

EM S1 Handins

Christopher Shen

Winter 2023

Contents

Handin 1	3
Handin 2	17

Handin 1

1. Consider set T_{ijk} with 3^3 elements, satisfying:

$$v_i = T_{ijk} R_{jk}$$

where v_i is a vector and R_{jk} a rank 2 tensor.

Show that T_{ijk} is a rank 3 tensor.

Proof. Direct proof.

Firstly define T'_{ijk} in frame S' and T_{ijk} in frame S , where our frames are related by $e'_i = \ell_{ij} e_j$. We then have that:

$$\begin{aligned} v'_i &= T'_{ijk} R'_{jk} \\ &= T'_{ijk} \ell_{jl} \ell_{km} R_{lm} \end{aligned}$$

and

$$\begin{aligned} v'_i &= \ell_{ij} v_j \\ &= \ell_{ij} T_{jkl} R_{kl}. \end{aligned}$$

$$\therefore T'_{ijk} \ell_{jl} \ell_{km} R_{lm} = \ell_{ij} T_{jkl} R_{kl}$$

Using the fact that R_{lm} is a tensor, we multiply both sides by vector a_m :

$$T'_{ijk} \ell_{jl} \ell_{km} R_{lm} a_m = \ell_{ij} T_{jkl} R_{kl} a_m.$$

The left hand side:

$$\begin{aligned} T'_{ijk} \ell_{jl} \ell_{km} R_{lm} a_m &= T'_{ijk} \ell_{jl} \ell_{km} a_l \\ &= T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} a_k. \end{aligned}$$

The right hand side:

$$\begin{aligned} \ell_{ij} T_{jkl} R_{kl} a_m &= \ell_{ij} T_{jkl} R_{kl} a_l \delta_{lm} \\ &= \ell_{ij} T_{jkl} \delta_{lm} a_k. \end{aligned}$$

Since equality still holds:

$$\begin{aligned} T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} a_k &= \ell_{ij} T_{jkl} \delta_{lm} a_k \\ \therefore (T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} - \ell_{ij} T_{jkl} \delta_{lm}) a_k &= 0 \end{aligned}$$

Because a_k is a vector that is not always zero:

$$\begin{aligned} T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} &= \ell_{ij} T_{jkl} \delta_{lm} \\ \therefore T'_{ijk} \ell_{jk} \ell_{km} &= \ell_{ij} T_{jkm} \end{aligned}$$

Then multiply both sides by ℓ_{in} :

$$T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = \ell_{ij}\ell_{in}T_{jkm}.$$

$$\therefore T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = \delta_{jn}T_{jkm}$$

$$\therefore T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = T_{nkm}$$

$$\therefore T'_{ijk}\ell_{in}\ell_{jk}\ell_{km} = T_{nkm}$$

And this is by definition a third rank tensor. □

2. Define

$$d\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial u_i} du_i$$

and the volume of the infinitesimal parallelepiped with $d\mathbf{r}_1$, $d\mathbf{r}_2$ and $d\mathbf{r}_3$:

$$dV = |d\mathbf{r}_1 \cdot (d\mathbf{r}_2 \times d\mathbf{r}_3)|.$$

For part (i) show that:

$$dV = |J| du_1 du_2 du_3$$

for $J = \det M$ where:

$$M_{ij} = \frac{\partial x_i}{\partial u_j}.$$

Using our definition of $d\mathbf{r}_i$:

$$\begin{aligned} dV &= |d\mathbf{r}_1 \cdot (d\mathbf{r}_2 \times d\mathbf{r}_3)| \\ &= \left| \frac{\partial \mathbf{r}}{\partial u_1} du_1 \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2} du_2 \times \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right) \right| \\ &= \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right) \right| du_1 du_2 du_3 \end{aligned}$$

Since $\mathbf{r} = x_i \mathbf{e}_i$:

$$\frac{\partial \mathbf{r}}{\partial u_j} = \frac{\partial x_i}{\partial u_j} \mathbf{e}_i.$$

Now the triple scalar product of three vectors is equivalent to the determinant of a matrix consisting of these three vectors, as either rows or columns. Therefore:

$$\begin{aligned} \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right) \right| &= \det \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \end{bmatrix} \\ &= \det M \\ &= J. \end{aligned}$$

Since volume is nonnegative:

$$\therefore dV = |J| du_1 du_2 du_3.$$

For part (ii) show:

- $(M^T M)_{ij} = g_{ij}$ for g_{ij} is the metric tensor.
- $dV = \sqrt{g} du_1 du_2 du_3$ for $g_{ij} = (G)_{ij}$ and $g = \det G$.

