@ European Tensor Network School, San Sebastian (2019)

An Introduction to Symmetric Tensor Networks

Sukhbinder Singh

Albert Einstein Institute, Potsdam

www.heptnseminar.org

(online seminar series on tensor networks)

GOALS:

(i) What does the global symmetry imply for the individual tensors?

(ii) How to protect & exploit the symmetry in numerical simulations?

(There are also conceptual benefits - classification of phases, holography etc.)

We will focus on symmetries described by groups

Restrict to compact and completely reducible groups

This includes most of the favourite groups in condensed matter

Finite groups: $Z_n, \; Z_2 \times Z_2$, the Pauli group $\{\sigma_x, \sigma_y, \sigma_z, I\}$

Continuous groups: U(N), SU(N), SO(N)

(Abelian and non-Abelian groups)

This talk: will use SU(2) as an example

It illustrates most of the concepts required to implement symmetries

Irreps can be labelled systematically:

$$a, b, c, \dots$$

Each of these label a whole set of matrices.

Denote the Irrep basis: $|a, m_a\rangle$

More general representations of the group can be obtained by combining irreps in two different ways

1) Direct sum (reducible representations): $a \oplus b$, $\quad$ $\mathrm{b} \oplus \mathrm{c} \oplus \mathrm{d} \oplus \mathrm{b}$, $\quad$ $\mathrm{a} \oplus \mathrm{a}$

basis: $|a, m_a, t_a\rangle$

2) Tensor product: $a \otimes b$

A tensor product of irreps is equivalent to a direct sum representation

$$a \otimes b = x \oplus y \oplus z \oplus \ \ldots$$

Fusion rules: Which irreps appear in the tensor product of any two irreps is completely determined by the group properties.

If irrep x appears in the tensor product of irreps a and b, we say that the three irreps (x,a,b) are compatible.

There exists a change of basis between the two representations:

$$|x, m_x\rangle = \sum_{m_a, m_b} C^{x m_x}_{a m_a, b m_b} \; |a, m_a\rangle \otimes |b, m_b\rangle$$

"Coupled/Fusion basis"

"Clebsch-Gordan coefficients"

The CG coefficients are completely determined by the group

(These coefficient are all equal to 1 for Abelian groups)

To summarize,

In the end we will need only the following data about the symmetry G

1) List of irreps (their labels/quantum numbers)

2) Fusion rules

3) Clebsch-Gordan coefficients

Representation Data of the symmetry

Outline

PART 1 (Conceptual)

(1) Setup
(2) Basic building blocks: symmetric tensors
(3) Symmetric tensors are sparse!   Example: SU(2) symmetry

PART 2 (Practical)

(1) Application I: SU(2) TEBD algorithm
(2) Application II: Using the symmetric MPS to detect gapped phases
(without using local or string order parameters)

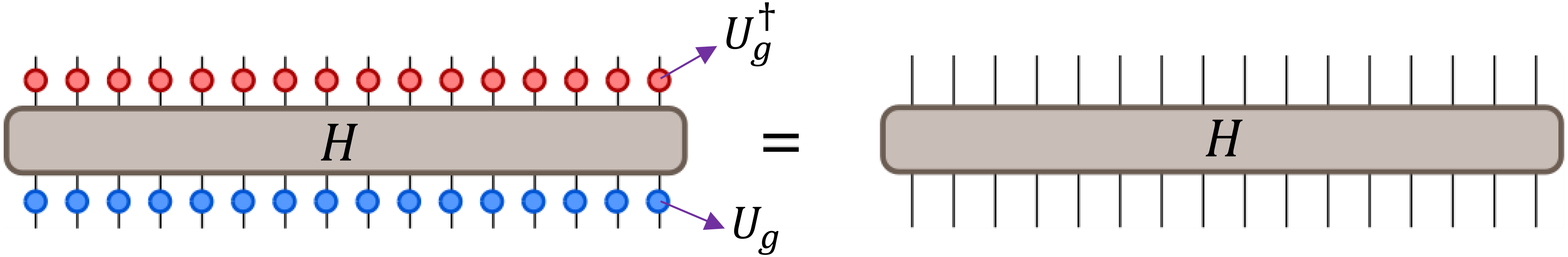
GENERALIZATIONS

Setup

Lattice

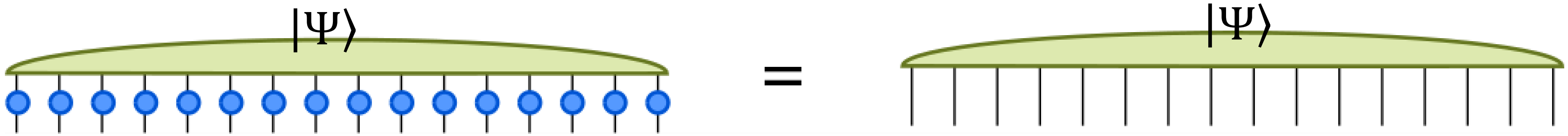
Global action of a symmetry $U_g \quad U_g \quad U_g \quad U_g \quad U_g \quad U_g \quad U_g \quad U_g$

Local Hamiltonian $[H, U_g] = 0$ for all g

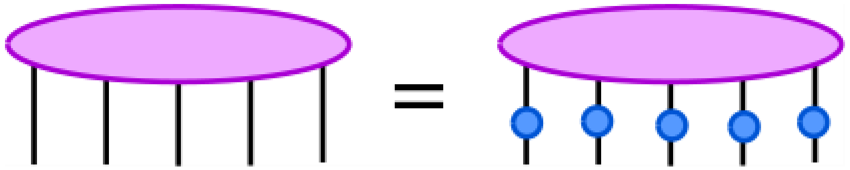


$|\Psi\rangle$

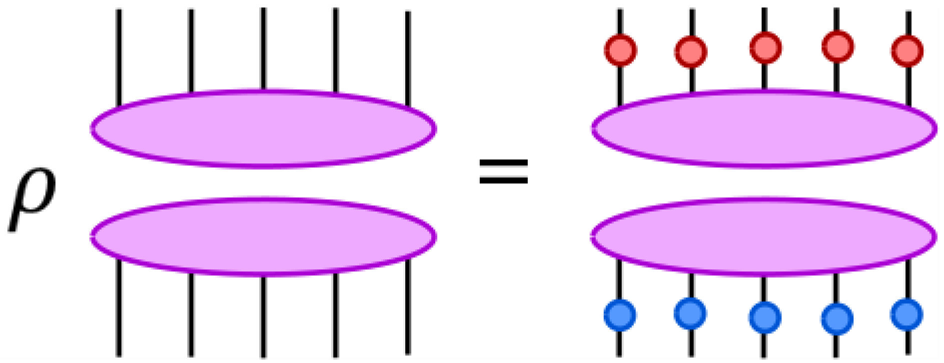
Ground state $|\Psi\rangle = U_g|\Psi\rangle$ for all g



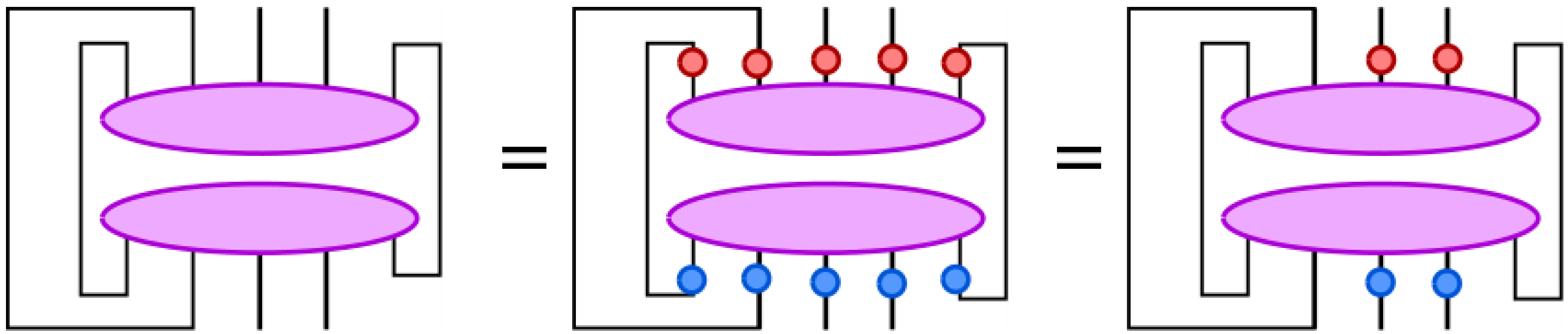
Density matrix of any subset of sites also commutes with the symmetry



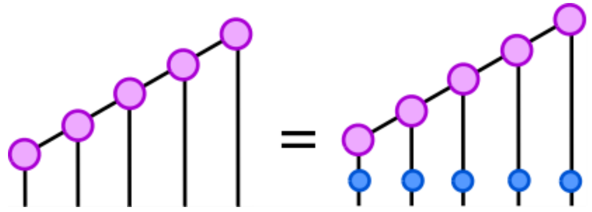
Density matrix of any subset of sites also commutes with the symmetry



Density matrix of any subset of sites also commutes with the symmetry

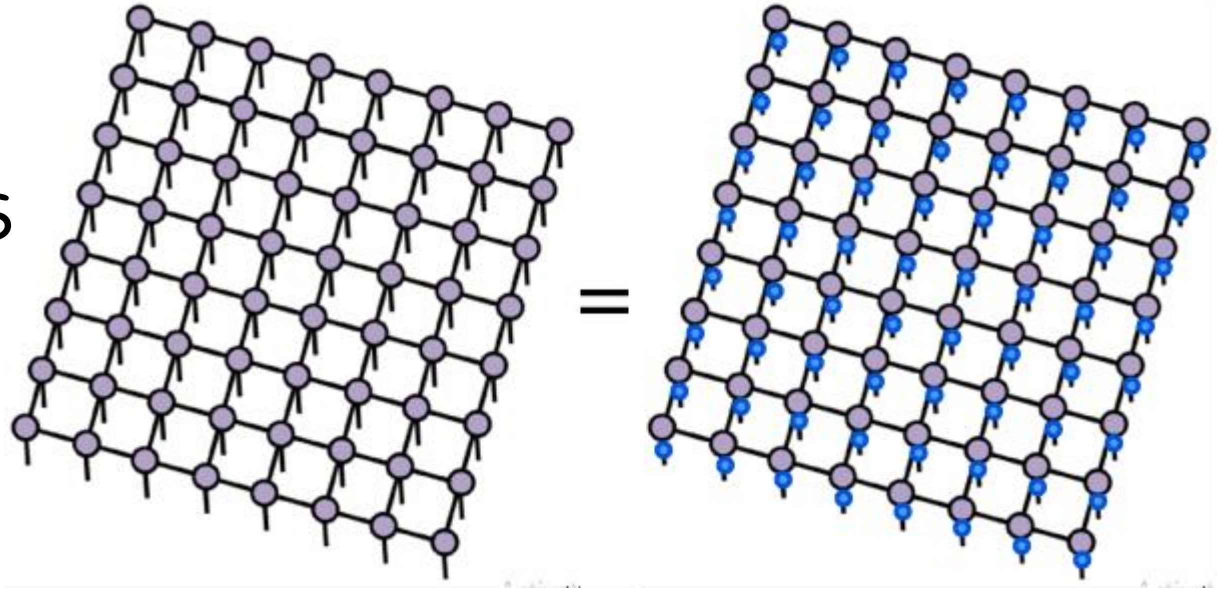


$|\Psi\rangle$ is described by a tensor network

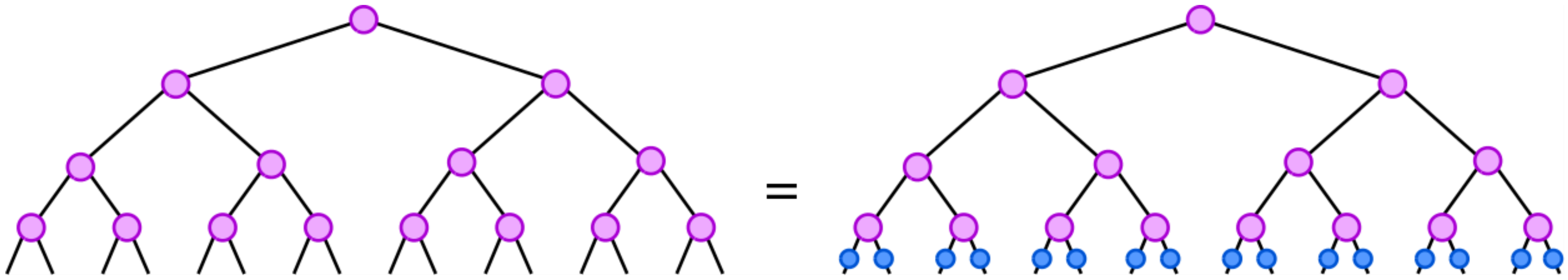


MPS

PEPS

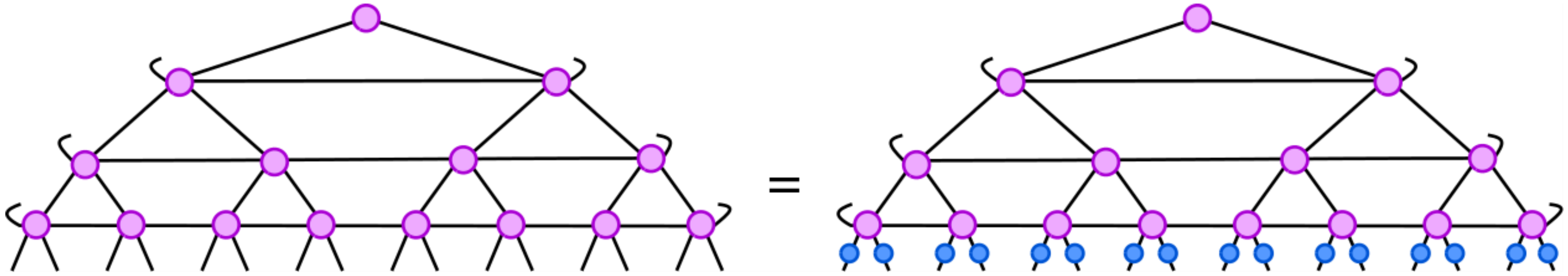


TTN



$\forall g$

MERA



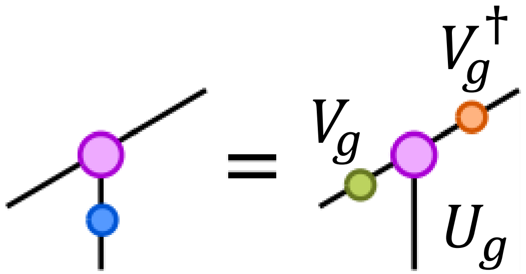
GOALS:

(i) What does the global symmetry imply for the individual tensors?

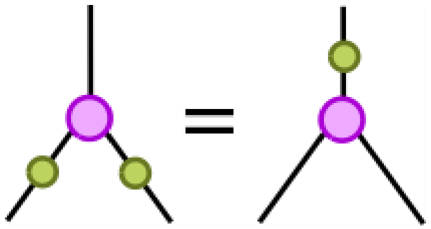
(ii) How to protect & exploit the symmetry in numerical simulations?

