page 239

Ex.2

In each part, C denotes the positively oriented circle $|z| = 3$.

(a) To evaluate $\int_C \frac{\exp(-z)}{z^2} dz$, we need the residue of the integrand at $z = 0$. From the Laurent series

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right) = \frac{1}{z^2} - \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots \qquad (0 < |z| < \infty),$$

we see that the required residue is -1. Thus

$$\int_C \frac{\exp(-z)}{z^2} dz = 2\pi i(-1) = -2\pi i.$$

(b)
$f(z) = \frac{\exp(-z)}{(z-1)^2}$在$C$中孤立奇点有$z = 1$,

$$\exp(-z) = \exp(-(z-1)-1) = \frac{1}{e}\exp(-(z-1)) = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{n!}(|z-1| < \infty)$$

$$f(z) = \frac{1}{e(z-1)^2} - \frac{1}{e(z-1)} + \frac{1}{e2!} - \ldots (0 < |z-1| < \infty)$$

$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = -\frac{2\pi i}{e}$$

(c) Likewise, to evaluate the integral $\int_C z^2 \exp\left(\frac{1}{z}\right) dz$, we must find the residue of the integrand at $z = 0$. The Laurent series

$$z^2 \exp\left(\frac{1}{z}\right) = z^2\left(1 + \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \frac{1}{4!}\cdot\frac{1}{z^4} + \cdots\right)$$

$$= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^2} + \cdots,$$

which is valid for $0 < |z| < \infty$, tells us that the needed residue is $\frac{1}{6}$. Hence

$$\int_C z^2 \exp\left(\frac{1}{z}\right) dz = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}.$$

(d) As for the integral $\int_C \frac{z+1}{z^2-2z}dz$, we need the two residues of

$$\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)},$$

one at $z=0$ and one at $z=2$. The residue at $z=0$ can be found by writing

$$\frac{z+1}{z(z-2)} = \left(\frac{z+1}{z}\right)\left(\frac{1}{z-2}\right) = \left(-\frac{1}{2}\right)\left(1+\frac{1}{z}\right)\cdot\frac{1}{1-(z/2)}$$

$$= \left(-\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{z}\right)\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right),$$

which is valid when $0<|z|<2$, and observing that the coefficient of $\frac{1}{z}$ in this last product is $-\frac{1}{2}$. To obtain the residue at $z=2$, we write

$$\frac{z+1}{z(z-2)} = \frac{(z-2)+3}{z-2}\cdot\frac{1}{2+(z-2)} = \frac{1}{2}\left(1+\frac{3}{z-2}\right)\cdot\frac{1}{1+(z-2)/2}$$

$$= \frac{1}{2}\left(1+\frac{3}{z-2}\right)\left[1-\frac{z-2}{2}+\frac{(z-2)^2}{2^2}-\cdots\right],$$

which is valid when $0<|z-2|<2$, and note that the coefficient of $\frac{1}{z-2}$ in this product is $\frac{3}{2}$. Finally, then, by the residue theorem,

$$\int_C \frac{z+1}{z^2-2z}dz = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = 2\pi i.$$

Ex.3

In each part of this problem, C is the positively oriented circle $|z|=2$.

(a) If $f(z)=\frac{z^5}{1-z^3}$, then

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^7-z^4} = -\frac{1}{z^4}\cdot\frac{1}{1-z^3} = -\frac{1}{z^4}(1+z^3+z^6+\cdots) = -\frac{1}{z^4}-\frac{1}{z}-z^2-\cdots$$

when $0<|z|<1$. This tells us that

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right) = 2\pi i(-1) = -2\pi i.$$

(b) When $f(z)=\frac{1}{1+z^2}$, we have

$$\frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}=1-z^2+z^4-\cdots \qquad (0<|z|<1).$$

Thus

$$\int_C f(z)\,dz=2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(0)=0.$$

(c) If $f(z)=\frac{1}{z}$, it follows that $\frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{1}{z}$. Evidently, then,

$$\int_C f(z)\,dz=2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(1)=2\pi i.$$

Ex.5

证：令简单闭曲线C包围n个奇点，取正方向，由柯西留数定理有

$$\int_C f(z)dz=2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k}f(z)$$

又有

$$\int_C f(z)dz=2\pi i\operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f(\frac{1}{z})\right]=-2\pi i\operatorname*{Res}_{z=\infty}f(z)$$

$$\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k}f(z)+\operatorname*{Res}_{z=\infty}f(z)=0$$

证毕。

Ex.6

We are given two polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0),$$

where $m \geq n+2$.

