Honours Differential Equations

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Contents

1 ODE systems

1.1 Integrating factors

Consider linear DE of form

$$y' + P(x)y = Q(x)$$

The integrating factor for this DE is:

$$I(x) = \exp\left(\int P(x)dx\right)$$

and the solution to the linear DE is:

$$y(x) = \frac{1}{I(x)}\int Q(x)I(x)dx + \frac{\alpha}{I(x)}$$

where here α is a constant.

1.2 Change of variables

For higher order differential equations of form

$$y^{(n)} = F(y, y', \ldots, y^{(n-1)}, t),$$

consider **change of variables** $x_{i+1} = y^{(i)}$ for $i \in \{0, 1, \ldots, n-1\}$.

Taking derivatives with respect to time yields a first order matrix ODE system:

$$x_j' = F_j(t, x_1, \ldots, x_n)$$

for $j = 1, \ldots, n$. We either immediately write this as a matrix system or linearise near a critical point.

1.3 Existence and uniqueness for IVPs

An initial value problem (**IVP**) is defined as

$$\frac{dx}{dt} = f(x, t)$$

for **initial** condition $x(t_0) = x_0$. A solution $x : I \to \mathbb{R}$ is a differentiable function that satisfies the IVP. Similarly for a first order system

$$x_i' = F_i(t, x_1, \ldots, x_n)$$

to have a **unique** solution, F_i and $\frac{\partial F_i}{\partial x_j}$ must be continuous in a region. Here $i, j \in \{1, \ldots, n\}$. This is known as the Picard-Lindelöf theorem.

1.4 Homogeneous systems

1.4.1 Unique eigenvalues

Now consider $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x}$, where $\mathbf{A}$ is a $n \times n$ matrix.
Substituting $\boldsymbol{x} = e^{rt}\boldsymbol{\xi}$ results in an eigenvalue problem:

$$(\mathbf{A} - r_i\mathbf{I_n})\boldsymbol{\xi^{(i)}} = \mathbf{0}$$

where $i \in \{1, 2, \ldots, n\}$. Our general solution is then:

$$\begin{aligned}\boldsymbol{x}(t) &= \sum_{i=1}^{n} c_i e^{r_i t}\boldsymbol{\xi}^{(i)} \\ &= \sum_{i=1}^{n} c_i \boldsymbol{x}^{(i)} \\ &= \boldsymbol{\Psi}(t)\boldsymbol{c}\end{aligned}$$

where $\boldsymbol{\Psi}(t)$ is our fundamental matrix satisfying $\boldsymbol{\Psi}' = \mathbf{A}\boldsymbol{\Psi}$ and that:

$$\boldsymbol{\Psi}(t) = [\boldsymbol{x^{(1)}}, \ldots, \boldsymbol{x^{(n)}}].$$

Furthermore if initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ are given we then have that:

$$\boldsymbol{c} = \boldsymbol{\Psi}^{-1}(t_0)\boldsymbol{x}(t_0)$$

and

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\boldsymbol{x}_0.$$

1.4.2 Matrix exponentials

We can also write our solutions as a matrix exponential, defined as such:

$$\begin{aligned}e^{\boldsymbol{A}t} &= \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}t)^n}{n!} \\ &= \boldsymbol{I}_n + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2 t^2 + \ldots\end{aligned}$$

and since an exponential power series is infinitely differentiable:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t}.$$

Therefore it is then deduced that the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0).$$

and that $e^{\boldsymbol{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$ where we are finding the coefficients to the general solution $\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{c}$.

1.4.3 Diagonalisation

Consider again $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ where $\boldsymbol{A}$ is a $n \times n$ matrix that is diagonalisable:

$$\boldsymbol{A}\boldsymbol{\xi}^{(i)} = r_i\boldsymbol{\xi}^{(i)}$$

$$\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$$

for here $\boldsymbol{D}$ is our diagonal matrix containing our eigenvalues r_i and

$$\boldsymbol{T} = [\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}].$$

Then let $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$. After some algebra we have that:

$$\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y}$$

which have particular solutions $\boldsymbol{y} = e^{r_i t}\boldsymbol{e}^{(i)}$ for $i \in \{1, \dots, n\}$.

Since our fundamental matrix with respect to $\boldsymbol{y}$ is $\boldsymbol{Q} = e^{\boldsymbol{D}t}$, the fundamental matrix with respect to $\boldsymbol{x}$ is:

$$\boldsymbol{\Psi}(t) = \boldsymbol{T}e^{\boldsymbol{D}t}$$

and we get an expression for the matrix exponential of $\boldsymbol{A}$:

$$e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{D}t}\boldsymbol{T}^{-1}$$

where $e^{\boldsymbol{D}t}$ is a diagonal matrix with entries $e^{r_i t}$.

