

**D1.1.1: Complex numbers**

Let  $z = x + iy$  and  $w = a + ib$  where  $x, y, a, b \in \mathbb{R}$ . Then  $z$  and  $w$  are complex numbers. Furthermore:

1.  $z = w$  **iff**  $x = a$  and  $y = b$ .
2.  $\operatorname{Re}(z) := x$  and  $\operatorname{Im}(z) := y$ .
3.  $|z| := \sqrt{x^2 + y^2}$
4. The **complex conjugate** of  $z$  is:  
$$z^* := x - iy.$$
5. Addition and multiplication:  
$$(x + iy) + (a + ib) = (x + a) + i(y + b)$$
$$(x + iy)(a + ib) = (xa - yb) + i(xb + ya).$$
6.  $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$

with rule  $i^2 = -1$ .

**L1.1.3**

Let  $u, w, z \in \mathbb{C}$  where  $z = x + iy$ . Then:

1.  $z + w = w + z$  and  $zw = wz$ .
2.  $u + (z + w) = (u + z) + w$
3.  $u(zw) = (uz)w$
4.  $u(z + w) = uz + uw$
5.  $z + 0 = z$  and  $1z = z$ .
6.  $\exists(-z := -x + i(-y)) : z + (-z) = 0$ .
7.  $\exists z^{-1} : zz^{-1} = 1$  where:

$$z^{-1} := \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}.$$

**D1.1.5 and D1.1.7: Polar form**

Let  $z \in \mathbb{C}$  and  $z = x + iy$ . Then:

$$z = r(\cos \theta + i \sin \theta)$$

$$= re^{i\theta}$$

for  $r = |z| = \sqrt{x^2 + y^2}$  and  $\theta \in (-\pi, \pi]$  is given by  $\tan \theta = y/x$ .

**L1.1.6**

Let  $\theta, \phi \in \mathbb{R}$  and  $n \in \mathbb{Z}$ . Then:

1.  $e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$
2.  $e^{in\theta} = (e^{i\theta})^n$

due to de Moivre's formula:

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$$

**L1.1.9**

Let  $z, w \in \mathbb{C}$ . Then:

1.  $|z| = 0$  **iff**  $z = 0$ .
2.  $|\bar{z}| = |z|$
3.  $|zw| = |z||w|$
4.  $(z^*)^* = z$
5.  $|z|^2 = zz^*$  and  $|z|^2 = |z|^2$ .
6.  $(z + w)^* = z^* + w^*$
7.  $(zw)^* = z^* w^*$
8.  $|\operatorname{Re}(z)| \leq |z|$  and  $|\operatorname{Im}(z)| \leq |z|$ .
9.  $\operatorname{Re}(z) = \frac{1}{2}(z + z^*)$
10.  $\operatorname{Im}(z) = \frac{1}{2i}(z - z^*)$ .

**L1.1.10 – 11: Triangle inequalities**

Let  $z, w \in \mathbb{C}$ . Then:

1.  $|z + w| \leq |z| + |w|$
2.  $||z| - |w|| \leq |z - w|$ .

**D1.1.12: Argument of  $z$** 

The set of all arguments of  $z$  is:

$$\arg(z) := \{\theta : z = |z|e^{i\theta}\}$$

$$= \{\operatorname{Arg}(z) + 2\pi k : k \in \mathbb{Z}\}.$$

The **principle argument of  $z$**  satisfies  $z = |z|e^{i\operatorname{Arg}(z)}$  with  $-\pi < \operatorname{Arg}(z) \leq \pi$ .

$$\therefore \operatorname{Arg}(z) \equiv \arg(z) \pmod{2\pi}$$

$\operatorname{Arg}(z)$  is discontinuous on the negative real axis since  $\forall x, \epsilon > 0; -x \pm i\epsilon \rightarrow x$ :

$$\lim_{\epsilon \rightarrow 0} \operatorname{Arg}(-x \pm i\epsilon) = \pm\pi.$$

**P1.1.14**

Let  $z, w \in \mathbb{C}$ . Then:

1.  $\arg(zw) = \arg(z) + \arg(w)$
2.  $\arg(z^*) = -\arg(z)$

for these are set operations.

**D1.2.1: Open and closed  $\epsilon$ -discs**

Let  $z_0 \in \mathbb{C}$  and  $\epsilon > 0$ .

1. An **open**  $\epsilon$ -disc centred at  $z_0$  is:

$$D_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}.$$

2. A **closed**  $\epsilon$ -disc centred at  $z_0$  is:

$$\bar{D}_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}.$$

A **punctured**  $\epsilon$ -disc centred at  $z_0$  is:

$$D'_\epsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}.$$

**D1.2.2: Open and closed sets**

Let  $U \subset \mathbb{C}$ . Set  $U$  is **open** if:

$$\forall z_0 \in U; \exists \epsilon > 0 : D_\epsilon(z_0) \subseteq U.$$

Subset  $F$  is **closed** if  $\mathbb{C} \setminus F$  is open.

