D1.1.1: Complex numbers

Let $z = x + iy$ and $w = a + ib$ where $x, y, a, b \in \mathbb{R}$. Then z and w are complex numbers. Furthermore:

1. $z = w$ **iff** $x = a$ and $y = b$.
2. $\mathrm{Re}(z) := x$ and $\mathrm{Im}(z) := y$.
3. $|z| := \sqrt{x^2 + y^2}$
4. The **complex conjugate** of z is:
$$\overline{z} := x - iy.$$
5. Addition and multiplication:
$$(x+iy)+(a+ib) = (x+a)+i(y+b)$$
$$(x+iy)(a+ib) = (xa-yb)+i(xb+ya).$$
6. $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$

with rule $i^2 = -1$.

L1.1.3

Let $u, w, z \in \mathbb{C}$ where $z = x + iy$. Then:

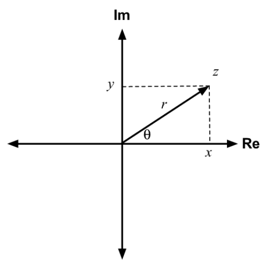
1. $z + w = w + z$ and $zw = wz$.
2. $u + (z + w) = (u + z) + w$
3. $u(zw) = (uz)w$
4. $u(z + w) = uz + uw$
5. $z + 0 = z$ and $1z = z$.
6. $\exists\big(-z := -x+i(-y)\big)$: $z+(-z) = 0$.
7. $\exists z^{-1} : zz^{-1} = 1$ where:
$$z^{-1} := \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

D1.1.5 and D1.1.7: Polar form

Let $z \in \mathbb{C}$ and $z = x + iy$. Then:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

for $r = \sqrt{x^2 + y^2}$ in complex plane.



L1.1.6

Let $\theta, \phi \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then:

1. $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$
2. $e^{in\theta} = (e^{i\theta})^n$

due to de Moivre's formula:

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

L1.1.9

Let $z, w \in \mathbb{C}$. Then:

1. $|z| = 0$ **iff** $z = 0$.
2. $|\overline{z}| = |z|$
3. $|zw| = |z||w|$
4. $\overline{\overline{z}} = z$
5. $|z|^2 = z\overline{z}$
6. $\overline{z + w} = \overline{z} + \overline{w}$
7. $\overline{zw} = \overline{z}\,\overline{w}$
8. $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.
9. $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$
10. $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$.

L1.1.10 – 11: Triangle inequalities

Let $z, w \in \mathbb{C}$. Then:

1. $|z + w| \leq |z| + |w|$
2. $||z| - |w|| \leq |z - w|$.

D1.1.12: Argument of z

Let $z = |z|e^{i\theta}$. Then:

$$\arg(z) := \theta \in (-\pi, \pi]$$

with period 2π.

P1.1.14

Let $z, w \in \mathbb{C}$. Then:

1. $\arg(zw) = \arg(z) + \arg(w)$
2. $\arg(\overline{z}) = -\arg(z)$

and holds under modulo 2π.

D1.2.1: Open and closed ϵ-discs

Let $z_0 \in \mathbb{C}$ and $\epsilon > 0$.

1. An **open** ϵ-disc centred at z_0 is:
$$D_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}.$$
2. A **closed** ϵ-disc centred at z_0 is:
$$\overline{D}_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}.$$

A **punctured** ϵ-disc centred at z_0 is:

$$D'_\epsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}.$$

D1.2.2: Open sets

Let $U \subset \mathbb{C}$. Set U is **open** if:

$$\forall z_0 \in U; \exists\epsilon > 0 : D_\epsilon(z_0) \subseteq U.$$

Subset F is **closed** if $\mathbb{C} \setminus F$ is open.

A **neighbourhood** of point $z_0 \in \mathbb{C}$ is an open set that contains z_0.

L1.2.3

Punctured disc $D'_\epsilon(z_0)$ is open.

D1.2.4: Limit points

Let $S \subseteq \mathbb{C}$. z_0 is a **limit point** of S if:

$$\forall\epsilon > 0; D'_\epsilon(z_0) \cap S \neq \emptyset.$$

The **closure** of S is set $\overline{S}$ and contains S and **all** its limit points.

L1.2.6

Let $S \subseteq \mathbb{C}$. S is closed **iff** $S = \overline{S}$.

D1.2.7: Bounded sets

Let $S \subseteq \mathbb{C}$. Set S is **bounded** if:

$$\forall z \in S; \exists M > 0 : |z| \leq S.$$

D1.2.8: ϵ-N convergence

Let $\mathbb{N} = \{0, 1, 2, \dots\}$.

Let $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ be a sequence and $z \in \mathbb{C}$. Then $\lim\limits_{n\to\infty} z_n = z$ if:

$$\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |z_n - z| < \epsilon.$$

L1.2.9

Let $z_n, z \in \mathbb{C}$ where $z_n = a_n + ib_n$.

Then $\lim\limits_{n\to\infty} z_n = z$ **iff**:

$$\mathrm{Re}(z) = \lim_{n\to\infty} a_n \text{ and } \mathrm{Im}(z) = \lim_{n\to\infty} b_n.$$

L1.2.10

Let $S \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. Then $z \in \overline{S}$ **iff**:

$$\exists z_n \in S : z = \lim_{n\to\infty} z_n.$$

D1.2.11: Cauchy sequences

z_n is a Cauchy sequence if:

$$\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \implies |z_n - z_m| < \epsilon.$$

L1.2.12

z_n is convergent **iff** z_n is Cauchy.

