T1. $T_{2n} = \frac{T_n}{2} + \frac{h}{2}\sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}})$

$T_m^{(k)} = \frac{4^m}{4^m-1} T_{m-1}^{(k+1)} - \frac{1}{4^m-1} T_{m-1}^{(k)}$

$T_0^{(0)} = T_1 = (\frac{1}{2}f(a) + \frac{1}{2}f(b))(b-a) = 0.6\times(-0.166667-0.555556) = -0.433334$

$T_0^{(1)} = T_2 = \frac{T_1}{2} + \frac{0.6}{2}\times f(1.3) = \frac{-0.433334}{2} + 0.3\times -0.562771 = -0.385498$

$T_1^{(0)} = \frac{4}{4-1}T_0^{(1)} - \frac{1}{4-1}T_0^{(0)} = -0.369542$

$T_0^{(2)} = T_4 = \frac{T_2}{2} + \frac{0.3}{2}(f(1.15)+f(1.45)) = \frac{-0.385498}{2} + 0.3\times(-0.429505-0.764163) = -0.371799$

$T_1^{(1)} = \frac{4^1}{4^1-1}T_0^{(2)} - \frac{1}{4^1-1}T_0^{(1)} = -0.367233$

$T_2^{(0)} = \frac{4^2}{4^2-1}T_1^{(1)} - \frac{1}{4^2-1}T_1^{(0)} = -0.367079$

$T_0^{(3)} = T_8 = \frac{T_4}{2} + \frac{0.15}{2}(f(1.075)+f(1.225)+f(1.375)+f(1.525))$

$= \frac{-0.371799}{2} + \frac{3}{40}(-0.377939-0.490123-0.651852-0.910788) = -0.368202$

$T_1^{(2)} = \frac{4^1}{4^1-1}T_0^{(3)} - \frac{1}{4^1-1}T_0^{(2)} = -0.367003$

$T_2^{(1)} = \frac{4^2}{4^2-1}T_1^{(2)} - \frac{1}{4^2-1}T_1^{(1)} = -0.366988$

$T_3^{(0)} = \frac{4^3}{4^3-1}T_2^{(1)} - \frac{1}{4^3-1}T_2^{(0)} = -0.366987$

$$\begin{array}{llll}
T_0^{(0)} & & & \\
T_0^{(1)} \to & T_1^{(0)} & & \\
T_0^{(2)} \to & T_1^{(1)} \to & T_2^{(0)} & \\
T_0^{(3)} \to & T_1^{(2)} \to & T_2^{(1)} \to & T_3^{(0)}
\end{array}$$

T2. 解. $\int_0^1 \frac{2}{\sqrt[3]{x}}dx = \int_0^1 2x^{-\frac{1}{3}}dx = 3x^{\frac{2}{3}}\Big|_0^1 = 3$

$\int_0^1 \frac{\cos 2x}{\sqrt[3]{x}}dx = \int_0^1 \frac{1-2\sin^2 x}{\sqrt[3]{x}}dx = \int_0^1 \frac{1}{\sqrt[3]{x}}dx - \int_0^1 \frac{2\sin^2 x}{\sqrt[3]{x}}dx = \frac{3}{2} - \int_0^1 \frac{2\sin^2 x}{\sqrt[3]{x}}dx$

设 $f(x) = \frac{2\sin^2 x}{\sqrt[3]{x}}$ $\therefore \int_0^1 f(x)dx = \frac{h}{6}[f(0)+2(f(\frac{1}{3})+f(\frac{2}{3}))+4(f(\frac{1}{6})+f(\frac{1}{2})+f(\frac{5}{6}))+f(1)]$

其中 $f(0) = \lim_{x\to 0}\frac{2\sin^2 x}{\sqrt[3]{x}} = \lim_{x\to 0}\frac{2\sin 2x}{\frac{1}{3x^{\frac{2}{3}}}} = 0$

$\therefore \int_0^1 f(x)dx = \frac{1}{18}[0+2(0.308804+0.895434)+4(0.100020+0.579183+1.164380)+1.416147]$

$= 0.619942$

$\therefore \int_0^1 \frac{\cos 2x}{\sqrt[3]{x}} = 1.5 - 0.619942 = 0.880058$

T3. (1) $f(x)=1$ $\int_0^1 f(x)dx \approx \frac{1}{2}+C_1 = 1$ $\Rightarrow$ $C_1=\frac{1}{2}$

$f(x)=x$ $\int_0^1 f(x)dx \approx \frac{1}{2}x_0+C_1x_1 = \frac{1}{2}$ $\Rightarrow$ $x_0+x_1=1$

$f(x)=x^2$ $\int_0^1 f(x)dx \approx \frac{1}{2}x_0^2+C_1x_1^2 = \frac{1}{3}$ $\Rightarrow$ $x_0^2+x_1^2=\frac{2}{3}$

$\Rightarrow \begin{cases} x_0=\frac{3-\sqrt{3}}{6} \\ x_1=\frac{3+\sqrt{3}}{6} \end{cases}$

$f(x)=x^3$ $\int_0^1 f(x)dx \approx \frac{1}{4}$. $\frac{1}{2}\left(\frac{3-\sqrt{3}}{6}\right)^3+\frac{1}{2}\left(\frac{3+\sqrt{3}}{6}\right)^3=\frac{1}{4}$ 满足

$f(x)=x^4$ $\int_0^1 f(x)dx \approx \frac{1}{5}$. $\frac{1}{2}\left(\frac{3-\sqrt{3}}{6}\right)^4+\frac{1}{2}\left(\frac{3+\sqrt{3}}{6}\right)^4 \neq \frac{7}{36}$ 不满足

$\therefore$ 求积公式为 $\frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}\right)+\frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}\right)$. 具有3次代数精度

