Study Guide for Section 1.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

It is important to understand how mathematicians use statements and it is especially important to have a thorough understanding of how conditional statements are used in mathematics. In fact, much of what we study in this text will depend upon a thorough understanding of conditional statements.

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is a statement? How do mathematicians decide if a statement is true or false?

2. What is a conditional statement? What is the hypothesis of a conditional statement and what is the conclusion of a conditional statement?

3. Under what conditions is a conditional statement false? Under what conditions is it true?

4. What does it mean to say that a set of numbers is closed under a specified operation (such as addition or multiplication)?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 for Section 1.1 and then study the answers for these beginning activities.

2. Study Section 1.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the Screencasts for Section 1.1 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Statements and Non-Statements (Screencast 1.1.1) (5:28)
- How Do We Know if a Statement is True? (Screencast 1.1.2) (6:54)
- Conditional Statements (Screencast 1.1.3)(5:33)
- When Are Conditional Statements True? (Screencast 1.1.4)(5:40)
- Truth Tables for Condional Statements (Screencast 1.1.5) (4:57)

4. Work on exercises 1, 2, 6(a, b, c), and 9 in Section 1.1 to test your understanding of the material.

Study Guide for Section 1.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

The main topic in this section is constructing and writing proofs in mathematics. In particular, we will learn the basics of writing a proof of a conditional statement. So it is essential that we understand conditional statements as studied in Section 1.1. In addition, properties of number systems and the definition of even and odd integers will be important as we learn how to construct and write proofs. In particular, students probably have not used precise, formal definitions of mathematical concepts in the way that we must do when writing proofs. So pay particular attention to the definitions in this section and understand them thoroughly.

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the formal definition of an even integer? What is the formal definition of an odd integer?
2. What is a "backward question" and how is one used to help prove a conditional statement? What is a "forward question" and how is one used to help prove a conditional statement?
3. What is a mathematical proof? What are some writing guidelines we should use when writing a mathematical proof?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 for Section 1.2 and then study the answers for these beginning activities.
2. Study Section 1.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the Screencasts for Section 1.2 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Working with Definitions (Screencast 1.2.1) (9:26)
   - Working with Definitions (Screencast 1.2.1 part (b)) (8:42)
   - Direct Proofs of Conditional Statements part (1) (Screencast 1.2.2) (8:24)
   - Direct Proofs of Conditional Statements part (1) (Screencast 1.2.3) (8:32)
   - Writing Up a Proof into a Paragraph (Screencast 1.2.4) (9:16)
4. Work on exercises 1(a), 2(c), 3(a, b), 4, 5(b), and 9 in Section 1.2 to test your understanding of the material.

Study Guide for Section 2.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Before learning more about how to construct mathematical proofs, we are going to learn some basic rules of logic. We will focus on those parts of logic that can be used to justify some of the methods of proof that we will study in Chapter 3. We actually started this in Chapter 1 by studying the conditional statement and its truth table. In this section, we will learn about other ways to form new statements from old statements. This is done with the use of what we call logical operators or connectives, which are often simple words such as "and", "or", and "not." In many ways, we use these connectives in the same way we use them in day-to-day life. One exception might be how mathematicians use the word "or." So pay close attention to how the connective "or" is used in mathematics.

Focus Questions

By the end of Section 2.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the conjunction of two statements? What is the truth table for the conjunction of statements P and Q?
2. What is the disjunction of two statements? What is the truth table for the disjunction of statements P and Q?
3. What is the negation of a statement? What is the truth table for the negation of the statement P?
4. What is a conditional statement? What is the truth table for a conditional statement?
5. What is a biconditional statement? What is the truth table for a biconditional statement?
6. What is a tautology? What is a contradiction?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 2.1 and then study the answers for these beginning activities.
2. Study Section 2.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the screencasts for Section 2.1 on the GVSU YouTube Math Channel:
   `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Negations of Simple Statements (Screencast 2.1.1) (3:34)
   - Truth Tables, Part 1 (Screencast 2.1.2) (6:07)

(Continued on the next page)

- Truth Tables, Part 2 (Screencast 2.1.3) (5:51)
- Truth Tables, Part 3 (Screencast 2.1.4) (6:17)
- Truth Tables, Part 4 (Screencast 2.1.5) (7:04)
- Truth Tables, Part 5 (Screencast 2.1.6) (3:25)
- Tautologies and Contradictions, Part 1 (Screencast 2.1.7) (6:44)
- Tautologies and Contradictions, Part 2 (Screencast 2.1.8)(4:18)

4. Work on exercises 1, 2, 5, 7, and 11 in Section 2.1 to test your understanding of the material.

Study Guide for Section 2.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

The important concept of logical equivalence is introduced in Preview Activity 1. Logical equivalencies will be the basis for many of the methods of proof that we will study in Chapter 3. Some important logical equivalencies are listed at the end of the section in Theorem 2.8. Pay particular attention to the logical equivalencies in Theorem 2.5 (De Morgan's Laws) and the logical equivalencies in Theorem 2.6. Another important concept introduced in this section is the negation of a conditional statement.

Focus Questions

By the end of Section 2.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What are logically equivalent statements? How can we use truth tables to show that two statements are logically equivalent?

2. What is the converse of a conditional statement? What is the contrapositive of a conditional statement?

3. How do we write the negation of a conditional statement?

4. What are some of the important logical equivalencies?

5. How can we use previously proven logical equivalencies to prove new logical equivalenices?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 for Section 2.2 and then study the answers for these beginning activities.