1.4.4 Generalised eigenvectors

If eigenvalues do not generate enough eigenvectors to form n linearly independent solutions for a $n \times n$ matrix $\boldsymbol{A}$ we try the following ansatz:

$$\boldsymbol{x} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}$$

which gives

$$(\mathbf{A} - r_i\mathbf{I_n})\boldsymbol{\eta}^{\boldsymbol{(i)}} = \boldsymbol{\xi}^{\boldsymbol{(i)}}.$$

Therefore we end up with:

$$\boldsymbol{x}^{(1)} = e^{rt}\boldsymbol{\xi}$$

and

$$\boldsymbol{x}^{(2)} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

1.5 Non-homogeneous systems

Consider non-homogeneous ODE system:

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}.$$

There a couple of different approaches we can take to solve such a system.

- **Change of basis**

  Let $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$, where $\boldsymbol{T}$ is our eigenvector matrix from diagonalisation.
  So $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$, and after some algebra we obtain:

  $$\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y} + \boldsymbol{T}^{-1}\boldsymbol{g}$$

  which can be solved by integrating factors. Finally revert back to $\boldsymbol{x}$.

- **Variation of parameters**

  So $\boldsymbol{x}_H = \boldsymbol{\Psi}\boldsymbol{c}$ solves the $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{c}$ is a constant vector.

  We then assume that the solution to our non-homogeneous system takes the form:

  $$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{u}$$

  for here $\boldsymbol{u} = \boldsymbol{u}(t)$. We then get $\boldsymbol{\Psi}\boldsymbol{u}' = \boldsymbol{g}$, which can be solved by eliminating variables and integrating.

- **Method of undetermined coefficients**

  Our non-homogeneous ODE system has solutions of form:

  $$\boldsymbol{x} = \boldsymbol{x}_H + \boldsymbol{x}_p$$

  Solving the homogeneous ODE gives us $\boldsymbol{x}_H$.

  On the other hand we just need to find a **particular solution** $\boldsymbol{x}_p$ that satisfies our non-homogeneous ODE. Then our solution is complete.

  Whilst the fastest, this method is not guaranteed to work.

1.6 Critical points & linearisation

Consider non-linear ODE system

$$x' = F(x, y),$$

$$y' = G(x, y).$$

We define $\boldsymbol{x^0} = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$ as a **critical point** when $F(\boldsymbol{x^0}) = G(\boldsymbol{x^0}) = 0$.

Non-linear systems may then be linearised by Taylor expanding them around a critical point $\boldsymbol{x^0}$, and discarding higher order terms.
i.e. let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_1 = x - x^0$ and $u_2 = y - y^0$.

$$\begin{aligned} \therefore u_1' &= x' \\ &\approx F(x^0, y^0) + \left(\frac{\partial F}{\partial x}\right)_{x^0} (x - x^0) + \left(\frac{\partial F}{\partial y}\right)_{y^0} (y - y^0) \end{aligned}$$

$$\begin{aligned} \therefore u_2' &= y' \\ &\approx G(x^0, y^0) + \left(\frac{\partial G}{\partial x}\right)_{x^0} (x - x^0) + \left(\frac{\partial G}{\partial y}\right)_{y^0} (y - y^0) \end{aligned}$$

Then we end up with the following linear system:

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u}$$

where $\boldsymbol{A} = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}_{\boldsymbol{x}=\boldsymbol{x^0}}$ and is a 2×2 Jacobian matrix.

Our critical points $\boldsymbol{x^0}$ may also be classified:

Eigenvalues	**Critical Points**	**Stability**
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asymp. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asymp. stable
$r_1,\ r_2 = \lambda \pm i\mu\ (\lambda > 0)$	focus	unstable
$r_1,\ r_2 = \lambda \pm i\mu\ (\lambda < 0)$	focus	asymp. stable
$r_1 = i\mu, r_2 = -i\mu$	center	stable

Linearisation preserves critical point behaviour **except** when eigenvalues are of $r = \pm i\mu$ form, for which then classification is unknown.

1.7 Stability of critical points

Stable critical points $\boldsymbol{x^0}$: All solutions start and stay near $\boldsymbol{x^0}$.

$\forall \epsilon > 0; \exists \delta > 0; \forall \boldsymbol{x}_{solution}$ to $\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x}, t)$:
$|\boldsymbol{x}(0) - \boldsymbol{x^0}| < \delta \implies |\boldsymbol{x}(t) - \boldsymbol{x^0}| < \epsilon$ for $\forall t \geq 0$

Attracting critical points $\boldsymbol{x^0}$: All solutions tends to $\boldsymbol{x^0}$.

