T1. 用梯形法解初值问题. $y'=-2y+2x^2+2x$, $y(0)=1$, 取步长 $h=0.1$, 计算到 $x=0.5$

并列出数值解和解析解 $y=e^{-2x}+x^2$ 的误差

改进欧拉的迭代公式.
$$\begin{cases} \bar{y}_{n+1}=y_n+hf(x_n,y_n) \\ y_{n+1}=y_n+\frac{h}{2}[f(x_n,y_n)+f(x_{n+1},\bar{y}_{n+1})] \end{cases}$$

直接整理改进欧拉法公式. $y_{n+1}=y_n+\frac{h}{2}[-2y_n+2x_n^2+2x_n-2y_{n+1}+2x_{n+1}^2+2x_{n+1}]$

$$(1+h)y_{n+1}=y_n-hy_n+hx_n^2+hx_n+hx_{n+1}^2+hx_{n+1}$$

$$y_{n+1}=\frac{1-h}{1+h}y_n+\frac{h}{1+h}x_n(x_n+1)+\frac{h}{1+h}x_{n+1}(x_{n+1}+1)$$

解方程计算 y_n: $x_0=0$ $y_0=1$ $x_1=0.1$ $y_1=0.82818$. $x_2=0.2$ $y_2=0.70942$

$x_3=0.3$ $y_3=0.63771$. $x_4=0.4$ $y_4=0.60813$. $x_5=0.5$ $y_5=0.61665$

计算解析解值: $y(0)=1$ $y(0.1)=0.82873$ $y(0.2)=0.71032$ $y(0.3)=0.63881$

$y(0.4)=0.60933$ $y(0.5)=0.61788$

∴数值解与解析解误差为: $\Delta_0=0$ $\Delta_1=5.5\times10^{-4}$ $\Delta_2=9\times10^{-4}$ $\Delta_3=1.1\times10^{-3}$

$\Delta_4=1.2\times10^{-3}$ $\Delta_5=1.23\times10^{-3}$

T2 证明. $y_{n+2}+(b-1)y_{n+1}-by_n=\frac{1}{4}h[(b+3)y'_{n+2}+(3b+1)y'_n]$

当 $b\neq-1$ 时是二阶的, 当 $b=-1$ 时是三阶的

$$y_{n+2}=y(x_n)+2hy'(x_n)+2h^2y''(x_n)+\frac{4}{3}h^3y'''(x_n)+\frac{2}{3}h^4y^{(4)}(x_n)+\cdots \qquad y_n=y(x_n)$$

$$y_{n+1}=y(x_n)+hy'(x_n)+\frac{h^2}{2}y''(x_n)+\frac{h^3}{6}y'''(x_n)+\frac{h^4}{24}y^{(4)}(x_n)+\cdots \qquad y'_n=y'(x_n)$$

$$y'_{n+2}=y'(x_n)+2hy''(x_n)+2h^2y'''(x_n)+\frac{4}{3}h^3y^{(4)}(x_n)+\cdots$$

代入原式: $y(x_n)\cdot(1+b-1-b)+hy'(x_n)(2+b-1)+\frac{h^2}{2}y''(x_n)(4+b-1)+\frac{h^3}{6}y^{(3)}(x_n)(8+b-1)+\frac{h^4}{24}y^{(4)}(x_n)(16+b-1)+\cdots$

$$=hy'(x_n)\left(\frac{3b+1}{4}+\frac{b+3}{4}\right)+h^2y''(x_n)\cdot\frac{b+3}{2}+h^3y'''(x_n)\cdot\frac{b+3}{2}+h^4y^{(4)}(x_n)\cdot\frac{b+3}{3}+\cdots$$

对应项比较. 对 $y(x_n)$ 项系数, 等式两边均为0. 对 $hy'(x_n)$ 项系数, 两边均为 $(1+b)$. 对 $h^2y''(x_n)$ 项系数, 两边均为 $\left(\frac{b+3}{2}\right)$

但对于 $h^3y^{(3)}(x_n)$ 项系数, 左 $=\frac{7+b}{6}$ 右 $=\frac{b+3}{2}$ 当且仅当 $b=-1$ 时, 两边相等. 而 $b=-1$ 时对 $h^4y^{(4)}(x_n)$ 两边系数不等

∴当 $b\neq-1$ 时, 原式是二阶的, 当 $b=-1$ 时, 原式是三阶的

T3. 试推导 $y_{n+3}=\frac{1}{8}(9y_{n+2}-y_n)+\frac{3}{8}h(y'_{n+3}+2y'_{n+2}-y'_{n+1})$ 的局部截断误差主项

$y(x_{n+3})=y(x_n)+3hy'(x_n)+\frac{9}{2}h^2y''(x_n)+\frac{9}{2}h^3y'''(x_n)+\frac{27}{8}h^4y^{(4)}(x_n)+\frac{81}{40}h^5y^{(5)}(x_n)+\cdots$

$y_{n+2}=y(x_n)+2hy'(x_n)+2h^2y''(x_n)+\frac{4}{3}h^3y'''(x_n)+\frac{2}{3}h^4y^{(4)}(x_n)+\frac{4}{15}h^5y^{(5)}(x_n)+\cdots$ $\quad y_n=y(x_n)$

$y'_{n+3}=y'(x_n)+3hy''(x_n)+\frac{9h^2}{2}y'''(x_n)+\frac{9}{2}h^3y^{(4)}(x_n)+\frac{27}{8}h^4y^{(5)}(x_n)+\cdots$

