Honours Differential Equations

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Contents

1 ODE systems

1.1 Integrating factors

Consider linear DE of form

$$y' + P(x)y = Q(x)$$

The integrating factor for this DE is:

$$I(x) = \exp\left(\int P(x)dx\right)$$

and the solution to the linear DE is:

$$y(x) = \frac{1}{I(x)}\int Q(x)I(x)dx + \frac{\alpha}{I(x)}$$

where here α is a constant.

1.2 Change of variables

For higher order differential equations of form

$$y^{(n)} = F(y, y', \ldots, y^{(n-1)}, t),$$

consider **change of variables** $x_{i+1} = y^{(i)}$ for $i \in \{0, 1, \ldots, n-1\}$.

Taking derivatives with respect to time yields a first order matrix ODE system:

$$x_j' = F_j(t, x_1, \ldots, x_n)$$

for $j = 1, \ldots, n$. We either immediately write this as a matrix system or linearise near a critical point.

1.3 Existence and uniqueness for IVPs

An initial value problem (**IVP**) is defined as

$$\frac{dx}{dt} = f(x, t)$$

for **initial** condition $x(t_0) = x_0$. A solution $x : I \to \mathbb{R}$ is a differentiable function that satisfies the IVP. Similarly for a first order system

$$x_i' = F_i(t, x_1, \ldots, x_n)$$

to have a **unique** solution, F_i and $\frac{\partial F_i}{\partial x_j}$ must be continuous in a region. Here $i, j \in \{1, \ldots, n\}$. This is known as the Picard-Lindelöf theorem.

1.4 Homogeneous systems

1.4.1 Unique eigenvalues

Now consider $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x}$, where $\mathbf{A}$ is a $n \times n$ matrix.
Substituting $\boldsymbol{x} = e^{rt}\boldsymbol{\xi}$ results in an eigenvalue problem:

$$(\mathbf{A} - r_i\mathbf{I_n})\boldsymbol{\xi^{(i)}} = \mathbf{0}$$

where $i \in \{1, 2, \ldots, n\}$. Our general solution is then:

$$\begin{aligned}\boldsymbol{x}(t) &= \sum_{i=1}^{n} c_i e^{r_i t}\boldsymbol{\xi}^{(i)} \\ &= \sum_{i=1}^{n} c_i \boldsymbol{x}^{(i)} \\ &= \boldsymbol{\Psi}(t)\boldsymbol{c}\end{aligned}$$

where $\boldsymbol{\Psi}(t)$ is our fundamental matrix satisfying $\boldsymbol{\Psi'} = \mathbf{A}\boldsymbol{\Psi}$ and that:

$$\boldsymbol{\Psi}(t) = [\boldsymbol{x^{(1)}}, \ldots, \boldsymbol{x^{(n)}}].$$

Furthermore if initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ are given we then have that:

$$\boldsymbol{c} = \boldsymbol{\Psi}^{-1}(t_0)\boldsymbol{x}(t_0)$$

and

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\boldsymbol{x}_0.$$

1.4.2 Matrix exponentials

We can also write our solutions as a matrix exponential, defined as such:

$$\begin{aligned}e^{\boldsymbol{A}t} &= \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}t)^n}{n!} \\ &= \boldsymbol{I}_n + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \ldots\end{aligned}$$

and since an exponential power series is infinitely differentiable:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}.$$

Therefore it is then deduced that the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0).$$

and that $e^{\boldsymbol{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$ where we are finding the coefficients to the general solution $\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{c}$.

1.4.3 Diagonalisation

Consider again $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ where $\boldsymbol{A}$ is a $n \times n$ matrix that is diagonalisable:

$$\boldsymbol{A}\boldsymbol{\xi}^{(i)} = r_i \boldsymbol{\xi}^{(i)}$$

$$\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$$

for here $\boldsymbol{D}$ is our diagonal matrix containing our eigenvalues r_i and

$$\boldsymbol{T} = [\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}].$$

Then let $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$. After some algebra we have that:

$$\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y}$$

which have particular solutions $\boldsymbol{y} = e^{r_i t}\boldsymbol{e}^{(i)}$ for $i \in \{1, \ldots, n\}$.

Since our fundamental matrix with respect to $\boldsymbol{y}$ is $\boldsymbol{Q} = e^{\boldsymbol{D}t}$, the fundamental matrix with respect to $\boldsymbol{x}$ is:

$$\boldsymbol{\Psi}(t) = \boldsymbol{T}e^{\boldsymbol{D}t}$$

and we get an expression for the matrix exponential of $\boldsymbol{A}$:

$$e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{D}t}\boldsymbol{T}^{-1}$$

where $e^{\boldsymbol{D}t}$ is a diagonal matrix with entries $e^{r_i t}$.

1.4.4 Generalised eigenvectors

If eigenvalues do not generate enough eigenvectors to form n linearly independent solutions for a $n \times n$ matrix $\boldsymbol{A}$ we try the following ansatz:

$$\boldsymbol{x} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}$$

which gives

$$(\mathbf{A} - r_i\mathbf{I_n})\boldsymbol{\eta}^{\boldsymbol{(i)}} = \boldsymbol{\xi}^{\boldsymbol{(i)}}.$$

Therefore we end up with:

$$\boldsymbol{x}^{(1)} = e^{rt}\boldsymbol{\xi}$$

and

$$\boldsymbol{x}^{(2)} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

The process is algorithmic — we try again until it works.

1.5 Non-homogeneous systems

Consider non-homogeneous ODE system:

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}.$$

There a couple of different approaches we can take to solve such a system.

- **Change of basis**

  Let $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$, where $\boldsymbol{T}$ is our eigenvector matrix from diagonalisation.
  So $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$, and after some algebra we obtain:

  $$\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y} + \boldsymbol{T}^{-1}\boldsymbol{g}$$

  which can be solved by integrating factors. Finally revert back to $\boldsymbol{x}$.

- **Variation of parameters**

  So $\boldsymbol{x}_H = \boldsymbol{\Psi}\boldsymbol{c}$ solves the $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{c}$ is a constant vector.

  We then assume that the solution to our non-homogeneous system takes the form:

  $$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{u}$$

  for here $\boldsymbol{u} = \boldsymbol{u}(t)$. We then get $\boldsymbol{\Psi}\boldsymbol{u}' = \boldsymbol{g}$, which can be solved by eliminating variables and integrating.

