(Title WIP)

Math classes are highly structured, and they just hand you results. It's way more fun to explore math on your own. I want to take you on three journeys, showing what it feels like to (re)invent math. The pre-reqs for this doc can change based on where in the doc you are, but not much prior knowledge is needed.

1 Roadside Gem

I'm in my AP Calculus AB class, and we've just learned about partial fraction decomposition. Here's a reminder of what that is: if you have a function that is the ratio of two polynomials, you can write it as a sum of simpler fractions. For example, $\frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)} = \frac{1}{x-2} - \frac{1}{x-3}$. Anyway, I'm facing this problem:

$$\int \frac{1}{x^2+1}\,\mathrm{d}x$$

... and I'm now a bit stuck, because I can't really factor x^2+1. Or... perhaps I can.

$$x^2+1 = x^2 - i^2 = (x-i)(x+i)$$

(By now, some of you might be screaming at the page about the integral being related to a certain trig function or whatever, but hey shh for now). Anyway, let's apply partial fraction decomposition:

$$\begin{aligned}
\frac{1}{x^2+1} &= \frac{A}{x-i} + \frac{B}{x+i} \\
1 &= A(x+i) + B(x-i) \\
\text{Let } x = i \implies 1 &= 2iA \\
\text{Let } x = -i \implies 1 &= -2iB \\
\therefore A &= \frac{1}{2i} = -\frac{i}{2} \\
\therefore B &= -\frac{1}{2i} = \frac{i}{2} \\
\frac{1}{x^2+1} &= \frac{i}{2}\left(\frac{1}{x+i} - \frac{1}{x-i}\right)
\end{aligned}$$

Now we can evaluate the integral:

$$\begin{aligned}\int \frac{1}{x^2+1}\,\mathrm{d}x &= \frac{i}{2}\int\left(\frac{1}{x+i}-\frac{1}{x-i}\right)\mathrm{d}x\\ &= \frac{i}{2}\left(\ln|x+i|-\ln|x-i|\right)+C\\ &= \frac{i}{2}\ln\left|\frac{x+i}{x-i}\right|+C\end{aligned}$$

You would be right to question this. What does it mean to take the natural log of a complex number? I have no clue. But hey, let's just assume this is valid and as a bit of a joke, we submit this wacky answer as homework and move on to the other problems. (Turns out you still get full points, but you suspect this is because your teacher does not look too closely)

Also, let's drop the absolute value signs. Like, at this point we're plugging in complex numbers, so negative numbers are the least of our worries.

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \frac{i}{2}\ln\left(\frac{x+i}{x-i}\right)+C$$

Later in class, I find out that the integral is actually a standard one and that:

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan(x)+C$$

I didn't see this, and so now I had my own answer to the problem. Let's take a leap of faith and assume that my answer is valid. What happens if we equate the two answers?

$$\frac{i}{2}\ln\left(\frac{x+i}{x-i}\right)+C = \arctan(x) \qquad \text{(only one constant is needed)}$$

Hmm, very interesting. Something involving logs and complex numbers on one side equals something involving inverse trig on the other. Maybe if we could find the inverse of this function, we could find a new way to represent $\tan(x)$. That would be interesting! But we need to find that constant C first.

(To be honest, at this point I put that expression into WolframAlpha to find what C is, but let's pretend I didn't do that and use a semi-rigorous argument instead.)

Since the two expressions are equal, their limits to infinity must be equal. Let's take the limit of both sides as $x \to \infty$:

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{i}{2}\ln\left(\frac{x+i}{x-i}\right)+C\right) &= \lim_{x\to\infty}\arctan(x)\\ C+\frac{i}{2}\lim_{x\to\infty}\ln\left(\frac{x+i}{x-i}\right) &= \frac{\pi}{2}\end{aligned}$$

Hmm, we don't actually know what that limit on the LHS is, but let's make the argument that as $x \to \infty$, the difference in imaginary part "matters" less and less. So:

$$\begin{aligned} C + \frac{i}{2} \lim_{x\to\infty} \ln\left(\frac{x+i}{x-i}\right) &= \frac{\pi}{2} \\ C + \frac{i}{2} \lim_{x\to\infty} \ln\left(\frac{x}{x}\right) &= \frac{\pi}{2} \\ C + \frac{i}{2} \ln(1) &= \frac{\pi}{2} \\ C + 0 &= \frac{\pi}{2} \\ \therefore C &= \frac{\pi}{2} \end{aligned}$$

