Vector products

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab \cos\theta$$

$$\boldsymbol{a} \times \boldsymbol{b} = ab \sin\theta \hat{\boldsymbol{n}}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \text{ and } \boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

Suffix notation

1. A suffix that appears <u>twice</u> implies a summation.

2. Any suffix <u>cannot appear</u> **more than twice** in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$
$$= -\epsilon_{ijk} = -\epsilon_{ijk} = -\epsilon_{ijk}$$

and we have the following identities:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i = a_i \boldsymbol{e}_i$$

$$A\boldsymbol{x} = a_{ij} x_j \boldsymbol{e}_i \text{ for } m \times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\ldots]_j \delta_{jk} = [\ldots]_k$$

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk} \boldsymbol{e}_k$$

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk} a_j b_k \boldsymbol{e}_i$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk} \epsilon_{ijk} = 6.$$

Transformations

Let matrix L relate basis $\{\boldsymbol{e}_i\}$ to basis $\{\boldsymbol{e}'_i\}$ with rule:

$$\boldsymbol{e}'_i = \ell_{ij} \boldsymbol{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then $L^T L = LL^T = I$, and:

$$\ell_{ik} \ell_{jk} = \ell_{ki} \ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij} p_j \text{ for } \boldsymbol{p} = p_i \boldsymbol{e}_i = p'_i \boldsymbol{e}'_i.$$

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip} \ell_{jq} \ell_{kr} T_{pqr}$$

which relates frame S in $\{\boldsymbol{e}_i\}$ to frame S' in $\{\boldsymbol{e}'_i\}$ with rule $\boldsymbol{e}'_i = \ell_{ij} \boldsymbol{e}_j$, etc.

Properties of tensors:

1. The <u>addition</u> of two rank n tensors is also a rank n tensor.

2. The <u>multiplication</u> of a rank m tensor with a rank n tensor yields a rank $m + n$ tensor.

3. If $T_{ijk\ldots s}$ is a rank m tensor then $T_{iik\ldots s}$ is a rank $m - 2$ tensor.

4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$T'_{ij} = \ell_{ip} \ell_{jq} T_{pq} \implies T' = LTL^T$$
$$T'_{ji} = \ell_{jp} \ell_{iq} T_{pq}.$$

Symmetric tensors

T_{ij} is a <u>symmetric</u> tensor when $T_{ij} = T_{ji}$ in frame S. Then $T'_{ij} = T'_{ji}$ in frame S'.

Similarly T_{ij} is an <u>anti-symmetric</u> tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally any tensor can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S'. Let $b_i = T_{ij} a_j$ where a_j is a vector. If b_i <u>always</u> transforms as a vector then T_{ij} is a <u>rank 2 tensor</u>.

Generalising, let $R_{ijk\ldots r}$ be a rank m tensor and $T_{ijk\ldots s}$ a set of 3^n numbers where $n > m$. If $T_{ijk\ldots s} R_{ijk\ldots r}$ is a rank $n - m$ tensor then $T_{ijk\ldots s}$ is a rank n tensor.

Matrices

We define a $m \times n$ matrix A as $(A)_{ij} = a_{ij}$ where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- $\operatorname{Tr} A = a_{ii}$

- $(A^T)_{ij} = a_{ji}$

- $(AB)^T = B^T A^T$

- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= \epsilon_{lmn} a_{1l} a_{2m} a_{3n}$$
$$= \epsilon_{lmn} a_{l1} a_{m2} a_{n3}.$$

Furthermore:

$$\epsilon_{ijk} \det A = \epsilon_{lmn} a_{il} a_{jm} a_{kn}$$

$$\epsilon_{lmn} \det A = \epsilon_{ijk} a_{il} a_{jkm} a_{kn}$$

$$\det A = \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} a_{il} a_{jm} a_{kn}.$$

Properties of determinants:

1. Adding rows to each other <u>does not</u> change the determinant.

2. Interchanging two rows <u>changes determinant signs</u>.

3. $\det A = \det A^T$

4. $\det(AB) = \det A \cdot \det B$

These also apply to columns. Finally:

$$\epsilon_{ijk} \epsilon_{lmn} \det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting $A = I$ yields:

$$\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

Linear equations

Let $\boldsymbol{y} = A\boldsymbol{x}$. Then $x_i = A_{ij}^{-1} y_i$ with:

$$A_{ij}^{-1} = \frac{1}{2} \frac{1}{\det A} \epsilon_{imn} \epsilon_{jpq} a_{pm} a_{qn}$$
$$= \frac{1}{\det A} C_{ij}^T$$

where C is the cofactor matrix of A.

Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L) \ell_{ip} \ell_{jq} T_{pq}$$

where $(L)_{ij} = \ell_{ij}$ and $\det L = \pm 1$.

Pseudovectors are rank 1 pseudotensors.

Invariant tensors

Tensor T is <u>invariant</u> or isotropic if:

$$T_{ijk\ldots} = \ell_{i\alpha} \ell_{j\beta} \ell_{k\gamma} \cdots T_{\alpha\beta\gamma\ldots}$$

for every orthogonal matrix L.

- If a_{ij} is a rank 2 invariant tensor then $a_{ij} = \lambda \delta_{ij}$.

- The most general rank 3 invariant <u>pseudotensor</u> is $a_{ijk} = \lambda \epsilon_{ijk}$. There are <u>no</u> rank 3 invariant true tensors.

- Invariant true tensors can only be even ranked.

- Invariant pseudotensors can only be odd ranked.

Rotation tensors

The clockwise <u>rotation</u> of position vector $\boldsymbol{x}$ to $\boldsymbol{y}$ about unit vector $\hat{\boldsymbol{n}}$ is given by:

$$y_i = R_{ij}(\theta, \hat{\boldsymbol{n}})x_j$$

$$R_{ij}(\theta, \hat{\boldsymbol{n}}) = \delta_{ij}\cos\theta + (1 - \cos\theta)n_i n_j$$
$$- \epsilon_{ijk}n_k \sin\theta$$

and is the rotation tensor.

