D: Supremum and infimum

Let $\alpha = \sup S$. Then:

1. $\forall s \in S; \alpha \geq s$
2. $\forall a \in \mathbb{R} : \forall s \in S; a \geq s;$
   $a \geq \alpha$

and similarly for infimum.

T: Approximation property

Consider bounded $E \subset \mathbb{R}$. Then:

$$\forall \epsilon > 0; \exists a \in E : \sup E - \epsilon < a \leq \sup E.$$

D: Completeness of $\mathbb{R}$

Every nonempty bounded subset of $\mathbb{R}$ has an infimum and supremum.

T: Archimedean property

$\forall a, b \in \mathbb{R}; a > 0; \exists n \in \mathbb{N} : na > b$

D1.1: Nested intervals

A sequence of sets $(I_n)_{n\in\mathbb{N}}$ is nested if $I_1 \supset I_2 \supset I_3 \ldots$.

T1.1: Nested interval property

Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of nonempty, closed and bounded nested intervals. Then:

$$E = \bigcap_{n\in\mathbb{N}} I_n \neq \emptyset.$$

If $\lambda(I_n) \to 0$ then E contains one number, where λ denotes length.

T1.2

Let $E = [a, b]$ and that there exists an open collection of nested intervals $(I_\alpha)_{\alpha\in A}$ such that:

$$E \subset \bigcup_{\alpha\in A} I_\alpha.$$

Then $\exists\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset A$ such that:

$$E \subset I_{\alpha_1} \cup I_{\alpha_2} \cup \cdots \cup I_{\alpha_n}.$$

D1.2: ϵ-N convergence

Let $\lim_{n\to\infty} x_n = a$. Then:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |x_n - a| < \epsilon.$$

D1.3: Cauchy sequences

The sequence (x_n) is Cauchy if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \implies |x_n - x_m| < \epsilon.$$

T1.3 and T1.4

Cauchy $\iff$ ϵ-N convergent.

D1.4: Subsequences

The subsequence of $(x_n)_{n\in\mathbb{N}}$ is a sequence of form $(x_{n_k})_{k\in\mathbb{N}}$ and is a selection of the original sequence **taken in order**.

T1.5: Bolzano-Weierstrass

Every bounded real sequence has **a** convergent subsequence.

D1.5: Limit inferior and superior

Let (x_n) be a bounded real sequence. Then:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right).$$

T1.6

The real sequence (x_n) is convergent **iff**:

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

D1.6: Convergence of infinite series

Let $S = \sum_{k=1}^{\infty} a_k$ is convergent if:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k < \infty$$

Series S is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ is also convergent.

Otherwise S is conditionally convergent.

T1.7: Cauchy criterion for series

$S = \sum_{k=1}^{\infty} a_k$ is convergent **iff**:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall m \geq n \geq N \implies \left|\sum_{k=n+1}^{m} a_k\right| < \epsilon.$$

T1.8

Let $S = \sum_{k=1}^{\infty} a_k$ be absolutely convergent. Let $z : \mathbb{N} \to \mathbb{N}$ be a bijection. Then:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}.$$

T1.9: Riemann rearrangement

Let $S = \sum_{k=1}^{\infty} a_k$ be conditionally convergent. Then there exists rearrangements such that S can take on any value.

D1.7: Sequential continuity

Let $f : \mathrm{dom}(f) \to \mathbb{R}$ where $\mathrm{dom}(f) \subset \mathbb{R}$. f is continuous at $\alpha \in \mathrm{dom}(f)$ if:

$$\forall (x_n)_{n\in\mathbb{N}} \subset \mathrm{dom}(f) : \lim_{n\to\infty} x_n = \alpha \implies \lim_{n\to\infty} f(x_n) = f(\alpha).$$

T1.10

Let $\alpha \in \mathbb{R}$ and f, g continuous on D. Then αf, $f + g$, fg are continuous on D.