Symmetric tensors as building blocks

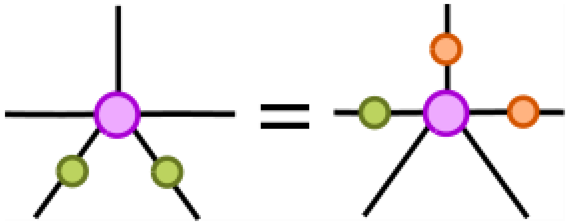
Suppose that the individual tensors are symmetric



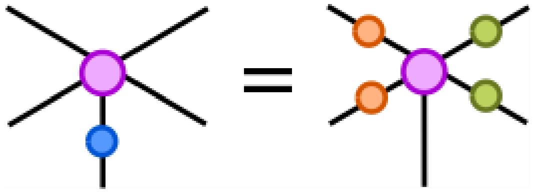
(MPS)



(Tree)

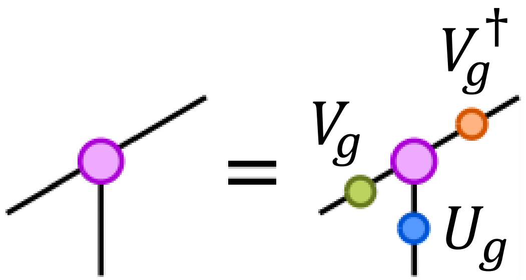


(MERA)

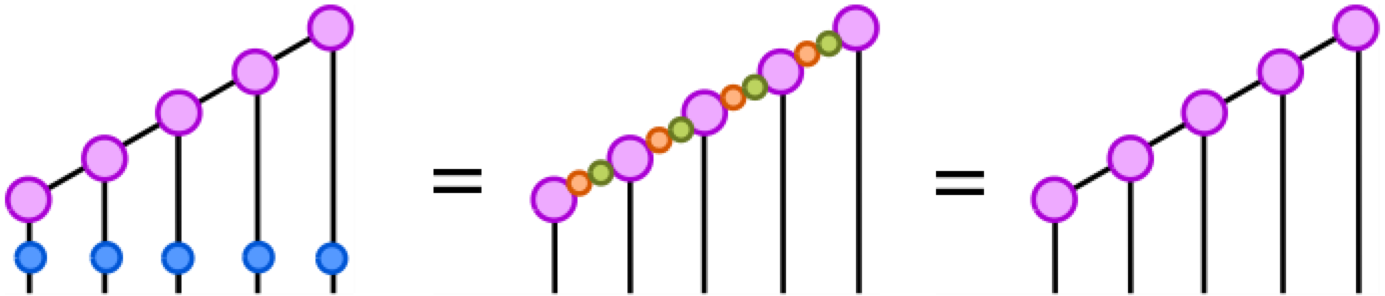


(PEPS)

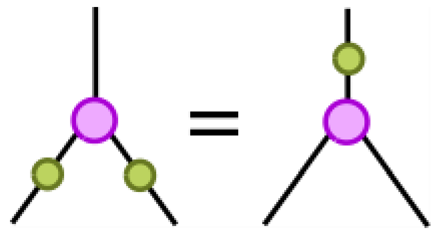
A tensor network made of symmetric tensors has a global symmetry



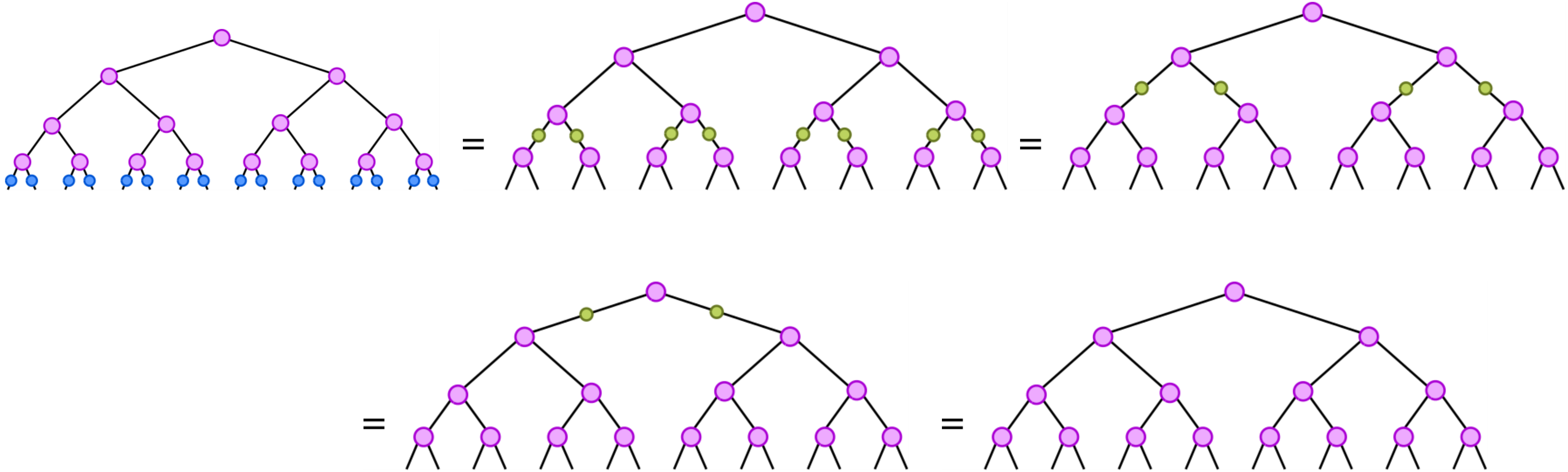
(MPS)



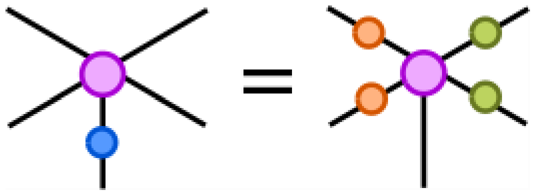
A tensor network made of symmetric tensors has a global symmetry



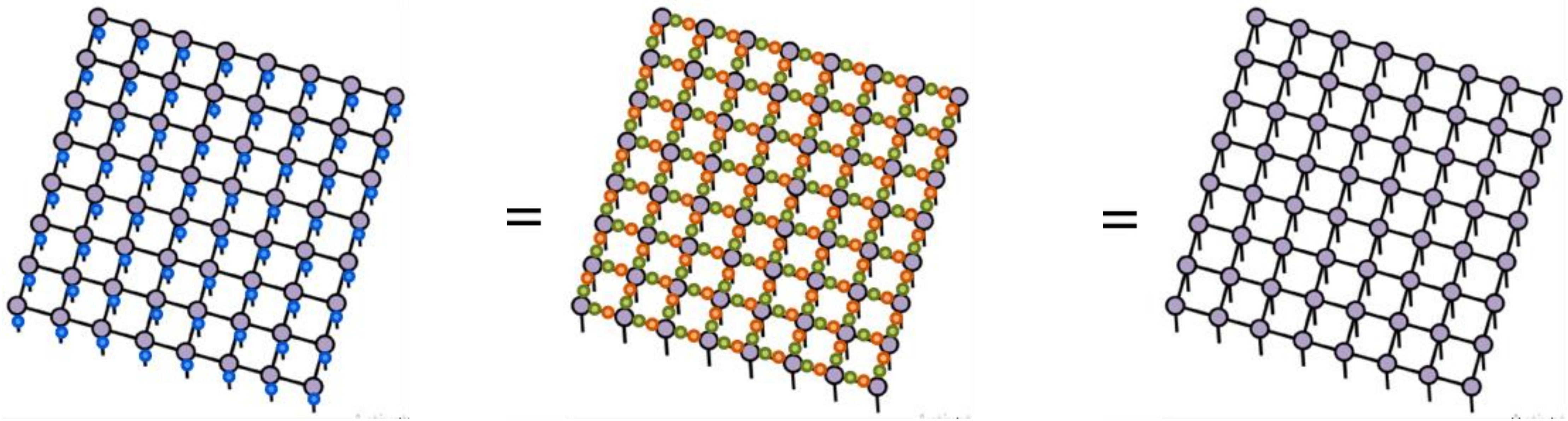
(Tree)



A tensor network made of symmetric tensors has a global symmetry



(PEPS)

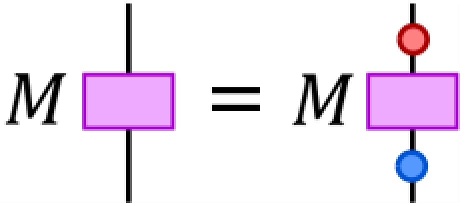


So

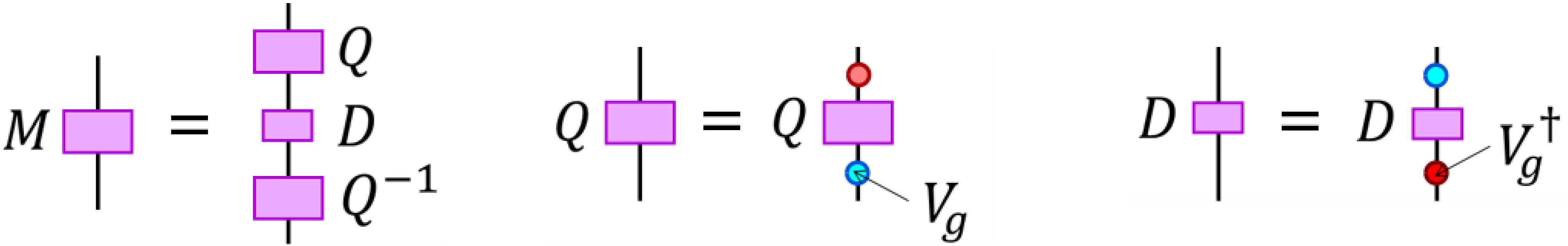
symmetric tensors $\Rightarrow$ global symmetry

But is the reverse true?

A useful fact

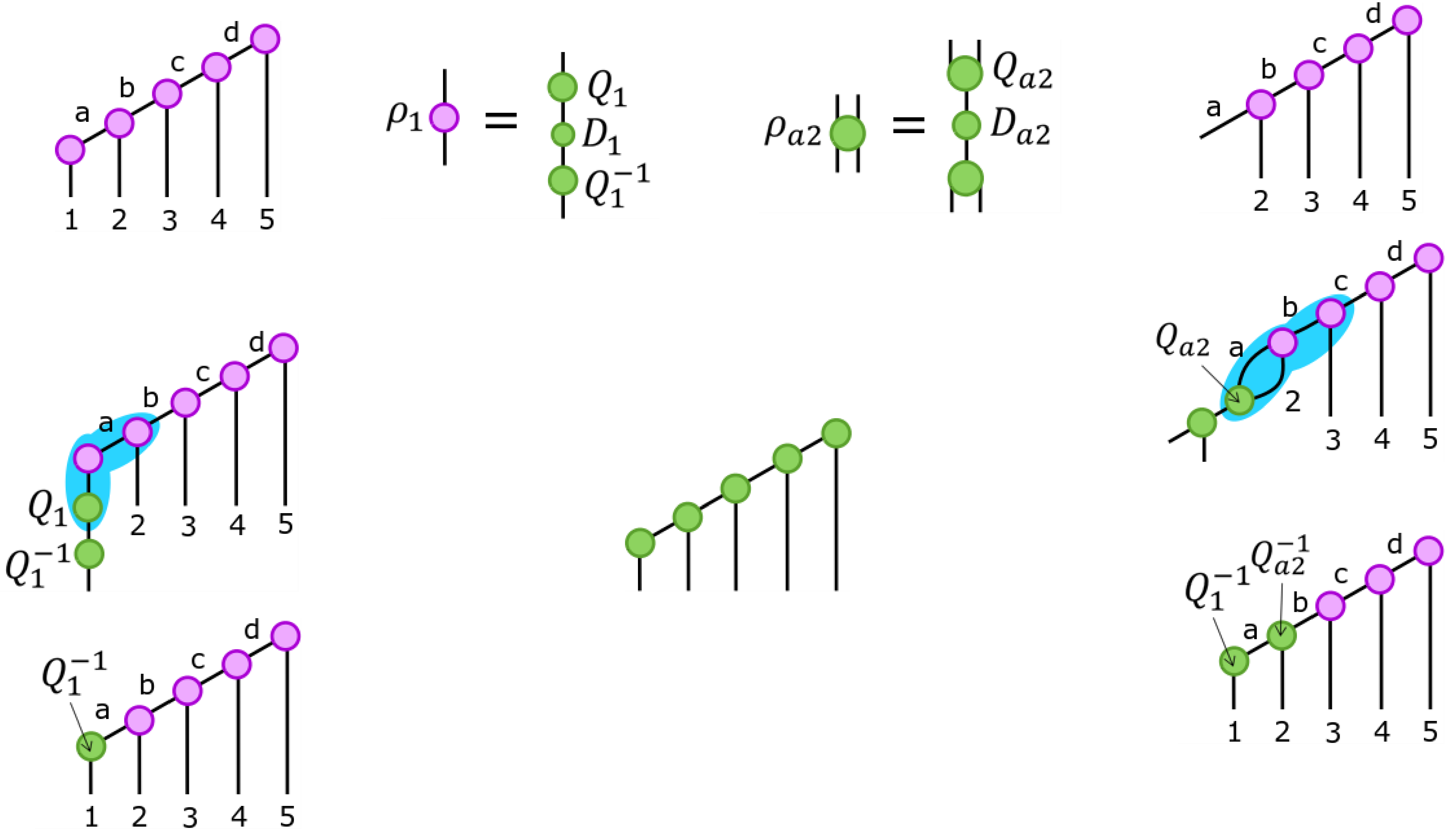


A symmetric matrix can always be eigenvalue decomposed as a product of symmetric matrices



V_g : Some representation of the group

Matrix Product States (OBC)

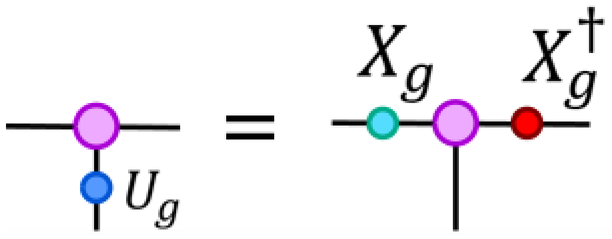


Matrix Product States (PBC & translation-invariance)

Previous proof does not work because of loops

But can use the fundamental theorem of MPS (more general proof. Covers the OBC case too)

Assume MPS is in a canonical form (ref. David's talk)



X_g : Some representation of the group

Can be projective!

