9.26. $\langle x_n \rangle$ is a sequence bounded above

$$y_j = \sup\{x_j, x_{j+1}, \ldots\}$$

$$y_j \geq y_{j+1} \geq y_{j+2}$$

$y^* = \lim_{j\to\infty} y_j$ exists

Want to prove that $\exists$ a subsequence $\langle x_{n_j} \rangle$

s.t. $\lim_{j\to\infty} x_{n_j} = y^* = \lim_{j\to\infty} \sup x_j$

$E = \{ j : \exists n_j \text{ with } n_j \geq j \text{ and } y_j = x_{n_j} \}$

$\{x_j, x_{j+1}, \ldots, x_{n_j} \ldots\}$

$\{j, j+1, \ldots, n_j, \ldots\}$

It is possible that $x_{n_j} = x_{n_{j'}}$ given $j \neq j'$

Hence we need to find j_{l_2} such that $j_{l_2} > n_j$,

$| x_{n_j} \quad x_{n_{j_{l_2}}} \quad x_{n_{j_{l_3}}} \ldots x_{n_{j_{l_i}}} \ldots$

when $j_{l_{k+1}} > n_{j_{l_k}}$

j_0 is the largest index in E

$j_1 = j_0 + 1$

$\nexists\ j_1 \leq n$ s.t. $x_n = y_{j_1}$

$\forall j > j_0 \ \nexists\ x_n = y_j \Rightarrow$ contradiction.

Since the # in the set $\{k \geq j : x_k > y_j - \frac{1}{l}\}$ is infinite.

$j_1 = j_0 + 1$

Let $l = 1$. $\{j_1 \leq k ; x_k > y_{j_1} - \frac{1}{1}\}$

Choose one of these k, $k_1 \geq j_1$

$$x_{k_1} \geq y_{j_1} - \frac{1}{1}$$

Choose $j_2 > k_1$ and $k_2 > j_2$ s.t. $x_{k_2} \geq y_{j_2} - \frac{1}{2}$

$\vdots$

$$j_1 \leq k_1 < j_2 \leq k_2 < j_3 \cdots \leq j_l \leq k_l$$

s.t. $x_{k_i} \geq y_{j_i} - \frac{1}{i}$ $\quad i = 1, 2, \ldots, l$

$y_{j_l} - \frac{1}{l} < x_{k_l} < y_{j_l}$ Then by squeeze theorem $\lim_{l\to\infty} x_{k_l} = y^*$

Squeeze Thm: If $x_n \le y_n \le z_n$ are sequences.
and $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, then
$\lim_{n\to\infty} y_n$ exists and equal to ~~eq~~ x^*

Pf: $0 \le y_n - x_n \le z_n - x_n$
Given N $\exists M$ s.t.
$|y_n - x_n| \le |z_n - x_n| \le \frac{1}{N}$ if $n \ge M$
Therefore $\lim_{n\to\infty} (y_n - x_n) = 0$
Since $y_n = x_n + y_n - x_n$
Hence $\lim_{n\to\infty} y_n = \cancel{x_n + 0}\ \lim_{n\to\infty} x_n$ and y_n convergent.

$\langle x_n \rangle$ is a bounded sequence
$y_j = \sup\{x_j, x_{j+1}, \dots\}$
$z_j = \inf\{x_j, x_{j+1}, \dots\}$.
$z_j \le x_j \le y_j$
We've proved that $\lim_{j\to\infty} y_j = y^*$
is a limit point of $\langle x_n \rangle$
① Any bounded sequence has limit points
$\Leftrightarrow$ $\exists$ a convergent subsequence.
$\exists$ a subsequence x_{n_j} s.t. $\lim_{j\to\infty} x_{n_j} = \lim_{j\to\infty} z_j$

Thm: Suppose that x^* is a limit point of $\langle x_n \rangle$,
then $\lim_{j\to\infty} \inf \le x^* \le \lim_{j\to\infty} \sup$.

Pf: If x^* is a limit point, then there is a subsequence $\langle x_{n_j} \rangle$ s.t.
$\lim_{j\to\infty} x_{n_j} = x^*$
$z_{n_j} \le x_{n_j} \le y_{n_j}$
Hence $x_{n_j} - z_{n_j} \ge 0$ and $y_{n_j} - x_{n_j} \ge 0$
$\lim_{j\to\infty} (x_{n_j} - z_{n_j}) \ge 0 \Rightarrow \lim_{j\to\infty} x_{n_j} \ge \lim_{j\to\infty} z_{n_j}$
$\Rightarrow x^* \ge \lim_{j\to\infty} \inf$
Similarly, $\lim_{j\to\infty} \sup \ge x^*$

Bolzano – Weierstrass Theorem

If $\langle x_n \rangle$ is a bounded sequence of real #s, then $\langle x_n \rangle$ has a convergent subsequence.