- **Method of undetermined coefficients**

  Our non-homogeneous ODE system has solutions of form:

  $$\boldsymbol{x} = \boldsymbol{x}_H + \boldsymbol{x}_p$$

  Solving the homogeneous ODE gives us $\boldsymbol{x}_H$.

  On the other hand we just need to find a **particular solution** $\boldsymbol{x}_p$ that satisfies our non-homogeneous ODE. Then our solution is complete.

  Whilst the fastest, this method is not guaranteed to work.

1.6 Critical points & linearisation

Consider non-linear ODE system

$$x' = F(x, y),$$
$$y' = G(x, y).$$

We define $\boldsymbol{x^0} = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$ as a **critical point** when $F(\boldsymbol{x^0}) = G(\boldsymbol{x^0}) = 0$.

Non-linear systems may then be linearised by Taylor expanding them around a critical point $\boldsymbol{x^0}$, and discarding higher order terms.
i.e. let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_1 = x - x^0$ and $u_2 = y - y^0$.

$$\begin{aligned} \therefore u_1' &= x' \\ &\approx F(x^0, y^0) + \left(\frac{\partial F}{\partial x}\right)_{x^0} (x - x^0) + \left(\frac{\partial F}{\partial y}\right)_{y^0} (y - y^0) \end{aligned}$$

$$\begin{aligned} \therefore u_2' &= y' \\ &\approx G(x^0, y^0) + \left(\frac{\partial G}{\partial x}\right)_{x^0} (x - x^0) + \left(\frac{\partial G}{\partial y}\right)_{y^0} (y - y^0) \end{aligned}$$

Then we end up with the following linear system:

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u}$$

where $\boldsymbol{A} = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}_{\boldsymbol{x}=\boldsymbol{x^0}}$ and is a 2×2 Jacobian matrix.

Our critical points $\boldsymbol{x^0}$ may also be classified:

Eigenvalues	**Critical Points**	**Stability**
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asymp. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asymp. stable
$r_1,\ r_2 = \lambda \pm i\mu\ (\lambda > 0)$	focus	unstable
$r_1,\ r_2 = \lambda \pm i\mu\ (\lambda < 0)$	focus	asymp. stable
$r_1 = i\mu\,, r_2 = -i\mu$	center	stable

Linearisation preserves critical point behaviour **except** when eigenvalues are of $r = \pm i\mu$ form, for which then classification is unknown.

1.7 Stability of critical points

Stable critical points $\boldsymbol{x^0}$: All solutions start and stay near $\boldsymbol{x^0}$.

$\forall \epsilon > 0; \exists \delta > 0; \forall \boldsymbol{x}_{solution}$ to $\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x}, t)$:
$|\boldsymbol{x}(0) - \boldsymbol{x^0}| < \delta \implies |\boldsymbol{x}(t) - \boldsymbol{x^0}| < \epsilon$ for $\forall t \geq 0$

Attracting critical points $\boldsymbol{x^0}$: All solutions tends to $\boldsymbol{x^0}$.

$\forall \delta > 0 : |\boldsymbol{x}(0) - \boldsymbol{x^0}| < \delta \implies \lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x^0}$

Asymptotically stable critical points $\boldsymbol{x^0}$: Attracting **and** stable

1.8 Lyapunov's theory and limit cycles

In this section $\dot{\boldsymbol{x}}$ means its first time derivative. So consider:

$$\dot{x} = F(x, y),$$

$$\dot{y} = G(x, y)$$

defined in $\mathbb{R}^2$. Let $\boldsymbol{x^0} \in D$ be a critical point.

The function $E : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a Lyapunov function where $E(x^0, y^0) = 0$, whenever it exists. Note that the time derivative of E is:

$$\frac{dE}{dt} = \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G$$

- Let $E > 0$ for $\forall \boldsymbol{x} \neq \boldsymbol{x^0}$.

  If $\frac{dE}{dt} \leq 0$ then $\boldsymbol{x^0}$ is stable.

  If $\frac{dE}{dt} < 0$ then $\boldsymbol{x^0}$ is asymptotically stable.
- If every neighbourhood of $\boldsymbol{x^0}$ contains $\boldsymbol{x^*}$ such that $E(\boldsymbol{x^*}) > 0$
  and if $\frac{dE}{dt} > 0$ then $\boldsymbol{x^0}$ is unstable.

Now **limit cycles** are defined as periodic solutions such that at least one other non-closed trajectory approaches the limit cycle as $t \to \infty$.

Comment on stability of limit cycles.

2 Fourier series

2.1 Real Fourier series

Let $f(x)$ and $f'(x)$ be **piecewise continuous** in $[-L, L]$ with **period** $2L$. i.e. $f(x) = f(x + \boldsymbol{2L})$ for $\forall x$. Then the Fourier series for $f(x)$ is

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}).$$

The **convergence** of our Fourier series depends on the continuity of $f(x)$:

- If $f(x)$ is <u>continuous</u> then $f_{FS}(x) = f(x)$.
- If $f(\alpha)$ is <u>discontinuous</u> then at point α we have

$$f_{FS}(\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}.$$

Note that $f(x)$ is continuous at α if $f(\alpha) = \lim_{x\to\alpha} f(x)$ and we define:

$$f(\alpha^-) = \lim_{x\to\alpha^-} f(x)$$

and

$$f(\alpha^+) = \lim_{x\to\alpha^+} f(x),$$

i.e. limits from left and right respectively. It is important to also note that the <u>derivative</u> of a Fourier series is **not necessarily convergent**.

Now consider $S_n = \sin\frac{n\pi x}{L}$ and $C_n = \cos\frac{n\pi x}{L}$. We then have the following **orthogonality relations**:

$$\langle S_n, S_m\rangle = \langle C_n, C_m\rangle = L\delta_{mn}$$

$$\langle S_n, C_m\rangle = 0$$

where we define the inner product as:

$$\langle u(x), v(x)\rangle = \int_{-L}^{L} u(x)v(x)dx$$

and use the following trigonometric identities:

$$\cos A\cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)].$$

Now integrating the following expression:

$$\int_{-L}^{L}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L})\right)\mathrm{d}x=\int_{-L}^{L}f(x)\mathrm{d}x$$

gives:

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx.$$

Similarly:

$$a_n=\frac{1}{L}\int_{-L}^{L}\cos\frac{n\pi x}{L}f(x)dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L}\sin\frac{n\pi x}{L}f(x)dx.$$

Note that δ_{mn} is the **Kronecker delta** and is defined as:

$$\delta_{mn}=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases}$$

Furthermore notice that:

- The Fourier series of **even** functions contains <u>only</u> **cosines**.
- The Fourier series of **odd** functions contains <u>only</u> **sines**.

