

D: Functions

A function $f : X \rightarrow Y$ is an assignment of an element of Y to each element of X .

1. f is **injective** if:

$$\begin{aligned} \forall x_1, x_2 \in X; f(x_1) = f(x_2) \\ \implies x_1 = x_2 \end{aligned}$$

and this implies that $|X| \leq |Y|$.

2. f is **surjective** if:

$$\forall y \in Y; \exists x \in X : y = f(x)$$

and this implies that $|X| \geq |Y|$.

3. f is **bijective** if it is injective and surjective.



D: Groups

A group G is a set with a composition operator (\circ) such that $\forall x, y, z, \in G$:

1. $x \circ y = xy \in G$
2. $(xy)z = x(yz)$
3. $\exists e \in G : ex = xe = x$
4. $\exists x^{-1} \in G : xx^{-1} = x^{-1}x = e$.

G is **Abelian** if $\forall x, y \in G; xy = yx$.

D1.2.1(i): Fields

A field F is a set defined with addition and multiplication such that:

1. $(+) : F \times F \rightarrow F; (\lambda, \mu) \mapsto \lambda + \mu$
2. $(\cdot) : F \times F \rightarrow F; (\lambda, \mu) \mapsto \lambda \cdot \mu$
3. $\exists(-\lambda) \in F : \lambda + (-\lambda) = 0_F$
4. $\exists(\lambda^{-1}) \in F : \lambda \cdot (\lambda^{-1}) = 1_F$ except when $\lambda = 0$.
5. $(+)$ and (\cdot) are associative, commutative and distributive.

Remark

$(F, +)$ and $(F \setminus \{0_F\}, \cdot)$ are groups.

Remark

Let n be prime or a prime power. Then \mathbb{F}_n is a finite field with n elements under modulo n . Also, \mathbb{Q} and \mathbb{R} are fields.

D1.2.1(ii): Vector spaces

A vector space V over a field F is an **Abelian group** $V := (V, +)$ with mapping:

$$F \times V \rightarrow V; (\lambda, v) \mapsto \lambda v$$

where for $\forall \lambda, \mu \in F$ and $\forall v, w \in V$:

1. $\lambda(v + w) = (\lambda v) + (\lambda w)$
2. $(\lambda + \mu)v = (\lambda v) + (\mu v)$
3. $\lambda(\mu v) = (\lambda\mu)v$
4. $1_F v = v$

and is known as a **F -vector space**.

Remark

Let V be a F -vector space and $v \in V$.

1. $0v = 0$
2. $(-1)v = -v$
3. $\lambda 0 = 0$ for $\forall \lambda \in F$.

D: Cartesian products

The **cross product** of sets X_1, \dots, X_n is:

$$X_1 \times \dots \times X_n := \{(x_1, \dots, x_n) : x_i \in X_i\}$$

with bijection $X^n \times X^m \rightarrow X^{n+m}$.

The **projection** of a cross product is:

$$\begin{aligned} \text{pr}_i : X_1 \times \dots \times X_n &\rightarrow X_i; \\ (x_1, \dots, x_n) &\mapsto x_i. \end{aligned}$$

D1.4.1: Vector subspaces

A vector subspace U of F -vector space V has the following properties:

1. $U \subset V$ and $0 \in U$.
2. Let $u, v \in U$ and $\lambda \in F$. Then $u + v \in U$ and $\lambda u \in U$.

and is also a vector space.

P1.4.5

Let $T \subset V$ where V is a F -vector space. Then for all vector subspaces containing T , there exists a smallest vector subspace:

$$\text{span}(T) = \langle T \rangle_F \subset V$$

known as the vector subspace generated by T , or the span of T .

D1.4.7: Generating set

Let $T \subset V$ where V is a F -vector space. Set T is a **generating set** of V if:

$$\text{span}(T) = V$$

and is the linear combination of vectors in T over field F . V is **finitely generated** if its generating set T is finite.

D1.4.9: Power sets

The power set of set X is:

$$\mathcal{P}(X) := \{U : U \subseteq X\}.$$

Let $\mathcal{U} \subseteq \mathcal{P}(X)$. Then:

$$\begin{aligned} \bigcup_{U \in \mathcal{U}} U &:= \{x \in X : (\exists U \in \mathcal{U} : x \in U)\} \\ \bigcap_{U \in \mathcal{U}} U &:= \{x \in X : \forall U \in \mathcal{U}; x \in U\}. \end{aligned}$$

D1.5.1: Linear independence

Let V be a F -vector space and $L \subseteq V$. Subset L is **linearly independent** if:

$$\begin{aligned} \alpha_1 v_1 + \dots + \alpha_r v_r &= 0 \\ \implies \alpha_1 = \dots = \alpha_r &= 0 \end{aligned}$$

for $v_i \in L$ and is pairwise distinct.

Remark

L is linearly dependent if some $\alpha_i \neq 0$.

D1.5.8: Basis

A basis of a vector space V is a **linearly independent generating set** in V .

T1.5.11: Basis evaluation mappings

Let V be a F -vector space.

Then $A = \{v_1, \dots, v_r\}$ is a basis of V **iff** the following **evaluation mapping**:

$$\Phi_A : F^r \rightarrow V;$$

$$(\alpha_1, \dots, \alpha_r) \mapsto \alpha_1 v_1 + \dots + \alpha_r v_r$$

is a bijection.

Remark

Φ is surjective if A is generating.