It is straightforward to show that

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = \frac{a_0 z^{m-2} + a_1 z^{m-3} + a_2 z^{m-4} + \cdots + a_n z^{m-n-2}}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_m} \qquad (z \neq 0).$$

Observe that the numerator here is, in fact, a polynomial since $m-n-2 \geq 0$. Also, since $b_m \neq 0$, the quotient of these polynomials is represented by a series of the form $d_0 + d_1 z + d_2 z^2 + \cdots$. That is,

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = d_0 + d_1 z + d_2 z^2 + \cdots \qquad (0 < |z| < R_2);$$

and we see that $\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)}$ has residue 0 at $z = 0$.

Suppose now that all of the zeros of $Q(z)$ lie inside a simple closed contour C, and assume that C is positively oriented. Since $P(z)/Q(z)$ is analytic everywhere in the finite plane except at the zeros of $Q(z)$, it follows from the theorem in Sec. 64 and the residue just obtained that

$$\int_C \frac{P(z)}{Q(z)} dz = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} \right] = 2\pi i \cdot 0 = 0.$$

If C is negatively oriented, this result is still true since then

$$\int_C \frac{P(z)}{Q(z)} dz = -\int_{-C} \frac{P(z)}{Q(z)} dz = 0.$$

page 243

Ex.1

(a) From the expansion

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we see that

$$z \exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{3!} \cdot \frac{1}{z^2} + \cdots \qquad (0 < |z| < \infty).$$

The principal part of $z\exp\left(\frac{1}{z}\right)$ at the isolated singular point $z=0$ is, then,

$$\frac{1}{2!}\cdot\frac{1}{z}+\frac{1}{3!}\cdot\frac{1}{z^2}+\cdots;$$

and $z=0$ is an essential singular point of that function.

(b) The isolated singular point of $\frac{z^2}{1+z}$ is at $z=-1$. Since the principal part at $z=-1$ involves powers of $z+1$, we begin by observing that

$$z^2=(z+1)^2-2z-1=(z+1)^2-2(z+1)+1.$$

This enables us to write

$$\frac{z^2}{1+z}=\frac{(z+1)^2-2(z+1)+1}{z+1}=(z+1)-2+\frac{1}{z+1}.$$

Since the principal part is $\frac{1}{z+1}$, the point $z=-1$ is a (simple) pole.

(c) The point $z=0$ is the isolated singular point of $\frac{\sin z}{z}$, and we can write

$$\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots \qquad (0<|z|<\infty).$$

The principal part here is evidently 0, and so $z=0$ is a removable singular point of the function $\frac{\sin z}{z}$.

(d) The isolated singular point of $\frac{\cos z}{z}$ is $z=0$. Since

$$\frac{\cos z}{z}=\frac{1}{z}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\cdots \qquad (0<|z|<\infty),$$

the principal part is $\frac{1}{z}$. This means that $z=0$ is a (simple) pole of $\frac{\cos z}{z}$.

(e) Upon writing $\frac{1}{(2-z)^3}=\frac{-1}{(z-2)^3}$, we find that the principal part of $\frac{1}{(2-z)^3}$ at its isolated singular point $z=2$ is simply the function itself. That point is evidently a pole (of order 3).

Ex.2

(a) The singular point is $z=0$. Since

$$\frac{1-\cosh z}{z^3}=\frac{1}{z^3}\left[1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots\right)\right]=-\frac{1}{2!}\cdot\frac{1}{z}-\frac{z}{4!}-\frac{z^3}{6!}-\cdots$$

when $0<|z|<\infty$, we have $m=1$ and $B=-\frac{1}{2!}=-\frac{1}{2}$.

(b) Here the singular point is also $z=0$. Since

$$\frac{1-\exp(2z)}{z^4}=\frac{1}{z^4}\left[1-\left(1+\frac{2z}{1!}+\frac{2^2z^2}{2!}+\frac{2^3z^3}{3!}+\frac{2^4z^4}{4!}+\frac{2^5z^5}{5!}+\cdots\right)\right]$$

$$=-\frac{2}{1!}\cdot\frac{1}{z^3}-\frac{2^2}{2!}\cdot\frac{1}{z^2}-\frac{2^3}{3!}\cdot\frac{1}{z}-\frac{2^4}{4!}-\frac{2^5}{5!}z-\cdots$$

when $0<|z|<\infty$, we have $m=3$ and $B=-\frac{2^3}{3!}=-\frac{4}{3}$.

