Assignment 3

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Problem 1

Problem 1 (12 marks)

Consider the following two algorithms that naïvely compute the sum and product of two $n \times n$ matrices.

```
sum(A,B):                          product(A,B):
  for i ∈ [0,n):                     for i ∈ [0,n):
    for j ∈ [0,n):                     for j ∈ [0,n):
      C[i,j] = A[i,j] + B[i,j]           C[i,j] = add{A[i,k] * B[k,j] : k ∈ [0,n)}
    end for                            end for
  end for                            end for
  return C                           return C
```

Assuming that adding and multiplying matrix elements can be carried out in $O(1)$ time, and add will add the elements of a set S in $O(|S|)$ time:

(a) Give an asymptotic upper bound, in terms of n, for the running time of sum. (3 marks)

(b) Give an asymptotic upper bound, in terms of n, for the running time of product. (3 marks)

When n is even, we can define a recursive procedure for multiplying two $n \times n$ matrices as follows. First, break the matrices into smaller submatrices:

$$A = \begin{pmatrix} S & T \\ U & V \end{pmatrix} \qquad B = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

where S, T, U, V, W, X, Y, Z are $\frac{n}{2} \times \frac{n}{2}$ matrices. Then it is possible to show:

$$AB = \begin{pmatrix} SW + TY & SX + TZ \\ UW + VY & UX + VZ \end{pmatrix}$$

where $SW + TY$, $SX + TZ$, etc. are sums of products of the smaller matrices. If n is a power of 2, each smaller product (SW, TY, etc) can be computed recursively, until the product of 1×1 matrices needs to be computed – which is nothing more than a simple multiplication, taking $O(1)$ time.

Assume n is a power of 2, and let $T(n)$ be the worst-case running time for computing the product of two $n \times n$ matrices using this method.

(c) With justification, give a recurrence equation for $T(n)$. (4 marks)

(d) Find an asymptotic upper bound for $T(n)$. (2 marks)

(a)

Given that two matrices are both $n \times n$, and adding elements take $O(1)$ time,

In the inner loop of $sum(A, B)$, each loop takes $O(1)$ time.

Then, the inner loop can be carried out in $O(n)$ time.

For the whole function which is the inner loop nested with an outer one, it can be carried out in $O(n^2)$ time.

Therefore, the asymptotic upper bound for the running time of sum is $O(n^2)$.

(b)

Given that two matrices are both $n \times n$, and multiplying elements take $O(1)$ time,

$A[i, k] * B[k, j]$ for $k \in [0, n)$ can be carried out in $O(1)$ time.

'Add' instruction adds all elements in the set above which is of size n.

The time complexity for 'add' instruction in function $product(A, B)$ is $O(n)$.

Same as question(a), 'add' instruction is also nested with two for loops.

Each of them runs for n times.

Therefore, the asymptotic upper bound for the running time of product is $O(n^3)$.

(c)

$$AB = \begin{pmatrix} SW + TY & SX + TZ \\ UW + VY & UX + VZ \end{pmatrix}$$

In the above figure, it is easy to count that, in each recursion,

8 multiplications are done for matrices which are of size $\frac{n}{2} \times \frac{n}{2}$ and 4 additions as well.

The time complexity analysis for multiplication is passed to the next recursion.

While, addition of two takes $O(n^2)$ time, given in question(a).

Therefore, the recurrence equation for $T(n)$ is defined below:

$$T(n) = 8 \cdot T(\frac{n}{2}) \; + \; O(n^2) \tag{1}$$

(d)

Theorem

Suppose

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

where $f(n) \in \Theta(n^c(\log n)^k)$.

Let $d = \log_b(a)$. *Then:*

- **Case 1:** *If* $c < d$ *then* $T(n) = \Theta(n^d)$
- **Case 2:** *If* $c = d$ *then* $T(n) = \Theta(n^c(\log n)^{k+1})$
- **Case 3:** *If* $c > d$ *then* $T(n) = \Theta(f(n))$

From the definition of Master Theorem, for recurrence equation in question(c):

$$\begin{aligned} a &= 8 \\ b &= 2 \\ c &= 2 \\ k &= 0 \end{aligned} \tag{2}$$

Which is Case 1, $T(n) = \Theta(n^3)$,

Therefore, the asymptotic upper bound for $T(n)$ is $O(n^3)$.