T1.5.12

Let V be a vector space and $E \subseteq V$. Then the following statements are equivalent:

1. E is a basis of V .
2. E is minimal among all generating sets, or that $E \setminus \{e\}$ is not a basis for $\forall e \in E$.
3. E is maximal amongst all linearly independent subsets. i.e. $E \cup \{v\}$ is linearly dependent for $\forall v \in V$.

C1.5.13

Every finitely generated vector space has a finite basis.

T1.5.14

Let V be a vector space.

1. Let $L \subseteq V$ be linearly independent and set E be minimal amongst all generating sets of V . Let $L \subseteq E$. Then E is a basis of V .
2. Let $E \subseteq V$ be a generating set and L be maximal amongst all linearly independent subsets of V .

Let $L \subseteq E$. Then E is a basis of V .

D1.5.15

Let X be a set and F be a field. Then:

$$\text{maps}(X, F) := \{f : (\forall f : X \rightarrow F)\}$$

and is a F -vector space under pointwise addition and multiplication via scalars.

Let $F\langle X \rangle \subseteq \text{maps}(X, F)$ be the subset of all mappings that sends all but finitely many elements of X to 0:

$$F\langle X \rangle := \{f : (\forall f : X \rightarrow \{0\})\}.$$

It contains all linear combinations of X in F and forms a vector subspace.

T1.5.16

Let V be a F -vector space.

Then $(v_i)_{i \in I}$ is a basis for V iff:

$$\forall v \in V; \exists! (a_i)_{i \in I} \subseteq F : v = \sum_{i \in I} a_i v_i.$$

T1.6.1

Let V be a vector space. Let $L \subset V$ be a linearly independent subset and $E \subseteq V$ a generating set. Then $|L| \leq |E|$.

T1.6.2: Steinitz exchange theorem

Let V be a vector space, $L \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set.

Then there exists an **injective** function $\phi : L \rightarrow E$ such that:

$$(E \setminus \phi(L)) \cup L \text{ is a generating set for } V.$$

L1.6.3: Exchange lemma

Let V be a vector space. Let $M \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set where $M \subseteq E$.

If $\exists w \in E \setminus M$ such that set $M \cup \{w\}$ is linearly independent then:

$$\exists e \in E \setminus M : (E \setminus e) \cup \{w\} \text{ is generating.}$$

C1.6.4

Let V be a finitely generated vector space.

1. V has finite basis.
2. V cannot have infinite basis.
3. Any two basis of V have the same number of elements.

D1.6.5: Dimension

The dimension of finite F -vector space V is the cardinality of one of its basis.

For infinite vector spaces: $\dim(V) = \infty$. We also define $\dim(\{0\}) := 0$.

C1.6.7

Let V be a finitely generated vector space.

1. Every linearly independent $L \subseteq V$ has **at most** $\dim(V)$ elements and if $|L| = \dim(V)$ then L is a basis.
2. Every generating set $E \subseteq V$ has **at least** $\dim(V)$ elements and if $|E| = \dim(V)$ then E is a basis.

C1.6.8

A proper vector subspace of a vector space with finite dimension has itself a strictly smaller dimension.

T1.6.10: Dimension theorem

Let V be a vector space and $U, W \subseteq V$ be vector subspaces. Then:

$$\begin{aligned} \dim(U + W) + \dim(U \cap W) \\ = \dim(U) + \dim(W) \end{aligned}$$

where $U + W := \langle U \cup W \rangle \subseteq V$.

D1.7.1: Linear mappings

Let V and W be F -vector spaces.

A mapping $f : V \rightarrow W$ is **F -linear** or a **homomorphism** of vector spaces if for $\forall v_1, v_2 \in V$ and $\forall \lambda \in F$:

1. $f(v_1 + v_2) = f(v_1) + f(v_2)$
2. $f(\lambda v_1) = \lambda f(v_1)$.

Furthermore bijective linear mappings are an **isomorphism** of vector spaces.

A homomorphism from a vector space to itself is an **endomorphism**.

An isomorphism of a vector space to itself is an **automorphism**.

D1.7.5: Fixed points

In a linear mapping a fixed point is sent to itself. Given mapping $f : X \rightarrow X$ the **set of fixed points** is:

$$X^f = \{x \in X : f(x) = x\}.$$

D1.7.6: Complementary subspaces

Vector subspaces V_1, V_2 of vector space V are **complementary** if the **direct sum** of vector subspaces is bijective:

$$\oplus : V_1 \times V_2 \rightarrow V; (v_1, v_2) \mapsto v_1 + v_2.$$

i.e. $V_1 \oplus V_2 = V$.

T1.7.7

Let $n \in \mathbb{N}$ and V a F -vector space. V is isomorphic to F^n **iff** $\dim(V) = n$.

L1.7.8

Let V, W be F -vector spaces and let B be a basis of V . Then the following mapping:

$$\text{hom}_F(V, W) \rightarrow \text{maps}(B, W); f \mapsto f_B$$

is a bijection. The set of all linear maps or homomorphisms from V to W is:

$$\text{hom}_F(V, W) \subseteq \text{maps}(B, W).$$

P1.7.9

Let $f : V \rightarrow W$ be a linear mapping, where V, W are vector spaces.

1. If f is injective, there exists map $g : W \rightarrow V$ such that $g \circ f = \text{id}_V$. i.e. it has a **left inverse**.
2. If f is surjective, there exists map $g : W \rightarrow V$ such that $f \circ g = \text{id}_W$. i.e. it has a **right inverse**.

D1.8.1: Image and kernel

Let $f : V \rightarrow W$ be a linear mapping. The **image** of this linear mapping f is:

$$\begin{aligned} \text{im}(f) &:= f(V) \\ &= \{w \in W : \forall v \in V; w = f(v)\} \end{aligned}$$

and is a vector subspace of W .