2. Study Section 2.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the screencasts for Section 2.2 on the GVSU YouTube Math Channel:
`http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Logical Equivalence (Screencast 2.2.1) (8:03)
- Converses and Contrapositives (Screencast 2.2.2) (7:00)
- Negations of Conditional Statements (Screencast 2.2.3) (4:29)
- Proving Logical Equivalence without Truth Tables (Screencast 2.2.4) (7:08)

4. Work on exercises 1, 2, 3(a, b, c, d, e), and 7 in Section 2.2 to test your understanding of the material.

Study Guide for Section 2.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Set theory is important in many areas of mathematics and basic concepts of sets and set notation will be introduced in this section. One thing that sometimes causes difficulty when learning to reason and write in mathematics is the proper use of mathematical notation. It is important to understand and use the notation of set theory correctly. This section also introduces the important concept of open sentences and then how an open sentence can be used to define a set. This is done through something called set builder notation, which is a very important way to define sets that is used throughout mathematics.

Focus Questions

By the end of Section 2.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. How do we specify the elements of a set using the roster method?
2. What does it mean to say that two sets are equal? What does it mean to say that one set is a subset of another set?
3. What is the difference between an open sentence and a statement?
4. What is the truth set of an open sentence?
5. How do we specify the elements of a set using set builder notation?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 for Section 2.3 and then study the answers for these beginning activities.
2. Study Section 2.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the Screencasts for Section 2.3 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Sets and Set Notation (Screencast 2.3.1) (5:35)
   - Open Sentences and Truth Sets (Screencast 2.3.2) (7:24)
   - Elements, Subsets, and Set Equality (Screencast 2.3.3) (7:02)
   - Set Builder Notation (Screencast 2.3.4) (9:25)
4. Work on exercises 1, 2, 3, 4(a, b), and 5(a, c) in Section 2.3 to test your understanding of the material.

Study Guide for Section 2.4

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In the previous section, the concept of an open sentence was introduced. In this section, we will use open sentences along with something called a quantifier to create statements. For example, the sentence, "$x^2 > 0$" is an open sentence. However, the sentence, "For every real number x, $x^2 > 0$" is a statement. The phrase " for every" is a quantifier (called a universal quantifier). Most of the statements and propositions we will encounter in the rest of the text will be quantified statements. So every mathematician must understand quantifiers and quantified statements. Other important ideas introduced in this section are negations of quantified statements, counterexamples, and negations of conditional statements.

Focus Questions

By the end of Section 2.4, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is a universal quantifier? What is an existential quantifier?

2. What are some of the common forms of quantified statements?

3. How do we write negations of quantified statements? How are quantifiers used in definitions?

4. What is a counterexample? For what reason do we use counterexamples?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 for Section 2.4 and then study the answers for these beginning activities.

2. Study Section 2.4 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the Screencasts for Section 2.4 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Quantified Statements (Screencast 2.4.1) (10:04)
- Negating Quantified Statements (Screencast 2.4.2) (10:01)

4. Work on exercises 1, 2(a, b, f), 3(a, c, e, h), 4(a, e), 5(a, e), and 6 in Section 2.4 to test your understanding of the material.

Study Guide for Section 3.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In one sense, this section is a continuation of Section 1.2 in that we will continue our study of constructing and writing direct proofs of conditional statements. One difference is that the conditional statements will now be in the form of universally quantified conditional statements. So we need to be sure that we understand quantifiers from Section 2.4. Another difference is that we will be constructing and writing proofs about some new mathematical concepts. So we will be learning about the concepts of "divides" and "congruence" in the integers. It is imperative to understand these definitions since we will be using these concepts throughout the rest of the text. In this section, we will also learn how to prove that a statement or proposition is false with the use of a counterexample. The ability to construct and use a counterexample is a very important skill for mathematicians.

Focus Questions

By the end of Section 3.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What do we mean when we say that a nonzero integer divides another integer?
2. How do we use definitions to help us construct direct proofs of conditional statements?
3. How do we use counterexamples in mathematics?
4. What do we mean when we say that an integer a is congruent to an integer b modulo a natural number n? How do we use this definition in constructing proofs about congruence in the integers?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 3.1 and then study the answers for these beginning activities.
2. Study Section 3.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the Screencasts for Section 3.1 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Integer Divisibility (Screencast 3.1.1) (8:53)
   - Direct Proof Involving Divisibility (Screencast 3.1.2) (8:50)
   - Integer Congruence (Screencast 3.1.3) (9:28)
   - Reducing an Integer Modulo n (Screencast 3.1.4) (9:44)
   - Proofs Involving Integer Congruence (Screencast 3.1.5) (8:01)
4. Work on exercises 1(a, c), 2(a, d), 3(b, d, g), and 8 in Section 3.1 to test your understanding of the material.

Study Guide for Section 3.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Most of the statements we prove in mathematics can be written in the form of a conditional statement. However, it is difficult (or sometimes impossible) to write a direct proof of a true conditional statement. So instead of proving a conditional statement with a direct proof, mathematicians often prove a statement that is logically equivalent to the given statement. In this section, we will learn how to prove a conditional statement by proving its contrapositive or by proving another logically equivalent statement. So before studying this section, it would be a good idea to review material in Section 2.2. Pay particular attention to the distinction between the contrapositive and the converse of a conditional statement and the logical equivalencies associated with conditional statements in Theorem 2.8 on page 48.

Focus Questions

By the end of Section 3.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. Why can we prove a conditional statement is true by proving that its contrapositive is true?
2. What is a standard method for proving a biconditional statement is true?
3. Why can we use logical equivalencies to help prove a given statement?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 3.2 and then study the answers for these beginning activities.
2. Study Section 3.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 3.2 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Proof by Contraposition (Screencast 3.2.1) (6:50)
   - Proof by Contraposition, Part 2 (Screencast 3.2.2) (6:48)
   - Proof of Biconditional Statements (Screencast 3.2.3) (7:13)
   - Proof of Biconditional Statements, Part 2 (Screencast 3.2.4) (7:48)
4. Work on exercises 1, 3, 4, 6, and 9 in Section 3.2 to test your understanding of the material.

Study Guide for Section 3.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Proof by contradiction is a standard proof technique used by mathematicians. The basic idea is to assume that a statement is false and show that this leads to a contradiction. This will be explained thoroughly in this section. The most important part of a proof by contradiction is the start of the proof, and this almost always involves forming the negation of a quantified conditional statement. So it is essential to review Section 2.4 before beginning this section. Pay particular attention to Theorem 2.16 and the discussion surrounding this theorem. It is also important to be able to accurately (and quickly) write the correct negation of a quantified conditional statement. (See page 69.)