Problem 2

Problem 2 (18 marks)

Recall from Assignment 2 the neighbourhood of eight houses:

As before, each house wants to set up its own wi-fi network, but the wireless networks of neighbouring houses – that is, houses that are either next to each other (ignoring trees) or over the road from one another (directly opposite) – can interfere, and must therefore be on different channels. Houses that are sufficiently far away may use the same wi-fi channel. Again we would like to solve the problem of finding the minimum number of channels needed, but this time we will solve it using techniques from logic and from probability. Rather than directly asking for the minimum number of channels required, we ask if it is possible to solve it with just 2 channels. So suppose each wi-fi network can either be on channel hi or on channel lo. Is it possible to assign channels to networks so that there is no interference?

(a) Formulate this problem as a problem in propositional logic. Specifically:

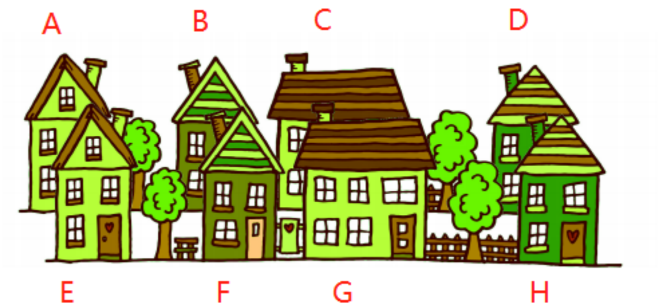
- (i) Define your propositional variables (4 marks)
- (ii) Define any propositional formulas that are appropriate and indicate what propositions they represent. (4 marks)
- (iii) Indicate how you would solve the problem (or show that it cannot be done) using propositional logic. It is sufficient to explain the method, you do not need to provide a solution. (2 marks)
- (iv)* Explain how to modify your answer(s) to (i) and (ii) if the goal was to see if it is possible to solve with 3 channels rather than 2. (4 marks)

(b) Suppose each house chooses, uniformly at random, one of the two network channels. What is the probability that there will be no interference? (4 marks)

(a)

(i)

Mark the 8 houses by 8 characters as below:



Each character represent 'House X uses channel 1'.

True means it is using channel 1 while false means it is using channel 2.

(ii)

Then the constraints, in this problem interference, are:

$$p \equiv A \rightarrow (\neg B \wedge \neg E) \tag{3}$$

$$q \equiv B \rightarrow ((\neg A \wedge \neg C) \wedge \neg F) \tag{4}$$

$$r \equiv C \rightarrow ((\neg B \wedge \neg D) \wedge \neg G) \tag{5}$$

$$s \equiv D \rightarrow (\neg C \wedge \neg H) \tag{6}$$

$$t \equiv E \rightarrow (\neg A \wedge \neg F) \tag{7}$$

$$u \equiv F \rightarrow ((\neg B \wedge \neg E) \wedge \neg G) \tag{8}$$

$$v \equiv G \rightarrow ((\neg C \wedge \neg F) \wedge \neg H) \tag{9}$$

$$w \equiv H \rightarrow (\neg D \wedge \neg G) \tag{10}$$

(iii)

To solve the problem, define ϕ as below:

$$\phi \equiv p \wedge q \wedge r \wedge s \wedge t \wedge u \wedge v \wedge w \tag{11}$$

If ϕ is satisfiable, then the problem is solved. Otherwise, it cannot be done.

To list a truth table of A, B, C, D, E, F, G, H and their constraints,

Omit those which turn their constraints to be all false.

That would also brings a satisfying result of ϕ, but it is not the problem asking for.

(iv)

Since problem changed from solving with 3 channels rather than 2,

It becomes insufficient if only declaring A, B, C, D, E, F, G, H.

Each of them can be expanded to the form of A_1, A_2, A_3,

Which represents 'House A is using channel 1 (2, 3)'.

Constraints should also time 3:

$$A_1 \rightarrow (\neg B_1 \ \wedge \neg E_1) \tag{12}$$

$$A_2 \rightarrow (\neg B_2 \ \wedge \neg E_2) \tag{13}$$

$$A_3 \rightarrow (\neg B_3 \ \wedge \neg E_3) \tag{14}$$

Similarly,

$$\phi_1 \equiv p_1 \ \wedge q_1 \ \wedge r_1 \ \wedge s_1 \ \wedge t_1 \ \wedge u_1 \ \wedge v_1 \ \wedge w_1 \tag{15}$$

$$\phi_2 \equiv p_2 \ \wedge q_2 \ \wedge r_2 \ \wedge s_2 \ \wedge t_2 \ \wedge u_2 \ \wedge v_2 \ \wedge w_2 \tag{16}$$

$$\phi_3 \equiv p_3 \ \wedge q_3 \ \wedge r_3 \ \wedge s_3 \ \wedge t_3 \ \wedge u_3 \ \wedge v_3 \ \wedge w_3 \tag{17}$$

The problem becomes to find truth values which make these three true.