Reflections and inversions

The <u>reflection</u> of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ in plane with unit vector $\hat{\boldsymbol{n}}$ is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The <u>inversion</u> of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ is given by $\boldsymbol{y} = -\boldsymbol{x}$ and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

Projections

We define P to be a <u>parallel</u> projection operator to vector $\boldsymbol{u}$ if:

$$P\boldsymbol{u} = \boldsymbol{u} \text{ and } P\boldsymbol{v} = \boldsymbol{0}$$

where $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{0}$. Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define Q to be an <u>orthogonal</u> projection to vector $\boldsymbol{u}$ if:

$$Q\boldsymbol{u} = \boldsymbol{0} \text{ and } Q\boldsymbol{v} = \boldsymbol{v}.$$

Here $Q = I - P$.

Inertia tensors

Let $\boldsymbol{L}$ denote the angular momentum of a rigid body about the origin of mass m, volume V and density ρ at position $\boldsymbol{r}$ with velocity $\boldsymbol{v}$. Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2\delta_{ij} - x_i x_j)\mathrm{d}V$$

where $I_{ij}(O)$ is the inertia tensor about the origin. The <u>kinetic energy</u> of such a body is:

$$T = \frac{1}{2}I_{ij}\omega_i\omega_j = \frac{1}{2}\boldsymbol{L} \cdot \boldsymbol{\omega}.$$

Parallel axis theorem

Consider the same rigid body now with centre of mass G and let $\overrightarrow{OG} = \boldsymbol{R}$. Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2\delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\boldsymbol{r}')\mathrm{d}V'.$$

Diagonalisation

Let $\boldsymbol{L} = I_{ij}\omega_j$ where I_{ij} is a rank 2 tensor and let $\boldsymbol{L} = \lambda\boldsymbol{\omega}$. Then:

$$(I_{ij} - \lambda\delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda\delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for $P = \det I$, $Q = \frac{1}{2}[(\operatorname{tr} I)^2 - \operatorname{tr}(I^2)]$ and $R = \operatorname{tr} I$ given <u>tensor</u> I.

Real symmetric tensors

Let rank 2 real symmetric tensor T be diagonalisable with real eigenvalues $\lambda^{(i)}$ and orthonormal eigenvectors $\boldsymbol{\ell}^{(i)}$ where $i = 1, 2, 3$. Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that $\det L = +1$ which transforms frame $S \to S'$.

Then since $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq}$$

$$= \lambda^{(i)}\delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ij}.$$

Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

and is f expanded about $x = a$.

Trignometric expansions are in radians!

$$\therefore f(x + a) = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(x)a^n$$

$$= \exp\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x)$$

Then for three dimensions:

$$\phi(\boldsymbol{r} + \boldsymbol{a}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{a} \cdot \boldsymbol{\nabla}_r)^n\phi(\boldsymbol{r})$$

$$= \exp(\boldsymbol{a} \cdot \boldsymbol{\nabla}_r)\phi(\boldsymbol{r}).$$

Curvilinear coordinates

Let x_i denote Cartesian coordinates and u_i denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \to (u_1, u_2, u_3)$$

where each $u_i = u_i(x_1, x_2, x_3)$ and:

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$$
$$= u_1\boldsymbol{e}_{u_1} + u_2\boldsymbol{e}_{u_2} + u_3\boldsymbol{e}_{u_3}.$$

Scale factors

Let $u_1 \to u_1 + \mathrm{d}u_1$ in $\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, u_3)$. Then $\mathrm{d}\boldsymbol{r}$ in $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 := h_1\boldsymbol{e}_1\mathrm{d}u_1.$$

h_1 is the scale factor of unit vector $\boldsymbol{e}_1$:

$$h_1 = \left|\frac{\partial \boldsymbol{r}}{\partial u_1}\right| \text{ and } \boldsymbol{e}_1 = \frac{1}{h_1}\frac{\partial \boldsymbol{r}}{\partial u_1}.$$

If $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ then u_i is an **orthogonal** curvilinear coordinate system.

Vector and arc length

The <u>vector length</u> $\mathrm{d}\boldsymbol{r}$ of $\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3} h_i\mathrm{d}u_i\boldsymbol{e}_i$$

where $u_i \to u_i + \mathrm{d}u_i$ for $\forall i = 1, 2, 3$.

Then the <u>arc length</u> $\mathrm{d}s$ is defined as:

$$(\mathrm{d}s)^2 = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r}$$
$$= g_{ij}\mathrm{d}u_i\mathrm{d}u_j$$

where g_{ij} is the metric tensor:

$$g_{ij} = g_{ji} = \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j}$$
$$= h_i h_j(\boldsymbol{e}_i \cdot \boldsymbol{e}_j).$$

Area and volume

Let $\mathrm{d}\boldsymbol{r}_i = h_i\boldsymbol{e}_i\mathrm{d}u_i$ denote vector length when $u_i \to u_i + \mathrm{d}u_i$. (**No sum!**)

The infinitesimal <u>vector area</u> or **surface element** formed by $\mathrm{d}\boldsymbol{r}_1$ and $\mathrm{d}\boldsymbol{r}_2$ is:

$$\mathrm{d}\boldsymbol{S} = (h_1\mathrm{d}u_1\boldsymbol{e}_1) \times (h_2\mathrm{d}u_2\boldsymbol{e}_2).$$

Similarly the infinitesimal <u>volume</u> formed by edges $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$ is:

$$\mathrm{d}V = |(\mathrm{d}\boldsymbol{r}_1 \times \mathrm{d}\boldsymbol{r}_2) \cdot \mathrm{d}\boldsymbol{r}_3|$$
$$= \sqrt{g}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3$$

where $g = \det(g_{ij})$.