In the process of studying this section, we will also learn the definitions of rational numbers and irrational numbers and will study the proof of one of the "classic" proofs in mathematics, which is the proof that the square root of 2 is an irrational number.

Focus Questions

By the end of Section 3.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the basic strategy for constructing a proof by contradiction for a conditional statement?

2. What do we assume when we begin a proof by contradiction? What is our goal when using a proof by contradiction?

3. Why might it be useful to use a proof by contradiction to prove that a given real number is irrational? Why might it be useful to use a proof by contradiction to prove that something does not exist?

4. How do we prove that the square root of 2 is irrational?

Resources for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 3.3 and then study the answers for these beginning activities.

2. Study Section 3.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 3.3 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Proof by Contradiction (Screencast 3.3.1) (6:58)
- Proof by Contradiction, Part 2 (Screencast 3.3.2) (5:10)
- Proof by Contradiction, Part 3 (Screencast 3.3.3) (3:45)
- Proof by Contradiction: Irrationality of Sqrt(2) (Screencast 3.3.4) (4:52)

4. Work on exercises 2(a, b, d), 3, 5 and 6(a, b) in Section 3.3 to test your understanding of the material.

Study Guide for Section 3.4

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In this section, we will introduce the strategy of using cases in a proof. This is a valuable tool that is frequently used when constructing a proof. The logical basis for a proof using cases is that $(P \vee Q) \to R$ is logically equivalent to $(P \to R) \wedge (Q \to R)$. This equivalency was introduced in Section 2.2 and is reviewed in Beginning Activity 1. As is suggested by this equivalency, a proof by cases is often used when there is a disjunction in the hypothesis.

The formal definition of the absolute value of a real number is introduced in this section, and as we shall see, when we want to prove something about absolute value, we often use a proof that uses cases.

Focus Questions

By the end of Section 3.4, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the basic concept and strategy for using cases in a proof?

2. What are some common situations in which a proof using cases could be used?

3. What are some additional writing guidelines when using a proof that uses cases?

4. How is the absolute value of a real number formally defined?

5. Why do we often use cases when proving something about absolute value?

Activties for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 3.4 and then study the answers for these beginning activities.

2. Study Section 3.4 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 3.4 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Proof by Cases, Part 1 (Screencast 3.4.1) (4:48)
- Proof by Cases, Part 2 (Screencast 3.4.2) (8:41)
- Proof by Cases, Part 3 (Screencast 3.4.3) (7:51)
- Proof by Cases, Part 4 (Screencast 3.4.4) (7:53)

4. Work on exercises 1, 2, 3, 4, and 6(a) in Section 3.4 to test your understanding of the material.

Study Guide for Section 3.5

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In this section, we continue our study of proofs that use cases, but the focus will be on proofs dealing with integers. We will be setting up cases for integers using the concept of the remainder when one integer is divided by another. The basis for this will be the very important result known as the Division Algorithm. It is essential to have a complete understanding of this result. Even though we may have worked with quotients and remainders in the past, the formal statement of the Division Algorithm is extremely important in setting up cases for the integers.

Another very important idea introduced in this section is how to use congruence modulo n as another way to set up cases for the integers. Pay particular attention to Theorem 3.28, Theorem 3.30, Theorem 3.31, and Corollary 3.32.

Focus Questions

By the end of Section 3.5, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the Division Algorithm and how is it used to write a given integer as a multiple of a quotient plus a remainder?
2. How can we use the Division Algorithm to construct a proof using cases when we are trying to prove something about integers?
3. What are the properties of congruence modulo n as expressed in Theorem 3.28 and Theorem 3.30?
4. How can we use congruence modulo n to construct a proof using cases when we are trying to prove something about integers?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 3.5 and then study the answers for these beginning activities.
2. Study Section 3.5 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the contents of the screencasts for Section 3.5 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - The Division Algorithm (Screencast 3.5.1) (6:52)
   - Using the Division Algorithm to Set Up Proof Cases (Screencast 3.5.2) (6:45)
   - The Division Algorithm and Integer Congruence (Screencast 3.5.3) (6:18)
   - Application to Cryptography (Screencast 3.5.4) (6:25)
4. Work on exercises 2, 4, 5, and 6 in Section 3.5 to test your understanding of the material.

Study Guide for Section 3.6

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

This section is different than the other sections in the text. There are no beginning activities, no progress checks, and no screencasts. This section is intended as a review of the proof methods studied in Chapter 3. The most important part of this section are the exercises.

Focus Questions

By the end of Section 3.6, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. Why does the process of constructing a direct proof of a conditional statement make sense?

2. What is the logical basis for proving the contrapositive of a conditional statement?

3. How do we start a proof by contradiction of a conditional statement and why does this proof method make sense?

4. What is the method of using cases to prove a proposition or theorem?

5. What is a constructive proof? What is a nonconstructive proof?

Activities for Learning

1. Study Section 3.6 of the textbook.

Study Guide for Section 4.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In Chapter 4, we will learn a new method of proof, called mathematical induction, that is often used to prove universally quantified statements about the set of natural numbers. In Section 4.1, the concept of an inductive set will be introduced followed by a statement of the Principle of Mathematical Induction. We will then learn how to use this principle to prove statements about the natural numbers.

Focus Questions

By the end of Section 4.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is an inductive set and how is such a set a part of the Principle of Mathematical Induction?

2. What is the Principle of Mathematical Induction?

3. What is the basic strategy for a proof by mathematical induction?

4. What is the basis step for a proof by induction and what is the inductive step in a proof by induction?

Resources for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 4.1 and then study the answers for these beginning activities.