Even functions are defined $f(-x)=f(x)$, and:

$$\int_{-L}^{L}f_{even}dx=2\int_{0}^{L}f_{even}dx.$$

Similarly **odd** functions are defined $f(-x)=-f(x)$, and:

$$\int_{-L}^{L}f_{odd}dx=0.$$

We can also extend a function defined in $[0,L]$ in several ways:

1. Define <u>even</u> function:

$$g(x)=\begin{cases}f(x) & x\in[0,L]\\ f(-x) & x\in(-L,0)\end{cases}$$

where its Fourier expansion is a cosine series.

2. Define <u>odd</u> function:

$$h(x)=\begin{cases}f(x) & x\in(0,L)\\ 0 & x=0,L\\ -f(-x) & x\in(-L,0)\end{cases}$$

where its Fourier expansion is a sine series.

2.2 Complex Fourier series

Expanding $f(x)$ defined in $[-L, L]$ with period $2L$:

$$f_{FS}(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi}{L}x\right)$$

using Euler's formula $e^{i\theta} = \sin\theta + i\cos\theta$. Its coefficients are:

$$c_n = \frac{1}{2L}\int_{-L}^{L} \exp\left(-\frac{in\pi}{L}x\right) f(x)dx$$

for $\forall n \in \mathbb{Z}$ and:

$$c_n = \begin{cases} (a_n - ib_n)/2 & n > 0 \\ (a_0)/2 & n = 0 \\ (a_n + ib_n)/2 & n < 0. \end{cases}$$

Here we define the **inner product** for complex functions as

$$\langle f, g\rangle = \int_{\alpha}^{\beta} f^*(x)g(x)dx$$

where $f^*(x)$ is the complex conjugate of $f(x)$. Then:

$$\begin{aligned} \langle \exp\left(\frac{i\boldsymbol{m}\pi}{L}x\right), \exp\left(\frac{i\boldsymbol{n}\pi}{L}x\right)\rangle &= \int_{-L}^{L} \exp\left(-\frac{i\boldsymbol{m}\pi}{L}x\right)\exp\left(\frac{i\boldsymbol{n}\pi}{L}x\right)dx \\ &= 2L\delta_{mn} \end{aligned}$$

and since $f(x) = f_{FS}(x)$ we obtain our formula.

2.3 Parseval's theorem

Parseval's theorem states that given a periodic $f(x)$ with convergent Fourier series we have that

$$\begin{aligned} \langle f, f\rangle &= \int_{-L}^{L} |f(x)|^2 dx \\ &= 2L\sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= L\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)\right] \end{aligned}$$

and is derived by orthogonality.

3 PDEs

3.1 Separation of variables

The only methodology considered is separation of variables. So for PDE:

$$\hat{D}[u(x_1, \ldots, x_n)] = 0$$

where $\hat{D}$ is our differential operator, we look for solutions of form:

$$u(x_1, \ldots, x_n) = X_1(x_1) \cdots X_n(x_n)$$

subject to **initial** and **boundary** conditions.

3.2 Heat equation

The heat equation is an equation of the following form:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

where α^2 is the thermal diffusivity constant.

3.2.1 Standard boundary conditions

We firstly define:

- **Initial condition**: $u(x, 0) = f(x)$ for $0 \leq x \leq L$
- **Boundary condition**: $u(0, t) = u(L, t) = 0$ for $\forall t > 0$

Let solutions be of form:

$$u(x, t) = X(x) \cdot T(t)$$
$$\therefore X(x) \cdot \dot{T}(t) = \alpha^2 X''(x) \cdot T(t)$$

Only a constant function may satisfy the first equality:

$$\frac{1}{\alpha^2} \frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda.$$

Writing this as two ODEs:

$$\dot{T} + \alpha^2 \lambda T = 0$$
$$X'' + \lambda X = 0.$$

The first one we can directly integrate, yielding:

$$T(t) = a_1 \exp\left(-\alpha^2 \lambda t\right).$$

The second ODE is a spring system, hence it has solution of form:

$$X(x) = b_1 \cos \lambda^{1/2} x + b_2 \sin \lambda^{1/2} x$$

where $\lambda > 0$ and all other cases yields at best constant solutions.

However this time before proceeding we need to consider boundary conditions:

$$X(0) = X(L) = 0.$$

We find $X(0) = b_1 = 0$ and $X(L) = b_2 \sin \lambda^{1/2} L = 0$.

The second equation implies that λ must of the following form:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad \forall n \in \mathbb{N}$$

and so

$$X'' + \lambda X = 0 \implies X_n = b_2 \sin \lambda_n^{1/2} x.$$

Since λ is discretised:

$$\therefore T_n = a_1 \exp\left(-\alpha^2 \lambda_n t\right).$$

Our general solution must then be:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\alpha^2 \lambda_n t\right) \sin \lambda_n^{1/2} x$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Using initial condition $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x$$

and we recognise this as an odd Fourier series with period $2L$.

$$\therefore \int_{-L}^{L} \sin\left(\lambda_n^{1/2} x\right) f(x) dx = \sum_{n=1}^{\infty} c_n \int_{-L}^{L} \left(\sin\left(\lambda_n^{1/2} x\right)\right)^2 \mathrm{d}x$$

$$\therefore 2 \int_0^L \sin\left(\lambda_n^{1/2} x\right) f(x) \mathrm{d}x = c_n L$$

We can do this as the product of two odd functions is even.

$$\therefore c_n = \frac{2}{L} \int_0^L \sin\left(\lambda_n^{1/2} x\right) f(x) \mathrm{d}x$$

This is fine because we can extend $u(x,t)$ via reflection for negative x.

3.2.2 Fixed boundary temperatures

We reconsider the heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

but now with the following **non-homogeneous boundary conditions**:

- $u(0,t) = T_1$
- $u(L,t) = T_2$
- $u(x,0) = f(x)$

Physically our rod has fixed boundary temperatures, namely T_1 and T_2.