$\forall \delta > 0 : |\boldsymbol{x}(0) - \boldsymbol{x^0}| < \delta \implies \lim_{t \to \infty} \boldsymbol{x}(t) = \boldsymbol{x^0}$

Asymptotically stable critical points $\boldsymbol{x^0}$: Attracting **and** stable

1.8 Lyapunov's theory and limit cycles

In this section $\dot{\boldsymbol{x}}$ means its first time derivative. So consider:

$$\dot{x} = F(x, y),$$

$$\dot{y} = G(x, y)$$

defined in $\mathbb{R}^2$. Let $\boldsymbol{x^0} \in D$ be a critical point.

The function $E : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a Lyapunov function where $E(x^0, y^0) = 0$, whenever it exists. Note that the time derivative of E is:

$$\frac{dE}{dt} = \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G$$

- Let $E > 0$ for $\forall \boldsymbol{x} \neq \boldsymbol{x^0}$.

  If $\frac{dE}{dt} \leq 0$ then $\boldsymbol{x^0}$ is stable.

  If $\frac{dE}{dt} < 0$ then $\boldsymbol{x^0}$ is asymptotically stable.

- If every neighbourhood of $\boldsymbol{x^0}$ contains $\boldsymbol{x^*}$ such that $E(\boldsymbol{x^*}) > 0$
  and if $\frac{dE}{dt} > 0$ then $\boldsymbol{x^0}$ is unstable.

Now **limit cycles** are defined as periodic solutions such that at least one other non-closed trajectory approaches the limit cycle as $t \to \infty$.

2 Fourier series

2.1 Real Fourier series

Let $f(x)$ and $f'(x)$ be **piecewise continuous** in $[-L, L]$ with **period** $2L$. i.e. $f(x) = f(x + \mathbf{2L})$ for $\forall x$. Then the Fourier series for $f(x)$ is

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}).$$

The **convergence** of our Fourier series depends on the continuity of $f(x)$:

- If $f(x)$ is <u>continuous</u> then $f_{FS}(x) = f(x)$.
- If $f(\alpha)$ is <u>discontinuous</u> then at point α we have

$$f_{FS}(\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}.$$

Note that $f(x)$ is continuous at α if $f(\alpha) = \lim_{x\to\alpha} f(x)$ and we define:

$$f(\alpha^-) = \lim_{x\to\alpha^-} f(x)$$

and

$$f(\alpha^+) = \lim_{x\to\alpha^+} f(x),$$

i.e. limits from left and right respectively. It is important to also note that the <u>derivative</u> of a Fourier series is **not necessarily convergent**.

Now consider $S_n = \sin\frac{n\pi x}{L}$ and $C_n = \cos\frac{n\pi x}{L}$. We then have the following **orthogonality relations**:

$$\langle S_n, S_m\rangle = \langle C_n, C_m\rangle = L\delta_{mn}$$

$$\langle S_n, C_m\rangle = 0$$

where we define the inner product as:

$$\langle u(x), v(x)\rangle = \int_{-L}^{L} u(x)v(x)dx$$

and use the following trigonometric identities:

$$\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)].$$

Now integrating the following expression:

$$\int_{-L}^{L}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L})\right)\mathrm{d}x=\int_{-L}^{L}f(x)\mathrm{d}x$$

gives:

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx.$$

Similarly:

$$a_n=\frac{1}{L}\int_{-L}^{L}\cos\frac{n\pi x}{L}f(x)dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L}\sin\frac{n\pi x}{L}f(x)dx.$$

Note that δ_{mn} is the **Kronecker delta** and is defined as:

$$\delta_{mn}=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases}$$

Furthermore notice that:

- The Fourier series of **even** functions contains <u>only</u> **cosines**.
- The Fourier series of **odd** functions contains <u>only</u> **sines**.

Even functions are defined $f(-x)=f(x)$, and:

$$\int_{-L}^{L}f_{even}dx=2\int_{0}^{L}f_{even}dx.$$

Similarly **odd** functions are defined $f(-x)=-f(x)$, and:

$$\int_{-L}^{L}f_{odd}dx=0.$$

We can also extend a function defined in $[0,L]$ in several ways:

1. Define <u>even</u> function:

$$g(x)=\begin{cases}f(x) & x\in[0,L]\\ f(-x) & x\in(-L,0)\end{cases}$$

where its Fourier expansion is a cosine series.

