Electromagnetism and Relativity

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2.1 True tensors

tensor algebra

2.1.1 Rank 2 quotient theorem

The **quotient theorem** is as an alternative definition for tensors. In the context of rank 2 tensors it states that if b_i always transforms as a vector in

$$b_i = T_{ij} a_j$$

and that a_j is also a vector then T_{ij} is a rank 2 tensor.

Proof. We egregiously define entity T_{ij} in frame S and T'_{ij} in frame S'.

The usual transformation laws apply, namely $\boldsymbol{e}'_i = \ell_{ij} \boldsymbol{e}_j$. By definition:

$$\begin{aligned} b'_i &= T'_{ij} a'_j \\ &= T'_{ij} \ell_{jk} a_k \end{aligned}$$

Also directly from transformation laws:

$$\begin{aligned} b'_i &= \ell_{ij} b_j \\ &= \ell_{ij} T_{jk} a_k \end{aligned}$$

$$\therefore \left(T'_{ij} \ell_{jk} - \ell_{ij} T_{jk}\right) a_k = 0$$

Since a_k are constants of our vector it must then be that:

$$T'_{ij} \ell_{jk} = \ell_{ij} T_{jk}$$

$$\therefore T'_{ij} \ell_{jk} \ell_{mk} = \ell_{ij} \ell_{mk} T_{jk}$$

Where here we aim to eliminate the first two ℓs. Finally:

$$T'_{im} = \ell_{ij} \ell_{mk} T_{jk}$$

□

2.1.2 General quotient theorem

Let $R_{ij\ldots r}$ be a rank m tensor, and $T_{ij\ldots s}$ be a set of 3^n numbers where $n > m$.

If $R_{ij\ldots r} T_{ij\ldots s}$ is a rank $n - m$ tensor then $T_{ij\ldots s}$ is a rank n tensor.

symmetric and anti symmetric tensors

2.2 Matrices as tensors

2.3 Pseudotensors

Firstly note that $\det L = +1$ for rotations, and $\det L = -1$ for reflections and inversions. Recall the transformation law $\boldsymbol{e}_i' = \ell_{ij}\boldsymbol{e}_j$.

A second rank **pseudotensor** is defined:

$$T_{ij}' = (\det L)\ell_{ip}\ell_{jq}T_{pq}.$$

Furthermore a rank 1 pseudotensor is a **pseudovector** and is defined as:

$$T_i' = (\det L)\ell_{ip}T_p.$$

Finally a **pseudoscalar** is a rank 0 pseudotensor:

$$a' = (\det L) \cdot a,$$

and changes sign under transformation.

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active and passive transformations

maybe merge with rotations?

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4.2.2 Divergence theorem

Consider closed surface S with volume V. We have that:

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{E} \mathrm{d}V = \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}.$$

4.2.3 Stokes's theorem

Consider open surface S with line C as its boundary. We then have that:

$$\int_S \boldsymbol{\nabla} \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}.$$

5 Curvilinear coordinates

5.1 Orthogonal curvilinear coordinates

5.1.1 Scale factors and basis vectors

Consider change of variables:

$$(x_1, x_2, x_3) \leftrightarrow (u_1, u_2, u_3)$$

where u_i are our curvilinear coordinates, and

$$u_i = u_i(x_1, x_2, x_3)$$
$$x_i = x_i(u_1, u_2, u_3).$$

Then we define:

$$\begin{aligned} \mathrm{d}\boldsymbol{r}_i &= \frac{\partial \boldsymbol{r}}{\partial u_i}\mathrm{d}u_i \\ &= h_i \boldsymbol{e}_i \mathrm{d}u_i \end{aligned}$$

where $h_i = |\frac{\partial \boldsymbol{r}}{\partial u_i}|$ is our **scale factor** and

$$\boldsymbol{e}_i = \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial u_i}$$

is our **basis vector** of <u>unit length</u> for a specific set of curvilinear coordinates.

Now if the basis vectors satisfy

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

we have an <u>orthogonal</u> set of curvilinear coordinates.

