(Title WIP)

Math classes are highly structured, and they just hand you results. It's way more fun to explore math on your own. I want to take you on three journeys, showing what it feels like to (re)invent math. The pre-reqs for this doc can change based on where in the doc you are, but not much prior knowledge is needed.

1 Roadside Gem

I'm in my AP Calculus AB class, and we've just learned about partial fraction decomposition. Here's a reminder of what that is: if you have a function that is the ratio of two polynomials, you can write it as a sum of simpler fractions. For example, $\frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)} = \frac{1}{x-2} - \frac{1}{x-3}$. Anyway, I'm facing this problem:

$$\int \frac{1}{x^2+1}\,\mathrm{d}x$$

... and I'm now a bit stuck, because I can't really factor x^2+1. Or... perhaps I can.

$$x^2+1 = x^2 - i^2 = (x-i)(x+i)$$

(By now, some of you might be screaming at the page about the integral being related to a certain trig function or whatever, but hey shh for now). Anyway, let's apply partial fraction decomposition:

$$\begin{aligned}
\frac{1}{x^2+1} &= \frac{A}{x-i} + \frac{B}{x+i} \\
1 &= A(x+i) + B(x-i) \\
\text{Let } x = i \implies 1 &= 2iA \\
\text{Let } x = -i \implies 1 &= -2iB \\
\therefore A &= \frac{1}{2i} = -\frac{i}{2} \\
\therefore B &= -\frac{1}{2i} = \frac{i}{2} \\
\frac{1}{x^2+1} &= \frac{i}{2}\left(\frac{1}{x+i} - \frac{1}{x-i}\right)
\end{aligned}$$

Now we can evaluate the integral:

$$\begin{aligned}\int \frac{1}{x^2+1}\,\mathrm{d}x &= \frac{i}{2}\int\left(\frac{1}{x+i}-\frac{1}{x-i}\right)\mathrm{d}x\\ &= \frac{i}{2}\left(\ln|x+i|-\ln|x-i|\right)+C\\ &= \frac{i}{2}\ln\left|\frac{x+i}{x-i}\right|+C\end{aligned}$$

You would be right to question this. What does it mean to take the natural log of a complex number? I have no clue. But hey, let's just assume this is valid and as a bit of a joke, we submit this wacky answer as homework and move on to the other problems. (Turns out you still get full points, but you suspect this is because your teacher does not look too closely)

Also, let's drop the absolute value signs. Like, at this point we're plugging in complex numbers, so negative numbers are the least of our worries.

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \frac{i}{2}\ln\left(\frac{x+i}{x-i}\right)+C$$

Later in class, I find out that the integral is actually a standard one and that:

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan(x)+C$$

I didn't see this, and so now I had my own answer to the problem. Let's take a leap of faith and assume that my answer is valid. What happens if we equate the two answers?

$$\frac{i}{2}\ln\left(\frac{x+i}{x-i}\right)+C = \arctan(x) \qquad \text{(only one constant is needed)}$$

Hmm, very interesting. Something involving logs and complex numbers on one side equals something involving inverse trig on the other. Maybe if we could find the inverse of this function, we could find a new way to represent $\tan(x)$. That would be interesting! But we need to find that constant C first.

(To be honest, at this point I put that expression into WolframAlpha to find what C is, but let's pretend I didn't do that and use a semi-rigorous argument instead.)

Since the two expressions are equal, their limits to infinity must be equal. Let's take the limit of both sides as $x \to \infty$:

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{i}{2}\ln\left(\frac{x+i}{x-i}\right)+C\right) &= \lim_{x\to\infty}\arctan(x)\\ C+\frac{i}{2}\lim_{x\to\infty}\ln\left(\frac{x+i}{x-i}\right) &= \frac{\pi}{2}\end{aligned}$$