2. Define <u>odd</u> function:

$$h(x)=\begin{cases}f(x) & x\in(0,L)\\ 0 & x=0,L\\ -f(-x) & x\in(-L,0)\end{cases}$$

where its Fourier expansion is a sine series.

2.2 Complex Fourier series

Expanding $f(x)$ defined in $[-L, L]$ with period $2L$:

$$f_{FS}(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi}{L}x\right)$$

using Euler's formula $e^{i\theta} = \sin\theta + i\cos\theta$. Its coefficients are:

$$c_n = \frac{1}{2L}\int_{-L}^{L} \exp\left(-\frac{in\pi}{L}x\right) f(x)dx$$

for $\forall n \in \mathbb{Z}$ and:

$$c_n = \begin{cases} (a_n - ib_n)/2 & n > 0 \\ (a_0)/2 & n = 0 \\ (a_n + ib_n)/2 & n < 0. \end{cases}$$

Here we define the **inner product** for complex functions as

$$\langle f, g\rangle = \int_{\alpha}^{\beta} f^*(x)g(x)dx$$

where $f^*(x)$ is the complex conjugate of $f(x)$. Then:

$$\begin{aligned} \langle \exp\left(\frac{\boldsymbol{im}\pi}{L}x\right), \exp\left(\frac{\boldsymbol{in}\pi}{L}x\right)\rangle &= \int_{-L}^{L} \exp\left(-\frac{\boldsymbol{im}\pi}{L}x\right)\exp\left(\frac{\boldsymbol{in}\pi}{L}x\right)dx \\ &= 2L\delta_{mn} \end{aligned}$$

and since $f(x) = f_{FS}(x)$ we obtain our formula.

2.3 Parseval's theorem

Parseval's theorem states that given a periodic $f(x)$ with convergent Fourier series we have that

$$\begin{aligned} \langle f, f\rangle &= \int_{-L}^{L} |f(x)|^2 dx \\ &= 2L\sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= L\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)\right] \end{aligned}$$

and is derived by orthogonality.

3 PDEs

3.1 Separation of variables

The only methodology considered is separation of variables. So for PDE:

$$\hat{D}[u(x_1, \ldots, x_n)] = 0$$

where $\hat{D}$ is our differential operator, we look for solutions of form:

$$u(x_1, \ldots, x_n) = X_1(x_1) \cdots X_n(x_n)$$

subject to **initial** and **boundary** conditions.

3.2 Heat equation

The heat equation is an equation of the following form:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

where α^2 is the thermal diffusivity constant.

3.2.1 Standard boundary conditions

We firstly define:

- **Initial condition**: $u(x, 0) = f(x)$ for $0 \leq x \leq L$
- **Boundary condition**: $u(0, t) = u(L, t) = 0$ for $\forall t > 0$

Let solutions be of form:

$$u(x, t) = X(x) \cdot T(t)$$
$$\therefore X(x) \cdot \dot{T}(t) = \alpha^2 X''(x) \cdot T(t)$$

Only a constant function may satisfy the first equality:

$$\frac{1}{\alpha^2} \frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda.$$

Writing this as two ODEs:

$$\dot{T} + \alpha^2 \lambda T = 0$$
$$X'' + \lambda X = 0.$$

The first one we can directly integrate, yielding:

$$T(t) = a_1 \exp\left(-\alpha^2 \lambda t\right).$$

The second ODE is a spring system, hence it has solution of form:

$$X(x) = b_1 \cos \lambda^{1/2} x + b_2 \sin \lambda^{1/2} x.$$

However this time before proceeding we need to consider boundary conditions:

$$X(0) = X(L) = 0.$$

We find $X(0) = b_1 = 0$ and $X(L) = b_2 \sin \lambda^{1/2} L = 0$.

The second equation implies that λ must of the following form:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad \forall n \in \mathbb{N}$$

and so

$$X'' + \lambda X = 0 \implies X_n = b_2 \sin \lambda_n^{1/2} x.$$

Since λ is discretised:

$$\therefore T_n = a_1 \exp\left(-\alpha^2 \lambda_n t\right).$$

Our general solution must then be:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\alpha^2 \lambda_n t\right) \sin \lambda_n^{1/2} x$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Using initial condition $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x$$

and we recognise this as an odd Fourier series with period $2L$.

$$\therefore \int_{-L}^{L} \sin\left(\lambda_n^{1/2} x\right) f(x) dx = \sum_{n=1}^{\infty} c_n \int_{-L}^{L} \left(\sin\left(\lambda_n^{1/2} x\right)\right)^2 dx$$

$$\therefore 2 \int_0^L \sin\left(\lambda_n^{1/2} x\right) f(x) dx = c_n L$$

The final step we split the integration range and use $x = -x^*$.

$$\therefore c_n = \frac{2}{L} \int_0^L \sin\left(\lambda_n^{1/2} x\right) f(x) dx$$

This is fine because we can extend $u(x,t)$ via reflection for negative x.