The **kernel** of this linear mapping f is:

$$\ker(f) := f^{-1}(0) = \{v \in V : f(v) = 0\}$$

and is a vector subspace of V .

$0 = e_H$ for group homomorphisms.

L1.8.2

A linear mapping $f : V \rightarrow W$ is injective **iff** $\ker(f) = \{0\}$.

T1.8.4: Rank-nullity theorem

Let $f : V \rightarrow W$ be a linear mapping and V, W are vector spaces. Then:

$$\dim(V) = \dim(\ker(f)) + \dim(\operatorname{im}(f)).$$

T2.1.1: Matrix mappings

Let F be a field and $m, n \in \mathbb{N}$.

Then there exists a bijection:

$$M : \operatorname{hom}_F(F^m, F^n) \rightarrow \operatorname{mat}(n \times m; F);$$

$$f \mapsto [f]$$

and attaches each linear mapping f with its **representing matrix** $M(f) := [f]$.

Remark

The set of $n \times m$ matrices in F is defined:

$$\operatorname{mat}(n \times m; F).$$

i.e. matrices with **n rows** and **m columns**.

D2.1.6: Matrix products

The product $A \circ B = AB$ for A is $n \times m$, B is $m \times \ell$ and AB is $n \times \ell$ is defined as:

$$(AB)_{ik} = \sum_{j=1}^m A_{ij}B_{jk}$$

with the following mapping:

$$\begin{aligned} \operatorname{mat}(n \times m; F) \times \operatorname{mat}(m \times \ell; F) \\ \rightarrow \operatorname{mat}(n \times \ell; F); (A, B) \mapsto AB. \end{aligned}$$

T2.1.8

Let $g : F^\ell \rightarrow F^m$ and $f : F^m \rightarrow F^n$ be linear mappings. Then $[f \circ g] = [f] \circ [g]$.

P2.1.9

Let A, A' be $n \times m$, B, B' be $m \times \ell$ and C, C' be $\ell \times k$. Denote $I = I_m$ as the $m \times m$ identity matrix. Then:

1. $(A + A')B = AB + A'B$
2. $A(B + B') = AB + AB'$
3. $IB = B$
4. $AI = A$
5. $(AB)C = A(BC)$.

D2.2.1: Invertible matrices

A matrix A is **invertible** if:

$$\exists B, C : BA = I \text{ and } AC = I.$$

D2.2.2: Elementary matrices

Elementary matrices are square matrices that differs from the identity matrix by at most one entry.

T2.2.3

Every square matrix with entries in a field can be written as a product of elementary matrices.

D2.2.4: Smith normal form

Matrices with **only** non-zero entries along the diagonal are in Smith normal form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

T2.2.5

Let A be an $n \times m$ matrix. Then:

$$PAQ \text{ is of Smith normal form}$$

where P and Q are invertible.

Remark

$$\operatorname{rank}(A) = \operatorname{rank}(PAQ).$$

D2.2.7: Column and row rank

Let matrix $A \in \operatorname{mat}(n \times m; F)$.

The column rank of A is the dimension of the subspace of F^n generated by the columns of A .

Similarly the row rank of A is the dimension of the subspace of F^m generated by the rows of A .

T2.2.8

Column and row ranks are equal.

D2.2.9: Full rank matrices

Let A be $n \times m$ with entries in F . A is **full rank** if $\operatorname{rank}(A) = \min(m, n)$.

Let $A = [a]$ with mapping $a : F^m \rightarrow F^n$. Then $\dim(\operatorname{im}(a)) := \operatorname{rank}(A)$.

T2.3.1: Representing matrices

Let V and W be F -vector spaces with bases $A = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ s.t. $\langle A \rangle = V$ and $B = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ s.t. $\langle B \rangle = W$.

Then for every linear map $f : V \rightarrow W$ there exists a **representing matrix**:

$$({}_B[f]_A)_{ij} = a_{ij}$$

$$f(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + \dots + a_{nj}\mathbf{w}_n \in W$$

which produces the following bijection:

$$M_B^A : \operatorname{hom}_F(V, W) \rightarrow \operatorname{mat}(n \times m; F);$$

$$f \mapsto {}_B[f]_A$$

and $M_B^A(f) = {}_B[f]_A$ is the representing matrix of linear mapping f with respect to bases A and B .

If A and B are standard bases then $[f]$.

T2.3.2

Let U, V, W be F -vector spaces with finite dimension and bases A, B, C respectively.

If $f : U \rightarrow V$ and $g : V \rightarrow W$ are linear mappings then ${}_C[g \circ f]_A = {}_C[g]_B \circ {}_B[f]_A$.

D2.3.3: Vector representations

Let V be a finite dimensional vector space with basis $A = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. Then:

$$\Phi_A^{-1} : V \rightarrow F^r; \mathbf{v} \mapsto {}_A[\mathbf{v}]$$

is a bijection and the column vector ${}_A[\mathbf{v}]$ is known as the **representation of vector \mathbf{v} with respect to basis A** .

T2.3.4

Let V, W be finite dimensional F -vector spaces with bases A and B respectively.

Let $f : V \rightarrow W$ be a linear mapping. Then ${}_B[f(\mathbf{v})] = {}_B[f]_A \circ {}_A[\mathbf{v}]$ for $\forall \mathbf{v} \in V$.