By the definition of the metric tensor:

$$\begin{aligned} g_{ij} &= \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \\ &= M_{ki} M_{kj} \\ &= (M^T)_{ik} (M)_{kj} \\ &= (M^T M)_{ij}. \end{aligned}$$

Since $G = M^T M$ taking the determinants gives:

$$\begin{aligned} \det G &= \det(M^T M) \\ &= \det M^T \det M \\ &= (\det M)^2 \\ &= J^2. \end{aligned}$$

Then from part (i):

$$\therefore J = \pm \sqrt{g}$$

$$\begin{aligned} \therefore dV &= |J| du_1 du_2 du_3 \\ &= \sqrt{g} du_1 du_2 du_3 \end{aligned}$$

For part (iii) show that given orthogonal curvilinear coordinates we have:

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$

For OCCs, $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ and therefore:

$$\begin{aligned} dV &= |\mathbf{dr}_1 \cdot (\mathbf{dr}_2 \times \mathbf{dr}_3)| \\ &= \left| \frac{\partial \mathbf{r}}{\partial u_1} du_1 \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2} du_2 \times \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right) \right| \\ &= |\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)| h_1 h_2 h_3 du_1 du_2 du_3 \\ &= |\mathbf{e}_1 \cdot \mathbf{e}_1| h_1 h_2 h_3 du_1 du_2 du_3 \\ &= h_1 h_2 h_3 du_1 du_2 du_3 \end{aligned}$$

where $h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|$ and $du_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_i}$.

3. For the first part of this problem we need to find the electric field $E(x)$ generated our rod of length $2a$ centered at $x = 0$ with total charge Q .

The thin rod is also assumed to have uniform charge density of ρ .

Then we can use the formula $F(x) = qE(x)$ to find force on our point charge q at $x = R$.

By Coulomb's law:

$$\begin{aligned} E(x) &= \int_{-a}^a \frac{\rho}{4\pi\epsilon_0} \frac{x - x'}{(x - x')^3} dx' \\ &= \frac{\rho}{4\pi\epsilon_0} \left[\frac{1}{x - x'} \right]_{x'=-a}^{x'=a} \\ &= \frac{\rho}{4\pi\epsilon_0} \frac{2a}{x^2 - a^2}. \end{aligned}$$

Because we have a uniform charge density across $2a$:

$$\rho = \frac{Q}{2a}$$

then the electric field generated by the thin rod becomes:

$$\begin{aligned} E(x) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{2a} \frac{2a}{x^2 - a^2} \\ &= \frac{Q}{4\pi\epsilon_0} \frac{1}{x^2 - a^2}. \end{aligned}$$

The force on our point charge q at $x = R$ is then:

$$\begin{aligned} F(R) &= qE(R) \\ &= \frac{qQ}{4\pi\epsilon_0} \frac{1}{R^2 - a^2}. \end{aligned}$$

The force on charge q at $x = R$ by charge Q at $x_1 = 0$ is given by:

$$\begin{aligned} F(x) &= F(R) \\ &= \frac{qQ}{4\pi\epsilon_0} \frac{x - x_1}{|x - x_1|^3} \\ &= \frac{qQ}{4\pi\epsilon_0} \frac{R}{|R|^3} \\ &= \frac{qQ}{4\pi\epsilon_0} \frac{1}{R^2}. \end{aligned}$$

Comparing these two forces:

$$F_{rod} = \frac{qQ}{4\pi\epsilon_0} \frac{1}{R^2 - a^2}$$

$$F_{point} = \frac{qQ}{4\pi\epsilon_0} \frac{1}{R^2},$$

since

$$\frac{1}{R^2 - a^2} > \frac{1}{R^2}$$

therefore $F_{rod} > F_{point}$.

