Honours DE Workshops

Christopher Shen

Winter 2023

Contents

1 Workshop 6

1.

2 Workshop 7

1. Find solutions to **BVPs**.

   For part (i) we have that:

$$y'' + y = x \quad \text{for} \quad y(0) = y(\pi) = 0.$$

   We first find the homogeneous solution:

$$y'' + y = 0,$$

   and this gives: $y_H = \alpha \cos x + \beta \sin x$. The particular solution for the differential equation is $y_p = x$, and therefore our general solution is:

$$y = x + \alpha \cos x + \beta \sin x.$$

   However if we substitute our **boundary conditions** we find that:

$$y(0) = \alpha = 0$$

   yet

$$y(\pi) = \pi - \alpha = 0.$$

   This is clearly a contradiction and there are **no solutions** to this problem.

   For part (ii) we are asked:

$$y'' + 4y = \cos x \quad \text{for} \quad y'(0) = y'(\pi) = 0.$$

   The homogeneous equation is:

$$y'' + 4y = 0$$

   and has eigenvalues $\lambda = \pm 2i$, which corresponds to solution:

$$y_H = \alpha \cos 2x + \beta \sin 2x.$$

   We try for a particular solution of form $y_p = \gamma \cos x + \eta \sin x$ and after substituting our general solution takes the form:

$$y = \alpha \cos 2x + \beta \sin 2x + \frac{1}{3} \cos x.$$

   Taking derivatives and substituting boundary conditions we find $\beta = 0$, and so for this **BVP** there are infinitely many solutions of form:

$$y = \alpha \cos 2x + \frac{1}{3} \cos x,$$

   where $\alpha \in \mathbb{R}$.

2. Consider the following function:

$$g(x) = \begin{cases} 1+x & x \in [-1,0) \\ 1-x & x \in [0,1). \end{cases}$$

Find its Fourier series with period $2L$, where $L = 1$.

Firstly our function g is an <u>even</u> function and so its Fourier series is purely in cosine form:

$$g_{FS}(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos n\pi x$$

since $L = 1$. Its coefficients are:

$$\begin{aligned} c_0 &= \frac{1}{L}\int_{-L}^{L} g(x)\mathrm{d}x \\ &= \int_{-1}^{1} g(x)\mathrm{d}x \\ &= \int_{-1}^{0} (1+x)\mathrm{d}x + \int_{0}^{1} (1-x)\mathrm{d}x \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} c_n &= \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} g(x)\mathrm{d}x \\ &= \int_{-1}^{1} \cos(n\pi x) g(x)\mathrm{d}x \\ &= \frac{2}{(n\pi)^2}\Big(1 - (-1)^n\Big) \end{aligned}$$

for $n \in \{1, 2, \dots\}$ and we note that c_n is nonzero in only <u>odd</u> terms.

$$\therefore g_{FS}(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi x}{(2m+1)^2}$$

3. For part (i) find the Fourier series of the following:

$$f(x) = \begin{cases} -1 & x \in [-L, 0) \\ 1 & x \in (0, L] \end{cases}$$

with period $2L$ and $L = \pi$.

Firstly our function is <u>odd</u> as hence:

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=}^{\infty} a_n \sin \frac{n\pi x}{L}$$

where its coefficients are:

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{0} (-1) \mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} \mathrm{d}x \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} \sin \frac{n\pi x}{L} f(x) \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{0} -\sin(nx) \mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} \sin(nx) \mathrm{d}x \\ &= \frac{2}{n\pi} \big[1 - (-1)^n\big]. \end{aligned}$$

Then our Fourier series takes the following form:

$$f_{FS}(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{2m+1}.$$

For part (ii) consider the following partial sums:

$$f_N(x) = \frac{4}{\pi}\sum_{m=0}^{N-1}\frac{\sin(2m+1)x}{2m+1}.$$

Show that its derivative is the following:

$$f_N'(x) = \frac{2}{\pi}\frac{\sin 2Nx}{\sin x}.$$

Starting from our partial sums we take derivatives:

$$\therefore f_N'(x) = \frac{4}{\pi}\sum_{m=0}^{N-1}\Big[\cos(2m+1)x\Big].$$

I had the idea of taking the real component of its complex exponential, but the algebra proved too tedious. Following the solutions:

$$\cos x = \frac{1}{2}\big(e^{ix} + e^{-ix}\big)$$

and therefore

$$\cos(2m+1)x = \frac{1}{2}\big(e^{i(2m+1)x} + e^{-i(2m+1)x}\big).$$

Substituting this into our partial sums:

$$\begin{aligned}
f_N'(x) &= \frac{4}{\pi}\sum_{m=0}^{N-1}\Big[\cos(2m+1)x\Big] \\
&= \frac{2}{\pi}\sum_{m=0}^{N-1}\Big[e^{i(2m+1)x} + e^{-i(2m+1)x}\Big] \\
&= \frac{2}{\pi}\sum_{m=0}^{N-1}\Big[e^{i(2m+1)x} + e^{-i(2m+1)x}\Big] \\
&= \frac{2}{\pi}\Big(e^{ix}\sum_{m=0}^{N-1}\big(e^{i\cdot 2x}\big)^m + e^{-ix}\sum_{m=0}^{N-1}\big(e^{-i\cdot 2x}\big)^m\Big).
\end{aligned}$$

Recall the geometric series formula:

$$\sum_{m=0}^{N-1} r^m = \frac{1-r^N}{1-r}.$$

$$\therefore \sum_{m=0}^{N-1} \left(e^{i\cdot 2x}\right)^m = \frac{1-e^{iN2x}}{1-e^{i2x}}$$

$$\therefore \sum_{m=0}^{N-1} \left(e^{-i\cdot 2x}\right)^m = \frac{1-e^{-iN2x}}{1-e^{-i2x}}$$

Then:

$$\begin{aligned}
f_N'(x) &= \frac{2}{\pi}\Big(e^{ix}\sum_{m=0}^{N-1}\left(e^{i\cdot 2x}\right)^m + e^{-ix}\sum_{m=0}^{N-1}\left(e^{-i\cdot 2x}\right)^m\Big) \\
&= \frac{2}{\pi}\Big(e^{ix}\frac{1-e^{iN2x}}{1-e^{i2x}} + e^{-ix}\frac{1-e^{-iN2x}}{1-e^{-i2x}}\Big) \\
&= \frac{2}{\pi}\Big(\frac{1-e^{iN2x}}{e^{-ix}-e^{ix}} + \frac{1-e^{-iN2x}}{e^{ix}-e^{-ix}}\Big) \\
&= \frac{2}{\pi}\Big(\frac{e^{iN2x}-e^{-iN2x}}{e^{ix}-e^{-ix}}\Big)
\end{aligned}$$

and because:

$$\sin x = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

we finally have that:

$$f_N'(x) = \frac{2}{\pi}\frac{\sin 2Nx}{\sin x}.$$

For part (iii) set $f_N'(x) = 0$ and it is evident that:

$$2Nx = m\pi$$

solves the equation.

$$\therefore x = \frac{m\pi}{2N}$$

where $m \in \mathbb{Z}$.

For part (iv) to show that $x = \dfrac{\pi}{2N}$ is a maximum we take second derivatives:

$$f''(x) = \frac{4N}{\pi}\cos 2Nx(\sin x)^{-1} - \frac{2}{\pi}\sin 2Nx\cos x(\sin x)^{-2}$$

and since $f''(\frac{\pi}{2N}) < 0$ this is a maximum.

For part (v) show that for large N:

$$f_N(\frac{\pi}{2N}) \approx \frac{2}{\pi}\int_0^{\pi} \frac{\sin\beta}{\beta}\mathrm{d}\beta.$$

