DeePC and DRO Notes

Alessandro Quattrociocchi

December 4, 2024

Contents

1 Problem Formulation

Consider a deterministic LTI system whose model is unknown. Let $T_f \in \mathbb{Z}_{>0}$ be the prediction horizon. Let $f_1 : \mathbb{R}^{mT_f} \to \mathbb{R}_{\geq 0}$, and $f_2 : \mathbb{R}^{pT_f} \to \mathbb{R}_{\geq 0}$ be a cost function on the future inputs and, respectively, outputs. We aim at finding a input sequece $\mathrm{col}(u_t, \dots, u_{t+T_f-1}) \in \mathbb{R}^{mT_f}$, such that $\mathrm{col}(y_t, \dots, y_{t+T_f-1}) \in \mathbb{R}^{pT_f}$ minimizes the cost $f_1 + f_2$, and the contraints satisfied, i.e., $u \in \mathcal{U}$ and $y \in \mathcal{Y}$, with $\mathcal{U} \subseteq \mathcal{R}^{mT_f}$ and $\mathcal{Y} \subseteq \mathcal{R}^{pT_f}$.

1.1 Conventional Formulation

Using conventional input output state space representation.

$$
\begin{aligned}
\min_{u,y,x} \quad & f_1(u) + f_2(y) \\
\text{s.t.} \quad & \forall k \in \{0, \dots, T_f - 1\} \\
& x_{k+1} = Ax_k + Bu_k \\
& y_k = Cx_k + Du_k \\
& x_0 = \hat{x}_t \\
& u \in \mathcal{U},\ y \in \mathcal{Y}
\end{aligned} \tag{1}
$$

1.2 DeePC Formulation

From data we contruct $\hat{U}_p, \hat{U}_f, \hat{Y}_p, \hat{Y}_f$.

$$\begin{aligned} \min_{g} \quad & f_1(u) + f_2(y) \\ \text{s.t.} \quad & \begin{bmatrix} \hat{U}_p \\ \hat{Y}_p \\ \hat{U}_f \\ \hat{Y}_f \end{bmatrix} g = \begin{bmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \\ y \end{bmatrix} \\ & u \in \mathcal{U},\ y \in \mathcal{Y} \end{aligned} \tag{2}$$

- U_p, Y_p represent the past input-output trajectories.
- U_f, Y_f represent the future input-output trajectories.

1.2.1 Remark I: *DeepC representation of predictor.*

$$\begin{bmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \\ y \end{bmatrix} = H(w)g, \quad \text{with} \quad H(w) = \begin{bmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{bmatrix} \tag{3}$$

where $H(w)$ is the suitably partitioned Hankel matrix, or

$$y = Y_f g \tag{4}$$

where g is computed from:

$$\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix} g = \begin{bmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \end{bmatrix}.$$

The explicit solution for y is the superposition of a particolar solition and the homogenous term:

$$y = Y_f \cdot \left(\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix}^+ \begin{bmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \end{bmatrix} + \ker\left(\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix} \right) \right),$$

where $\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix}^+$ is the pseudoinverse of the Hankel matrix, and ker represents the kernel of the matrix.

1.2.2 Remark: *Regularization in DeePC(rank-deficient Hankel matrices)*

When the Hankel matrix H is rank-deficient (i.e., H has more columns than rows), there are infinitely many solutions g that satisfy the system equation:

$$Hg = \begin{bmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \end{bmatrix}. \tag{5}$$

To find a noise-robust prediction, the Projection Regularizer is introduced. The regularizer penalizes the portion of g that lies in the nullspace of H. This is achieved by minimizing:

$$\|(I - \Pi)g\|_2^2, \tag{6}$$

where:

- Π is the orthogonal projection matrix onto the row space of H,
- $(I - \Pi)g$ isolates the component of g in $\ker(H)$, the nullspace of H.

This ensures that the computed g is the smallest norm solution to:

$$y = Y_f g. \tag{7}$$

In the noiseless case ($\lambda_s = 0$) with regularization constant $\lambda_p \geq 0$, the optimization problem ensures that both u and y match the reference trajectory u_r, y_r, making it beneficial to use the regularizer with a sufficiently large λ_p.

2 Stochastic DeePC

Following the formulation in [1], we now extend the DeePC definition of the unknow LTI system by considering a disturbance vector $w_t \in \mathbb{R}^q$, such as:

$$\begin{cases} x_{t+1} = Ax_t + Bu_t + Ew_t \\ y_t = Cx_t + Du_t + Fw_t \end{cases} \tag{8}$$

where $E \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{p\times q}$. Therefore, the system is subject to an **unknow** and **uncontrollable** disturbance, whose past trajectory can be measured but whose future evolutions are unknown. Similar to u^d and y^d, we can build the Hankel matrix of the disturbance $\mathcal{H}_{T_{ini+N}}$, splitted as follows:

$$\mathcal{H} \tag{9}$$

References

[1] Linbin Huang, Jeremy Coulson, John Lygeros, and Florian Dörfler. Decentralized data-enabled predictive control for power system oscillation damping. *IEEE Transactions on Control Systems Technology*, 30(3):1065–1077, 2021.