(b)

All possible situation is $2^8 = 256$.

The situation with no interference is 2.

Therefore, the probability is $\frac{1}{128}$.

Problem 3

Problem 3 (12 marks)
Prove the following results hold in all Boolean Algebras:

(a) For all x: $(x \wedge 1') \vee (x' \wedge 1) = x'$ (4 marks)

(b) For all x, y: $(x \wedge y) \vee x = x$ (4 marks)

(c) For all x, y: $y' \wedge ((x \vee y) \wedge x') = 0$ (4 marks)

Proof assistant

https://cgi.cse.unsw.edu.au/~cs9020/cgi-bin/logic/21T3/boolean_algebra/assignment3a

(a)

$$\begin{aligned}
(x \wedge 1') \vee (x' \wedge 1) &= ((x \wedge 1') \vee x') \wedge ((x \wedge 1') \vee 1) && \text{(Distributivity of } \vee \text{ over } \wedge\text{)} \\
&= (x' \vee (x \wedge 1')) \wedge ((x \wedge 1') \vee 1) && \text{(Commutatitivity of } \vee\text{)} \\
&= (x' \vee (x \wedge 1')) \wedge (1 \vee (x \wedge 1')) && \text{(Commutatitivity of } \vee\text{)} \\
&= ((x' \vee x) \wedge (x' \vee 1')) \wedge (1 \vee (x \wedge 1')) && \text{(Distributivity of } \vee \text{ over } \wedge\text{)} \\
&= ((x' \vee x) \wedge (x' \vee 1')) \wedge ((1 \vee x) \wedge (1 \vee 1')) && \text{(Distributivity of } \vee \text{ over } \wedge\text{)} \\
&= ((x \vee x') \wedge (x' \vee 1')) \wedge ((1 \vee x) \wedge (1 \vee 1')) && \text{(Commutatitivity of } \vee\text{)} \\
&= (1 \wedge (x' \vee 1')) \wedge ((1 \vee x) \wedge (1 \vee 1')) && \text{(Complement with } \vee\text{)} \\
&= (1 \wedge (x' \vee 1')) \wedge ((1 \vee x) \wedge 1) && \text{(Complement with } \vee\text{)} \\
&= ((x' \vee 1') \wedge 1) \wedge ((1 \vee x) \wedge 1) && \text{(Commutatitivity of } \wedge\text{)} \\
&= (x' \vee 1') \wedge ((1 \vee x) \wedge 1) && \text{(Identity of } \wedge\text{)} \\
&= (x' \vee 1') \wedge (1 \vee x) && \text{(Identity of } \wedge\text{)} \\
&= (x \wedge 1)' \wedge (1 \vee x) && \text{(De Morgan's, } ' \text{ over } \wedge\text{)}
\end{aligned} \tag{18}$$

$$
\begin{aligned}
&= x' \wedge (1 \vee x) && \text{(Identity of } \wedge \text{)} \\
&= x' \wedge (x \vee 1) && \text{(Commutatitivity of } \vee \text{)} \\
&= x' \wedge 1 && \text{(Annihilation of } \vee \text{)} \\
&= x' && \text{(Identity of } \wedge \text{)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
(x \wedge y) \vee x &= (x \wedge y) \vee (x \wedge 1) && \text{(Identity of } \wedge \text{)} \\
&= (x \wedge 1) \vee (x \wedge y) && \text{(Commutatitivity of } \vee \text{)} \\
&= x \wedge (1 \vee y) && \text{(Distributivity of } \wedge \text{ over } \vee \text{)} \\
&= x \wedge (y \vee 1) && \text{(Commutatitivity of } \vee \text{)} \\
&= x \wedge 1 && \text{(Annihilation of } \vee \text{)} \\
&= x && \text{(Identity of } \wedge \text{)}
\end{aligned} \tag{19}
$$