So begin by integrating the following expression:

$$\int_0^{\frac{\pi}{2N}} \frac{2}{\pi}\frac{\sin 2Nx}{\sin x}\mathrm{d}x.$$

By substituting $\beta = 2Nx$ we get:

$$\begin{aligned}\int_{x=0}^{x=\frac{\pi}{2N}} \frac{2}{\pi}\frac{\sin 2Nx}{\sin x}\mathrm{d}x &= \int_{\beta=0}^{\beta=\pi} \frac{2}{\pi}\frac{\sin\beta}{\sin\frac{\beta}{2N}}\frac{\mathrm{d}\beta}{2N} \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2N}\frac{\sin\beta}{\sin\frac{\beta}{2N}}\mathrm{d}\beta \\ &\approx \frac{2}{\pi}\int_0^{\pi} \frac{\sin\beta}{\beta}\mathrm{d}\beta\end{aligned}$$

since when $N \gg \beta$ by small angle approximations we have that:

$$\sin\frac{\beta}{2N} \approx \frac{\beta}{2N}.$$

Finally for part (vi) consider the power series expansion for:

$$\frac{\sin\beta}{\beta} = \sum_{m=0}^{\infty}(-1)^m\frac{\beta^{2m}}{(2m+1)!}.$$

Using this series in our integral:

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi}\frac{\sin\beta}{\beta}\mathrm{d}\beta &= \frac{2}{\pi}\sum_{m=0}^{\infty}(-1)^m\int_0^{\pi}\frac{\beta^{2m}}{(2m+1)!}\mathrm{d}\beta \\ &= 2\sum_{m=0}^{\infty}(-1)^m\frac{\pi^{2m}}{(2m)!(2m+1)^2} \\ &\approx 1.18.\end{aligned}$$

3 Workshop 8

1. For part (a) we want to use the method of separation of variables to find ODE solutions to:
   - $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial t} + \dfrac{\partial u}{\partial t} = 0$
   - $\dfrac{\partial^2 u}{\partial x^2} + (x+y)\dfrac{\partial^2 u}{\partial y^2} = 0$

   For the first PDE let
   $$u = X(x)T(t)$$
   and after differentiating we get:
   $$X''T + X'\dot{T} + X\dot{T} = 0.$$
   Rearranging this equation and introducing a separation constant:
   $$\frac{X''}{X' + X} = -\frac{\dot{T}}{T} = \lambda.$$
   This is clearly two ODEs:
   $$X'' - \lambda(X' + X) = 0$$
   $$\dot{T} + \lambda T = 0.$$

   For the second equation let
   $$u = X(x)Y(y)$$
   and differentiating this gives
   $$X''Y + (x+y)XY'' = 0$$
   which is of non-separable form.

For part $(b)(i)$ consider the heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

We solve this problem via separation of variables:

$$u = X(x)Y(y)T(t).$$

Differentiating and substituting:

$$XY\dot{T} = \alpha^2 \left(X''YT + XY''T \right)$$

and dividing both side by XYT gives:

$$\frac{1}{\alpha^2}\frac{\dot{T}}{T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda.$$

We get our first ODE:

$$\dot{T} - \lambda\alpha^2 T = 0.$$

Clearly we need another separation constant:

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu$$

and this yields the other two ODEs:

$$X'' - \mu X = 0$$

$$Y'' + (\mu - \lambda)Y = 0.$$

For part $(b)(ii)$ reconsider the heat equation in polar coordinates:

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} \right).$$

We again use separation of variables:

$$u = R(r)\Phi(\phi)T(t)$$

and after differentiating we get:

$$R\Phi\dot{T} = \alpha^2 \left(R''\Phi T + \frac{1}{r}R'\Phi T + \frac{1}{r^2}R\Phi'' T \right).$$

Dividing through by $R\Phi T$ and setting a separation constant:

$$\frac{1}{\alpha^2}\frac{\dot{T}}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = \lambda$$

and we obtain our first ODE:

$$\dot{T} - \lambda\alpha^2 T = 0.$$

Multiplying both sides by r^2 gives:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} = r^2\lambda.$$

Rearranging and introducing another separation constant:

$$r^2\frac{R''}{R} + r\frac{R'}{R} - r^2\lambda = -\frac{\Phi''}{\Phi} = \mu$$

and clearly we have two ODEs.

$$\therefore \Phi'' + \mu\Phi = 0$$

$$\therefore r^2R'' + rR' - (\mu + \lambda r^2)R = 0$$

2. The solution to the heat equation is:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \exp\left(-n^2\alpha^2\pi^2 t/L^2\right) \cos\frac{n\pi x}{L}$$

where we have initial condition $u(x,0) = f(x)$ and boundary condition:

$$\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(L,t) = 0.$$

Firstly set $t = 0$. We then have that:

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{L} = f(x)$$

for $\forall x \in [0, L]$. Taking the integral over this interval:

$$\int_0^L f(x)\mathrm{d}x = \int_0^L \frac{c_0}{2}\mathrm{d}x = \frac{L}{2}c_0.$$

$$\therefore c_0 = \frac{2}{L}\int_0^L f(x)\mathrm{d}x$$

Similarly we have that:

$$c_n = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L} f(x)\mathrm{d}x.$$

Now if

$$f(x) = 3\cos\frac{2\pi x}{L}$$

then:

$$\begin{aligned} c_0 &= \frac{2}{L}\int_0^L 3\cos\frac{2\pi x}{L}\mathrm{d}x \\ &= \frac{6}{L}\left[\frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_0^L \\ &= 0. \end{aligned}$$

To find c_n we need the following identity:

$$\cos A \cos B = \frac{1}{2}\Big(\cos(A-B) + \cos(A+B)\Big).$$

So **if** $\underline{n \neq 2}$:

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\cdot 3\cos\frac{2\pi x}{L}\mathrm{d}x \\
&= \frac{6}{L}\int_0^L \cos\frac{n\pi x}{L}\cos\frac{2\pi x}{L}\mathrm{d}x \\
&= \frac{6}{L}\int_0^L \frac{1}{2}\Big[\cos(2-n)\frac{\pi x}{L} + \cos(n+2)\frac{\pi x}{L}\Big]\mathrm{d}x \\
&= \frac{3}{\pi}\Big[\frac{1}{2-n}\sin\frac{(2-n)\pi x}{L} + \frac{1}{n+2}\sin\frac{(n+2)\pi x}{L}\Big]_0^L \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\therefore c_2 &= \frac{6}{L}\int_0^L \left(\cos\frac{2\pi x}{L}\right)^2 \mathrm{d}x \\
&= \frac{6}{L}\int_0^L \frac{1}{2}\left(1+\cos\frac{4\pi x}{L}\right)\mathrm{d}x \\
&= \frac{3}{L}\left[x + \frac{L}{4\pi}\sin\frac{4\pi x}{L}\right]_0^L \\
&= 3
\end{aligned}$$

Therefore our solution becomes:

$$u(x,t) = 3\exp\left(-n^2\alpha^2\pi^2 t/L^2\right)\cos\frac{2\pi x}{L}$$

and tends to zero over time.

3. For part (a) consider the solutions to the wave equation:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L}$$

with <u>no initial velocity</u> and

$$u(x,0) = f(x).$$

Integrating this over $[0, L]$ and using trigonometric identities gives:

$$c_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} f(x) \mathrm{d}x$$

Now we set:

$$f(x) = \begin{cases} 2x/L & x \in [0, \frac{L}{2}] \\ 2(L-x)/L & x \in [\frac{L}{2}, L] \end{cases}$$

and the coefficients are:

$$\begin{aligned} c_n &= \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} f(x) \mathrm{d}x \\ &= \frac{2}{L} \int_0^{L/2} \frac{2}{L} x \sin \frac{n\pi x}{L} \mathrm{d}x + \frac{2}{L} \int_{L/2}^{L} (2 - \frac{2}{L} x) \sin \frac{n\pi x}{L} \mathrm{d}x \\ &= \Big(-\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \Big) \\ &\quad + \Big(\frac{2}{n\pi} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \sin n\pi + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \Big) \\ &= \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \sin n\pi \\ &= \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \end{aligned}$$

since the last expression is always zero.

For part (b) the solution to boundary condition:

$$\frac{\partial}{\partial t}u(x,0) = g(x)$$

is the following expression:

$$u(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

So we have that:

$$\begin{aligned}\frac{\partial}{\partial t}u(x,0) &= \sum_{n=1}^{\infty}\frac{\pi a}{L} n k_n \sin\frac{n\pi x}{L} \\ &= g(x)\end{aligned}$$

and integrating this expression gives:

$$k_m = \frac{L}{m\pi a}\frac{2}{L}\int_0^L \sin\frac{m\pi x}{L} g(x)\mathrm{d}x.$$

Now if $g(x) = f(x)$ we then have:

$$\begin{aligned}k_m &= \frac{L}{m\pi a}\frac{2}{L}\int_0^L \sin\frac{m\pi x}{L} f(x)\mathrm{d}x \\ &= \frac{L}{m\pi a}c_m \\ &= \frac{8L}{m^3\pi^3 a}\sin\frac{m\pi}{2}\end{aligned}$$

and we are finished.