4. For part (i) show that:

$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

The Lie brackets are defined:

$$[x, y] = xy - yx$$

and so

$$[T_i, T_j] = T_i T_j - T_j T_i.$$

Because

$$(T_k)_{ij} = -i\epsilon_{ijk}$$

then we have:

$$(T_i)_{lk} = -i\epsilon_{lki}$$

$$(T_j)_{km} = -i\epsilon_{kmj}$$

and swapping order gives:

$$(T_j)_{lk} = -i\epsilon_{lkj}$$

$$(T_i)_{km} = -i\epsilon_{kmi}.$$

It is important to use more indices here:

$$\begin{aligned} (T_i T_j - T_j T_i)_{lm} &= (T_i)_{lk} (T_j)_{km} - (T_j)_{lk} (T_i)_{km} \\ &= -\epsilon_{lki} \epsilon_{kmj} + \epsilon_{lkj} \epsilon_{kmi} \\ &= -\epsilon_{ilk} \epsilon_{kmj} + \epsilon_{jlk} \epsilon_{kmi}. \end{aligned}$$

We first consider $-\epsilon_{ilk} \epsilon_{kmj}$. Because we have

$$\begin{aligned} \epsilon_{ijk} \epsilon_{klm} &= \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \\ &= \delta_{im} \delta_{lj} - \delta_{ij} \delta_{lm} \\ &= \epsilon_{ilk} \epsilon_{kmj} \end{aligned}$$

where we swap $j \rightarrow l$, $l \rightarrow m$ and $m \rightarrow j$:

$$\therefore -\epsilon_{ilk} \epsilon_{kmj} = -(\delta_{im} \delta_{lj} - \delta_{ij} \delta_{lm})$$

$$\therefore \epsilon_{jlk} \epsilon_{kmi} = \delta_{jm} \delta_{li} - \delta_{ij} \delta_{lm}$$

Then:

$$\begin{aligned}
 (T_i T_j - T_j T_i)_{lm} &= -\epsilon_{ilk} \epsilon_{kmj} + \epsilon_{jlk} \epsilon_{kmi} \\
 &= -(\delta_{im} \delta_{lj} - \delta_{ij} \delta_{lm}) + \delta_{jm} \delta_{li} - \delta_{ij} \delta_{lm} \\
 &= -\delta_{im} \delta_{lj} + \delta_{jm} \delta_{li}.
 \end{aligned}$$

Due to the symmetry of the Kronecker delta:

$$\begin{aligned}
 (T_i T_j - T_j T_i)_{lm} &= -\delta_{im} \delta_{lj} + \delta_{jm} \delta_{li} \\
 &= \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \\
 &= \epsilon_{ijk} \epsilon_{klm} \\
 &= \epsilon_{ijk} \epsilon_{lmk} \\
 &= -i \epsilon_{ijk} \cdot -i \epsilon_{lmk} \\
 &= -i \epsilon_{ijk} \cdot (T_k)_{lm}.
 \end{aligned}$$

$$\therefore [T_i, T_j]_{lm} = -i \epsilon_{ijk} \cdot (T_k)_{lm}$$

$$\therefore [T_i, T_j] = -i \epsilon_{ijk} T_k$$

For part (ii) we want to show:

- T_i is hermitian
- $\text{Tr}(T_i T_j) = 2\delta_{ij}$

The definition of a hermitian matrix is as such:

$$H = (H^T)^*$$

$$H_{ij} = (H^T)^*_{ij}$$

where $*$ denotes the complex conjugate. So:

$$(T_i)_{jk} = -i\epsilon_{ijk}.$$

$$\begin{aligned} \therefore (T_i^T)_{jk} &= (T_i)_{kj} \\ &= -i\epsilon_{kji} \\ &= -i\epsilon_{ikj} \\ &= i\epsilon_{ijk} \end{aligned}$$

Then by the definition of the complex conjugate:

$$\therefore (T_i^T)^*_{jk} = -i\epsilon_{ijk}.$$

Since $(T_i)_{jk} = (T_i^T)^*_{jk}$ our matrix T_i is hermitian.

For the second part we firstly define:

$$(T_i)_{lk} = -i\epsilon_{lki}$$

and

$$(T_j)_{km} = -i\epsilon_{kmj}.$$

$$\begin{aligned} \therefore (T_i)_{lk}(T_j)_{km} &= (T_i T_j)_{lm} \\ &= -\epsilon_{lki}\epsilon_{kmj} \\ &= -\epsilon_{ilk}\epsilon_{kmj} \end{aligned}$$

Now:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\ &= \delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm} \\ &= \epsilon_{ilk}\epsilon_{kmj} \end{aligned}$$

if we swap $j \rightarrow l$, $l \rightarrow m$ and $m \rightarrow j$.

$$\begin{aligned}\therefore (T_i T_j)_{lm} &= -\epsilon_{ilk} \epsilon_{kmj} \\ &= -(\delta_{im} \delta_{lj} - \delta_{ij} \delta_{lm})\end{aligned}$$

Taking the trace of a matrix is summing up its diagonals:

$$\begin{aligned}\text{Tr}(T_i T_j) &= (T_i T_j)_{ii} \\ &= -(\delta_{ii} \delta_{ij} - \delta_{ij} \delta_{ii}) \\ &= -(\delta_{ij} - 3\delta_{ij}) \\ &= 2\delta_{ij}.\end{aligned}$$

Finally for part (iii) consider:

$$R(\alpha, \mathbf{n}) = \exp(-i\alpha \mathbf{n} \cdot \mathbf{T})$$

where α is our rotation angle about unit axis vector \mathbf{n} .