D2.4.1

Let V be a F -vector space and let sets $A = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $B = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be bases of V . The representation matrix of the identity mapping:

$$\operatorname{id}_V : V \rightarrow V; \mathbf{v} \mapsto \mathbf{v}$$

is a **change of basis matrix** ${}_B[\operatorname{id}_V]_A$ with entries a_{ij} given by definition:

$$\mathbf{v}_j = \sum_{i=1}^n a_{ij}\mathbf{w}_i.$$

T2.4.3: Change of basis

Let V and W be finite dimensional vector spaces with linear mapping $f : V \rightarrow W$. Let A, A' be ordered bases of V and B, B' be ordered bases of W . Then:

$$B'[f]_{A'} = B'[\operatorname{id}_W]_B \circ {}_B[f]_A \circ {}_A[\operatorname{id}_V]_{A'}.$$

C2.4.4

Let V be a finite dimensional vector space and let $f : V \rightarrow V$ be an endomorphism. Let A, A' be bases of V . Then:

$$A'[f]_{A'} = A[\operatorname{id}_V]_{A'}^{-1} \circ A[f]_A \circ A[\operatorname{id}_V]_{A'}.$$

T2.4.5

Let V and W be finite dimensional vector spaces and let $f : V \rightarrow W$ be linear.

Then there exists a basis A of V and a basis B of W such that the representing matrix ${}_B[f]_A$ has nonzero entries only on the diagonal.

D2.4.6: Trace

The trace of a $n \times n$ matrix A is the sum of its diagonal entries:

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}.$$

D3.1.1: Rings

A **ring** R is a set equipped with addition and multiplication that satisfy:

1. $(R, +)$ is an **Abelian group** with additive identity $0_R \in R$.
2. (R, \cdot) is a **monoid**, meaning that:

$$(\cdot) : R \times R \rightarrow R; (a, b) \mapsto a \cdot b$$

is associative with identity element $1 = 1_R \in R$ such that:

$$\forall a, b, c \in R; (a \cdot b) \cdot c = a \cdot (b \cdot c),$$

$$a \cdot 1 = 1 \cdot a = a$$

yet $a \cdot b \neq b \cdot a$ in general.

3. Multiplication in R with respect to addition is distributive:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$$

for $\forall a, b, c \in R$.

For **nonzero** rings $0_R \neq 1_R$.

Division rings are rings where nonzero elements have multiplicative inverses.

P3.1.7

A natural number is divisible by 3 if the sum of its digits is divisible by 3.

D3.1.8: Fields

A **field** F is a nonzero **commutative** ring with **multiplicative** inverses to every **nonzero** element:

$$\forall a \in F; \exists a^{-1} \in F : aa^{-1} = a^{-1}a = 1.$$

i.e. a commutative division ring.

P3.1.11

$\mathbb{Z}/m\mathbb{Z}$ is a field **iff** m is prime.

L3.2.1

Let R be a ring and $a, b \in R$. Then:

1. $0a = a0 = 0$
2. $(-a)b = -(ab) = a(-b)$
3. $(-a)(-b) = ab$.

D3.2.3

Let $m \in \mathbb{Z}$. Then m th multiple ma of $a \in (R, +)$ is $ma = \underbrace{a + \cdots + a}_{m \text{ times}}$ if $m > 0$.

$0a := 0$ and if $m < 0$, $(-m)a = -(ma)$.

L3.2.4

Let R be a ring where $a, b \in R$ and $m, n \in \mathbb{Z}$. Then:

1. $m(a + b) = ma + mb$
2. $(m + n)a = ma + na$
3. $m(na) = (mn)a$
4. $m(ab) = (ma)b = a(mb)$
5. $(ma)(nb) = (mn)(ab)$.

D3.2.6: Units

Let R be a ring. An element $r \in R$ is a **unit** if it has a **multiplicative inverse**:

$$\exists r^{-1} \in R : rr^{-1} = r^{-1}r = 1_R.$$

P3.2.9: Group of units

R^\times is the **set of units** in ring R and forms a group under multiplication.

D3.2.11: Divisor of zero

Let $r \neq 0_R \in R$ where R is a ring. Then element r is a **divisor of zero** if:

$$\exists s \in R : rs = 0_R \text{ or } sr = 0_R.$$

D3.2.12: Integral domains

Integral domains are commutative rings with **no** divisors of zeros.

P3.2.15: Cancellation law

Let $a, b, c \in R$ for R is an integral domain. If $ab = ac$ and $a \neq 0$ then $b = c$.

P3.2.16

Let $m \in \mathbb{N}$. Then $\mathbb{Z}/m\mathbb{Z}$ is an integral domain **iff** m is prime.

T3.2.17

Every finite integral domain is a field.

Remark

If $|R| < \infty$ then $f : R \rightarrow R$ is surjective.

D3.3.2: Polynomial rings

$R[X]$ is a ring of polynomials over R with zero and identity: $0, 1 \in R$. If $P \in R[X]$:

$$P = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m$$

with $\deg(P) = m \geq 0$ and $a_i \in R$.

L3.3.3

Let R be a ring and let $P, Q \neq 0 \in R[X]$.

1. $\deg(PQ) = \deg(P) + \deg(Q)$
2. If R is an integral domain then so is polynomial ring $R[X]$.

T3.3.4

Let R be an integral domain and let $P, Q \in R[X]$ where $\deg(Q) \leq \deg(P)$ and that polynomial Q is a **monic**.

Then $\exists! A, B \in R[X] : P = AQ + B$ and either $\deg(B) < \deg(Q)$ or $B = 0$.

Remark

A polynomial Q is monic if:

$$Q = q_0 + \cdots + q_mX^m$$

where $q_m = 1$.

