D1.1.1: Complex numbers

Let $z = x + iy$ and $w = a + ib$ where $x, y, a, b \in \mathbb{R}$. Then z and w are complex numbers. Furthermore:

1. $z = w$ **iff** $x = a$ and $y = b$.
2. $\text{Re}(z) := x$ and $\text{Im}(z) := y$.
3. $|z| := \sqrt{x^2 + y^2}$
4. The **complex conjugate** of z is:
$$z^* := x - iy.$$
5. Addition and multiplication:
$$(x+iy)+(a+ib) = (x+a)+i(y+b)$$
$$(x+iy)(a+ib) = (xa-yb)+i(xb+ya).$$
6. $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$

with rule $i^2 = -1$.

L1.1.3

Let $u, w, z \in \mathbb{C}$ where $z = x + iy$. Then:

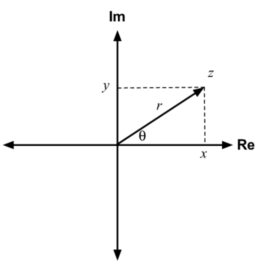
1. $z + w = w + z$ and $zw = wz$.
2. $u + (z + w) = (u + z) + w$
3. $u(zw) = (uz)w$
4. $u(z + w) = uz + uw$
5. $z + 0 = z$ and $1z = z$.
6. $\exists\big(-z := -x + i(-y)\big)$: $z + (-z) = 0$.
7. $\exists z^{-1} : zz^{-1} = 1$ where:
$$z^{-1} := \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

D1.1.5 and D1.1.7: Polar form

Let $z \in \mathbb{C}$ and $z = x + iy$. Then:

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

for $r = |z| = \sqrt{x^2 + y^2}$ and $\theta \in (-\pi, \pi]$ is given by $\tan\theta = y/x$.



L1.1.6

Let $\theta, \phi \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then:

1. $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$
2. $e^{in\theta} = (e^{i\theta})^n$

due to de Moivre's formula:

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

L1.1.9

Let $z, w \in \mathbb{C}$. Then:

1. $|z| = 0$ **iff** $z = 0$.
2. $|\overline{z}| = |z|$
3. $|zw| = |z||w|$
4. $(z^*)^* = z$
5. $|z|^2 = zz^*$ and $|z|^2 = |z|^2$.
6. $(z + w)^* = z^* + w^*$
7. $(zw)^* = z^*w^*$
8. $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$.
9. $\text{Re}(z) = \frac{1}{2}(z + z^*)$
10. $\text{Im}(z) = \frac{1}{2i}(z - z^*)$.

L1.1.10 – 11: Triangle inequalities

Let $z, w \in \mathbb{C}$. Then:

1. $|z + w| \leq |z| + |w|$
2. $||z| - |w|| \leq |z - w|$.

D1.1.12: Argument of z

The set of all arguments of z is:

$$\begin{aligned} \arg(z) &:= \{\theta : z = |z|e^{i\theta}\} \\ &= \{\text{Arg}(z) + 2\pi k : k \in \mathbb{Z}\} \end{aligned}$$

for $z = |z|e^{i\text{Arg}(z)}$ with $-\pi < \text{Arg}(z) \leq \pi$ and is the principle argument of z.

$\therefore \text{Arg}(z) \equiv \arg(z) \mod 2\pi$

P1.1.14

Let $z, w \in \mathbb{C}$. Then:

1. $\arg(zw) = \arg(z) + \arg(w)$
2. $\arg(z^*) = -\arg(z)$.

D1.2.1: Open and closed ϵ-discs

Let $z_0 \in \mathbb{C}$ and $\epsilon > 0$.

1. An **open** ϵ-disc centred at z_0 is:
$$D_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}.$$
2. A **closed** ϵ-disc centred at z_0 is:
$$\overline{D}_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}.$$

A **punctured** ϵ-disc centred at z_0 is:

$$D'_\epsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}.$$

D1.2.2: Open and closed sets

Let $U \subset \mathbb{C}$. Set U is **open** if:

$$\forall z_0 \in U; \exists\epsilon > 0 : D_\epsilon(z_0) \subseteq U.$$

Subset F is **closed** if $\mathbb{C} \setminus F$ is open.