Hmm, we don't actually know what that limit on the LHS is, but let's make the argument that as $x \to \infty$, the difference in imaginary part "matters" less and less. So:

$$\begin{aligned} C + \frac{i}{2} \lim_{x\to\infty} \ln\left(\frac{x+i}{x-i}\right) &= \frac{\pi}{2} \\ C + \frac{i}{2} \lim_{x\to\infty} \ln\left(\frac{x}{x}\right) &= \frac{\pi}{2} \\ C + \frac{i}{2} \ln(1) &= \frac{\pi}{2} \\ C + 0 &= \frac{\pi}{2} \\ \therefore C &= \frac{\pi}{2} \end{aligned}$$

Finally, we now have a solid new representation for $\arctan(x)$:

$$\arctan(x) = \frac{i}{2} \ln\left(\frac{x+i}{x-i}\right) + \frac{\pi}{2}$$

Isn't that kinda cool? Yes we made some mildly shady arguments. But they're reasonable, and this is how discovery works. Come on, let's just see what happens. Let's try and find what tan is. Let's start by introducing two new variables:

$$\begin{aligned} \text{Define } u, v \text{ such that } \tan(u) &= v \\ \text{Then, } \arctan(v) &= u \\ \frac{i}{2} \ln\left(\frac{v+i}{v-i}\right) + \frac{\pi}{2} &= u \end{aligned}$$

If we isolate v in the above equation, we'll have a new representation for $\tan(x)$. Let's do that:

$$\begin{aligned} u &= \frac{i}{2} \ln\left(\frac{v+i}{v-i}\right) + \frac{\pi}{2} \\ 2u &= i \ln\left(\frac{v+i}{v-i}\right) + \pi \\ 2u - \pi &= i \ln\left(\frac{v+i}{v-i}\right) \\ -i(2u - \pi) &= \ln\left(\frac{v+i}{v-i}\right) \\ (\pi - 2u)i &= \ln\left(\frac{v+i}{v-i}\right) \\ e^{i\pi - 2ui} &= \frac{v+i}{v-i} \end{aligned}$$

Wait. Hold on a sec. Do you see that? If only we didn't have that pesky $2ui$ we may be able to find out the value of $e^{i\pi}$! And that would be quite a gem.

Well let's try setting $u = 0$ and see what happens:

$$e^{\pi i - 2ui} = \frac{v+i}{v-i}$$
$$\text{Set } u = 0 \implies e^{i\pi} = \frac{v+i}{v-i}$$

Welp. We don't really know what v is. So we can't find $e^{i\pi}$. Right? Wrong! We know what v is since we defined u and v to be related by tan. Since $\tan(u) = v$, we know that when $u = 0$, $v = tan(0) = 0$. So:

$$e^{i\pi} = \frac{0+i}{0-i}$$
$$e^{i\pi} = \frac{i}{-i}$$

$$e^{i\pi} = -1$$

And there, we have found the gem. But the road goes on, and so I strongly encourage you to carry on finding what tan is. It's a fun journey, and you rediscover Euler's formula among other things along the way.

2 Exping a matrix

3 $\exp(D)f(x) = ?$

I remember watching a 3Blue1Brown video that ended on the massive cliffhanger of what $e^{\frac{\mathrm{d}}{\mathrm{d}x}}$ is (in words, the exponential of the $\frac{\mathrm{d}}{\mathrm{d}x}$ operator), and so let's explore that.

To simplify things, let's use Heaviside's notation for the derivative operator.

$$D = \frac{\mathrm{d}}{\mathrm{d}x}$$

What's an operator, you ask? It's just something that takes in a function and spits out another function. For example, applying D (same as $\frac{\mathrm{d}}{\mathrm{d}x}$) to the function x^2 gives $2x$.

As usual, let's start by expanding e^D:

$$e^D = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \frac{D^4}{4!} + \cdots$$

Does this maybe feel like an abuse of notation? Like, we're just using this Taylor series expansion in a way it's not meant to? It should. You could understandably scoff and say that this is complete nonsense. But again, let's just be reasonable whenever we run into problems, and see what happens.