We approach this problem with a change of variables:

$$v(x) = \lim_{t\to\infty} u(x,t).$$

Using our boundary conditions v must be linear:

$$\therefore v(x) = \frac{T_2 - T_1}{L} x + T_1$$

since $v'' = 0$, $v(0) = T_1$ and $v(L) = T_2$. We then deduce that:

$$u(x,t) = v(x) + \omega(x,t)$$

for $\omega(x,t)$ satisfies the same heat equation with initial conditions:

- $\omega(0,t) = \omega(L,t) = 0$
- $\omega(x,0) = f(x) - v(x)$

Recognising this as our initial example:

$$\omega(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\alpha^2 \lambda_n t\right) \sin \lambda_n^{1/2} x$$

where again $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and because $\omega(x,t)$ is a Fourier series with period $2L$:

$$c_n = \frac{2}{L} \int_0^L \sin\Big(\lambda_n^{1/2} x\Big) \Big(f(x) - v(x)\Big) \mathrm{d}x.$$

3.2.3 Insulated rod ends

For the final example we consider:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

and define the following conditions:

- $\frac{\partial}{\partial x} u(0,t) = \frac{\partial}{\partial x} u(L,t) = 0$
- $u(x,0) = f(x)$.

Let $u(x,t) = X(x)T(t)$ and we get:

$$X'' + \lambda X = 0$$

where $X'(0) = X'(L) = 0$ and

$$\dot{T} + \alpha^2 \lambda T = 0.$$

If $\lambda \leq 0$ we get at best constant solutions. So let $\lambda > 0$ and we get solutions:

$$X(x) = c_1 \cos \lambda^{1/2} x + c_2 \sin \lambda^{1/2} x.$$

After substituting for boundary conditions we have that $c_2 = 0$ and:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

for $n \in \mathbb{N}$. Solving the time component ODE yields:

$$T_n(t) = k \exp\left(-\alpha^2 \lambda_n t\right)$$

and hence we have general solutions of form:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\alpha^2 \lambda_n t\right) \cos \lambda_n^{1/2} x.$$

Finally using our initial conditions:

$$c_n = \frac{2}{L} \int_0^L \cos\left(\lambda_n^{1/2} x\right) f(x) \mathrm{d}x$$

where we define an even extension of $u(x,t)$ in negative x to obtain this formula.

3.3 Wave equation

3.3.1 Derivation

Consider string with length x and time dependent vertical displacement $u(x,t)$. Let $\theta(x,t)$ denote the angle formed by the string. By Newton's second law:

$$\begin{aligned}\rho\Delta x\frac{\partial^2 u}{\partial t^2} &= T\Big[\sin\theta(x+\Delta x,t)-\sin\theta(x,t)\Big]\\ &\approx T\Big[\theta(x+\Delta x,t)-\theta(x,t)\Big]\\ &= T\frac{\partial\theta}{\partial x}\Delta x+\mathcal{O}(\Delta x^2)\\ &\approx T\frac{\partial^2 u}{\partial x^2}\Delta x\end{aligned}$$

where ρ is mass per unit length and T our string tension. If $\Delta x\to 0$ then:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

for $c^2=T/\rho$ and is our wave speed.

3.3.2 Plucked string

So consider the wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

for $t\geq 0$ and $x\in[0,L]$ subject to the following conditions:

- $u(0,t)=u(L,t)=0$
- $\dfrac{\partial}{\partial t}u(x,0)=0$
- $u(x,0)=f(x)$.

Let $u(x,t)=X(x)T(t)$ and substituting gives:

$$\frac{X''}{X}=\frac{\ddot{T}}{c^2T}=-\lambda$$

where this is two ODEs:

$$X''+\lambda X=0 \text{ with } X(0)=X(L)=0$$

and

$$\ddot{T}+c^2\lambda T=0 \text{ with } \dot{T}(0)=0.$$

The first spatial ODE we have solved before:

$$X_n(x)=b_2\sin\lambda_n^{1/2}x \text{ with } \lambda_n=\frac{n^2\pi^2}{L^2}.$$

The time component ODE is yet another SHM system:

$$T_n(t) = a_n \cos c\lambda_n^{1/2} t + b_n \sin c\lambda_n^{1/2} t$$

and hence by initial conditions we obtain:

$$T_n(t) = a_n \cos c\lambda_n^{1/2} t.$$

Combining our results the general solution takes the following form:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x \cos c\lambda_n^{1/2} t$$

where $\lambda_n^{1/2} = \dfrac{n\pi}{L}$. Since the initial string shape is $u(x,0) = f(x)$:

$$c_n = \frac{2}{L} \int_0^L \sin\left(\lambda_n^{1/2} x\right) f(x) \mathrm{d}x.$$

Our displacement $u(x,t)$ may also be written as a sum of travelling waves:

$$\begin{aligned} u(x,t) &= \frac{1}{2} \sum_{n=1}^{\infty} c_n \left[\sin \lambda_n^{1/2}(x - at) + \sin \lambda_n^{1/2}(x + at)\right] \\ &= \frac{1}{2}\big[\underbrace{h(x - at)}_{+x \text{ direction}} + \underbrace{h(x - at)}_{-x \text{ direction}}\big]. \end{aligned}$$

3.3.3 d'Alembert's solution

Consider the 1D wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with change of variables $\xi = x - ct$ and $\eta = x + ct$. Using the chain rule:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$u(\xi, \eta) = p(\eta) + q(\xi)$$

or that we have solutions:

$$u(x,t) = p(x + ct) + q(x - ct)$$

where functions p and q represent travelling waves in $\pm x$ directions.

3.3.4 General initial conditions

Again consider:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

but now with the following conditions:

- $u(0,t) = u(L,t) = 0$
- $\dfrac{\partial}{\partial t} u(x,0) = g(x)$
- $u(x,0) = f(x)$.