$y'_{n+2}=y'(x_n)+2hy''(x_n)+2h^2y'''(x_n)+\frac{4}{3}h^3y^{(4)}(x_n)+\frac{2}{3}h^4y^{(5)}(x_n)+\cdots$

$y'_{n+1}=y'(x_n)+hy''(x_n)+\frac{h^2}{2}y'''(x_n)+\frac{h^3}{6}y^{(4)}(x_n)+\frac{1}{24}h^4y^{(5)}(x_n)+\cdots$

$\therefore y_{n+3}=\frac{1}{8}(9y_{n+2}-y_n)+\frac{3}{8}h(y'_{n+3}+2y'_{n+2}-y'_{n+1})$

$=y(x_n)\cdot(\frac{9}{8}-\frac{1}{8})+hy'(x_n)\cdot(\frac{9}{4}+\frac{3}{8}\times2)+h^2y''(x_n)(\frac{9}{4}+\frac{9}{4})+h^3y'''(x_n)(\frac{3}{2}+3)+h^4y^{(4)}(x_n)(\frac{3}{4}+\frac{21}{8})$

$+h^5y^{(5)}(x_n)(\frac{3}{10}+\frac{7}{4})+\cdots$

$=y(x_n)+3hy'(x_n)+\frac{9}{2}h^2y''(x_n)+\frac{9}{2}h^3y'''(x_n)+\frac{27}{8}h^4y^{(4)}(x_n)+\frac{82}{40}h^5y^{(5)}(x_n)+\cdots$

$\therefore R=y(x_{n+3})-y_{n+3}\approx-\frac{1}{40}h^5y^{(5)}(x_n)=O(h^5)$

$\therefore$ 局部截断误差主项为 $-\frac{1}{40}h^5y^{(5)}(x_n)$

T4. 已知 $y'=x^2+y^2$ 和 $y(1)=4$ 采用改进欧拉法求取 $y(0)$. 结果应精确到小数点后第8位. 假设每次迭代运算中加减乘除运算不产生误差, 每次迭代计算的结果均用精确到小数点后某位的小数进行保存. 试给出具体的算法（有推导过程）

[考虑 $x\in[0,1]$ 上 y 的取值范围，无需计算出步长 h 的具体值]

设 $x\in[0,1]$ 上迭代计算的步长为 h $\because y'>0$ $\therefore y<4$ $\therefore y'<y'(1)=17$

$\therefore \left|\frac{\partial f}{\partial y}(x,y)\right|\le M=8$, $|y^{(2)}(x)|\le L=138$, $|y^{(3)}(x)|\le T=1684$

$x_0=1$, $y_0=4$ 由步长, 可知迭代次数 $n=\frac{1}{h}$

由改进欧拉法的方法误差: $\Delta_{n+1}\le(1+hM+\frac{h^2}{2}M^2)\Delta_n+(\frac{ML}{4}+\frac{T}{12})h^3$

$\therefore \Delta_{n+1}+\frac{1}{hM+\frac{h^2M^2}{2}}(\frac{ML}{4}+\frac{T}{12})h^3\le(1+hM+\frac{h^2}{2}M^2)[\Delta_n+\frac{1}{hM+\frac{h^2M^2}{2}}(\frac{ML}{4}+\frac{T}{12})h^3]$

$\cdots\le(1+hM+\frac{h^2M^2}{2})^{n+1}[\Delta_0+\frac{1}{hM+\frac{h^2M^2}{2}}(\frac{ML}{4}+\frac{T}{12})h^3]$

由改进欧拉法的舍入误差: $\delta_{n+1}\le(1+hM+\frac{h^2}{2}M^2)\delta_n+(1+\frac{hM}{2})\cdot\frac{1}{2}\cdot10^{-m}$ （m 为迭代计算时精确到小数点后位数）

$\therefore \delta_{n+1}+\frac{1}{hM+\frac{h^2M^2}{2}}(1+\frac{hM}{2})\cdot\frac{1}{2}\cdot10^{-m}\le(1+hM+\frac{h^2M^2}{2})[\delta_n+\frac{1}{hM+\frac{h^2M^2}{2}}(1+\frac{hM}{2})\cdot\frac{1}{2}\cdot10^{-m}]$

$\because$ 整体误差为 $\Delta_{n+1}+\delta_{n+1}$ 而由题目要求有. $\Delta_{n+1}+\delta_{n+1}\leq \frac{1}{2}\times 10^{-8}$

不失一般性可令 $\delta_{n+1}\leq \frac{1}{4}\times 10^{-8}$ $\quad \Delta_{n+1}\leq \frac{1}{4}\times 10^{-8}$

$\therefore$ 有 $\begin{cases} \left[\left(1+hM+\frac{h^2M^2}{2}\right)^{n+1}-1\right]\left[\frac{1}{hM+\frac{h^2M^2}{2}}\left(\frac{ML}{4}+\frac{T}{12}\right)h^3\right]\leq \frac{1}{4}\times 10^{-8}. \text{ 其中 } M=8.\ L=138.\ T=1684.\ h=\frac{1}{n} \\ \left[\left(1+hM+\frac{h^2M^2}{2}\right)^{n+1}-1\right]\left[\frac{1}{hM+\frac{h^2M^2}{2}}\left(1+\frac{hM}{2}\right)\cdot\frac{1}{2}\cdot 10^{-m}\right]\leq \frac{1}{4}\times 10^{-8} \text{ 其中 } M=8.\ h=\frac{1}{n}.\ \text{求 } m. \end{cases}$