Cylindrical coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$ where ρ represents the radial distance from the origin and ϕ is the anticlockwise rotation angle on the x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_1 + \rho\sin\phi\boldsymbol{e}_2 + z\boldsymbol{e}_3$$

$$h_\rho = 1, \quad \boldsymbol{e}_\rho = \cos\phi\boldsymbol{e}_1 + \sin\phi\boldsymbol{e}_2$$
$$h_\phi = \rho, \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$
$$h_z = 1, \quad \boldsymbol{e}_z = \boldsymbol{e}_3$$

and forms an orthogonal set.

Spherical coordinates

$(u_1, u_2, u_3) = (r, \theta, \phi)$ where θ represents the clockwise rotation angle in y-z plane and ϕ the anticlockwise rotation angle in x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = r\sin\theta\cos\phi\boldsymbol{e}_1 + r\sin\theta\sin\phi\boldsymbol{e}_2 + r\cos\theta\boldsymbol{e}_3$$

$$h_r = 1, \ h_\theta = r, \ h_\phi = r\sin\theta$$

$$\boldsymbol{e}_r = \sin\theta\cos\phi\boldsymbol{e}_1 + \sin\theta\sin\phi\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\boldsymbol{e}_1 + \cos\theta\sin\phi\boldsymbol{e}_2 - \sin\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

and also forms an orthogonal set.

Gradient

The gradient of a <u>scalar</u> field $f(\boldsymbol{r})$ is:

$$\mathrm{d}f(\boldsymbol{r}) := \boldsymbol{\nabla}f(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}$$

when $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r} \implies f \to f + \mathrm{d}f$. Taking the total differential of f yields:

$$\boldsymbol{\nabla}f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i$$

where $\{\boldsymbol{e}_i\}$ is <u>orthogonal</u>.

Divergence

The divergence of a <u>vector</u> field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} := \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

for surface δS bounds infinitesimal δV. In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left\{ \frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 F_2 h_3) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3) \right\}.$$

Curl

The curl of a <u>vector</u> field $\boldsymbol{F}$ in the direction of unit vector $\hat{\boldsymbol{n}}$ is:

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) := \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

where curve δC encloses plane δS. In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{vmatrix}.$$

Laplacian

The Laplacian of a <u>scalar</u> field f is:

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}f)$$

and in orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla}^2 f = \frac{1}{h_1 h_2 h_3}\left\{ \frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right) \right\}.$$

The Laplacian of a <u>vector</u> field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla}^2 \boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}).$$

Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\boldsymbol{\nabla}f = \frac{\partial f}{\partial x_i}\boldsymbol{e}_i$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_i}{\partial x_i}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}\boldsymbol{e}_i$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

The following operator acts on **both** scalar and vector fields:

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{F} = u_j \frac{\partial}{\partial x_j}F_i.$$

If ψ is a scalar field and $\boldsymbol{v}$ a vector field:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi) = \boldsymbol{0}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{0}$$

$$\boldsymbol{\nabla} \cdot (\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi \cdot \boldsymbol{v} + \psi\boldsymbol{\nabla} \cdot \boldsymbol{v}$$

$$\boldsymbol{\nabla} \times (\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi \times \boldsymbol{v} + \psi\boldsymbol{\nabla} \times \boldsymbol{v}.$$

Let $\boldsymbol{r} = x_i\boldsymbol{e}_i$ and $r = (x_i^2)^{1/2}$. Then:

- $\boldsymbol{\nabla}r = \dfrac{\boldsymbol{r}}{r}$ and $\boldsymbol{\nabla}\left(\dfrac{1}{r}\right) = -\dfrac{\boldsymbol{r}}{r^3}$

- $\boldsymbol{\nabla}r^n = nr^{n-2}\boldsymbol{r}$

- $\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3$ and $\boldsymbol{\nabla} \times \boldsymbol{r} = \boldsymbol{0}$

- $\boldsymbol{\nabla} \times (\boldsymbol{c} \times \boldsymbol{r}) = 2\boldsymbol{c}$

- $\boldsymbol{\nabla} \cdot (\boldsymbol{c} \times \boldsymbol{r}) = \boldsymbol{0}$ for constant $\boldsymbol{c}$.

Divergence theorem

Let surface S enclose volume V. Then:

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{F}\mathrm{d}V = \oint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

where $\boldsymbol{F}$ is a vector field.

Stokes' theorem

Let closed curve C bound <u>open</u> surface S and let $\boldsymbol{F}$ be a vector field. Then:

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

for C is traversed in anticlockwise sense.

Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x - a) := \begin{cases} \infty & x = a \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions this becomes:

$$\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0) := \delta(\boldsymbol{r} - \boldsymbol{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

where (x, y, z) are Cartesian coordinates.

In orthogonal curvilinear coordinates:

$$\delta(\boldsymbol{r} - \boldsymbol{a}) = \frac{1}{h_1 h_2 h_3}\delta(u_1 - a_1) \cdot \delta(u_2 - a_2)\delta(u_3 - a_3).$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x)\delta(x - a)\mathrm{d}x = f(a)$$

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

which yields:

$$x\delta(x) = 0 \text{ and } \delta(cx) = \frac{1}{|c|}\delta(x).$$

If simple solutions of $g(x) = 0$ are x_i:

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))\mathrm{d}x = \sum_i \frac{f(x_i)}{|g'(x_i)|}.$$

Coulomb's law

Consider charges q and q_1 at positions $\boldsymbol{r}$ and $\boldsymbol{r}_1$. The force <u>on</u> charge q at $\boldsymbol{r}$ due to charge q_1 at $\boldsymbol{r}_1$ is:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{qq_1(\boldsymbol{r} - \boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}$$

where $qq_1 > 0$ denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419 \times 10^{-12}\mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}.$$

Charge has units Coulombs (C) and an electron has charge $-1.60218 \times 10^{-19}\mathrm{C}$.