Using $u(x,t) = X(x)T(t)$ we get two ODEs:

$$X'' + \lambda X = 0 \text{ with } X(0) = X(L) = 0$$

$$\ddot{T} + c^2 \lambda T = 0.$$

We have solved the spatial component ODE:

$$X_n(x) = b_2 \sin \lambda_n^{1/2} x \text{ with } \lambda_n = \frac{n^2\pi^2}{L^2}.$$

However now the time component ODE yields:

$$T_n(t) = a_n \cos c\lambda_n^{1/2} t + b_n \sin c\lambda_n^{1/2} t$$

which gives us the following general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin \lambda_n^{1/2} x \Big(a_n \cos c\lambda_n^{1/2} t + b_n \sin c\lambda_n^{1/2} t \Big).$$

To find our coefficients we use our initial conditions:

$$a_n = \frac{2}{L} \int_0^L \sin\Big(\lambda_n^{1/2} x\Big) f(x) \mathrm{d}x$$

and

$$b_n = \frac{1}{c\lambda_n^{1/2}} \frac{2}{L} \int_0^L \sin\Big(\lambda_n^{1/2} x\Big) g(x) \mathrm{d}x.$$

3.4 Laplace's equation

Laplace's equation takes the form $\boldsymbol{\nabla}^2 u = 0$. In two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and we only consider boundary conditions. (Dirichlet conditions)

3.4.1 Rectangular boundary conditions

We open with the following example:

- **Boundary for y**: $u(x,0) = u(x,b) = 0$
- **Boundary for x**: $u(0,y) = 0$ and $u(a,y) = f(y)$

where $x \in [0,a]$ and $y \in [0,b]$. Begin by separation of variables:

$$u(x,y) = X(x) \cdot Y(y)$$

$$\therefore \frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

Recognising the previous statement as two ODEs:

$$X'' - \lambda X = 0 \text{ for } X(0) = 0$$

$$Y'' + \lambda Y = 0 \text{ for } Y(0) = Y(b) = 0$$

The second ODE we have already solved in the heat equation. It has solution:

$$Y_n = a_1 \sin\left(\lambda_n^{1/2} y\right) \text{ for } \lambda_n = \frac{n^2\pi^2}{b^2}.$$

The first ODE has solutions of form:

$$X_n = a_2 \cosh\left(\lambda_n^{1/2} x\right) + a_3 \sinh\left(\lambda_n^{1/2} x\right)$$

where these are the hyperbolic functions:

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}).$$

Using our boundary condition $X(0) = 0$ gives:

$$X_n = a_3 \sinh\left(\lambda_n^{1/2} x\right).$$

Now putting all of this together we get:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\lambda_n^{1/2} x\right) \sin\left(\lambda_n^{1/2} y\right)$$

To find coefficients c_n we use $u(a, y) = f(y)$.

$$\therefore f(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\lambda_n^{1/2} a\right) \sin\left(\lambda_n^{1/2} y\right)$$

Since we have a Fourier series with period $2b$:

$$\begin{aligned}\int_{-b}^{b} \sin\left(\lambda_n^{1/2} y\right) f(y) dy &= \sum_{n=1}^{\infty} c_n \sinh\left(\lambda_n^{1/2} a\right) \int_{-b}^{b} \sin\left(\lambda_n^{1/2} y\right) dy \\ &= c_n \sinh\left(\lambda_n^{1/2} a\right) \cdot b\end{aligned}$$

We can split the first integral to give us:

$$c_n = \frac{2}{b \sinh\left(\lambda_n^{1/2} a\right)} \int_{0}^{b} \sin\left(\lambda_n^{1/2} y\right) f(y) dy$$

where $\lambda_n = \left(\frac{n\pi}{b}\right)^2$ and our solution is complete.

3.4.2 Circular boundary conditions

Now we solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

but with a circular boundary. In polar coordinates (r, θ):

- $u(a, \theta) = f(\theta)$
- $u(r, \theta)$ is bounded

where $r \in [0, a]$ and $\theta \in [0, 2\pi]$. Since $u = u(x, y)$:

$$\begin{aligned} u'_\theta &= u'_x x'_\theta + u'_y y'_\theta \\ u''_{\theta\theta} &= (u''_{xx} x'_\theta + u''_{xy} y'_\theta) x'_\theta + u'_x x''_{\theta\theta} + (u''_{yy} y'_\theta + u''_{xy} x'_\theta) x'_\theta + u'_y y''_{\theta\theta} \\ u'_r &= u'_x x'_r + u'_y y'_r \\ u''_{rr} &= (u''_{xx} x'_r + u''_{xy} y'_r) x'_r + u'_x x''_{rr} + (u''_{xy} x'_r + u''_{yy} y'_r) y'_r + u'_y y''_{rr} \end{aligned}$$

and here we have used the chain rule.

Applying these derivatives we obtain the following equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0.$$

Using separation of variables:

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\therefore r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \lambda$$

where λ is our separation constant.

$$\therefore \ddot{\Theta} + \lambda\Theta = 0$$

$$\therefore r^2R'' + rR' = \lambda R$$

For the first ODE if $\lambda \leq 0$ then we get at best constant solutions. If $\lambda > 0$:

$$\Theta(\theta) = a_1\cos\lambda^{1/2}\theta + a_2\sin\lambda^{1/2}\theta$$

and since periodicity must be preserved:

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\therefore \lambda_n^{1/2} = n \text{ or } \lambda_n^{1/2} = 0$$

where $n \in \mathbb{N}$. So when $\lambda_n = 0$:

$$r^2R'' + rR' = 0$$

and since $u(r,\theta)$ is bounded we get only constant solutions. If $\lambda_n = n^2$ then:

$$r^2R'' + rR' - n^2R = 0$$

with solutions of form $R(r) = r^\alpha$ which yields $R_n(r) = c_nr^n$. Then:

$$u(r,\theta) = \frac{p_0}{2} + \sum_{n=1}^{\infty} r^n\Big(q_n\cos\lambda_n^{1/2}\theta + r_n\sin\lambda_n^{1/2}\theta\Big)$$

$$p_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta$$

$$q_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi}\cos\Big(\lambda_n^{1/2}\theta\Big)f(\theta)\mathrm{d}\theta$$

$$r_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi}\sin\Big(\lambda_n^{1/2}\theta\Big)f(\theta)\mathrm{d}\theta$$

since this is a Fourier series with period 2π.

4 Sturm-Liouville theory

4.1 Regular S-L problems

Sturm-Liouville theory is a general theory for 2nd order ODEs.

Consider the following eigenvalue ODE:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+q(x)y=\lambda r(x)y$$

where $r(x)$ is our weight function. We define the following boundary conditions:

1. $a_1y(0)+a_2y'(0)=0$
2. $b_1y(1)+b_2y'(1)=0$.

This is a **regular Sturm-Liouville** problem, where $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous functions and $p(x)$, $r(x)$ are strictly positive functions for $\forall x\in[0,1]$.

Eigenvalues λ_n yield **eigenfunctions** $\phi_n(x)$ which are nontrivial solutions to our S-L problem. Important consequences include:

- Eigenvalues λ_n of a S-L problem are **real**.

