9.28 Midterm Oct. 19. In class
Final Dec. 18

Last time: 1) Cauchy Sequence
If $\langle x_n \rangle$ is a sequence of numbers s.t. given $\varepsilon > 0$ $\exists N$ s.t. $|x_n - x_m| < \varepsilon$ if $n, m > N$ then we say that $\langle x_n \rangle$ is a Cauchy Sequence.

2) Thm: If $\langle x_n \rangle$ is a Cauchy Sequence, then $\lim_{n\to\infty} x_n$ exists.
This property is called completeness.

3) The set of Cauchy sequence of rational numbers, on which we define an equivalence relation $\langle x_n \rangle \sim \langle x_n' \rangle$ if $\lim_{n\to\infty}(x_n - x_n') = 0$

4) Real numbers, as a set, is defined to be the equivalence classes of Cauchy sequences of rational numbers.

Arithmatic of Real Numbers.

$x \longleftrightarrow \langle x_n \rangle \quad y \longleftrightarrow \langle y_n \rangle$:

$-x \longleftrightarrow \langle -x_n \rangle \quad x+y \longleftrightarrow \langle x_n + y_n \rangle \quad x \cdot y \longleftrightarrow \langle x_n \cdot y_n \rangle$

Easy to prove these arithmatics are well-defined, i.e. if $\langle x_n \rangle \sim \langle x_n' \rangle$ and $\langle y_n \rangle \sim \langle y_n' \rangle$, then $\langle -x_n \rangle \sim \langle -x_n' \rangle$, $\langle x_n + y_n \rangle \sim \langle x_n' + y_n' \rangle$ and $\langle x_n \cdot y_n \rangle \sim \langle x_n' \cdot y_n' \rangle$ and these sequences are all Cauchy sequences.

Now we consider division:

$y \neq 0 \iff \exists$ a N s.t. $\forall \langle y_n \rangle$ that $\exists M$ s.t. $|y_n| \geq \frac{1}{N}$ for $n \geq M$

If $y \neq 0$, then we can find a representative $\langle y_n \rangle$ s.t. $|y_n| \geq \frac{1}{N}$ $\forall n$.

$\frac{1}{y} \longleftrightarrow \langle \frac{1}{y_n} \rangle$ is also well-defined.

In addition, $y \cdot \frac{1}{y} \longleftrightarrow \langle y_n \cdot \frac{1}{y_n} \rangle = \langle 1 \rangle$

Order Relation:

We say $x \leq y$ if for any representatives $\langle x_n \rangle, \langle y_n \rangle$ $\exists M$ s.t. if $n \geq M$ then $x_n \leq y_n$

Distance:

$d(x,y) = |x-y| = \langle |x_n - y_n| \rangle$

We need to show that real numbers are a complete orded field. First, prove the completeness, i.e. If $\langle x_n \rangle$ is a Cauchy sequence of real numbers, then $\exists$ a real number x^* ~~+x~~ s.t. $|x_n - x^*| \to 0$ as $n \to \infty$.

Pf: $\langle x_k^{(n)} \rangle$ is a Cauchy sequence represents x_n.

We need to construct $\langle y_n \rangle$ of rational numbers that represents $\lim_{n\to\infty} x_n$

Given $N \; \exists M$ s.t. $|x_n - x_m| \leq \frac{1}{N}$ if $m, n \geq M$

$$\begin{gathered} \| \\ \langle |x_k^{(n)} - x_k^{(m)}| \rangle \leq \frac{1}{N} \\ \Downarrow \end{gathered}$$

given $\ell > 0 \; \exists K_\ell$ s.t. $|x_k^{(n)} - x_k^{(m)}| \leq \frac{1}{N} + \frac{1}{\ell}$ if $k \geq K_\ell$

