D: Functions

A function $f: X \to Y$ is an assignment of an element of Y to each element of X.

1. f is **injective** if:
$$\forall x_1, x_2 \in X; f(x_1) = f(x_2) \implies x_1 = x_2.$$
2. f is **surjective** if:
$$\forall y \in Y; \exists x \in X : y = f(x).$$
3. f is **bijective** if it is injective and surjective.

T: Triangle inequalities

Let $\alpha, \beta \in \mathbb{R}$. We then have that:

1. $|\alpha| + |\beta| \geq |\alpha + \beta|$
2. $\big||\alpha| - |\beta|\big| \leq |\alpha - \beta|$.

D: Supremum and infimum

Let $\alpha = \sup S$. Then:

1. $\forall s \in S; \alpha \geq s$
2. $\forall a \in \mathbb{R} : \forall s \in S; a \geq s;$ $a \geq \alpha$

and similarly for infimum.

T: Approximation property

Consider bounded $E \subset \mathbb{R}$. Then:
$$\forall \epsilon > 0; \exists a \in E : \sup E - \epsilon < a \leq \sup E.$$

D: Completeness of $\mathbb{R}$

Every nonempty bounded subset of $\mathbb{R}$ has an infimum and supremum.

T: Archimedean property

$\forall a, b \in \mathbb{R}; a > 0; \exists n \in \mathbb{N} : na > b$

D1.1: Nested intervals

A sequence of sets $(I_n)_{n\in\mathbb{N}}$ is nested if $I_1 \supset I_2 \supset I_3 \ldots$.

T1.1: Nested interval property

Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of nonempty, closed and bounded nested intervals. Then:
$$E = \bigcap_{n\in\mathbb{N}} I_n \neq \emptyset.$$

If $\lambda(I_n) \to 0$ then E contains one number, where λ denotes length.

T1.2

Let $E = [a, b]$ and that there exists an open collection of nested intervals $(I_\alpha)_{\alpha\in A}$ such that:
$$E \subset \bigcup_{\alpha\in A} I_\alpha.$$
Then $\exists\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset A$ such that:
$$E \subset I_{\alpha_1} \cup I_{\alpha_2} \cup \cdots \cup I_{\alpha_n}.$$

D1.2: ϵ-N convergence

Let $\lim_{n\to\infty} x_n = a$. Then:
$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |x_n - a| < \epsilon.$$

D1.3: Cauchy sequences

The sequence (x_n) is Cauchy if:
$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \implies |x_n - x_m| < \epsilon.$$

T1.3 and T1.4

Cauchy $\iff$ ϵ-N convergent.

T: Monotone convergence

Let $(x_n)_{n\in\mathbb{N}}$ be increasing and bounded above. Then:
$$\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}}\{x_n\}$$
and similarly for sequences that are decreasing and bounded below.

D1.4: Subsequences

The subsequence of $(x_n)_{n\in\mathbb{N}}$ is a sequence of form $(x_{n_k})_{k\in\mathbb{N}}$ and is a selection of the original sequence **taken in order**.

T1.5: Bolzano-Weierstrass

Every bounded real sequence has **a** convergent subsequence.

D1.5: Limit inferior and superior

Let (x_n) be a bounded real sequence. Then:
$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right).$$

T1.6

The real sequence (x_n) is convergent if and only if:
$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

D1.6: Convergence of infinite series

Series $S = \sum_{k=1}^{\infty} a_k$ is convergent if:
$$\lim_{n\to\infty}\sum_{k=1}^{n} a_k < \infty.$$

Series S is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ is also convergent.

Otherwise S is conditionally convergent.

T1.7: Cauchy criterion for series

$S = \sum_{k=1}^{\infty} a_k$ is convergent **iff**:
$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall m \geq n \geq N \implies \left|\sum_{k=n+1}^{m} a_k\right| < \epsilon.$$

T1.8

Let $S = \sum_{k=1}^{\infty} a_k$ be absolutely convergent.

