EM S1 Handins

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Contents

Handin 1

1. Consider set T_{ijk} with 3^3 elements, satisfying:

$$v_i = T_{ijk} R_{jk}$$

where v_i is a vector and R_{jk} a rank 2 tensor.

Show that T_{ijk} is a rank 3 tensor.

Proof. Direct proof.

Firstly define T'_{ijk} in frame S' and T_{ijk} in frame S, where our frames are related by $\boldsymbol{e}'_i = \ell_{ij} \boldsymbol{e}_j$. We then have that:

$$\begin{aligned} v'_i &= T'_{ijk} R'_{jk} \\ &= T'_{ijk} \ell_{jl} \ell_{km} R_{lm} \end{aligned}$$

and

$$\begin{aligned} v'_i &= \ell_{ij} v_j \\ &= \ell_{ij} T_{jkl} R_{kl}. \end{aligned}$$

$$\therefore T'_{ijk} \ell_{jl} \ell_{km} R_{lm} = \ell_{ij} T_{jkl} R_{kl}$$

Using the fact that R_{lm} is a tensor, we multiply both sides by vector a_m:

$$T'_{ijk} \ell_{jl} \ell_{km} R_{lm} a_m = \ell_{ij} T_{jkl} R_{kl} a_m.$$

The left hand side:

$$\begin{aligned} T'_{ijk} \ell_{jl} \ell_{km} R_{lm} a_m &= T'_{ijk} \ell_{jl} \ell_{km} a_l \\ &= T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} a_k. \end{aligned}$$

The right hand side:

$$\begin{aligned} \ell_{ij} T_{jkl} R_{kl} a_m &= \ell_{ij} T_{jkl} R_{kl} a_l \delta_{lm} \\ &= \ell_{ij} T_{jkl} \delta_{lm} a_k. \end{aligned}$$

Since equality still holds:

$$T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} a_k = \ell_{ij} T_{jkl} \delta_{lm} a_k.$$

$$\therefore \left(T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} - \ell_{ij} T_{jkl} \delta_{lm} \right) a_k = 0$$

Because a_k is a vector that is not always zero:

$$T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} = \ell_{ij} T_{jkl} \delta_{lm}.$$

$$\therefore T'_{ijk} \ell_{jk} \ell_{km} = \ell_{ij} T_{jkm}$$

Then multiply both sides by ℓ_{in}:

$$T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = \ell_{ij}\ell_{in}T_{jkm}.$$

$$\therefore T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = \delta_{jn}T_{jkm}$$

$$\therefore T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = T_{nkm}$$

$$\therefore T'_{ijk}\ell_{in}\ell_{jk}\ell_{km} = T_{nkm}$$

And this is by definition a third rank tensor. □

2. Define
$$\mathrm{d}\boldsymbol{r}_i = \frac{\partial \boldsymbol{r}}{\partial u_i}\mathrm{d}u_i$$
and the volume of the infinitesimal parallelepiped with $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$:
$$\mathrm{d}V = |\mathrm{d}\boldsymbol{r}_1 \cdot (\mathrm{d}\boldsymbol{r}_2 \times \mathrm{d}\boldsymbol{r}_3)|.$$
For part (i) show that:
$$\mathrm{d}V = |J|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$$
for $J = \det M$ where:
$$M_{ij} = \frac{\partial x_i}{\partial u_j}.$$