D3.3.6

Let R be a commutative ring and let $P \in R[X]$ be a polynomial. Then:

$$R[X] \rightarrow \text{maps}(R, R)$$

where we **evaluate** $P(\lambda)$ for $\lambda \in R$:

$$P(X) \mapsto \{P_\lambda : R \rightarrow R; \lambda \mapsto P(\lambda)\}.$$

If $P(\lambda) = 0$ then λ is a **root** of P .

P3.3.9

Let R be a commutative ring, let $\lambda \in R$ and $P(X) \in R[X]$. Then $P(\lambda) = 0$ **iff**:

$$P(X) = (X - \lambda)Q(X)$$

where $Q(X) \in R[X]$.

T3.3.10

Polynomial $P \neq 0 \in R[X] \setminus \{0\}$ has at most $\deg(P)$ roots in integral domain R .

D3.3.11: Algebraically closed field

A field F is algebraically closed if every $P \in F[X] \setminus F$ has a root in field F .

T3.3.13: FTA

Field \mathbb{C} is algebraically closed.

T3.3.14

Let field F be algebraically closed. Then every $P \in F[X] \setminus \{0\}$ decomposes into:

$$P = c(X - \lambda_1) \dots (X - \lambda_n)$$

where $c \in F^\times$ and $\lambda_1, \dots, \lambda_n \in F$.

D3.4.1: Ring homomorphisms

Let R and S be rings. $f : R \rightarrow S$ is a ring homomorphism if for all $x, y \in R$:

$$f(x + y) = f(x) + f(y)$$

$$f(xy) = f(x)f(y).$$

Remark

If $f : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ is a homomorphism and $f(1) = x \in \mathbb{Z}/n\mathbb{Z}$ then:

1. $x^2 = x \pmod{n}$
2. $mx = 0 \pmod{n}$, with $\gcd(m, n)$ evenly spaced solutions.

L3.4.5

Let R and S be rings. Let $f : R \rightarrow S$ be a **ring homomorphism**. Then for all $x, y \in R$ and $m \in \mathbb{Z}$:

1. $f(0_R) = 0_S$
2. $f(-x) = -f(x)$
3. $f(x - y) = f(x) - f(y)$
4. $f(mx) = mf(x)$

since $(R, +)$ is a group.

D3.4.7: Ideals

Let $I \subset R$ where R is a ring. Then I is an **ideal** of ring R if:

1. $I \neq \emptyset$ and $0_R \in I$
2. I is closed under subtraction.
3. $\forall i \in I; \forall r \in R; ri, ir \in I$

and we denote $I \trianglelefteq R$.

D3.4.11: Ideal of R generated by T

Let R be a commutative ring and $T \subset R$. Then the ideal of R generated by T is:

$${}_R\langle T \rangle = \left\{ \sum_i r_i t_i : t_i \in T; \forall r_i \in R \right\}$$

where $i \in \{1, \dots, m\}$ and $m \leq |T|$.

P3.4.14

${}_R\langle T \rangle$ is the smallest ideal containing T .

D3.4.15: Principle ideal

An ideal of a commutative ring R is the **principle ideal** if:

$$I = {}_R\langle t \rangle \text{ where } t \in R.$$

P3.4.18

Let $f : R \rightarrow S$ be a ring homomorphism. Then $\ker(f)$ is an **ideal** of ring R where:

$$\ker(f) = \{r \in R : f(r) = 0_S\}$$

and is a subgroup of $(R, +)$.

L3.4.21 and L3.4.22

The set intersection and addition of ideals also form ideals.

D3.4.23: Subrings

A subset $R' \subseteq R$ is a subring of ring R if R' also satisfies D3.1.1.

P3.4.26: Subring test

$R' \subseteq R$ is a subring of R iff $\forall a, b \in R'$:

1. R' has multiplicative identity.
2. $a - b \in R'$
3. $ab, ba \in R'$

i.e. that R' is closed under subtraction and multiplication.

P3.4.28

Let $f : R \rightarrow S$ be a ring homomorphism.

1. If R' is a subring of R then $f(R')$ and $\text{im}(f)$ are subrings of S .
2. Let $f(1_R) = 1_S$. Then:

$$x \in R^\times \implies f(x) \in S^\times.$$

D3.5.1: Relations

A **relation** R on set X is a subset of $X \times X$. We denote $(x, y) = xRy \in X \times X$.

R is an **equivalence relation** on set X if $\forall x, y, z \in X$ the following is true:

1. Reflexive: xRx
2. Symmetric: $xRy \iff yRx$
3. Transitive: $(xRy \wedge yRz) \implies xRz$.

D3.5.3: Equivalence classes

Let \sim be an equivalence relation on X . Then the **equivalence class** of $x \in X$ is:

$$E(x) = \{z \in X : z \sim x\} \subseteq X$$

where an element of an equivalence class is a **representative** of the class.

D3.5.5

Given an equivalence relation \sim on set X , the **set of equivalence classes** is:

$$(X \setminus \sim) := \{E(x) : x \in X\} \subseteq \mathcal{P}(X).$$

We also define a **surjective** map:

$$\text{can} : X \rightarrow (X / \sim); x \mapsto E(x)$$

known as the **canonical mapping**.

Remark

A mapping $f : X \rightarrow Z$ is **well-defined** if there is an equivalence relation \sim on X such that $x \sim y \implies f(x) = f(y)$.

Then there exists a unique mapping \bar{f} :

$$\bar{f} : (X / \sim) \rightarrow Z; E(x) \mapsto f(x)$$

where $f = \bar{f} \circ \text{can}$.