T1.11

Let f be continuous at $\alpha \in \mathbb{R}$ and g at $f(\alpha)$. Then $g \circ f$ is continuous at α.

D1.12: ϵ-δ continuity

Let $f : \mathrm{dom}(f) \to \mathbb{R}$ where $\mathrm{dom}(f) \subset \mathbb{R}$. Then f is continuous at $\alpha \in \mathrm{dom}(f)$ if:

$$\forall \epsilon > 0; \exists \delta > 0 : |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon.$$

T1.13: Intermediate value theorem

Let f be continuous on $[a, b]$.
If $f(a)f(b) < 0$ then:

$$\exists c \in (a, b) : f(c) = 0.$$

T1.14: Extreme value theorem

Let f be continuous on $[a, b]$.
Then $\exists c, d \in [a, b]$ such that:

$$f(c) = \inf\{f(x) : x \in [a, b]\}$$

$$f(d) = \sup\{f(x) : x \in [a, b]\}.$$

T: Mean value theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then:

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

D: Differentiability

f is differentiable at α if:

$$f'(\alpha) = \lim_{h\to 0} \frac{f(\alpha + h) - f(\alpha)}{h}.$$

T: Continuity test

f is continuous at α if:

$$\lim_{x\to\alpha} f(x) = f(\alpha)$$

where the limit from left/right must both exist and be equal to each other.

D2.1: Pointwise convergence

$f_n \to f$ pointwise on E if:

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Here $f_n : E \to \mathbb{R}$ and:

$$\forall x \in E; \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

D2.2: Uniform convergence

$f_n \to f$ uniformly on E if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \text{ and } \forall x \in E \implies |f_n(x) - f(x)| < \epsilon$$

P2.1

The following statements are equivalent.

1. $f_n \to f$ uniformly on E
2. $\lim\limits_{n\to\infty} \sup\limits_{x\in E} |f_n(x) - f(x)| = 0$
3. $\exists a_n \to 0$ s.t. $|f_n(x) - f(x)| \leq a_n$ for $\forall x \in E$.

T2.1

If f_n is continuous on E **and** $f_n \to f$ uniformly on E then f is continuous on E.

Remark

If f is not continuous on E then f_n cannot be uniform on E.

T2.5: Weierstrass M-test

Let $E \subset \mathbb{R}$ and $f_k : E \to \mathbb{R}$.
$\exists M_k > 0 : \sum\limits_{k=1}^{\infty} M_k < \infty$.
If $\forall k \in \mathbb{N}$ and $\forall x \in E$; $|f_k(x)| \leq M_k$ then:
$\sum\limits_{k=1}^{\infty} f_k(x)$ converges uniformly on E.

D: Power series

Let (a_n) be a real sequence and $c \in \mathbb{R}$. Then:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

is a power series centered at c, with **radius of convergence**:

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

where $R = \infty$ implying that series converges everywhere.

T3.1: Convergence of power series

Let $0 < R < \infty$. If $|x - c| < R$ then $f_{PS}(x)$ converges absolutely.

If $|x - c| > R$ then $f_{PS}(x)$ diverges.

T3.2: Continuity of power series

Let $0 < r < R$ where R is the radius of convergence of $f_{PS}(x)$.

Then for $|x - c| \leq r$, $f_{PS}(x)$ converges absolutely and uniformly to a continuous function $f(x)$.

L3.1

$\sum\limits_{n=1}^{\infty} a_n (x-c)^n$ and $\sum\limits_{n=1}^{\infty} n a_n (x-c)^{n-1}$ have the same radius of convergence.

T3.3

Let R be the radius of convegence of $f_{PS}(x)$. Then for $\forall x : |x-c| < R$, $f_{PS}(x)$ is **infinitely differentiable**.

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$\therefore a_n = \frac{f^{(n)}(c)}{n!}$$

Elementary expansions

- $E(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$
- $S(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$