Electric fields

The electric field is induced by a charge distribution and defined in terms of the force on a small positive test charge q:

$$\boldsymbol{E}(\boldsymbol{r}) := \lim_{q \to 0} \frac{1}{q} \boldsymbol{F}.$$

Then for our two charges q and q_1:

$$\boldsymbol{F}_1(\boldsymbol{r}) = q \boldsymbol{E}_1(\boldsymbol{r})$$

where q_1 produces electric field $\boldsymbol{E}_1$.

$$\therefore \boldsymbol{E}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1(\boldsymbol{r} - \boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}$$

$$\therefore \phi_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\boldsymbol{r} - \boldsymbol{r}_1|}$$

Principle of superposition

For a set of charges q_i at position $\boldsymbol{r}_i$ the total electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i(\boldsymbol{r} - \boldsymbol{r}_i)}{|\boldsymbol{r} - \boldsymbol{r}_i|^3}.$$

For object with **charge density** $\rho(\boldsymbol{r}')$ its overall electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}V'$$

where $\rho(\boldsymbol{r}')$ is charge divided by volume. The **type** of integral (surface or line) is dependent on the object in consideration.

Electrostatic Maxwell's equations

Because $\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V'\right)$$

and therefore for all <u>static</u> electric fields:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}.$$

$\boldsymbol{E}$ is a **conservative** vector field where its line integral is independent of path. Furthermore it may be written as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$$

for $\phi(\boldsymbol{r})$ is the potential of $\boldsymbol{E}$.

$$\therefore \phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V'$$
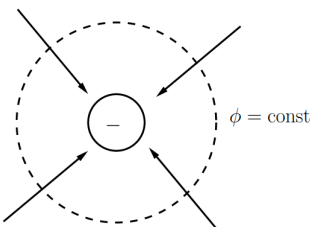
The **potential difference** between two points A and B is the energy per unit charge needed to move a small charge q from A to B:

$$\begin{aligned} V_{A \to B} &= \lim_{q \to 0} \frac{1}{q} W_{A \to B} \\ &= -\frac{1}{q} \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \\ &= \phi_B - \phi_A. \end{aligned}$$

A charge distribution $\rho(\boldsymbol{r}')$ in an <u>external</u> electric field has potential energy:

$$W = \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}') \mathrm{d}V'.$$

Because $\boldsymbol{\nabla}^2\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}')$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

Electric dipoles

An electric dipole at $\boldsymbol{r}_0$ is defined as two charges $-q$ at $\boldsymbol{r}_0$ and $+q$ at $\boldsymbol{r}_0 + \boldsymbol{d}$ which generates **dipole moment**:

$$\boldsymbol{p} = q\boldsymbol{d}$$

and in the <u>dipole limit</u> this is defined as:

$$\boldsymbol{p} := \lim_{\substack{q \to \infty \\ \boldsymbol{d} \to \boldsymbol{0}}} q\boldsymbol{d}$$

known as an ideal dipole.
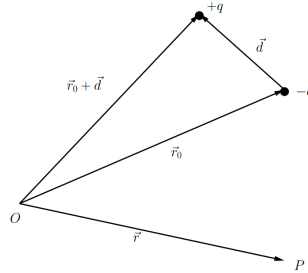
The electrostatic potential generated by this ideal dipole at $\boldsymbol{r}_0$ is given by:

$$\begin{aligned} \phi(\boldsymbol{r}) &= \phi_q + \phi_{-q} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0 - \boldsymbol{d}|} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|}\right) \\ &\approx \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}_0|^3} \end{aligned}$$

for the first term is expanded in powers of $-\boldsymbol{d}$ about $\boldsymbol{r} - \boldsymbol{r}_0$.

The electric field generated is:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\bigg[&-\frac{\boldsymbol{p}}{|\boldsymbol{r} - \boldsymbol{r}_0|^3} \\ &+ \frac{3\boldsymbol{p} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}_0|^5}(\boldsymbol{r} - \boldsymbol{r}_0)\bigg]. \end{aligned}$$

If the ideal dipole is at the origin:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^3}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\boldsymbol{p} \cdot \boldsymbol{r}}{r^5}\boldsymbol{r} - \frac{\boldsymbol{p}}{r^3}\right).$$

Let ideal dipole moment $\boldsymbol{p}$ be parallel to the z-axis. Then in spherical coordinates (r, θ, χ), $\boldsymbol{r} = r\boldsymbol{e}_r$, $\boldsymbol{p} = p\boldsymbol{e}_z$ and:

$$\phi(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0}\left(\frac{2\cos\theta}{r^3}\boldsymbol{e}_r + \frac{\sin\theta}{r^3}\boldsymbol{e}_\theta\right).$$

Force, torque and energy

The **force** <u>on</u> a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} \boldsymbol{F} &= -q\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{E}_{ext}(\boldsymbol{r} + \boldsymbol{d}) \\ &\approx (\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{E}_{ext}(\boldsymbol{r}). \end{aligned}$$

The **torque** <u>on</u> a dipole at $\boldsymbol{r}$ about the axis $\boldsymbol{r}$ due to $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} \boldsymbol{G} &= \boldsymbol{\tau}_{-q} + \boldsymbol{\tau}_q \\ &= -q\boldsymbol{0} \times \boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{d} \times \boldsymbol{E}_{ext}(\boldsymbol{r} + \boldsymbol{d}) \\ &\approx \boldsymbol{p} \times \boldsymbol{E}_{ext}(\boldsymbol{r}). \end{aligned}$$

The **energy** of a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} W &= -q\phi_{ext}(\boldsymbol{r}) + q\phi_{ext}(\boldsymbol{r} + \boldsymbol{d}) \\ &\approx -\boldsymbol{p} \cdot \boldsymbol{E}_{ext}(\boldsymbol{r}) \end{aligned}$$

and $\boldsymbol{F} = -\boldsymbol{\nabla}W$.