D3.6.1: Cosets

Let I be an ideal of ring R . Then:

$$x + I = \{x + i : i \in I\} \subseteq R$$

is the coset of x with respect to I in R .

Remark

1. $x + I$ is both a left and right coset of x since $(R, +)$ is Abelian.
2. Ideals of rings are subgroups.

D3.6.3: Factor rings

Let I be an ideal of ring R and define an equivalence relation on R where:

$$x \sim y \iff x - y \in I.$$

Then the **factor ring** of R by I is the **set of cosets** of I in R and denoted as R/I :

$$R/I = (R / \sim)$$

for each element is an equivalence class:

$$\begin{aligned} E(x) &= \{z \in R : z - x \in I\} \\ &= \{x + i \in R : i \in I\} \\ &= x + I. \end{aligned}$$

T3.6.4

Let I be an ideal of ring R . Then R/I is a ring where $\forall x, y \in R$:

$$(x + I) + (y + I) = (x + y) + I$$

$$(x + I) \cdot (y + I) = xy + I$$

where $x + I, y + I \in R/I$.

T3.6.7

Let I be an ideal of ring R . Then:

1. $\text{can} : R \rightarrow R/I$ is a surjective ring homomorphism with kernel I .
2. Let $f : R \rightarrow S$ where $f(I) = \{0_S\}$ and that f is a ring homomorphism.

Then there is a unique $\bar{f} : R/I \rightarrow S$ such that $f = \bar{f} \circ \text{can}$ and that \bar{f} is also a ring homomorphism.

$$\begin{array}{ccc} R & \xrightarrow{\text{can}} & R/I \\ & \searrow f & \downarrow \bar{f} \\ & & S \end{array}$$

T3.6.9: FIT for rings

Every ring homomorphism $f : R \rightarrow S$ induces a ring isomorphism:

$$\bar{f} : R/\ker(f) \rightarrow \text{im}(f)$$

where \bar{f} is a **bijection**. This is the first isomorphism theorem for rings.

D3.7.1: Left modules

A left module M over a ring R is a **pair** consisting of an **Abelian group** $(M, +)$ and the following mapping:

$$R \times M \rightarrow M; (r, a) \mapsto ra$$

such that $\forall r, s \in R$ and $\forall a, b \in M$:

$$r(a + b) = (ra) + (rb)$$

$$(r + s)a = (ra) + (sa)$$

$$r(sa) = (rs)a$$

$$1_R a = a$$

also known as an **R -module**.

L3.7.8

Let R be a ring and M be a R -module. Then $\forall r \in R$ and $\forall a \in M$:

1. $0_R a = 0_M$
2. $r 0_M = 0_M$
3. $(-r)a = r(-a) = -(ra)$.

D3.7.11: Module homomorphisms

Let R be a ring, M and N be R -modules. Then $f : M \rightarrow N$ is an **R -homomorphism** if $\forall r \in R$ and $\forall a, b \in M$:

$$f(a + b) = f(a) + f(b)$$

$$f(ra) = rf(a).$$

f is an **R -isomorphism** if it is bijjective and we denote that $M \cong N$.

D3.7.15: Submodules

$M' \subseteq M$ is a submodule if it also satisfies D3.7.11 but restricted to itself.

P3.7.20: Submodule test

Let M be a R -module. M' is a submodule of M **iff** $\forall a, b \in M$ and $\forall r \in R$:

1. $0_M \in M'$
2. $a - b \in M'$
3. $ra \in M'$.

L3.7.21

Let $f : M \rightarrow N$ be an R -homomorphism. Then $\ker(f)$ and $\text{im}(f)$ are submodules.

D3.7.23: Submodule generated by T

Let $T \subseteq M$ for M is a R -module. Then:

$${}_R\langle T \rangle = \left\{ \sum_i r_i t_i : t_i \in T; \forall r_i \in R \right\}$$

where $i \in \{1, \dots, m\}$ and $m \leq |T|$.

Module N is **cyclic** if $N = {}_R\langle t \rangle$.

L3.7.29 and L3.7.30

Intersecting and adding collections of submodules also form submodules.

D3.7.31: Factor modules

Let R be a ring, M be a R -module and N a submodule of M . Let $a \in M$. Then the **coset of a with respect to N in M** is:

$$a + N = \{a + b : b \in N\}$$

Every coset is an equivalent class of the following equivalence relation:

$$\forall a, b \in M; a \sim b \iff a - b \in N$$

and we define the **factor module** of M by the submodule N as:

$$M/N = (M/\sim)$$

with the following operators:

$$(a + N) + (b + N) = (a + b) + N$$

$$r(a + N) = ra + N$$

and additive identity $0_{M/N} = 0_M + N$.

T3.7.33: FIT for modules

Let M and N be R -modules. Then every R -homomorphism $f : M \rightarrow N$ induces a R -isomorphism:

$$\bar{f} : M/\ker(f) \rightarrow \text{im}(f)$$

where \bar{f} is a bijection.

D4.1.1: Symmetric group S_n

The set of bijections from $\{1, \dots, n\}$ to itself is a group under composition with $n!$ elements, denoted by S_n .

Each bijection $\sigma \in S_n$ is a **permutation**.

Transpositions are permutations that swap **only two elements** in $\{1, \dots, n\}$.

