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Ex.1

(a) $f(z)=\bar{z}=x-iy$. So $u=x, v=-y$.

Inasmuch as $u_x=v_y \Rightarrow 1=-1$, the Cauchy-Riemann equations are not satisfied anywhere.

(b) $f(z)=z-\bar{z}=(x+iy)-(x-iy)=0+i2y$. So $u=0, v=2y$.

Since $u_x=v_y \Rightarrow 0=2$, the Cauchy-Riemann equations are not satisfied anywhere.

(c) $f(z)=2x+ixy^2$. Here $u=2x, v=xy^2$.

$u_x=v_y \Rightarrow 2=2xy \Rightarrow xy=1$.

$u_y=-v_x \Rightarrow 0=-y^2 \Rightarrow y=0$.

Substituting $y=0$ into $xy=1$, we have $0=1$. Thus the Cauchy-Riemann equations do not hold anywhere.

(d) $f(z)=e^x e^{-iy}=e^x(\cos y-i\sin y)=e^x\cos y-ie^x\sin y$. So $u=e^x\cos y, v=-e^x\sin y$.

$u_x=v_y \Rightarrow e^x\cos y=-e^x\cos y \Rightarrow 2e^x\cos y=0 \Rightarrow \cos y=0$. Thus

$$y=\frac{\pi}{2}+n\pi \qquad (n=0,\pm1,\pm2,\ldots).$$

$u_y=-v_x \Rightarrow -e^x\sin y=e^x\sin y \Rightarrow 2e^x\sin y=0 \Rightarrow \sin y=0$. Hence

$$y=n\pi \qquad (n=0,\pm1,\pm2,\ldots).$$

Since these are two different sets of values of y, the Cauchy-Riemann equations cannot be satisfied anywhere.

Ex.2

(a) $f(z)=iz+2=2-y+ix, u(x,y)=2-y, v(x,y)=x$，一阶偏导 $u_x=0, u_y=-1, v_x=1, v_y=0$ 在复平面上处处连续，并且柯西方程 $u_x=v_y, u_y=-v_x$ 在复平面上任意点成立，因此 $f'(z)=i$ 在复平面上处处存在，同理 $f''(z)=0$ 也存在于整个复平面。

(b) $f(z)=e^{-x}e^{-iy}=e^{-x}\cos y-ie^{-x}\sin y$,一阶偏导 $u_x=-e^{-x}\cos y, u_y=-e^{-x}\sin y, v_x=e^{-x}\sin y, v_y=-e^{-x}\cos y$ 在复平面上处处连续，并且柯西方程 $u_x=v_y, u_y=-v_x$ 在复平面上任意点成立，因此 $f'(z)=-e^{-x}\cos y+ie^{-x}\sin y=-f(z)$ 在复平面上处处存在，同理 $f''(z)=-f'(z)=-(-f(z))=f(z)=e^{-x}e^{-iy}=e^{-x}\cos y-ie^{-x}\sin y$ 在复平面上处处存在。

(c) $f(z)=z^3=x^3-3xy^2+i(3x^2y-y^3)$，一阶偏导 $u_x=3x^2-3y^2, u_y=-6xy, v_x=$

$6xy, v_y = 3x^2 - 3y^2$ 在整个复平面上连续，并且柯西方程 $u_x = v_y, u_y = -v_x$ 在复平面上任意点成立，因此 $f'(z) = 3x^2 - 3y^2 + i6xy = 3z^2$ 在复平面上处处存在，同理 $f''(z) = (3z^2)' = 6z$ 在复平面上处处存在。

(d) $f(z) = \cos x \cosh y - i \sin x \sinh y$, 一阶偏导 $u_x = -\sin x \cosh y, u_y = \cos x \sinh y, v_x = -\cos x \sinh y, v_y = -\sin x \cosh y$ 在整个复平面上连续，并且柯西方程 $u_x = v_y, u_y = -v_x$ 在复平面上任意点成立，因此 $f'(z) = -\sin x \cosh y - i \cos x \sinh y$ 在复平面上处处存在，同理 $f''(z) = -\cos x \cosh y + i \sin x \sinh y = -f(z)$ 在复平面上处处存在。