(c)

$$
\begin{aligned}
y' \wedge ((x \vee y) \wedge x') &= ((x \vee y) \wedge x') \wedge y' && \text{(Commutatitivity of } \wedge \text{)} \\
&= (x \vee y) \wedge (x' \wedge y') && \text{(Associativity of } \wedge \text{)} \\
&= (x \vee y) \wedge (x \vee y)' && \text{(De Morgan's, } ' \text{ over } \vee \text{)} \\
&= 0 && \text{(Complement with } \wedge \text{)}
\end{aligned} \tag{20}
$$

Problem 4

Definition

A **Boolean algebra** is a structure $(T, \vee, \wedge, ', \mathbb{0}, \mathbb{1})$ where

- $\mathbb{0}, \mathbb{1} \in T$
- $\vee, \wedge : T \times T \to T$ (called **join** and **meet** respectively)
- $' : T \to T$ (called **complementation**)

and the following laws hold for all $x, y, z \in T$:

Commutativity:	$x \vee y = y \vee x, \quad x \wedge y = y \wedge x$
Associativity:	$(x \vee y) \vee z = x \vee (y \vee z)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
Distributivity:	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
Identity:	$x \vee \mathbb{0} = x, \quad x \wedge \mathbb{1} = x$
Complementation:	$x \vee x' = \mathbb{1}, \quad x \wedge x' = \mathbb{0}$

Let a Boolean Algebra be $(T, \vee, \wedge, ', \mathbb{0}, \mathbb{1})$ where $T = \{\mathbb{0}, \mathbb{1}\}$

From the definition of complementation:

$$
\begin{aligned}
\mathbb{0}' &= \mathbb{1} \\
\mathbb{1}' &= \mathbb{0}
\end{aligned} \tag{21}
$$

Now, add a third element x to this structure, where x is distinct from $\mathbb{0}$ and $\mathbb{1}$.

To meet complementation requirement, x' should also in T.

1. If $x' = \mathbb{0}$, then $x = \mathbb{1}$, x is not distinct.
2. if $x' = \mathbb{1}$, then $x = \mathbb{0}$, x is not distinct.
3. if $x' = x$, then $x \vee x' = x$, complementation not holds.

It is either not a Boolean Algebra or another new element should be added.

Therefore, there are no three element Boolean Algebras.

Problem 5

Problem 5 (12 marks)
Prove or disprove the following logical equivalences:

(a) $\neg(p \to q) \equiv (\neg p \to \neg q)$ (4 marks)

(b) $((p \wedge q) \to r) \equiv (p \to (q \to r))$ (4 marks)

(c) $((p \vee (q \vee r)) \wedge (r \vee p)) \equiv ((p \wedge q) \vee (r \vee p))$ (4 marks)

Proof assistant

https://cgi.cse.unsw.edu.au/~cs9020/cgi-bin/logic/21T3/prop_logic/assignment3b

(a)

Left side of the equation can be changed to the form as below:

$$\begin{aligned}\neg(p \to q) &\equiv \neg(\neg p \vee q) && \text{(Implication)}\\ &\equiv \neg\neg p \wedge \neg q && \text{(De Morgan's, } \neg \text{ over } \vee\text{)}\\ &\equiv p \wedge \neg q && \text{(Double negation)}\end{aligned} \tag{22}$$

Right side of the equation can be changed to the form as below:

$$(\neg p \to \neg q) \equiv p \vee \neg q \qquad \text{(Implication)} \tag{23}$$

Equation(22) is equivalent to equation(23) if and only if $p \equiv q$, that is:

$$p \wedge \neg q \equiv p \equiv q \equiv p \vee \neg q \tag{24}$$

Therefore, question(a) disproved.

(b)

$$\begin{aligned}((p \wedge q) \to r) &\equiv \neg(p \wedge q) \vee r && \text{(Implication)}\\ &\equiv (\neg p \vee \neg q) \vee r && \text{(De Morgan's, } \neg \text{ over } \wedge\text{)}\\ &\equiv \neg p \vee (\neg q \vee r) && \text{(Associativity of } \vee\text{)}\\ &\equiv p \to (\neg q \vee r) && \text{(Implication)}\\ &\equiv (p \to (q \to r)) && \text{(Implication)}\end{aligned} \tag{25}$$

(c)