5.1.2 Cylindrical coordinates

We define cylindrical coordinates as

$$(u_1, u_2, u_3) = (\rho, \phi, z)$$

and with the following relation to Cartesian coordinates:

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_x + \rho\sin\phi\boldsymbol{e}_y + z\boldsymbol{e}_z.$$

Furthermore:

$$h_\rho = 1 \quad \text{and} \quad \boldsymbol{e}_\rho = \cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y$$
$$h_\phi = \rho \quad \text{and} \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y$$
$$h_z = 1 \quad \text{and} \quad \boldsymbol{e}_z = \boldsymbol{e}_z.$$

Here ϕ is the <u>anticlockwise</u> rotation of the xy-plane.

5.1.3 Spherical coordinates

We define the spherical coordinates as

$$(u_1, u_2, u_3) = (r, \theta, \phi)$$

$$\boldsymbol{r} = r \sin\theta \cos\phi \boldsymbol{e}_x + r \sin\theta \sin\phi \boldsymbol{e}_y + r \cos\theta \boldsymbol{e}_z$$

where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ represent the Cartesian unit vectors.

Now $\phi \in [0, 2\pi]$ is the rotation angle in xy-plane, and $\theta \in [0, \pi]$ in z-plane. We also have that:

$$h_r = 1 \quad \text{and} \quad \boldsymbol{e}_r = \sin\theta \cos\phi \boldsymbol{e}_x + \sin\theta \sin\phi \boldsymbol{e}_y + \cos\theta \boldsymbol{e}_z$$

$$h_\theta = r \quad \text{and} \quad \boldsymbol{e}_\theta = \cos\theta \cos\phi \boldsymbol{e}_x + \cos\theta \sin\phi \boldsymbol{e}_y - \sin\theta \boldsymbol{e}_z$$

$$h_\phi = r \sin\theta \quad \text{and} \quad \boldsymbol{e}_\phi = -\sin\phi \boldsymbol{e}_x + \cos\phi \boldsymbol{e}_y.$$

We note that spherical coordinates are orthongonal:

$$\boldsymbol{e}_r \times \boldsymbol{e}_\theta = \boldsymbol{e}_\phi, \quad \boldsymbol{e}_\theta \times \boldsymbol{e}_\phi = \boldsymbol{e}_r \quad \text{and} \quad \boldsymbol{e}_\phi \times \boldsymbol{e}_r = \boldsymbol{e}_\theta.$$

5.2 Length, area and volume

5.2.1 Vector and arc length

Firstly the **vector length** due to infinitesimal change in all directions is

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3} h_i \mathrm{d}u_i \boldsymbol{e}_i.$$

It is important to note that summation notation does not work here.

Now the **arc length** of $\mathrm{d}\boldsymbol{r}$ is:

$$\begin{aligned}\mathrm{d}s &= |\mathrm{d}\boldsymbol{r}| \\ &= \sqrt{\mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r}}\end{aligned}$$

and we define the **metric tensor** as

$$\begin{aligned}g_{ij} &= \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \\ &= \frac{\partial \boldsymbol{r}}{\partial u_i} \cdot \frac{\partial \boldsymbol{r}}{\partial u_j}.\end{aligned}$$

Since $\mathrm{d}\boldsymbol{r} = \mathrm{d}x_k$ we then the following relation:

$$(\mathrm{d}s)^2 = g_{ij}\mathrm{d}u_i \mathrm{d}u_j.$$

5.2.2 Vector area

5.2.3 Volume

The volume of the infinitesimal parallelepiped defined by $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$ is:

$$\begin{aligned}\mathrm{d}V &= |(\mathrm{d}\boldsymbol{r}_1 \times \mathrm{d}\boldsymbol{r}_2) \cdot \mathrm{d}\boldsymbol{r}_3| \\ &= h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3\, |(\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot \boldsymbol{e}_3| \\ &= \sqrt{g}\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3\end{aligned}$$

where g is the determinant of the metric tensor.