Ex.6

Here u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z)=\begin{cases}\overline{z}^2/z & \text{when } z\neq 0,\\ 0 & \text{when } z=0.\end{cases}$$

Now

$$f(z)=\underbrace{\frac{x^3-3xy^2}{x^2+y^2}}_{u}+i\underbrace{\frac{y^3-3x^2y}{x^2+y^2}}_{v}$$

when $z\neq 0$, and the following calculations show that

$$u_x(0,0)=v_y(0,0) \quad \text{and} \quad u_y(0,0)=-v_x(0,0):$$

$$u_x(0,0)=\lim_{\Delta x\to 0}\frac{u(0+\Delta x,0)-u(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1,$$

$$u_y(0,0)=\lim_{\Delta y\to 0}\frac{u(0,0+\Delta y)-u(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0}{\Delta y}=0,$$

$$v_x(0,0)=\lim_{\Delta x\to 0}\frac{v(0+\Delta x,0)-v(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0,$$

$$v_y(0,0)=\lim_{\Delta y\to 0}\frac{v(0,0+\Delta y)-v(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\Delta y}{\Delta y}=1.$$

Ex.7

Equations (2), Sec. 23, are

$$u_x \cos\theta + u_y \sin\theta = u_r,$$
$$-u_x r\sin\theta + u_y r\cos\theta = u_\theta.$$

Solving these simultaneous linear equations for u_x and u_y, we find that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \quad \text{and} \quad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Likewise,

$$v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r} \quad \text{and} \quad v_y = v_r \sin\theta + v_\theta \frac{\cos\theta}{r}.$$

Assume now that the Cauchy-Riemann equations in polar form,

$$ru_r = v_\theta, \quad u_\theta = -rv_r,$$

are satisfied at z_0. It follows that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} = v_\theta \frac{\cos\theta}{r} + v_r \sin\theta = v_r \sin\theta + v_\theta \frac{\cos\theta}{r} = v_y,$$

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r} = v_\theta \frac{\sin\theta}{r} - v_r \cos\theta = -\left(v_r \cos\theta - v_\theta \frac{\sin\theta}{r} \right) = -v_x.$$

Ex.8

$$
\begin{aligned}
f'(z_0) &= u_x + iv_x \\
&= u_r\cos\theta - u_\theta\frac{\sin\theta}{r} + i(v_r\cos\theta - v_\theta\frac{\sin\theta}{r}) \\
&= u_r\cos\theta - iv_\theta\frac{\sin\theta}{r} + i(v_r\cos\theta + iu_\theta\frac{\sin\theta}{r}) \\
&= u_r(\cos\theta - i\sin\theta) + iv_r(\cos\theta - i\sin\theta) \\
&= e^{-i\theta}(u_r + iv_r)
\end{aligned}
$$

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Ex.2

(a) $f(z)=\underbrace{xy}_{u}+i\underbrace{y}_{v}$ is nowhere analytic since

$$u_x = v_y \Rightarrow y=1 \text{ and } u_y = -v_x \Rightarrow x=0,$$

which means that the Cauchy-Riemann equations hold only at the point $z=(0,1)=i$.

(c) $f(z)=e^y e^{ix} = e^y(\cos x + i\sin x) = \underbrace{e^y\cos x}_{u} + i\underbrace{e^y\sin x}_{v}$ is nowhere analytic since

$$u_x = v_y \Rightarrow -e^y\sin x = e^y \sin x \Rightarrow 2e^y\sin x = 0 \Rightarrow \sin x = 0$$

and

$$u_y = -v_x \Rightarrow e^y\cos x = -e^y\cos x \Rightarrow 2e^y\cos x = 0 \Rightarrow \cos x = 0.$$

More precisely, the roots of the equation $\sin x = 0$ are $n\pi$ $(n=0,\pm1,\pm2,\ldots)$, and $\cos n\pi = (-1)^n \neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.