2. Study Section 4.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 4.1 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- The Traveler and the Strange Suitcase (Screencast 4.1.1.) (3:19)
- Mathematical Induction, Part 1 (Screencast 4.1.2) (5:23)
- Mathematical Induction, Part 2 (Screencast 4.1.3) (6:15)
- Mathematical Induction: Example with Integer Division (Screencast 4.1.4) (7:05)
- Mathematical Induction: Example with Inequality (Screencast 4.1.5) (4:45)
- Mathematical Induction: Example with Calculus (Screencast 4.1.6) (8:26)

4. Work on exercises 1, 3(a), 6, and 7 in Section 4.1 to test your understanding of the material.

Study Guide for Section 4.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

The Principle of Mathematical Induction was introduced in Section 4.1. In this section, we will learn about The Extended Principle of Mathematical Induction and The Second Principle of Mathematical Induction. Near the end of the section, we will prove the important result in number that every natural number greater than 1 is either a prime number or the product of prime numbers. (Theorem 4.9.)

Focus Questions

By the end of Section 4.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the definition of n factorial for a natural number n? Why is it true that for each natural number k, $(k+1)! = (k+1)k!$?
2. What is the definition of a prime number? What is the definition of a composite number?
3. What is the Extended Principle of Mathematical Induction and how is it different than the Principle of Mathematical Induction? Under what conditions would we consider using the Extended Principle of Mathematical Induction in a proof by induction?
4. What is the Second Principle of Mathematical Induction and how is it different than the Principle of Mathematical Induction?
5. Under what conditions would we consider using the Second Principle of Mathematical Induction in a proof by induction?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 4.2 and then study the answers for these beginning activities.
2. Study Section 4.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 4.2 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - The Extended Principle of Mathematical Induction (Screencast 4.2.1) (9:16)
   - The Extended Principle of Mathematical Induction: Example Using Computational Geoemtry (Screencast 4.2.2) (7:56)
   - The Second Principle of Mathematical Induction (Screencast 4.2.3) (8:49)
4. Work on exercises 1(a), 2, and 8 in Section 4.2 to test your understanding of the material.

Study Guide for Section 4.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Many sequences in mathematics are defined recursively. This means that the first term (or first few terms) are defined as specific values and the rest of the terms are defined in terms of the previously defined terms. This idea will be introduced in Beginning Activity 1. In Beginning Activity 2, the Fibonacci Numbers will be introduced. The Fibonacci numbers form one of the most famous sequence in mathematics and many interesting properties of the Fibonacci numbers can be proved used mathematical induction.

Focus Questions

By the end of Section 4.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. How do we determine the elements of a sequence whose terms are defined recursively.
2. How are the Fibonacci numbers defined?
3. How do we use mathematical induction to prove propositions involving recursively-defined functions?
4. What is a geometric sequence? What is a geometric series?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 4.3 and then study the answers for these beginning activities.
2. Study Section 4.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts forSection 4.3 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Recursively Defined Sequences (Screencast 4.3.1) (9:22)
   - The Fibonacci Sequence (Screencast 4.3.2) (9:07)
   - Proving Propositions about Fibonacci Numbers (Screencast 4.3.3) (11:43)
4. Work on exercises 1, 2(a, f), and 12 in Section 4.3 to test your understanding of the material.

Study Guide for Section 5.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Before beginning this section, it would be a good idea to review sets and set notation, including the roster method and set builder notation, in Section 2.3. In Section 2.1, we used logical operators (conjunction, disjunction, negation) to form new statements from existing statements. In a similar manner, there are several ways to create new sets from sets that have already been defined. In fact, we will form these new sets using the logical operators of conjunction (and), disjunction (or), and negation (not). The definitions of these new sets is in Beginning Activity 1.

In Section 2.3, we also introduced the definition of set equality and what it means to say that one set is a subset of another set. These definitions should be reviewed before beginning this section.

Focus Questions

By the end of Section 5.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What are the definitions of the intersection of two sets, the union of two sets, the set difference of two sets, and the complement of a set?
2. What are the definitions of equal sets, of subset, and of proper subset?
3. How are Venn diagrams used to explore the relationships between sets?
4. What is the power set of a set?
5. What is the cardinality of a finite set?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 5.1 and then study the answers for these beginning activities.
2. Study Section 5.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 5.1 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Sets and Set Operations (Screencast 5.1.1) (8:18)
   - Operations Using Infinite Sets (Screencast 5.1.2) (9:40)
   - Subsets and Set Equality (Screencast 5.1.3) (9:41)
   - Cardinality (Screencast 5.1.4) (7:50)
4. Work on exercises 1, 3, 5, 6(a), and 7 in Section 5.1 to test your understanding of the material.

Study Guide for Section 5.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In Section 5.1, we learned some of the basic definitions and notations used in set theory. In this section, we will learn how to prove certain relationships about sets. The main type of proof we will study will be how to prove that one set is a subset of another set. In doing so, it will be very helpful to use the definition of subset written as a conditinal statement. That is, think of a set A as being a subset of the set B if an only if the following conditional statement is true:

For each x in the universal set, if $x \in A$, then $x \in B$.

Focus Questions

By the end of Section 5.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the choose-an-element method and how do we use it to prove that one set is a subset of another set?
2. What is a standard way to prove that two sets are equal when using the choose-an-element method?
3. What does it mean to say that two sets are disjoint?
4. How do we prove that two sets are disjoint?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 5.2 and then study the answers for these beginning activities.
2. Study Section 5.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 5.2 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Proving Subset Inclusion (Screencast 5.2.1) (7:06)
   - Proving Set Equality (Screencast 5.2.2) (15:00)
   - Disjoint Sets (Screencast 5.2.3) (5:37)
4. Work on exercises 1, 3, 5(a, c), 7(a, b, e), and 12(a) in Section 5.2 to test your understanding of the material.

Study Guide for Section 5.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

One purpose of this section is to prove certain properties of set operations. Since the primary set operations (intersection, union, complement) are closed related to the logical connectives (and, or, not), we will see that many of these properties of set operations are closely related to some of the logical equivalencies from Section 2.2. Many of the important properties of set operations are given in Theorem 5.18 and Theorem 5.20.