Using our definition of $\mathrm{d}\boldsymbol{r}_i$:
$$\begin{aligned}
\mathrm{d}V &= |\mathrm{d}\boldsymbol{r}_1 \cdot (\mathrm{d}\boldsymbol{r}_2 \times \mathrm{d}\boldsymbol{r}_3)| \\
&= |\frac{\partial \boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2}\mathrm{d}u_2 \times \frac{\partial \boldsymbol{r}}{\partial u_3}\mathrm{d}u_3)| \\
&= |\frac{\partial \boldsymbol{r}}{\partial u_1} \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2} \times \frac{\partial \boldsymbol{r}}{\partial u_3})|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3
\end{aligned}$$
Since $\boldsymbol{r} = x_i\boldsymbol{e}_i$:
$$\frac{\partial \boldsymbol{r}}{\partial u_j} = \frac{\partial x_i}{\partial u_j}\boldsymbol{e}_i.$$
Now the triple scalar product of three vectors is equivalent to the determinant of a matrix consisting of these three vectors, as either rows or columns. Therefore:
$$\begin{aligned}
|\frac{\partial \boldsymbol{r}}{\partial u_1} \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2} \times \frac{\partial \boldsymbol{r}}{\partial u_3})| &= \det\begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \end{bmatrix} \\
&= \det M \\
&= J.
\end{aligned}$$
Since volume is nonnegative:
$$\therefore \mathrm{d}V = |J|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3.$$

For part (ii) show:

- $(M^T M)_{ij} = g_{ij}$ for g_{ij} is the metric tensor.
- $\mathrm{d}V = \sqrt{g}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$ for $g_{ij} = (G)_{ij}$ and $g = \det G$.

By the definition of the metric tensor:

$$\begin{aligned} g_{ij} &= \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \\ &= M_{ki}M_{kj} \\ &= (M^T)_{ik}(M)_{kj} \\ &= (M^T M)_{ij}. \end{aligned}$$

Since $G = M^T M$ taking the determinants gives:

$$\begin{aligned} \det G &= \det\bigl(M^T M\bigr) \\ &= \det M^T \det M \\ &= (\det M)^2 \\ &= J^2. \end{aligned}$$

Then from part (i):

$$\therefore J = \pm\sqrt{g}$$

$$\begin{aligned} \therefore \mathrm{d}V &= |J|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 \\ &= \sqrt{g}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 \end{aligned}$$

For part (*iii*) show that given orthogonal curvilinear coordinates we have:

$$\mathrm{d}V = h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3$$

For OCCs, $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ and therefore:

$$\begin{aligned}
\mathrm{d}V &= |\mathrm{d}\boldsymbol{r}_1 \cdot (\mathrm{d}\boldsymbol{r}_2 \times \mathrm{d}\boldsymbol{r}_3)| \\
&= |\frac{\partial \boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2}\mathrm{d}u_2 \times \frac{\partial \boldsymbol{r}}{\partial u_3}\mathrm{d}u_3)| \\
&= |\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)|\, h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3 \\
&= |\boldsymbol{e}_1 \cdot \boldsymbol{e}_1|\, h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3 \\
&= h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3
\end{aligned}$$

where $h_i = |\frac{\partial \boldsymbol{r}}{\partial u_i}|$ and $\mathrm{d}u_i = \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial u_i}$.

3. For the first part of this problem we need to find the electric field $E(x)$ generated our rod of length $2a$ centered at $x = 0$ with total charge Q.

   The thin rod is also assumed to have <u>uniform</u> charge density of ρ.

   Then we can use the formula $F(x) = qE(x)$ to find force on our point charge q at $x = R$.

   By Coulomb's law:

$$\begin{aligned} E(x) &= \int_{-a}^{a} \frac{\rho}{4\pi\epsilon_0} \frac{x - x'}{(x - x')^3} \mathrm{d}x' \\ &= \frac{\rho}{4\pi\epsilon_0} \left[\frac{1}{x - x'} \right]_{x'=-a}^{x'=a} \\ &= \frac{\rho}{4\pi\epsilon_0} \frac{2a}{x^2 - a^2}. \end{aligned}$$

   Because we have a uniform charge density across $2a$:

$$\rho = \frac{Q}{2a}$$

   then the electric field generated by the thin rod becomes:

$$\begin{aligned} E(x) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{2a} \frac{2a}{x^2 - a^2} \\ &= \frac{Q}{4\pi\epsilon_0} \frac{1}{x^2 - a^2}. \end{aligned}$$