D1.2.14: Bounded sequences

z_n is bounded if:

$$\forall n \in \mathbb{N}; \exists M > 0 : |z_n| \leq M.$$

L1.2.15: Bolzano-Weierstrass

Let z_n be a bounded sequence. Then:

$$\exists(z_{n_k})_{k,n_k\in\mathbb{N}} : \lim_{k\to\infty} z_{n_k} = z \in \mathbb{C}$$

or that z_n has a convergent subsequence.

A selection of a sequence is a subsequence.

D1.3.1: Bounded functions

Let $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$. Then f is a bounded function if:

$$\forall z \in S; \exists M > 0 : |f(z)| \leq M.$$

D1.3.2: ϵ-δ convergence

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}$ and $a_0 \in \mathbb{C}$. Then $\lim_{z \to z_0} f(z) = a_0$ if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : 0 < |z - z_0| < \delta \implies |f(z) - a_0| < \epsilon.$$

L1.3.3

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}$ and $a_0 \in \mathbb{C}$ where $z_0 = x_0 + iy_0$ and $f = u + iv$.

Then $\lim_{z \to z_0} f(z) = a_0$ **iff**:

$$\mathrm{Re}(a_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y)$$

and

$$\mathrm{Im}(a_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y).$$

L1.3.4

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}, a_0 \in \mathbb{C}$ and sequence $w_n \in S \setminus \{z_0\}$.

If $\lim_{z \to z_0} f(z) = a_0$ and $\lim_{n \to \infty} w_n = z_0$ then:

$$\lim_{n \to \infty} f(w_n) = a_0.$$

L1.3.5: Limit identities

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}$ and $a_0, b_0 \in \mathbb{C}$.
Let $f, g : S \to \mathbb{C}$.

If $\lim_{z \to z_0} f(z) = a_0$ and $\lim_{z \to z_0} g(z) = b_0$ then:

1. $\lim_{z \to z_0} (f(z) + g(z)) = a_0 + b_0$
2. $\lim_{z \to z_0} (f(z)g(z)) = a_0 b_0$
3. $\lim_{z \to z_0} \left(\frac{f(z)}{g(z)} \right) = \frac{a_0}{b_0}$ if $b_0 \neq 0$.

D1.3.6: ϵ-δ continuity

Let $S \subseteq \mathbb{C}, f : S \to \mathbb{C}$ and $z_0 \in S$. Then f is continuous at z_0 if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

L1.3.7

Let $f : \mathbb{C} \to \mathbb{C}$ with rule $f = u + iv$ and $z_0 = x_0 + iy_0 \in \mathbb{C}$.

Then f is continuous at z_0 **iff** u and v are continuous at (x_0, y_0).

L1.3.8

If $f, g : \mathbb{C} \to \mathbb{C}$ are continuous at z_0 then:

1. $f + g$ is continuous at z_0.
2. fg is continuous at z_0.
3. f/g is continuous at z_0. ($g \neq 0$)

D: Image and preimage

Let $f : X \to Y$ where $A \subseteq X$ and $B \subseteq Y$. The image of A is:

$$f(A) = \{f(x) : x \in A\}$$

and the preimage of B is:

$$f^{-1}(B) = \{x : f(x) \in B\}.$$

L1.3.9

Let $U \subseteq \mathbb{C}$ be an open set. $f : \mathbb{C} \to \mathbb{C}$ is continuous **iff** $\forall U \subseteq \mathbb{C}; f^{-1}(U)$ is open for $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\}$.

L1.3.10

Let $f : S \to \mathbb{C}$ be continuous. Let $S \subseteq \mathbb{C}$ be closed and bounded.

Then $f(S)$ is closed and bounded.

D1.4.1: Differentiability

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $f : U \to \mathbb{C}$. Then f is differentiable at z_0 if the following limit exists:

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

L1.4.3

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. If $f : U \to \mathbb{C}$ is differentiable at z_0 then f is continuous at z_0.

L1.4.4

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $f, g : U \to \mathbb{C}$ be differentiable at z_0. Then $f + g$, fg and f/g (where $g(z_0) \neq 0$) are all differentiable at z_0.

L1.4.5: Chain rule

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $g : U \to \mathbb{C}$ be such that $g(U)$ is a neighbourhood of $g(z_0)$. Assume that g is differentiable at z_0 and f is differentiable at $g(z_0)$. Then $f \circ g$ is differentiable at z_0:

$$(f \circ g)'(z_0) = f(g(z_0))g'(z_0).$$

T1.4.6: Cauchy-Riemann equations

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0.
Let $f : U \to \mathbb{C}$ be differentiable at z_0.
Let $z_0 = x_0 + iy_0$ and $f = u + iv$. Then:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

T1.4.8

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0 for $z_0 = x_0 + iy_0$. Let $f : U \to \mathbb{C}$ where $f = u + iv$. Assume that real functions u and v are continuously differentiable on a neighbourhood of (x_0, y_0).

Then f is differentiable at z_0.

Remark

f is continuously differentiable if its first derivatives are continuous.

D1.4.9: Holomorphic functions

f is **holomorphic** at z_0 if there exists a neighbourhood U of z_0 such that f is defined and differentiable.

D1.4.13: Harmonic equations

$h(x, y)$ is harmonic if for $\forall (x, y) \in \mathbb{R}^2$ it satisfies Laplace's equation:

$$\frac{\partial^2 h}{\partial x^2}(x, y) + \frac{\partial^2 h}{\partial y^2}(x, y) = 0.$$

L1.4.14

Let $u(x, y), v(x, y)$ be twice continuously differentiable and that $f(x + iy) = u + iy$ is holomorphic on $\mathbb{C}$.

Then u and v are harmonic.

D1.4.15: Harmonic conjugates

D1.5.1: Complex polynomials

L1.5.2