\mathbf{T} is a vector of generator matrices.

Our aims are:

- Show $(\mathbf{n} \cdot \mathbf{T})_{ij}^2 = \delta_{ij} - n_i n_j$.
- Show $(\mathbf{n} \cdot \mathbf{T})_{ij}^3 = (\mathbf{n} \cdot \mathbf{T})_{ij}$.
- General formula for $(\mathbf{n} \cdot \mathbf{T})_{ij}^m$ where $m > 3$.
- Expand $\exp(-i\alpha \mathbf{n} \cdot \mathbf{T})$ as a power series.
- Recover standard rotation tensor of form:

$$R_{ij}(\alpha, \mathbf{n}) = \delta_{ij} \cos \alpha + n_i n_j (1 - \cos \alpha) - \epsilon_{ijk} n_k \sin \alpha.$$

Firstly define:

$$(\mathbf{n} \cdot \mathbf{T})_{ik} = n_\alpha (T_\alpha)_{ik} = n_\alpha \cdot -i\epsilon_{ik\alpha}$$

$$(\mathbf{n} \cdot \mathbf{T})_{kj} = n_\beta (T_\beta)_{kj} = n_\beta \cdot -i\epsilon_{kj\beta}$$

then we have that

$$\begin{aligned} (\mathbf{n} \cdot \mathbf{T})_{ij}^2 &= n_\alpha (T_\alpha)_{ik} \cdot n_\beta (T_\beta)_{kj} \\ &= n_\alpha n_\beta \cdot -1 \cdot \epsilon_{ik\alpha} \epsilon_{kj\beta}. \end{aligned}$$

Using the standard identity we get:

$$\begin{aligned} \epsilon_{ijk} \epsilon_{klm} &= \epsilon_{jki} \epsilon_{klm} \\ &= \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \\ &= \delta_{\alpha j} \delta_{i\beta} - \delta_{\alpha\beta} \delta_{ij} \\ &= \epsilon_{ik\alpha} \epsilon_{kj\beta} \end{aligned}$$

where $j \rightarrow i$, $i \rightarrow \alpha$, $l \rightarrow j$ and $m \rightarrow \beta$.

Then substituting back into our equation:

$$\begin{aligned} (\mathbf{n} \cdot \mathbf{T})_{ij}^2 &= n_\alpha n_\beta \cdot -1 \cdot \epsilon_{ik\alpha} \epsilon_{kj\beta} \\ &= n_\alpha n_\beta (\delta_{\alpha\beta} \delta_{ij} - \delta_{\alpha j} \delta_{i\beta}) \\ &= \delta_{ij} - n_i n_j \end{aligned}$$

Now we show that $(\mathbf{n} \cdot \mathbf{T})_{ij}^3 = (\mathbf{n} \cdot \mathbf{T})_{ij}$. Define:

$$(\mathbf{n} \cdot \mathbf{T})_{ik}^2 = \delta_{ik} - n_i n_k$$

$$(\mathbf{n} \cdot \mathbf{T})_{kj} = n_l (T_l)_{kj} = n_l \cdot -i\epsilon_{kjl}$$

and so multiplying them together gives:

$$\begin{aligned} (\mathbf{n} \cdot \mathbf{T})_{ij}^3 &= (\mathbf{n} \cdot \mathbf{T})_{ik}^2 (\mathbf{n} \cdot \mathbf{T})_{kj} \\ &= (\delta_{ik} - n_i n_k) n_l \cdot -i\epsilon_{kjl} \\ &= -in_l (\epsilon_{ijl} - n_i n_k \epsilon_{kjl}) \\ &= -in_l \epsilon_{ijl} + in_l n_i n_k \epsilon_{kjl} \\ &= -in_l \epsilon_{ijl} + in_i \delta_{lk} \epsilon_{kjl} \\ &= -in_l \epsilon_{ijl} \\ &= (\mathbf{n} \cdot \mathbf{T})_{ij}. \end{aligned}$$

