Assignment 2

Sixiang Qiu
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Problem 1

Problem 1 (15 marks)
Let S be a set.

(a) Show that for any set T and any function $f : S \to T$, the relation $R_f \subseteq S \times S$, defined as:

$$(s, s') \in R_f \text{ if and only if } f(s) = f(s')$$

is an equivalence relation. (9 marks)

(b) Show that if $R \subseteq S \times S$ is an equivalence relation, then there exists a set T and a function $f_R : S \to T$ such that:

$$(s, s') \in R \text{ if and only if } f_R(s) = f_R(s')$$

(6 marks)

(a)

Reflexivity:

$$\begin{aligned} \forall x \in S : f(x) = f(x)\ , \\ (x, x) \in R_f \end{aligned} \tag{1}$$

Symmetry:

$$\begin{aligned} \forall x, y \ \ that \ \ (x, y) \in R_f : f(x) = f(y), \\ f(y) = f(x), \\ (y, x) \in R_f \end{aligned} \tag{2}$$

Transitivity:

$$\begin{aligned} \forall x, y, z \ \ that \ \ (x, y) \in R_f \ \ and \ \ (y, z) \in R_f : \\ f(x) = f(y) = f(z), \\ (x, z) \in R_f \end{aligned} \tag{3}$$

Therefore, R_f is an equivalence relation.

(b)

Assume that there exists a set T and a function $f_R : S \to T$ such that:

$$(s, s') \in R \ \textit{if and only if} \ f_R(s) = f_R(s') \tag{4}$$

in which $R \subseteq S \times S$ is an equivalence relation.

Then, $(s, f_R(s)) \in f_R$ and $(s', f_R(s')) \in f_R$.

According to Reflexivity:

$$\exists s' = s \ \textit{such that} \ (s, s') \in R \tag{5}$$

Then, $(s, f_R(s)) \in f_R$ and $(s, f_R(s')) \in f_R$.

Since $f_R(s) = f_R(s')$, satisfy (Fun) and (Tot) in definition of function.

Definition

(Fun)	functional	For all $s \in S$ there is at most one $t \in T$ such that $(s, t) \in R$
(Tot)	total	For all $s \in S$ there is at least one $t \in T$ such that $(s, t) \in R$
(Inj)	injective	For all $t \in T$ there is at most one $s \in S$ such that $(s, t) \in R$
(Sur)	surjective	For all $t \in T$ there is at least $s \in S$ such that $(s, t) \in R$
(Bij)	bijective	Injective and surjective

Therefore, function f_R exists.

Problem 2

Problem 2 (20 marks)

Let $\mathbb{B} = \{0, 1\}$ and consider the function $f : \mathbb{N} \to \mathbb{B}$ given by

$$f(n) = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that for all $a, b \in \mathbb{N}$:

(i) $f(a+b) = \max\{f(a), f(b)\}$

(ii) $f(ab) = \min\{f(a), f(b)\}$

(6 marks)

From Problem 1, we know that $R_f \subseteq \mathbb{N} \times \mathbb{N}$, the relation given by:

$$(m, n) \in R_f \text{ if and only if } f(m) = f(n)$$

is an equivalence relation. Let $\mathbb{E} \subseteq \mathrm{Pow}(\mathbb{N})$ be the set of equivalence classes of R_f, and for $n \in \mathbb{N}$, let $[n] \in \mathbb{E}$ denote the equivalence class of n.

We would like to define binary operations, $\boxplus$ and $\boxdot$, on $\mathbb{E}$ as follows:

$$[x] \boxplus [y] := [x+y]$$
$$[x] \boxdot [y] := [xy].$$

The difficulty is that the operands $[x]$ and $[y]$ can have multiple representations (e.g. if $z \in [x]$ then $[x] = [z]$), and so it is not clear that such a definition makes sense: if we take a different representation of the operands, do we still end up with the same result? That is, if $[x] = [x']$ and $[y] = [y']$ is it the case that $[x+y] = [x'+y']$ and $[xy] = [x'y']$? Our next step is to show that such a definition makes sense.

(b) Define relations $\boxplus, \boxdot \subseteq \mathbb{E}^2 \times \mathbb{E}$ as follows:

$$((X, Y), Z) \in \boxplus \text{ if and only if there is } x \in X \text{ and } y \in Y \text{ such that } x + y \in Z$$
$$((X, Y), Z) \in \boxdot \text{ if and only if there is } x \in X \text{ and } y \in Y \text{ such that } xy \in Z$$

(i) Show that $\boxplus$ is a function.

(ii) Show that $\boxdot$ is a function.

(6 marks)

Part (b) shows that the informal definition of $\boxplus$ and $\boxdot$ given earlier is *well-defined*, so from now we will view $\boxplus$ and $\boxdot$ as **binary operations** on $\mathbb{E}$, that is $\boxplus, \boxdot : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$.

(c) Show that for all $A, B, C \in \mathbb{E}$:

(i) $A \boxdot [1] = A$

(ii) $A \boxplus B = B \boxplus A$

(iii) $A \boxdot (B \boxplus C) = (A \boxdot B) \boxplus (A \boxdot C)$

(8 marks)

(a)

(i)

Domain of $f(n)$ is partitioned by $n = 0$ or $n > 0$.