Another purpose of this section is to learn how to use the results about set operations to prove other results about sets. We will do this with a process that is called "the algebra of sets."

Focus Questions

By the end of Section 5.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the choose-an-element method and how do we use it to prove that one set is a subset of another set?
2. What is a standard way to prove that two sets are equal when using the choose-an-element method?
3. What does it mean to say that two sets are disjoint?
4. How do we prove that two sets are disjoint?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 5.3 and then study the answers for these beginning activities.
2. Study Section 5.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 5.3 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Proving Subset Inclusion (Screencast 5.3.1) (7:06)
   - Proving Set Equality (Screencast 5.3.2) (15:00)
   - Disjoint Sets (Screencast 5.3.3) (5:37)
4. Work on exercises 1(a, c), 2, 4(a, c), and 6(a) in Section 5.3 to test your understanding of the material.

Study Guide for Section 5.4

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

We have worked with the concept of an ordered pair of real numbers in previous mathematics courses. In particular, we used an ordered pair to designate a solution of an equation such as $3x + 4y = 12$. For example, we consider the ordered pair $(4, 0)$ to be a solution of this equation since $3 \cdot 4 + 4 \cdot 0 = 12$.

In this section, we will generalize the concept of an ordered pair so that the coordinates of the ordered pair can come from any given set (not just the set of real numbers). In the process, we will show how to construct a set of ordered pairs from two given sets. Given two sets A and B, we form the new set $A \times B$ (called the Cartesian product of A and B), which is the set of all ordered pairs whose first coordinate is an element of A and whose second coordinate is an element of B.

We will also prove many properties about Cartesian products. Although we will not use the results later in the book, proving these results provides excellent practice at using the choose-an-element method of proof. We will,however, use the definition of the Cartesian product of two sets when we study functions in Chapter 6.

Focus Questions

By the end of Section 5.4, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is an ordered pair and what are the coordinates of an ordered pair?
2. What is the Cartesian product of two sets?
3. What is the Cartesian plane?
4. How do we prove results about Cartesian products of sets?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 5.4 and then study the answers for these beginning activities.
2. Study Section 5.4 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 5.4 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Cartesian Products (Screencast 5.4.1) (11:31)
   - Proofs Involving Cartesian Products (Screencast 5.4.2) (8:13)
4. Work on exercises 1, 3, and 4 in Section 5.4 to test your understanding of the material.

Study Guide for Section 5.5

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In the previous of sections of Chapter 5, we learned about the union and intersection of two sets and the complement of a set. In Section 5.3, we learned about various properties of set operations. In this section, we will extend the ideas of union and intersection to situations that involve more than two sets.

Focus Questions

By the end of Section 5.5, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the union of a family of sets? What is the intersection of a family of sets?

2. What is an indexed family of sets?

3. What is the union of an indexed family of sets? What is the intersection of an indexed family of sets?

4. What are De Morgan's Laws for an indexed family of sets?

5. What is a pairwise disjoint family of sets?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 5.5 and then study the answers for these beginning activities.

2. Study Section 5.5 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. There are no screencasts for Section 5.5.

4. Work on exercises 1(a, d), 2(a, c, d, f), 3(a, b), 5(a), and 8(a) in Section 5.5 to test your understanding of the material.

Study Guide for Section 6.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

One of the most important concepts in modern mathematics is that of a function. In Section 6.1, we will discuss functions in both an informal way and a formal way. The informal way is to help you build your inuition about functions and to help you better understand the formal definitons. Mathematicians must be able to work with functions on both an informal level and a formal level. So Chapter 6 is very important and you need to understand Section 6.1 to continue with the rest of Chapter 6. Due to its importance, there are 8 screencasts for Section 6.1. However, you may view only the first 5 in the list for Section 6.1 and save the last three in the list for the study of Section 6.2.

Focus Questions

By the end of Section 6.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the definition of a function from a set A to a set B? What is the domain of a function and what is the codomain of a function?

2. What is some of the standard terminology used with functions such as the image of a under f and the preimage of b under f ?

3. What is the range of a function and how is it related to the codomain of the function?

4. How can we use the graph of a real function to help us answer questions about real functions?

5. How can we use arrow diagrams to help us visualize functions with a finite domain and a finite codomain?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 6.1 and then study the answers for these beginning activities.

2. Study Section 6.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 6.1 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Functions: The Big Concepts (Screencast 6.1.1) (6:12)
- Functions: Terminology (Screencast 6.1.2) (15:44)
- Function Example: Names to Initials (Screencast 6.1.3) (7:57)

(continued on the next page)

- Function Example: Counting Primes (Screencast 6.1.4) (8:29)
- Non-Examples of Functions (Screencast 6.1.8) (6:41)
- Function Example: Congruence Functions (Screencast 6.1.5) (7:17)
- Function Example: Derivatives (Screencast 6.1.6) (6:56)
- Function Example: Averages (Screencast 6.1.7) (5:07)

4. Work on exercises 1, 3, 5(b, d), and 6 in Section 6.1 to test your understanding of the material.

Study Guide for Section 6.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In many ways, this section is a continuation of Section 6.1. In this section, we will introduce some new types of functions, some of which we may not have seen in previous mathematics courses. The important concept of equality of functions will be introduced, and functions of two variables will be introduced at the end of the section.

Focus Questions

By the end of Section 6.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. How can congruences be used to define functions?

2. What does it mean to say that two functions are equal?

3. How is it possible to view certain mathematical processes as functions?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 6.2 and then study the answers for these beginning activities.