Multipole expansion

Consider object with volume V and charge distribution $\rho(\boldsymbol{r}')$. Let origin be in the object. Then the potential at $\boldsymbol{r}$ is:

$$\begin{aligned} \phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' \\ &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^3} + \frac{Q_{ij}x_i x_j}{2r^5}\right) \end{aligned}$$

where Q is the **total charge** in V:

$$Q = \int_V \rho(\boldsymbol{r}') \mathrm{d}V'$$

$\boldsymbol{p}$ the **dipole moment** about the origin:

$$\boldsymbol{p} = \int_V \boldsymbol{r}' \rho(\boldsymbol{r}') \mathrm{d}V'$$

and Q_{ij} the **quadrupole tensor**:

$$Q_{ij} = \int_V \rho(\boldsymbol{r}')\Big[3x_i' x_j' - (r')^2\delta_{ij}\Big] \mathrm{d}V'.$$

If $Q \neq 0$ then in the far zone ($r \gg r_0$) the first term (monopole term) dominates.

If $Q = 0$ and $\boldsymbol{p} = \boldsymbol{0}$ then the third term (quadruple term) dominates in the far zone and etc.

Interaction energy

By expanding $\phi_{ext}(\boldsymbol{r})$ about $\boldsymbol{r} = \boldsymbol{0}$:

$$
\begin{aligned}
W &= \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}')\mathrm{d}V' \\
&= Q\phi_{ext}(\boldsymbol{0}) - \boldsymbol{p} \cdot \boldsymbol{E}_{ext}(\boldsymbol{0}) \\
&\quad - \frac{1}{6}Q_{ij}\frac{\partial\big(\boldsymbol{E}_{ext}(\boldsymbol{0})\big)_i}{\partial x_j} + \dots
\end{aligned}
$$

and is the potential energy of a charge distribution $\rho(\boldsymbol{r})$ in $\boldsymbol{E}_{ext}$.

Gauss' law

For object with charge distribution $\rho(\boldsymbol{r}')$ and volume V enclosed by surface S:

$$
\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}
$$

where Q_{enc} is total charge enclosed by V:

$$
Q_{enc} = \int_V \rho(\boldsymbol{r}')\mathrm{d}V'
$$

and is useful for symmetric problems.

Boundaries in electrostatics

Let σ be the charge density of a surface separating electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$.

1. Normal component of electric field is <u>discontinuous</u> across surface by:

$$
\hat{\boldsymbol{n}} \cdot (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \frac{\sigma}{\epsilon_0}.
$$

2. Tangential component of electric field is continuous across surface:

$$
\boldsymbol{E}_\parallel := \hat{\boldsymbol{n}} \times \boldsymbol{E}_1 = \hat{\boldsymbol{n}} \times \boldsymbol{E}_2.
$$

Conductors

Conductors have surplus electrons that can move freely when an electric field is applied. **In electrostatics**:

1. Conductors are in equilibrium, all charges are at rest and reside on the <u>surface</u> of the conductor.

   Hence inside a conductor $\rho(\boldsymbol{r}) = 0$, $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}$ and $\phi = $ constant.

2. An electric field is always <u>normal</u> to the surface of a conductor:

$$
E_\perp = \frac{\sigma}{\epsilon_0} \text{ and } E_\parallel = 0.
$$

The presence of an external electric field induces a charge distribution σ on the surface of our conductor. This changes the external electric field as it needs to be normal to the surface of the conductor.

Poisson's equation

Because $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho(\boldsymbol{r})/\epsilon_0$:

$$
\boldsymbol{\nabla}^2\phi = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}.
$$

We can solve this by direct integration or using the **method of images**.

Given volume under consideration place fictitious charge <u>outside</u> the volume such that the system still satisfies Poisson's equation with boundary conditions.

This potential is our solution.

Electrostatic energy

The work needed to move point charge q from $\boldsymbol{r}_A$ to $\boldsymbol{r}_B$ in $\boldsymbol{E}(\boldsymbol{r})$ is:

$$
W_{A \to B} = qV_{A \to B}.
$$

Then $W_{\infty \to B} = q\phi(\boldsymbol{r}_B)$ since potential ϕ vanishes at infinity.

Generalising, the work needed to move a system of n charges q_i from infinity to $\boldsymbol{r}$ is a double sum with overcounting as each charge contributes to the electric field:

$$
W_e = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i,j(i \neq j)}^n \frac{q_i q_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}.
$$

Furthermore the energy needed to move a continuous charge distribution $\rho(\boldsymbol{r}')$ from infinity to position $\boldsymbol{r}$ is:

$$
\begin{aligned}
W_e &= \frac{1}{2}\int_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V \\
&= \frac{\epsilon_0}{2}\int_V |\boldsymbol{E}(\boldsymbol{r})|^2 \mathrm{d}V.
\end{aligned}
$$

Capacitors

A capacitor is formed by two conductors 1 and 2 with equal and opposite charges Q and $-Q$. The **capacitance** $(1\mathrm{CV}^{-1})$ of a capacitor is defined as:

$$
C := \frac{Q}{V}
$$

where $Q = \sigma A$ for A is the surface area of one conductor and potential difference $V = \phi_1 - \phi_2$ from the conductors.

The energy stored in a capacitor is the amount of work done to move charge across the two conductors. So to move charge $\mathrm{d}q$ from conductor with $+q$:

$$
\mathrm{d}W = \left(\frac{q}{C}\right)\mathrm{d}q
$$

and integrating this up to Q gives:

$$
W = \frac{1}{2}\frac{Q^2}{C}.
$$

Currents

An elementary current is generated by a charge q moving at velocity $\boldsymbol{v}$.

The **bulk current density** is:

$$
\boldsymbol{J}(\boldsymbol{r}) := \rho(\boldsymbol{r})\boldsymbol{v}
$$

for $\rho(\boldsymbol{r})$ is the volume charge density.

The **surface current density** is:

$$
\boldsymbol{K}(\boldsymbol{r}) := \sigma(\boldsymbol{r})\boldsymbol{v}
$$

for $\sigma(\boldsymbol{r})$ is the surface charge density.