5.3 Vector operators in OCCs

5.3.1 Gradient

Let $\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, u_3)$. The gradient of a scalar field in terms of this OCC is:

$$\boldsymbol{\nabla} f(\boldsymbol{r}) = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\boldsymbol{e}_3.$$

So let there be the following infinitesimal change:

$$u_1 \to u_1 + \mathrm{d}u_1, \quad u_2 \to u_2 + \mathrm{d}u_2, \quad u_3 \to u_3 + \mathrm{d}u_3$$

and consider the following:

$$\begin{aligned}
\mathrm{d}f(\boldsymbol{r}) &= \boldsymbol{\nabla} f(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} \\
&= \frac{\partial f}{\partial u_1}\mathrm{d}u_1 + \frac{\partial f}{\partial u_2}\mathrm{d}u_2 + \frac{\partial f}{\partial u_3}\mathrm{d}u_3 \\
&= \left[\frac{1}{h_1}\frac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\boldsymbol{e}_3\right] \cdot \left[h_1\boldsymbol{e}_1\mathrm{d}u_1 + h_2\boldsymbol{e}_2\mathrm{d}u_2 + h_3\boldsymbol{e}_3\mathrm{d}u_3\right] \\
&= \left[\frac{1}{h_1}\frac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\boldsymbol{e}_3\right] \cdot \mathrm{d}\boldsymbol{r}
\end{aligned}$$

and our claim follows from equating terms.

Here h_i are our scale factors and $\boldsymbol{e}_i$ our **orthogonal** basis vectors.

6 Electrostatics

6.1 Dirac delta function

The one dimensional **Dirac delta** is defined:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0, \end{cases}$$

and can be thought of as infinitely sharp at $x = 0$ and zero elsewhere.

It satisfies some useful properties:

- $\delta(x-a) = \lim_{\sigma \to 0} \left[\frac{1}{|\sigma|\sqrt{\pi}} \exp\left(-\frac{(x-a)^2}{\sigma^2} \right) \right]$

  i.e. an infinitely sharp Gaussian. (generalised functions)
- **Sift property**

$$\int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a)$$

- Let x_i be the solutions to $g(x_i) = 0$. Then:

$$\int_{\mathbb{R}} f(x)\delta[g(x)]dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

Now we consider the **3D Dirac delta**, which is defined as follows:

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

given Cartesian coordinates (x_1, x_2, x_3). It also satisfies the **sift** property:

$$\int_{\mathbb{R}^3} f(\boldsymbol{r})\delta(\boldsymbol{r} - \boldsymbol{r}_0) = f(\boldsymbol{r}_0).$$

The three dimensional Dirac delta defined in a orthogonal curvilinear coordinate system (u_1, u_2, u_3) is as follows:

$$\delta(\boldsymbol{r} - \boldsymbol{a}) = \frac{1}{h_1 h_2 h_3}\delta(u_1 - a_1)\delta(u_2 - a_2)\delta(u_3 - a_3)$$

for h_1, h_2 and h_3 are the scale factors.

6.2 Coulomb's law

Consider the force on charge q at $\boldsymbol{r}$ due to charge q_1 at $\boldsymbol{r}_1$:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{qq_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3},$$

for here $\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$ in vacuum.

Physically, like charges ($qq_1 > 0$) repel while opposite charges ($qq_1 < 0$) attract.

We then define an **electric field** as the force on a small positive test charge:

$$\boldsymbol{E}(\boldsymbol{r}) = \lim_{q\to 0}\left(\frac{1}{q}\boldsymbol{F}(\boldsymbol{r})\right).$$

The force on a charge q at $\boldsymbol{r}$ from the origin in this electric field is:

$$\boldsymbol{F}(\boldsymbol{r}) = q\boldsymbol{E}(\boldsymbol{r}).$$

A negative point charge is a sink whereas a positive point charge is a source.

Consider a collection of charges q_i at position $\boldsymbol{r}_i$. The **principle of superposition** tells us that:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i\left(\frac{q_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right).$$

Now consider a continuous charged object with volume V and **charge density** $\rho(\boldsymbol{r}')$. It generates the following electric field:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}dV'.$$

Returning to the electric field generated by a point charge q_1 at position $\boldsymbol{r}_1$:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^3},$$

this is a **conservative field**, and we may write it as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r}),$$

where:

$$\phi(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1|}.$$

Conservative fields have zero curl, and their line integrals are path independent. This namely applies to finding work done.

6.3 Electrostatic Maxwell's equations

6.3.1 Curl equation

For a continuous charge distribution:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= -\frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V' \\ &= -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right)\end{aligned}$$

and therefore $\boldsymbol{\nabla}\times\boldsymbol{E}=\boldsymbol{0}$ for <u>static electric fields</u>.