(2). $f(x)=1$. $\int_0^h f(x)dx \approx h+0=h$ 满足

$f(x)=x$ $\int_0^h f(x)dx \approx \frac{h^2}{2}+0=\frac{h^2}{2}$ 满足

$f(x)=x^2$ $\int_0^h f(x)dx \approx \frac{h^3}{2}-2ah^3=\frac{h^3}{3}$ $\Rightarrow$ $a=\frac{1}{12}$

$f(x)=x^3$ $\int_0^h f(x)dx \approx \frac{h^4}{2}-\frac{1}{4}h^4=\frac{h^4}{4}$ 满足

$f(x)=x^4$ $\int_0^h f(x)dx \approx \frac{h^5}{2}-\frac{h^5}{3} \neq \frac{h^5}{5}$ 不满足

$\therefore a=\frac{1}{12}$. 代数精度为3

T4. $n=4$ 分为四个小段 $[0,\frac{\pi}{4}]$, $[\frac{\pi}{4},\frac{\pi}{2}]$ $[\frac{\pi}{2},\frac{3\pi}{4}]$. $[\frac{3\pi}{4},\pi]$

① $\varphi=\frac{\frac{\pi}{4}}{2}+\frac{t}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}+\frac{\pi}{8}t$ $t\in[-1,1]$

$\int_0^{\frac{\pi}{4}} f(\varphi)d\varphi=\frac{\pi}{8}[A_0F(t_0)+A_1F(t_1)]$ $F(t)=f\left(\frac{\pi}{8}+\frac{\pi}{8}t\right)$

$\therefore t_0=-\frac{1}{\sqrt{3}}$, $t_1=\frac{1}{\sqrt{3}}$, $A_0=1$. $A_1=1$

$\int_0^{\frac{\pi}{4}} f(\varphi)d\varphi=\frac{\pi}{8}F\left(-\frac{1}{\sqrt{3}}\right)+\frac{\pi}{8}F\left(\frac{1}{\sqrt{3}}\right)=0.089263$

② $\varphi=\frac{3}{8}\pi+\frac{\pi}{8}t$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(\varphi)d\varphi=\frac{\pi}{8}f\left(\frac{3}{8}\pi-\frac{\pi}{8\sqrt{3}}\right)+\frac{\pi}{8}f\left(\frac{3\pi}{8}+\frac{\pi}{8\sqrt{3}}\right)=1.053751$

③ $\varphi=\frac{5}{8}\pi+\frac{\pi}{8}t$

$\int_{\frac{\pi}{2}}^{\frac{3}{4}\pi} f(\varphi)d\varphi=\frac{\pi}{8}f\left(\frac{5}{8}\pi-\frac{\pi}{8\sqrt{3}}\right)+\frac{\pi}{8}f\left(\frac{5}{8}\pi+\frac{\pi}{8\sqrt{3}}\right)=2.702065$

④ $\varphi=\frac{7}{8}\pi+\frac{\pi}{8}t$

$\int_{\frac{3}{4}\pi}^{\pi} f(\varphi)d\varphi=\frac{\pi}{8}f\left(\frac{7}{8}\pi-\frac{\pi}{8\sqrt{3}}\right)+\frac{\pi}{8}f\left(\frac{7}{8}\pi+\frac{\pi}{8\sqrt{3}}\right)=2.024768$

$\therefore \int_0^{\pi}\varphi^2\sin\varphi\, d\varphi=5.869847$

T5. $L=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sqrt{1+\frac{1}{\cos^4 x}}\,dx$ 积分表达式

令 $f(x)=\sqrt{1+\frac{1}{\cos^4 x}}=\frac{\sqrt{1+\cos^4 x}}{\cos^2 x}$，将区间分为n份. $h=\frac{\pi}{2n}$. $(-\frac{\pi}{4},\frac{\pi}{4})$

$T_n=\sum_{k=0}^{n-1}\frac{h}{2}(f(x_k)+f(x_{k+1}))=(f(a)+f(b)+\sum_{k=1}^{n-1}f(x_k))\frac{\pi}{4n}$

$=\frac{\pi}{4n}\left(f(\frac{\pi}{4})+f(\frac{\pi}{4})+\sum_{k=1}^{n-1}f(x_k)\right)$

$x_k=-\frac{\pi}{4}+k\cdot\frac{\pi}{2n}\quad k=1.2.\cdots n-1$

$R_n(f)=-\frac{b-a}{12}h^2f''(\eta)=-\frac{\pi^3}{96n^2}f''(\eta)$

$f'(x)=\frac{2\sin x}{\cos^5 x\sqrt{1+\frac{1}{\cos^4 x}}}\qquad f''(x)=\frac{\cos^4 x(8\sin^2 x+2)+2+4\sin^2 x}{(\cos^8 x+\cos^4 x)\sqrt{1+\cos^4 x}}$

$\therefore f''(x)$ 在 $-\frac{\pi}{4}, \frac{\pi}{4}$ 时最大. $f''(\frac{\pi}{4})=f''(-\frac{\pi}{4})=\frac{176}{5\sqrt{5}}=15.7419$

方法误差 $|R_n(f)|=\frac{5.0840}{n^2}$

舍入误差: $T_n=\frac{h}{2}(f(a)+f(b)+2\sum_{k=1}^{n-1}f(x_k))$

$\delta\le\left|\frac{\partial T}{\partial n}\right|\delta h+\sum_{0\le k\le n}\left|\frac{\partial T}{\partial f(x_k)}\right|\delta f(x_k)+\frac{1}{2}\times10^{-m}$

其中. $\delta f(x_k)\le|f'(x_k)|\,\delta x_k+\frac{1}{2}\times10^{-m}$

$\delta_n\ .\ \delta x_k\le\frac{1}{2}\times10^{-m}$