

EM S1 Tutorials

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Tutorial 3

1. Let T_{ij} be a rank 2 tensor, and S_{ij} a rank 2 pseudotensor.

Show that:

- $T_{ij}S_{ik}$ is a pseudotensor.
- $T_{ij}S_{ij}$ is a pseudoscalar.

So by definition we have the following:

$$\begin{aligned} T'_{ij} &= \ell_{ip}\ell_{jq}T_{pq} \\ S'_{ij} &= (\det L)\ell_{ip}\ell_{jq}S_{pq}. \end{aligned}$$

Changing indices for the pseudotensor:

$$S'_{ik} = (\det L)\ell_{ir}\ell_{ks}S_{rs}$$

We must swap for r and s here! Then:

$$\begin{aligned} T'_{ij}S'_{ik} &= (\det L)\ell_{ip}\ell_{jq}\ell_{ir}\ell_{ks}T_{pq}S_{rs} \\ &= (\det L)\delta_{pr}\ell_{jq}\ell_{ks}T_{pq}S_{rs} \\ &= (\det L)\ell_{jq}\ell_{ks}T_{rq}S_{rs} \end{aligned}$$

Then set $\alpha_{qs} = T_{rq}S_{rs}$, and therefore $T_{ij}S_{ik}$ is a pseudotensor. i.e:

$$\alpha'_{jk} = (\det L)\ell_{jq}\ell_{ks}\alpha_{qs}.$$

For the second part set $j = k$. Therefore:

$$\begin{aligned} T'_{ij}S'_{ij} &= (\det L)\ell_{jq}\ell_{js}T_{rq}S_{rs} \\ &= (\det L)\delta_{qs}T_{rq}S_{rs} \\ &= (\det L)T_{rq}S_{rq}, \end{aligned}$$

which is the transformation law for pseudoscalars.

Notice how the equation is similar to a dot product, yielding a scalar quantity.

Tutorial 5

1. Current in crystal from applied electric field is explored. Consider:

$$j_i = \sigma_{ij} E_j,$$

where j_i is the current density, σ_{ij} is the conductivity of our crystal and E_j the electric field we subject our crystal to.

Now by the **quotient theorem** clearly if j_i and E_j are vectors then σ_{ij} is a rank 2 tensor.

The conductivity tensor of our crystal is defined as:

$$\sigma_{ij} = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{ij}$$

and to find electric field directions where no current flows we need to solve $\sigma_{ij} E_j = 0$. This can be done by writing our equation as an augmented matrix and then row reducing, which yields:

$$\mathbf{E} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}.$$

Tutorial 7

1. For part (i), we are asked to find $\nabla(\mathbf{c} \cdot \mathbf{r})^n$.

This is the **gradient** operator and acts only on scalar fields $\phi(\mathbf{r})$. Note:

$$\nabla\phi(\mathbf{r}) = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i,$$

where $\mathbf{r} = x_i\mathbf{e}_i$. Using the chain rule we get:

$$\begin{aligned}\nabla(\mathbf{c} \cdot \mathbf{r})^n &= n(\mathbf{c} \cdot \mathbf{r})^{n-1}\nabla(\mathbf{c} \cdot \mathbf{r}) \\ &= n\mathbf{c}(\mathbf{c} \cdot \mathbf{r})^{n-1}.\end{aligned}$$

Our solution makes sense since we obtain a vector quantity. Now however our question asks us to show this using suffix notation:

$$\begin{aligned}\nabla(\mathbf{c} \cdot \mathbf{r})^n &= \mathbf{e}_i \frac{\partial}{\partial x_i}(\mathbf{c} \cdot \mathbf{r})^n \\ &= n(\mathbf{c} \cdot \mathbf{r})^{n-1}\mathbf{e}_i \frac{\partial}{\partial x_i}(c_j x_j) \\ &= n(\mathbf{c} \cdot \mathbf{r})^{n-1}\mathbf{e}_i c_j \delta_{ij} \\ &= n(\mathbf{c} \cdot \mathbf{r})^{n-1}\mathbf{c}.\end{aligned}$$

For part (ii) we want ∇r^n .