The general formula for $(\mathbf{n} \cdot \mathbf{T})_{ij}^m$ takes the form:

$$(\mathbf{n} \cdot \mathbf{T})_{ij}^m = \begin{cases} \delta_{ij} - n_i n_j & m \text{ even} \\ (\mathbf{n} \cdot \mathbf{T})_{ij} & m \text{ odd.} \end{cases}$$

The power series for an exponential is:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and so:

$$\begin{aligned} \left(\exp(-i\alpha \mathbf{n} \cdot \mathbf{T}) \right)_{ij} &= \sum_{k=0}^{\infty} \left(\frac{1}{k!} [-i\alpha]^k (\mathbf{n} \cdot \mathbf{T})_{ij}^k \right) \\ &= 1 + [-i\alpha] (\mathbf{n} \cdot \mathbf{T})_{ij}^1 \\ &\quad + \sum_{k=2,4,\dots} \frac{1}{k!} [-i\alpha]^k (\delta_{ij} - n_i n_j) \\ &\quad + \sum_{k=3,5,\dots} \frac{1}{k!} [-i\alpha]^k (\mathbf{n} \cdot \mathbf{T})_{ij}. \end{aligned}$$

Set $k = 2n$ for the first sum and $k = 2m+1$ for the second. Here $n, m \in \mathbb{N}$.

$$\begin{aligned}
\therefore \left(\exp(-i\alpha \mathbf{n} \cdot \mathbf{T}) \right)_{ij} &= 1 + [-i\alpha](\mathbf{n} \cdot \mathbf{T})_{ij}^k \\
&\quad + (\delta_{ij} - n_i n_j) \left(\left(\sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} \right) - 1 \right) \\
&\quad + (\mathbf{n} \cdot \mathbf{T})_{ij} \cdot -i \left(\left(\sum_{m=0}^{\infty} (-1)^m \frac{\alpha^{2m+1}}{(2m+1)!} \right) - \alpha \right)
\end{aligned}$$

Recognising this as the series expansion for cosine and sine:

$$\begin{aligned}
\therefore \left(\exp(-i\alpha \mathbf{n} \cdot \mathbf{T}) \right)_{ij} &= 1 + [-i\alpha](\mathbf{n} \cdot \mathbf{T})_{ij}^k \\
&\quad + (\delta_{ij} - n_i n_j) (\cos \alpha - 1) \\
&\quad + (\mathbf{n} \cdot \mathbf{T})_{ij} \cdot -i (\sin \alpha - \alpha)
\end{aligned}$$

Since $(\mathbf{n} \cdot \mathbf{T})_{ij} = n_k (T_k)_{ij} = -i n_k \epsilon_{ijk}$:

$$\begin{aligned}
\left(\exp(-i\alpha \mathbf{n} \cdot \mathbf{T}) \right)_{ij} &= 1 - \alpha n_k \epsilon_{ijk} + (\delta_{ij} - n_i n_j) (\cos \alpha - 1) - n_k \epsilon_{ijk} (\sin \alpha - \alpha) \\
&= \delta_{ij} \cos \alpha + n_i n_j (1 - \cos \alpha) - \epsilon_{ijk} n_k \sin \alpha
\end{aligned}$$

where we use the bogus argument:

$$\begin{aligned}
1 - \delta_{ij} &= \frac{1}{3} \delta_{ii} - \frac{1}{3} \delta_{jj} \delta_{ij} \\
&= \frac{1}{3} \delta_{ii} - \frac{1}{3} \delta_{ii} \\
&= 0.
\end{aligned}$$

Handin 2

1. For part (i) consider uniformly charged region bounded by two spheres, of radius b and a where $b > a$. Let the charge density of this region be ρ . Find its potential $\phi(\mathbf{r})$.