Finally, we now have a solid new representation for $\arctan(x)$:

$$\arctan(x) = \frac{i}{2} \ln\left(\frac{x+i}{x-i}\right) + \frac{\pi}{2}$$

Isn't that kinda cool? Yes we made some mildly shady arguments. But they're reasonable, and this is how discovery works. Come on, let's just see what happens. Let's try and find what tan is. Let's start by introducing two new variables:

$$\begin{aligned} \text{Define } u, v \text{ such that } \tan(u) &= v \\ \text{Then, } \arctan(v) &= u \\ \frac{i}{2} \ln\left(\frac{v+i}{v-i}\right) + \frac{\pi}{2} &= u \end{aligned}$$

If we isolate v in the above equation, we'll have a new representation for $\tan(x)$. Let's do that:

$$\begin{aligned} u &= \frac{i}{2} \ln\left(\frac{v+i}{v-i}\right) + \frac{\pi}{2} \\ 2u &= i \ln\left(\frac{v+i}{v-i}\right) + \pi \\ 2u - \pi &= i \ln\left(\frac{v+i}{v-i}\right) \\ -i(2u - \pi) &= \ln\left(\frac{v+i}{v-i}\right) \\ (\pi - 2u)i &= \ln\left(\frac{v+i}{v-i}\right) \\ e^{i\pi - 2ui} &= \frac{v+i}{v-i} \end{aligned}$$

Wait. Hold on a sec. Do you see that? If only we didn't have that pesky $2ui$ we may be able to find out the value of $e^{i\pi}$! And that would be quite a gem.

Well let's try setting $u = 0$ and see what happens:

$$e^{\pi i - 2ui} = \frac{v+i}{v-i}$$
$$\text{Set } u = 0 \implies e^{i\pi} = \frac{v+i}{v-i}$$

Welp. We don't really know what v is. So we can't find $e^{i\pi}$. Right? Wrong! We know what v is since we defined u and v to be related by tan. Since $\tan(u) = v$, we know that when $u = 0$, $v = tan(0) = 0$. So:

$$e^{i\pi} = \frac{0+i}{0-i}$$
$$e^{i\pi} = \frac{i}{-i}$$
$$e^{i\pi} = -1$$

And there, we have found the gem. But the road goes on, and so I strongly encourage you to carry on finding what tan is. It's a fun journey, and you rediscover Euler's formula among other things along the way.

2 Matrix Flow

3 Taylor Might

I remember watching a 3Blue1Brown video that ended on the massive cliffhanger of what $e^{\frac{\mathrm{d}}{\mathrm{d}x}}$ is (in words, the exponential of the $\frac{\mathrm{d}}{\mathrm{d}x}$ operator), and so let's explore that.

To simplify things, let's use Heaviside's notation for the derivative operator.

$$D = \frac{\mathrm{d}}{\mathrm{d}x}$$

What's an operator, you ask? It's just something that takes in a function and spits out another function. For example, applying D (same as $\frac{\mathrm{d}}{\mathrm{d}x}$) to the function x^2 gives $2x$.

As usual, let's start by expanding e^D:

$$e^D = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \frac{D^4}{4!} + \cdots$$

Does this maybe feel like an abuse of notation? Like, we're just using this Taylor series expansion in a way it's not meant to? It should. You could understandably scoff and say that this is complete nonsense. But again, let's just be reasonable whenever we run into problems, and see what happens.

Here's a bunch of questions you might reasonably ask (in order of decreasing obviousness)

1. What does multiplying two operators mean???
2. What's addition??
3. And what does multiplying by a scalar do?

Okay now let's come up with some reasonable answers.

1. Let's say that multiplying operators means applying them in sequence.

$$D^2 = DD = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

2. Let's say that adding operators means applying them and adding the results.

$$\begin{aligned}(D^2 + 2D + 1)\sin x &= D^2 \sin x + 2D\sin x + \sin x \\ &= -\sin x + 2\cos x + \sin x \\ &= 2\cos x\end{aligned}$$

Ah, but also $(D^2 + 2D + 1) = (D + 1)^2$, so let's confirm that gives us the same answer:

$$\begin{aligned}(D+1)^2 \sin x &= (D+1)(D+1)\sin x \\ &= (D+1)(\cos x + \sin x) \\ &= (-\sin x + \cos x) + (\cos x + \sin x) \\ &= 2\cos x\end{aligned}$$

Wow, maybe we have some good stuff here.

