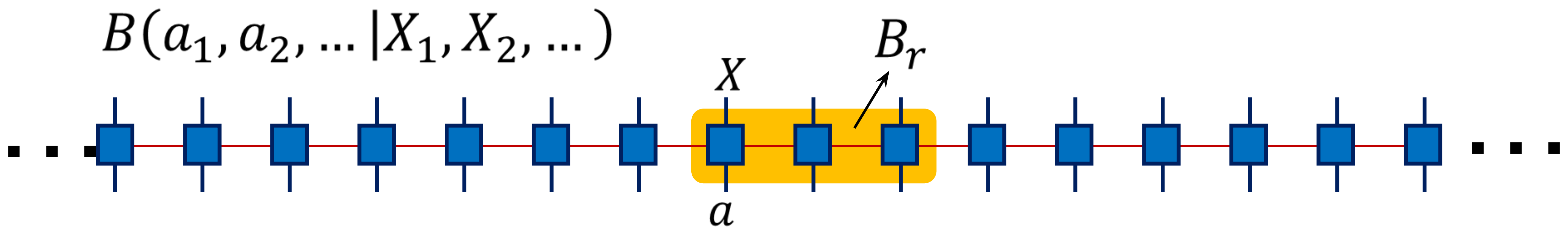
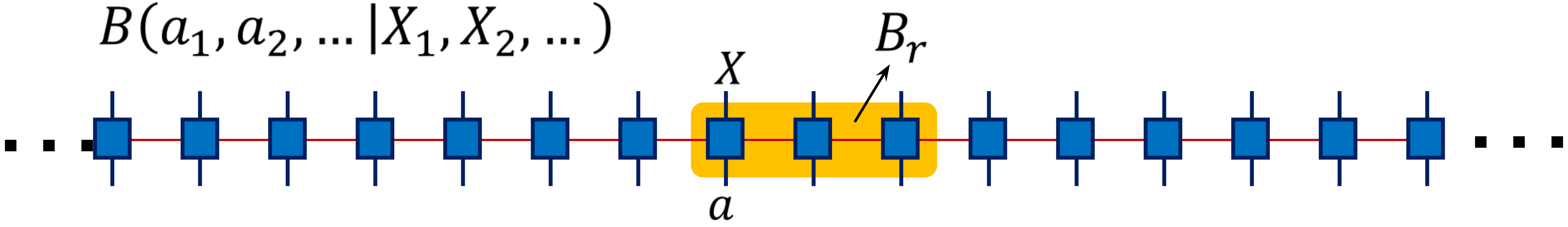
  $B(a_1, a_2, \dots | X_1, X_2, \dots)$ (no signaling)
- Box B is <u>translation invariant</u>: the marginal box B_r of any r sites is the same anywhere along the chain (for all r).
- Have access to only a marginal box B_r. Can we detect if the total TI box B is classical or not?



- A classical box is one that has a local hidden variable model, namely, its statistics can be modelled by sharing classical randomness.



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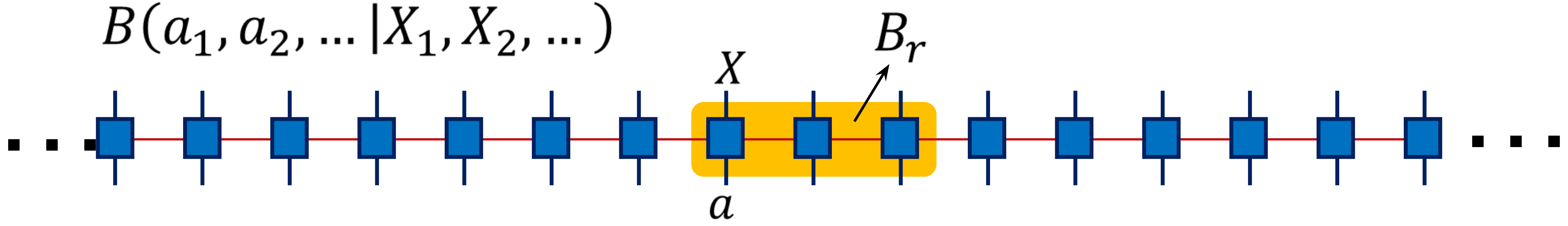