The **line charge density** is:

$$
\boldsymbol{I}(\boldsymbol{r}) := \lambda(\boldsymbol{r})\boldsymbol{v}
$$

for $\lambda(\boldsymbol{r})$ is the line charge density.

The infinitesimal **current element** is:

$$
\mathrm{d}\boldsymbol{I}(\boldsymbol{r}) := \begin{cases} \boldsymbol{J}(\boldsymbol{r})\mathrm{d}V & \text{volume current} \\ \boldsymbol{K}(\boldsymbol{r})\mathrm{d}S & \text{surface current} \\ \boldsymbol{I}(\boldsymbol{r})\mathrm{d}r & \text{line current} \end{cases}
$$

Current I has units Ampères (A) but the infinitesimal current element $\mathrm{d}\boldsymbol{I}(\boldsymbol{r})$ has units of current/volume, etc.

Note: $1\mathrm{A} := 1\mathrm{Cs}^{-1}$

Consider volume V bounded by surface S with total charge Q. Because the total charge is conserved:

$$
\frac{\partial\rho(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}(\boldsymbol{r},t) = 0
$$

$$
Q = \int_V \rho(\boldsymbol{r},t)\mathrm{d}V.
$$

An electric field can induce a current in a conductor, via **Ohm's law**:

$$
\boldsymbol{J} \approx \sigma_{ij}E_j
$$

where σ_{ij} is the conductivity tensor.

1. Perfect conductors:
   $\sigma \to \infty$ and $\boldsymbol{E} = \boldsymbol{0}$.

2. Insulators: $\sigma = 0$.

The **electromotive force** (emf) is:

$$
\begin{aligned}
\mathcal{E}_{1 \to 2} &:= \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} \\
&= \phi(\boldsymbol{r}_1) - \phi(\boldsymbol{r}_2)
\end{aligned}
$$

since $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$. (static case)

Biot-Savart law

The magnetic field at $\boldsymbol{r}$ generated by a static current loop carrying current I is:
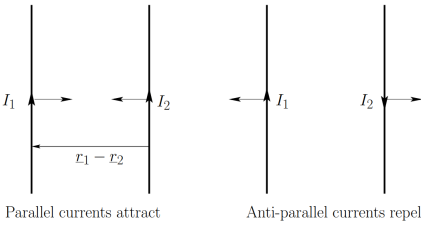
$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

and is the right grip rule.

electric current
conductor
magnetic field lines

Magnetic fields have units Teslas (T), where $\mathrm{NC^{-1}m^{-1}s = T}$ and:

$$\mu_0 = 1.25664 \cdots \times 10^{-6} \mathrm{NA^{-2}}.$$

I_1 I_2 I_1 I_2

$r_1 - r_2$

Parallel currents attract   Anti-parallel currents repel

No force is induced from perpendicular currents.

Lozentz force

Through physical experiments the **force density** on a charge distribution $\rho(\boldsymbol{r})$ in electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ is:

$$\boldsymbol{f} := \rho\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}$$

where $\boldsymbol{J}$ is the **current density** and integrating yields the **right hand rule**.

Current is the movement of charge.

For point charge q at $\boldsymbol{r}'$ with velocity $\boldsymbol{v}$ in $\boldsymbol{E}$ and $\boldsymbol{B}$ its net force is the integral of the force density over volume V:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

since $\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r} - \boldsymbol{r}')$ and $\boldsymbol{J} = \rho(\boldsymbol{r})\boldsymbol{v}$.

Magnetostatic Maxwell's equations

Because $\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$:

$$\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi} \int_V \boldsymbol{J}(\boldsymbol{r}') \times \boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) \mathrm{d}V'$$

therefore $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{0}$ due to:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{v} - (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \boldsymbol{u}.$$

This implies that magnetic fields always form **closed loops** — there are no sources or sinks for magnetic fields.

Similarly using the following identity:

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{v}(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$

and $\boldsymbol{\nabla}^2\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}')$:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0\boldsymbol{J}.$$

Ampère's law

Using the divergence theorem, there is no magnetic flux through a **closed** surface:

$$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0.$$

From Stokes' theorem:

$$\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \mu_0 I$$

which implies that the circulation of a magnetic field $\boldsymbol{B}$ around a closed loop C is proportional to the <u>total</u> current I that passes through the enclosed surface.

B
S
J
C
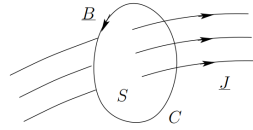
Boundaries in magnetostatics

Let conductor with current density $\boldsymbol{K}$ separate magnetic fields $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.

1. Normal component of magnetic field is continuous across surface:

$$\boldsymbol{B}_\perp := \boldsymbol{B}_1 \cdot \hat{\boldsymbol{n}} = \boldsymbol{B}_2 \cdot \hat{\boldsymbol{n}}.$$

2. Tangential component of magnetic field is discontinuous across surface:

$$\hat{\boldsymbol{n}} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \mu_0\boldsymbol{K}.$$

Magnetic vector potentials

Because $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$:

$$\exists \boldsymbol{A} : \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

known as the **vector potential**. Since $\boldsymbol{A}$ is invariant under transformation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$$

and is the Coulomb gauge. Therefore:

$$\boldsymbol{\nabla}^2\boldsymbol{A} = -\mu_0\boldsymbol{J}$$

with solutions:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V'$$

and boundary condition $\lim_{r\to\infty} \boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{0}$.
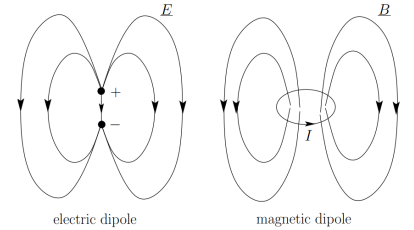
Magnetic dipoles

The vector potential for a current loop positioned at the origin in the far zone is:

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi} \oint_C \frac{I\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \\ &= \frac{\mu_0}{4\pi} \oint_C \frac{1}{r}\left(1 + \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{r^2} + \dots\right)\mathrm{d}\boldsymbol{r}' \\ &\approx \frac{\mu_0}{4\pi}\frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}\end{aligned}$$

where $\boldsymbol{m}$ is the magnetic dipole moment:

$$\begin{aligned}\boldsymbol{m} &= I\int_S \mathrm{d}\boldsymbol{S} \\ &= \frac{1}{2}\oint_C \boldsymbol{r} \times \mathrm{d}\boldsymbol{I}\end{aligned}$$

via Stokes' theorem.