3. I kind of already used it for the second one's answer. It's pretty obvious:

$$
\begin{aligned}
(1)f(x) &= f(x) \\
(\pi D)f(x) &= \pi(Df(x))
\end{aligned}
$$

Finally, armed with these reasonable definitions, we now know what this operation means. We don't yet know what it actually does, but we can evaluate it.

$$
e^D = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \frac{D^4}{4!} + \cdots
$$

(Oh, btw, that 1 is an operator. Consider it the identity operator. Don't be fooled!)

Let's apply it to a couple functions and see what happens.

$$
\begin{aligned}
e^D x &= (1)x + (D)x + \frac{(D^2)x}{2!} + \frac{(D^3)x}{3!} + \cdots \\
&= x + 1 + 0 + 0 + \ldots \\
&= x + 1
\end{aligned}
$$

$$
\begin{aligned}
e^D x^2 &= (1)x^2 + (D)x^2 + \frac{(D^2)x^2}{2!} + \frac{(D^3)x^2}{3!} + \cdots \\
&= x^2 + 2x + 1 + 0 + \ldots \\
&= (x+1)^2
\end{aligned}
$$

$$
\begin{aligned}
e^D e^x &= (1)e^x + (D)e^x + \frac{(D^2)e^x}{2!} + \frac{(D^3)e^x}{3!} + \cdots \\
&= e^x \cdot \left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) \\
&= e^x e \\
&= e^{x+1}
\end{aligned}
$$

Perhaps you are noticing a pattern by now...

Let's try something a bit harder. Let's try $e^D \sin x$.

$$e^D \sin x = (1)\sin x + (D)\sin x + \frac{(D^2)\sin x}{2!} + \frac{(D^3)\sin x}{3!} + \cdots$$

$$\begin{aligned} &= \frac{\sin x}{0!} + \frac{\cos x}{1!} + \frac{-\sin x}{2!} + \frac{-\cos x}{3!} + \\ &\quad \frac{\sin x}{4!} + \frac{\cos x}{5!} + \frac{-\sin x}{6!} + \frac{-\cos x}{7!} + \cdots \end{aligned}$$

$$= \sin x \left(\frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \right) + \cos x \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \right)$$

Hmm, do those infinite sums look a bit familar? Let's remind ourselves of the Taylor series expansions of sin and cos:

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \therefore \sin 1 &= \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \\ \therefore \cos 1 &= \frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \end{aligned}$$

Thus:

$$\begin{aligned} e^D \sin x &= \sin x \cos 1 + \cos x \sin 1 \\ \sin(x+a) &= \sin x \cos a + \cos x \sin a && \text{(angle sum)} \\ &\therefore \\ e^D \sin x &= \sin(x+1) \end{aligned}$$

Wow. Look at that simplification! That's crazy. Wow it really does seem like:

$$e^D f(x) = f(x+1)$$

Pretty odd. And it seems like Taylor series play a big role. Maybe we can figure out if this fact is generally true by assuming we have a series representation of some function $f(x)$ and then applying e^D to it. Let's try that.

Let's say that $f(x)$ has a certain series representation:

$$\begin{aligned}
f(x) &= a_0 + a_1 x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \cdots \\
f'(x) &= a_1 + a_2 x + \frac{a_3}{2!}x^2 + \frac{a_4}{3!}x^3 + \cdots \\
f''(x) &= a_2 + a_3 x + \frac{a_4}{2!}x^2 + \frac{a_5}{3!}x^3 + \cdots \\
&\cdots \\
\therefore f^{(k)}(x) &= \sum_{n=0}^{\infty} \frac{a_{n+k}}{n!}x^n
\end{aligned}$$

Now let's apply e^D to $f(x)$:

$$\begin{aligned}
e^D f(x) &= f(x) + f'(x) + \frac{f''(x)}{2!} + \frac{f'''(x)}{3!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n + \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!}x^n + \frac{1}{2!}\sum_{n=0}^{\infty} \frac{a_{n+2}}{n!}x^n + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!}\left(a_n + a_{n+1} + \frac{a_{n+2}}{2!} + \frac{a_{n+3}}{3!} + \cdots\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\sum_{k=0}^{\infty} \frac{a_{n+k}}{k!}\right)
\end{aligned}$$