D4.1.2: Inversions

Given a permutation $\sigma \in S_n$, an inversion is a pair (i, j) that is interchanged:

$$i < j \text{ and } \sigma(j) < \sigma(i)$$

where $i, j \in \{1, \dots, n\}$. The **length** of σ is the number of inversions:

$$\ell(\sigma) := |\{(i, j) : i < j \text{ and } \sigma(j) < \sigma(i)\}|$$

or the number of crossings in a diagram. We define the **sign** of σ :

$$\text{sgn}(\sigma) = (-1)^{\ell(\sigma)} = \begin{cases} +1 & \text{even } \sigma \\ -1 & \text{odd } \sigma \end{cases}$$

Remark

A cycle $(a_1 \dots a_r)$ sends each element to its **right** **except** a_r which is sent to a_1 .

$$(a_1 a_2 \dots a_r) = (a_1 a_r)(a_1 a_{r-1}) \dots (a_1 a_2)$$

Note that order may not be interchanged.

D: Group homomorphisms

Let G and H be groups. Then $f : G \rightarrow H$ is a group homomorphism if $\forall a, b \in G$:

$$f(ab) = f(a)f(b).$$

L4.1.5

$\text{sgn} : S_n \rightarrow \{+1, -1\}$, which is the sign of permutations, is a group homomorphism:

$$\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$$

for all $\sigma, \tau \in S_n$.

D4.1.6: Alternating groups

The alternating group is a subgroup of S_n and is the kernel of sgn function:

$$\begin{aligned} A_n &:= \ker(\text{sgn}) \\ &= \{\sigma \in S_n : \text{sgn} = e_{\{\pm 1\}} = +1\}. \end{aligned}$$

D4.2.1: Determinants

Let R be a commutative ring and $n \in \mathbb{N}$. Then the determinant map is defined as:

$$\det : \text{mat}(n \times n; R) \rightarrow R;$$

$$A \mapsto \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

where $(A)_{ij} = a_{ij}$.

D4.3.1: Bilinear forms

Let U, V and W be F -vector spaces. Then a **bilinear form** on $U \times V$ with values in W is the **mapping** $H : U \times V \rightarrow W$ where $\forall \mathbf{u}_1, \mathbf{u}_2 \in U, \forall \mathbf{v}_1, \mathbf{v}_2 \in V$ and $\forall \lambda \in F$:

$$H(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1) = H(\mathbf{u}_1, \mathbf{v}_1) + H(\mathbf{u}_2, \mathbf{v}_1)$$

$$H(\lambda \mathbf{u}_1, \mathbf{v}_1) = \lambda H(\mathbf{u}_1, \mathbf{v}_1)$$

$$H(\mathbf{u}_1, \mathbf{v}_1 + \mathbf{v}_2) = H(\mathbf{u}_1, \mathbf{v}_1) + H(\mathbf{u}_1, \mathbf{v}_2)$$

$$H(\mathbf{u}_1, \lambda \mathbf{u}_1) = \lambda H(\mathbf{v}_1, \mathbf{u}_1)$$

or that every entry of H is linear.

Consider a bilinear form H where:

$$H : U \times U \rightarrow W.$$

H is **symmetric** if $H(\mathbf{u}, \mathbf{v}) = H(\mathbf{v}, \mathbf{u})$.

H is **alternating** if $H(\mathbf{u}, \mathbf{u}) = 0$, and these must hold for all $\mathbf{u}, \mathbf{v} \in U$.

Remark

A bilinear form H that satisfies:

$$H(\mathbf{u}, \mathbf{u}) + H(\mathbf{u}, \mathbf{u}) = 0$$

is alternating provided $1_F + 1_F \neq 0_F$.

D4.3.3: Multilinear forms

Let V_1, \dots, V_n, W be F -vector spaces. Then the map $H : V_1 \times \dots \times V_n \rightarrow W$ is a **multilinear form** if every $V_j \rightarrow W$ is **linear** for all $\mathbf{m}_j \in V_j$.

H is **alternating** if $\exists i \neq j : \mathbf{v}_i = \mathbf{v}_j$ and:

$$H(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) = 0.$$

T4.3.6

Let F be a field. Then the mapping:

$$\det : \text{mat}(n \times n; F) \rightarrow F$$

is the unique alternating multilinear form on n -tuples of ordered column vectors of the matrix under mapping. Furthermore:

$$\det(I_n) = 1_F.$$

T4.4.1

Let $A, B \in \text{mat}(n \times n; R)$ where R is a commutative ring. Then:

$$\det(AB) = \det(A) \det(B).$$

T4.4.2

A square matrix A with entries in a field F is **invertible** iff $\det(A) \neq 0$.

L4.4.4

Let R be a commutative ring. Then: $\det(A^T) = \det(A)$ for $A \in \text{mat}(n \times n; R)$.

D4.4.6: Cofactors

Let R be a commutative ring and that $A \in \text{mat}(n \times n; R)$. Let $i, j \in \{1, \dots, n\}$. We define a **cofactor** of A as:

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is a *minor* of A , obtained from removing the i^{th} row and j^{th} column of A .

T4.4.7: Laplace's expansion

Let R be a commutative ring, $A = (a_{ij})$ where $A \in \text{mat}(n \times n; R)$. Then:

$$\begin{aligned} \det(A) &= \sum_{j=1}^n a_{ij} C_{ij} \quad (i^{\text{th}} \text{ row}) \\ &= \sum_{i=1}^n a_{ij} C_{ij} \quad (j^{\text{th}} \text{ column}) \end{aligned}$$

D4.4.8: Adjugate matrices

Let R be a commutative ring and let $A \in \text{mat}(n \times n; R)$. Then the **adjugate matrix** $\text{adj}(A)$ is the $n \times n$ matrix where:

$$\text{adj}(A)_{ij} = C_{ji}.$$

T4.4.9: Cramer's rule

Let R be a commutative ring and let $A \in \text{mat}(n \times n; R)$. Then:

$$A \cdot \text{adj}(A) = (\det A) I_n.$$

C4.4.11

Let R be a commutative ring and let $A \in \text{mat}(n \times n; R)$. Then A is **invertible** iff $\det(A) \in R^\times$.