4. Let's now consider Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the following boundary conditions:

- x-boundary: $u(0, y) = 0$ and $u(a, y) = 0$
- y-boundary: $u(x, 0) = h(x)$ and $u(x, b) = 0$

for $\forall x \in [0, a]$ and $\forall y \in (0, b)$.

Note here that $u = u(x, y)$ maps a plane to a line.

Begin by separation of variables:

$$u(x, y) = X(x)Y(y).$$

$$\therefore X''Y + XY'' = 0$$

Now choosing a separation constant depends on our boundary conditions:

$$X(0) = X(a) = Y(b) = 0.$$

$$\therefore \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

We identify the two ODEs:

$$X'' + \lambda X = 0 \implies X(x) = a_1 \cos \lambda^{1/2} x + a_2 \sin \lambda^{1/2} x$$

$$Y'' - \lambda Y = 0 \implies Y(y) = b_1 \cosh \lambda^{1/2} y + b_2 \sinh \lambda^{1/2} y$$

and using $X(0) = X(a) = 0$ we eliminate a_1:

$$X(a) = a_2 \sin \lambda^{1/2} a = 0.$$

$$\therefore \lambda_n = \frac{n^2\pi^2}{a^2}$$

$$\therefore X_n(x) = a_n \sin \frac{n\pi x}{a}$$

Turning our attention to the other ODE:

$$Y(b) = b_1 \cosh \lambda_n^{1/2} b + b_2 \sinh \lambda_n^{1/2} b = 0$$

and so

$$b_2 = -b_1 \frac{\cosh \lambda_n^{1/2} b}{\sinh \lambda_n^{1/2} b}.$$

$$\therefore Y_n(y) = b_n \sinh \left(\frac{n\pi b}{a} - \frac{n\pi y}{a} \right)$$

Putting all these together we obtain the general solution:

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{a} x \, \sinh \frac{n\pi}{a} (b - y)$$

and finally since $u(x, 0) = h(x)$ for $\forall x \in [0, a]$:

$$\sum_{n=1}^{\infty} \sin \frac{n\pi}{a} x \, \sinh \frac{n\pi b}{a} = h(x).$$

Integrating this over $[0, a]$ gives:

$$c_n = \frac{2}{a} \frac{1}{\sinh \frac{n\pi b}{a}} \int_0^a \sinh \frac{n\pi x}{a} h(x) \mathrm{d}x.$$

4 Workshop 9

1. For part (a) consider the following boundary value problem:

$$y'' + \lambda y = 0$$

with:

$$y(0) - y'(0) = 0$$
$$y(1) + y'(1) = 0.$$

Find the eigenvalues and eigenfunctions to this problem.

Firstly this BVP is in S-L form, and therefore $\lambda \in \mathbb{R}$.

Consider solutions when $\lambda = 0$.

$$\therefore y'' = 0$$

Clearly our solutions are linear in this case:

$$y = a_1 x + a_2.$$

Now substituting this into our boundary conditions gives:

$$y(0) - y'(0) = a_2 - a_1 = 0$$

$$y(1) + y'(1) = a_1 + 2a_2 = 0$$

which implies that $a_1 = a_2 = 0$, so only trivial solutions remain.

So when $\lambda > 0$:

$$y = b_1 \sin \lambda^{1/2} x + b_2 \cos \lambda^{1/2} x$$
$$y' = \lambda^{1/2}\big(b_1 \cos \lambda^{1/2} x - b_2 \sin \lambda^{1/2} x\big)$$

and substituting this gives:

$$y(0) - y'(0) = b_2 - \lambda^{1/2} b_1 = 0.$$

$$\therefore b_2 = \lambda^{1/2} b_1$$

Our solution can then be written as:

$$y = b_1\big(\sin \lambda^{1/2} x + \lambda^{1/2} \cos \lambda^{1/2} x\big)$$

and its derivative:

$$y' = b_1\big(\lambda^{1/2} \cos \lambda^{1/2} x - \lambda \sin \lambda^{1/2} x\big).$$