  Furthermore each eigenvalue corresponds to one eigenfunction.

- Eigenfunctions $\phi_n(x)$ are orthogonal:

  $$\langle\phi_m,\phi_n\rangle=\int_0^1 r(x)\phi_m(x)\phi_n(x)\mathrm{d}x=\delta_{mn}$$

  in Hilbert space $L^2([0,1],r(x)\mathrm{d}x)$.

Note that our eigenfunctions are orthonormal and so we define:

$$\phi_n(x)=k_ny_n(x)$$

where k_n is our scale factor. Since $\langle\phi_n,\phi_n\rangle=1$:

$$\therefore\int_0^1 r(x)k_n^2y_n^2(x)\mathrm{d}x=1$$

and so we have that:

$$\begin{aligned}k_n&=\frac{1}{\sqrt{\langle y_n,y_n\rangle}}\\&=\left(\int_0^1 r(x)y_n^2(x)\mathrm{d}x\right)^{-1/2}.\end{aligned}$$

4.1.1 General 2nd order ODEs

Consider the following general 2nd order eigenvalue ODE:

$$-P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \omega(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = \lambda r(x)y.$$

Multiply this by the following integrating factor:

$$F(x) = \exp\Big[\int_0^x \frac{\omega(s) - p'(s)}{p(s)}\mathrm{d}s\Big]$$

yields an ODE of S-L form:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big[F(x)P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\Big] + F(x)q(x)y = \lambda F(x)r(x)y.$$

4.1.2 Lagrange's identity

Our previous definition is motivated by the **Lagrange's identity**:

$$\begin{aligned}\langle \mathcal{L}[u], v\rangle - \langle u, \mathcal{L}[v]\rangle &= -\Big[p\Big(u'v^* - u(v^*)'\Big)\Big]_0^1 \\ &= -\Big[p(x)(\frac{\mathrm{d}u}{\mathrm{d}x}\cdot v^* - u\cdot\frac{\mathrm{d}v^*}{\mathrm{d}x})\Big]_0^1\end{aligned}$$

where $u = u(x)$, $v = v(x)$ are complex functions and

$$\mathcal{L}[u] = -\frac{\mathrm{d}}{\mathrm{d}x}\Big[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\Big] + q(x)u.$$

Here the inner product is defined as

$$\langle u, v\rangle = \int_0^1 uv^*\mathrm{d}x$$

and we have integrated by parts using the following identities:

$$\big[pu'v^*\big]' = \big(pu'\big)'v^* + pu'(v^*)'$$

$$\big[pu(v^*)'\big]' = \Big(p(v^*)'\Big)'u + pu'(v^*)'.$$

For a S-L problem its properties follow from:

$$\langle \mathcal{L}[u], v\rangle = \langle u, \mathcal{L}[v]\rangle$$

where functions u and v satisfy its boundary conditions.

Note: self-adjoint operators

4.1.3 Series expansion

Now the set of orthonormal eigenfunctions $\{\phi_n(x)\}$ from a S-L problem with boundary conditions may be used to expand function $f(x)$:

$$f_\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

for $\forall x \in [0,1]$. Integrating this on both sides:

$$\begin{aligned}\int_0^1 r(x)\phi_m(x)f(x)\mathrm{d}x &= \int_0^1 r(x)\phi_m(x)\sum_{n=1}^{\infty} c_n\phi_n(x)\mathrm{d}x \\ &= \sum_{n=1}^{\infty} c_n \int_0^1 r(x)\phi_m(x)\phi_n(x)\mathrm{d}x \\ &= \sum_{n=1}^{\infty} c_n \delta_{mn} \\ &= c_m\end{aligned}$$

and so we get a general formula for our coefficients:

$$c_n = \int_0^1 r(x)\phi_n(x)f(x)\mathrm{d}x.$$

If $f(x)$ and $f'(x)$ are piecewise continuous on $x \in [0,1]$ then:

$$\forall x \in (0,1); \sum_{n=1}^{\infty} c_n\phi_n(x) = \frac{f(x^-)+f(x^+)}{2}$$

where these are the left and right handed limits.

4.1.4 General Parseval's identity for S-L problems

We have that:

$$\int_0^1 r(x)\big[f(x)\big]^2\mathrm{d}x = \sum_{n=1}^{\infty} c_n^2$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

and

$$c_n = \int_0^1 r(x)\phi_n(x)f(x)\mathrm{d}x.$$

4.2 Non-homogeneous S-L problems

Consider:

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

where $f(x)$ is our non-homogeneous term, and that we define

$$\mathcal{L}[y] = -\frac{\mathrm{d}}{\mathrm{d}x}\left[P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y.$$

Assume that this ODE satisfies regular S-L boundary conditions.

Firstly we solve the corresponding homogeneous S-L problem:

$$\mathcal{L}[y] = \lambda r(x)y$$

for non-trivial λ_n and $\phi_n(x)$.

Now let the general solution to our non-homogeneous S-L problem be:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

and substituting this yields:

$$\begin{aligned}\sum_{n=1}^{\infty} b_n \mathcal{L}[\phi_n(x)] &= r(x)\sum_{n=1}^{\infty} b_n \lambda_n \phi_n(x) \\ &= \mu r(x)\sum_{n=1}^{\infty} b_n \phi_n(x) + f(x).\end{aligned}$$

Rearranging then gives:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} b_n(\lambda_n - \mu)\phi_n(x).$$

Now we can also expand $\dfrac{f(x)}{r(x)}$ in terms of our eigenfunctions:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

with

$$\begin{aligned}c_n &= \int_0^1 r(x)\phi_n(x)\frac{f(x)}{r(x)}\mathrm{d}x \\ &= \int_0^1 \phi_n(x)f(x)\mathrm{d}x\end{aligned}$$

and so equating yields the following relation

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

4.3 Non-homogeneous PDEs

Consider the following non-homogeneous PDE:

$$r(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big[p(x)\frac{\partial u}{\partial x}\Big] - q(x)u(x,t) + F(x,t)$$

with boundary and initial conditions:

- $\frac{\partial}{\partial x}u(0,t) - h_1 u(0,t) = 0$
- $\frac{\partial}{\partial x}u(1,t) - h_2 u(1,t) = 0$
- $u(x,0) = f(x)$.