2. Study Section 6.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 6.2 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Function Example: Congruence Functions (Screencast 6.1.5) (7:17)
- Function Example: Derivatives (Screencast 6.1.6) (6:56)
- Function Example: Averages (Screencast 6.1.7) (5:07)
- Equality of Functions (Screencast 6.2.1) (11:25)
- Functions Involving Congruences (Screencast 6.2.2) (10:19)

4. Work on exercises 1, 3, 5(a, c), 7, 8(a, c) in Section 6.2 to test your understanding of the material.

Study Guide for Section 6.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

We should now have a good understanding of what a function is and how to define functions. Always remember that in order to define a function, we must specify the domain, specify the codomain, and specify the rule that describes how to calculate the value (output) of the function for a given element of the domain (input).

In this section, we will learn about some special types of functions that occur quite frequently. The primary ones will be injections (one-to-one functions) and surjections (onto functions). In order to understand these special types of functions, we will need to be able to work with quantifiers and to quickly form the correct negation of a quantified statement. If necessary, Section 2.4 and in particular, Theorem 2.16, should be reviewed before studying this section.

Focus Questions

By the end of Section 6.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What does it mean to say that a function is an injection? How do we prove that a function is an injection?
2. What does it mean to say that a function is not an injection? How do we prove that a function is not an injection?
3. What does it mean to say that a function is a surjection? How do we prove that a function is a surjection?
4. What does it mean to say that a function is not a surjection? How do we prove that a function is not a surjection?
5. Why are the domain and codomain important when determining if a function is an injection or a surjection?
6. What is a bijection or a one-to-one correspondence?

Resources for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 6.3 and then study the answers for these beginning activities.
2. Study Section 6.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

(next page)

Study Guide for Section 6.3

Mathematical Reasoning: Writing and Proof – Version 3

3. View and study the content of the screencasts for Section 6.3 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Injective Functions (Screencast 6.3.1) (6:49)
- How to Prove a Function Is an Injection (Screencast 6.3.2) (9:05)
- Surjective Functions (Screencast 6.3.3) (9:03)
- How to Prove that a Function Is a Surjection (Screencast 6.3.4) (15:49)

4. Work on exercises 2(a, c), 3(a, b, h), 4(a, b), 7, 9 in Section 6.3 to test your understanding of the material.

Study Guide for Section 6.4

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In this section, we will learn about how to form a new function from two given functions using a process called composition of functions. This topic is often studied in precalculus courses and is used in calculus since the so-called chain rule gives a method of finding the derivative of a composite function. In this course, we will focus on the formal definition of the composition of two functions. Please remember that to define a function, we need to specify its domain, its codomain, and the rule for determining values of the function. We will also discuss results about composite functions using the concepts of injections and surjections from Section 6.3.

Focus Questions

By the end of Section 6.4, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the composition of two functions and what is a composite function?
2. How can we decompose a function into the composition of two functions?
3. What are some important theorems about composite functions?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 6.4 and then study the answers for these beginning activities.
2. Study Section 6.4 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 6.4 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Composition of Functions (Screencast 6.4.1) (8:42)
   - Proving Results Involving Compositions (Screencasts 6.4.2) (7:54)
4. Work on exercises 2, 3, 5(a), and 7(a, b, f) in Section 6.4 to test your understanding of the material.

Study Guide for Section 6.5

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

The following should be reviewed before beginning this section.

- The Cartesian product of two sets. See Beginning Activity 2 in Section 5.4.
- The definition of a function on page 284.
- The definition of an injection on page 310 and the definition of a surjection on page 311.
- The definition of the composition of two functions on page 325.

This list indicates that we will be using several concepts we have studied in Chapter 6. We will also be describing the ordered pair representation of a function. This will be used to define the important concept of the inverse of a function.

Focus Questions

By the end of Section 6.5, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. How can a function be represented by a set of ordered pairs?
2. What is the inverse of a function?
3. If $f : A \to B$, under what conditions is the inverse of f a function from B to A?
4. If a function is a bijection, what is the composition of f with its inverse ?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 6.5 and then study the answers for these beginning activities.
2. Study Section 6.5 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. View and study the content of the screencasts for Section 6.5 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`
   - Functions as Sets of Ordered Pairs (Screencast 6.5.1) (8:46)
   - Sets of Ordered Pairs as Functions (Screencast 6.5.2) (4:47)
   - Inverses of Functions (Screencast 6.5.3) (10:41)
   - Working with Inverse Functions (Screencast 6.5.4) (9:04)
4. Work on exercises 2, 3, 4, and 7 in Section 6.5 to test your understanding of the material.

Study Guide for Section 6.6

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In Chapter 5, we studied set theory, and we have studied functions in the first five sections of Chapter 6. In this section, we will use concepts from Chapter 5 and Chapter 6. In our study of functions so far, we have focused on how a function "maps" an individual element of its domain to its codomain. In this section, we will learn how a function "maps" a subset of the domain to a subset of the codomain, and we will learn how to associate a subset of the domain with a given subset of the codomain. Once that is done, we will learn how to combine these ideas with the set operations of union and intersection.

Focus Questions

By the end of Section 6.6, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the image of a subset of the domain of a function?
2. What is the preimage (inverse image) of a subset of the codomain of a function?
3. What are the important results about images and preimages of intersections and unions of two sets given in Theorem 6.34 and Theorem 6.35?
4. If A is a subset of the domain of a function f, what is the relationship between A and $f^{-1}(f(A))$?
5. If C is a subset of the codomain of a function f, what is the relationship between C and $f\left(f^{-1}(C)\right)$?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 6.6 and then study the answers for these beginning activities.
2. Study Section 6.6 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. There are no screencasts for Section 6.6.
4. Work on exercises 1(a, d, f, h), 2(b, d, e, f), 3(a, b), and 5 in Section 6.6 to test your understanding of the material.

Study Guide for Section 7.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

We saw how to represent a function as a set of ordered pairs in Section 6.5. In this section, we will study the concept of a relation from one set to another set using this idea of relating the elements of one set to those of another set using ordered pairs. We will see that functions are a special type of relation but not every relation is a function.

Focus Questions

By the end of Section 7.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is a relation from a set A to a set B? What is a relation on a set A?