Let $z : \mathbb{N} \to \mathbb{N}$ be a bijection. Then:
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}.$$

T1.9: Riemann rearrangement

Let $S = \sum_{k=1}^{\infty} a_k$ be conditionally convergent. Then there exists rearrangements such that S can take on any value.

T: Geometric series

Let $a \in \mathbb{R}$ and $|r| < 1$. Then:
$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$
$$\sum_{k=m}^{n} ar^{k-1} = \begin{cases} \frac{a(r^{m-1}-r^n)}{1-r} & r \neq 1 \\ a(n-m+1) & r = 1 \end{cases}$$
where $m, n \in \mathbb{N}$.

D1.7: Sequential continuity

Let $f : \mathrm{dom}(f) \to \mathbb{R}$ where $\mathrm{dom}(f) \subset \mathbb{R}$. f is continuous at $\alpha \in \mathrm{dom}(f)$ if:
$$\forall (x_n)_{n\in\mathbb{N}} \subset \mathrm{dom}(f) : \lim_{n\to\infty} x_n = \alpha \implies \lim_{n\to\infty} f(x_n) = f(\alpha).$$

T1.10

Let $\alpha \in \mathbb{R}$ and f, g continuous on D. Then αf, $f + g$, fg are continuous on D.

T1.11

Let f be continuous at $\alpha \in \mathbb{R}$ and g at $f(\alpha)$. Then $g \circ f$ is continuous at α.

D1.12: ϵ-δ continuity

Let $f : \mathrm{dom}(f) \to \mathbb{R}$ where $\mathrm{dom}(f) \subset \mathbb{R}$. Then f is continuous at $\alpha \in \mathrm{dom}(f)$ if:

$$\forall \epsilon > 0; \exists \delta > 0 : |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon.$$

T: Continuity test

f is continuous at α if:

$$\lim_{x \to \alpha} f(x) = f(\alpha)$$

where the limit from left/right must both exist and be equal to each other.

D: Uniform continuity

f is uniformly continuous on I if:

$$\forall \epsilon > 0; \exists \delta > 0 : \forall x, y \in I; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Remark

f is **not** uniformly continuous on I **iff**:

$$\exists \epsilon > 0; \exists (x_n)_{n\in\mathbb{N}} \wedge (y_n)_{n\in\mathbb{N}} \subset I : \lim_{n\to\infty} |x_n - y_n| = 0 \wedge |f(x_n) - f(y_n)| \geq \epsilon \text{ for } \forall n \in \mathbb{N}.$$

Functions on closed bounded intervals are always uniformly continuous.

D: Differentiability

f is differentiable at α if:

$$f'(\alpha) = \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h}.$$

Remark

Differentiability implies continuity.

T1.13: Intermediate value theorem

Let f be continuous on $[a, b]$.
If $f(a)f(b) < 0$ then:

$$\exists c \in (a, b) : f(c) = 0.$$

T1.14: Extreme value theorem

Let f be continuous on $[a, b]$.
Then $\exists c, d \in [a, b]$ such that:

$$f(c) = \inf\{f(x) : x \in [a, b]\}$$

$$f(d) = \sup\{f(x) : x \in [a, b]\}.$$

T: Mean value theorem

Let f be continuous on $[a, b]$
and differentiable on (a, b). Then:

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

D2.1: Pointwise convergence

$f_n \to f$ pointwise on E if:

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Here $f_n : E \to \mathbb{R}$ and:

$$\forall x \in E; \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

D2.2: Uniform convergence

$f_n \to f$ uniformly on E if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \text{ and } \forall x \in E \implies |f_n(x) - f(x)| < \epsilon.$$

P2.1

The following statements are equivalent.

1. $f_n \to f$ uniformly on E
2. $\lim_{n\to\infty} \sup_{x \in E} |f_n(x) - f(x)| = 0$
3. $\exists a_n \to 0$ s.t. $|f_n(x) - f(x)| \leq a_n$ for $\forall x \in E$.