A **neighbourhood** of point  $z_0 \in \mathbb{C}$  is an open set that contains  $z_0$ .

**Remark**

$\emptyset$  is **vacuously** open. Therefore  $\mathbb{C}$  is open **and** closed. A set like  $D_2(0) \setminus D_1(0)$  is **neither closed nor open**.

The union and intersection of open sets is also an open set.

**L1.2.3**

Punctured disc  $D'_\epsilon(z_0)$  is open.

**D1.2.4: Limit points**

Let  $S \subseteq \mathbb{C}$ .  $z_0$  is a **limit point** of  $S$  if:

$$\forall \epsilon > 0; D'_\epsilon(z_0) \cap S \neq \emptyset.$$

The **closure** of  $S$  is set  $\bar{S}$  and contains  $S$  and **all** its limit points.

**L1.2.6**

Let  $S \subseteq \mathbb{C}$ .  $S$  is closed **iff**  $S = \bar{S}$ .

**D1.2.7: Bounded sets**

Let  $S \subseteq \mathbb{C}$ . Set  $S$  is **bounded** if:

$$\forall z \in S; \exists M > 0 : |z| \leq M.$$

**D1.2.8:  $\epsilon$ -N convergence**

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

Let  $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$  be a sequence and  $z \in \mathbb{C}$ . Then  $\lim_{n \rightarrow \infty} z_n = z$  if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N$$

$$\implies |z_n - z| < \epsilon.$$

**L1.2.9**

Let  $z_n, z \in \mathbb{C}$  where  $z_n = a_n + ib_n$ .

Then  $\lim_{n \rightarrow \infty} z_n = z$  **iff**:

$$\operatorname{Re}(z) = \lim_{n \rightarrow \infty} a_n \text{ and } \operatorname{Im}(z) = \lim_{n \rightarrow \infty} b_n.$$

**L1.2.10**

Let  $S \subseteq \mathbb{C}$  and  $z \in \mathbb{C}$ . Then  $z \in \bar{S}$  **iff**:

$$\exists z_n \in S : z = \lim_{n \rightarrow \infty} z_n.$$

**D1.2.11: Cauchy sequences**

$z_n$  is a Cauchy sequence if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \\ \implies |z_n - z_m| < \epsilon.$$

**L1.2.12**

$z_n$  is convergent **iff**  $z_n$  is Cauchy.

**D1.2.14: Bounded sequences**

$z_n$  is bounded if:

$$\forall n \in \mathbb{N}; \exists M > 0 : |z_n| \leq M.$$

**L1.2.15: Bolzano-Weierstrass**

Let  $z_n$  be a bounded sequence. Then:

$$\exists (z_{n_k})_{k, n_k \in \mathbb{N}} : \lim_{k \rightarrow \infty} z_{n_k} = z \in \mathbb{C}$$

or that  $z_n$  has a convergent subsequence.

A selection of a sequence is a subsequence.

**D1.3.1: Bounded functions**

Let  $S \subseteq \mathbb{C}$  and  $f : S \rightarrow \mathbb{C}$ . Then  $f$  is a bounded function if:

$$\forall z \in S; \exists M > 0 : |f(z)| \leq M.$$

**D1.3.2:  $\epsilon$ - $\delta$  convergence**

Let  $S \subseteq \mathbb{C}, z_0 \in \bar{S}, f : S \rightarrow \mathbb{C}$  and  $a_0 \in \mathbb{C}$ . Then  $\lim_{z \rightarrow z_0} f(z) = a_0$  if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : 0 < |z - z_0| < \delta \\ \implies |f(z) - a_0| < \epsilon.$$

**L1.3.3**

Let  $S \subseteq \mathbb{C}, z_0 \in \bar{S}, f : S \rightarrow \mathbb{C}$  and  $a_0 \in \mathbb{C}$  where  $z_0 = x_0 + iy_0$  and  $f = u + iv$ .

Then  $\lim_{z \rightarrow z_0} f(z) = a_0$  **iff**:

$$\operatorname{Re}(a_0) = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} u(x, y)$$

and

$$\operatorname{Im}(a_0) = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} v(x, y).$$

**L1.3.4**

Let  $S \subseteq \mathbb{C}, z_0 \in \bar{S}, f : S \rightarrow \mathbb{C}, a_0 \in \mathbb{C}$  and sequence  $w_n \in S \setminus \{z_0\}$ .

If  $\lim_{z \rightarrow z_0} f(z) = a_0$  and  $\lim_{n \rightarrow \infty} w_n = z_0$  then:

$$\lim_{n \rightarrow \infty} f(w_n) = a_0.$$

**L1.3.5: Limit identities**

Let  $S \subseteq \mathbb{C}, z_0 \in \bar{S}$  and  $a_0, b_0 \in \mathbb{C}$ .