Firstly we need to prove an important result, namely that:

$$\nabla r = \frac{1}{r}\mathbf{r},$$

where $r = |\mathbf{r}| = \sqrt{x_j^2}$ for $\mathbf{r} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

So consider the following:

$$\begin{aligned}\nabla r &= \frac{\partial r}{\partial x_i}\mathbf{e}_i \\ &= \mathbf{e}_i \frac{\partial}{\partial x_i} \sqrt{x_j^2} \\ &= \mathbf{e}_i \cdot \frac{1}{2} \frac{1}{\sqrt{x_j^2}} \cdot 2x_j \frac{\partial x_j}{\partial x_i} \\ &= \mathbf{e}_i \cdot \frac{1}{r} \cdot x_j \delta_{ij} \\ &\Rightarrow \frac{1}{r}\mathbf{r}.\end{aligned}$$

Now that this is established we may use it:

$$\begin{aligned}\nabla r^n &= \frac{\partial}{\partial \mathbf{r}}(r^n) \cdot \nabla r \\ &= n \cdot r^{n-2} \mathbf{r},\end{aligned}$$

and we are finished. We can also do this with suffix notation:

$$\begin{aligned}\nabla r^n &= e_i \frac{\partial}{\partial x_i} [x_j^2]^{n/2} \\ &= \frac{n}{2} [x_j^2]^{(n-2)/2} e_i \frac{\partial}{\partial x_i} [x_k^2] \\ &= \frac{n}{2} [x_j^2]^{(n-2)/2} e_i \cdot 2x_k \delta_{ik} \\ &= nr^{n-2} \mathbf{r}.\end{aligned}$$

For part (iii), we need to find $\nabla \cdot (r^n \mathbf{r})$.

This is the **divergence** operator and is defined:

$$\nabla \cdot \mathbf{E} = \frac{\partial E_i}{\partial x_i},$$

where $\mathbf{E} = \mathbf{E}(\mathbf{r})$ and is a vector field. Therefore:

$$\nabla \cdot \mathbf{r} = 3.$$

We also need the chain rule for divergence:

$$\begin{aligned}\nabla \cdot (\phi \mathbf{a}) &= \frac{\partial}{\partial x_i} (\phi a_i) \\ \implies (\nabla \phi) \cdot \mathbf{a} + \phi (\nabla \cdot \mathbf{a}).\end{aligned}$$

Putting all this together we get:

$$\begin{aligned}\nabla \cdot (r^n \mathbf{r}) &= (\nabla r^n) \cdot \mathbf{r} + r^n (\nabla \cdot \mathbf{r}) \\ &= (n+3)r^n.\end{aligned}$$

We can also do this in suffix notation:

$$\begin{aligned}\nabla \cdot (r^n \mathbf{r}) &= \frac{\partial}{\partial x_i} [r^n x_i] \\ &= \left(\frac{\partial}{\partial x_i} r^n \right) x_i + r^n \left(\frac{\partial}{\partial x_i} x_i \right) \\ &= \left(nr^{n-1} \frac{\partial}{\partial x_i} [x_k^2]^{1/2} \right) x_i + r^n \delta_{ii} \\ &= (n+3)r^n.\end{aligned}$$

For part (iv) we want $\nabla \times (r^n \mathbf{r})$.

So note the chain rule for curl:

$$\nabla \times (\phi \mathbf{a}) = \nabla \phi \times \mathbf{a} + \phi \nabla \times \mathbf{a}.$$

Therefore:

$$\begin{aligned} \nabla \times (r^n \mathbf{r}) &= \nabla r^n \times \mathbf{r} + r^n \nabla \times \mathbf{r} \\ &= n \cdot r^{n-2} \cdot \mathbf{r} \times \mathbf{r} + r^n \cdot \mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

We can also do this with suffix notation:

$$\begin{aligned} \nabla \times (r^n \mathbf{r}) &= e_i \epsilon_{ijk} \frac{\partial}{\partial x_j} [r^n \mathbf{r}]_k \\ &= e_i \epsilon_{ijk} \frac{\partial}{\partial x_j} [r^n x_k] \\ &= e_i \epsilon_{ijk} \left(\frac{\partial}{\partial x_j} [r^n] x_k + r^n \frac{\partial}{\partial x_j} x_k \right) \\ &= e_i \epsilon_{ijk} (n r^{n-2} x_i x_k + r^n \delta_{jk}) \\ &= 0. \end{aligned}$$