Here's a bunch of questions you might reasonably ask (in order of decreasing obviousness)

1. What does multiplying two operators mean???
2. What's addition??
3. And what does multiplying by a scalar do?

Okay now let's come up with some reasonable answers.

1. Let's say that multiplying operators means applying them in sequence.

$$D^2 = DD = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

2. Let's say that adding operators means applying the operator and adding the results. Here's some examples.

$$\begin{aligned}(D^2 + 2D + 1)\sin x &= D^2 \sin x + 2D \sin x + \sin x \\ &= -\sin x + 2\cos x + \sin x \\ &= 2\cos x\end{aligned}$$

Ah but also $(D^2+2D+1) = (D+1)^2$, so let's confirm that gives us the same answer:

$$\begin{aligned}(D+1)^2 \sin x &= (D+1)(D+1)\sin x \\ &= (D+1)(\cos x + \sin x) \\ &= (-\sin x + \cos x) + (\cos x + \sin x) \\ &= 2\cos x\end{aligned}$$

Wow, maybe we have some good stuff here.

3. I kind of already used it for the second one's answer. It's pretty obvious:

$$\begin{aligned}(1)f(x) &= f(x) \\ (\pi D)f(x) &= \pi(Df(x))\end{aligned}$$

Finally, armed with those reasonably definitions, we now know what this means. We don't yet know what it actually does, but we can evaluate it.

$$e^D = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \frac{D^4}{4!} + \cdots$$

(Oh, btw, that 1 is an operator. Consider it the identity operator. Don't be fooled!)

Let's apply it to a couple functions and see what happens.

$$\begin{aligned}e^D x &= (1)x + (D)x + \frac{(D^2)x}{2!} + \frac{(D^3)x}{3!} + \cdots \\ &= x + 1 + 0 + 0 + \ldots \\ &= x + 1\end{aligned}$$

$$\begin{aligned}e^D x^2 &= (1)x^2 + (D)x^2 + \frac{(D^2)x^2}{2!} + \frac{(D^3)x^2}{3!} + \cdots \\ &= x^2 + 2x + 1 + 0 + \ldots \\ &= (x+1)^2\end{aligned}$$

$$\begin{aligned}e^D e^x &= (1)e^x + (D)e^x + \frac{(D^2)e^x}{2!} + \frac{(D^3)e^x}{3!} + \cdots \\ &= e^x \cdot \left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) \\ &= e^x e \\ &= e^{x+1}\end{aligned}$$

Perhaps you are noticing a pattern by now...

Let's try something a bit harder. Let's try $e^D \sin x$.

$$
\begin{aligned}
e^D \sin x &= (1)\sin x + (D)\sin x + \frac{(D^2)\sin x}{2!} + \frac{(D^3)\sin x}{3!} + \cdots \\
&= \frac{\sin x}{0!} + \frac{\cos x}{1!} + \frac{-\sin x}{2!} + \frac{-\cos x}{3!} + \\
&\quad \frac{\sin x}{4!} + \frac{\cos x}{5!} + \frac{-\sin x}{6!} + \frac{-\cos x}{7!} + \cdots \\
&= \sin x \left(\frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \right) + \cos x \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \right)
\end{aligned}
$$

Hmm, do those infinite sums look a bit familar? Let's remind ourselves of the Taylor series expansions of sin and cos:

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}
$$

Thus:

$$
\begin{aligned}
e^D \sin x &= \sin x \cos 1 + \cos x \sin 1 \\
\sin(x + a) &= \sin x \cos a + \cos x \sin a && \text{(angle sum)} \\
&\therefore \\
e^D \sin x &= \sin(x + 1)
\end{aligned}
$$

Wow. Look at that simplification! That's crazy. Wow it really does seem like:

$$
e^D f(x) = f(x + 1)
$$

Looks like it takes in a function and shifts it by one. Pretty odd. And it seems like Taylor series play a big role. So maybe we can figure out if this fact is generally true by assuming we have a series representation of some function $f(x)$ and then applying e^D to it. Let's try that.