   The force on our point charge q at $x = R$ is then:

$$\begin{aligned} F(R) &= qE(R) \\ &= \frac{qQ}{4\pi\epsilon_0} \frac{1}{R^2 - a^2}. \end{aligned}$$

The force on charge q at $x = R$ by charge Q at $x_1 = 0$ is given by:

$$\begin{aligned} F(x) &= F(R) \\ &= \frac{qQ}{4\pi\epsilon_0}\frac{x - x_1}{|x - x_1|^3} \\ &= \frac{qQ}{4\pi\epsilon_0}\frac{R}{|R|^3} \\ &= \frac{qQ}{4\pi\epsilon_0}\frac{1}{R^2}. \end{aligned}$$

Comparing these two forces:

$$F_{rod} = \frac{qQ}{4\pi\epsilon_0}\frac{1}{R^2 - a^2}$$

$$F_{point} = \frac{qQ}{4\pi\epsilon_0}\frac{1}{R^2},$$

since

$$\frac{1}{R^2 - a^2} > \frac{1}{R^2}$$

therefore $F_{rod} > F_{point}$.

4. For part (i) show that:
$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

The Lie brackets are defined:

$$[x, y] = xy - yx$$

and so
$$[T_i, T_j] = T_iT_j - T_jT_i.$$

Because
$$(T_k)_{ij} = -i\epsilon_{ijk}$$

then we have:
$$(T_i)_{lk} = -i\epsilon_{lki}$$
$$(T_j)_{km} = -i\epsilon_{kmj}$$

and swapping order gives:

$$(T_j)_{lk} = -i\epsilon_{lkj}$$
$$(T_i)_{km} = -i\epsilon_{kmi}.$$

It is important to use more indices here:

$$\begin{aligned}\big(T_iT_j - T_jT_i\big)_{lm} &= (T_i)_{lk}(T_j)_{km} - (T_j)_{lk}(T_i)_{km} \\ &= -\epsilon_{lki}\epsilon_{kmj} + \epsilon_{lkj}\epsilon_{kmi} \\ &= -\epsilon_{ilk}\epsilon_{kmj} + \epsilon_{jlk}\epsilon_{kmi}.\end{aligned}$$

We first consider $-\epsilon_{ilk}\epsilon_{kmj}$. Because we have

$$\begin{aligned}\epsilon_{ijk}\epsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\ &= \delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm} \\ &= \epsilon_{ilk}\epsilon_{kmj}\end{aligned}$$

where we swap $j \to l$, $l \to m$ and $m \to j$:

$$\therefore -\epsilon_{ilk}\epsilon_{kmj} = -\big(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\big)$$

$$\therefore \epsilon_{jlk}\epsilon_{kmi} = \delta_{jm}\delta_{li} - \delta_{ij}\delta_{lm}$$

Then:

$$\begin{aligned}
\big(T_iT_j - T_jT_i\big)_{lm} &= -\epsilon_{ilk}\epsilon_{kmj} + \epsilon_{jlk}\epsilon_{kmi} \\
&= -\big(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\big) + \delta_{jm}\delta_{li} - \delta_{ij}\delta_{lm} \\
&= -\delta_{im}\delta_{lj} + \delta_{jm}\delta_{li}.
\end{aligned}$$

Due to the symmetry of the Kronecker delta:

$$\begin{aligned}
\big(T_iT_j - T_jT_i\big)_{lm} &= -\delta_{im}\delta_{lj} + \delta_{jm}\delta_{li} \\
&= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\
&= \epsilon_{ijk}\epsilon_{klm} \\
&= \epsilon_{ijk}\epsilon_{lmk} \\
&= -i\epsilon_{ijk} \cdot -i\epsilon_{lmk} \\
&= -i\epsilon_{ijk} \cdot (T_k)_{lm}.
\end{aligned}$$

$$\begin{aligned}
\therefore [T_i, T_j]_{lm} &= -i\epsilon_{ijk} \cdot (T_k)_{lm} \\
\therefore [T_i, T_j] &= -i\epsilon_{ijk} T_k
\end{aligned}$$