Let $\mathbf{r} = r\mathbf{e}_r + \theta\mathbf{e}_\theta + \phi\mathbf{e}_\phi$ where $r \in (a, b)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ be our parametrisation of the region. We have Gauss's law:

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\epsilon_0}.$$

Because of spherical symmetry, our electric field is of only radial form:

$$\mathbf{E}(\mathbf{r}) = E_r(r)\mathbf{e}_r.$$

Evaluating our surface integral:

$$\begin{aligned} \int_S \mathbf{E} \cdot d\mathbf{S} &= \iint \mathbf{E} \cdot \left(\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} \right) d\theta d\phi \\ &= \iint E_r(r)\mathbf{e}_r \cdot (r^2 \sin \theta \mathbf{e}_r) d\theta d\phi \\ &= r^2 E_r(r) \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin \theta d\theta d\phi \\ &= 4\pi r^2 E_r(r). \end{aligned}$$

Now the total charge enclosed is $Q_{enc} = \rho V$ where V is enclosed volume.

$$\therefore 4\pi r^2 E_r(r) = \begin{cases} \frac{\rho}{\epsilon_0} \frac{4}{3} \pi (b^3 - a^3) & r > b \\ \frac{\rho}{\epsilon_0} \frac{4}{3} \pi (r^3 - a^3) & r \in [a, b] \\ 0 & r < a \end{cases}$$

Then rearranging:

$$E_r(r) = \begin{cases} \frac{1}{3} \frac{\rho}{\epsilon_0} (b^3 - a^3) \frac{1}{r^2} & r > b \\ \frac{1}{3} \frac{\rho}{\epsilon_0} \left(r - \frac{a^3}{r^2} \right) & r \in [a, b] \\ 0 & r < a. \end{cases}$$

Since $\mathbf{E} = -\nabla\phi(\mathbf{r})$ we then have that $E_r(r) = -\frac{\partial\phi}{\partial r}$ and:

$$\phi(r) = - \int E_r(r) dr.$$

So when $r \in [a, b]$:

$$\begin{aligned}\phi(r) &= - \int \frac{\rho}{\epsilon_0} \frac{1}{3} (r - a^3 r^{-2}) dr \\ &= -\frac{1}{3} \frac{\rho}{\epsilon_0} \left[\frac{r^2}{2} + \frac{a^3}{r} \right] + C_1\end{aligned}$$

and when $r > b$:

$$\begin{aligned}\phi(r) &= -\frac{1}{3} \frac{\rho}{\epsilon_0} (b^3 - a^3) \int r^{-2} dr \\ &= \frac{1}{3} \frac{\rho}{\epsilon_0} (b^3 - a^3) \frac{1}{r} + C_2\end{aligned}$$

where $C_2 = 0$ so that $\phi \rightarrow 0$. For continuity, $C_1 = \frac{1}{2} \frac{\rho}{\epsilon_0} b^2$.

$$\therefore \phi(r) = \begin{cases} \frac{1}{3} \frac{\rho}{\epsilon_0} (b^3 - a^3) \frac{1}{r} & r > b \\ -\frac{1}{3} \frac{\rho}{\epsilon_0} \left[\frac{r^2}{2} + \frac{a^3}{r} \right] + \frac{1}{2} \frac{\rho}{\epsilon_0} b^2 & r \in [a, b] \\ 0 & r < a. \end{cases}$$

For part (ii):

$$\begin{aligned}E_r(r) &= \frac{1}{3} \frac{\rho}{\epsilon_0} \left(r - \frac{a^3}{r^2} \right) \\ &= \frac{1}{3} \frac{\rho}{\epsilon_0} \frac{1}{r^2} (r^3 - a^3) \\ &= \frac{\rho}{\epsilon_0} (r - a) \frac{1}{3r^2} (r^2 + a^2 + ra) \\ &= \frac{\rho}{\epsilon_0} (b - a) \frac{1}{3b^2} (b^2 + b^2 + rb)\end{aligned}$$

but since $b \rightarrow a$ we then have that:

$$E_r(r) = \frac{\rho}{\epsilon_0} (b - a) = \frac{\sigma}{\epsilon_0}.$$

2. For part (i) a dipole at \mathbf{r}_1 with moment \mathbf{p}_1 generates an electric field:

$$\mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}_1 \cdot (\mathbf{r} - \mathbf{r}_1))(\mathbf{r} - \mathbf{r}_1) - |\mathbf{r} - \mathbf{r}_1|^2 \mathbf{p}_1}{|\mathbf{r} - \mathbf{r}_1|^5}$$

and so at point \mathbf{r}_2 we have:

$$\mathbf{E}_1(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1))(\mathbf{r}_2 - \mathbf{r}_1) - |\mathbf{r}_2 - \mathbf{r}_1|^2 \mathbf{p}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^5}.$$