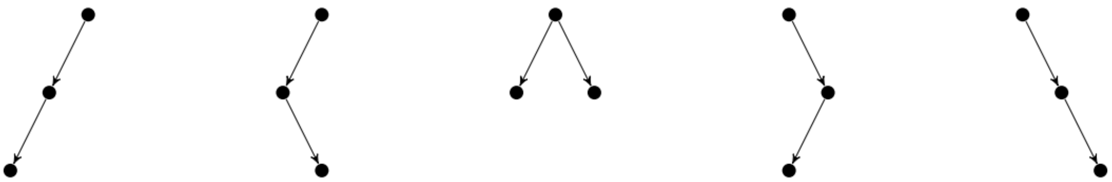
$$
\begin{aligned}
((p \vee (q \vee r)) \wedge (r \vee p)) &\equiv ((p \vee q) \vee r) \wedge (r \vee p) && \text{(Associativity of } \vee \text{)} \\
&\equiv (r \vee (p \vee q)) \wedge (r \vee p) && \text{(Commutatitivity of } \vee \text{)} \\
&\equiv r \vee ((p \vee q) \wedge p) && \text{(Distributivity of } \vee \text{ over } \wedge \text{)} \\
&\equiv r \vee (p \wedge (p \vee q)) && \text{(Commutatitivity of } \wedge \text{)} \\
&\equiv r \vee ((p \vee p) \wedge (p \vee q)) && \text{(Idempotence of } \vee \text{)} \\
&\equiv r \vee (p \vee (p \wedge q)) && \text{(Distributivity of } \vee \text{ over } \wedge \text{)} \\
&\equiv (r \vee p) \vee (p \wedge q) && \text{(Associativity of } \vee \text{)} \\
&\equiv ((p \wedge q) \vee (r \vee p)) && \text{(Commutatitivity of } \vee \text{)}
\end{aligned} \tag{26}
$$

Problem 6

Problem 6 (16 marks)

Recall from Assignment 2 the definition of a binary tree data structure: either an empty tree, or a node with two children that are trees.

Let $T(n)$ denote the number of binary trees with n nodes. For example $T(3) = 5$ because there are five binary trees with three nodes:



(a) Using the recursive definition of a binary tree structure, or otherwise, derive a recurrence equation for $T(n)$. (6 marks)

A **full binary tree** is a non-empty binary tree where every node has either two non-empty children (i.e. is a fully-internal node) or two empty children (i.e. is a leaf).

(b) Using observations from Assignment 2, or otherwise, explain why a full binary tree must have an odd number of nodes. (2 marks)

(c) Let $B(n)$ denote the number of full binary trees with n nodes. Derive an expression for $B(n)$, involving $T(n')$ where $n' \leq n$. *Hint: Relate the internal nodes of a full binary tree to $T(n)$.* (4 marks)

A well-formed formula is in **Negated normal form** if it consists of just $\wedge$, $\vee$, and literals (i.e. propositional variables or negations of propositional variables). For example, $(p \vee (\neg q \wedge \neg r))$ is in negated normal form; but $(p \vee \neg(q \vee r))$ is not.

Let $F(n)$ denote the number of well-formed, negated normal form formulas[1] there are that use precisely n propositional variables exactly one time each. So $F(1) = 2$, $F(2) = 16$, and $F(4) = 15360$.

(d) Using your answer for part (c), give an expression for $F(n)$. (4 marks)

(a)

The recursive definition of a binary tree structure shows that,

To count the number of nodes of a binary tree, it is to add up the number of nodes in its left tree and right tree.

Reversely, to depict a binary tree with n nodes, after fixing its root node, There lefts $n-1$ nodes to be distributed.

For $n=3$, there are two nodes regardless of the root, to distribute these two nodes,

Either put them to the left tree or right tree. Otherwise, one in left tree and one in right tree, that is:

$$T(3)=T(2)T(0)+T(1)T(1)+T(0)T(2) \tag{27}$$

For n, this function can be written in the form as below:

$$T(n)=T(n-1)T(0)+T(n-2)T(1)+\ldots+T(1)T(n-2)+T(0)T(n-1) \tag{28}$$

Therefore, the recurrence equation for T(n) is:

$$T(0)=1 \tag{29}$$

$$T(n)=\sum_{k=0}^{n-1} T(k)T(n-k-1) \qquad (n\geq 1) \tag{30}$$

(b)

(d) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves(T)} = \mathsf{internal(T)} + 1$. Prove that $P(\mathsf{T})$ holds for all binary trees T. Your proof should be based on your answers given in (b) and (c). (8 marks)

The nodes in a full binary tree is either a fully-internal node or a leaf.