Ex.3

1. 假定 $g(z), f(z)$ 为整函数，由于 $f(z)$ 的值域是复平面的子集，因此 $g[f(z)]$ 在整个复平面也是解析的，即 $g[f(z)]$ 为整函数。

2. 假定 $f_1(z), f_2(z)$ 为整函数，乘上一个不为 0 的常数 c 不影响函数的解析性，则 $c_1f_1(z), c_2f_2(z)$ 仍然为整函数，两函数之和的解析域为两函数自身解析域的交集，因此 $c_1f_1(z) + c_2f_2(z)$ 在复平面上是解析的，即为整函数。

Ex.4

(a)

$$f(z) = \frac{P(z)}{Q(z)} = \frac{2z+1}{z(z^2+1)}$$

$z(z^2+1)=0 \Rightarrow z=0,\pm i$，$P(z),Q(z)$ 是整函数，因此 $f(z)=P(z)/Q(z)$ 在除 $Q(z)=0$ 的点以外解析。

(b)

$$f(z) = \frac{P(z)}{Q(z)} = \frac{z^3+i}{z^2-3z+2}$$

$z^2-3z+2=0 \Rightarrow z=1,2$，$P(z),Q(z)$ 是整函数，因此 $f(z)=P(z)/Q(z)$ 在除 $Q(z)=0$ 的点以外解析。

(c)

$$f(z) = \frac{P(z)}{Q(z)} = \frac{z^2+1}{(z+2)(z^2+2z+2)}$$

$(z+2)(z^2+2z+2)=0 \Rightarrow z=-2,-1\pm i$，$P(z),Q(z)$ 是整函数，因此 $f(z)=P(z)/Q(z)$ 在除 $Q(z)=0$ 的点以外解析。

Ex.6

$g(z)=\ln r+i\theta$ 在 $D=\{r>0, 0<\theta<2\pi\}$ 上有定义，一阶偏导 $u_r=\frac{1}{r}, u_\theta=0, v_r=0, v_\theta=1$ 在 D 上连续，并且柯西方程 $ru_r=v_\theta, u_\theta=-rv_r$ 成立，因此 $g(z)$ 在 D 上解析，

$$g'(z)=e^{-i\theta}(u_r+iv_r)=(re^{i\theta})^{-1}=z^{-1}$$

令 $f(z)=z^2+1=x^2-y^2+1+i2xy$，$f(z)$ 为整函数，在 $x>0,y>0$ 象限上有 $\mathrm{Im}[f(z)]>0$，因此该象限内 $f(z)$ 的值域是 D 的子集，$G(z)=g[f(z)]$ 在 $x>0,y>0$ 上解析，

$$G'(z)=g'[f(z)]f'(z)=\frac{f'(z)}{f(z)}=\frac{2z}{z^2+1}$$

Ex.7

Suppose that a function $f(z)=u(x,y)+iv(x,y)$ is analytic and real-valued in a domain D. Since $f(z)$ is real-valued, it has the form $f(z)=u(x,y)+i0$. The Cauchy-Riemann equations $u_x=v_y, u_y=-v_x$ thus become $u_x=0, u_y=0$; and this means that $u(x,y)=a$, where a is a (real) constant. (See the proof of the theorem in Sec. 24.) Evidently, then, $f(z)=a$. That is, f is constant in D.