Categorized discussion:

1. If $a + b = 0$ which stands for $a = b = 0$:

$$f(a+b) = f(0) = 0 = max\{0,0\} = max\{f(0), f(0)\} = max\{f(a), f(b)\} \tag{6}$$

2. If $a + b > 0$ in which $b = 0$ and $a > 0$:

$$f(a+b) = 1 = max\{1,0\} = max\{f(a), f(b)\} \tag{7}$$

3. If $a + b > 0$ in which $a = 0$ and $b > 0$, the same as above.
4. If $a + b > 0$ in which $a > 0$ and $b > 0$:

$$f(a+b) = 1 = max\{1,1\} = max\{f(a), f(b)\} \tag{8}$$

Therefore, $f(a+b) = max\{f(a), f(b)\}$.

(ii)

Categorized discussion:

1. If $ab = 0$ in which $a = 0$ and $b = 0$:

$$f(a+b) = 0 = min\{0,0\} = min\{f(a), f(b)\} \tag{9}$$

2. If $ab = 0$ in which $a = 0$ and $b > 0$:

$$f(a+b) = 0 = min\{0,1\} = min\{f(a), f(b)\} \tag{10}$$

3. If $ab = 0$ in which $b = 0$ and $a > 0$, the same as above.
4. If $ab > 0$ which stands for $a > 0$ and $b > 0$:

$$f(a+b) = 1 = min\{1,1\} = min\{f(a), f(b)\} \tag{11}$$

(b)

(i)

Let X, Y, Z be one of the elements in $\boxplus$, that is $[X] \boxplus [Y] := [X+Y] = [Z]$.

Let X', Y', Z' be one of the elements in $\boxplus$, in which $[X'] = [X], [Y'] = [Y]$ and $[X'] \boxplus [Y'] := [X' + Y'] = [Z']$.

From the question, define $x \in X$ and $y \in Y$ such that $x + y \in Z$.

Since $[X'] = [X], [Y'] = [Y]$, apparently, $x \in X'$ and $y \in Y'$ such that $x + y \in Z'$ too.

Therefore, $[Z] = [X+Y] = [X'+Y'] = [Z']$. It satisfies (Fun) and (Tot) which are the definition of a function.

Therefore, $\boxplus$ is a function.

(ii)

Let X, Y, Z be one of the elements in $\boxdot$, that is $[X] \boxdot [Y] := [XY] = [Z]$.

Let X', Y', Z' be one of the elements in $\boxdot$, in which $[X'] = [X], [Y'] = [Y]$ and $[X'] \boxdot [Y'] := [X'Y'] = [Z']$.

From the question, define $x \in X$ and $y \in Y$ such that $xy \in Z$.

Since $[X'] = [X], [Y'] = [Y]$, apparently, $x \in X'$ and $y \in Y'$ such that $xy \in Z'$ too.

Therefore, $[Z] = [XY] = [X'Y'] = [Z']$. It satisfies (Fun) and (Tot) which are the definition of a function.

Therefore, $\boxdot$ is a function.

(c)

(i)

$$\begin{gathered} Let\ A = [n] \in E, \\ A \boxdot [1] = [n] \boxdot [1] = [n] = A \end{gathered} \tag{12}$$

(ii)

$$\begin{gathered} Let\ A = [n_1] \in E,\ B = [n_2] \in E, \\ A \boxplus B = [n_1] \boxplus [n_2] = [n_1 + n_2] = [n_2 + n_1] = [n_2] \boxplus [n_1] = B \boxplus A \end{gathered} \tag{13}$$

(iii)

$$\begin{gathered} Let\ A = [a] \in E,\ B = [b] \in E,\ C = [c] \in E, \\ A \boxdot (B \boxplus C) = [a] \boxdot ([b] \boxplus [c]) = [a] \boxdot [b + c] = [a(b + c)] \\ = [ab + ac] = [(ab) + (ac)] = ([a] \boxdot [b]) \boxplus ([a] \boxdot [c]) \\ = (A \boxdot B) \boxplus (A \boxdot C) \end{gathered} \tag{14}$$

Problem 3

Problem 3 (12 marks)

Eight houses are lined up on a street, with four on each side of the road as shown:

Each house wants to set up its own wi-fi network, but the wireless networks of neighbouring houses – that is, houses that are either next to each other (ignoring trees) or over the road from one another (directly opposite) – can interfere, and must therefore be on different channels. Houses that are sufficiently far away may use the same wi-fi channel. Your goal is to find the minimum number of different channels the neighbourhood requires.