Firstly we solve the homogenous case via separation of variables. Let:

$$u(x,t) = X(x)T(t)$$

and after some algebra:

$$\frac{1}{rX}\Big[p'X' + pX'' - qX\Big] = \frac{\dot{T}}{T} = -\lambda.$$

This yields two ODEs:

$$\dot{T} + \lambda T = 0$$

$$-\big[pX'\big]' + qX = \lambda r X$$

with boundary conditions:

$$X'(0) - h_1 X(0) = 0$$

$$X'(1) - h_2 X(1) = 0$$

Clearly our second ODE is of S-L form. We are now going to assume that this regular S-L problem has non-trivial λ_n and orthonormal eigenfunctions $\phi_n(x)$.

Let the general solution to our PDE be:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x).$$

Substituting this into our PDE yields:

$$r(x)\sum_{n=1}^{\infty}\dot{b_n}(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Big[\Big(p(x)\phi_n'(x)\Big)' - q(x)\phi_n(x)\Big] + F(x,t).$$

Now since we have a S-L problem:

$$\Big(p(x)\phi_n'(x)\Big)' - q(x)\phi_n(x) = -\lambda_n\phi_n(x)r(x)$$

and after dividing through our PDE by $r(x)$ we get:

$$\sum_{n=1}^{\infty}\dot{b_n}(t)\phi_n(x) = \sum_{n=1}^{\infty}b_n(t)\Big[-\lambda_n\phi_n(x)\Big] + \frac{F(x,t)}{r(x)}.$$

But we can also expand our non-homogeneous term as a sum:

$$\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)$$

and its coefficients are:

$$\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)\mathrm{d}x$$

in $L^2([0,1], r(x))$. Therefore after some algebra:

$$\sum_{n=1}^{\infty}\Big[\dot{b_n}(t) + \lambda_n b_n(t) - \gamma_n(t)\Big]\phi_n(x) = 0$$

and yields the following ODE:

$$\dot{b_n}(t) + \lambda_n b_n(t) - \gamma_n(t) = 0.$$

Solution is via integrating factors:

$$b_n(t) = e^{-\lambda_n t}\int_0^t \gamma_n(s)e^{\lambda_n s}\mathrm{d}s + e^{-\lambda_n t}b_n(0).$$

Finally using $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty}b_n(0)\phi_n(x) = f(x)$$

and we get

$$b_n(0) = \int_0^1 r(x)f(x)\phi_n(x)\mathrm{d}x.$$

4.4 Singular S-L problems

Consider the following ODE:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \lambda r(x)y$$

but now $p(x)$, $q(x)$ and $r(x)$ are discontinuous at $x = 0$ and/or $x = 1$.
This is a **singular** S-L problem with boundary condition:

$$b_1 y(1) + b_2 y'(1) = 0$$

or

$$a_1 y(0) + a_2 y'(0) = 0.$$

Now singular S-L problems at $x = 0$ may be self-adjoint or that they yield:

- $\lambda_n \in \mathbb{R}$ (Real eigenvalues)
- $\langle \phi_m, \phi_n \rangle = \delta_{mn}$ (Orthogonal eigenfunctions)

if they satisfy Lagrange's identity. Consider singular S-L problem at $x = 0$:

$$\begin{aligned}\int_\epsilon^1 \Big(\mathcal{L}[u]v - u\mathcal{L}[v]\Big)\mathrm{d}x &= \Big[-p(x)\Big(u'(x)v(x) - u(x)v'(x)\Big)\Big]_\epsilon^1 \\ &= p(\epsilon)\Big(u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\Big)\end{aligned}$$

and tends to zero if and only if:

$$\lim_{\epsilon\to\infty}\Big[p(\epsilon)\Big(u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\Big)\Big] = 0,$$

where we define:

$$\mathcal{L}[u] = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u.$$

Therefore this is the condition for our singular S-L problem to have real eigenvalues and orthogonal eigenfunctions.

Similarly for singular S-L problem at $x = 1$ it is self-adjoint if:

$$\lim_{\epsilon\to\infty}\Big[p(1-\epsilon)\Big(u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon)\Big)\Big] = 0.$$

Finally whilst eigenvalues may be real they are not necessarily discrete!

5 Laplace transforms

So let $f(t)$ be defined for $t \in [0, \infty)$. Its Laplace transform is:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

Let **functions of exponential order** be defined as:

$$\forall t \in [0, \infty); \exists A > 0 \text{ and } B \in \mathbb{R} : |f(t)| \leq Ae^{Bt}$$

where f is piecewise continous and we denote such functions as $f \in E$.

For sufficiently large s, the Laplace transform $f \in E$ converges.

5.1 Properties

5.1.1 Inversion formula

Now let $F(s) = \mathcal{L}[f(t)]$. We have the following inversion formula:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) \mathrm{d}s. \end{aligned}$$

5.1.2 Reduction of order

Applying the Laplace transform to derivatives:

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

for $\forall f, f' \in E$ and generalising this via induction gives:

$$\begin{aligned} \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f^{(0)}(0) - s^{(n-2)} f^{(1)}(0) \\ - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0). \end{aligned}$$

5.1.3 Shifts, scaling and derivatives

Let $F(s) = \mathcal{L}[f(t)](s)$. We then have that:

1. **s-shift**:
$$\mathcal{L}[e^{-ct} f(t)](s) = F(s + c)$$
where $s + c > \gamma$.
2. **t-shift**:
Let $c \geq 0$ and $f(t) = 0$ if $t < 0$. Then:
$$\mathcal{L}[f(t - c)](s) = e^{-sc} F(s).$$
Furthermore utilising the unit step function:
$$\mathcal{L}[g(t - c) u_c(t)](s) = e^{-sc} G(s)$$
where $G(s) = \mathcal{L}[g(t)](s)$ and $g(t)$ any normal function.

3. **s-derivative**:
$$\mathcal{L}[tf(t)](s) = -\frac{\mathrm{d}}{\mathrm{d}s}F(s).$$
We can also extend this to the nth derivative:
$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s).$$

4. **scaling**:
$$\mathcal{L}[f(ct)] = \frac{1}{c}F(\frac{s}{c})$$
and
$$\frac{1}{c}\mathcal{L}[f(\frac{t}{c})] = F(cs)$$
where $c > 0$.

5.2 Applications

5.2.1 Higher order ODEs

So consider the following nth order ODE:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = f(t)$$

for $f \in E$. After taking the Laplace transform of both sides we get:

$$Z(s)\mathcal{L}[y(t)] = \mathcal{L}[f(t)] + Z_0(s)$$

where $Z(s)$ is a degree n polynomial and $Z_0(s)$ a degree $n-1$ polynomial dependent on our initial conditions.