T2.1

If f_n is continuous on E **and** $f_n \to f$ uniformly on E then f is continuous on E.

Remark

If f is <u>not continuous</u> on E then f_n <u>cannot</u> be uniform on E.

T2.5: Weierstrass M-test

Let $E \subset \mathbb{R}$ and $f_k : E \to \mathbb{R}$.
$\exists M_k > 0 : \sum_{k=1}^{\infty} M_k < \infty$.
If $\forall k \in \mathbb{N}$ and $\forall x \in E$; $|f_k(x)| \leq M_k$ then:
$\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on E.

D: Power series

Let (a_n) be a real sequence and $c \in \mathbb{R}$. Then:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

is a power series centered at c,
with **radius of convergence**:

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

where $R = \infty$ implying that series converges everywhere.

T3.1: Convergence of power series

Let $0 < R < \infty$. If $|x - c| < R$ then $f_{PS}(x)$ converges absolutely.

If $|x - c| > R$ then $f_{PS}(x)$ diverges.

T3.2: Continuity of power series

Let $0 < r < R$ where R is the radius of convergence of $f_{PS}(x)$.

Then for $|x - c| \leq r$, $f_{PS}(x)$ converges absolutely and uniformly to a <u>continuous</u> function $f(x)$.

L3.1

$\sum_{n=1}^{\infty} a_n (x-c)^n$ and $\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$ have the same radius of convergence.

T: Root and ratio tests

Let $S = \sum_{n=1}^{\infty} \alpha_n$ and consider:

1. Ratio test: $\rho = \lim_{n\to\infty} \left| \frac{\alpha_{n+1}}{\alpha_n} \right|$
2. Root test: $\rho = \lim_{n\to\infty} |\alpha_n|^{1/n}$.

Then:

- $\rho < 1$: S converges absolutely
- $\rho > 1$: S diverges
- $\rho = 1$: test is inconclusive.

T3.3

Let R be the radius of convegence of $f_{PS}(x)$. Then for $\forall x : |x - c| < R$, $f_{PS}(x)$ is **infinitely differentiable** and:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

T: Taylor's theorem

Let f be n times differentiable at $\alpha \in \mathbb{R}$ where $n \in \mathbb{N}$. Then:

$$f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(\alpha)}{k!}(x - \alpha)^k + h_n(x)(x - \alpha)^n$$

where $\lim_{x\to\alpha} h_n(x) = 0$.

Elementary expansions

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

D: Characteristic functions

Let $E \subset \mathbb{R}$. The characteristic function is defined as a real function such that:

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise.} \end{cases}$$

D4.1 and D4.2: Step functions

The step function with respect to finite set $\{x_0, \dots, x_n\}$ for some $n \in \mathbb{N}$ is:

$$\phi(x) = \begin{cases} 0 & x < x_0 \text{ or } x > x_n \\ c_j & x \in (x_{j-1}, x_j);\ 1 \le j \le n \end{cases}$$

and its integral is defined as:

$$\int \phi = \sum_{j=1}^{n} c_j (x_j - x_{j-1}).$$

D4.3: Lebesgue integrable

$f : I \to \mathbb{R}$ is Lebesgue integrable on I if:

1. $\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$
2. $\forall x \in I; f(x) = \sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$

Here $c_j \in \mathbb{R}$, $J_i \subset I$ and is bounded for $j \in \{1, 2, 3, \dots\}$. Then:

$$\int_I f = \sum_{j=1}^{\infty} |c_j| \lambda(J_j).$$

T4.1

Let $c_j, d_j \in \mathbb{R}$ and J_j, K_j be bounded intervals where $j \in \{1, 2, \dots\}$. Let:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

$$\sum_{j=1}^{\infty} |d_j| \lambda(K_j) < \infty.$$

If:

$$\forall x; \sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) :$$

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty \text{ and}$$

$$\sum_{j=1}^{\infty} |d_j| \chi_{K_j}(x) < \infty$$

then $\sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j)$.