(a) Model this as a graph problem. Remember to:

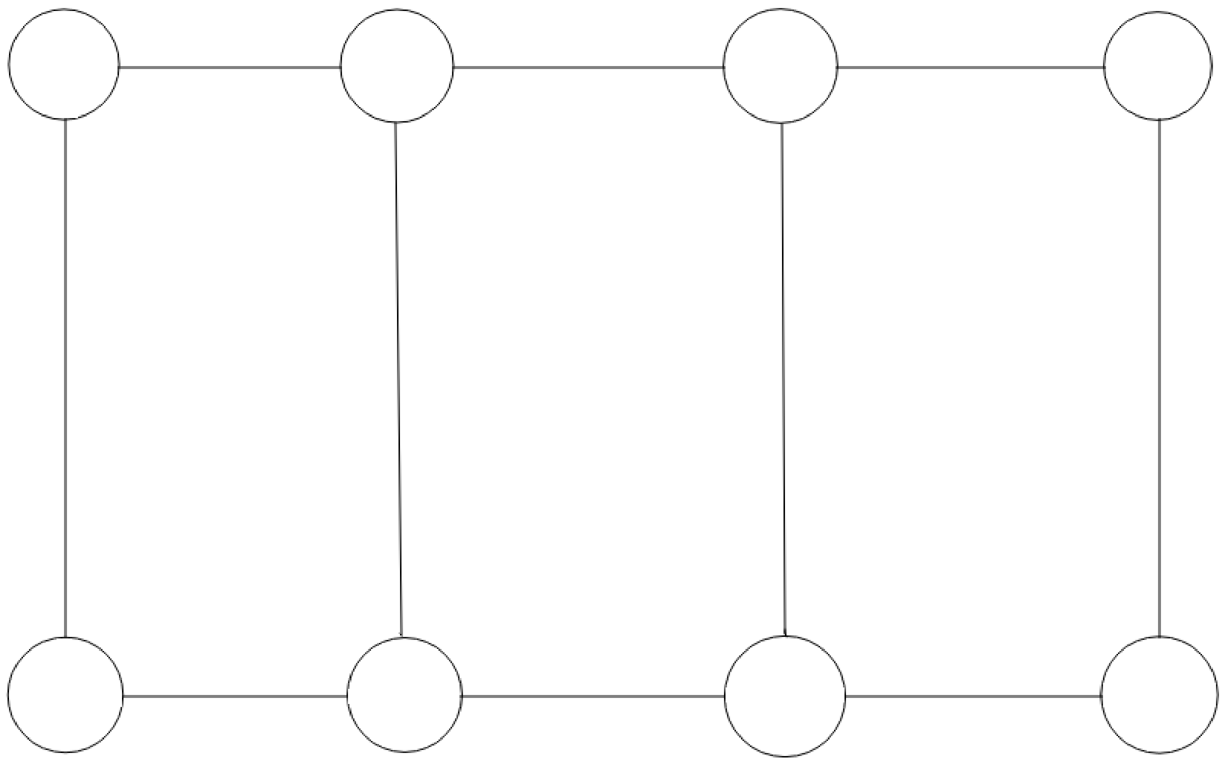
(i) Clearly define the vertices and edges of your graph. (4 marks)

(ii) State the associated graph problem that you need to solve. (2 marks)

(b) Give the solution to the graph problem corresponding to this scenario; and determine the minimum number of wi-fi channels required for the neighbourhood? (2 marks)

(c) How do your answers to (a) and (b) change if a house's wireless network can also interfere with those of the houses to the left and right of the house over the road? (4 marks)

(a)



(i)

