Vector products

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$$

$$\boldsymbol{a}\times\boldsymbol{b} = ab\sin\theta\hat{\boldsymbol{n}}$$

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear **more than twice** in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{ijk} = -\epsilon_{ijk} = -\epsilon_{ijk} \end{aligned}$$

and we have the following identities:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i\boldsymbol{e}_i = a_i\boldsymbol{e}_i$$

$$\delta_{ii} = 3$$

$$[\dots]_j\delta_{jk} = [\dots]_k$$

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$$

$$\boldsymbol{e}_i\times\boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k$$

$$\boldsymbol{a}\times\boldsymbol{b} = \epsilon_{ijk}a_jb_k\boldsymbol{e}_i$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \epsilon_{ijk}a_ib_jc_k$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk}\epsilon_{ijk} = 6.$$

Transformations

Let matrix L relate basis $\{\boldsymbol{e}_i\}$ to basis $\{\boldsymbol{e}'_i\}$ with rule:

$$\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then $L^TL = LL^T = I$, and:

$$\ell_{ik}\ell_{jk} = \ell_{ki}\ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij}p_j \text{ for } \boldsymbol{p} = p_i\boldsymbol{e}_i = p'_i\boldsymbol{e}'_i.$$

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip}\ell_{jq}\ell_{kr}T_{pqr}$$

which relates frame S in $\{\boldsymbol{e}_i\}$ to frame S' in $\{\boldsymbol{e}'_i\}$ with rule $\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j$, etc.

Properties of tensors:

1. The addition of two rank n tensors is also a rank n tensor.
2. The multiplication of a rank m tensor with a rank n tensor yields a rank $m+n$ tensor.
3. If $T_{ijk\dots s}$ is a rank m tensor then $T_{iik\dots s}$ is a rank $m-2$ tensor.
4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq}$$
$$T'_{ji} = \ell_{jp}\ell_{iq}T_{pq}.$$

Symmetric tensors

T_{ij} is a symmetric tensor when $T_{ij} = T_{ji}$ in frame S. Then $T'_{ij} = T'_{ji}$ in frame S'.

Similarly T_{ij} is an anti-symmetric tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally any tensor can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S'. Let $b_i = T_{ij}a_j$ where a_j is a vector. If b_i always transforms as a vector then T_{ij} is a rank 2 tensor.

Generalising, let $R_{ijk\dots r}$ be a rank m tensor and $T_{ijk\dots s}$ a set of 3^n numbers where $n > m$. If $T_{ijk\dots s}R_{ijk\dots r}$ is a rank $n-m$ tensor then $T_{ijk\dots s}$ is a rank n tensor.

Matrices

We define a $m\times n$ matrix A as $(A)_{ij} = a_{ij}$ where $i = 1,\dots,m$ and $j = 1,\dots,n$.

- $\mathrm{Tr}\,A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^TA^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn}a_{1l}a_{2m}a_{3n} \\ &= \epsilon_{lmn}a_{l1}a_{m2}a_{n3}. \end{aligned}$$

Furthermore:

$$\epsilon_{ijk}\det A = \epsilon_{lmn}a_{il}a_{jm}a_{kn}$$

$$\epsilon_{lmn}\det A = \epsilon_{ijk}a_{il}a_{jkm}a_{kn}$$

$$\det A = \frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}a_{il}a_{jm}a_{kn}.$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3. $\det A = \det A^T$
4. $\det(AB) = \det A\cdot\det B$

These also apply to columns. Finally:

$$\epsilon_{ijk}\epsilon_{lmn}\det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting $A = I$ yields:

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

Linear equations

Let $\boldsymbol{y} = A\boldsymbol{x}$. Then $x_i = A^{-1}_{ij}y_i$ with:

$$\begin{aligned} A^{-1}_{ij} &= \frac{1}{2}\frac{1}{\det A}\epsilon_{imn}\epsilon_{jpq}a_{pm}a_{qn} \\ &= \frac{1}{\det A}C^T_{ij} \end{aligned}$$

where C is the cofactor matrix of A.

Orthogonal matrices

Pseudotensors

Invariant tensors