Probability distributions

The probablity of an event in a trial is:

$$\mathbb{P}(\text{event}) := \lim_{N\to\infty} \frac{n}{N}$$

given n occurrences in N trials.
For discrete probabilities:

$$\sum_{i=1}^{q} \mathbb{P}(i) = 1$$

$$\mathbb{P}(i \text{ or } j) = \mathbb{P}(i) + \mathbb{P}(j)$$

$$\mathbb{P}(i \text{ and } j) = \mathbb{P}(i)\mathbb{P}(j).$$

Given continuous random variables:

$$\mathbb{P}([x, x + \mathrm{d}x]) = P(x)\mathrm{d}x$$

for P is the probability density function:

$$\int_{-\infty}^{\infty} P(x)\mathrm{d}x = 1.$$

We define the **mean** and **variance** as:

$$\overline{x} = \sum_{i=1}^{q} x_i P_i \text{ or } \int_{-\infty}^{\infty} xP(x)\mathrm{d}x$$

$$\begin{aligned}\overline{\Delta x^2} &= \sum_{i=1}^{q} (x_i - \overline{x})^2 P_i \\ &= \int_{-\infty}^{\infty} (x - \overline{x})^2 P(x)\mathrm{d}x \\ &= \overline{x^2} - (\overline{x})^2.\end{aligned}$$

The **standard deviation** is the square root of the variance $\left(\overline{\Delta x^2}\right)^{1/2}$ and:

$$\overline{f(x)} = \int_{-\infty}^{\infty} f(x)P(x)\mathrm{d}x.$$

Binomial distribution

The probability of observing n events each with probability p in N trials is:

$$P_n = \binom{N}{n} p^n (1-p)^{N-n}$$

where $\binom{N}{n} = \dfrac{N!}{n!(N-n)!}$ with:

$$\overline{n} = Np \text{ and } \overline{\Delta n^2} = Np(1-p)$$

since we have that:

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}$$

$$\begin{aligned}f(\alpha) &= \sum_{n=0}^{N} \binom{N}{n} (p\alpha)^n (1-p)^{N-n} \\ &= (p\alpha + 1 - p)^N.\end{aligned}$$

Note that $\binom{N}{n}$ denotes ways to pick n items from N items. For large N:

$$\ln(N!) \approx N\ln(N) - N$$

known as **Stirling's approximation**.

We also define the **fractional deviation** as the deviation on the scale of the mean:

$$\frac{\left(\overline{\Delta x^2}\right)^{1/2}}{\overline{n}} = \frac{1}{N^{1/2}}.$$

Taylor expansions

Let $s(n)$ be expanded at $n = a$:

$$\begin{aligned}s(n) &= s(a) + s'(a)(n-a) \\ &\quad + \frac{1}{2}s''(a)(n-a)^2 + \mathcal{O}[(n-a)^3].\end{aligned}$$

Poisson distribution

Let $N \gg n$ and let p be the probability of an event in a trial. Assume that as $N \to \infty$, $p \to 0$. Under such conditions the binomial probability of observing n events in N trials is:

$$P_n \approx (\overline{n})^n \frac{\exp(-\overline{n})}{n!}$$

with mean and variance Np.

Gaussian distribution

Let N be very large. Then the binomial distribution becomes Gaussian:

$$P_n \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(n-Np)^2}{2\sigma^2}\right)$$

via Stirling's approximation and Taylor expansions with variance $\sigma^2 = Np(1-p)$ and mean $\mu = Np$.

Microstates

A microstate is a complete specification of all degrees of freedoms in a system, with respect to a microscopic model.

Macrostate

A macrostate is a limited description by the values of observables, like pressure.

Given any macrostate there correspond many microstates.