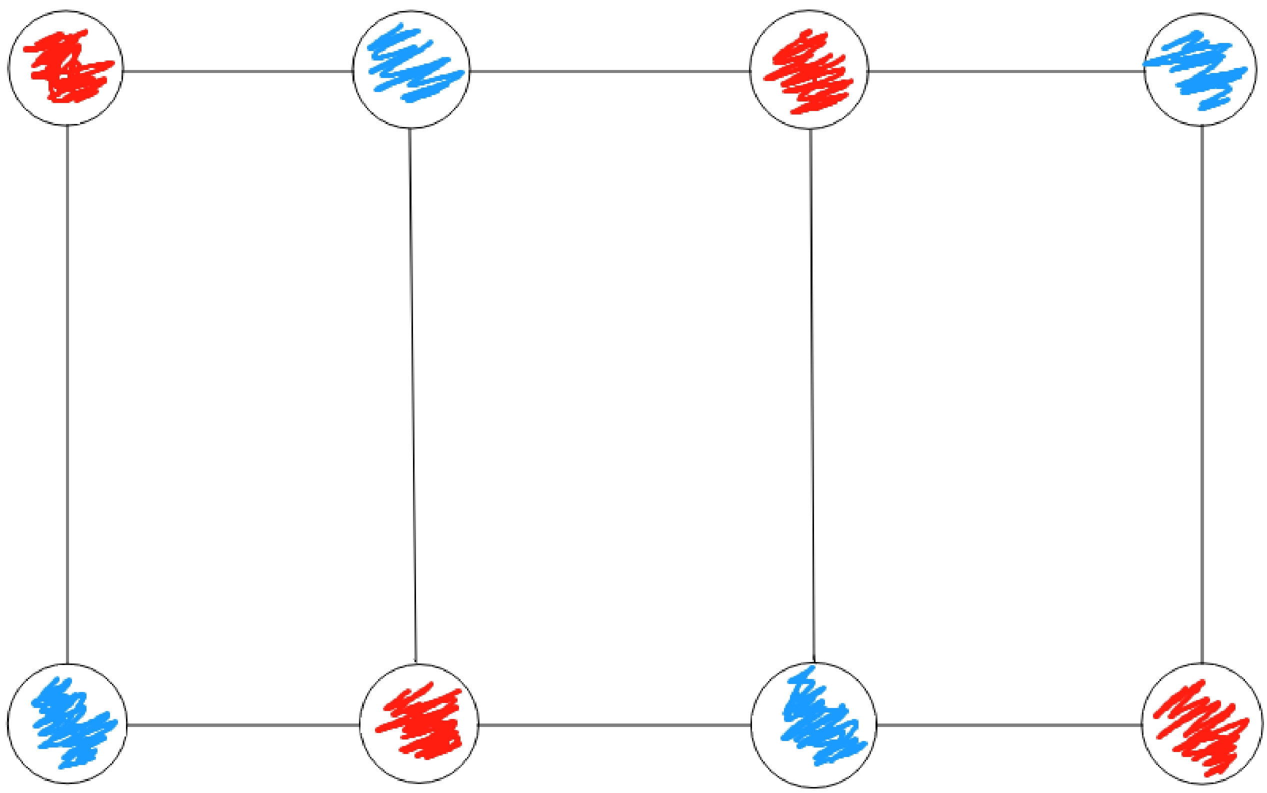
Vertex: Houses

Edge: Neighboring relation among houses

(ii)

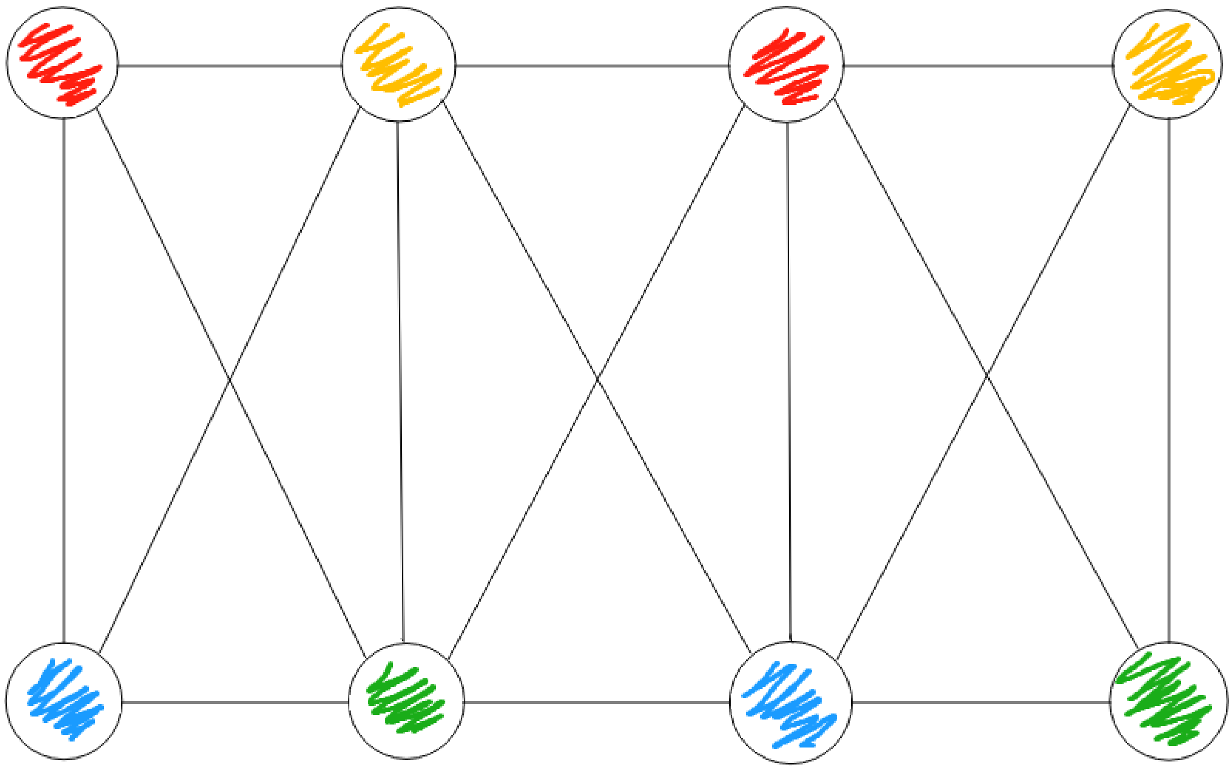
Problem: Find the chromatic number of this graph

(b)



Minimum number: 2

(c)

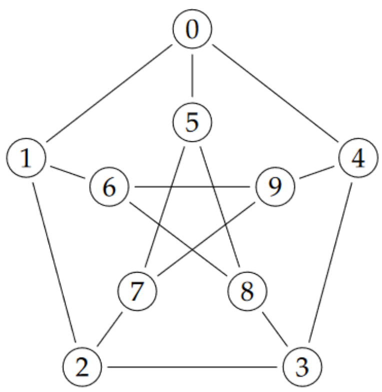


Definition of the graph stays the same while more edges added.

The minimum number becomes 4.