Hence electrostatic fields are <u>conversative fields</u>:

$$\int_{C_1}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r} = \int_{C_2}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r}$$

and we have a generalisation of the fundamental theorem of calculus:

$$-\int_a^b \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r} = \phi(b)-\phi(a)$$

where $\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$. Therefore our <u>potential</u> takes the expression:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'$$

and has units <u>Volts</u> V or JC^{-1}. We define the **potential difference**:

$$\begin{aligned}V_{A\to B} &= \phi_B-\phi_A \\ &= -\int_C \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r}\end{aligned}$$

and is the energy per unit charge to move <u>small</u> test charge from A to B:

$$V_{A\to B} = \lim_{q\to 0}\frac{1}{q}W_{A\to B} = -\frac{1}{q}\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$$

for here work is done <u>against</u> force.

Now consider charge q at $\boldsymbol{r}$ subject to <u>external</u> <u>electrostatic</u> field.

$$\therefore \boldsymbol{E}_{ext}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi_{ext}(\boldsymbol{r})$$

$$\therefore W_{ext} = \int_V \rho(\boldsymbol{r})\phi_{ext}(\boldsymbol{r})\mathrm{d}V$$

Note that W_{ext} is the <u>interaction energy</u>.

6.3.2 Divergence equation

Now consider:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \boldsymbol{\nabla}\cdot\left[-\boldsymbol{\nabla}\phi(\boldsymbol{r})\right] \\
&= -\boldsymbol{\nabla}^2\phi(\boldsymbol{r}) \\
&= -\boldsymbol{\nabla}^2\left(\frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right) \\
&= -\frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\mathrm{d}V'\left[\boldsymbol{\nabla}^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\right] \\
&= -\frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\mathrm{d}V'\left[-4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')\right] \\
&= \frac{\rho(\boldsymbol{r})}{\epsilon_0}
\end{aligned}$$

due to the <u>sift</u> and <u>symmetric</u> properties of the delta delta function.

Now previously we also used the following result:

$$\boldsymbol{\nabla}^2\left(\frac{1}{\boldsymbol{r}}\right) = -4\pi\delta(\boldsymbol{r}).$$

When $\boldsymbol{r}\neq\boldsymbol{0}$ we have that:

$$\begin{aligned}
\boldsymbol{\nabla}^2\left[\frac{1}{r}\right] &= \frac{\partial}{\partial x_i}\left[-\frac{x_i}{r^3}\right] \\
&= \left[-\frac{1}{r^3}\delta_{ii} - x_i\frac{3}{2}r^{-5}2x_i\right] \\
&= 0.
\end{aligned}$$

Now if $\boldsymbol{r}=\boldsymbol{0}$ consider the following volume integral of an ϵ sized sphere:

$$\begin{aligned}
\int_{V_\epsilon}\boldsymbol{\nabla}^2\left[\frac{1}{r}\right]\mathrm{d}V &= -\int_{V_\epsilon}\boldsymbol{\nabla}\cdot\left[\frac{\boldsymbol{r}}{r^3}\right]\mathrm{d}V \\
&= -\int_{S_\epsilon}\frac{\boldsymbol{r}}{r^3}\cdot\mathrm{d}\boldsymbol{S}
\end{aligned}$$

where in the final step we used the divergence theorem. On the surface of our sphere, $\boldsymbol{r}=\epsilon\boldsymbol{e}_r$ and since:

$$\begin{aligned}
\mathrm{d}\boldsymbol{S} &= \left(\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right)\mathrm{d}u\mathrm{d}v \\
&= r^2\sin\theta\boldsymbol{e}_r\mathrm{d}\theta\mathrm{d}\phi
\end{aligned}$$

evaluating our surface integral yields -4π. We then deduce the Dirac delta because at $\boldsymbol{r}=\boldsymbol{0}$ the charge is unbounded.

6.4 Electric dipoles

6.4.1 Potential and electric field

Dipoles consist of two equal and **opposite point charges** that are $\boldsymbol{d}$ apart.