For part (ii) we want to show:

- T_i is hermitian
- $\text{Tr}(T_i T_j) = 2\delta_{ij}$

The definition of a hermitian matrix is as such:

$$H = (H^T)^*$$
$$H_{ij} = (H^T)^*_{ij}$$

where $*$ denotes the complex conjugate. So:

$$(T_i)_{jk} = -i\epsilon_{ijk}.$$

$$\begin{aligned} \therefore (T_i^T)_{jk} &= (T_i)_{kj} \\ &= -i\epsilon_{kji} \\ &= -i\epsilon_{ikj} \\ &= i\epsilon_{ijk} \end{aligned}$$

Then by the definition of the complex conjugate:

$$\therefore (T_i^T)^*_{jk} = -i\epsilon_{ijk}.$$

Since $(T_i)_{jk} = (T_i^T)^*_{jk}$ our matrix T_i is hermitian.

For the second part we firstly define:

$$(T_i)_{lk} = -i\epsilon_{lki}$$

and

$$(T_j)_{km} = -i\epsilon_{kmj}.$$

$$\begin{aligned} \therefore (T_i)_{lk}(T_j)_{km} &= (T_i T_j)_{lm} \\ &= -\epsilon_{lki}\epsilon_{kmj} \\ &= -\epsilon_{ilk}\epsilon_{kmj} \end{aligned}$$

Now:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\ &= \delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm} \\ &= \epsilon_{ilk}\epsilon_{kmj} \end{aligned}$$

if we swap $j \to l$, $l \to m$ and $m \to j$.

$$\begin{aligned}
\therefore (T_i T_j)_{lm} &= -\epsilon_{ilk}\epsilon_{kmj} \\
&= -\big(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\big)
\end{aligned}$$

Taking the trace of a matrix is summing up its diagonals:

$$\begin{aligned}
\mathrm{Tr}(T_i T_j) &= (T_i T_j)_{ll} \\
&= -\big(\delta_{il}\delta_{lj} - \delta_{ij}\delta_{ll}\big) \\
&= -\big(\delta_{ij} - 3\delta_{ij}\big) \\
&= 2\delta_{ij}.
\end{aligned}$$

Finally for part (*iii*) consider:

$$R(\alpha, \boldsymbol{n}) = \exp\bigl(-i\alpha \boldsymbol{n} \cdot \boldsymbol{T}\bigr)$$

where α is our rotation angle about <u>unit</u> axis vector $\boldsymbol{n}$.

$\boldsymbol{T}$ is a vector of generator matrices.

Our aims are:

- Show $\bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)^2_{ij} = \delta_{ij} - n_i n_j$.
- Show $\bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)^3_{ij} = \bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)_{ij}$.
- General formula for $\bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)^m_{ij}$ where $m > 3$.
- Expand $\exp\bigl(-i\alpha \boldsymbol{n} \cdot \boldsymbol{T}\bigr)$ as a power series.
- Recover standard rotation tensor of form:

$$R_{ij}(\alpha, \boldsymbol{n}) = \delta_{ij} \cos\alpha + n_i n_j \bigl(1 - \cos\alpha\bigr) - \epsilon_{ijk} n_k \sin\alpha.$$