Problem 4

Problem 4 (12 marks)
This is the Petersen graph:



(a) Give an argument to show that the Petersen graph does not contain a subdivision of K_5. (6 marks)

(b) Show that the Petersen graph contains a subdivision of $K_{3,3}$. (6 marks)

(a)

To show G contains a subdivision of H:

Strategy II:

- Start at G
- Perform the following operations as many times as you need:
  1. Delete an edge
  2. Delete a vertex (and all adjacent edges)
  3. Replace a vertex of degree 2 with an edge connecting its neighbours (contracting a vertex)
- Finish with H

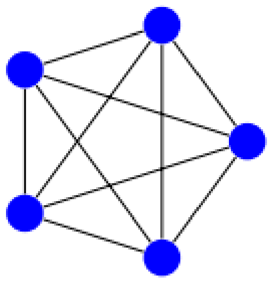
From the definition above, to show G contains a subdivision of H needs to perform the three steps several times.

However, all of the steps either decreases the adjacent edges of certain vertex, or remains the same number of adjacent edges of this vertex.

As shown below, Graph K_5 contains 5 vertexes each of which possesses a degree of 4.

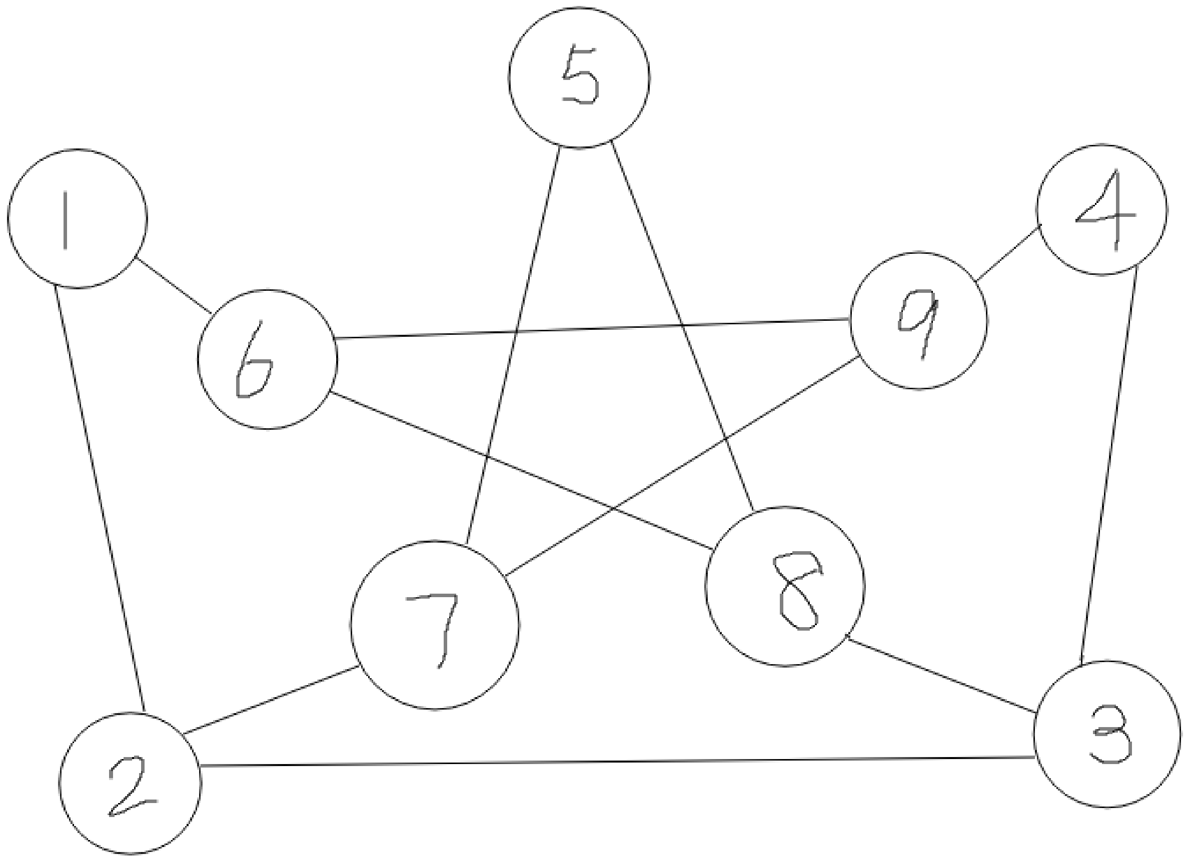
While in the Petersen graph, each vertex only possesses a degree of 3.

K_5 :