For each $\ell \in \mathbb{N}$, we know that there is an M_ℓ

s.t. $|x_n - x_m| \leq \frac{1}{2^\ell}$ $n, m \geq M_\ell$

$|x_k^{(n)} - x_k^{(m)}| \leq \frac{2}{2^\ell}$ with $k \geq K_{n,m}$

$|x_k^{(n)} - x_k^{(M_\ell)}| \leq \frac{2}{2^\ell}$ with $n \geq M_\ell$

~~Throw away $x_1^{(M_\ell)} \cdots x_{K-1}^{(M_\ell)}$~~

$\langle x_k^{(M_\ell)} \rangle$ is a Cauchy sequence:

$\exists K$ s.t. $|x_m^{(M_\ell)} - x_n^{(M_\ell)}| < \frac{1}{2^\ell}$ if $m, n \geq K_\ell$

Need to show there is a sequence of rational numbers $\langle y_n \rangle$ s.t. $\forall N \exists M$ $|y_n - y_m| \leq \frac{1}{N}$ if $n, m > M$ that represent $\lim_{n\to\infty} x_n$.

Choose $y_\ell = x_k^{(M_\ell)}$ for $k > K_{n,M_\ell}$ and $k \geq K'_\ell$

$\exists M_{\ell+1}$ s.t.
$|x_n - x_m| < \frac{1}{2^{\ell+1}}$ for $n, m \geq M_{\ell+1} > M_\ell$

$$x_{M_{\ell+1}} \to \langle x_k^{(M_{\ell+1})} \rangle$$

Choose $K'_{\ell+1}$ s.t.
$|x_n^{(M_{\ell+1})} - x_m^{(M_{\ell+1})}| \leq \frac{1}{2^{\ell+1}}$ $n, m \geq K'_{\ell+1}$

Let $y_{\ell+1} = x_{K'_{\ell+1}}^{(M_{\ell+1})}$

Easy to show $\langle y_n \rangle$ is a Cauchy sequence.
Now we have to show $\langle y_n \rangle$ represents the Limit.

$$x_n \to \langle x_k^{(n)} \rangle$$
$$\Updownarrow$$
$$\exists M_n \text{ s.t. } |x_p^{(n)} - x_q^{(n)}| \leq \frac{1}{2^n} \text{ if } p, q \leq M_n.$$

Then we replace $\langle x_k^{(n)} \rangle$ with $\langle x_{k+M_n}^{(n)} \rangle$.

N.T.S.: $|\langle y_j \rangle - \langle x^{(M_\ell)} \rangle|$
$$= \langle |y_j - x_j^{(M_\ell)}| \rangle \to 0 \text{ as } M_\ell \to \infty.$$

Noted that $|y_j - x_j^{(M_\ell)}| \leq |y_j - y_\ell| + |y_\ell - x_j^{(M_\ell)}|$
$$= |y_j - y_\ell| + |x_{K'_\ell}^{(M_\ell)} - x_j^{(M_\ell)}|$$

Since $\langle y_n \rangle$ and $\langle x_n^{(M_\ell)} \rangle$ are Cauchy sequence, we can choose j, ℓ large enough such that
$$|y_j - y_\ell| + |x_{K'_\ell}^{(M_\ell)} - x_j^{(M_\ell)}| \leq \frac{1}{2^\ell} + \frac{1}{2^{M_\ell}}$$

Now we have shown that $\langle y_j \rangle$ represents $\lim_{\ell \to \infty} x_{M_\ell}$

We still need to show if subsequence $\langle x_{n_j} \rangle$ of Cauchy Sequence $\langle x_n \rangle$ converges

Then $\langle x_n \rangle$ converges

Given N $\exists M$ s.t. $|x_{n_j} - x^*| < \frac{1}{N}$ if $j \geq M$
$\exists M'$ s.t. $n, m \geq M'$, $|x_n - x_m| < \frac{1}{N}$ for $n, m > M'$
$$M'' = \max\{M, n_M\}$$

If $n \geq M''$, then

$$|x_n - x^*| \leq |x_n - x_{n_j}| + |x_{n_j} - x^*| \leq \frac{1}{2^r}$$

Hence $\langle x_n \rangle$ is convergent.

It follows that $\lim_{\ell \to \infty} x_{M_\ell} = \lim_{\ell \to \infty} x_\ell$ is represented by $\langle y_j \rangle$.