Firstly define:

$$(\boldsymbol{n} \cdot \boldsymbol{T})_{ik} = n_\alpha (T_\alpha)_{ik} = n_\alpha \cdot -i\epsilon_{ik\alpha}$$
$$(\boldsymbol{n} \cdot \boldsymbol{T})_{kj} = n_\beta (T_\beta)_{kj} = n_\beta \cdot -i\epsilon_{kj\beta}$$

then we have that

$$\begin{aligned}(\boldsymbol{n} \cdot \boldsymbol{T})^2_{ij} &= n_\alpha (T_\alpha)_{ik} \cdot n_\beta (T_\beta)_{kj} \\ &= n_\alpha n_\beta \cdot -1 \cdot \epsilon_{ik\alpha} \epsilon_{kj\beta}.\end{aligned}$$

Using the standard identity we get:

$$\begin{aligned}\epsilon_{ijk}\epsilon_{klm} &= \epsilon_{jki}\epsilon_{klm} \\ &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\ &= \delta_{\alpha j}\delta_{i\beta} - \delta_{\alpha\beta}\delta_{ij} \\ &= \epsilon_{ik\alpha}\epsilon_{kj\beta}\end{aligned}$$

where $j \to i$, $i \to \alpha$, $l \to j$ and $m \to \beta$.

Then substituting back into our equation:

$$\begin{aligned}(\boldsymbol{n} \cdot \boldsymbol{T})^2_{ij} &= n_\alpha n_\beta \cdot -1 \cdot \epsilon_{ik\alpha}\epsilon_{kj\beta} \\ &= n_\alpha n_\beta \bigl(\delta_{\alpha\beta}\delta_{ij} - \delta_{\alpha j}\delta_{i\beta}\bigr) \\ &= \delta_{ij} - n_i n_j\end{aligned}$$

Now we show that $\big(\boldsymbol{n}\cdot\boldsymbol{T}\big)^3_{ij} = \big(\boldsymbol{n}\cdot\boldsymbol{T}\big)_{ij}$. Define:

$$(\boldsymbol{n}\cdot\boldsymbol{T})^2_{ik} = \delta_{ik} - n_i n_k$$

$$(\boldsymbol{n}\cdot\boldsymbol{T})_{kj} = n_l (T_l)_{kj} = n_l \cdot -i\epsilon_{kjl}$$

and so multiplying them together gives:

$$\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{T})^3_{ij} &= (\boldsymbol{n}\cdot\boldsymbol{T})^2_{ik}(\boldsymbol{n}\cdot\boldsymbol{T})_{kj} \\
&= (\delta_{ik} - n_i n_k) n_l \cdot -i\epsilon_{kjl} \\
&= -in_l\big(\epsilon_{ijl} - n_i n_k \epsilon_{kjl}\big) \\
&= -in_l\epsilon_{ijl} + in_l n_i n_k \epsilon_{kjl} \\
&= -in_l\epsilon_{ijl} + in_i \delta_{lk}\epsilon_{kjl} \\
&= -in_l\epsilon_{ijl} \\
&= (\boldsymbol{n}\cdot\boldsymbol{T})_{ij}.
\end{aligned}$$

The general formula for $\big(\boldsymbol{n}\cdot\boldsymbol{T}\big)^m_{ij}$ takes the form:

$$\big(\boldsymbol{n}\cdot\boldsymbol{T}\big)^m_{ij} = \begin{cases} \delta_{ij} - n_i n_j & m \text{ even} \\ \big(\boldsymbol{n}\cdot\boldsymbol{T}\big)_{ij} & m \text{ odd.} \end{cases}$$

The power series for an exponential is:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and so:

$$\begin{aligned}
\Big(\exp\big(-i\alpha\boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= \sum_{k=0}^{\infty}\Big(\frac{1}{k!}[-i\alpha]^k(\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij}\Big) \\
&= 1 + [-i\alpha](\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij} \\
&\quad + \sum_{k=2,4,\ldots} \frac{1}{k!}[-i\alpha]^k(\delta_{ij} - n_i n_j) \\
&\quad + \sum_{k=3,5,\ldots} \frac{1}{k!}[-i\alpha]^k(\boldsymbol{n}\cdot\boldsymbol{T})_{ij}.
\end{aligned}$$