The other boundary condition gives:

$$\begin{aligned} y(1) + y'(1) &= b_1\big(\sin \lambda^{1/2} + \lambda^{1/2} \cos \lambda^{1/2}\big) + b_1\big(\lambda^{1/2} \cos \lambda^{1/2} - \lambda \sin \lambda^{1/2}\big) \\ &= b_1\Big((1-\lambda)\sin \lambda^{1/2} + 2\lambda^{1/2} \cos \lambda^{1/2}\Big) \\ &= 0 \end{aligned}$$

which implies the following relation:

$$\tan \lambda_n^{1/2} = \frac{2\lambda_n^{1/2}}{\lambda_n - 1}$$

with its corresponding eigenfunction:

$$\phi_n(x) = k_n\big(\sin \lambda_n^{1/2} x + \lambda_n^{1/2} \cos \lambda_n^{1/2} x\big).$$

Now consider solutions when $\lambda < 0$:

$$y = c_1 \cosh \lambda^{1/2} x + c_2 \sinh \lambda^{1/2} x.$$

Taking derivatives gives the following:

$$y' = \lambda^{1/2}\big(c_1 \sinh \lambda^{1/2} x + c_2 \cosh \lambda^{1/2} x\big).$$

Now the first boundary condition gives:

$$y(0) - y'(0) = c_1 - \lambda^{1/2} c_2 = 0$$

or that

$$c_1 = \lambda^{1/2} c_2.$$

Rewriting our solutions:

$$y = c_2\big(\lambda^{1/2} \cosh \lambda^{1/2} x + \sinh \lambda^{1/2} x\big)$$

$$y' = c_2\big(\lambda \sinh \lambda^{1/2} x + \lambda^{1/2} \cosh \lambda^{1/2} x\big).$$

Using the second boundary condition:

$$\begin{aligned} y(1) + y'(1) &= c_2\big(\lambda^{1/2} \cosh \lambda^{1/2} + \sinh \lambda^{1/2}\big) + c_2\big(\lambda \sinh \lambda^{1/2} + \lambda^{1/2} \cosh \lambda^{1/2}\big) \\ &= c_2\big(2\lambda^{1/2} \cosh \lambda^{1/2} + (\lambda + 1)\sinh \lambda^{1/2}\big) \\ &= 0 \end{aligned}$$

and so we end up with the following relation:

$$\tanh \lambda_n^{1/2} = -\frac{2\lambda_n^{1/2}}{\lambda_n + 1}$$

with eigenfunction:

$$y = c_2\big(\lambda_n^{1/2} \cosh \lambda_n^{1/2} x + \sinh \lambda_n^{1/2} x\big).$$

For part $(b)(i)$ we examine integrating factors. Define:

$$\mu(x) = \frac{1}{P(x)} \exp\left[\int_{x_0}^{x} \frac{Q(s)}{P(s)} \mathrm{d}s\right]$$

to be the integrating factor to the following ODE:

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

Because by construction we have that:

$$\left[\mu P y'\right]' = \mu P y'' + \mu Q y'$$

therefore multiplying our original ODE yields the following:

$$\left[\mu P y'\right]' + \mu R y = 0.$$

We now consider some examples of this method. For part $(b)(ii)(1)$:

$$y'' - 2xy' + \lambda y = 0$$

may be converted into the following form:

$$\left[\exp\left(-x^2\right)y'\right]' + \lambda \exp\left(-x^2\right)y = 0.$$

For part $(b)(ii)(2)$:

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

gives:

$$\left[(x^2 - v^2)^{1/2}y'\right]' + \frac{(x^2 - v^2)^{3/2}}{x^2}y = 0.$$

Finally for part $(b)(ii)(3)$:

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0$$

we have that

$$\left[(1 - x^2)^{1/2}y'\right]' + \alpha^2(1 - x^2)^{-1/2}y = 0.$$

For part (c) firstly find the <u>normalised</u> eigenfunctions to:

$$y'' + \lambda y = 0$$

with boundary conditions $y'(0) = 0$ and

$$y(1) + y'(1) = 0.$$

This is of S-L form:

$$-y'' = \lambda y$$

with $r(x) = 1$. (This will be important for inner products.)