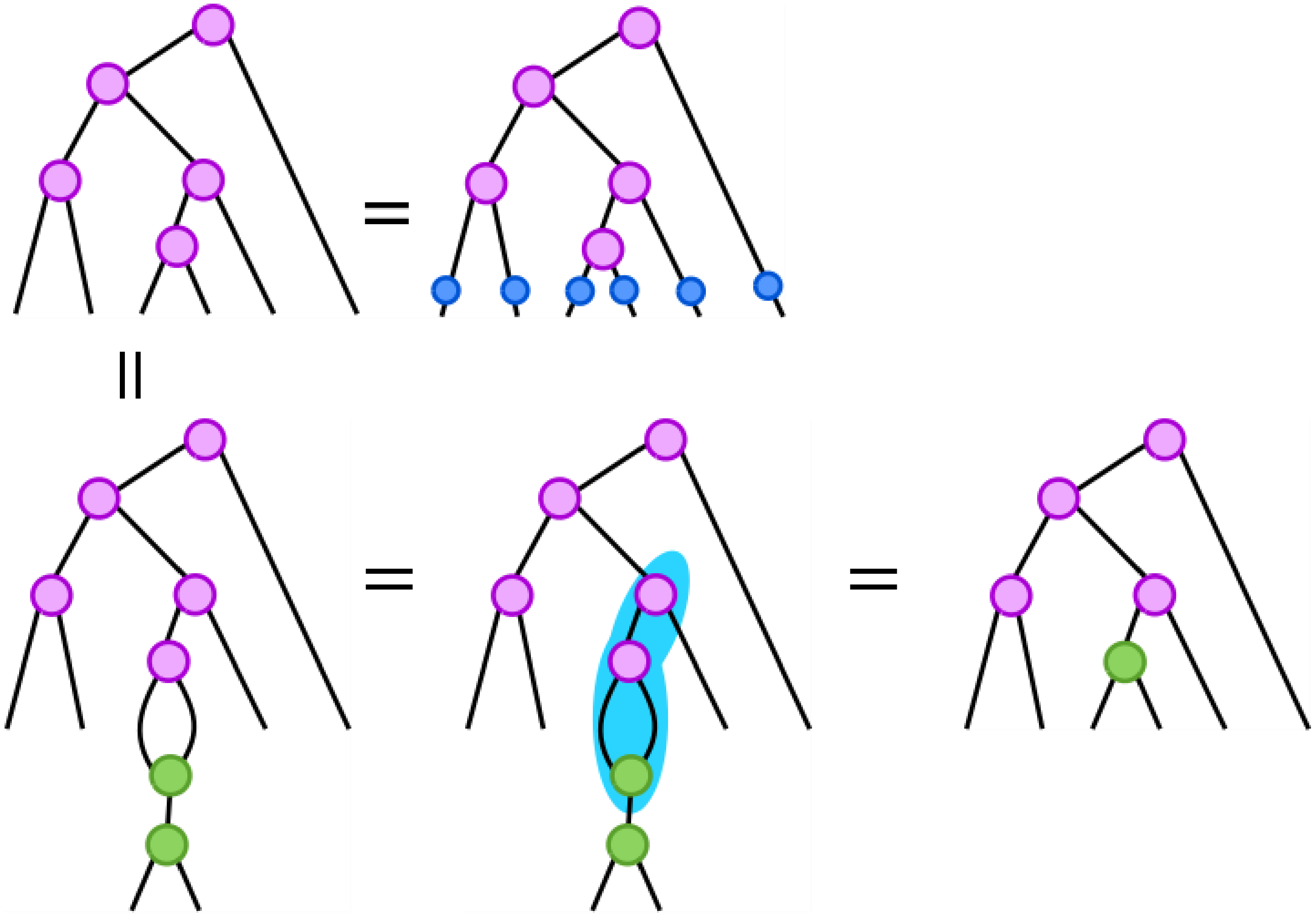
X_g is unitary, if MPS is in the canonical form

Thus, for the MPS (with OBC or PBC)

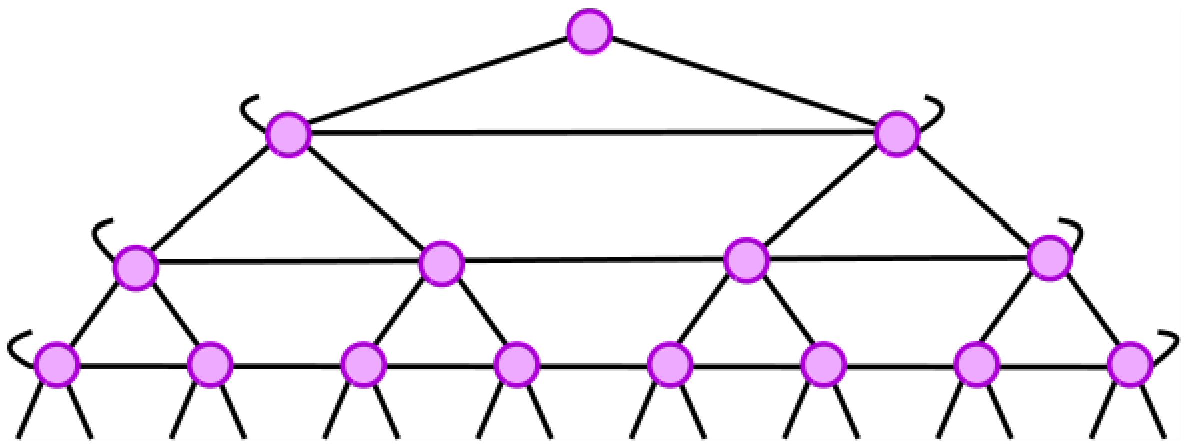
global symmetry $\Rightarrow$ symmetric tensors

Tree tensor network states

The same proof technique from MPS with OBC works
(MPS with OBC is simply a special case of trees)

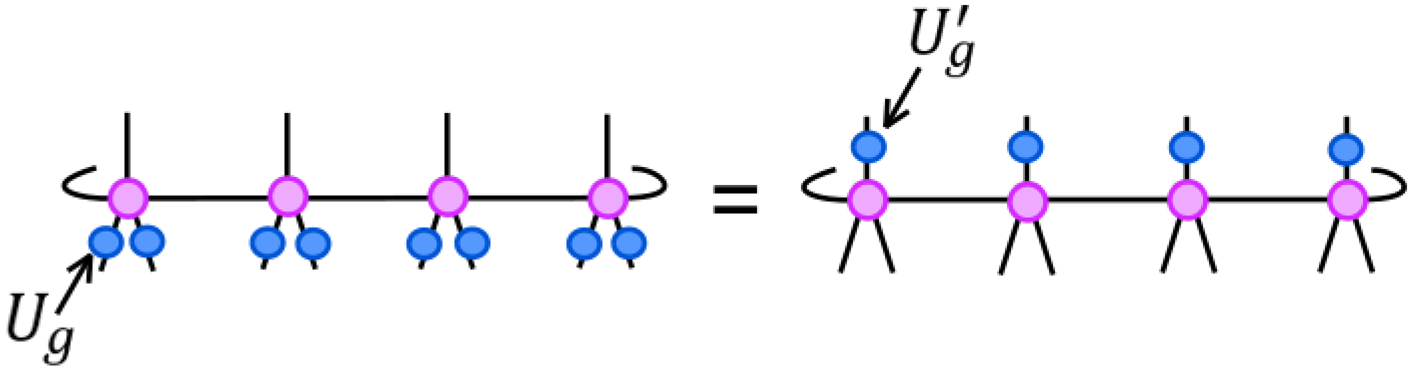


MERA

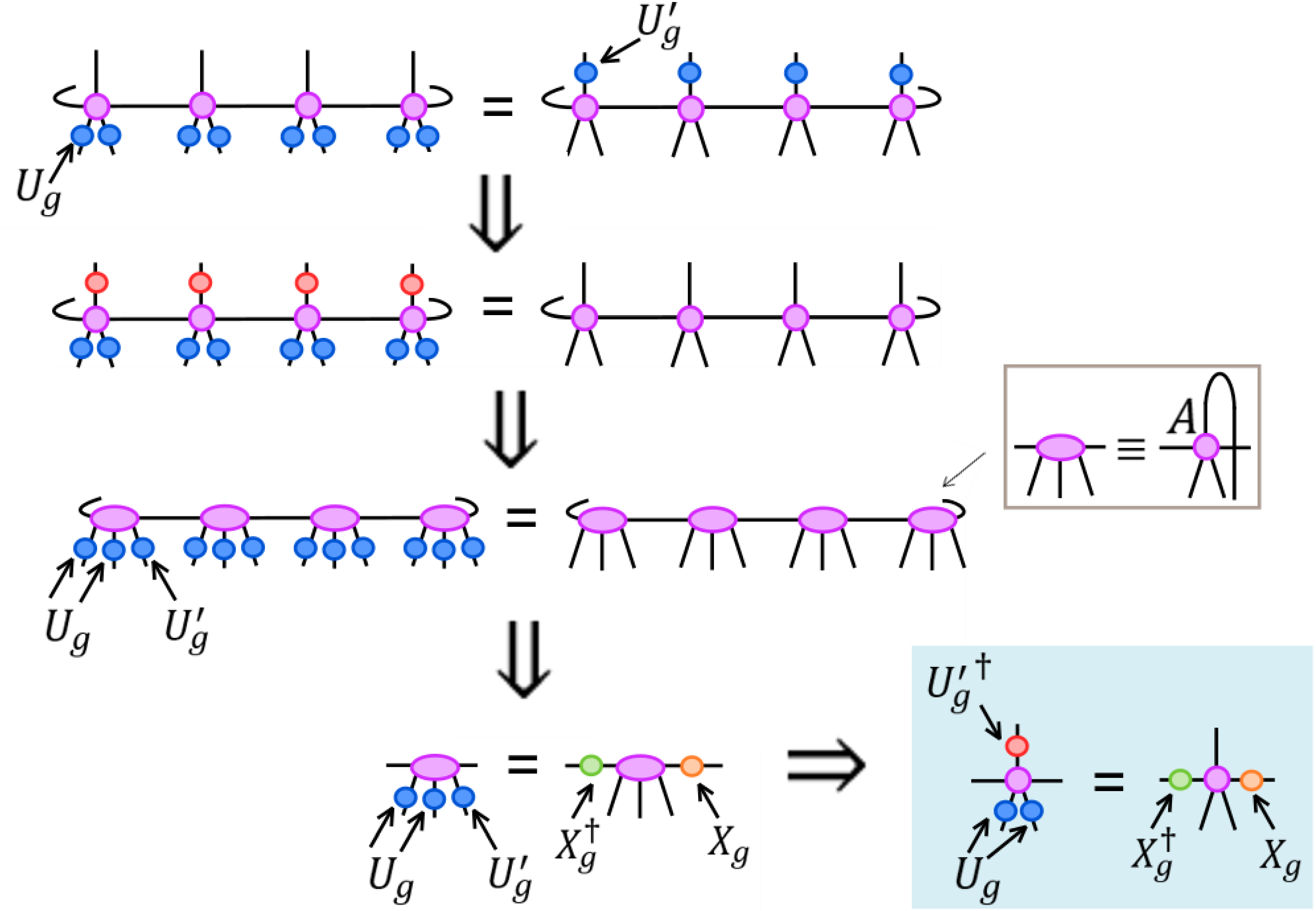


Can't say much very generally.

But we can prove symmetric tensors if we assume the following:



(Heuristic motivation: anomaly matching condition of RG in QFT)



U'_g
U_g
A
U_g
U'_g
U_g
U'_g
$X_g^\dagger$
X_g
${U'_g}^\dagger$
U_g
$X_g^\dagger$
X_g

Symmetric tensors are sparse

Representation data of SU(2)

Irrep labels: $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (possible values of total spin)

Irrep basis: $|j, m_j\rangle$ (diagonalizes generator Z of SU(2))

the spin projection along z-axis (takes values: $-j, -j+1, \ldots, j$)

Dimension of irrep j: 2j+1 **e.g.** dim(0) = 1, dim(1/2) = 2, dim(1) = 3

Example: For total spin $\frac{1}{2}$ the basis consists of 2 states:

$$|j = \frac{1}{2}, m = -\frac{1}{2}\rangle \qquad |j = \frac{1}{2}, m = \frac{1}{2}\rangle$$

Example: For spin 1 the basis consists of 3 states:

$$|j = 1, m = -1\rangle \qquad |j = 1, m = 0\rangle \qquad |j = 1, m = -1\rangle$$

Representation data of SU(2)

Direct sum of irreps.

Examples: $(i)\ 0 \oplus 1,\quad (ii)\ \ 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1,\quad (iii)\ \ 1 \oplus \frac{1}{2}$

Basis. $\quad (i)\ \ |j=0, m_0=0\rangle, |j=1, m_1=-1\rangle, |j=1, m_1=0\rangle|j=1, m_1=-1\rangle$

$(ii)\ \ |j=0, m_0=0, t_0=1\rangle, |j=0, m_0=0, t_0=2\rangle, \ldots$

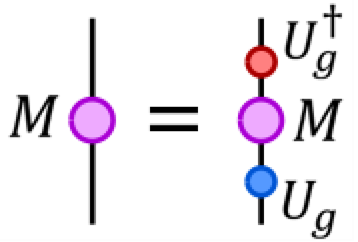
Fusion rules

Examples: $\quad 0 \otimes j = j \otimes 0 = j \qquad \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \qquad \frac{1}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2}$

Generally,

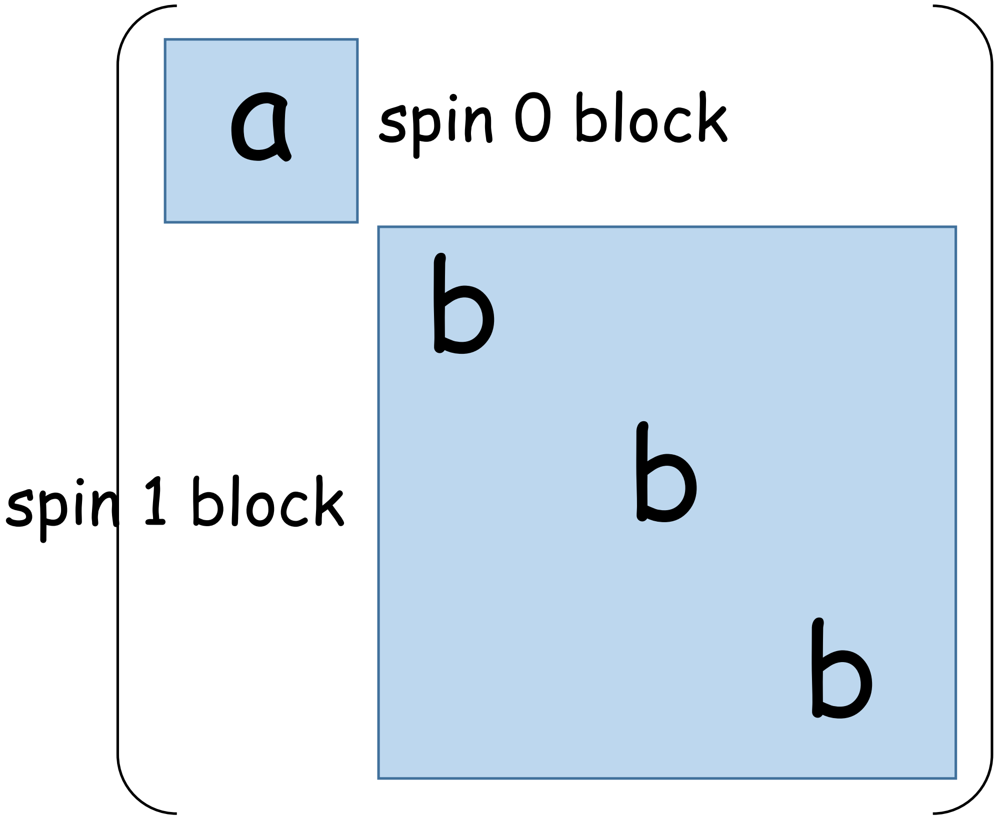
$$a \otimes b = |a-b| \oplus \cdots \oplus a+b$$

SU(2)-symmetric matrix



1) If U_g is an irrep then M is proportional to the Identity (Schur's Lemma).