A **neighbourhood** of point $z_0 \in \mathbb{C}$ is an open set that contains z_0.

Remark

$\emptyset$ is vacuously open. Therefore $\mathbb{C}$ is open **and** closed. A set like $D_2(0) \setminus D_1(0)$ is **neither closed nor open**.

The union and intersection of open sets is also an open set.

L1.2.3

Punctured disc $D'_\epsilon(z_0)$ is open.

D1.2.4: Limit points

Let $S \subseteq \mathbb{C}$. z_0 is a **limit point** of S if:

$$\forall\epsilon > 0; D'_\epsilon(z_0) \cap S \neq \emptyset.$$

The **closure** of S is set $\overline{S}$ and contains S and **all** its limit points.

L1.2.6

Let $S \subseteq \mathbb{C}$. S is closed **iff** $S = \overline{S}$.

D1.2.7: Bounded sets

Let $S \subseteq \mathbb{C}$. Set S is **bounded** if:

$$\forall z \in S; \exists M > 0 : |z| \leq S.$$

D1.2.8: ϵ-N convergence

Let $\mathbb{N} = \{0, 1, 2, \dots\}$.

Let $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ be a sequence and $z \in \mathbb{C}$. Then $\lim\limits_{n\to\infty} z_n = z$ if:

$$\begin{aligned} &\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \\ &\implies |z_n - z| < \epsilon. \end{aligned}$$

L1.2.9

Let $z_n, z \in \mathbb{C}$ where $z_n = a_n + ib_n$.

Then $\lim\limits_{n\to\infty} z_n = z$ **iff**:

$$\text{Re}(z) = \lim_{n\to\infty} a_n \text{ and } \text{Im}(z) = \lim_{n\to\infty} b_n.$$

L1.2.10

Let $S \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. Then $z \in \overline{S}$ **iff**:

$$\exists z_n \in S : z = \lim_{n\to\infty} z_n.$$

D1.2.11: Cauchy sequences

z_n is a Cauchy sequence if:

$$\begin{aligned} &\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \\ &\implies |z_n - z_m| < \epsilon. \end{aligned}$$

L1.2.12

z_n is convergent **iff** z_n is Cauchy.

D1.2.14: Bounded sequences

z_n is bounded if:

$$\forall n \in \mathbb{N}; \exists M > 0 : |z_n| \leq M.$$

L1.2.15: Bolzano-Weierstrass

Let z_n be a bounded sequence. Then:

$$\exists (z_{n_k})_{k, n_k \in \mathbb{N}} : \lim_{k \to \infty} z_{n_k} = z \in \mathbb{C}$$

or that z_n has a convergent subsequence.

A selection of a sequence is a subsequence.

D1.3.1: Bounded functions

Let $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$. Then f is a bounded function if:

$$\forall z \in S; \exists M > 0 : |f(z)| \leq M.$$

D1.3.2: ϵ-δ convergence

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}$ and $a_0 \in \mathbb{C}$. Then $\lim\limits_{z \to z_0} f(z) = a_0$ if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : 0 < |z - z_0| < \delta \implies |f(z) - a_0| < \epsilon.$$

L1.3.3

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}$ and $a_0 \in \mathbb{C}$ where $z_0 = x_0 + iy_0$ and $f = u + iv$.

Then $\lim\limits_{z \to z_0} f(z) = a_0$ **iff**:

$$\mathrm{Re}(a_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y)$$

and

$$\mathrm{Im}(a_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y).$$

L1.3.4

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}, a_0 \in \mathbb{C}$ and sequence $w_n \in S \setminus \{z_0\}$.

If $\lim\limits_{z \to z_0} f(z) = a_0$ and $\lim\limits_{n \to \infty} w_n = z_0$ then:

$$\lim_{n \to \infty} f(w_n) = a_0.$$

L1.3.5: Limit identities

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}$ and $a_0, b_0 \in \mathbb{C}$.
Let $f, g : S \to \mathbb{C}$.