Hmm, we don't really know what that innermost series is. I spent a couple minutes looking at it until I realized the following:

$$\begin{aligned}
\text{(As shown)} \quad f^{(k)}(x) &= \sum_{n=0}^{\infty} \frac{a_{n+k}}{n!}x^n \\
f^{(k)}(1) &= \sum_{n=0}^{\infty} \frac{a_{n+k}}{n!}(1)^n \\
&= \sum_{n=0}^{\infty} \frac{a_{n+k}}{n!}
\end{aligned}$$

This is almost the form we want. Now let's rename n to k and k to n to get this:

$$f^{(n)}(1) = \sum_{k=0}^{\infty} \frac{a_{k+n}}{k!}$$

Convince yourself that renaming is a valid move. Now we can substitute this into our expression for $e^D f(x)$:

$$\begin{aligned}
e^D f(x) &= \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\sum_{k=0}^{\infty} \frac{a_{n+k}}{k!}\right) \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(1)
\end{aligned}$$

This is beginning to look a lot like a Taylor series expansion, except we seem to be missing the shift. Or are we?! Look:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(1) &= \sum_{n=0}^{\infty} \frac{(x+1-1)^n}{n!} f^{(n)}(1) \\ &= \sum_{n=0}^{\infty} \frac{((x+1)-1)^n}{n!} f^{(n)}(1) \\ &= f(x+1)\end{aligned}$$

That series perfectly matches the Taylor series expansion of $f(x+1)$ centered at 1. So it seems like we have a general result. If $f(x)$ has a series expansion with center 0, then:

$$e^D f(x) = f(x+1)$$

It's easy to see that we can extend this to any center c by constructing a new function $g(x) = f(x+c)$ and then applying the above result to $g(x)$. Thus, if $f(x)$ has a Taylor series expansion, applying the exponential of the derivative operator to it shifts the function by one!

A natural question to ask at this point is whether we can create any shift. For example, it's easy to see that applying e^D twice should shift $f(x)$ by 2. But also:

$$\begin{aligned}e^D e^D f(x) &= \left(e^D\right)^2 f(x) \\ f(x+2) &\overset{!?}{=} e^{2D} f(x)\end{aligned}$$

Huh. Based on this, we can see that it could be reasonable to conjecture that:

$$e^{sD} f(x) = f(x+s)$$

If we go through the earlier proof again but with s this time, it's not hard to see that:

$$e^{sD} f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\sum_{k=0}^{\infty} \frac{a_{n+k}}{k!} \cdot s^k \right)$$

Also not too hard to see that:

$$f^{(n)}(s) = \sum_{k=0}^{\infty} \frac{a_{k+n}}{k!} \cdot s^k$$

And so:

$$\begin{aligned}e^{sD} f(x) &= \sum_{n=0}^{\infty} \frac{((x+s)-s)^n}{n!} f^{(n)}(s) \\ &= f(x+s)\end{aligned}$$

Thus we have this general result. If $f(x)$ has a Taylor series expansion, then:

$$e^{s\frac{\mathrm{d}}{\mathrm{d}x}} f(x) = f(x+s)$$

Wow! Scaling the derivative operator by some number, then applying the exponential of that operator to a function, shifts the function by that number. Isn't that beautiful?

Why might this be useful? Great question. I guess, for example, you could derive the angle sum formula. Do let me know if you think of something. But also it's kind of just pretty.

More abuses of notation to consider:

$$e^{iD} f(x) \stackrel{?}{=} (\cos D + i \sin D)\, f(x) \stackrel{?}{=} f(x+i) \qquad \text{Pretty sure this is fine}$$

$$\left(e^D\right)^{\int} f(x) \stackrel{?}{=} \left(e^{D\int}\right) f(x) \stackrel{?}{=} e f(x) \qquad \text{Likely nonsense}$$

$$e^D e^{-D} f(x) = f(x) \qquad \text{Definitely true}$$

Try the first one! But I do want to remind you that notation abuse doesn't always work out:

$$\begin{aligned} e^{i\pi} &= e^{-i\pi} = -1 \\ e^{i\pi D} &\stackrel{?}{=} e^{-i\pi D} \end{aligned}$$

One side shifts functions by $i\pi$ and the other by $-i\pi$, so it's definitely not true. (Do you see why it works in one case but not the other? Hint: inverse.)

I still think most people are too abuse-of-notation averse, so I encourage you to try that. Have fun and play responsibly!