For part (ii) the force induced by our dipole at \mathbf{r}_1 with moment \mathbf{p}_1 on another dipole at \mathbf{r}_2 with moment \mathbf{p}_2 is:

$$\begin{aligned} \mathbf{F}_2(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0} \bigg[& -\frac{15}{|\mathbf{r}_2 - \mathbf{r}_1|^7} (\mathbf{p}_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1)) (\mathbf{p}_2 \cdot (\mathbf{r}_2 - \mathbf{r}_1)) (\mathbf{r}_2 - \mathbf{r}_1) \\ & + \frac{3}{|\mathbf{r}_2 - \mathbf{r}_1|^5} \left((\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{r}_2 - \mathbf{r}_1) + (\mathbf{p}_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1)) \mathbf{p}_2 \right. \\ & \left. + (\mathbf{p}_2 \cdot (\mathbf{r}_2 - \mathbf{r}_1)) \mathbf{p}_1 \right) \bigg]. \end{aligned}$$

For part (iii) find the torque \mathbf{G}_1 on dipole \mathbf{p}_1 at \mathbf{r}_1 due to the electric field \mathbf{E}_2 generated by dipole \mathbf{p}_2 at \mathbf{r}_2 .

So we have that:

$$\mathbf{G}_1(\mathbf{r}_1) = \mathbf{p}_1 \times \mathbf{E}_2(\mathbf{r}_1)$$

where

$$\mathbf{E}_2(\mathbf{r}_1) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2))(\mathbf{r}_1 - \mathbf{r}_2) - |\mathbf{r}_1 - \mathbf{r}_2|^2 \mathbf{p}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^5}$$

and evaluating this expression:

$$\mathbf{G}_1(\mathbf{r}_1) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)) \mathbf{p}_1 \times (\mathbf{r}_1 - \mathbf{r}_2) - |\mathbf{r}_1 - \mathbf{r}_2|^2 \mathbf{p}_1 \times \mathbf{p}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^5}.$$

For part (iv) find the torque \mathbf{G}_2 on dipole \mathbf{p}_2 at \mathbf{r}_1 from the electric field \mathbf{E}_1 generated by \mathbf{p}_1 at \mathbf{r}_1 .

Now the torque on \mathbf{p}_2 at \mathbf{r}_2 is then:

$$\begin{aligned}
 \mathbf{G}_2(\mathbf{r}_2) &= \tau_{+q} + \tau_{-q} \\
 &= q(\mathbf{0} + \mathbf{d}) \times \mathbf{E}_1(\mathbf{r}_2 + \mathbf{d}) - q\mathbf{0} \times \mathbf{E}_1(\mathbf{r}_2) \\
 &= q\mathbf{d} \times \left[\mathbf{E}_1(\mathbf{r}_2) + (\mathbf{d} \cdot \nabla) \mathbf{E}_1(\mathbf{r}_2) + \dots \right] \\
 &\approx q\mathbf{d} \times \mathbf{E}_1(\mathbf{r}_2) + \mathbf{d} \times (q\mathbf{d} \cdot \nabla) \mathbf{E}_1(\mathbf{r}_2) \\
 &= \mathbf{p}_2 \times \mathbf{E}_1(\mathbf{r}_2) + \mathbf{d} \times (\mathbf{p}_2 \cdot \nabla) \mathbf{E}_1(\mathbf{r}_2) \\
 &= \mathbf{p}_2 \times \mathbf{E}_1(\mathbf{r}_2) + \mathbf{d} \times \mathbf{F}_2(\mathbf{r}_2)
 \end{aligned}$$

where dipole $\mathbf{p}_2 = q\mathbf{d}$ has $-q$ charge at \mathbf{r}_2 and $+q$ charge at $\mathbf{r}_2 + \mathbf{d}$ with the limit $\mathbf{d} \rightarrow \mathbf{0}$.

In the dipole limit our two dipoles \mathbf{p}_1 and \mathbf{p}_2 are infinitely close. There probably exists dipole interactions between these two dipoles. So let $\mathbf{r}_1 = \mathbf{r}_2 - \mathbf{d}$, and in the limit $\mathbf{d} \rightarrow \mathbf{0}$ we get:

$$\mathbf{G}_2(\mathbf{r}_1) = \mathbf{p}_2 \times \mathbf{E}_1(\mathbf{r}_2) + (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_2(\mathbf{r}_2)$$

where $\mathbf{r}_2 \rightarrow \mathbf{r}_1$.