From the observation from Assignment 2, number of leaves equals to number of internal nodes plus one.

Therefore, number of nodes in a full binary tree can be expressed as below:

$$\begin{aligned} \text{count}(n) &= \text{internal}(n)+\text{leaves}(n) \\ &= \text{internal}(n)+\text{internal}(n)+1 \\ &= 2\cdot\text{internal}(n)+1 \end{aligned} \tag{31}$$

Therefore, a full binary tree must have an odd number of nodes.

(c)

Let n' represent the number of internal nodes in a full binary tree with number of nodes n.

Then, there are $T(n')$ ways to form a tree with these internal nodes alone.

Meanwhile, for each way, there is exactly one corresponding way to insert the leaf nodes so that the whole binary tree remains a full one.

Therefore,

$$B(n)=T(n') \tag{32}$$

From equation(31) that:

$$n=2\cdot n'+1 \tag{33}$$

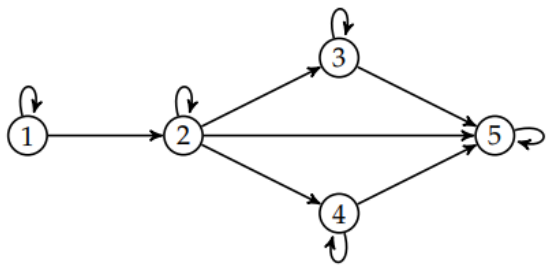
Therefore,

$$B(n)=T(\frac{n-1}{2}) \tag{34}$$

Problem 7

Problem 7 (16 marks)

Consider the following directed graph:



and consider the following process:

- Initially, start at 1.
- At each time step, choose one of the outgoing edges from your current location uniformly at random, and follow it to the next location. For example, if your current location was 2, then with probability $\frac{1}{4}$ you would move to 3; with probability $\frac{1}{4}$ you would move to 4; with probability $\frac{1}{4}$ you would move to 5; and with probability $\frac{1}{4}$ you would stay at 2.

Let $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$, $p_5(n)$ be the probability your location after n time steps is 1, 2, 3, 4, or 5 respectively. So $p_1(0) = 1$ and $p_2(0) = p_3(0) = p_4(0) = p_5(0) = 0$.

(a) Express $p_1(n+1)$, $p_2(n+1)$, $p_3(n+1)$, $p_4(n+1)$, and $p_5(n+1)$ in terms of $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$, and $p_5(n)$. (5 marks)

(b) Prove ONE of the following:

(i) For all $n \in \mathbb{N}$: $p_1(n) = \frac{1}{2^n}$ (5 marks)

(ii) For all $n \in \mathbb{N}$: $p_2(n) = 2\left(\frac{1}{2^n} - \frac{1}{4^n}\right)$ (6 marks)

(iii) For all $n \in \mathbb{N}$: $p_3(n) = p_4(n) = (n-2)\frac{1}{2^n} + \frac{2}{4^n}$ (7 marks)

(iv) For all $n \in \mathbb{N}$: $p_5(n) = 1 - (2n-1)\frac{1}{2^n} - \frac{2}{4^n}$ (8 marks)

Note

Clearly state which identity you are proving. A maximum of 8 marks is available for this question and marks will be awarded based on level of technical ability demonstrated. You may assume the identities which you are not proving.

(c) For each $n \in \mathbb{N}$ let X_n be the random variable that has value:

- 0 if your location at time n is 1;
- 1 if your location at time n is 2;
- 2 if your location at time n is 3 or 4; and
- 3 if your location at time n is 5

(i. e. X_n is the length of the longest path from 1 to your location at time n).

What is the expected value of X_3? (3 marks)

Remark

This is an example of a Markov chain – a very useful model for stochastic processes.

(a)

$$p_1(n+1) = \frac{1}{2}\, p_1(n) \tag{35}$$

$$p_2(n+1) = \frac{1}{2}\, p_1(n) \;+\; \frac{1}{4}\, p_2(n) \tag{36}$$

$$p_3(n+1) = \frac{1}{4}\, p_2(n) \;+\; \frac{1}{2}\, p_3(n) \tag{37}$$

$$p_4(n+1) = \frac{1}{4}\, p_2(n) \;+\; \frac{1}{2}\, p_4(n) \tag{38}$$

$$p_5(n+1) = \frac{1}{4}\, p_2(n) \;+\; \frac{1}{2}\, p_3(n) \;+\; \frac{1}{2}\, p_4(n) \;+\; p_5(n) \tag{39}$$