2) If U_g is a direct sum of irreps



Example: $U_g = 0 \bigoplus 1$

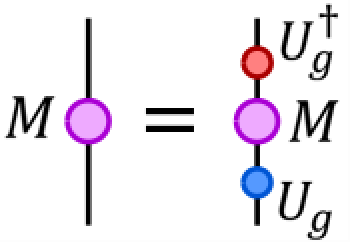
Total dimension of M = 4×4

Number of free parameters = 2

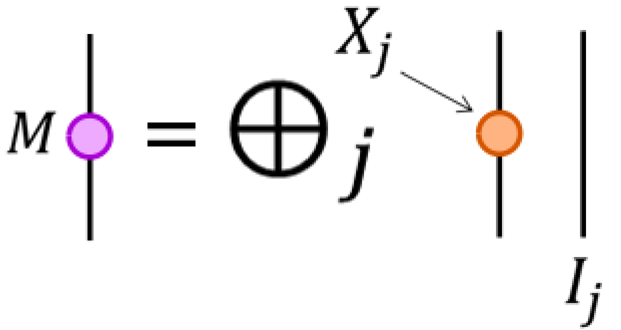
SU(2)-symmetric matrix

"Degeneracy matrix"

3) If U_g is a sum of degenerate irreps



$\Rightarrow \quad M = \oplus_j \ (X_j \otimes I_j)$



Example: $U_g = 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1$

Total dimension of $\mathrm{M} = 11 \times 11$

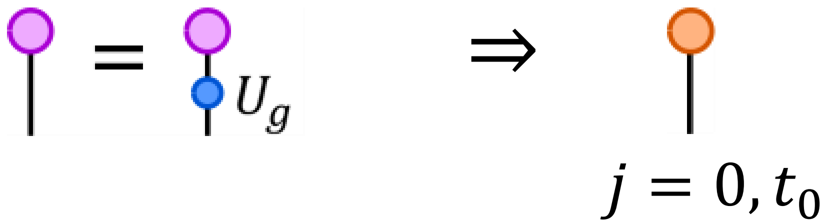
Total dimension of $\mathrm{X}_0 = 2 \times 2$

Total dimension of $\mathrm{X}_1 = 3 \times 3$

Total number of free parameters $4 + 9 = 13$

$$M = \ (X_0 \otimes I_0) \oplus (X_1 \otimes I_1)$$

SU(2)-symmetric vector



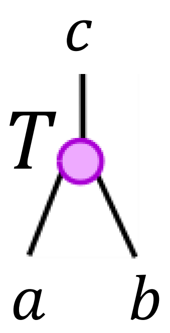
Example: $U_g = 0 \oplus 0 \oplus 1 \oplus 2$

Total dimension of $U_g = 1 + 1 + 3 + 5 = 10$

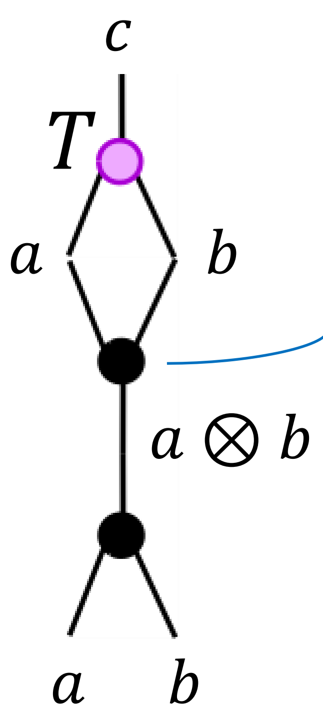
Dimension of spin 0 subspace $= 1 + 1 = 2$

$$\begin{pmatrix} \left.\begin{matrix} \sqrt{\frac{1}{4}} \\ \sqrt{\frac{3}{4}} \end{matrix}\right\} j = 0 \\ \hline \left.\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\} 1 \oplus 2 \end{pmatrix}$$

3-index symmetric tensor

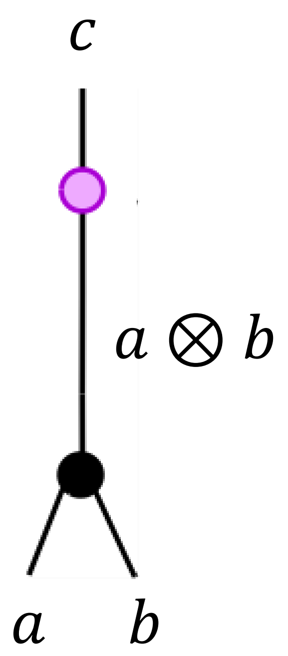


3-index symmetric tensor

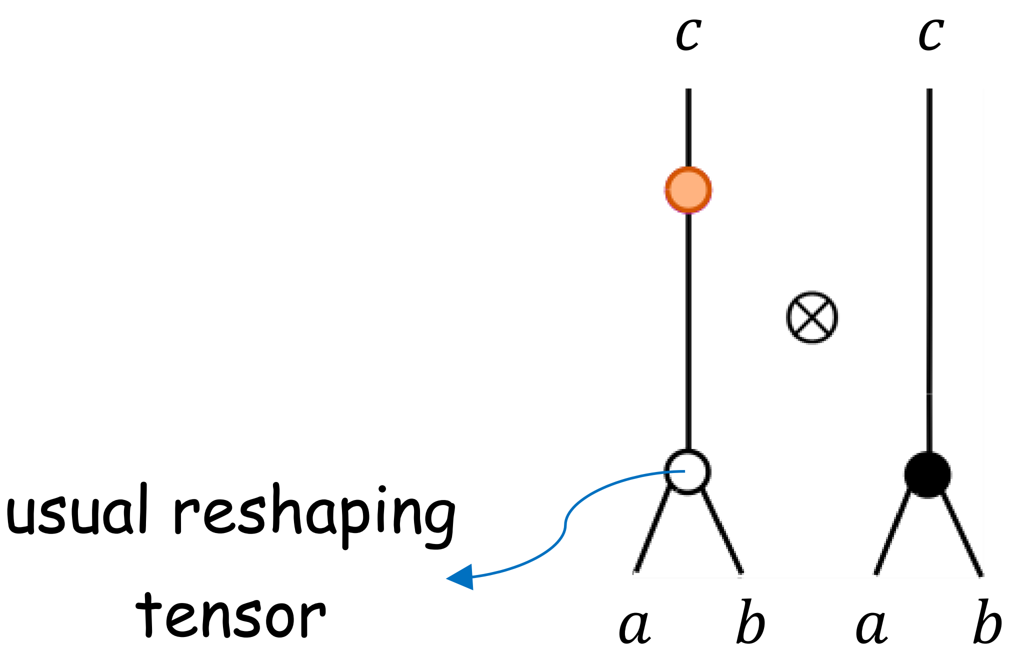


Clebsch-Gordan tensor

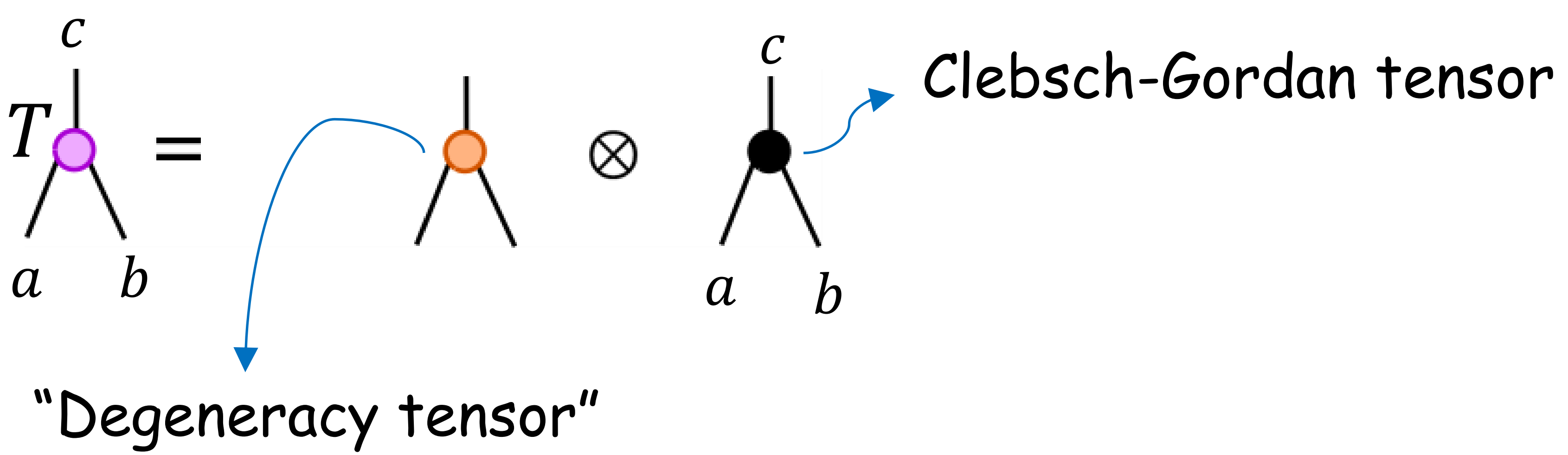
3-index symmetric tensor



3-index symmetric tensor



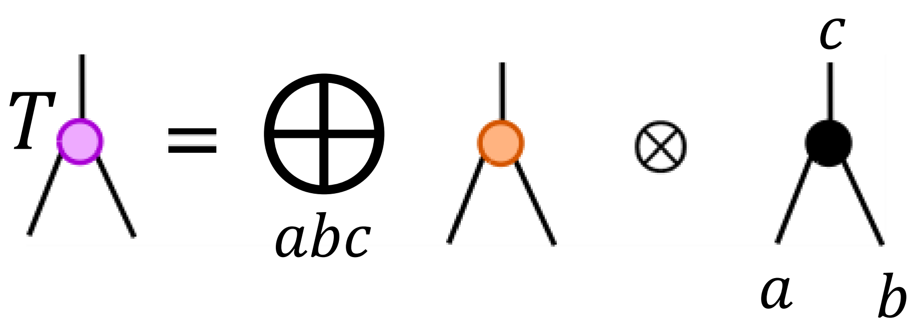
3-index symmetric tensor



T is identically zero if a,b,c are incompatible

That is, if irrep c does not appear in the tensor product of a & b

3-index symmetric tensor



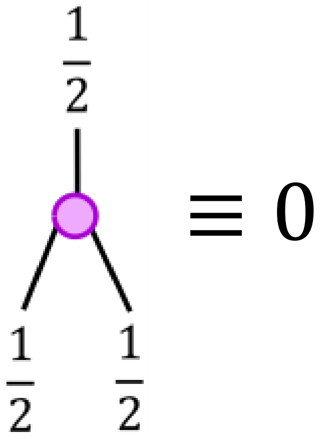
If more than one irrep appears on any index

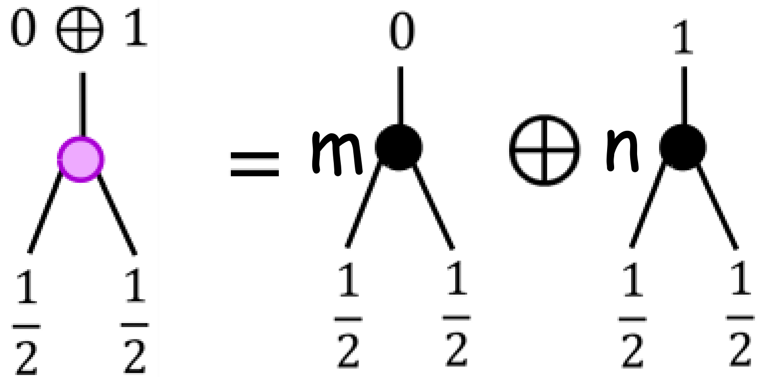
T consists of blocks

Each block decomposes as in the previous slide

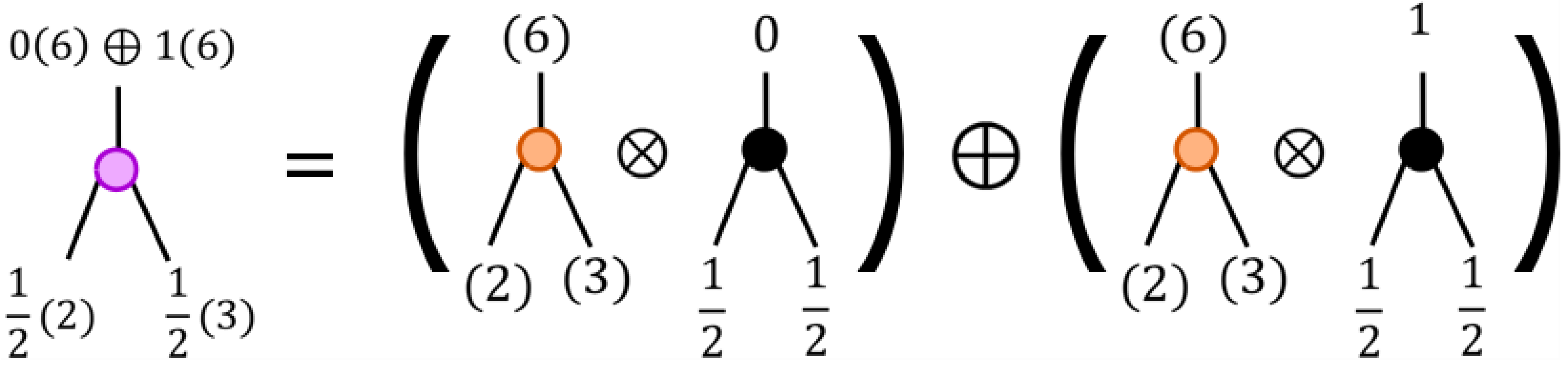
3-index symmetric tensor

Examples:





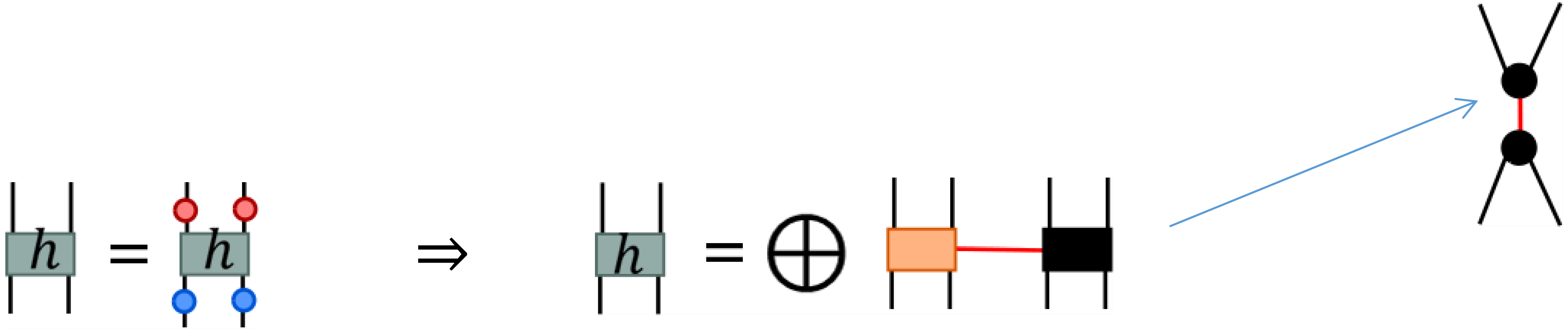
2 free parameters: m,n



4x6x24 = 576

2x3x6 + 2x3x6 = 72

4-index symmetric tensors

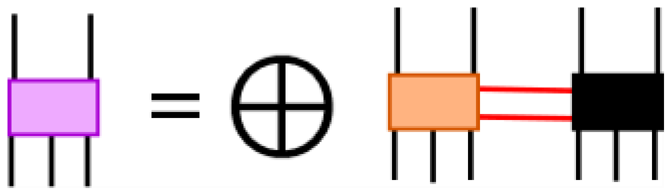


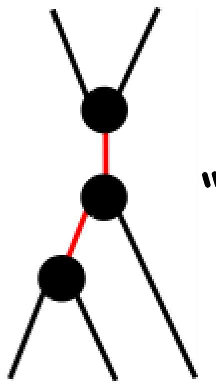
Symmetric tensors with more than 3 indices also decompose into degeneracy tensors and tensors that are determined by the symmetry.

But notice that an additional red index appears.