T4.2: Basic properties

Let f, g be integrable on I and $\alpha, \beta \in \mathbb{R}$.

1. $\alpha f + \beta g$ is integrable on I and:
$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g.$$
2. If $f \ge g$ on I then $\int_I f \ge \int_I g$.
3. $|f|$ is integrable on I and:
$$\int_I |f| \ge \left| \int_I f \right|.$$
4. If f or g is bounded on I then fg is integrable on I.
5. If $f \ge 0$ and $\int_I f = 0$, then $\forall h$ such that $0 \le h \le f$ is also integrable on I.

T4.3

Let f_n be integrable on I where:

$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty.$$

1. Let f be defined as:
$$\forall x \in I; f(x) = \sum_{n=1}^{\infty} f_n(x) :$$
$$\sum_{n=1}^{\infty} |f_n(x)| < \infty.$$
Then f is integrable on I and:
$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n.$$
2. Let each $f_n \ge 0$ and:
$$\forall x \in I; f(x) = \sum_{n=1}^{\infty} f_n(x).$$
Then f is integrable on I **iff**:
$$\sum_{n=1}^{\infty} \int_I f_n < \infty.$$

T4.4: MCT for integrals

Let f_n be monotone increasing sequence of functions on I and that:

$$\forall x \in I; f(x) = \lim_{n \to \infty} f_n(x).$$

Then f is integrable on I **iff**:

$$\sup_{n \in \mathbb{N}} \int_I f_n = \lim_{n \to \infty} \int_I f_n < \infty.$$

Furthermore:

$$\int_I f = \lim_{n \to \infty} \int_I f_n.$$

D4.4: Riemann integrable

f is Riemann-integrable on $[a, b]$ if:

$$\forall \epsilon > 0; \exists \phi, \psi : \phi \le f \le \psi$$
$$\text{and } \int \psi - \int \phi < \epsilon$$

where ϕ and ψ are step functions, i.e. the bounded support of f.

T4.5

f is Riemann-integrable if and only if:

$$\sup \left\{ \int \phi : \phi \le f \right\} = \inf \left\{ \int \psi : f \le \psi \right\}$$

where ϕ and ψ are step functions.

T4.6

If f is Riemann-integrable on I then f is also Lebesgue-integrable on I.

Remark

The converse of T4.6 is not true.

Remark

If f is Riemann-integrable on I then $|f|$ is also Riemann-integrable on I, but reverse is not true!

L4.1

Let f be a bounded function with bounded support on $[a, b]$. The following statements are equivalent:

1. f is Riemann-integrable.
2. $\forall \epsilon > 0; \exists \{a = x_0 < \dots < x_n = b\}$:
$$\sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon$$
where we define:
$$M_j = \sup_{x \in (x_{j-1}, x_j)} \left\{ f(x) \right\}$$
$$m_j = \inf_{x \in (x_{j-1}, x_j)} \left\{ f(x) \right\}$$
and $n \in \mathbb{N}$. (i.e. finite partition)
3. $\forall \epsilon > 0; \exists \{a = x_0 < \dots < x_n = b\}$:
$$\sum_{j=1}^{n} \sup_{x, y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$
where $I_j = (x_{j-1}, x_j)$.

T4.7

Let $g : [a, b] \to \mathbb{R}$ and f be such that $f(x) = g(x)$ if $x \in [a, b]$ and $f(x) = 0$ otherwise.

1. If g is continuous on $[a, b]$ then f is Riemann-integrable.
2. If g is a monotone function then f is Riemann-integrable.

T4.8

Let $J \subset I$.

1. If f is integrable on I then f is integrable on J.
2. If f is integrable on J and for $\forall x \in I \backslash J; f(x) = 0$ then f is integrable on I.

   Furthermore: $\displaystyle\int_J f = \int_I f.$
3. If f is integrable on I and $f(x) \geq 0$ for $\forall x \in I$ then:
$$\int_I f \geq \int_J f.$$
4. Assume that I can be written as the union of disjoint intervals I_n and that f is integrable on each I_n.