3.2.2 Fixed boundary temperatures

We reconsider the heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

but now with the following **non-homogeneous boundary conditions**:

- $u(0,t) = T_1$
- $u(L,t) = T_2$
- $u(x,0) = f(x)$

Physically our rod has fixed boundary temperatures, namely T_1 and T_2.

We approach this problem with a change of variables:

$$v(x) = \lim_{t\to\infty} u(x,t).$$

Using our boundary conditions v must be linear:

$$\therefore v(x) = \frac{T_2 - T_1}{L} x + T_1$$

since $v'' = 0$, $v(0) = T_1$ and $v(L) = T_2$. We then deduce that:

$$u(x,t) = v(x) + \omega(x,t)$$

for $\omega(x,t)$ satisfies the same heat equation with initial conditions:

- $\omega(0,t) = \omega(L,t) = 0$
- $\omega(x,0) = f(x) - v(x)$

Recognising this as our initial example:

$$\omega(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\alpha^2 \lambda_n t\right) \sin \lambda_n^{1/2} x$$

where again $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and because $\omega(x,t)$ is a Fourier series with period $2L$:

$$c_n = \frac{2}{L} \int_0^L \sin\left(\lambda_n^{1/2} x\right) \{f(x) - v(x)\} dx.$$

3.2.3 Insulated rod ends

For the final example we consider:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

and define the following conditions:

- $\dfrac{\partial}{\partial x} u(0, t) = \dfrac{\partial}{\partial x} u(L, t) = 0$
- $u(x, 0) = f(x)$

We begin again with a separation of variables:

3.3 Wave equation

3.4 Laplace's equation

Laplace's equation takes the form $\boldsymbol{\nabla}^2 u = 0$. In two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and we only consider boundary conditions. (Dirichlet conditions)

3.4.1 Rectangular boundary conditions

We open with the following example:

- **Boundary for y**: $u(x, 0) = u(x, b) = 0$
- **Boundary for x**: $u(0, y) = 0$ and $u(a, y) = f(y)$

where $x \in [0, a]$ and $y \in [0, b]$. Begin by separation of variables:

$$u(x, y) = X(x) \cdot Y(y)$$

$$\therefore \frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

Recognising the previous statement as two ODEs:

$$X'' - \lambda X = 0 \text{ for } X(0) = 0$$

$$Y'' + \lambda Y = 0 \text{ for } Y(0) = Y(b) = 0$$

The second ODE we have already solved in the heat equation. It has solution:

$$Y_n = a_1 \sin\left(\lambda_n^{1/2} y\right) \text{ for } \lambda_n = \frac{n^2\pi^2}{b^2}.$$

The first ODE has solutions of form:

$$X_n = a_2 \cosh\left(\lambda_n^{1/2} x\right) + a_3 \sinh\left(\lambda_n^{1/2} x\right)$$

where these are the hyperbolic functions:

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}).$$

Using our boundary condition $X(0) = 0$ gives:

$$X_n = a_3 \sinh\left(\lambda_n^{1/2} x\right).$$

Now putting all of this together we get:

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\lambda_n^{1/2} x\right) \sin\left(\lambda_n^{1/2} y\right)$$

To find coefficients c_n we use $u(a,y) = f(y)$.

$$\therefore f(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\lambda_n^{1/2} a\right) \sin\left(\lambda_n^{1/2} y\right)$$

Since we have a Fourier series with period $2b$:

$$\begin{aligned}\int_{-b}^{b} \sin\left(\lambda_n^{1/2} y\right) f(y) dy &= \sum_{n=1}^{\infty} c_n \sinh\left(\lambda_n^{1/2} a\right) \int_{-b}^{b} \sin\left(\lambda_n^{1/2} y\right) dy \\ &= c_n \sinh\left(\lambda_n^{1/2} a\right) \cdot b\end{aligned}$$

We can split the first integral to give us:

$$c_n = \frac{2}{b \sinh\left(\lambda_n^{1/2} a\right)} \int_{0}^{b} \sin\left(\lambda_n^{1/2} y\right) f(y) dy$$

where $\lambda_n = \left(\frac{n\pi}{b}\right)^2$ and our solution is complete.