Red index is special: takes only j values (no m and t)

5-index symmetric tensors





“Fusion tree”

Same story

But two red indices appears (in general, N-index tensor corresponds to N-3 red indices)

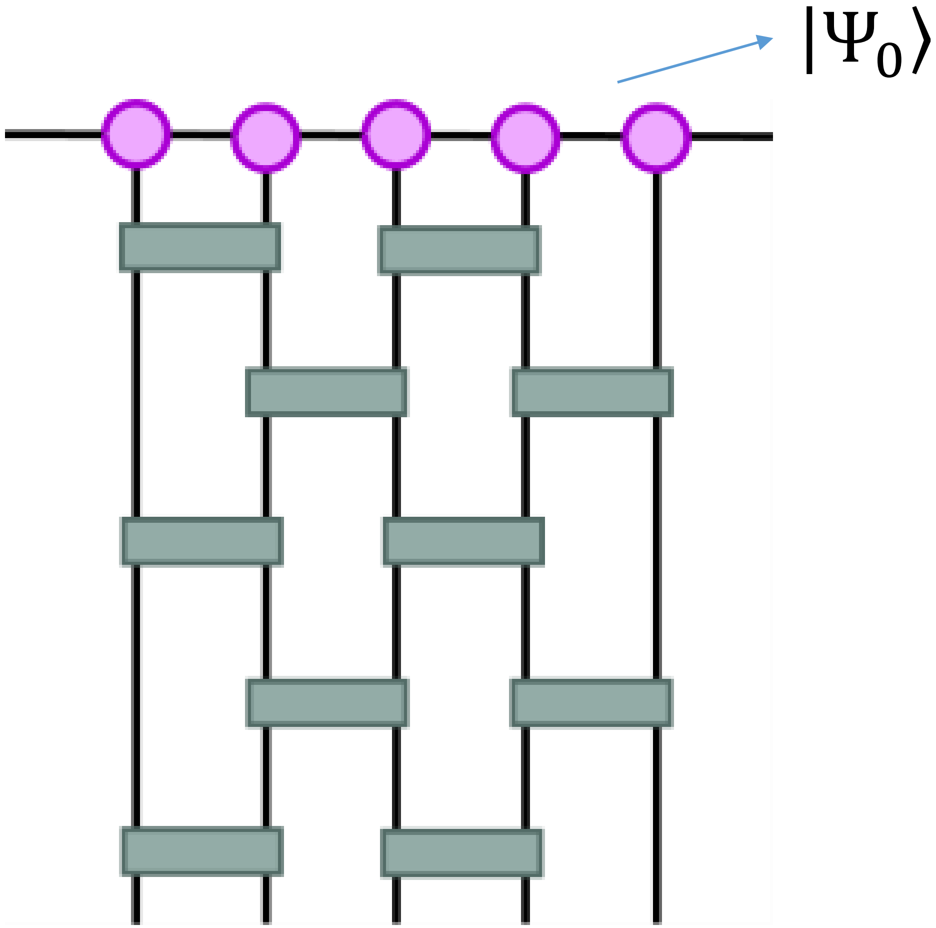
The black tensor decomposes as a tree of Clebsch-Gordan tensors.

And so on for tensors with more number of indices

The SU(2) TEBD algorithm

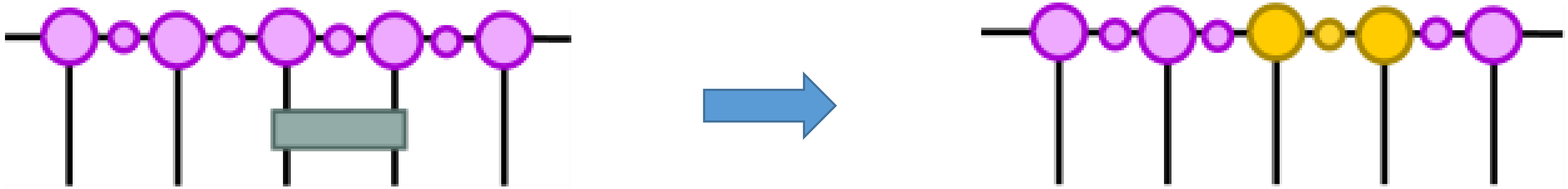
$$|\Psi\rangle = e^{iHt}|\Psi_0\rangle$$

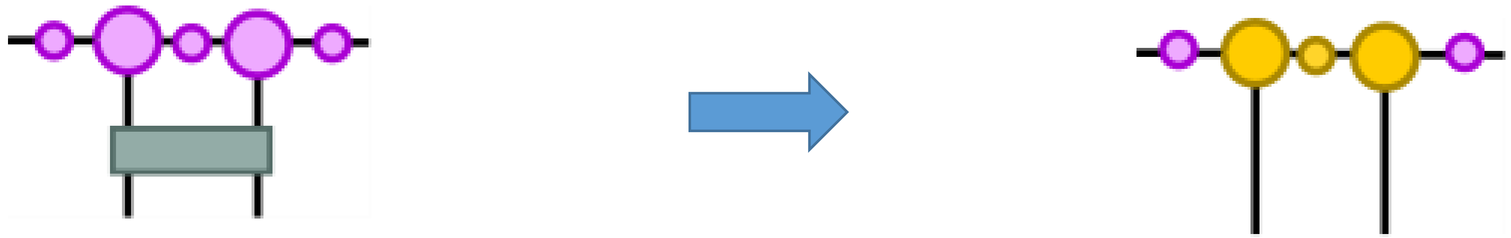
infinite lattice



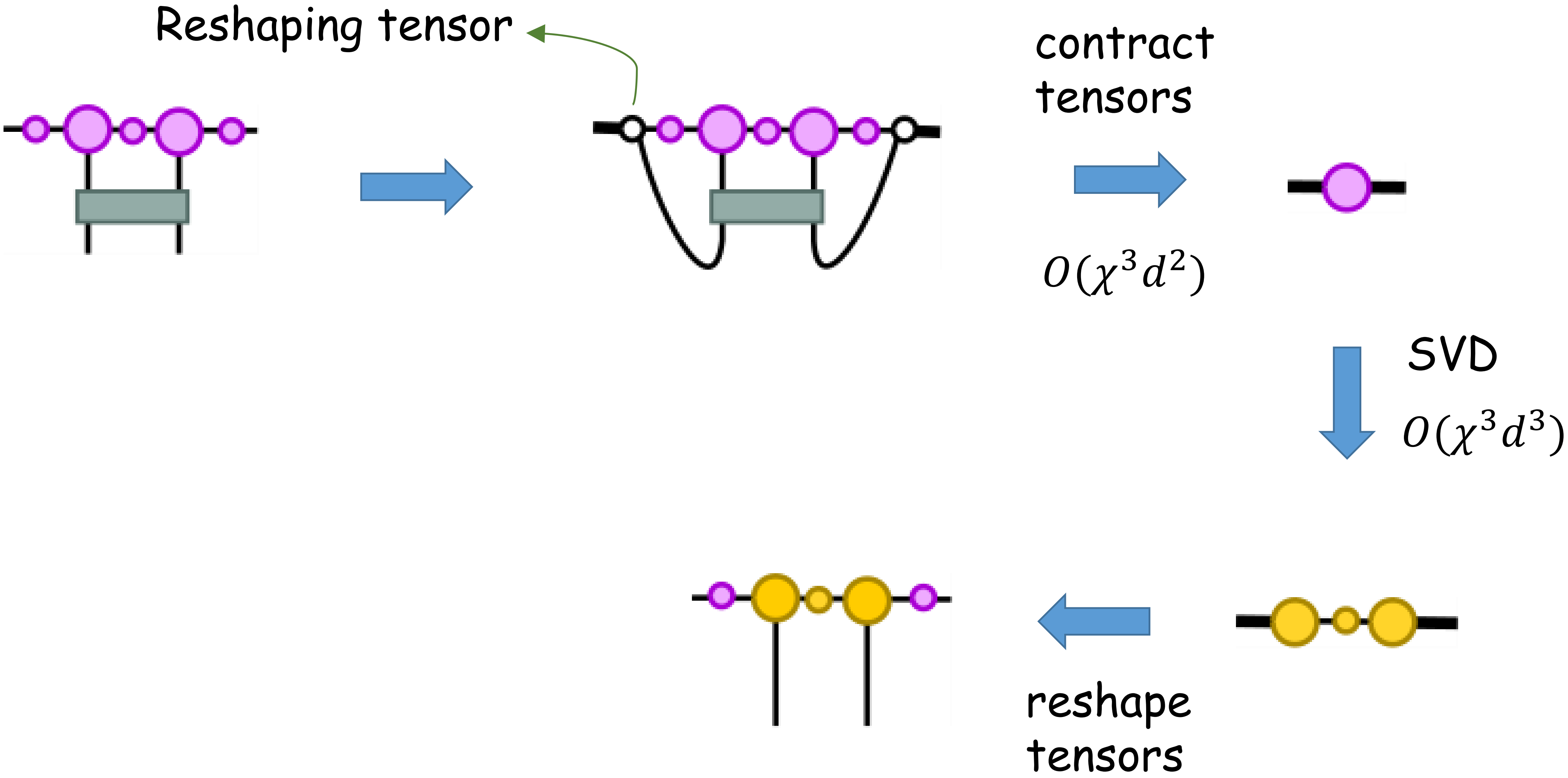
Main update in the TEBD algorithm

(MPS is in the canonical form)



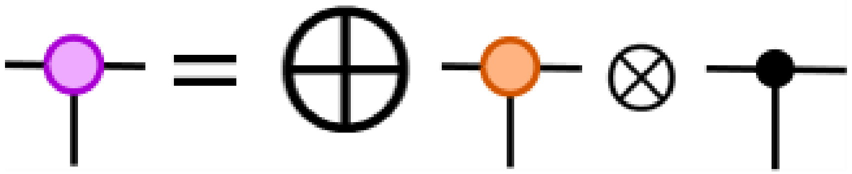


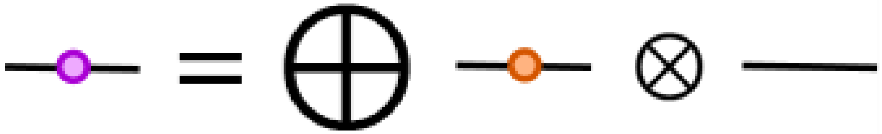
Main update in the TEBD algorithm



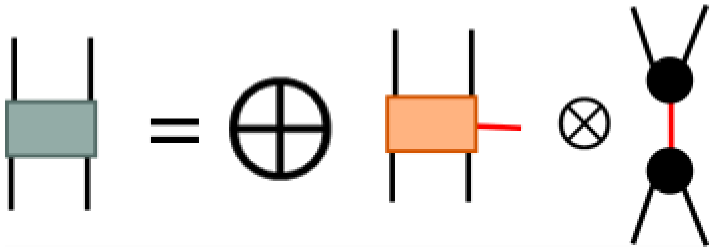
SU(2)-symmetric MPS

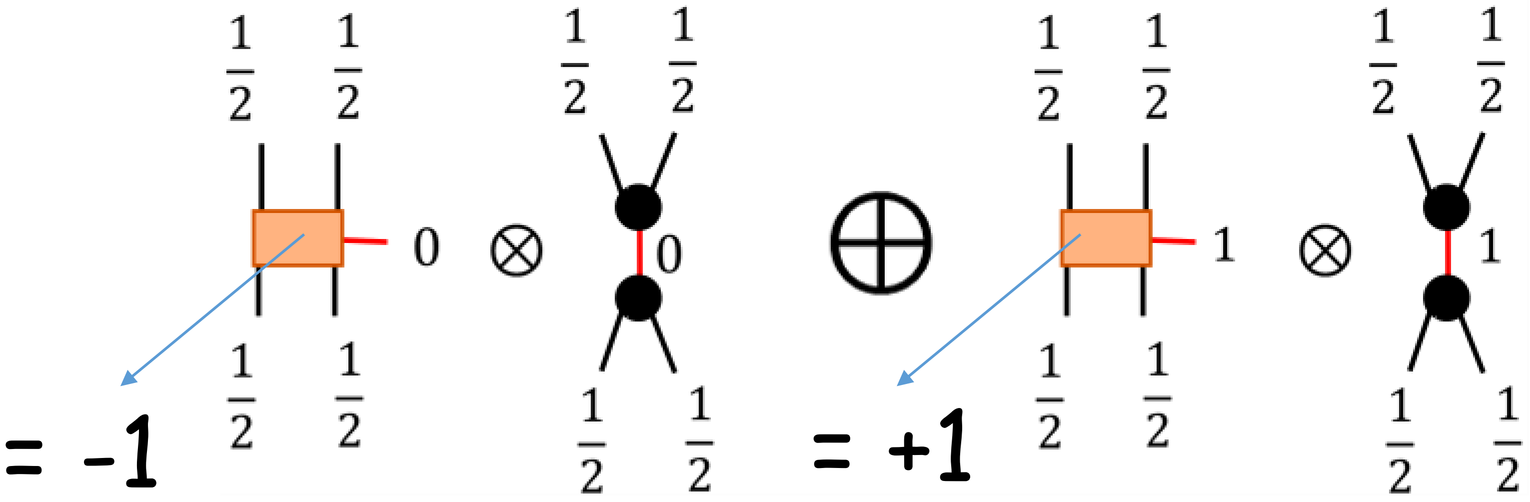
Express all tensors in the symmetric basis