E B

I

electric dipole   magnetic dipole

Force and torque

The Lorentz force on a current loop due to an external magnetic field $\boldsymbol{B}$ is:

$$\begin{aligned}\boldsymbol{F} &= I\oint_C \mathrm{d}\boldsymbol{r}' \times \boldsymbol{B}(\boldsymbol{r}') \\ &= \boldsymbol{\nabla}(\boldsymbol{m} \cdot \boldsymbol{B}).\end{aligned}$$

The torque on a current loop due to an external magnetic field $\boldsymbol{B}$ is:

$$\begin{aligned}\boldsymbol{G} &= \oint_C \boldsymbol{r}' \times [I\mathrm{d}\boldsymbol{r}' \times \boldsymbol{B}(\boldsymbol{r}')] \\ &= \boldsymbol{m} \times \boldsymbol{B}.\end{aligned}$$

Motional electromotive force

The electromotive force (emf) is the work needed for unit point charge to circulate around a conductor loop:

$$\begin{aligned}\mathcal{E} &= \oint_C \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{r} \\ &= \oint_C (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{r}\end{aligned}$$

for $\boldsymbol{v}$ is the velocity of the charge and $\boldsymbol{f}$ is the force density on point charge.

Magnetic induction

Faraday's law of induction states that a **change** in magnetic flux Φ induces an emf in a conductor loop:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

where **magnetic flux** is defined as the total magnetic field passing through the region bounded by the conductor loop:

$$\Phi = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$
$$= \boldsymbol{B} \cdot \boldsymbol{S} \text{ if constant } \boldsymbol{B}$$

and any surface S enclosing the region.

For static charges in a conductor loop C with time dependent magnetic field:

$$\mathcal{E} = \oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}$$

and using Stokes' theorem:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

In words, a time dependent magnetic field **always accompanies** a spatial and time dependent electric field.

Galilean relativity

If the velocity of frame S' in S is $\boldsymbol{v}$:

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t$$
$$\mathrm{d}t' = \mathrm{d}t$$

for point P has position $\boldsymbol{r}$ in frame S and position $\boldsymbol{r}'$ in frame S'.

Let circuit C be in motion in frame S with velocity $\boldsymbol{v}$ with respect to $\boldsymbol{B}(\boldsymbol{r}, t)$. Then let C be stationary in frame S'. Since the electromotive force generated is the same regardless of frames, in frame S:

$$\mathcal{E} = \oint_C (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{r}$$

and as $\boldsymbol{v} = \boldsymbol{0}$ in frame S':

$$\mathcal{E} = \oint_C \boldsymbol{E}' \cdot \mathrm{d}\boldsymbol{r}.$$

Equating the two statements:

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$$

but only applies at $v \ll c$.

Mutual and self inductance

Consider conductor loops 1 and 2 with current I_1 and I_2. The magnetic vector potential generated by loop 1 is:

$$\boldsymbol{A}_1(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 \mathrm{d}\boldsymbol{r}_1'}{|\boldsymbol{r} - \boldsymbol{r}_1'|}$$

and the magnetic flux in loop 2 is:

$$\Phi_{2\leftarrow 1} = \frac{\mu_0 I_1}{4\pi} \oint_{C_2} \mathrm{d}\boldsymbol{r}_2' \cdot \oint_{C_1} \frac{\mathrm{d}\boldsymbol{r}_1'}{|\boldsymbol{r}_2' - \boldsymbol{r}_1'|}$$
$$= M_{21} I_1$$

where M_{21} is the Neumann's formula for mutual induction. Then for two loops:

$$\begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix} = - \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}I_1}{\mathrm{d}t} \\ \frac{\mathrm{d}I_2}{\mathrm{d}t} \end{pmatrix}$$

where $M = M_{21} = M_{12}$. L_1 and L_2 are the self-inductance for each loop.

formula for self inductance

M_{21} only equal when we have equal areas

Magnetic field energy

Consider an inductor generating field $\boldsymbol{B}$ with current I and self-inductance L:

$$\mathrm{d}W_m = i\mathrm{d}t \cdot -\mathcal{E}$$
$$= i\mathrm{d}t \cdot \frac{\mathrm{d}\Phi_i}{\mathrm{d}t}$$
$$= iL\mathrm{d}i$$

and since $\Phi_{1\leftarrow 1} = IL$ the energy stored our inductor is an integral over current:

$$W_m = \int_0^I \mathrm{d}W_m = \frac{\Phi_{1\leftarrow 1}^2}{2L} \text{ or } \frac{1}{2}\Phi_{1\leftarrow 1}I.$$