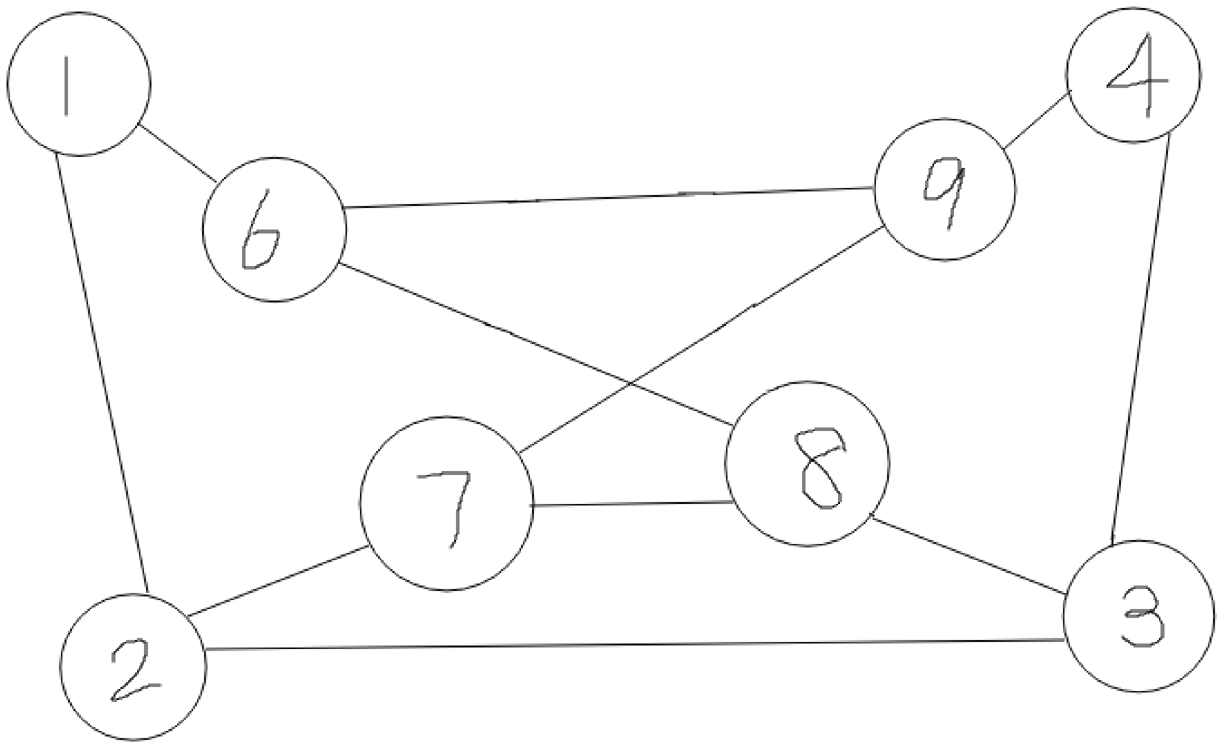


(b)

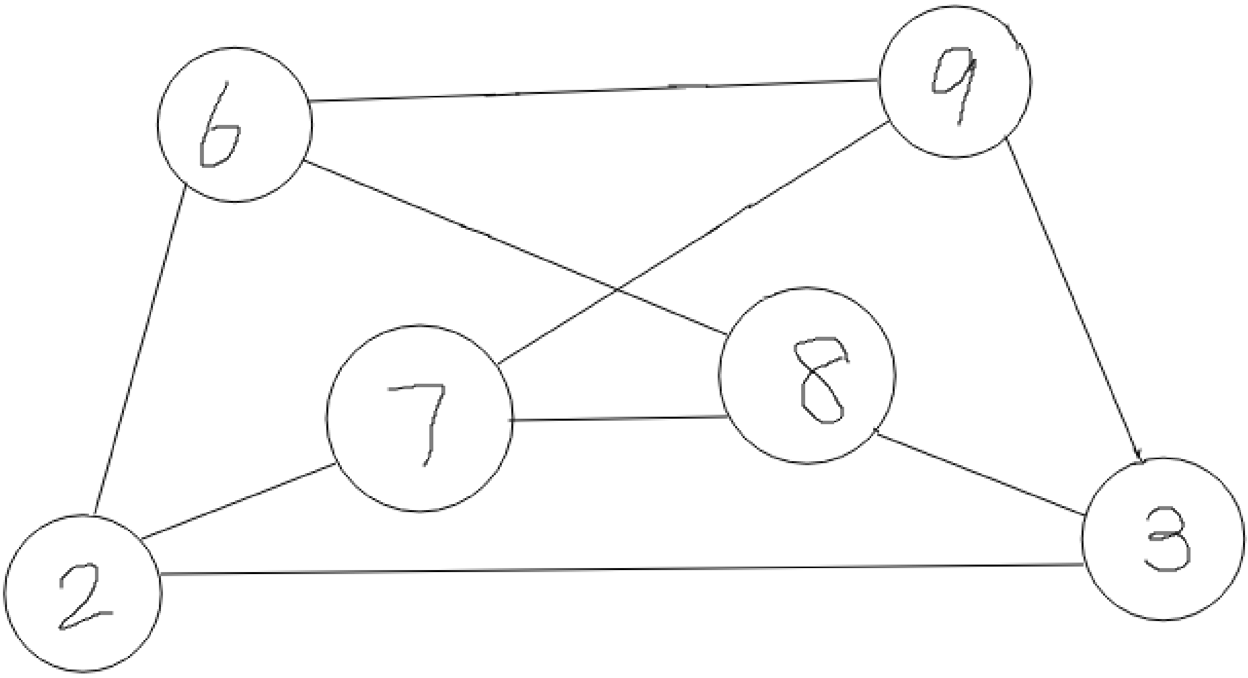
1. Delete Vertex 0:



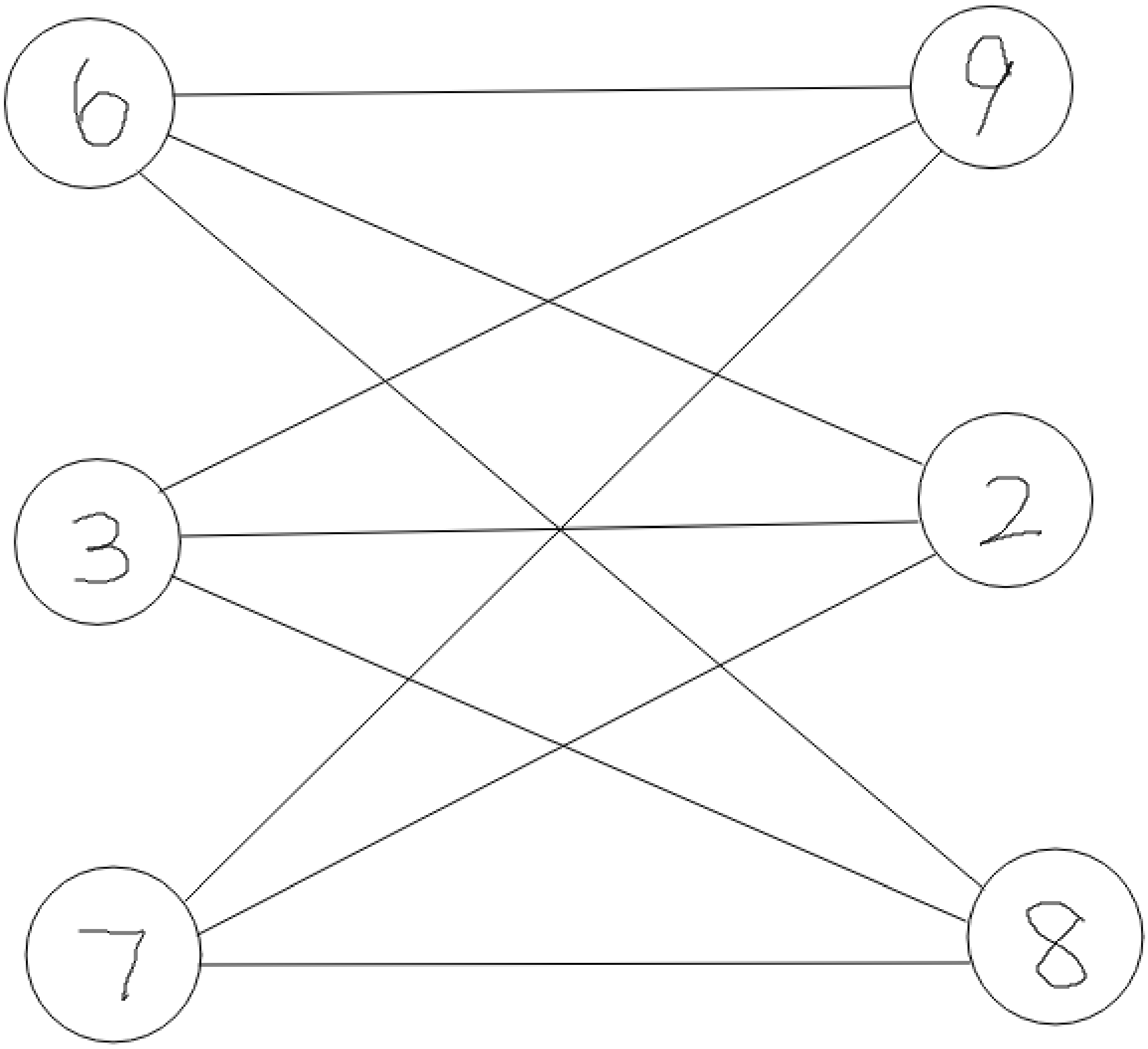
2. Replace Vertex 5:



3. Replace Vertex 1 and 4:



4. Put Vertex 2 to the right side while Vertex 3 to the left side:



This is exactly what $K_{3,3}$ looks like.

Therefore, the Petersen graph contains a subdivision of $K_{3,3}$.

Problem 5

Problem 5 (20 marks)
Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

$$\begin{aligned} R^0 &:= I = \{(x,x) : x \in S\}, \text{ and} \\ R^{n+1} &:= R^n \cup (R; R^n) \text{ for } n \geq 0 \end{aligned}$$

(a) Prove that for all $i, j \in \mathbb{N}$, if $i \leq j$ then $R^i \subseteq R^j$. *Hint: Let $P_i(j)$ be the proposition that $R^i \subseteq R^j$ and prove that $P_i(j)$ holds for all $j \geq i$.* (4 marks)

(b) Let $P(n)$ be the proposition that for all $m \in \mathbb{N}$: $R^n; R^m = R^{n+m}$. Prove that $P(n)$ holds for all $n \in \mathbb{N}$. *Hint: Use results from Assignment 1* (4 marks)

(c) Prove that if there exists $i \in \mathbb{N}$ such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$. (4 marks)

(d) If $|S| = k$, explain why $R^{k^2} = R^{k^2+1}$. (2 marks)

(e) If $|S| = k$, show that R^{k^2} is transitive. (2 marks)

(f)* If $|S| = k$ show that R^{k^2} is the minimum (with respect to $\subseteq$) of all reflexive and transitive relations that contain R. (4 marks)

(a)

Let $P_i(j)$ be the proposition that $R^i \subseteq R^j$.

$[B]$: when $j = i$, $R^i = R^j \subseteq R^j$,

$[I]$: when $j > i$, if $R^i \subseteq R^j$, then $R^i \subseteq R^j \subseteq R^j \cup (R; R^j) = R^{j+1}$.