An **ideal dipole** is defined as when the following **dipole limit** is finite and constant:

$$\boldsymbol{p} = \lim_{\substack{q\to\infty \\ \boldsymbol{d}\to 0}} q\boldsymbol{d}.$$

A **dipole moment** is simply $\boldsymbol{p} = q\boldsymbol{d}$. The **dipole potential** at $\boldsymbol{r}_0$ is:

$$\begin{aligned}\phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{d}|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3},\end{aligned}$$

where we have Taylor expanded the first term about $|\boldsymbol{r}-\boldsymbol{r}_0|$. For simplicity we set $\boldsymbol{r}_0 = \boldsymbol{0}$. Then the **electric field** generated by our dipole at the origin is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\boldsymbol{p}\cdot\boldsymbol{r}}{r^5}\boldsymbol{r} - \frac{1}{r^3}\boldsymbol{p}\right),$$

since $\boldsymbol{E} = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$. Note that these formulae are in Cartesian coordinates.

Now let our dipole with moment $\boldsymbol{p} = p\boldsymbol{e}_z$ be at:

$$\boldsymbol{r} = r\sin\theta\cos\phi\boldsymbol{e}_x + r\sin\theta\sin\phi\boldsymbol{e}_y + r\cos\theta\boldsymbol{e}_z.$$

Then in spherical coordinates (r,θ,χ) we have that:

$$\begin{aligned}\phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} \\ &= \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= -\boldsymbol{\nabla}\phi(\boldsymbol{r}) \\ &= -\left(\frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\chi}\boldsymbol{e}_\chi\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left[2\cos\theta\boldsymbol{e}_r + \sin\theta\boldsymbol{e}_\theta\right]\end{aligned}$$

where χ represents the anticlockwise rotation in the xy-plane.

6.4.2 Force, torque and energy

Consider a dipole at $\boldsymbol{r}$ with moment $\boldsymbol{p} = q\boldsymbol{d}$.

The force on this dipole due to an <u>external</u> electric field $\boldsymbol{E}_{ext}$ is:

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}) &= \boldsymbol{F}_{-q} + \boldsymbol{F}_{+q} \\ &= -q\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{E}_{ext}(\boldsymbol{r}+\boldsymbol{d})\end{aligned}$$

where we have $-q$ at $\boldsymbol{r}$ and $-q$ at $\boldsymbol{r}+\boldsymbol{d}$. Now in the dipole limit:

$$\boldsymbol{F}(\boldsymbol{r}) = (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E}_{ext}(\boldsymbol{r})$$

since $\boldsymbol{d} \to 0$ and we use the three dimensional Taylor expansion.

The <u>torque</u> on our dipole from external electric field is:

$$\boldsymbol{G}(\boldsymbol{r}) = \boldsymbol{p} \times \boldsymbol{E}_{ext}(\boldsymbol{r}).$$

The <u>interaction energy</u> is the following:

$$\begin{aligned}W &= -q\phi_{ext}(\boldsymbol{r}) + q\phi_{ext}(\boldsymbol{r}+\boldsymbol{d}) \\ &= -\boldsymbol{p}\cdot\boldsymbol{E}_{ext}(\boldsymbol{r})\end{aligned}$$

where we Taylor expand the second expression.

6.4.3 Multidipole expansion

potential

work done

6.5 Gauss's law

Gauss's law is the integral form of Maxwell's first equation:

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is the total charge enclosed by volume V. This result follows from the application of the divergence theorem and is useful in problems with symmetry.

6.5.1 Boundaries

Consider surface S with charge density σ separating electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. Firstly consider the normal component:

6.5.2 Conductors

special case for electrostatics

6.6 Poisson's equation

In electrostatics we have:

$$\boldsymbol{\nabla}^2\phi = \frac{\rho}{\epsilon_0}$$

where ρ is our charge density. This is the **Poisson's equation** and is a consequence of the fact that $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$.

6.6.1 Existence and uniqueness of solutions

The existence of solutions is given by the fact that:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi.$$

Poisson's equation has **unique** solution ϕ if we have volume V bounded by surface S and one of the following boundary conditions:

1.

6.6.2 Method of images

6.7 Electrostatic energy

6.8 Capacitors

6.8.1 Parallel plates

6.8.2 Concentric spheres

7 Magnetostatics

charge distribution $\implies$ electric field

current $\implies$ magnetic field

7.1 Currents

Elementary current

Bulk current density

Surface current density

Line current

units!

Infinitesimal current element (dependent on material)
units: $Cs^{-1}m = Am$

Note that $J = Am^{-2}$.

Current flowing through surface and line.