If $\lim\limits_{z \to z_0} f(z) = a_0$ and $\lim\limits_{z \to z_0} g(z) = b_0$ then:

1. $\lim\limits_{z \to z_0} (f(z) + g(z)) = a_0 + b_0$
2. $\lim\limits_{z \to z_0} (f(z)g(z)) = a_0 b_0$
3. $\lim\limits_{z \to z_0} \left(\dfrac{f(z)}{g(z)} \right) = \dfrac{a_0}{b_0}$ if $b_0 \neq 0$.

D1.3.6: ϵ-δ continuity

Let $S \subseteq \mathbb{C}, f : S \to \mathbb{C}$ and $z_0 \in S$. Then f is continuous at z_0 if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

L1.3.7

Let $f : \mathbb{C} \to \mathbb{C}$ with rule $f = u + iv$ and $z_0 = x_0 + iy_0 \in \mathbb{C}$.

Then f is continuous at z_0 **iff** u and v are continuous at (x_0, y_0).

L1.3.8

If $f, g : \mathbb{C} \to \mathbb{C}$ are continuous at z_0 then:

1. $f + g$ is continuous at z_0.
2. fg is continuous at z_0.
3. f/g is continuous at z_0. $(g \neq 0)$

D: Image and preimage

Let $f : X \to Y$ where $A \subseteq X$ and $B \subseteq Y$. The image of A is:

$$f(A) = \{f(x) : x \in A\}$$

and the preimage of B is:

$$f^{-1}(B) = \{x : f(x) \in B\}.$$

L1.3.9

Let $U \subseteq \mathbb{C}$ be an open set. $f : \mathbb{C} \to \mathbb{C}$ is continuous **iff** $\forall U \subseteq \mathbb{C}; f^{-1}(U)$ is open for $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\}$.

L1.3.10

Let $f : S \to \mathbb{C}$ be continuous. Let $S \subseteq \mathbb{C}$ be closed and bounded.

Then $f(S)$ is closed and bounded.

D1.4.1: Differentiability

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $f : U \to \mathbb{C}$. Then f is differentiable at z_0 if the following limit exists:

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

L1.4.3

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. If $f : U \to \mathbb{C}$ is differentiable at z_0 then f is continuous at z_0.

L1.4.4

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $f, g : U \to \mathbb{C}$ be differentiable at z_0. Then $f + g$, fg and f/g (where $g(z_0) \neq 0$) are all differentiable at z_0.

L1.4.5: Chain rule

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $g : U \to \mathbb{C}$ be such that $g(U)$ is a neighbourhood of $g(z_0)$. Assume that g is differentiable at z_0 and f is differentiable at $g(z_0)$. Then $f \circ g$ is differentiable at z_0:

$$(f \circ g)'(z_0) = f(g(z_0))g'(z_0).$$

T1.4.6: Cauchy-Riemann equations

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $f : U \to \mathbb{C}$ be differentiable at z_0. Let $z_0 = x_0 + iy_0$ and $f = u + iv$. Then:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

T1.4.8

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0 for $z_0 = x_0 + iy_0$. Let $f : U \to \mathbb{C}$ where $f = u + iv$.

Assume that u and v have continuous first derivatives on a neighbourhood of (x_0, y_0). Then f is differentiable at z_0.

Remark

For f to be differentiable we need to check T1.4.8 and the contrapositive of T1.4.6.

D1.4.9: Holomorphic functions

f is **holomorphic** at z_0 if there exists a neighbourhood U of z_0 such that f is defined and differentiable.

D1.4.13: Harmonic equations

$h(x, y)$ is harmonic if for $\forall (x, y) \in \mathbb{R}^2$ it satisfies Laplace's equation:

$$\frac{\partial^2 h}{\partial x^2}(x, y) + \frac{\partial^2 h}{\partial y^2}(x, y) = 0.$$

L1.4.14

Let $u(x, y), v(x, y)$ be twice continuously differentiable and that $f(x + iy) = u + iy$ is holomorphic on $\mathbb{C}$.

Then u and v are harmonic.

D1.4.15: Harmonic conjugates

Let $U \subseteq \mathbb{R}^2$ and $u : U \to \mathbb{R}$ be harmonic. Then harmonic function $v : U \to \mathbb{R}$ is a **harmonic conjugate** of u if complex function $f = u + iv$ is holomorphic on U.