   Then f is integrable on I **iff**:
$$\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty.$$
   If this is true then:
$$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} |f|.$$

T4.9

If any <u>two</u> of the following integrals exists:
$$\int_a^b f, \qquad \int_b^c f, \qquad \int_a^c f$$
then so does the third and:
$$\int_a^b f + \int_b^c f = \int_a^c f.$$

T4.10: FTC I

Let g be integrable on I and let:
$$G(x) = \int_{x_0}^{x} g(s)\mathrm{d}s$$
where $x, x_0 \in I$.

If g is continuous at x then:
$$\frac{\mathrm{d}}{\mathrm{d}x} G(x) = g(x).$$

T4.11: FTC II

Let $f'(x)$ be continuous on I. Then:
$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$$
where $a, b \in I$.

L4.2: Fatoux's lemma

Let $f_n \geq 0$ be integrable on I and:
$$\forall x \in I; f(x) = \liminf_{n\to\infty} f_n(x).$$
If $\displaystyle\liminf_{n\to\infty} \int_I f_n < \infty$ then:
$$\int_I f \leq \liminf_{n\to\infty} \int_I f_n.$$

T4.12: Dominated convergence

Let f_n, g be integrable on I and:
$$\forall x \in I; f(x) = \lim_{n\to\infty} f_n(x).$$
If $|f_n(x)| \leq g(x)$ for $\forall x \in I$ then f is integrable on I and:
$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

T4.13

Let f_n be integrable on (a, b) and that $f_n \to f$ uniformly on (a, b).

Then f is integrable on (a, b) and:
$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

D5.1: L^2 space

$f \in L^2([a, b])$ if:

1. $f : [a, b] \to \mathbb{C}$ is measurable
2. $x \mapsto |f(x)|^2$ is integrable:
$$||f||_2^2 := \int_a^b |f(x)|^2\mathrm{d}x < \infty$$

and $||f||_2$ is the L^2-norm of f.

Remark

If $z \in \mathbb{C}$ then $z\bar{z} := |z|^2$.

D5.2: Inner products

The inner product of $f, g \in L^2([a, b])$ is:
$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\mathrm{d}x.$$

T5.1: Cauchy-Schwarz inequality

Let $f, g \in L^2([a, b])$. Then:
$$|\langle f, g\rangle| \leq ||f||_2||g||_2.$$

C: Minkowski's inequality

Let $f, g \in L^2([a, b])$. Then:
$$||f + g||_2 \leq ||f||_2 + ||g||_2.$$

D5.3: L^2 convergence

$f_n \to f$ in L^2 if:
$$\lim_{n\to\infty} ||f_n - f||_2 = 0.$$
Here $f, f_1, f_2, \ldots \in L^2([a, b])$.

D5.4: Orthonormal systems

The sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ in L^2 is an orthonormal system on $[a, b]$ if:
$$\langle \phi_m, \phi_n\rangle = \delta_{mn}.$$

T5.2

Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal system on $[a, b]$ with **linear span** X_n.

Assume that $f \in L^2$ and:
$$s_N(x) = \sum_{n=1}^{N} \langle f, \phi_N\rangle \phi_n(x).$$
Then:
$$||f - s_N||_2 \leq ||f - g||_2$$
holds for $\forall g \in X_n$.

T5.3: Bessel's inequality

Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal system on $[a, b]$ and $f \in L^2$. Then:
$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \leq ||f||_2^2.$$

C: Riemann-Lebesgue lemma

Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal system on $[a, b]$ and $f \in L^2$. Then:
$$\lim_{n\to\infty} \langle f, \phi_n\rangle = 0.$$

D5.5: Completeness

The orthonormal system $(\phi_n)_{n\in\mathbb{N}}$ is complete if:
$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 = ||f||_2^2$$
for $\forall f \in L^2$.