For part (v) we want $\nabla \cdot (\mathbf{c} \times \mathbf{r})$.

$$\begin{aligned} \nabla \cdot (\mathbf{c} \times \mathbf{r}) &= \frac{\partial}{\partial x_i} [\epsilon_{ijk} c_j x_k] \\ &= \epsilon_{ijk} c_j \cdot \frac{\partial}{\partial x_i} x_k \\ &= \epsilon_{ijk} c_j \delta_{ik} = 0. \end{aligned}$$

For part (vi) we want $\nabla \times (\mathbf{c} \times \mathbf{r})$.

$$\begin{aligned} \nabla \times (\mathbf{c} \times \mathbf{r}) &= \epsilon_{ijk} \cdot \frac{\partial}{\partial x_j} [\epsilon_{klm} c_l x_m] \\ &= \epsilon_{ijk} \epsilon_{klm} \cdot c_l \delta_{mj} \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) c_l \delta_{mj} \\ &= 2c_i \\ &= \mathbf{c}. \end{aligned}$$

For part (vii) we want $(\mathbf{c} \cdot \nabla) \mathbf{r}$.

$$\begin{aligned} (\mathbf{c} \cdot \nabla) \mathbf{r} &= c_i \frac{\partial}{\partial x_i} x_j \mathbf{e}_j \\ &= c_i \delta_{ij} \mathbf{e}_j \\ &= \mathbf{c}. \end{aligned}$$

2.

3.

4. For (i) we are asked to show the following:

- $x\delta(x) = 0$
- $\delta(cx) = \frac{1}{|c|}\delta(x)$

So we have the **sift** property:

$$\int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a).$$

Taking $f(x) = x$ and $a = 0$ gives:

$$\int_{\mathbb{R}} x\delta(x)dx = 0.$$

Recall the **fundamental theorem of calculus**:

$$\frac{d}{dx} \int_{x_0}^x g(t)dt = g(x).$$

Let $g(t) = t\delta(t)$ and integrate over \mathbb{R} . $\therefore x\delta(x) = 0$.

Now consider the following identity:

$$\int_{\mathbb{R}} \delta(x)dx = 1.$$

We aim to write $\delta(cx)$ into this form. First assume that $c > 0$:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(cx)dx &= \int_{-\infty}^{\infty} \frac{1}{c}\delta(cx)d(cx) \\ &= \frac{1}{c}. \end{aligned}$$

If $c = 0$ then our equality holds trivially. Now assume that $c < 0$. We can write c as $c = -|c|$. or that:

$$\int_{\mathbb{R}} |c|\delta(cx)dx = 1.$$

It must then be that:

$$|c|\delta(cx) = \delta(x).$$

Tutorial 8

1. Firstly consider a point dipole with moment \mathbf{p} at position \mathbf{r} :

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}.$$

Show that it generates the following electric field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r} - r^2\mathbf{p}}{r^5}.$$

Since $\mathbf{E} = -\nabla\phi(\mathbf{r})$:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right) \\ &= -\frac{1}{4\pi\epsilon_0} \left(\nabla(\mathbf{p} \cdot \mathbf{r}) \frac{1}{r^3} + (\mathbf{p} \cdot \mathbf{r}) \nabla \left(\frac{1}{r^3} \right) \right). \end{aligned}$$

Let's separately deal with the dels:

$$\begin{aligned} \nabla(\mathbf{p} \cdot \mathbf{r}) &= \mathbf{e}_i \frac{\partial}{\partial x_i} (p_j x_j) \\ &= \mathbf{e}_i p_j \delta_{ij} \\ &= \mathbf{p} \end{aligned}$$

and

$$\begin{aligned} \nabla \left(\frac{1}{r^3} \right) &= \mathbf{e}_i \frac{\partial}{\partial x_i} \left[(x_j^2)^{-3/2} \right] \\ &= \mathbf{e}_i \left(-\frac{3}{2} (x_j^2)^{-5/2} \cdot 2x_j \frac{\partial x_j}{\partial x_i} \right) \\ &= -3r^{-5} \mathbf{r}. \end{aligned}$$

Combining these yields us the desired electric field.