D1.5.1: Polynomial degree

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. The **degree** of P is the highest power of the variable in P, denoted as $\deg(P)$.

L1.5.2

Let $z_0 \in \mathbb{C}$. Let complex functions f and g be holomorphic at z_0. Then $f+g$, fg and f/g $(g \neq 0)$ are holomorphic at z_0.

C1.5.3

Let $N \in \mathbb{N}$ and $a_0, \dots, a_N \in \mathbb{C}$.
Let $P(z) = \sum_{n=0}^{N} a_n z^n$.
Then $P(z)$ is holomorphic on $\mathbb{C}$ and:

$$P'(z_0) = \sum_{n=1}^{N} n a_n z_0^{n-1}.$$

L1.5.4

Let $P(z) = \sum_{n=0}^{N} a_n z^n$ where $a_i \in \mathbb{R}$ and $P(z_0) = 0$ for $z_0 \in \mathbb{C}$. Then $P(z_0^*) = 0$.

D1.5.5: Rational functions

Let $P, Q : \mathbb{C} \to \mathbb{C}$ be complex functions. Then $R : \{z \in \mathbb{C} : Q(z) \neq 0\} \to \mathbb{C}$ with $R(z) = P(z)/Q(z)$ is a rational function.

L1.5.7

The rational function $R(z) = P(z)/Q(z)$ is holomorphic on $\{z \in \mathbb{C} : Q(z) \neq 0\}$.

L1.5.8

Let $U \subseteq \mathbb{C}$ be open. Let g be holomorphic on U and f be holomorphic on $g(U)$.

Then $f \circ g$ is holomorphic on U.

L1.5.10

Let $U \subseteq \mathbb{R}^2$ be open and $u, v : U \to \mathbb{R}$. u and v satisfy the Cauchy-Riemann equations **iff** $\overline{\partial} f = 0$, where $f = u + iv$ with map $f : U \to \mathbb{C}$.

Remark

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

D1.6.1: Exponential function

The complex exponential function is a function defined as $\exp : \mathbb{C} \to \mathbb{C}$ and rule:

$$\exp(z) := e^x(\cos y + i \sin y)$$

where $z = x + iy$.

P1.6.2

Let $z, w \in \mathbb{C}$.

1. $\exp(z)$ is holomorphic on $\mathbb{C}$.
2. $\exp(z) = \exp'(z)$
3. $\exp(z+w) = \exp(z)\exp(w)$
4. $\exp(z + 2\pi i) = \exp(z)$

D1.6.6: Cosine and sine functions

$$\cos(z) := \frac{1}{2}\Big(\exp(iz) + \exp(-iz)\Big)$$
$$\sin(z) := \frac{1}{2i}\Big(\exp(iz) - \exp(-iz)\Big)$$

L1.6.7

Let $z \in \mathbb{C}$ where $z = x + iy$. Then:

1. $\cos(z)$ and $\sin(z)$ are holomorphic at z, where $\cos'(z) = -\sin(z)$ and $\sin'(z) = \cos(z)$.
2. $\cos^2(z) + \sin^2(z) = 1$
3. $\cos(z + 2\pi) = \cos(z)$
   $\sin(z + 2\pi) = \sin(z)$

L1.6.8

Let $z, w \in \mathbb{C}$. Then:

1. $\sin(z + \pi/2) = \cos(z)$
2. $\sin(z + w)$
   $= \sin(z)\cos(w) + \sin(w)\cos(z)$
3. $\cos(z + w)$
   $= \cos(z)\cos(w) - \sin(z)\sin(w)$.

L1.6.9

Let $z \in \mathbb{C}$ where $z = x + iy$. Then:

$$\sin(x + iy)$$
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$\cos(x + iy)$$
$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

D1.6.11: Hyperbolic functions

$$\cosh(z) := \frac{1}{2}\Big(\exp(z) + \exp(-z)\Big)$$
$$\sinh(z) := \frac{1}{2}\Big(\exp(z) - \exp(-z)\Big)$$

L1.6.12

Let $z \in \mathbb{C}$. Then $\sinh(iz) = i\sin(z)$ and $\cosh(iz) = \cos(z)$.