Because $\Phi_{1\leftarrow 1} = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$:

$$W_m = \frac{1}{2} \oint_C \boldsymbol{A} \cdot I \mathrm{d}\boldsymbol{r}$$

and generalising this to volume integrals:

$$W_m = \frac{1}{2} \int_V \boldsymbol{J} \cdot \boldsymbol{A} \mathrm{d}V$$

where $\boldsymbol{J}$ is the current density. Since:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{v} - (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \boldsymbol{u}$$

and assuming that the field vanish at ∞:

$$W_m = \frac{1}{2\mu_0} \int_V B^2 \mathrm{d}V$$

with **magnetic field density**:

$$w_m = \frac{1}{2\mu_0} B^2.$$

Circuits

Consider an LRC circuit with alternating current source and electromotive force:

$$\mathcal{E}_S = V_S \cos(\phi + \omega t)$$
$$= \mathrm{Re}[V_0 e^{i\omega t}]$$

where $V_0 = V_S e^{i\phi}$. Equating emfs:

$$\mathcal{E}_S = L\frac{\mathrm{d}I}{\mathrm{d}t} + V_C(t) + IR$$
$$I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}$$

and due to linearity we assume that:

$$I(t) = \mathrm{Re}[I_0 e^{i\omega t}]$$
$$V_C(t) = \mathrm{Re}[V_{C_0} e^{i\omega t}].$$

After subsituting into our equations:

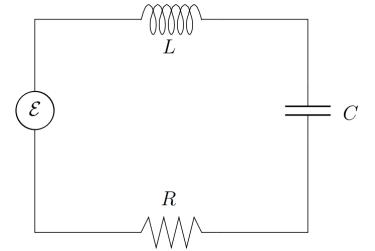$$V_0 = i\omega L I_0 + V_{C_0} + R I_0$$
$$I_0 = i\omega C V_0 \text{ and } Z := \frac{V_0}{I_0}$$

which yields solutions:

$$I(t) = \mathrm{Re}\left[\frac{V_0}{Z} e^{i\omega t}\right]$$
$$= \frac{V_S \cos(\omega t + \phi - \psi)}{\left[R^2 + (\omega L - \frac{1}{\omega C})^2\right]^{1/2}}$$

and $\psi = \arctan\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right]$.

current/voltage in series/parallel circuits

voltage of specific components

Electromagnetic waves

From considering the capacitor paradox:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

Since in vacuum $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \boldsymbol{E})$$

and using $\boldsymbol{\nabla} \times \boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t}$:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{\nabla}^2 \boldsymbol{B} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\boldsymbol{B}$$
$$\implies \boldsymbol{\nabla}^2 \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}.$$

Similarly for our electric field $\boldsymbol{E}$:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \boldsymbol{\nabla}^2 \boldsymbol{E} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \boldsymbol{E}$$

$$\implies \boldsymbol{\nabla}^2 \boldsymbol{E} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}.$$

Let F be any component of fields $\boldsymbol{E}(\boldsymbol{r}, t)$ or $\boldsymbol{B}(\boldsymbol{r}, t)$. Then it satisfies the following:

$$\boldsymbol{\nabla}^2 F = \mu_0 \epsilon_0 \frac{\partial^2 F}{\partial t^2}.$$

Substituting solutions of form:

$$F(t, x, y, z) = f(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$$
$$\implies |\boldsymbol{k}|^2 = \mu_0 \epsilon_0 \omega^2$$

and with phase velocity:

$$v_{phase} = \frac{\omega}{|\boldsymbol{k}|} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$
$$= 3.10 \times 10^8 \mathrm{ms}^{-1} = c$$

which is the speed of light and implies that light is also an electromagnetic wave!

Lorentz transformations

It is then postulated that:

- The **speed of light is universal**. c is frame invariant and classically only propagate <u>forwards</u> in time.
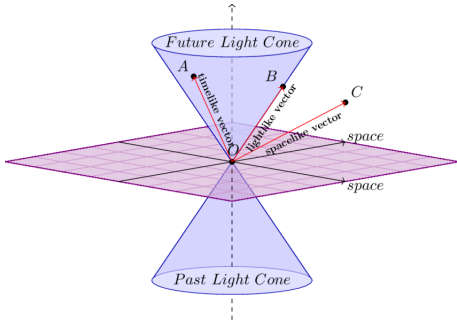
In Minkowsky spacetime we have that:

$$(\Delta S)^2 = (\Delta S')^2$$

between two frames S, S' and:

$$(\Delta S)^2 := (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

for events A and B, $\Delta x = x_B - x_A$, etc.

- $(\Delta S)^2 = 0$: light-like separated
- $(\Delta S)^2 > 0$: time-like separated
- $(\Delta S)^2 < 0$: space-like separated

A **boost** B_x in the $\boldsymbol{e}_x$ direction is defined:

$$\boldsymbol{x}' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \boldsymbol{x}$$

where $\boldsymbol{x} = (ct, x, y, z)^T$.

Generally we can relate reference frames S and S' by a composition of rotations and boosts. This forms a group, denoted by $\mathrm{SO}(3) = \{R_x, R_y, R_z, B_x, B_y, B_z\}$.

Practically if frame S' is moving at $v\boldsymbol{e}_x$ with respect to frame S then:

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(-\beta ct + x)$$
$$y' = y \text{ and } z' = z$$

where $\gamma = (1 - \beta^2)^{1/2}$ and $\beta = v/c$.

Time and length

Consider object in rest in frame S', which is moving with respect to frame S. Then:

- lifetime in S: $\gamma\tau$
- length in S: ℓ_0/γ

for time τ and length ℓ_0 are its physical quantities in frame S'.

Electromagnetic energy

By considering the Lorentz force with:

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \boldsymbol{F} \cdot \boldsymbol{v}\mathrm{d}t$$

and generalising to a charge distribution:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_V \boldsymbol{J} \cdot \boldsymbol{E}\,\mathrm{d}^3 r.$$

Using Maxwell's equations:

$$\frac{\mathrm{d}W}{\mathrm{d}t} + \int_{\partial V} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} + \frac{\mathrm{d}U_{em}}{\mathrm{d}t} = 0$$

$$U_{em} = \int_V \left(\frac{1}{\mu_0}|\boldsymbol{B}|^2 + \epsilon_0|\boldsymbol{E}|^2 \right) \mathrm{d}^3 r$$

where we define the **Poynting vector**:

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}.$$

If $\displaystyle\int_{\partial V} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{a} < 0$ then power flows **into** the volume bounded by a surface.