Now if our source term is of the following form:

$$f(t) = t^n e^{at}\big(A\cos bt + B\sin bt\big)$$

then $\mathcal{L}[f(t)]$ is rational and therefore:

$$\mathcal{L}[y(t)] = \frac{\mathcal{L}[f(t)]}{Z(s)} + \frac{Z_0(s)}{Z(s)}$$

where we can solve this via standard transforms.

5.2.2 Discontinuous source terms

Since the Laplace transform is convergent for all piecewise continuous functions, we can use it to solve ODEs with discontinous source terms:

$$Ay''(t) + By'(t) + Cy(t) = g(t)$$

where $g(t)$ is piecewise continuous and of the following form:

$$g(t) = f(t)[u_a(t) - u_b(t)] = \begin{cases} f(t) & t \in [a, b) \\ 0 & t \notin [a, b) \end{cases}$$

for $b > a$. This is the **unit step function**:

$$u_c(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq c \end{cases}$$

and is also known as the Heaviside function. Its Laplace transform is:

$$\mathcal{L}[u_c(t)](s) = \frac{e^{-sc}}{s}.$$

Furthermore we can define a shift of $f(t)$ by $c > 0$ to the right by:

$$f(t-c)u_c(t)$$

$$\mathcal{L}[f(t-c)u_c(t)] = e^{-sc}F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

5.2.3 Impulse functions

The **Dirac delta** is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

and physically a "strike" on a system in infinitely short time. We have that:

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) \mathrm{d}t = f(t_0)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[u_{t_0}(t)\Big] = \delta(t - t_0)$$

and

$$\mathcal{L}[\delta(t - t_0)] = e^{-st_0}.$$

If $t_0 = 0$ then $\mathcal{L}[\delta(t)] = \lim_{t_0 \to 0}\left(e^{-st_0}\right) = 1$.

Taking the Laplace transform of the following ODE

$$y''(t) + y(t) = \delta(t)$$

with initial conditions $y(0) = y'(0) = 0$ gives:

$$\mathcal{L}[y(t)] = \frac{1}{s^2+1} \lim_{t_0 \to 0} \left(e^{-st_0}\right).$$

It is important that we do not evaluate the limit here!

Finally by inspection we have that:

$$\begin{aligned} y(t) &= \lim_{t_0 \to 0} \Big(\sin(t - t_0) u_{t_0}(t) \Big) \\ &= \sin(t) u_0(t). \end{aligned}$$

5.2.4 Convolutions

Convolutions can also be helpful in solving ODEs. They are defined:

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(s) g(t-s) \mathrm{d}s \\ &= \int_0^t g(s) f(t-s) \mathrm{d}s \end{aligned}$$

for functions $f, g : [0, \infty) \to \mathbb{R}$ and has the following properties:

- $f * (g + h) = f * g + f * h$
- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * 1 \neq f$
- $f * f \neq f^2$.

Importantly we have the **convolution theorem**:

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

where if $f, g \in E$ then $f * g \in E$, or that they are functions of exponential order so that our Laplace transforms converge. Note that:

$$\int_0^t f(s) \mathrm{d}s = \int_0^t f(s) u_0(t-s) \mathrm{d}s$$

and therefore

$$\mathcal{L}\Big[\int_0^t f(s) \mathrm{d}s\Big] = \frac{\mathcal{L}[f(t)]}{s}.$$

Now consider the following ODE:

$$ay''(t) + by'(t) + cy(t) = g(t)$$

with initial conditions $y(0) = \alpha$ and $y'(0) = \beta$. Here $a, b, c, \alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} \therefore \mathcal{L}[y(t)] &= \Phi(s) + \Psi(s) \\ &= \frac{(as+b)\alpha + a\beta}{as^2+bs+c} + \frac{1}{as^2+bs+c}\mathcal{L}[g(t)] \end{aligned}$$

$$\begin{aligned} \therefore y(t) &= \mathcal{L}^{-1}[\Phi(s)] + \mathcal{L}^{-1}[\Psi(s)] \\ &= \phi(t) + \psi(t) \end{aligned}$$

The first expression $\phi(t)$ can be found via standard transforms. For the second expression we define the **transfer function**[1]:

$$H(s) = \frac{1}{as^2+bs+c}$$

where it is the Laplace transform of the following corresponding ODE:

$$ah''(t) + bh'(t) + ch(t) = \delta(t)$$

with initial conditions $h(0) = h'(0) = 0$. $\therefore H(s) = \mathcal{L}[h(t)]$

This is helpful since:

$$\begin{aligned} \Psi(s) &= \frac{1}{as^2+bs+c}\mathcal{L}[g(t)] \\ &= H(s)\mathcal{L}[g(t)] \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}[\psi(t)] &= \mathcal{L}[h(t)]\mathcal{L}[g(t)] \\ &= \mathcal{L}[h(t) * g(t)]. \end{aligned}$$

Applying the convolution theorem we get:

$$\begin{aligned} \psi(t) &= h(t) * g(t) \\ &= \int_0^t h(s)g(t-s)\mathrm{d}s. \end{aligned}$$

[1] Also known as a Green's function.

5.3 Standard transforms

- $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ where $n \in \mathbb{N}$ and $s > 0$.
- $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$ where $s > a$.
- $\mathcal{L}[t^n e^{at}] = \dfrac{n!}{(s-a)^{n+1}}$ where $n \in \mathbb{N}$ and $s > a$.
- $\mathcal{L}[\sin at] = \dfrac{a}{s^2+a^2}$,

  $\mathcal{L}[\cos at] = \dfrac{s}{s^2+a^2}$ where $s > 0$.
- $\mathcal{L}[\sinh at] = \dfrac{a}{s^2-a^2}$,

  $\mathcal{L}[\cosh at] = \dfrac{s}{s^2-a^2}$ where $s > |a|$.
- $\mathcal{L}[e^{at}\sin bt] = \dfrac{b}{(s-a)^2+b^2}$,

  $\mathcal{L}[e^{at}\cos bt] = \dfrac{s-a}{(s-a)^2+b^2}$ where $s > a$.
- $\mathcal{L}[u_c(t)] = \dfrac{e^{-cs}}{s}$ where $s > 0$.
- $\mathcal{L}[\delta(t-c)] = e^{-cs}$.