T4.4.14: Jacobi's formula

Let $A = (a_{ij})$ where $a_{ij} = a_{ij}(t)$. Then:

$$\frac{d}{dt} \det(A) = \text{tr}(\text{adj}(A) \frac{dA}{dt}).$$

D4.5.1: Eigenspaces

Let $f : V \rightarrow V$ be an endomorphism of F -vector space V . An **eigenvalue** $\lambda \in F$ of f is a scalar such that $f(\mathbf{v}) = \lambda \mathbf{v}$ with eigenvector $\mathbf{v} \neq \mathbf{0}$. The **eigenspace** of map f with eigenvalue λ is defined as:

$$E(\lambda, f) = \{\mathbf{v} \in V : f(\mathbf{v}) = \lambda \mathbf{v}\}.$$

T4.5.4

Let V be a nonzero *finite* F -vector space where F is an algebraically closed field. Then each endomorphism of the vector space V has an eigenvalue.

D4.5.6: Characteristic polynomials

Let R be a commutative ring and let $A \in \text{mat}(n \times n; R)$. The **characteristic polynomial** of matrix A is defined as:

$$\chi_A(x) := \det(xI_n - A) \in R[x].$$

T4.5.8

Let F be a field and $A \in \text{mat}(n \times n; F)$. Then the eigenvalues of the linear map $A : F^n \rightarrow F^n$ are the roots of χ_A .

P4.6.1: Triangularisability

Let $f : V \rightarrow V$ be an endomorphism of a *finite* F -vector space. Then the following two statements are equivalent:

1. Vector space V has an *ordered* basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that:

$$f(\mathbf{v}_1) = a_{11}\mathbf{v}_1$$

...

$$f(\mathbf{v}_n) = a_{1n}\mathbf{v}_1 + \dots + a_{nn}\mathbf{v}_n$$

or that its representation matrix is:

$${}_B[f]_B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

i.e. f is **triangularisable**.

2. $\chi_f(x)$ may be written as a *product* of *linear factors* in $F[x]$.

D4.6.6

Consider an endomorphism $f : V \rightarrow V$ where V is a F -vector space. Then f is **diagonalisable** if the set of eigenvectors of f forms a basis of V .

If V is finite dimensional then there exists an *ordered* basis B such that ${}_B[f]_B$ is a diagonal matrix of eigenvalues.

A square matrix $A \in \text{mat}(n \times n; F)$ is diagonalisable if its corresponding linear map is diagonalisable. This implies that A is **conjugate** to a diagonal matrix.

T4.6.10: Cayley-Hamilton

Let R be a commutative ring and let $A \in \text{mat}(n \times n; R)$. Then $\chi_A(x = A) = 0$.

D4.7.5: Markov matrices

A **Markov matrix** M has non-negative entries, and every column sums to 1.

L4.7.6

1 is an eigenvalue of any Markov matrix.

D5.1.1: Inner product spaces

Let V be a \mathbb{C} -vector space. An **inner product** on V is a mapping:

$$(-, -) : V \times V \rightarrow \mathbb{C}$$

such that $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\forall \lambda, \mu \in \mathbb{C}$:

1. $(\lambda \mathbf{x} + \mu \mathbf{y}, \mathbf{z}) = \lambda(\mathbf{x}, \mathbf{z}) + \mu(\mathbf{y}, \mathbf{z})$
2. $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})^*$
3. $(\mathbf{x}, \mathbf{x}) \geq 0$ iff $\mathbf{x} = \mathbf{0}$.

D5.1.5: Inner product norm

The **norm** of a vector $\mathbf{v} \in V$ in a real or complex inner product space is:

$$\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$$

D5.1.7: Orthonormal family

include explicit calculation

T5.1.10**D5.2.1****P5.2.2****D5.2.3: Orthogonal projections****P5.2.4****T5.2.5: Cauchy-Schwarz****C5.2.6****T5.2.7****Remark: Gram-Schmidt****D6.2.1: Jordan blocks**

Let F be an algebraically closed field. We define a **nilpotent Jordan block** as:

$$J_{ij}(r) = \begin{cases} 1 & j = i + 1 \\ 0 & \text{else} \end{cases}$$

where $1 \leq i, j \leq r$. For example:

$$J(r) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

i.e. $J(r) \in \text{mat}(r \times r; F)$. Then:

$$J(r, \lambda) := \lambda I_r + J(r) = D + N$$

given an eigenvalue $\lambda \in F$ such that:

$$DN = ND.$$

Remark

Matrix N is **nilpotent** if it is *square* and there exists $k \in \mathbb{N}$ such that $N^k = 0$.

T6.2.2: Jordan normal form

Let F be an algebraically closed field. Let V be a F -vector space of *finite* dimension. Let $\phi : V \rightarrow V$ be an endomorphism with the following characteristic polynomial:

$$\chi_\phi(x) = (x - \lambda_1)^{a_1} \cdots (x - \lambda_s)^{a_s} \in F[x]$$

$$a_i \geq 1 \text{ and } \sum_{i=1}^s a_i = \dim(V)$$

where $\lambda_1, \dots, \lambda_s \in F$ are *distinct*.

L6.3.1

There exists $Q_j(x) \in F[x]$ such that:

$$\sum_{j=1}^s P_j(x) Q_j(x) = 1$$