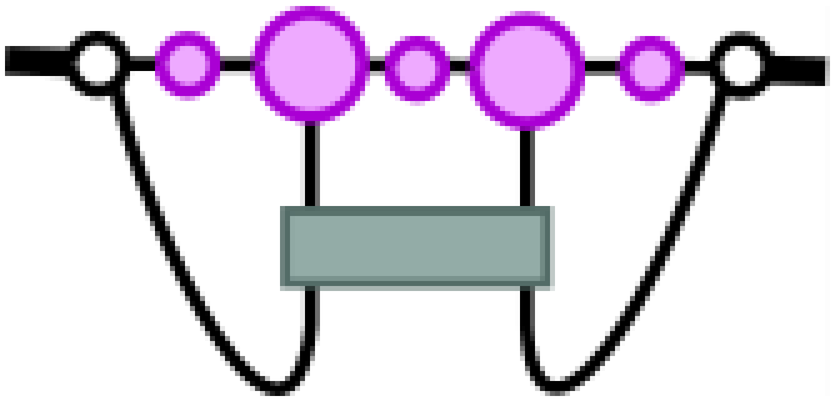


Hamiltonian. Example: Heisenberg model

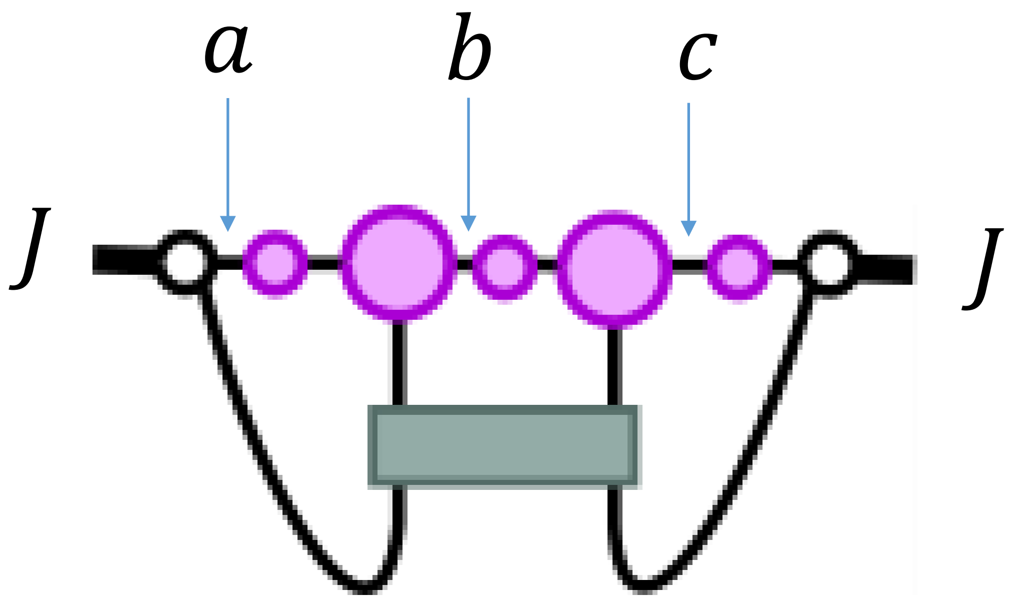




SU(2)-symmetric TEBD

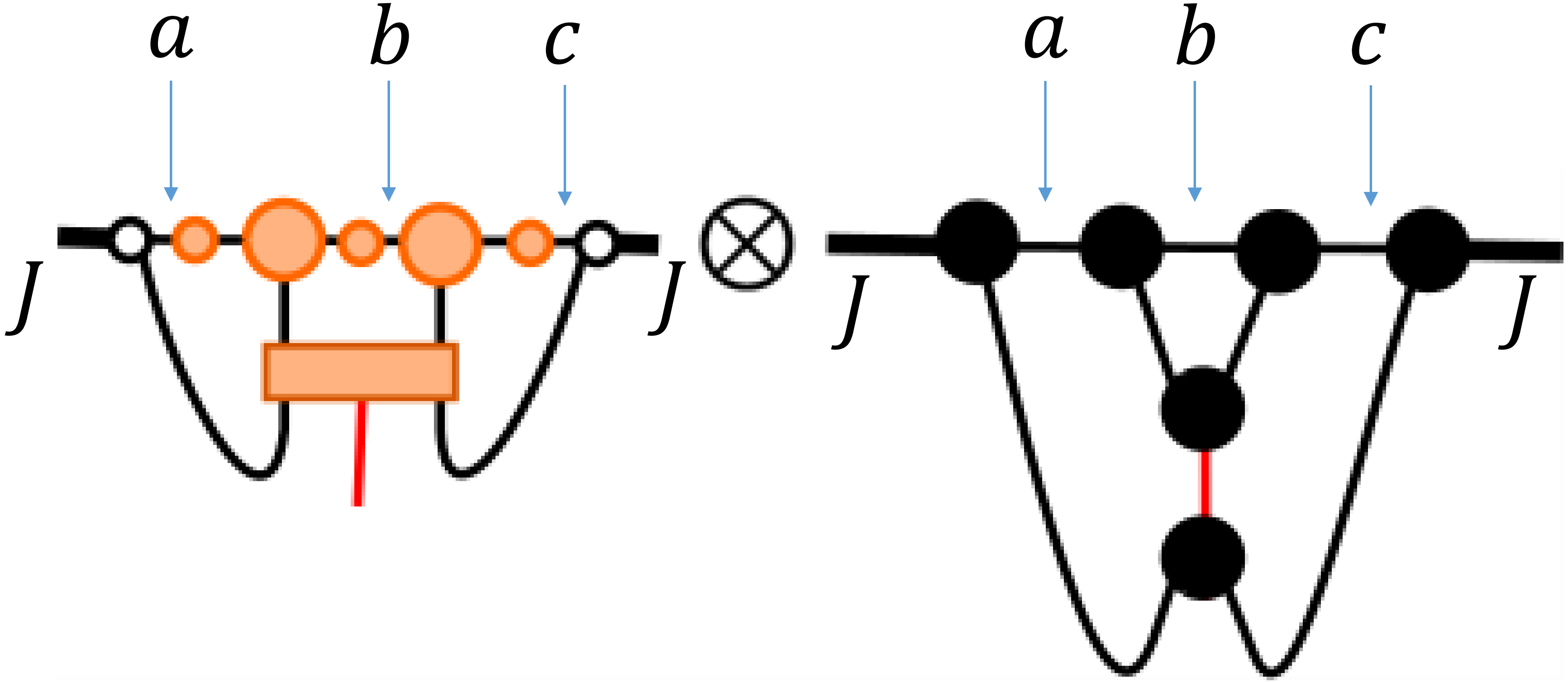


SU(2)-symmetric TEBD



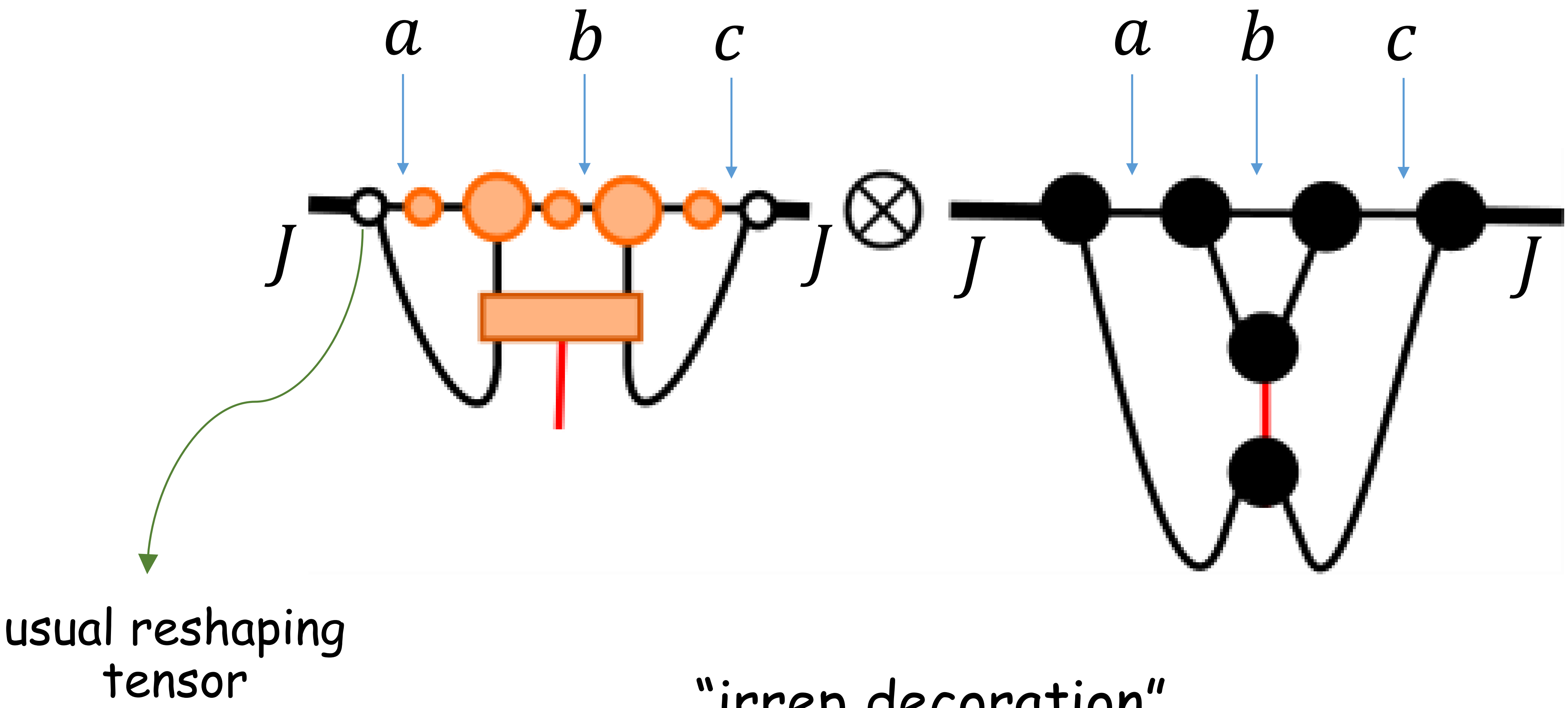
"irrep decoration"

SU(2)-symmetric TEBD

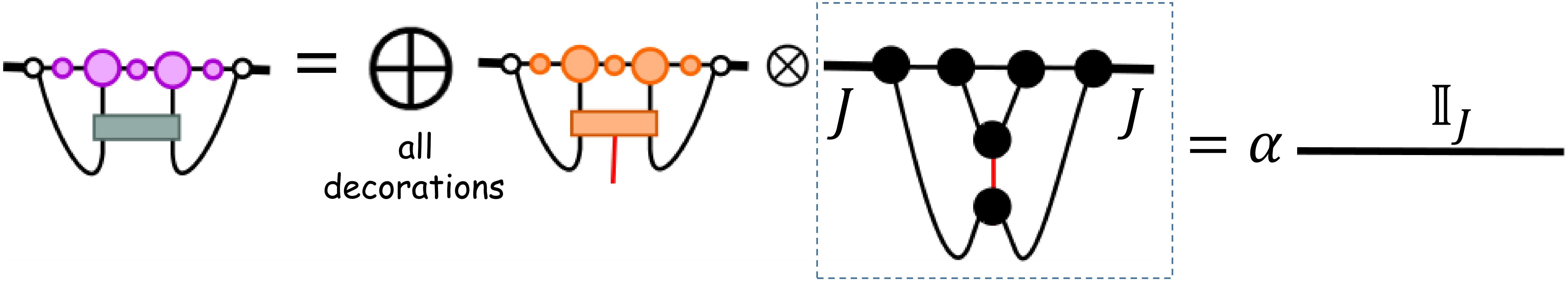


"irrep decoration"

SU(2)-symmetric TEBD



"irrep decoration"



The full contraction reduces to contracting only the much smaller degeneracy tensors

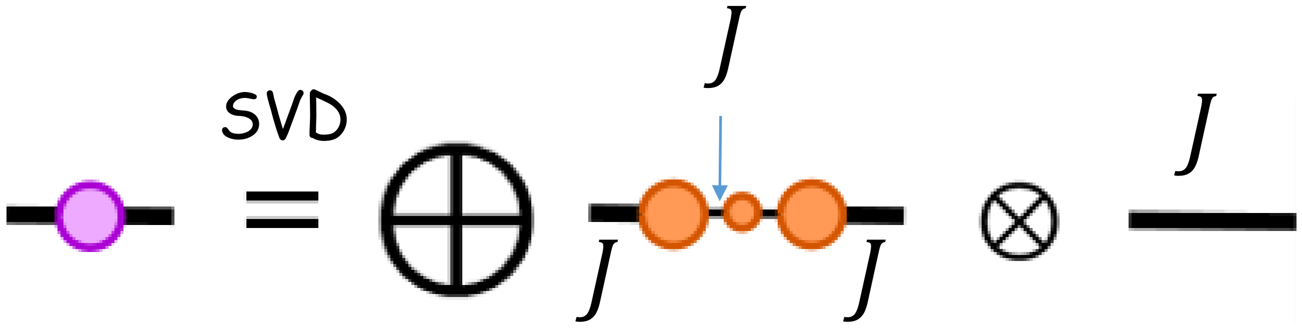
The black tensor network need not be contracted

Since the black tensor network is a symmetric matrix with one irrep on its (open) indices.

It is simply proportional to the Identity. (Schur's lemma)

The proportionality factor can be analytically obtained. It the product of two 6-j symbols here. Depends on the specific contraction.

SU(2)-symmetric TEBD



Can implement SVD blockwise for each J

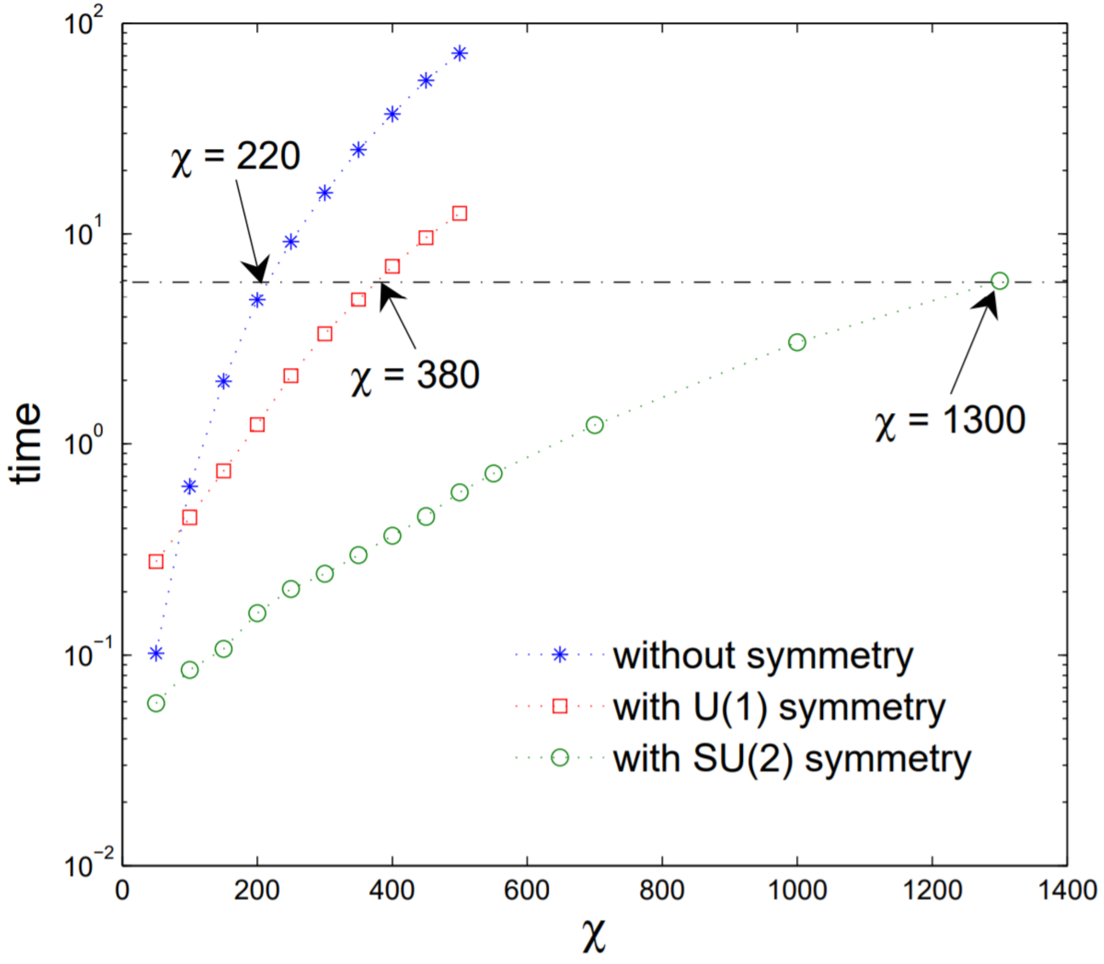
Only have to diagonalize the much smaller degeneracy matrices