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- Formally, a box B is classical if and only if it can be decomposed as

$$B(a_1, a_2, \dots | X_1, X_2, \dots) = \int d\,\vec{d}\; Q(d_1, d_2, \dots)\, d_1(a_1|X_1) \otimes d_2(a_2|X_2) \otimes \cdots$$

some probability distribution
(local hidden variable model for B)

deterministic box for site 1

(FINE'S THEOREM)

- Again, the interesting case is if the marginal box B_r is classical.

- We were more successful in solving this problem: We completely characterized all classical boxes that admit a TI classical extension.

- Using Fine’s theorem, we can map a box B (conditional probability distribution) to a simple probability distribution Q.

- **Observation 2:** If B admits a TI classical extension then Q must have a TI extension. We have already solved this problem. **(Lemma 1)**

- Thus, simply borrowing the result here we have that the classical boxes that have a classical TI extension form a polytope.

- This is itself surprising. In 2D the corresponding set is not necessarily a polytope!

- Using standard software can find the facets of this polytope, which correspond to linear inequalities (Bell inequalities).

- **Example inequalities** for 3 parties, 2 inputs and 2 outputs with restricted access. Parties can only access the distribution for sites 1,2 and/or 1,3 (we wanted to restrict to 2-site correlators).

$$I_T \equiv -2E_0 - 4E_1 - 2E_{00}^{1,2} + 2E_{01}^{1,2} + 2E_{10}^{1,2} + 2E_{11}^{1,2} + E_{00}^{1,3} + 2E_{11}^{1,3} \geq -4$$

$$I_G \equiv -2E_0 - 6E_1 - 3E_{00}^{1,2} + 2E_{01}^{1,2} + 3E_{10}^{1,2} + 2E_{11}^{1,2} + 2E_{00}^{1,3} + E_{10}^{1,3} + E_{11}^{1,3} \geq -6$$

where $E_x \equiv \langle A_x^1 \rangle = \sum_{a=0,1} P_1(a|x)(-1)^a$ and

$$E_{x,y}^{i,j} \equiv \langle A_x^i A_y^j \rangle = \sum_{a,b=0,1} P_{1,j}(a,b|x,y)(-1)^a(-1)^b$$

- Here A_x^i denotes the observable corresponding to measuring property x at site i and assigning it the numerical value $(-1)^a$.

- Next, we wanted to find out if we can violate these inequalities by TI quantum states.

- Considered a quantum lattice made of sites of dimension 4 (these inequalities cannot be violated by qubits)

- Estimating the quantum value of an inequality given by

$$I \equiv \sum_{x,y=0,1} 0.5\, C_x E_x + C_{xy}^{AB} E_{xy}^{1,2} + C_{xy}^{AC} E_{xy}^{1,3}$$

- Then identified observables A_0, A_1 at each site with the operators $A_0, \equiv M(0,0) \quad A_1 \equiv M(\theta, \phi)$ where

$$M(\theta,\phi) \equiv \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ \sin(\theta) & -\cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\phi) & \sin(\phi) \\ 0 & 0 & \sin(\phi) & -\cos(\phi) \end{pmatrix}$$

- Fixing θ, ϕ can map the original Bell inequality:

$$I \equiv \sum_{x,y=0,1} 0.5\, C_x E_x + C_{xy}^{AB} E_{xy}^{1,2} + C_{xy}^{AC} E_{xy}^{1,3}$$

- To a 3-body Hamiltonian

$$H \equiv \sum_{x,y=0,1} 0.5\, C_x A_x^i + C_{xy}^{AB} A_x^i \otimes A_y^{i+1} + C_{xy}^{AC} A_x^i \otimes A_y^{i+2}$$