(b)

(ii)*

To prove, for all $n \in \mathbb{N} : p_2(n) = 2(\frac{1}{2^n} - \frac{1}{4^n})$, substitute $p_1(n)$ in the recurrence equation of $p_2(n+1)$:

$$\begin{aligned} p_2(n+1) &= \frac{1}{2^{n+1}} + \frac{1}{4}\, p_2(n) \\ p_2(n) &= \frac{1}{2^n} + \frac{1}{4}\, p_2(n-1) \\ 2^n p_2(n) &= \frac{1}{2}\, 2^{n-1} p_2(n-1) + 1 \end{aligned} \tag{40}$$

Let $a_n = 2^n p_2(n)$, then:

$$\begin{aligned} a_n &= \frac{1}{2}\, a_{n-1} + 1 \\ a_n - 2 &= \frac{1}{2}\, (a_{n-1} - 2) \end{aligned} \tag{41}$$

Let $b_n = a_n - 2$, then:

$$\frac{b_n}{b_{n-1}} = \frac{1}{2} \tag{42}$$

$b_1 = a_1 - 2 = 2\, p_2(1) - 2$, then:

$$p_2(1) = \frac{1}{2} + \frac{1}{4}\, p_2(0) \tag{43}$$

$p_2(0) = 0$, hence, $b_1 = -1$, then:

$$\begin{aligned} b_n &= -\frac{1}{2^{n-1}} \\ a_n &= 2 - \frac{1}{2^{n-1}} \\ p_2(n) &= \frac{2 - \frac{1}{2^{n-1}}}{2^n} \\ &= \frac{2\,(1 - \frac{1}{2^n})}{2^n} \\ &= 2\,(\frac{1}{2^n} - \frac{1}{4^n}) \end{aligned} \tag{44}$$

(c)

At time 3, the probability for each location is:

$$
\begin{aligned}
p_1(3) &= \frac{1}{8} \\
p_2(3) &= \frac{7}{32} \\
p_3(3) &= \frac{5}{32} \\
p_4(3) &= \frac{5}{32} \\
p_5(3) &= \frac{11}{32}
\end{aligned} \tag{45}
$$

Then, the expected value of X_3 can be calculated as:

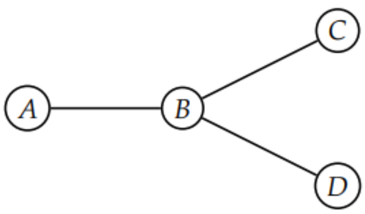
$$
\begin{aligned}
\text{expected value of } X_3 &= 0 \times p_1(3) + 1 \times p_2(3) + 2 \times (p_3(3) + p_4(3)) + 3 \times p_5(3) \\
&= \frac{7}{32} + \frac{20}{32} + \frac{33}{32} \\
&= \frac{15}{8}
\end{aligned} \tag{46}
$$

Problem 8

Problem 8 (10 marks)

In this question we are going to look at modelling the spread of a virus in a network (or news in a social network).

Consider the following graph:



and consider the following process:

- Initially, at time $n = 0$, A is infected and no other vertices are infected.
- At each time step, each infected vertex does the following:
  - for each uninfected neighbour, spread the infection to that vertex with probability $\frac{1}{2}$.

  So if A and B were infected and C and D were not, then in one time step, the virus would spread to both C and D with probability $\frac{1}{4}$; spread to C only with probability $\frac{1}{4}$; spread to D only with probability $\frac{1}{4}$; and not spread any further with probability $\frac{1}{4}$.

Let $p_A(n)$, $p_B(n)$, $p_C(n)$, $p_D(n)$ be the probability that A, B, C, D (respectively) are infected after n time steps. So $p_A(0) = 1$ and $p_B(0) = p_C(0) = p_D(0) = 0$.

(a) Express $p_D(n+1)$ in terms of $p_A(n)$, $p_B(n)$, $p_C(n)$ and $p_D(n)$. (4 marks)

(b) Find an expression for $p_D(n)$ in terms of n only. You do not need to prove the result, but you should briefly justify your answer. *Hint: Try to relate this system with Question 7* (4 marks)

(c)* What is the expected number of infected vertices after $n = 3$ time steps? (2 marks)

(a)

$$p_D(n+1) = \frac{1}{2}\, p_B(n) + p_D(n) \tag{47}$$