Ex.1

(a) $(1+i)^{100} = (\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)))^{100} = 2^{50}(\cos\pi + i\sin\pi) = -2^{50}$.

(b) e^2.

Ex.2

EXAMPLE 5. We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function

$$u(x, y) = y^3 - 3x^2y \tag{5}$$

is readily seen to be harmonic throughout the entire xy plane. Since a harmonic conjugate $v(x, y)$ is related to $u(x, y)$ by means of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x, \tag{6}$$

the first of these equations tells us that

$$v_y(x, y) = -6xy.$$

Holding x fixed and integrating each side here with respect to y, we find that

$$v(x, y) = -3xy^2 + \phi(x) \tag{7}$$

where ϕ is, at present, an arbitrary function of x. Using the second of equations (6), we have

$$3y^2 - 3x^2 = 3y^2 - \phi'(x),$$

or $\phi'(x) = 3x^2$. Thus $\phi(x) = x^3 + C$, where C is an arbitrary real number. According to equation (7), then, the function

$$v(x, y) = -3xy^2 + x^3 + C \tag{8}$$

is a harmonic conjugate of $u(x, y)$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + C). \tag{9}$$

The form $f(z) = i(z^3 + C)$ of this function is easily verified and is suggested by noting that when $y = 0$, expression (9) becomes $f(x) = i(x^3 + C)$.

Ex.3

假设 $f(z)$ 在 z_0 的一阶导存在，

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\mathrm{Re}\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\mathrm{Im}\frac{\Delta w}{\Delta z}\right)$$

令 $(\Delta x, \Delta y)$ 沿着实轴 $(\Delta x, 0)$ 逼近 $(0,0)$，则有

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left(\mathrm{Re}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} = u_x(x_0,y_0)$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left(\mathrm{Im}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} = v_x(x_0,y_0)$$

得到 $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$

令 $(\Delta x, \Delta y)$ 沿着虚轴 $(0, \Delta y)$ 逼近 $(0,0)$，则有

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i\frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y} \\ &= \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y} - i\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y}\end{aligned}$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left(\mathrm{Re}\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta y\to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{\Delta y} = v_y(x_0,y_0)$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left(\mathrm{Im}\frac{\Delta w}{\Delta z}\right) = -\lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{\Delta y} = -u_y(x_0,y_0)$$

得到 $f'(z_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$. 利用极限存在的必要条件，得到 $u_x(x_0,y_0) = v_y(x_0,y_0), u_y(x_0,y_0) = -v_x(x_0,y_0)$.

Ex.4

(a) $\mathrm{Log}(-ei) = 1 - i\frac{\pi}{2}$

(b)

EXAMPLE 2. The principal value of $(-i)^i$ is

$$\exp[i \operatorname{Log}(-i)] = \exp\left[i\left(\ln 1 - i\frac{\pi}{2}\right)\right] = \exp\frac{\pi}{2}.$$

That is,

$$\text{P.V. } (-i)^i = \exp\frac{\pi}{2}. \tag{6}$$

Ex.5

(a)

$$\begin{aligned}
\int_{C_1} f(z)dz &= \int_{OA} f(z)dz + \int_{AB} f(z)dz \\
&= \int_0^1 yidy + \int_0^1 (1 - x - i3x^2)dx \\
&= \frac{i}{2} + \frac{1}{2} - i \\
&= \frac{1-i}{2}
\end{aligned}$$

(b) C_2 为 $y = x$ 上的线段，$z = x + ix$

$$\begin{aligned}
\int_{C_2} f(z)dz &= \int_0^1 -i3x^2(1+i)dx \\
&= 3(1-i)\int_0^1 x^2dx \\
&= 1 - i
\end{aligned}$$

Ex.6

$$|z+4| \le |z| + 4 = 6$$

$$|z^3 - 1| \ge ||z|^3 - 1| = 7$$

则 $\left|\frac{z+4}{z^3-1}\right| \le \frac{6}{7}$，$C$ 的长度 $L = \pi$，因此有 $\left|\int_C \frac{z+4}{z^3-1}dz\right| \le \frac{6\pi}{7}$

Ex.7

Theorem. 设$z_n = x_n + iy_n (n = 1, 2, \cdots)$和$z = x + iy$，则

$$\lim_{n\to\infty} z_n = z \tag{4}$$

当且仅当

$$\lim_{n\to\infty} x_n = x \quad 和 \quad \lim_{n\to\infty} y_n = y. \tag{5}$$

To prove this theorem, we first assume that conditions (5) hold and obtain condition (4) from it. According to conditions (5), there exist, for each positive number ε, positive integers n_1 and n_2 such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_1$$

and

$$|y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_2.$$

Hence if n_0 is the larger of the two integers n_1 and n_2,

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_0.$$

Since

$$|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|,$$

then,

$$|z_n - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{whenever} \quad n > n_0.$$

Condition (4) thus holds.

Conversely, if we start with condition (4), we know that for each positive number ε, there exists a positive integer n_0 such that

$$|(x_n + iy_n) - (x + iy)| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

But

$$|x_n - x| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

and

$$|y_n - y| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|;$$

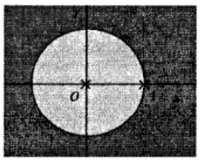
and this means that

$$|x_n - x| < \varepsilon \quad \text{and} \quad |y_n - y| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

That is, conditions (5) are satisfied.

Ex.8

The singularities of the function $f(z) = \dfrac{1}{z^2(1-z)}$ are at the points $z=0$ and $z=1$. Hence there are Laurent series in powers of z for the domains $0<|z|<1$ and $1<|z|<\infty$ (see the figure below).



To find the series when $0<|z|<1$, recall that $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ $(|z|<1)$ and write

$$f(z) = \frac{1}{z^2}\cdot\frac{1}{1-z} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2}+\frac{1}{z}+\sum_{n=2}^{\infty} z^{n-2} = \sum_{n=0}^{\infty} z^n + \frac{1}{z}+\frac{1}{z^2}.$$

As for the domain $1<|z|<\infty$, note that $|1/z|<1$ and write

$$f(z) = -\frac{1}{z^3}\cdot\frac{1}{1-(1/z)} = -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}} = -\sum_{n=3}^{\infty}\frac{1}{z^n}.$$

Ex.9

(a) $f(z) = \frac{1}{(z-i)(z+2)}$ 在 C 上以及内部解析，故 $\int_c f(z)dz = 0$

(b) C 中包含奇点 $z=2$，

$$\begin{aligned} f(z) &= \frac{1}{z(z-2)^4} \\ &= \frac{1}{1-(-\frac{z-2}{2})}\frac{1}{2(z-2)^4} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-4} \quad (0<|z-2|<2) \end{aligned}$$

因此 $f(z)$ 在 $z=2$ 处的留数为 $-1/16$，$\int_C f(z)dz = 2\pi i(-1/16) = -\pi i/8$

Ex.10

C 中包含两个奇点 $z=0$ 和 $z=1$，

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{2}{z}+\frac{3}{z-1}$$

因此 $f(z)$ 在 $z=0$ 的留数 $B_1=2$,在 $z=1$ 的留数 $B_2=3$, $\int_C f(z)dz=2\pi i(2+3)=10\pi i$