D: Supremum and infimum

T: Approximation lemma

D: Completeness of $\mathbb{R}$

Every nonempty <u>bounded</u> subset of $\mathbb{R}$ has an infimum and supremum.

T: Archimedean property

$\forall a, b \in \mathbb{R}; a > 0; \exists n \in \mathbb{N} : na > b$

D1.1: Nested intervals

A sequence of sets $(I_n)_{n\in\mathbb{N}}$ is nested if $I_1 \supset I_2 \supset I_3 \ldots$.

T1.1: Nested interval property

Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of <u>nonempty</u>, <u>closed</u> and <u>bounded</u> nested intervals. Then:

$$E = \bigcap_{n\in\mathbb{N}} I_n \neq \emptyset.$$

If $\lambda(I_n) \to 0$ then E contains one number, where λ denotes length.

T1.2

Let $E = [a, b]$ and that there exists an <u>open</u> collection of nested intervals $(I_\alpha)_{\alpha\in A}$ such that:

$$E \subset \bigcup_{\alpha\in A} I_\alpha.$$

Then $\exists\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset A$ such that:

$$E \subset I_{\alpha_1} \cup I_{\alpha_2} \cup \cdots \cup I_{\alpha_n}.$$

D1.2: ϵ-N convergence

Let $\lim_{n\to\infty} x_n = a$. Then:

$$\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |x_n - a| < \epsilon.$$

D1.3: Cauchy sequences

The sequence (x_n) is Cauchy if:

$$\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N$$
$$\implies |x_n - x_m| < \epsilon.$$

D2.1: Pointwise convergence

$f_n \to f$ pointwise on E if:

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Here $f_n : E \to \mathbb{R}$.

$$\forall x \in E; \forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N$$
$$\implies |f_n(x) - f(x)| < \epsilon$$

T1.3 and T1.4

Cauchy $\iff$ ϵ-N convergent.

D1.4: Subsequences

The subsequence of $(x_n)_{n\in\mathbb{N}}$ is a sequence of form $(x_{n_k})_{k\in\mathbb{N}}$ and is a selection of the original sequence **taken in order**.

T1.5: Bolzano-Weierstrass

Every <u>bounded</u> real sequence has **a** convergent subsequence.

D1.5: Limit inferior and superior

Let (x_n) be a bounded real sequence. Then:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right).$$

T1.6

The real sequence (x_n) is convergent **iff**:

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

D1.6: Convergence of infinite series

Let $S = \sum_{k=1}^{\infty} a_k$ is convergent if:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k < \infty$$

The infinite series S is **absolutely convergent** if $S = \sum_{k=1}^{\infty} |a_k|$ is also convergent. Otherwise S is conditionally convergent.

T1.7: Cauchy criterion for series

$S = \sum_{k=1}^{\infty} a_k$ is convergent **iff**:

$$\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall m \geq n \geq N$$
$$\implies \left|\sum_{k=n+1}^{m} a_k\right| < \epsilon.$$

T1.8

Let $S = \sum_{k=1}^{\infty} a_k$ be absolutely convergent. Let $z : \mathbb{N} \to \mathbb{N}$ be a bijection. Then:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}.$$

T1.9: Riemann rearrangement

Let $S = \sum_{k=1}^{\infty} a_k$ be conditionally convergent. Then there exists rearrangements such that S can take on any value.

D1.7: Sequential continuity

T1.10

D1.8: Composition of functions

T1.11

T1.12: ϵ-δ continuity

T1.13: Intermediate value theorem

T1.14: Extreme value theorem

T: Mean value theorem

D: Differentiability

T: Continuity test

D2.2: Uniform convergence

$f_n \to f$ uniformly on E if:

$$\forall\epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \text{ and } \forall x \in E$$
$$\implies |f_n(x) - f(x)| < \epsilon$$

P2.1

The following statements are equivalent.

1. $f_n \to f$ uniformly on E
2. $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$
3. $\exists a_n \to 0$ s.t. $|f_n(x) - f(x)| \leq a_n$ for $\forall x \in E$.

T2.1

If f_n is continuous on E **and** $f_n \to f$ uniformly on E then f is continuous on E.

Remark

If f is <u>not continuous</u> on E then f_n <u>cannot</u> be uniform on E.

T2.5: Weierstrass M-test

Let $E \subset \mathbb{R}$ and $f_k : E \to \mathbb{R}$.

$\exists M_k > 0 : \sum_{k=1}^{\infty} M_k < \infty$.

If $\forall k \in \mathbb{N}$ and $\forall x \in E$; $|f_k(x)| \leq M_k$ then: $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on E.

D: Power series

Let (a_n) be a real sequence and $c \in \mathbb{R}$. Then:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

is a power series centered at c, with **radius of convergence**:

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

where $R = \infty$ implying that series converges everywhere.

T3.1: Convergence of power series

Let $0 < R < \infty$. If $|x - c| < R$ then $f_{PS}(x)$ converges absolutely.

If $|x - c| > R$ then $f_{PS}(x)$ diverges.

T3.2: Continuity of power series

Let $0 < r < R$ where R is the radius of convergence of $f_{PS}(x)$.

Then for $|x - c| \leq r$, $f_{PS}(x)$ converges absolutely and uniformly to a <u>continuous</u> function $f(x)$.

L3.1

$\sum_{n=1}^{\infty} a_n(x-c)^n$ and $\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$ have the same radius of convergence.

T3.3

Let R be the radius of convegence of $f_{PS}(x)$. Then for $\forall x : |x - c| < R$, $f_{PS}(x)$ is **infinitely differentiable**.

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

$$\therefore a_n = \frac{f^{(n)}(c)}{n!}$$

Elementary expansions

- $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$