Now the eigenvalues with be real and we consider three cases.

If $\lambda = 0$:

$$y' = a_1 x + a_2$$

and using our boundary conditions only trival solutions remain.

If $\lambda > 0$:

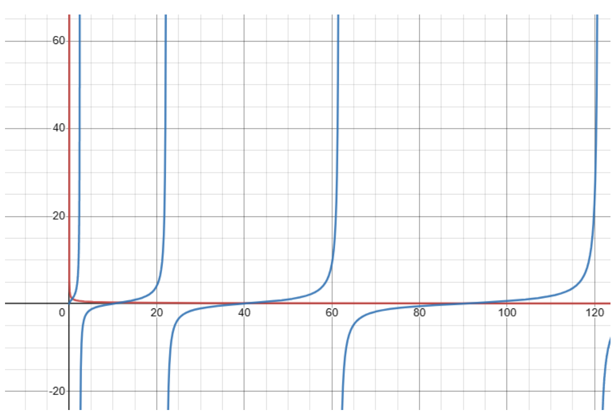
$$y = b_1 \sin \lambda^{1/2} x + b_2 \cos \lambda^{1/2} x$$
$$y' = \lambda^{1/2}\big(b_1 \cos \lambda^{1/2} x - b_2 \sin \lambda^{1/2} x\big)$$

which after solving for boundary conditions gives:

$$\lambda_n^{-1/2} = \tan \lambda^{1/2}$$

$$\phi_n(x) = k_n \cos \lambda^{1/2} x$$

which we can see have nontrivial eigenvalues.



Now if $\lambda < 0$ we have:

$$y = c_1 \cosh \lambda^{1/2} x + c_2 \sinh \lambda^{1/2} x.$$

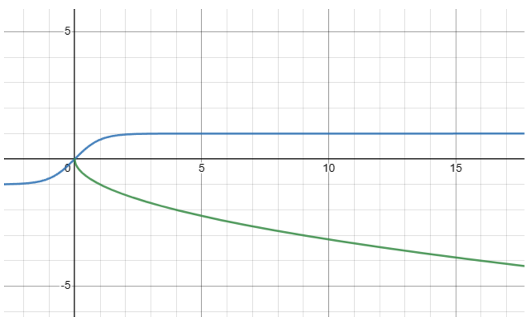
$$y' = \lambda^{1/2}\big(c_1 \sinh \lambda^{1/2} x + c_2 \cosh \lambda^{1/2} x\big).$$

and substituting for boundary conditions gives:

$$\tanh \lambda^{1/2} = -\lambda^{1/2}$$

$$\phi_n(x) = m_n \sinh \lambda^{1/2} x.$$

But a plot shows that only trivial eigenvalues exists for this case.



And therefore we conclude that the only nontrivial solutions to our S-L problem is:

$$\lambda_n^{-1/2} = \tan \lambda^{1/2}$$

$$\phi_n(x) = k_n \cos \lambda^{1/2} x.$$

We now need to find the normalisation constant k_n:

$$\begin{aligned}\langle \phi_n, \phi_n \rangle &= \int_0^1 k_n^2 \cos^2 \lambda_n^{1/2} x \mathrm{d}x \\ &= \frac{1}{2} k_n^2 \left[1 + \frac{1}{2\lambda_n^{1/2}} \sin 2\lambda_n^{1/2}\right] \\ &= 1\end{aligned}$$

in Hilbert space $L^2([0,1],1)$. Therefore we have that:

$$k_n = \left(\frac{4\lambda_n^{1/2}}{2\lambda_n^{1/2} + \sin 2\lambda_n^{1/2}}\right)^{1/2}.$$

The question now asks us to find a series expansion for $f(x) = x$ in terms of our orthonormal eigenfunction basis. So we have that:

$$f_\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

where $f_\phi = x$ and it remains to compute the coefficients.