(c) The singular point of $\frac{\exp(2z)}{(z-1)^2}$ is $z=1$. The Taylor series

$$\exp(2z)=e^{2(z-1)}e^2=e^2\left[1+\frac{2(z-1)}{1!}+\frac{2^2(z-1)^2}{2!}+\frac{2^3(z-1)^3}{3!}+\cdots\right] \qquad (|z|<\infty)$$

enables us to write the Laurent series

$$\frac{\exp(2z)}{(z-1)^2}=e^2\left[\frac{1}{(z-1)^2}+\frac{2}{1!}\cdot\frac{1}{z-1}+\frac{2^2}{2!}+\frac{2^2}{3!}(z-1)+\cdots\right] \qquad (0<|z-1|<\infty).$$

Thus $m=2$ and $B=e^2\frac{2}{1!}=2e^2$.

Ex.3

Since f is analytic at z_0, it has a Taylor series representation

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots \qquad (|z-z_0|<R_0).$$

Let g be defined by means of the equation

$$g(z)=\frac{f(z)}{z-z_0}.$$

(a) Suppose that $f(z_0) \neq 0$. Then

$$g(z) = \frac{1}{z-z_0}\left[f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots\right]$$

$$= \frac{f(z_0)}{z-z_0} + \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z-z_0) + \cdots \qquad (0 < |z-z_0| < R_0).$$

This shows that g has a simple pole at z_0, with residue $f(z_0)$.

(b) Suppose, on the other hand, that $f(z_0) = 0$. Then

$$g(z) = \frac{1}{z-z_0}\left[\frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots\right]$$

$$= \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z-z_0) + \cdots \qquad (0 < |z-z_0| < R_0).$$

Since the principal part of g at z_0 is just 0, the point $z=0$ is a removable singular point of g.

page 248

Ex.1

(a) The function $f(z) = \dfrac{z^2+2}{z-1}$ has an isolated singular point at $z=1$. Writing $f(z) = \dfrac{\phi(z)}{z-1}$, where $\phi(z) = z^2+2$, and observing that $\phi(z)$ is analytic and nonzero at $z=1$, we see that $z=1$ is a pole of order $m=1$ and that the residue there is $B = \phi(1) = 3$.

(b) If we write

$$f(z) = \left(\frac{z}{2z+1}\right)^3 = \frac{\phi(z)}{\left[z-\left(-\frac{1}{2}\right)\right]^3}, \quad \text{where} \quad \phi(z) = \frac{z^3}{8},$$

we see that $z = -\frac{1}{2}$ is a singular point of f. Since $\phi(z)$ is analytic and nonzero at that point, f has a pole of order $m=3$ there. The residue is

$$B = \frac{\phi''(-1/2)}{2!} = -\frac{3}{16}.$$

(c) The function

$$\frac{\exp z}{z^2+\pi^2} = \frac{\exp z}{(z-\pi i)(z+\pi i)}$$

has poles of order $m=1$ at the two points $z = \pm\pi i$. The residue at $z=\pi i$ is

$$B_1 = \frac{\exp \pi i}{2\pi i} = \frac{-1}{2\pi i} = \frac{i}{2\pi},$$

and the one at $z = -\pi i$ is

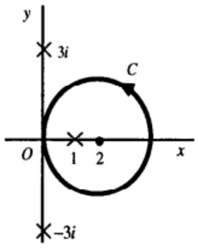
$$B_2 = \frac{\exp(-\pi i)}{-2\pi i} = \frac{-1}{-2\pi i} = -\frac{i}{2\pi}.$$

Ex.3

(a) We wish to evaluate the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

where C is the circle $|z-2|=2$, taken in the counterclockwise direction. That circle and the singularities $z=1, \pm 3i$ of the integrand are shown in the figure just below.