Thm: $\langle x_n \rangle$ is convergent if and only if
$$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n$$

Pf: If $\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n = x^*$
$$\inf x_n \le x_n \le \sup x_n$$
by sequeeze thm, $\lim_{n\to\infty} x_n = x^*$

If $\lim_{n\to\infty} x_n = x^*$

Then given l $\exists$ M_l s.t. $x^* - \frac{1}{l} \le x_n \le x^* + \frac{1}{l}$ if $n \ge M_l$

$x^* - \frac{1}{l} \le y_j \le x^* + \frac{1}{l}$ $\quad x^* - \frac{1}{l} \le z_j \le x^* + \frac{1}{l}$

if $j \ge M_l$

Choose $j_l \ge M_l$
$$x^* - \frac{1}{l} \le y_{j_l} \le x^* + \frac{1}{l}$$
$\exists$ M_{l+1} s.t. $|x_n - x^*| \le \frac{1}{l+1}$ if $n > M_{l+1}$

~~$y_{j_{l+1}}$~~ $j_{l+1} > \max\{M_{l+1}, j_l\}$
$$x^* - \frac{1}{l+1} \le y_{j_{l+1}} \le x^* + \frac{1}{l+1}$$

$$\lim_{l\to\infty} y_{j_l} = x^* = \lim_{l\to\infty} z_{j_l}$$

$\lim_{n\to\infty} x_n = +\infty$ if $\forall N$ $\exists M$ s.t. $x_n \ge N$ if $n \ge M$

$\lim_{n\to\infty} x_n = -\infty$ if $\forall N$ $\exists M$ s.t. $x_n \le -N$ if $n \ge M$.

Observation:

$\lim_{n\to\infty} x_n = x^*$

$|x_n - x_m| \le |x_n - x^*| + |x^* - x_m|$

Given N $\exists M$ s.t. $|x_n - x^*| \le \frac{1}{N}$

$|x_m - x^*| \le \frac{1}{N}$

if $n, m \ge M$

Hence $|x_n - x_m| < \frac{1}{2N}$ if $n, m \ge M$.

Def: A sequence $\langle x_n \rangle$ is called a Cauchy sequence if given N $\exists M$ s.t. $|x_n - x_m| < \frac{1}{N}$ if $n, m \geq M$.

Thm. A Cauchy Sequence is alway convergent.

Pf: Fix a numbe N $\exists M$ s.t.

$$|x_n - x_m| \leq \frac{1}{N} \quad \text{if } n, m \geq M$$

$$x_M - \frac{1}{N} \leq x_n \leq x_M + \frac{1}{N} \quad \forall n \geq M$$

$$y_n = \sup\{x_n, x_{n+1}, \cdots\}$$

$$z_n = \inf\{x_n, x_{n+1}, \ldots\}.$$

$$x_{M_N} - \frac{1}{N} \leq z_{M_N} \leq y_{M_N} \leq x_{M_N} + \frac{1}{N}$$

$$|y_{M_N} - z_{M_N}| \leq \frac{2}{N}$$

Hence $\lim_{N \to \infty} (y_{M_N} - z_{M_N}) = 0$

Since y_n, z_n convergent,

$$\lim_{n\to\infty} y_n = \lim_{N\to\infty} y_{M_N} = \lim_{N\to\infty} z_{M_N} = \lim_{n\to\infty} z_n.$$

Hence $\langle x_n \rangle$ is convergent.

This property 'Any Cauchy Seqence is convergent' is called completeness.

Completion of the rational numbers

$\exists$ $\langle x_n \rangle \subseteq \mathbb{Q}$ that are Cauchy sequence, which do not have rational limits.

To each Cauchy sequence we could add a new number to $\mathbb{Q}$.

But $\exists \langle x_n \rangle, \langle x_n' \rangle$ s.t. $x_n - x_n' \to 0$ and $\langle x_n \rangle - \langle x_n' \rangle \neq 0$

Hence we need to define an equivalence ration among all the Cauchy sequences.

$$\langle x_n \rangle \sim \langle x_n' \rangle \quad \text{if } \lim_{n\to\infty} (x_n - x_n') = 0.$$

Equivalence Relation satisfies:

1. Reflexive : $\langle x_n \rangle = \langle x_n \rangle$
2. Symmetric : If $\langle x_n \rangle \sim \langle x_n' \rangle$, then $\langle x_n' \rangle \sim \langle x_n \rangle$
3. Transitive : If $\langle x_n \rangle \sim \langle x_n' \rangle$ and $\langle x_n' \rangle \sim \langle x_n'' \rangle$, then $\langle x_n \rangle \sim \langle x_n'' \rangle$

We define equivalence class

$$[\langle x_n \rangle] = \{ \langle x_n' \rangle \mid \langle x_n' \rangle \sim \langle x_n \rangle \}.$$

Then $[\langle x_n \rangle] = [\langle y_n \rangle]$ or $[\langle x_n \rangle] \cap [\langle y_n \rangle] = \phi$.

This set of equivalence classes is defined to be the set underlying the real numbers.