New spins J can appear on the bonds of the updated MPS

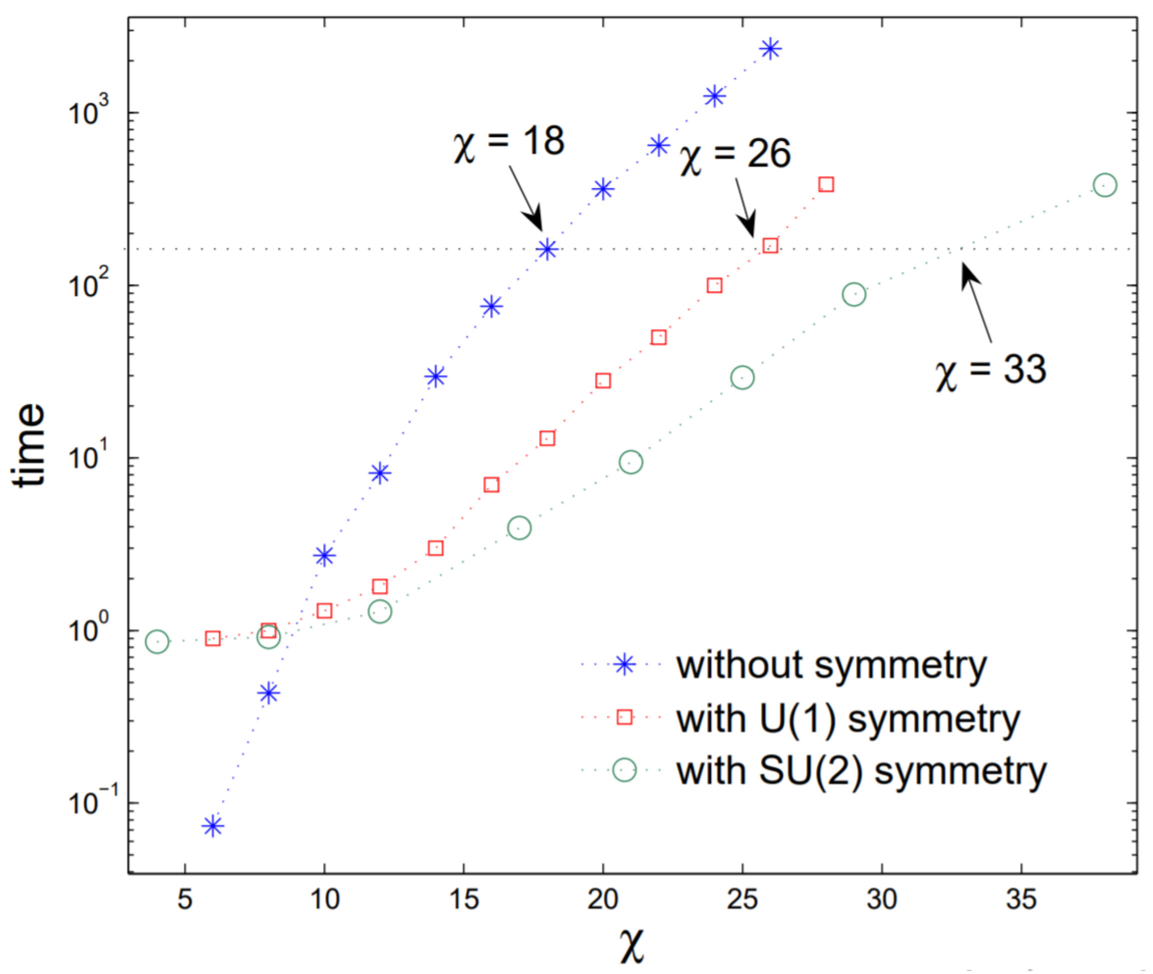
Significant cost reduction

Examples of computational speedup

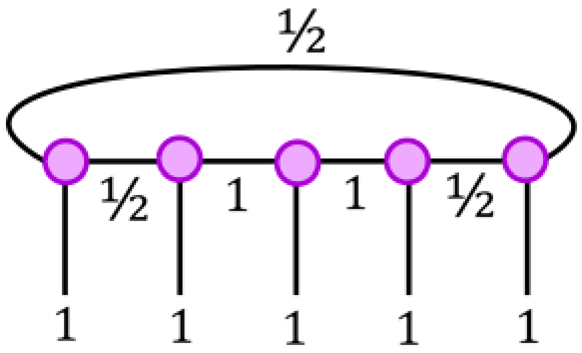
TEBD



MERA



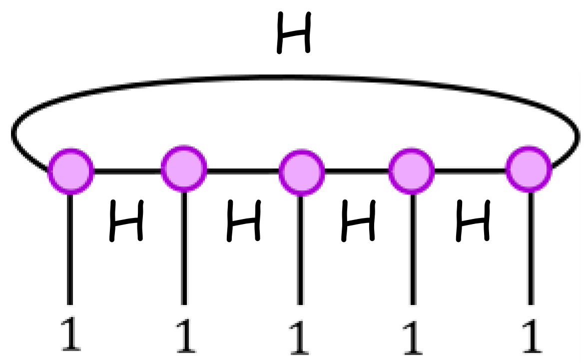
Using the symmetric MPS to detect phases

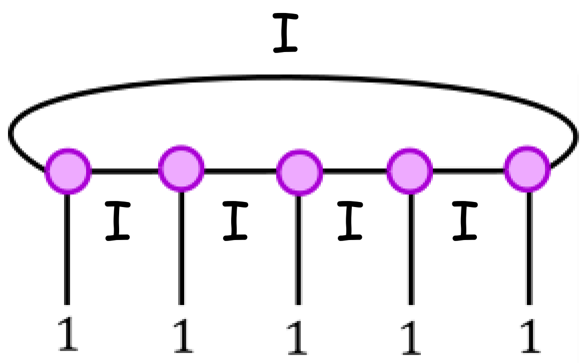


What's strange about this MPS? (All tensors are SU(2)-symmetric)

It is identically zero because of incompatible bond irreps

In fact, the only compatible decorations on an integer spin lattice are

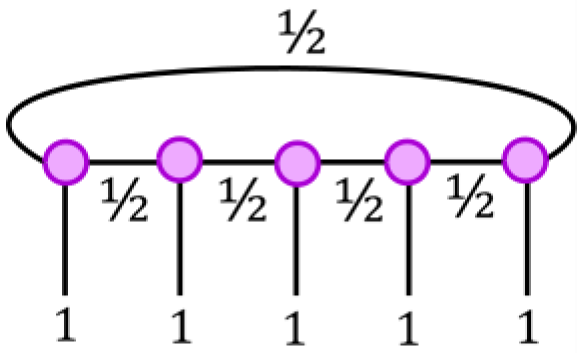




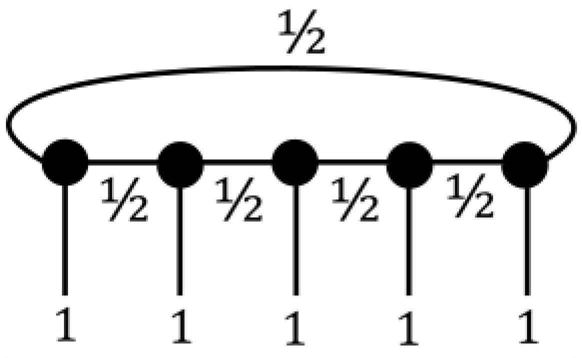
H : Half-integers

I : Integers

Simple example of state with half-integer bond irreps

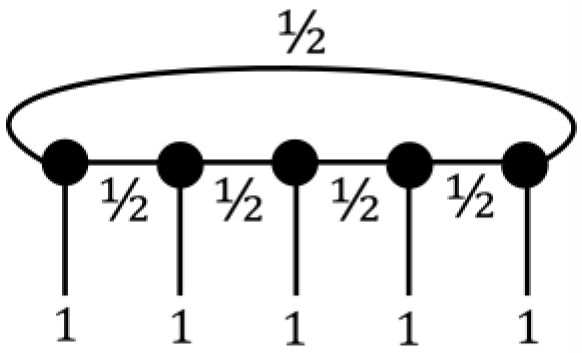


Simple example of state with half-integer bond irreps



The famous AKLT state

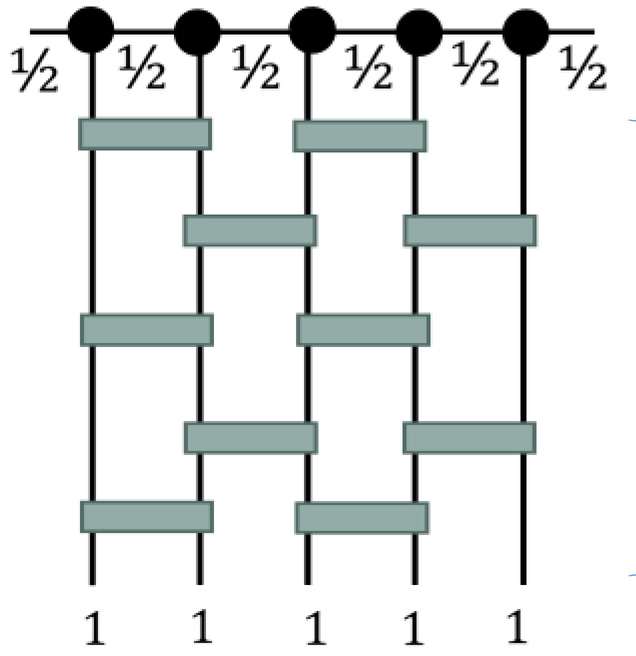
Simple example of state with half-integer bond irreps



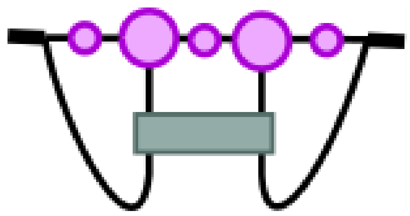
The famous AKLT state

Let us see what happens to the bond spins
if we time evolve for a finite time

infinite lattice

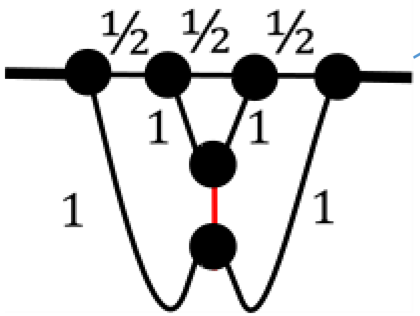


Finite depth unitary circuit



This is the update step where new bond spins can appear.

(We only want to track if the class of irreps changes)



new spins: $\frac{1}{2}, \frac{3}{2}$

(no degeneracy part)

(red index = 0,1,2)

This is the update step where new bond spins can appear.

The new spins can only be half-integer.

This means that the AKLT state (or any half-integer bond state) cannot be transformed by a finite depth circuit to an integer bond state (and vice-versa).

Provided the circuit is symmetric! (Otherwise you can)

This means that half-integer and integer bond states belong to different phases.

These phase are different from usual ones e.g. the ordered and disordered phase in the quantum Ising model.

In the Ising model, the phase transition breaks the Z_2 symmetry.

In our case, both half-integer and integer bond states are symmetric.

These are examples of symmetry protected topological phases.

Since on a spin 1 lattice, the action of SU(2) is isomorphic to SO(3).

So while the AKLT state (or any half-integer bond state) only has SO(3) symmetry, the tensors have an enhanced symmetry = SU(2)

Half-integer irreps are projective representations of SO(3)

This symmetry enhancement at the level of individual MPS tensors led to the classification of all 1d bosonic phases with a symmetry *G*

There is one phase for each equivalence class of projective representations of *G*
(For SO(3) we have two phases)

In 1d systems, the only possible phases are symmetry breaking or symmetry protected topological phases (e.g. no intrinsic topological order)

A relevant problem is to numerically detect which phases can manifest in a given 1d lattice model.

Usual approach: use local or string order parameters.

But we can also use the symmetric MPS to detect phases, without the need for identifying any local or string order parameters.

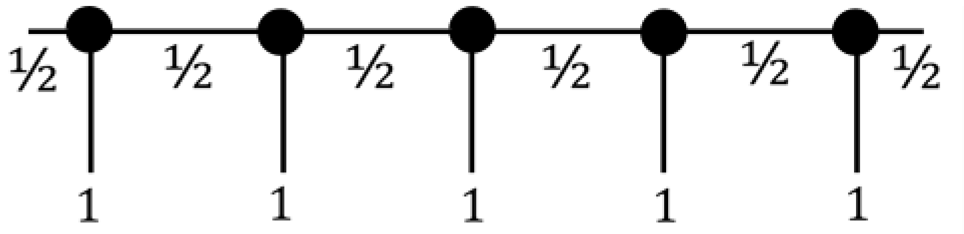
First, note that in e.g. TEBD the (equivalence) class of the bond irreps gets fixed by whatever is chosen in the initial state.

(Since we have seen that the update cannot change the class of the irreps, while preserving the symmetry.)

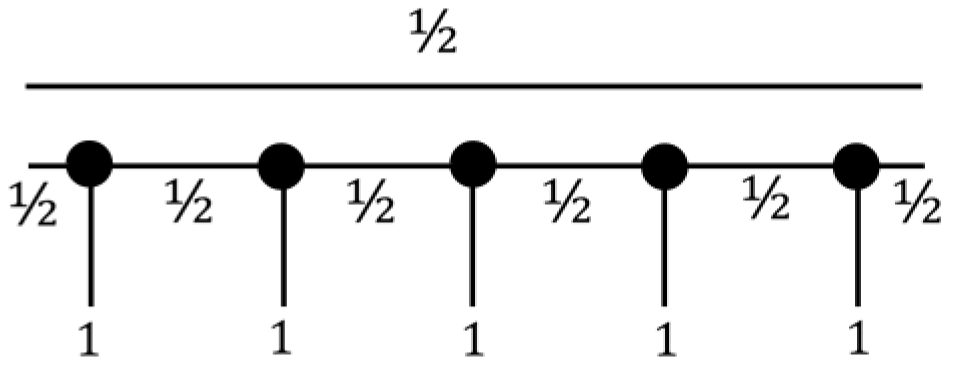
So what happens if we run a simulation for say the AKLT state (or any half-integer bond state) starting with an integer bond state!

Does the computer explode?

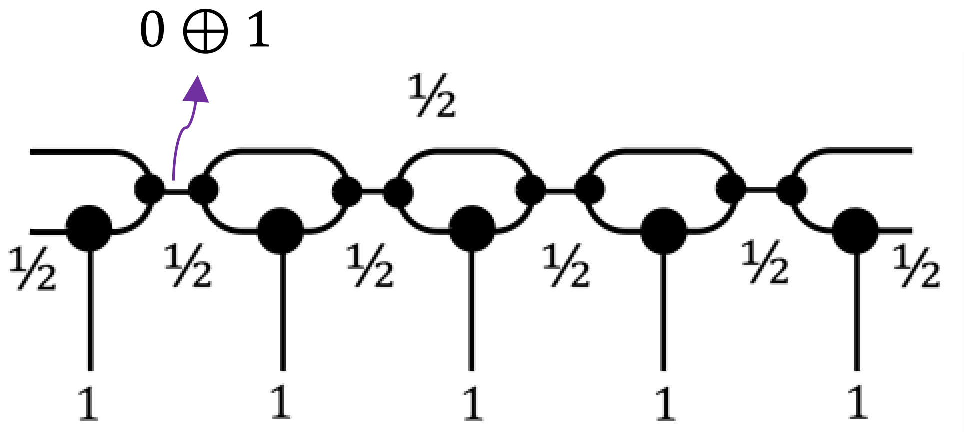
Thankfully no. Since the AKLT state can also be represented by integers bond spins.