3.4.2 Circular boundary conditions

Now we solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

but with a circular boundary. In polar coordinates (r, θ):

- $u(a, \theta) = f(\theta)$
- $u(r, \theta)$ is bounded

where a is the radius of our circle and $\theta \in [0, 2\pi]$. Since $u = u(x, y)$:

$$\begin{aligned} u'_\theta &= u'_x x'_\theta + u'_y y'_\theta \\ u''_{\theta\theta} &= (u''_{xx} x'_\theta + u''_{xy} y'_\theta) x'_\theta + u'_x x''_{\theta\theta} + (u''_{yy} y'_\theta + u''_{xy} x'_\theta) x'_\theta + u'_y y''_{\theta\theta} \\ u'_r &= u'_x x'_r + u'_y y'_r \\ u''_{rr} &= (u''_{xx} x'_r + u''_{xy} y'_r) x'_r + u'_x x''_{rr} + (u''_{xy} x'_r + u''_{yy} y'_r) y'_r + u'_y y''_{rr} \end{aligned}$$

and here we have used the chain rule.

Applying these derivatives we obtain the following equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0.$$

Using separation of variables:

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\therefore r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \lambda$$

where λ is our separation constant.

$$\therefore \ddot{\Theta} + \lambda\Theta = 0$$

$$\therefore r^2 R'' + rR' = \lambda R$$

4 Sturm-Liouville theory

4.1 Regular S-L problems

Sturm-Liouville theory is a general theory for 2nd order ODEs.

Consider the following eigenvalue ODE:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \lambda r(x)y$$

where $r(x)$ is our weight function. We define the following boundary conditions:

1. $a_1 y(0) + a_2 y'(0) = 0$
2. $b_1 y(1) + b_2 y'(1) = 0$.

This is a **regular Sturm-Liouville** problem, where $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous functions and $p(x)$, $r(x)$ are strictly positive functions for $\forall x \in [0, 1]$.

Eigenvalues λ_n yield **eigenfunctions** $\phi_n(x)$ which are nontrivial solutions to our S-L problem. Important consequences include:

- Eigenvalues λ_n of a S-L problem are **real**.

  Furthermore each eigenvalue corresponds to one eigenfunction.

- Eigenfunctions $\phi_n(x)$ are orthogonal:

  $$\langle \phi_m, \phi_n \rangle = \int_0^1 r(x)\phi_m(x)\phi_n(x)\mathrm{d}x = \delta_{mn}$$

  in Hilbert space $L^2([0, 1], r(x)\mathrm{d}x)$.

Note that our eigenfunctions are orthonormal and so we define:

$$\phi_n(x) = k_n y_n(x)$$

where k_n is our scale factor. Since $\langle \phi_n, \phi_n \rangle = 1$:

$$\therefore \int_0^1 r(x)k_n^2 y_n^2(x)\mathrm{d}x = 1$$

and so we have that:

$$\begin{aligned} k_n &= \frac{1}{\sqrt{\langle y_n, y_n \rangle}} \\ &= \left(\int_0^1 r(x)y_n^2(x)\mathrm{d}x\right)^{-1/2}. \end{aligned}$$

4.1.1 General 2nd order ODEs

Consider the following general 2nd order eigenvalue ODE:

$$-P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\omega(x)\frac{\mathrm{d}y}{\mathrm{d}x}+q(x)y=\lambda r(x)y.$$

Multiply this by the following integrating factor:

$$F(x)=\exp\Big[\int_0^x\frac{\omega(s)-p'(s)}{p(s)}\mathrm{d}s\Big]$$

yields an ODE of S-L form:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big[F(x)P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\Big]+F(x)q(x)y=\lambda F(x)r(x)y.$$

4.1.2 Lagrange's identity

Our previous definition is motivated by the **Lagrange's identity**:

$$\begin{aligned}\langle\mathcal{L}[u],v\rangle-\langle u,\mathcal{L}[v]\rangle&=-\Big[p\Big(u'v^*-u(v^*)'\Big)\Big]_0^1\\&=-\Big[p(x)(\frac{\mathrm{d}u}{\mathrm{d}x}\cdot v^*-u\cdot\frac{\mathrm{d}v^*}{\mathrm{d}x})\Big]_0^1\end{aligned}$$

where $u=u(x)$, $v=v(x)$ are complex functions and

$$\mathcal{L}[u]=-\frac{\mathrm{d}}{\mathrm{d}x}\Big[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\Big]+q(x)u.$$

Here the inner product is defined as

$$\langle u,v\rangle=\int_0^1uv^*\mathrm{d}x$$

and we have integrated by parts using the following identities:

$$\big[pu'v^*\big]'=\big(pu'\big)'v^*+pu'(v^*)'$$

$$\big[pu(v^*)'\big]'=\Big(p(v^*)'\Big)'u+pu'(v^*)'.$$

For a S-L problem its properties follow from:

$$\langle\mathcal{L}[u],v\rangle=\langle u,\mathcal{L}[v]\rangle$$

where functions u and v satisfy its boundary conditions.