2. What is the domain of a relation? What is the range of a relation?

3. What is the standard notation for relations?

4. What are some standard mathematical relations?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 7.1 and then study the answers for these beginning activities.

2. Study Section 7.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 7.1 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Relations (Screencast 7.1.1) (6:53)
- Directed Graphs for Relations (Screencast 7.1.2) (5:23)

4. Work on exercises 1, 2, 4, 6 in Section 7.1 to test your understanding of the material.

Study Guide for Section 7.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

When we studied functions in Chapter 6, we first defined the concept of a function and then studied special types of functions such as injections and surjections. Now that the definition of a relation on a set has been introduced, we can define special types of relations by defining certain properties that the relations might possess. This section introduces the concept of an equivalence relation, which is a very important topic for studying many different areas of mathematics. In one sense, the notion of an equivalence relation is a generalization of the concept of equality. In this section, we will focus on the properties of a relation that define an equivalence relation, and in the next section, we will see how an equivalence relation can be used to sort the elements of a set into certain classes.

Focus Questions

By the end of Section 7.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What does it mean to say that a relation R on a set A is reflexive on A? What does it mean to say that R is not reflexive on A?

2. What does it mean to say that a relation R on a set A is symmetric? What does it mean to say that R is not symmetric?

3. What does it mean to say that a relation R on a set A is transitive? What does it mean to say that R is not transitive?

4. What is an equivalence relation on a set A?

5. Why is congruence modulo n an equivalence relation on $\mathbb{Z}$?

6. What are some other examples of equivalence relations?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 7.2 and then study the answers for these beginning activities.

2. Study Section 7.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 7.2 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Properties of Relations (Screencast 7.2.1) (9:55)
- Equivalence Relations (Screencast 7.2.2) (14:03)

4. Work on exercises 1, 3, 4, 6, 10(a) in Section 7.2 to test your understanding of the material.

Study Guide for Section 7.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Given a relation $\sim$ on a set A, we can associate a subset of A with each element of A. This is explored in Beginning Activity 1. The main purpose of this beginning activity is to when the relation is an equivalence relation, these sets have certain properties that will help us "sort" the elements of A. Beginning Activity 2 also explores certain subsets of A associated with the equivalence relation of congruence modulo 3 on the integers. It might be helpful to review the definition of congruence on page 92 and the properties of congruence in Theorem 3.30 and Corollary 3.32.

Focus Questions

By the end of Section 7.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. If $\sim$ is an equivalence relation on a set A and a is an element of A, what is the equivalence class of a?

2. What is the congruence class of an integer modulo n?

3. What are the important properties of equivalence classes?

4. What is a partition and why do the equivalence classes for an equivalence relation form a partition of the set?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 7.3 and then study the answers for these beginning activities.

2. Study Section 7.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 7.3 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- Equivalence Classes (Screencast 7.3.1) (5:54)
- Properties of Equivalence Classes (Screencast 7.3.2) (15:14)
- Partitions (Screencast 7.3.3) (6:30)

4. Work on exercises 1, 2, 3, 5(a), 6(a) in Section 7.3 to test your understanding of the material.

Study Guide for Section 7.4

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

Everyone may be familiar with the operations of addition and multiplication in our standard number systems. In this section, we will learn how to add and multiply in new number systems called the integers modulo n. These new numbers will actually be equivalence classes involving the equivalence relation of congruence modulo n. A distinctive feature of these new number systems is that they will be defined on finite sets. For an interesting use of this material, please view the last screencast for this section (Affine Ciphers (Screencast 7.4.4).

Focus Questions

By the end of Section 7.4, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the set of integers modulo n?

2. How doe we define addition and multiplication in the integers modulo n?

3. Why is Corollary 7.19 important for the definitions of addition and multiplication in the integers modulo n?

4. How can we use addition and multiplication in the integers modulo n to prove divisibility tests such as the divisibility tests for division by 3 or 9?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 7.4 and then study the answers for these beginning activities.

2. Study Section 7.4 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. View and study the content of the screencasts for Section 7.4 on the GVSU YouTube Math Channel: `http://www.youtube.com/playlist?list=PL2419488168AE7001`

- The Integers Modulo n (Screencast 7.4.1) (6:40)
- Modular Arithmetic (Screencast 7.4.2) (11:07)
- Divisibility by 3 Test (Screencast 7.4.3) (13:09)
- Affine Ciphers (Screencast 7.4.4) (15:44)

4. Work on exercises 1(a, b), 2(a, e, g), 3, 5(a) in Section 7.4 to test your understanding of the material.

Study Guide for Section 8.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In this short chapter, we will continue our study of number theory, which is the study of the system of integers. We have already studied some elementary number in this text such as even and odd integers, divisibility of integers, the Division Algorithm, and congruence. In this section, we will study the greatest common divisor of two integers where at least one of them is not zero.

Focus Questions

By the end of Section 8.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is a common divisor of two integers, not both of which are zero?

2. What is the greatest common divisor of two integers, not both of which are zero?

3. How do we use the Euclidean Algorithm to find the greatest common divisor or two integers, not both of which are zero?

4. What is a linear combination of two integers?

5. How do we use the Euclidean Algorithm to write the greatest common divisor of two integers as a linear combination of those two integers?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 8.1 and then study the answers for these beginning activities.

2. Study Section 8.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. There are no screencasts for Section 8.1.

4. Work on exercises 1(a, b, c, d), 2, 5(a, b, e), 6(a) in Section 8.1 to test your understanding of the material.

Study Guide for Section 8.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In Section 8.1, we introduced the greatest common divisor of two integers and found that this greatest common divisor is a linear combination of the two integers. In this section, we will learn how this will help us prove the Fundamental Theorem of Arithmetic, which basically states that any natural number is either a prime number or has a unique factorization into a product of prime numbers.

Focus Questions

By the end of Section 8.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What are the important ways that the greatest common divisor of two integers is related to linear combinations of those two integers.