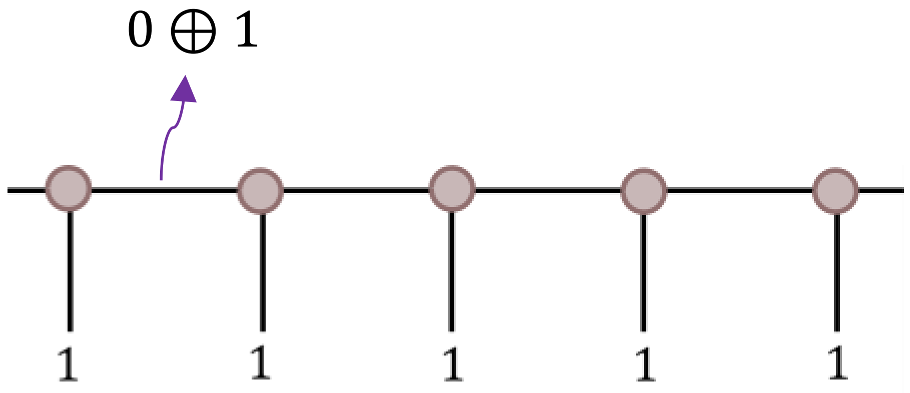
AKLT state on an infinite lattice



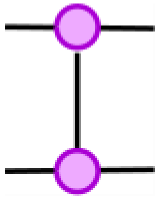
AKLT state on an infinite lattice



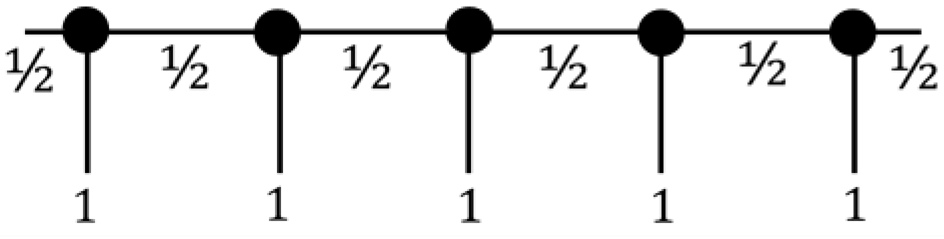
AKLT state on an infinite lattice



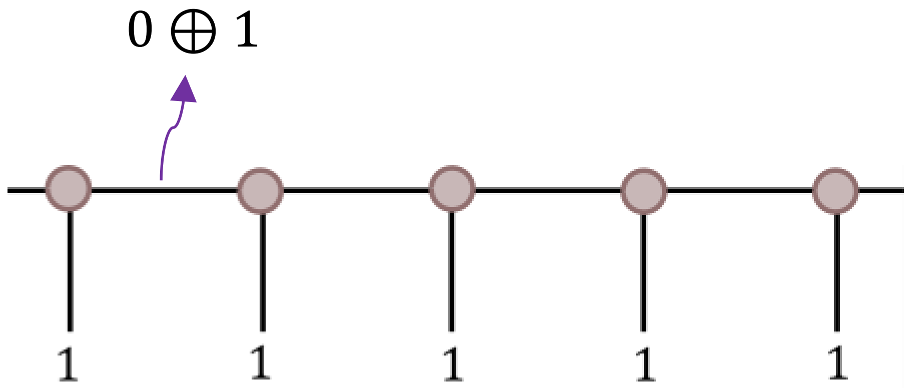
AKLT state on an infinite lattice



Transfer matrix



injective



inflated

AKLT state on an infinite lattice

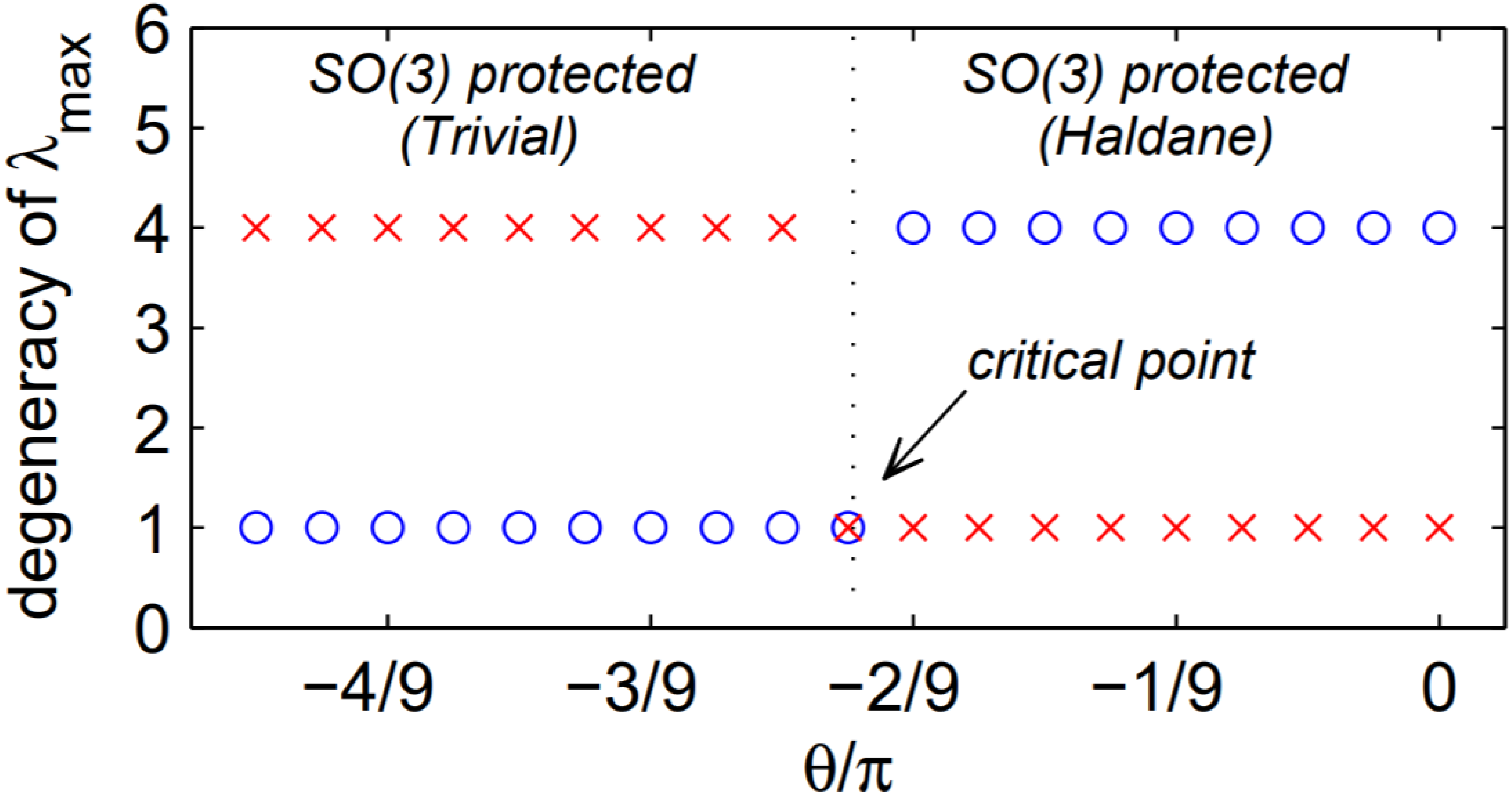
A simple phase detection algorithm for SO(3)

1) Begin with an initial state that is either an integer or half-integer bond state

2) In this way, find two MPS representations of the ground state: one with integer spins and one with half-integer spins (Recall that once we have chosen the class of irreps in the initial state, the algorithm preserves this choice)

3) The state belongs to the phase that corresponds to the bond irrep class that appears in the injective MPS.

$$\hat{H}^{\mathrm{BLBQ}} = \sum_{k\in\mathcal{L}} \cos\theta \left(\vec{S}_k \vec{S}_{k+1}\right) + \sin\theta \left(\vec{S}_k \vec{S}_{k+1}\right)^2$$



A simple phase detection algorithm for any group

1) Projective representations of group are linear representations of another group called the covering group (also called representation group)

2) Implement the covering group symmetry in the MPS.

3) Find all MPS representations of the given ground state (by iterating over all the second cohomology classes of symmetry)

4) Only one MPS representations will be injective (rest are inflated).

5) Read off phase (bond irreps) from the injective MPS

Detecting symmetry breaking phases

1) Turns out that even symmetry breaking phases can be detected using symmetric tensors

2) A symmetry breaking phase is characterized by the presence of degenerate ground states, not all of which are symmetric.

3) But there are always ground states that are symmetric.

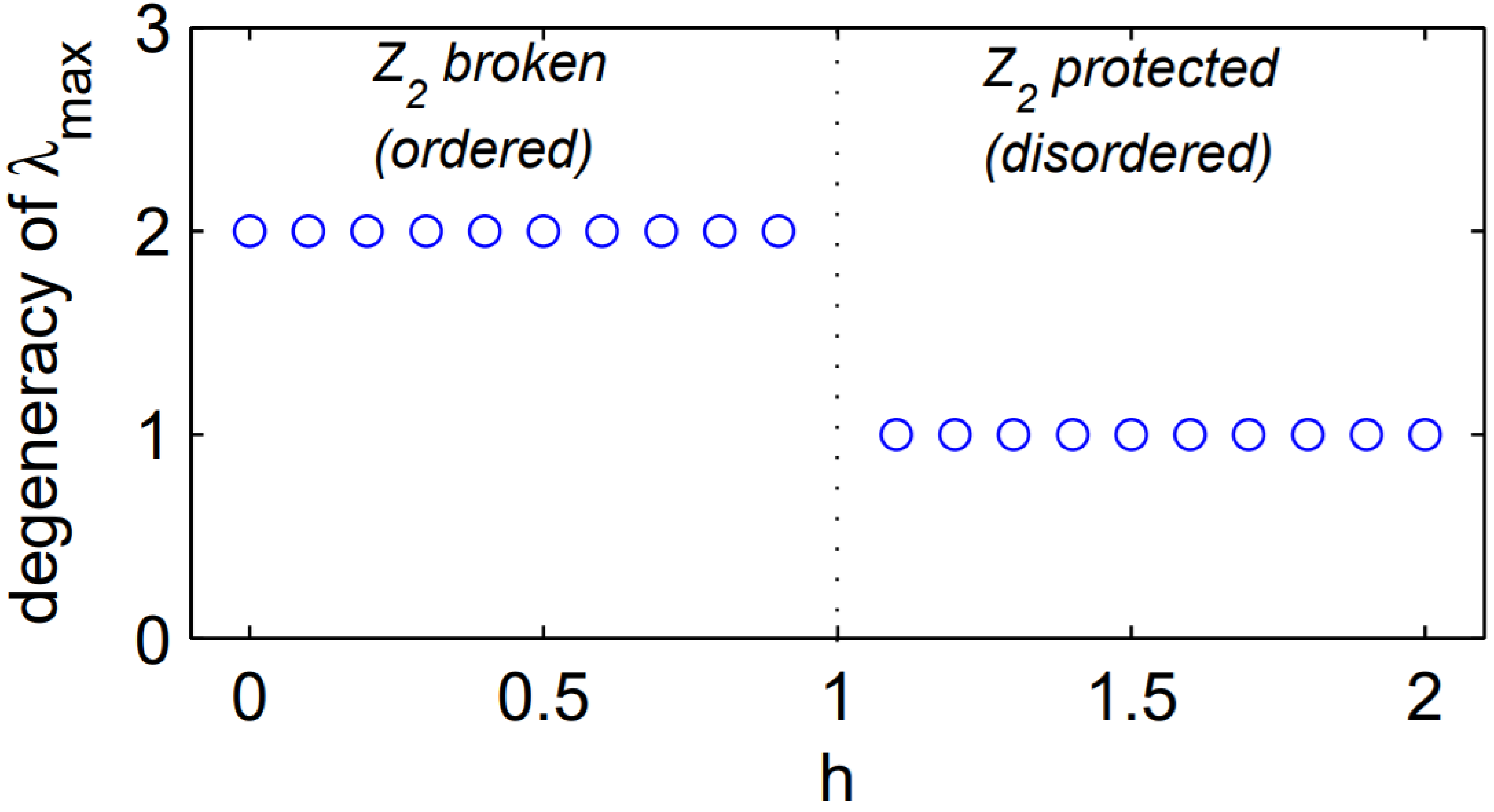
4) But the symmetric ground states are special in this case.

5) They are GHZ like states. $|0000\ \dots\rangle + |1111\ \dots\rangle$

6) Their MPS representation is non-injective! (Only the largest eigenvalue of the transfer matrix is degenerate.)

Example: 1d quantum Ising model

$$\hat{H}^{\mathrm{ISING}} = \sum_{k\in\mathcal{L}} \hat{\sigma}_k^z \hat{\sigma}_{k+1}^z + h\hat{\sigma}_k^x$$



The complete phase detection algorithm

1) Find a set of MPS description of the ground state (one for each different projective representation)

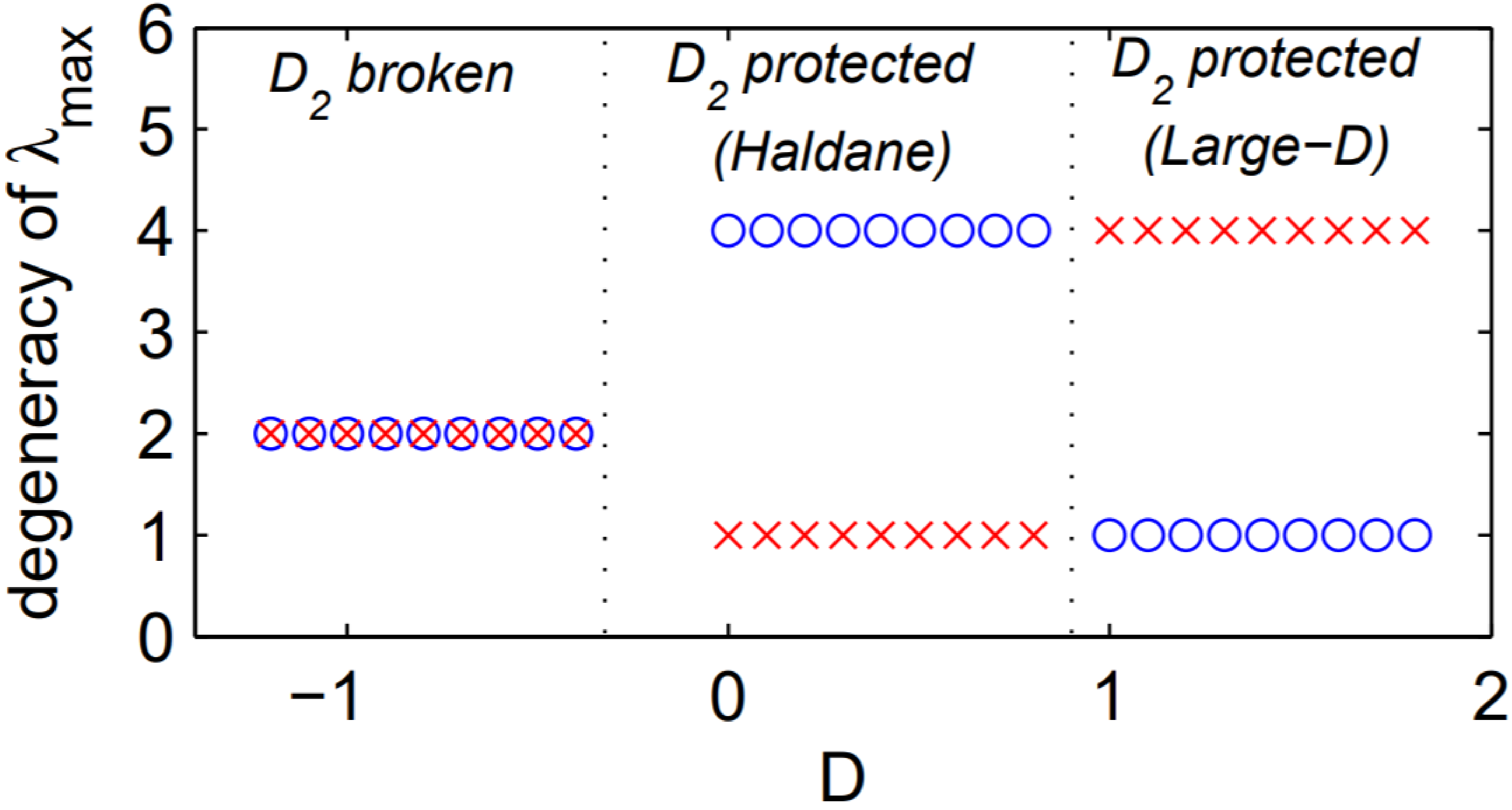
2) Only one MPS will be injective (rest are inflated)

3) Read off phase from the injective MPS

4) If the MPS belongs to a phase in which the symmetry is broken then the MPS representation will be non-injective (but not inflated) for any projective representation

Example with symmetry protected & symmetry breaking phases

$$\hat{H}^{\mathrm{HEIS}} = \sum_{k\in\mathcal{L}} \vec{S}_k \vec{S}_{k+1} + D(S_k^z)^2$$



Take home messages

1) Symmetric tensors are the basic building blocks in many symmetric tensor networks

2) Symmetric tensors are sparse in a particular basis (symmetry basis)

3) Only a small amount of data required from the symmetry: list of irreps, fusion rules, and F-symbols. **(The representation data of the symmetry.)**

4) Implementing symmetries has several benefits: speedup, targeting symmetry sectors, detection of phases etc.

Generalizations

Symmetries underlying Anyon models.

Not described by groups, but by fusion categories.

But a fusion category is simply a generalization of the representation data.

A fusion category is described by data similar to the representation data of a group

So the same code work by simply replacing the representation data.

Implementing gauge symmetries.

Implement global on-site symmetry first.

Then gauge the symmetry by inserting certain tensors in the tensor network.

Thanks!

References

1) S. Singh, H.-Q. Zhou, and G. Vidal, NJP (2010), arXiv:cond-mat/0701427

2) S. Singh, R. N. C. Pfeifer, and G. Vidal, PRA (R) (2010), arXiv:0907.2994

3) S. Singh, R. N. C. Pfeifer, and G. Vidal, PRB (2011), arXiv:1008.4774

4) S. Singh and G. Vidal, PRB (2012), arXiv:1208.3919

5) S. Singh and G. Vidal, PRB (2013), arXiv:1307.1522

6) S. Singh, R. N. C. Pfeifer, G. Vidal, and G. K. Brennen, PRB (2014), arXiv:1311.0967

7) S. Singh, PRB (2015), arXiv:1409.7873.