4.1.3 Series expansion

Now the set of orthonormal eigenfunctions $\{\phi_n(x)\}$ from a S-L problem with boundary conditions may be used to expand function $f(x)$:

$$f_\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

for $\forall x \in [0,1]$. Integrating this on both sides:

$$\begin{aligned}
\int_0^1 r(x)\phi_m(x)f(x)\mathrm{d}x &= \int_0^1 r(x)\phi_m(x)\sum_{n=1}^{\infty} c_n\phi_n(x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} c_n \int_0^1 r(x)\phi_m(x)\phi_n(x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} c_n \delta_{mn} \\
&= c_m
\end{aligned}$$

and so we get a general formula for our coefficients:

$$c_n = \int_0^1 r(x)\phi_n(x)f(x)\mathrm{d}x.$$

If $f(x)$ and $f'(x)$ are piecewise continuous on $x \in [0,1]$ then:

$$\forall x \in (0,1); \sum_{n=1}^{\infty} c_n\phi_n(x) = \frac{f(x^-)+f(x^+)}{2}$$

where these are the left and right handed limits.

4.1.4 General Parseval's identity for S-L problems

We have that:

$$\int_0^1 r(x)\big[f(x)\big]^2\mathrm{d}x = \sum_{n=1}^{\infty} c_n^2$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

and

$$c_n = \int_0^1 r(x)\phi_n(x)f(x)\mathrm{d}x.$$

4.2 Non-homogeneous S-L problems

Consider:

$$\mathcal{L}[y] = \mu r(x) y + f(x)$$

where $f(x)$ is our non-homogeneous term, and that we define

$$\mathcal{L}[y] = -\frac{\mathrm{d}}{\mathrm{d}x}\left[P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y.$$

Assume that this ODE satisfies regular S-L boundary conditions.

Firstly we solve the corresponding homogeneous S-L problem:

$$\mathcal{L}[y] = \lambda r(x) y$$

for non-trivial λ_n and $\phi_n(x)$.

Now let the general solution to our non-homogeneous S-L problem be:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

and substituting this yields:

$$\begin{aligned}\sum_{n=1}^{\infty} b_n \mathcal{L}[\phi_n(x)] &= r(x)\sum_{n=1}^{\infty} b_n \lambda_n \phi_n(x) \\ &= \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x).\end{aligned}$$

Rearranging then gives:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} b_n(\lambda_n - \mu)\phi_n(x).$$

Now we can also expand $\dfrac{f(x)}{r(x)}$ in terms of our eigenfunctions:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

with

$$\begin{aligned}c_n &= \int_0^1 r(x)\phi_n(x)\frac{f(x)}{r(x)}\mathrm{d}x \\ &= \int_0^1 \phi_n(x) f(x)\mathrm{d}x\end{aligned}$$

and so equating yields the following relation

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

4.3 Non-homogeneous PDEs

Consider the following non-homogeneous PDE:

$$r(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big[p(x)\frac{\partial u}{\partial x}\Big] - q(x)u(x,t) + F(x,t)$$

with boundary and initial conditions:

- $\dfrac{\partial}{\partial x}u(0,t) - h_1 u(0,t) = 0$
- $\dfrac{\partial}{\partial x}u(1,t) - h_2 u(1,t) = 0$
- $u(x,0) = f(x)$.

Firstly we solve the homogenous case via separation of variables. Let:

$$u(x,t) = X(x)T(t)$$

and after some algebra:

$$\frac{1}{rX}\Big[p'X' + pX'' - qX\Big] = \frac{\dot{T}}{T} = -\lambda.$$

This yields two ODEs:

$$\dot{T} + \lambda T = 0$$
$$-\big[pX'\big]' + qX = \lambda r X$$

with boundary conditions:

$$X'(0) - h_1 X(0) = 0$$

$$X'(1) - h_2 X(1) = 0$$

Clearly our second ODE is of S-L form. We are now going to assume that this regular S-L problem has non-trivial λ_n and orthonormal eigenfunctions $\phi_n(x)$.