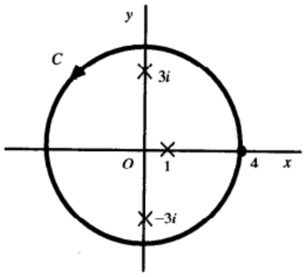
Observe that the point $z=1$, which is the only singularity inside C, is a simple pole of the integrand and that

$$\operatorname*{Res}_{z=1}\frac{3z^3+2}{(z-1)(z^2+9)}=\left.\frac{3z^3+2}{z^2+9}\right]_{z=1}=\frac{1}{2}.$$

According to the residue theorem, then,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz=2\pi i\left(\frac{1}{2}\right)=\pi i.$$

(b) Let us redo part *(a)*when C is changed to be the positively oriented circle $|z|=4$, shown in the figure below.



In this case, all three singularities $z=1, \pm 3i$ of the integrand are interior to C. We already know from part *(a)* that

$$\operatorname*{Res}_{z=1}\frac{3z^3+2}{(z-1)(z^2+9)}=\frac{1}{2}.$$

It is, moreover, straightforward to show that

$$\operatorname*{Res}_{z=3i}\frac{3z^3+2}{(z-1)(z^2+9)}=\left.\frac{3z^3+2}{(z-1)(z+3i)}\right]_{z=3i}=\frac{15+49i}{12}$$

and

$$\operatorname*{Res}_{z=-3i}\frac{3z^3+2}{(z-1)(z^2+9)}=\left.\frac{3z^3+2}{(z-1)(z-3i)}\right]_{z=-3i}=\frac{15-49i}{12}.$$

The residue theorem now tells us that

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz=2\pi i\left(\frac{1}{2}+\frac{15+49i}{12}+\frac{15-49i}{12}\right)=6\pi i.$$

Ex.6

In each part of this problem, C denotes the positively oriented circle $|z|=3$.

(a) It is straightforward to show that

$$\text{if } f(z)=\frac{(3z+2)^2}{z(z-1)(2z+5)}, \quad \text{then} \quad \frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

This function $\frac{1}{z^2}f\left(\frac{1}{z}\right)$ has a simple pole at $z=0$, and

$$\int_C \frac{(3z+2)^2}{z(z-1)(2z+5)}\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right]=2\pi i\left(\frac{9}{2}\right)=9\pi i.$$

(b) Likewise,

$$\text{if } f(z)=\frac{z^3(1-3z)}{(1+z)(1+2z^4)}, \quad \text{then} \quad \frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{z-3}{z(z+1)(z^4+2)}.$$

The function $\frac{1}{z^2}f\left(\frac{1}{z}\right)$ has a simple pole at $z=0$, and we find here that

$$\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right]=2\pi i\left(-\frac{3}{2}\right)=-3\pi i.$$

(c) Finally,

$$\text{if } f(z)=\frac{z^3 e^{1/z}}{1+z^3}, \quad \text{then} \quad \frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{e^z}{z^2(1+z^3)}.$$

The point $z=0$ is a pole of order 2 of $\frac{1}{z^2}f\left(\frac{1}{z}\right)$. The residue is $\phi'(0)$, where

$$\varphi(z)=\frac{e^z}{1+z^3}.$$

Since

$$\phi'(z)=\frac{(1+z^3)e^z-e^z 3z^2}{(1+z^3)^2},$$

the value of $\phi'(0)$ is 1. So

$$\int_C \frac{z^3 e^{1/z}}{1+z^3}\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right]=2\pi i(1)=2\pi i.$$

page 255

Ex.6

The path C here is the positively oriented boundary of the rectangle with vertices at the points ± 2 and $\pm 2 + i$. The problem is to evaluate the integral

$$\int_C \frac{dz}{(z^2-1)^2+3}.$$

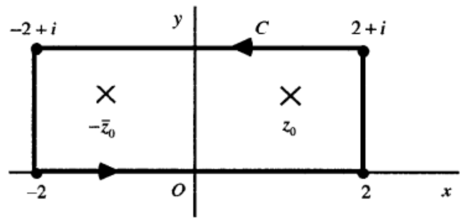
The isolated singularities of the integrand are the zeros of the polynomial

$$q(z) = (z^2-1)^2+3.$$

Setting this polynomial equal to zero and solving for z^2, we find that any zero z of $q(z)$ has the property $z^2 = 1 \pm \sqrt{3}i$. It is straightforward to find the two square roots of $1+\sqrt{3}i$ and also the two square roots of $1-\sqrt{3}i$. These are the four zeros of $q(z)$. Only two of those zeros,

$$z_0 = \sqrt{2}e^{i\pi/6} = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\bar{z}_0 = -\sqrt{2}e^{-i\pi/6} = \frac{-\sqrt{3}+i}{\sqrt{2}},$$

lie inside C. They are shown in the figure below.