2. What are relatively prime integers?

3. What can we conclude if an integer a divides a product bc of integers and a and b are relatively prime?

4. What can we conclude if a prime number divides a product of two integers?

5. What is the Fundamental Theorem of Arithmetic?

6. What are some of the famous open questions about prime numbers?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 8.2 and then study the answers for these beginning activities.

2. Study Section 8.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. There are no screencasts for Section 8.2.

4. Work on exercises 1, 2, 4, 7(a, b) in Section 8.2 to test your understanding of the material.

Study Guide for Section 8.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

One of the main parts of algebra is determining ways to solve certain types of algebraic equations. In this section, we will learn how to use our knowledge of number theory to help find solutions of some special types of equations called Diophantine equations. In particular, we will study equations of the form $ax + by = c$ where a, b, and c are integers.

Focus Questions

By the end of Section 8.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is a Diophantine equation?
2. What is a linear Diophantine equation in one variable? What is a linear Diophantine equation in two variables?
3. Under what conditions does a linear Diophantine equation in two variables have no solution?
4. Under what conditions does a linear Diophantine equation in two variables have infinitely many solutions?
5. Given a linear Diophantine equation in two variables ($ax + by = c$) that has one solution (x_0, y_0), how do write formulas for all of the solutions of this Diophantine equation in terms of x_0, y_0, a, b, and the greatest common divisor of a and b?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 8.3 and then study the answers for these beginning activities.
2. Study Section 8.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. There are no screencasts for Section 8.3.
4. Work on exercises 1, 2, 4, 7(a, b) in Section 8.3 to test your understanding of the material.

Study Guide for Section 9.1

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

We have studied sets in previous parts of the book, most notably in Chapter 5. During that study of sets, we may have made informal distinctions between finite sets and infinite sets. Mathematically, there are very important distinctions between finite and infinite sets, and this will be the focus of Chapter 9. We begin by studying finite sets in Section 9.1. Some of the work in this section may seem like a lot of work to prove what appear to be obvious results. The reason for being this careful with our work with finite sets is to better understand how mathematicians make distinctions between finite and infinite sets. This will be the focus of the other sections in this chapter. The basic concept used to study finite and infinite sets is that of a function, in particular, injections, surjections, and bijections. A good understanding of the material in Section 6.3 is necessary to study the material in this section.

Focus Questions

By the end of Section 9.1, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What do we mean when we say that one set is equivalent to another set?

2. What is a finite set? What is an infinite set?

3. What do we mean by the cardinality of a finite set?

4. What are some of the important properties of finite sets given in Theorem 9.6 and its corollaries? (Corollary 9.8 is quite important.)

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 9.1 and then study the answers for these beginning activities.

2. Study Section 9.1 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.

3. There are no screencasts for Section 9.1.

4. Work on exercises 2, 3, 5(a, b), 7(a) in Section 9.1 to test your understanding of the material.

Study Guide for Section 9.2

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In this section, we begin our study of infinite sets. We will use the concept of equivalent sets (and hence bijections) extensively. There may be some surprising results in this section. When we study infinite sets, our intuition may not be the best guide and we have to rely on precise mathematical definitions and be able to prove that certain functions are bijections. So keep an open mind and be able to adjust your intuition based on the mathematical results that you will see in this section and the next section.

Focus Questions

By the end of Section 9.2, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is one way to prove that a set is infinite according to Corollary 9.8.
2. What symbol do we use to represent the cardinality of the set of natural numbers?
3. What is a countably infinite set? What is a countable set? What is an uncountable set?
4. What is an example of a bijection from the set of the natural numbers to the set of the integers? What does this prove about the set of integers?
5. What is a diagram that can be used to exhibit a bijection from the set of natural numbers to the set of positive rational numbers? What does this prove about the set of positive rational numbers?
6. How do we prove that the set of rational numbers is countably infinite?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 9.2 and then study the answers for these beginning activities.
2. Study Section 9.2 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. There are no screencasts for Section 9.2.
4. Work on exercises 1, 2(a, e, f), 6, 7 in Section 9.2 to test your understanding of the material.

Study Guide for Section 9.3

Mathematical Reasoning: Writing and Proof – Version 3

Overview and Preliminaries

In Section 9.2, we introduced the concept of an infinite set and the concept of a countably infinite set. At that time, we also defined an uncountable set as a set that is not countable. However, we really did not give an example of an uncountable set. We will do so in this section with some very important results (Theorem 9.22, Theorem 9.24, and Theorem 9.26). For a surprising result, make sure you complete Exercise (2). We end this section with a very important theoretical theorem (Cantor's Theorem, Theorem 9.27).

Note: The definition of an uncountable set may seem like a strange definition, but it is similar to the way we define irrational numbers. (See page 122.) In both situations, we gave a very precise definition of one of the terms (rational number or uncountably infinite set) and then defined the other term (irrational number or uncountable set) as one that is not the first term. Just as we usually use a proof by contradiction to prove that a real number is irrational, we often use a proof by contradiction to prove that an infinite set is uncountable.

Focus Questions

By the end of Section 9.3, you should be able to give precise and thorough answers to each of the following questions. You should keep these questions in mind to focus your thoughts as you complete your study of this section.

1. What is the normalized for of the decimal expansion of a real number? Why is it important to use the normalized form of decimal expansions of real numbers?
2. How do we prove that the open interval $(0, 1)$ is an uncountable set?
3. What does it mean to say that a set has cardinality $\boldsymbol{c}$?
4. What is the cardinality of an open interval of real numbers and what is the cardinality of the set $\mathbb{R}$?
5. What is Cantor's Theorem and how do we prove Cantor's Theorem?

Activities for Learning

1. Complete Beginning Activity 1 and Beginning Activity 2 of Section 9.3 and then study the answers for these beginning activities.
2. Study Section 9.3 of the textbook. While studying the section, complete each progress check and then consult the answers in Appendix B before proceeding further in the section.
3. There are no screencasts for Section 9.3.
4. Work on exercises 1(a, b), 2, 3, 4 in Section 9.3 to test your understanding of the material.