Let the general solution to our PDE be:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x).$$

Substituting this into our PDE yields:

$$r(x)\sum_{n=1}^{\infty}\dot{b_n}(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Big[\Big(p(x)\phi_n'(x)\Big)' - q(x)\phi_n(x)\Big] + F(x,t).$$

Now since we have a S-L problem:

$$\Big(p(x)\phi_n'(x)\Big)' - q(x)\phi_n(x) = -\lambda_n \phi_n(x) r(x)$$

and after dividing through our PDE by $r(x)$ we get:

$$\sum_{n=1}^{\infty} \dot{b_n}(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Big[-\lambda_n\phi_n(x)\Big] + \frac{F(x,t)}{r(x)}.$$

But we can also expand our non-homogeneous term as a sum:

$$\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x)$$

and its coefficients are:

$$\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)\mathrm{d}x$$

in $L^2([0,1], r(x))$. Therefore after some algebra:

$$\sum_{n=1}^{\infty} \Big[\dot{b_n}(t) + \lambda_n b_n(t) - \gamma_n(t)\Big]\phi_n(x) = 0$$

and yields the following ODE:

$$\dot{b_n}(t) + \lambda_n b_n(t) - \gamma_n(t) = 0.$$

Solution is via integrating factors:

$$b_n(t) = e^{-\lambda_n t}\int_0^t \gamma_n(s)e^{\lambda_n s}\mathrm{d}s + e^{-\lambda_n t} b_n(0).$$

Finally using $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) = f(x)$$

and we get

$$b_n(0) = \int_0^1 r(x)f(x)\phi_n(x)\mathrm{d}x.$$

4.4 Singular S-L problems

Consider the following ODE:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+q(x)y=\lambda r(x)y$$

but now $p(x)$, $q(x)$ and $r(x)$ are discontinuous at $x=0$ and/or $x=1$.
This is a **singular** S-L problem with boundary condition:

$$b_1y(1)+b_2y'(1)=0$$

or

$$a_1y(0)+a_2y'(0)=0.$$

Now singular S-L problems at $x=0$ may be self-adjoint or that they yield:

- $\lambda_n\in\mathbb{R}$ (Real eigenvalues)
- $\langle\phi_m,\phi_n\rangle=\delta_{mn}$ (Orthogonal eigenfunctions)

if they satisfy Lagrange's identity. Consider singular S-L problem at $x=0$:

$$\begin{aligned}\int_\epsilon^1\Big(\mathcal{L}[u]v-u\mathcal{L}[v]\Big)\mathrm{d}x&=\Big[-p(x)\Big(u'(x)v(x)-u(x)v'(x)\Big)\Big]_\epsilon^1\\&=p(\epsilon)\Big(u'(\epsilon)v(\epsilon)-u(\epsilon)v'(\epsilon)\Big)\end{aligned}$$

and tends to zero if and only if:

$$\lim_{\epsilon\to\infty}\Big[p(\epsilon)\Big(u'(\epsilon)v(\epsilon)-u(\epsilon)v'(\epsilon)\Big)\Big]=0,$$

where we define:

$$\mathcal{L}[u]=-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right]+q(x)u.$$

Therefore this is the condition for our singular S-L problem to have real eigenvalues and orthogonal eigenfunctions.

Similarly for singular S-L problem at $x=1$ it is self-adjoint if:

$$\lim_{\epsilon\to\infty}\Big[p(1-\epsilon)\Big(u'(1-\epsilon)v(1-\epsilon)-u(1-\epsilon)v'(1-\epsilon)\Big)\Big]=0.$$

Finally whilst eigenvalues may be real they are not necessarily discrete!

5 Laplace transforms

So let $f(t)$ be defined for $t \in [0, \infty)$. Its Laplace transform is:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

Let functions of exponential order be defined as:

$$\forall t \in [0, \infty); \exists A > 0 \text{ and } B \in \mathbb{R} : |f(t)| \leq Ae^{Bt}$$

where f is piecewise continous and we denote such functions as $f \in E$.

For sufficiently large s, the Laplace transform $f \in E$ converges.

5.1 Properties

5.1.1 Inversion formula

Now let $F(s) = \mathcal{L}[f(t)]$. We have the following inversion formula:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) \mathrm{d}s. \end{aligned}$$

5.1.2 Reduction of order

Applying the Laplace transform to derivatives:

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

for $\forall f, f' \in E$ and generalising this via induction gives:

$$\begin{aligned} \mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{(n-1)} f^{(0)}(0) - s^{(n-2)} f^{(1)}(0) \\ - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0). \end{aligned}$$

5.1.3 Shifts, scaling and derivatives

5.2 Standard transforms

include partial fraction theory

5.3 Applications

5.3.1 Higher order ODEs

5.3.2 Piecewise continuous source term

5.3.3 Impulse functions

5.3.4 Convolutions