To find the residues at z_0 and $-\bar{z}_0$, we write the integrand of the integral to be evaluated as

$$\frac{1}{(z^2-1)^2+3} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = (z^2-1)^2+3.$$

This polynomial $q(z)$ is, of course, the same $q(z)$ as above; hence $q(z_0) = 0$. Note, too, that p and q are analytic at z_0 and that $p(z_0) \neq 0$. Finally, it is straightforward to show that $q'(z) = 4z\left(z^2-1\right)$ and hence that

$$q'(z_0) = 4z_0\left(z_0^2-1\right) = -2\sqrt{6}+6\sqrt{2}i \neq 0.$$

We may conclude, then, that z_0 is a simple pole of the integrand, with residue

$$\frac{p(z_0)}{q'(z_0)} = \frac{1}{-2\sqrt{6}+6\sqrt{2}i}.$$

Similar results are to be found at the singular point $-\bar{z}_0$. To be specific, it is easy to see that

$$q'(-\bar{z}_0) = -q'(\bar{z}_0) = -\overline{q'(z_0)} = 2\sqrt{6}+6\sqrt{2}i \neq 0,$$

the residue of the integrand at $-\bar{z}_0$ being

$$\frac{p(-\bar{z}_0)}{q'(-\bar{z}_0)} = \frac{1}{2\sqrt{6}+6\sqrt{2}i}.$$

Finally, by the residue theorem,

$$\int_C \frac{dz}{(z^2-1)^2+3} = 2\pi i\left(\frac{1}{-2\sqrt{6}+6\sqrt{2}i} + \frac{1}{2\sqrt{6}+6\sqrt{2}i}\right) = \frac{\pi}{2\sqrt{2}}.$$

Ex.7

We are given that $f(z) = 1/[q(z)]^2$, where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. These conditions on q tell us that q has a zero of order $m = 1$ at z_0. Hence $q(z) = (z - z_0)g(z)$, where g is a function that is analytic and nonzero at z_0; and this enables us to write

$$f(z) = \frac{\phi(z)}{(z-z_0)^2}, \quad \text{where} \quad \phi(z) = \frac{1}{[g(z)]^2}.$$

So f has a pole of order 2 at z_0, and

$$\operatorname*{Res}_{z=z_0} f(z) = \phi'(z_0) = -\frac{2g'(z_0)}{[g(z_0)]^3}.$$

But, since $q(z) = (z - z_0)g(z)$, we know that

$$q'(z) = (z-z_0)g'(z) + g(z) \quad \text{and} \quad q''(z) = (z-z_0)g''(z) + 2g'(z).$$

Then, by setting $z = z_0$ in these last two equations, we find that

$$q'(z_0) = g(z_0) \quad \text{and} \quad q''(z_0) = 2g'(z_0).$$

Consequently, our expression for the residue of f at z_0 can be put in the desired form:

$$\operatorname*{Res}_{z=0} f(z) = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

Ex.8

(a) To find the residue of the function $\csc^2 z$ at $z = 0$, we write

$$\csc^2 z = \frac{1}{[q(z)]^2}, \quad \text{where} \quad q(z) = \sin z.$$

Since q is entire, $q(0) = 0$, and $q'(0) = 1 \neq 0$, the result in Exercise 7 tells us that

$$\operatorname*{Res}_{z=0} \csc^2 z = -\frac{q''(0)}{[q'(0)]^3} = 0.$$

(*b*) The residue of the function $\dfrac{1}{\left(z+z^2\right)^2}$ at $z=0$ can be obtained by writing

$$\frac{1}{\left(z+z^2\right)^2}=\frac{1}{[q(z)]^2}, \quad \text{where} \quad q(z)=z+z^2.$$

Inasmuch as q is entire, $q(0)=0$, and $q'(0)=1\neq 0$, we know from Exercise 7 that

$$\operatorname*{Res}_{z=0}\frac{1}{\left(z+z^2\right)^2}=-\frac{q''(0)}{[q'(0)]^3}=-2.$$