Therefore, $P_i(j)$ holds for all $j \geq i$.

Therefore, $\forall i, j \in N$, if $i \leq j$ then $R^i \subseteq R^j$.

(b)

$$(R_1; R_2); R_3 = R_1; (R_2; R_3)$$

$$I; R_1 = R_1; I = R_1 \text{ where } I = \{(x,x) : x \in S\}$$

$$(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$$

Above is the results from Assignment 1.

Let $P(n)$ be the proposition that $\forall m \in N : R^n; R^m = R^{n+m}$.

$[B]$: when $n = 0$, $R^0; R^m = I; R^m = R^m$.

$[I]$: when $n > 0$, if $R^n; R^m = R^{n+m}$, then:

$$R^{n+1}; R^m = (R^n \cup (R; R^n)); R^m = (R^n; R^m) \cup ((R; R^n); R^m)$$

$$
\begin{aligned}
&= R^{n+m} \cup (R; (R^n; R^m)) \\
&= R^{n+m} \cup (R; (R^{n+m})) \\
&= R^{n+m+1}
\end{aligned} \tag{15}
$$

Therefore, $P(n)$ holds for all $n \in N$.

(c)

If there exists $i \in N$ such that $R^i = R^{i+1}$,

Let $P(j)$ be the proposition that $\forall j \geq i : R^j = R^i$.

$[B]$: when $j = i$, $R^j = R^i$.

$[I]$: when $j > i$, if $R^j = R^i$, then:

$$R^{j+1} = R^j \cup (R; R^j) = R^i \cup (R; R^i) = R^{i+1} = R^i \tag{16}$$

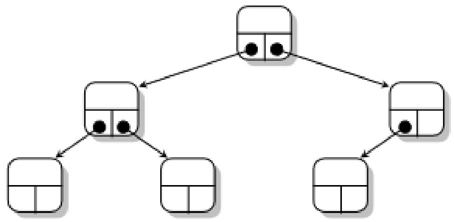
Therefore, $P(j)$ holds for all $j \geq i$.

Therefore, if there exists $i \in N$ such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$.

Problem 6

Problem 6 (20 marks)

A *binary tree* is a data structure where each node is linked to at most two successor nodes:



If we include empty binary trees (trees with no nodes) as part of the definition, then we can simplify the description of the data structure. Rather than saying a node has 0, 1, or 2 successor nodes, we can instead say that a node has exactly two *children*, where a child is a binary tree. That is, we can abstractly define the structure of a binary tree as follows:

- (B): An empty tree, τ
- (R): An ordered pair $(T_{\text{left}}, T_{\text{right}})$ where T_{left} and T_{right} are trees.

So, for example, the above tree would be defined as the tree T where:

$$\begin{aligned} T &= (T_1, T_2), \text{ where} \\ &T_1 = (T_3, T_4) \text{ and } T_2 = (T_5, \tau), \text{ where} \\ &\quad T_3 = T_4 = T_5 = (\tau, \tau) \end{aligned}$$

That is,

$$T = \Big(\big((\tau,\tau),(\tau,\tau)\big),\big((\tau,\tau),\tau\big)\Big)$$

A *leaf* in a binary tree is a node that has no successors (i.e. it is of the form (τ, τ)). A *fully-internal* node in a binary tree is a node that has exactly two successors (i.e. it is of the form (T_1, T_2) where $T_1, T_2 \neq \tau$). The example above has 3 leaves (T_3, T_4, and T_5) and 2 fully-internal nodes (T and T_1). For technical reasons (that will become apparent) we assume that an empty tree has 0 leaves and -1 fully-internal nodes.

(a) Based on the recursive definition above, recursively define a function $\mathsf{count}(\mathsf{T})$ that counts the number of nodes in a binary tree T. (4 marks)

(b) Based on the recursive definition above, recursively define a function $\mathsf{leaves}(\mathsf{T})$ that counts the number of leaves in a binary tree T. (4 marks)

(c) Based on the recursive definition above, recursively define a function $\mathsf{internal}(\mathsf{T})$ that counts the number of fully-internal nodes in a binary tree T. (4 marks)

(d) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves}(\mathsf{T}) = \mathsf{internal}(\mathsf{T}) + 1$. Prove that $P(\mathsf{T})$ holds for all binary trees T. Your proof should be based on your answers given in (b) and (c). (8 marks)

(a)

$count(T):$

$(B)\ if\ (T = \tau) : 0$

$(R)\ else:\ 1 + count(T_{left}) + count(T_{right})$

(b)

$leaves(T):$

$(B)\ if\ (T = \tau) : 0$

$(R)\ if\ (T_{left} = T_{right} = \tau) : 1$

$else:\ leaves(T_{left}) + leaves(T_{right})$

(c)

$internal(T):$

$(B)\ if\ (T = \tau) : -1$

$(R)\ if\ (T_{left} \neq \tau\ \ and\ \ T_{right} \neq \tau) : 1 + internal(T_{left}) + internal(T_{right})$

$if\ (T_{left} = T_{right} = \tau) : 0$

$if\ (T_{left} = \tau) : internal(T_{right})$

$if\ (T_{right} = \tau) : internal(T_{left})$