- The minimum quantum value of the Bell inequality corresponds to the ground state energy per site of H. Can use MPS algorithms! (DMRG, TEBD)

- **Examples of violations:**

$$\theta = 0.077, \phi = 1.874, I_T = -4.1847$$
$$\theta = 6.236, \phi = 4.175, I_G = -6.1789$$

Found many inequalities

No.	$\mathcal{L}$	C_0	C_1	C_{00}^{AB}	C_{01}^{AB}	C_{10}^{AB}	C_{11}^{AB}	C_{00}^{AC}	C_{01}^{AC}	C_{10}^{AC}	C_{11}^{AC}
1	-3	-2	-2	2	2	-1	1	0	1	0	0
2	-4	-2	-4	-2	2	2	2	1	0	0	1
3	-5	-3	-3	2	2	2	-3	1	0	-1	2
4	-6	-4	-6	-3	2	3	2	2	0	1	1
5	-11	-4	-12	-4	6	6	6	1	-1	-1	4
6	-7	-5	-5	2	3	2	-4	1	1	-1	3
7	-8	-6	-8	-4	3	3	2	3	1	1	1
8	-5	-2	2	2	-2	-2	-4	1	1	1	2
9	-3	-3	1	1	1	1	-1	1	0	-1	1
10	-6	-4	2	2	2	2	-4	1	-1	-1	3
11	-6	-6	0	2	3	3	-2	3	-1	-1	1

Inequalities

No.	$\mathcal{L}$	$\mathcal{Q}$	θ	ϕ	$\inf \mathcal{Q}$	$\inf \mathcal{NS}$	Genuine
1	-3	-3.111	6.236	1.501	-3.1907	-3.5	N
2	-4	-4.184	0.077	1.874	-4.38643	-4.8	N
3	-5	-5.098	2.17	6.275	-5.3502	-5.8	N
4	-6	-6.179	6.236	4.175	-6.35706	-6.8	Y
5	-11	-11.104	5.996	4.691	-11.7124	-12.87	N
6	-7	-7.073	4.093	0.29	-7.31685	-7.8	Y
7	-8	-8.191	4.359	6.197	-8.52433	-9.06	Y
8	-5	-5.039	3.169	5.226	-5.32177	-5.8	N
9	-3	-3.04	3.843	1.193	-3.22662	-3.5	Y
10	-6	-6.109	0.817	2.421	-6.37417	-7	Y
11	-6	-6.081	3.787	6.067	-6.36487	-7	Y

Violations detected

Summary

- Described how one can detect global properties of an infinite system from partial information (of a small number of neighbors).

Entanglement detection

- Provided a characterization of reduced density matrices of TI multi-separable states **(Observation 1)** and used it to derive **entanglement witnesses** for qubit chains.
- Constructed examples of TI states with separable 2-site reduced density matrix which admit only entangled TI extensions.

Bell non-locality detection

- Fully characterized the set of classical boxes admitting TI classical extensions.
- Identified a tripartite classical box which only admits non-classical TI extensions.
- Ground states of some few-body Hamiltonians are non-local! Can prepare these in the lab?

- **Subtlety:** the inequalities I_T and I_G indicate two types of violations
- Consider the set of all tripartite classical box B_{123} such that $B_{12} = B_{23}$ (not necessarily equipped with TI classical extensions)
- If we minimize the inequality I_T over this set we find the minimum value to be exactly -4 (i.e. the lower bound of I_T)
- So a violation of this inequality simply indicates that the violating box B_{123} is non-classical, without reference to TI extensions.
- But I_G is different. In the violating total box (obtained using MPS) we found that the marginal box B_{123} is non-classical. So we added some TI local noise to it such that the marginal box B_{123} becomes classical. But this B_{123} still violated inequality, indicating ‘genuine TI non-classicality’ (i.e. cannot be extended to TI classical box).

Thanks!