Set $k = 2n$ for the first sum and $k = 2m+1$ for the second. Here $n, m \in \mathbb{N}$.

$$\begin{aligned}\therefore \Big(\exp\big(-i\alpha \boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= 1 + [-i\alpha](\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij} \\ &\quad + (\delta_{ij} - n_i n_j)\Big(\big(\sum_{n=0}^{\infty}(-1)^n \frac{\alpha^{2n}}{(2n)!}\big) - 1\Big) \\ &\quad + (\boldsymbol{n}\cdot\boldsymbol{T})_{ij}\cdot -i\Big(\big(\sum_{m=0}^{\infty}(-1)^m \frac{\alpha^{2m+1}}{(2m+1)!}\big) - \alpha\Big)\end{aligned}$$

Recognising this as the series expansion for cosine and sine:

$$\begin{aligned}\therefore \Big(\exp\big(-i\alpha \boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= 1 + [-i\alpha](\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij} \\ &\quad + (\delta_{ij} - n_i n_j)\Big(\cos\alpha - 1\Big) \\ &\quad + (\boldsymbol{n}\cdot\boldsymbol{T})_{ij}\cdot -i\Big(\sin\alpha - \alpha\Big)\end{aligned}$$

Since $(\boldsymbol{n}\cdot\boldsymbol{T})_{ij} = n_k(T_k)_{ij} = -in_k\epsilon_{ijk}$:

$$\begin{aligned}\Big(\exp\big(-i\alpha \boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= 1 - \alpha n_k\epsilon_{ijk} + (\delta_{ij} - n_i n_j)\Big(\cos\alpha - 1\Big) - n_k\epsilon_{ijk}\Big(\sin\alpha - \alpha\Big) \\ &= \delta_{ij}\cos\alpha + n_i n_j\Big(1 - \cos\alpha\Big) - \epsilon_{ijk} n_k \sin\alpha\end{aligned}$$

where we use the bogus argument:

$$\begin{aligned}1 - \delta_{ij} &= \frac{1}{3}\delta_{ii} - \frac{1}{3}\delta_{jj}\delta_{ij} \\ &= \frac{1}{3}\delta_{ii} - \frac{1}{3}\delta_{ii} \\ &= 0.\end{aligned}$$

Handin 2

1. 1

2. For part (i) a dipole at $\boldsymbol{r}_1$ with moment $\boldsymbol{p}_1$ generates an electric field:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}_1\cdot(\boldsymbol{r}-\boldsymbol{r}_1))(\boldsymbol{r}-\boldsymbol{r}_1)-|\boldsymbol{r}-\boldsymbol{r}_1|^2\boldsymbol{p}_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^5}$$

and so at point $\boldsymbol{r}_2$ we have:

$$\boldsymbol{E}(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}_1\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1))(\boldsymbol{r}_2-\boldsymbol{r}_1)-|\boldsymbol{r}_2-\boldsymbol{r}_1|^2\boldsymbol{p}_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^5}.$$

For part (ii) the force induced by our dipole at $\boldsymbol{r}_1$ with moment $\boldsymbol{p}_1$ on <u>another</u> dipole at $\boldsymbol{r}_2$ with moment $\boldsymbol{p}_2$ is:

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0}\Big[&-\frac{15}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^7}\big(\boldsymbol{p}_1\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\big)\big(\boldsymbol{p}_2\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\big)(\boldsymbol{r}_2-\boldsymbol{r}_1)\\ &+\frac{3}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^5}\Big((\boldsymbol{p}_1\cdot\boldsymbol{p}_2)(\boldsymbol{r}_2-\boldsymbol{r}_1)+\Big(\boldsymbol{p}_1\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\Big)\boldsymbol{p}_2\\ &+\Big(\boldsymbol{p}_2\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\Big)\boldsymbol{p}_1\Big)\Big].\end{aligned}$$

For part (iii)

3.