Finally for part (v) verify that:

$$\mathbf{G}_2(\mathbf{r}_1) = -\mathbf{G}_1(\mathbf{r}_1).$$

Using previous parts we have that:

$$\mathbf{p}_2 \times \mathbf{E}_1(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1))\mathbf{p}_2 \times (\mathbf{r}_2 - \mathbf{r}_1) - |\mathbf{r}_2 - \mathbf{r}_1|^2 \mathbf{p}_2 \times \mathbf{p}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^5}$$

$$\begin{aligned}
 (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_2(\mathbf{r}_2) &= \frac{1}{4\pi\epsilon_0} \frac{3}{|\mathbf{r}_2 - \mathbf{r}_1|^5} \left[-(\mathbf{p}_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1))\mathbf{p}_2 \times (\mathbf{r}_2 - \mathbf{r}_1) \right. \\
 &\quad \left. + -(\mathbf{p}_2 \cdot (\mathbf{r}_2 - \mathbf{r}_1))\mathbf{p}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) \right]
 \end{aligned}$$

Adding these two up:

$$\begin{aligned}
 \mathbf{G}_2(\mathbf{r}_1) &= -\frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}_2 \cdot (\mathbf{r}_2 - \mathbf{r}_1))\mathbf{p}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) + |\mathbf{r}_2 - \mathbf{r}_1|^2 \mathbf{p}_2 \times \mathbf{p}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^5} \\
 &= -\mathbf{G}_1(\mathbf{r}_1).
 \end{aligned}$$

3. For part (i) using Gauss's law:

$$\begin{aligned}\int_S \mathbf{E} \cdot d\mathbf{S} &= \iint E_\rho(\rho) \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial z} \right) d\phi dz \\ &= E_\rho(\rho) \int_z \int_0^{2\pi} \rho d\phi dz \\ &= z E_\rho(\rho) \rho 2\pi\end{aligned}$$

and when $\rho \in (a, b)$ this is equivalent to:

$$z E_\rho(\rho) \rho 2\pi = -\frac{\lambda}{\epsilon_0} z.$$

When $\rho \notin (a, b)$ choose spherical shell with radius ρ . By definition it has no charge and hence:

$$E_\rho(\rho) = 0.$$

So we have that:

$$E_\rho(\rho) = \begin{cases} -\frac{\lambda}{2\pi\epsilon_0} \frac{1}{\rho} & \rho \in (a, b) \\ 0 & \rho \notin (a, b). \end{cases}$$

For part (ii) since $\mathbf{E} = -\nabla\phi(\mathbf{r})$ integrating gives:

$$\phi(\rho) = \begin{cases} \frac{\lambda}{2\pi\epsilon_0} \ln \rho & \rho \in (a, b) \\ 0 & \rho \notin (a, b) \end{cases}$$

The potential difference between the two plates is

$$\begin{aligned}V &= \phi(b) - \phi(a) \\ &= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a}\end{aligned}$$

and since $Q = \lambda$:

$$\begin{aligned}C &= \frac{Q}{V} \\ &= \frac{2\pi\epsilon_0}{\ln b/a}.\end{aligned}$$

For part (iii) the energy per unit length is:

$$\begin{aligned} W &= \frac{Q}{2}(\phi_b - \phi_a) \\ &= \frac{\lambda^2}{4\pi\epsilon_0} \ln \frac{b}{a}. \end{aligned}$$

For part (iv):

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \int dV |\mathbf{E}(\mathbf{r})|^2 \\ &= \frac{\epsilon_0}{2} \frac{\lambda^2}{4\pi^2\epsilon_0} \int_z \int_0^{2\pi} \int_a^b \frac{1}{\rho^2} |\rho| d\rho d\phi dz \\ &= \frac{\lambda^2}{4\pi\epsilon_0} \ln \frac{b}{a}. \end{aligned}$$

For part (v) show that when b is slightly larger than a we have that:

$$C = \frac{2\pi\epsilon_0}{\ln b/a} \approx \frac{2\pi a\epsilon_0}{d}$$

where a is the radius of a cylinder and d is plate separation distance.

Here capacitance in per unit length. Let $b - a = \epsilon$ where ϵ is a small quantity. Then consider the following:

$$\begin{aligned} \ln \frac{b}{a} &= \ln \left(1 + \frac{\epsilon}{a} \right) \\ &\approx \frac{\epsilon}{a} \end{aligned}$$

and

$$\begin{aligned} \frac{2\pi\epsilon_0}{\ln b/a} &\approx \frac{2\pi\epsilon_0}{\frac{\epsilon}{a}} \\ &= \frac{2\pi a\epsilon_0}{b-a} \end{aligned}$$

which is an expression for the capacitance of a thinly separated parallel-plate capacitor.