T5.4

Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal system on $[a,b]$ and let $(s_N)_{N\in\mathbb{N}}$ be a sequence of functions where:

$$s_N(x) = \sum_{n=1}^{N} \langle f, \phi_N \rangle \phi_n(x).$$

Then $(\phi_n)_{n\in\mathbb{N}}$ is complete **iff**:

$$\forall f \in L^2; s_N \to f \text{ in } L^2.$$

D5.6: Trigonometric polynomial

Trigonometric polynomials are functions of form:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$$

where $x \in \mathbb{R}$ and $c_n \in \mathbb{C}$.

L5.1

$(e^{2\pi i n x})_{n\in\mathbb{Z}}$ forms an orthonormal system on $[0,1]$. Furthermore:

1. $\displaystyle\int_0^1 e^{2\pi i n x}\mathrm{d}x = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$
2. If $f_{FS} = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$ then:

$$c_n = \langle f, e^{2\pi i n x} \rangle.$$

D5.7 and D5.8: Fourier series

The nth Fourier coefficient of integrable 1-periodic f where $n \in \mathbb{Z}$ is defined as:

$$\widehat{f}(n) = \langle f, \phi_n \rangle$$

and the Fourier series of f is:

$$f_{FS} = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}.$$

The Fourier partial sums is defined as:

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}$$

where $N \in \mathbb{Z}$.

D5.9: Convolutions

The convolution of 1-periodic functions $f, g \in L^2$ is:

$$f * g(x) = \int_0^1 f(t) g(x-t) \mathrm{d}t.$$

L5.2

For 1-periodic $f, g \in L^2$: $f * g = g * f$.

L5.3: Dirichlet kernel

The Dirichlet kernel is defined as:

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{2\pi i n x} \\ &= \frac{\sin(2N+1)\pi x}{\sin \pi x} \end{aligned}$$

where $N \in \mathbb{N}$.

L5.4: Fejér kernel

The Fejér kernel is defined as:

$$\begin{aligned} K_N(x) &= \frac{1}{N+1} \sum_{n=0}^{N} D_n(x) \\ &= \frac{1}{N+1} \left[\frac{\sin(N+1)\pi x}{\sin \pi x} \right]^2 \end{aligned}$$

where $N \in \mathbb{N}$.

T5.5: Fejér's theorem

In the limit $N \to \infty$:

$$K_N * f \to f \text{ uniformly on } \mathbb{R}$$

where f is 1-periodic and continuous.

C

For every 1-periodic continuous f:

$$\exists (f_n)_{n\in\mathbb{N}} : f_n \to f \text{ uniformly on } D$$

for f_n is a trigonometric polynomial and domain D subject to f.

D5.10: Approximation of unity

A sequence of 1-periodic integrable $(k_n)_{n\in\mathbb{N}}$ is an approximation of unity if for all 1-periodic continuous f:

$$f * k_n \to f \text{ uniformly on } \mathbb{R}$$

or that:

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f * k_n(x) - f(x)| = 0.$$

T5.6

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of 1-periodic integrable functions that satisfies:

1. $k_n(x) \geq 0$ for $\forall x \in \mathbb{R}$.
2. $\displaystyle\int_{-1/2}^{1/2} k_n(t)\mathrm{d}t = 1$
3. $\forall \delta \in (0, \frac{1}{2}]; \displaystyle\lim_{n\to\infty} \int_{-\delta}^{\delta} k_n(t)\mathrm{d}t = 1.$

Then $(k_n)_{n\in\mathbb{N}}$ is an approximation of unity.

C

The Fejér kernel $(K_N)_{N\in\mathbb{N}}$ is an approximation of unity.

L5.5

If f is 1-periodic continuous then:

$$\lim_{N\to\infty} ||S_N f - f||_2 = 0.$$

T5.7

For every 1-periodic $f \in L^2$:

$$S_N f \to f \text{ in } L^2$$

or that the Fourier series of f converges to f in the L^2 sense.

C: Parseval's theorem

Let $f, g \in L^2$ be 1-periodic. Then:

$$\langle f, g \rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}$$

and in particular:

$$||f||_2^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2.$$