1. 1
2. 2
3. 3
4. 4
5. 5
6. 6

7. Define $L(x) = \displaystyle\int_1^x \frac{\mathrm{d}t}{t}$ for $\forall x > 0$. Show:

   - $L(xy) = L(x) + L(y)$
   - $L'(x) = \dfrac{1}{x}$
   - $L_{inv}(x) = E(x)$, where we define $E(x) = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

   For the first part we want to show:

$$\int_1^{yx} \frac{\mathrm{d}t}{t} = \int_1^x \frac{\mathrm{d}t}{t} + \int_1^y \frac{\mathrm{d}t}{t}.$$

   Beginning from the left hand side let $t = x\alpha$.

$$\therefore \int_{t=1}^{t=xy} \implies \int_{\alpha=\frac{1}{x}}^{\alpha=y}$$

$$\therefore \mathrm{d}t = x\mathrm{d}\alpha$$

$$\therefore \frac{1}{t} = \frac{1}{x\alpha}$$

   Now splitting this integral via T4.9 gives:

$$\begin{aligned}\int_{t=1}^{t=yx} \frac{\mathrm{d}t}{t} &= \int_{\alpha=\frac{1}{x}}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} \\ &= \int_{\alpha=1}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} + \int_{\alpha=\frac{1}{x}}^{\alpha=1} \frac{\mathrm{d}\alpha}{\alpha} \\ &= \int_{\alpha=1}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} + \int_{\beta=1}^{\beta=x} \frac{\mathrm{d}\beta}{\beta}\end{aligned}$$

   where we set $\alpha = \frac{1}{x}\beta$ in the second integral.

$$\therefore L(xy) = L(x) + L(y)$$

   Using the fundamental theorem of calculus:

$$L(x) = \int_1^x \frac{\mathrm{d}t}{t} \implies \frac{\mathrm{d}}{\mathrm{d}x} L(x) = \frac{1}{x}$$

   since $\forall t > 0$, $\frac{1}{t}$ is continuous.

For the final part let's first define our functions:

$$E : \mathbb{R} \to \mathbb{R}$$

$$L : \mathbb{R}^+ \to \mathbb{R}$$

where $\mathbb{R}^+ = \mathbb{R}\backslash\{0, \dots\}$ represents the positive reals. Then define:

$$E(x) = z$$

for $x, z \in \mathbb{R}$ and:

$$L(y) = x$$

for $y \in \mathbb{R}^+$.

For these two functions to be inverses of each other we must show that:

$$E(L(y)) = y$$

and

$$L(E(x)) = x.$$

Consider

$$\frac{\mathrm{d}}{\mathrm{d}y} E(L(y)) = E(L(y))\frac{1}{y}.$$

Rearranging this and taking integrals:

$$\int_1^{E(L(y))} \frac{1}{E(L(y))} \mathrm{d}E(L(y)) = \int_1^y \frac{1}{y} \mathrm{d}y.$$

This gives:

$$\Big[L(E(L(y)))\Big]_{E(L(y))=1}^{E(L(y))=E(L(y))} = [L(y)]_1^y$$

or that:

$$L(E(L(y))) = L(y).$$

$$\therefore E(L(y)) = y$$

This is fine since $y \in \mathbb{R}^+ \subset \mathbb{R}$. Similarly consider the following:

$$\frac{\mathrm{d}}{\mathrm{d}x} L(E(x)) = \frac{1}{E(x)} E(x) = 1.$$

Here $L(E(x))$ is defined as $\forall x \in \mathbb{R}; E(x) > 0$.

Integrating our expression as an indefinite integral:

$$L(E(x)) = x + k$$

and we find that $k = 0$ by setting $x = 0$.

$$\therefore L(E(x)) = x$$

8. Let $g : [a, b] \to \mathbb{R}$ be continuous, and that $g \geq 0$ for $\forall x \in [a, b]$. Then let:

$$\int_a^b g(x)\mathrm{d}x = 0.$$

Show that $\forall x \in [a, b]$ we have $g(x) = 0$.

Firstly because $g \geq 0$ splitting the integral using T4.9:

$$\int_a^b g(x)\mathrm{d}x = \int_a^c g(x)\mathrm{d}x + \int_c^b g(x)\mathrm{d}x = 0$$

implies that $\forall c \in [a, b]$:

$$\int_a^c g(x)\mathrm{d}x = 0$$

as areas of positive functions are always positive.

Since $g(x)$ is continuous we can use the fundamental theorem of calculus. Let:

$$G(x) = \int_a^x g(t)\mathrm{d}t = 0$$

for $\forall x \in [a, b]$ as shown above. We then have that:

$$g(x) = \frac{\mathrm{d}}{\mathrm{d}x}G(x) = 0$$

for $\forall x \in [a, b]$.