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Ex.1

(a) It is straightforward to show that $u_{xx}+u_{yy}=0$ when $u(x,y)=2x(1-y)$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y)=2-2y$. Now

$$u_x=v_y \Rightarrow v_y=2-2y \Rightarrow v(x,y)=2y-y^2+\phi(x).$$

Then

$$u_y=-v_x \Rightarrow -2x=-\phi'(x) \Rightarrow \phi'(x)=2x \Rightarrow \phi(x)=x^2+c.$$

Consequently,

$$v(x,y)=2y-y^2+(x^2+c)=x^2-y^2+2y+c.$$

(b) It is straightforward to show that $u_{xx}+u_{yy}=0$ when $u(x,y)=2x-x^3+3xy^2$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y)=2-3x^2+3y^2$. Now

$$u_x=v_y \Rightarrow v_y=2-3x^2+3y^2 \Rightarrow v(x,y)=2y-3x^2y+y^3+\phi(x).$$

Then

$$u_y=-v_x \Rightarrow 6xy=6xy-\phi'(x) \Rightarrow \phi'(x)=0 \Rightarrow \phi(x)=c.$$

Consequently,

$$v(x,y) = 2y - 3x^2y + y^3 + c.$$

(c) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x,y) = \sinh x \sin y$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = \cosh x \sin y$. Now

$$u_x = v_y \Rightarrow v_y = \cosh x \sin y \Rightarrow v(x,y) = -\cosh x \cos y + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow \sinh x \cos y = \sinh x \cos y - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

Consequently,

$$v(x,y) = -\cosh x \cos y + c.$$

(d) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x,y) = \dfrac{y}{x^2+y^2}$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = -\dfrac{2xy}{(x^2+y^2)^2}$. Now

$$u_x = v_y \Rightarrow v_y = -\frac{2xy}{(x^2+y^2)^2} \Rightarrow v(x,y) = \frac{x}{x^2+y^2} + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

Consequently,

$$v(x,y) = \frac{x}{x^2+y^2} + c.$$

Ex.2

Suppose that v and V are harmonic conjugates of u in a domain D. This means that

$$u_x = v_y,\ u_y = -v_x \ \text{ and } \ u_x = V_y,\ u_y = -V_x.$$

If $w = v - V$, then,

$$w_x = v_x - V_x = -u_y + u_y = 0 \ \text{ and } \ w_y = v_y - V_y = u_x - u_x = 0.$$

Hence $w(x,y) = c$, where c is a (real) constant (compare the proof of the theorem in Sec. 24). That is, $v(x,y) - V(x,y) = c$.

Ex.3

Suppose that u and v are harmonic conjugates of each other in a domain D. Then

$$u_x = v_y, \; u_y = -v_x \;\text{ and }\; v_x = u_y, \; v_y = -u_x.$$

It follows readily from these equations that

$$u_x = 0, \; u_y = 0 \;\text{ and }\; v_x = 0, \; v_y = 0.$$

Consequently, $u(x,y)$ and $v(x,y)$ must be constant throughout D (compare the proof of the theorem in Sec. 24).

Ex.5

The Cauchy-Riemann equations in polar coordinates are

$$ru_r = v_\theta \;\text{ and }\; u_\theta = -rv_r.$$

Now

$$ru_r = v_\theta \Rightarrow ru_{rr} + u_r = v_{\theta r}$$

and

$$u_\theta = -rv_r \Rightarrow u_{\theta\theta} = -rv_{r\theta}.$$

Thus

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = rv_{\theta r} - rv_{r\theta};$$

and, since $v_{\theta r} = v_{r\theta}$, we have

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0,$$

which is the polar form of Laplace's equation. To show that v satisfies the same equation, we observe that

$$u_\theta = -rv_r \Rightarrow v_r = -\frac{1}{r}u_\theta \Rightarrow v_{rr} = \frac{1}{r^2}u_\theta - \frac{1}{r}u_{\theta r}$$

and

$$ru_r = v_\theta \Rightarrow v_{\theta\theta} = ru_{r\theta}.$$

Since $u_{\theta r} = u_{r\theta}$, then,

$$r^2 v_{rr} + rv_r + v_{\theta\theta} = u_\theta - ru_{\theta r} - u_\theta + ru_{r\theta} = 0.$$