D1.7.1: Logarithm function

Let $z \neq 0 \in \mathbb{C}$. Then:

$$\log(z) := \{w \in \mathbb{C} : z = \exp(w)\}.$$

L1.7.3

Let $z, w \in \mathbb{C}$ be nonzero. Then:

1. $\log(z) = \{\ln|z| + i\mathrm{Arg}(z) + i2\pi k\}$
2. $\log(zw) = \log(z) + \log(w)$
3. $\log(1/z) = -\log(z)$

where $k \in \mathbb{Z}$ and $\ln(x)$ denotes the real valued natural logarithm of x.

D1.7.5: Principle branch of $\log z$

The principle branch of the logarithm function is defined as:

$$\mathrm{Log} : \mathbb{C}\backslash\{0\} \to \mathbb{C};$$
$$\mathrm{Log}(z) := \ln|z| + i\mathrm{Arg}(z)$$

and is discontinuous on the negative real axis since $\forall x, \epsilon > 0; x \pm i\epsilon \to x$ yet:

$$\lim_{\epsilon \to 0} \mathrm{Log}(-x \pm i\epsilon) = \ln|z| \pm i\pi.$$

i.e. the limit on the axis does not exist.

D1.7.7: Branch cuts

A branch cut $L \subset \mathbb{C}$ is removed so that we may define a holomorphic branch of a multivalued function on $\mathbb{C}\backslash L$.

The half-line from z_0 at angle ϕ is:

$$L_{z_0,\phi} = \{z \in \mathbb{C} : z = z_0 + re^{i\phi}; r \geq 0\}$$

and $D_{z_0,\phi} = \mathbb{C}\backslash L_{z_0,\phi}$.

D1.7.9

Let $\phi \in \mathbb{R}$. Then:

$$\phi < \mathrm{Arg}_\phi(z) \leq \phi + 2\pi$$
$$\mathrm{Log}_\phi(z) := \ln|z| + i\mathrm{Arg}_\phi(z).$$

L1.7.10

Branch $\mathrm{Log}_\phi(z)$ is holomorphic on $D_{0,\phi}$:

$$\forall z \in D_{0,\phi}; \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Log}_\phi(z) = \frac{1}{z}.$$

Also $\mathrm{Log}_\phi\big[g(z)\big]$ is holomorphic on all points $z \in g^{-1}(D_{0,\phi})$.

L1.7.11

Let $\phi \in \mathbb{R}$, $U \subseteq \mathbb{C}$ be open and $g : U \to \mathbb{C}$ be holomorphic on U. Then $\mathrm{Log}_\phi\big(g(z)\big)$ is holomorphic on $U \cap g^{-1}(D_\phi)$.

D1.8.1: α-th power of z

Let $z, \alpha \in \mathbb{C}$. Then the α-th power of z is: $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\}$ for $z \neq 0$.

T1.8.4

Let $\alpha, z \neq 0 \in \mathbb{C}$.

1. If $\alpha \in \mathbb{Z}$ there is one value of z^α.
2. If $\alpha = p/q \in \mathbb{Q}$ for p, q are coprime then there are q values of z^α.
3. If α is irrational or complex then there are infinite values of z^α.

D1.8.5: Roots of unity

Let q be a positive integer. Then:

$$1^{1/q} = 1, \omega, \ldots, \omega^{q-1}; \omega := \exp(2\pi i/q)$$

are the q roots of unity.

D1.8.7: Principle branch of z^α

Let $z \in D$ such that $\mathrm{Log}(z)$ is defined. Then the principle branch of z^α is:

$$z^\alpha := \exp\big(\alpha \mathrm{Log}(z)\big).$$

L1.8.8

Let $\alpha, \beta, z \in \mathbb{C}$ for $z \neq 0$. Then:

$$z^\alpha z^\beta = z^{\alpha+\beta}.$$

L1.8.9

A branch of z^α is holomorphic on D_ϕ and:

$$\forall z \in D_\phi; (z^\alpha)' = \alpha z^{\alpha-1}.$$