So integrating on both sides gives:

$$\begin{aligned}
\int_0^1 r(x)\phi_m(x)f(x)\mathrm{d}x &= \int_0^1 r(x)\phi_m(x)\sum_{n=1}^{\infty} c_n\phi_n(x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} c_n \int_0^1 r(x)\phi_m(x)\phi_n(x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} c_n \delta_{mn} \\
&= c_m
\end{aligned}$$

and since $r(x) = 1$ we obtain the following:

$$\begin{aligned}
c_n &= \int_0^1 \phi_n(x)f(x)\mathrm{d}x \\
&= \int_0^1 k_n y_n(x)f(x)\mathrm{d}x \\
&= k_n \int_0^1 y_n(x)f(x)\mathrm{d}x.
\end{aligned}$$

From the previous page we have:

$$\phi_n(x) = k_n \cos \lambda^{1/2} x$$

and so substituting everything in:

$$\begin{aligned}
c_n &= k_n \int_0^1 x \cos \lambda^{1/2} x \mathrm{d}x \\
&= k_n \left(\frac{1}{\lambda_n^{1/2}} \sin \lambda_n^{1/2} + \frac{1}{\lambda_n} \cos \lambda_n^{1/2} - \frac{1}{\lambda_n} \right)
\end{aligned}$$

where we have:

$$k_n = \left(\frac{4\lambda_n^{1/2}}{2\lambda_n^{1/2} + \sin 2\lambda_n^{1/2}} \right)^{1/2}.$$

2. Solve the following:

$$y'' + 2y = -x$$

with boundary conditions: $y(0) = 0$ and $y'(1) = 0$.

First consider the corresponding homogeneous S-L problem:

$$-y'' = \lambda y$$

with the same boundary conditions. If $\lambda \leq 0$ we only have trivial solutions. Hence assume $\lambda > 0$. After solving and substituting boundary conditions we get:

$$\phi_n(x) = k_n \sin \lambda_n^{1/2}$$
$$\lambda_n^{1/2} = (2n+1)\frac{\pi}{2}$$

with normalisation constant:

$$k_n = \sqrt{2}\left(1 - \frac{1}{2\lambda^{1/2}} \sin 2\lambda^{1/2}\right)^{-1/2}.$$

Rewriting our original ODE in S-L form:

$$-y'' = \mu r(x) y + f(x)$$

and hence this ODE has $\mu = 2$, $r(x) = 1$ and $f(x) = x$. So using our homogeneous eigenfunctions we can write the solution as a sum of our eigenfunctions:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

which after substituting into our ODE we get:

$$\sum_{n=1}^{\infty} \lambda_n \phi_n(x) = \sum_{n=1}^{\infty} \mu \phi_n(x) + x.$$

$$\therefore \sum_{n=1}^{\infty} b_n(\lambda_n - \mu)\phi_n(x) = x.$$

But we can also expand $f(x) = x$ in terms of our eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

and so

$$c_n = \int_0^1 \phi_n(x) f(x) \mathrm{d}x$$

with the equality:

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

The only thing let is to compute c_n:

$$\begin{aligned} c_n &= \int_0^1 k_n y_n(x) f(x) \mathrm{d}x \\ &= k_n \int_0^1 x \sin \lambda_n^{1/2} x \mathrm{d}x \\ &= k_n \left(-\frac{1}{\lambda_n^{1/2}} \cos \lambda_n^{1/2} + \frac{1}{\lambda_n} \sin \lambda_n^{1/2} \right) \end{aligned}$$

with

$$k_n = \sqrt{2} \left(1 - \frac{1}{2\lambda_n^{1/2}} \sin 2\lambda_n^{1/2} \right)^{-1/2}.$$

Then the solution to our ODE is

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

where

$$\begin{aligned} b_n &= \frac{k_n}{\lambda_n - 2} \left(-\frac{1}{\lambda_n^{1/2}} \cos \lambda_n^{1/2} + \frac{1}{\lambda_n} \sin \lambda_n^{1/2} \right) \\ k_n &= \sqrt{2} \left(1 - \frac{1}{2\lambda_n^{1/2}} \sin 2\lambda_n^{1/2} \right)^{-1/2} \end{aligned}$$

and

$$\begin{aligned} \phi_n(x) &= k_n \sin \lambda_n^{1/2} \\ \lambda_n^{1/2} &= (2n+1)\frac{\pi}{2}. \end{aligned}$$