Let  $f, g : S \rightarrow \mathbb{C}$ .

If  $\lim_{z \rightarrow z_0} f(z) = a_0$  and  $\lim_{z \rightarrow z_0} g(z) = b_0$  then:

1.  $\lim_{z \rightarrow z_0} (f(z) + g(z)) = a_0 + b_0$
2.  $\lim_{z \rightarrow z_0} (f(z)g(z)) = a_0b_0$
3.  $\lim_{z \rightarrow z_0} \left( \frac{f(z)}{g(z)} \right) = \frac{a_0}{b_0}$  if  $b_0 \neq 0$ .

**D1.3.6:  $\epsilon$ - $\delta$  continuity**

Let  $S \subseteq \mathbb{C}, f : S \rightarrow \mathbb{C}$  and  $z_0 \in S$ . Then  $f$  is continuous at  $z_0$  if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : |z - z_0| < \delta \\ \implies |f(z) - f(z_0)| < \epsilon.$$

**L1.3.7**

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  with rule  $f = u + iv$  and  $z_0 = x_0 + iy_0 \in \mathbb{C}$ .

Then  $f$  is continuous at  $z_0$  **iff**  $u$  and  $v$  are continuous at  $(x_0, y_0)$ .

**L1.3.8**

If  $f, g : \mathbb{C} \rightarrow \mathbb{C}$  are continuous at  $z_0$  then:

1.  $f + g$  is continuous at  $z_0$ .
2.  $fg$  is continuous at  $z_0$ .
3.  $f/g$  is continuous at  $z_0$ . ( $g \neq 0$ )

**D: Image and preimage**

Let  $f : X \rightarrow Y$  where  $A \subseteq X$  and  $B \subseteq Y$ . The image of  $A$  is:

$$f(A) = \{f(x) : x \in A\}$$

and the preimage of  $B$  is:

$$f^{-1}(B) = \{x : f(x) \in B\}.$$

**L1.3.9**

Let  $U \subseteq \mathbb{C}$  be an open set.  $f : \mathbb{C} \rightarrow \mathbb{C}$  is continuous **iff**  $\forall U \subseteq \mathbb{C}; f^{-1}(U)$  is open for  $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\}$ .

**L1.3.10**

Let  $f : S \rightarrow \mathbb{C}$  be continuous. Let  $S \subseteq \mathbb{C}$  be closed and bounded.

Then  $f(S)$  is closed and bounded.

**D1.4.1: Differentiability**

Let  $z_0 \in \mathbb{C}$  and  $U$  a neighbourhood of  $z_0$ . Let  $f : U \rightarrow \mathbb{C}$ . Then  $f$  is differentiable at  $z_0$  if the following limit exists:

$$f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

**L1.4.3**

Differentiability  $\implies$  continuity.

Let  $z_0 \in \mathbb{C}$  and  $U$  a neighbourhood of  $z_0$ . If  $f : U \rightarrow \mathbb{C}$  is differentiable at  $z_0$  then  $f$  is continuous at  $z_0$ .

**L1.4.4**

Let  $z_0 \in \mathbb{C}$  and  $U$  a neighbourhood of  $z_0$ . Let  $f, g : U \rightarrow \mathbb{C}$  be differentiable at  $z_0$ . Then  $f + g, fg$  and  $f/g$  (where  $g(z_0) \neq 0$ ) are all differentiable at  $z_0$ .

**L1.4.5: Chain rule**

Let  $z_0 \in \mathbb{C}$  and  $U$  a neighbourhood of  $z_0$ . Let  $g : U \rightarrow \mathbb{C}$  be such that  $g(U)$  is a neighbourhood of  $g(z_0)$ . Assume that  $g$  is differentiable at  $z_0$  and  $f$  is differentiable at  $g(z_0)$ . Then  $f \circ g$  is differentiable at  $z_0$ :

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

**T1.4.6: Cauchy-Riemann equations**

Let  $z_0 \in \mathbb{C}$  and  $U$  a neighbourhood of  $z_0$ . Let  $f : U \rightarrow \mathbb{C}$  be differentiable at  $z_0$ . Let  $z_0 = x_0 + iy_0$  and  $f = u + iv$ . Then:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

and are the Cauchy-Riemann equations.

**T1.4.8**

Let  $z_0 \in \mathbb{C}$  and  $U$  a neighbourhood of  $z_0$  for  $z_0 = x_0 + iy_0$ . Let  $f : U \rightarrow \mathbb{C}$  where  $f = u + iv$ .

Assume that  $u$  and  $v$  have **continuous first derivatives** on a neighbourhood of  $(x_0, y_0)$  **and** also that they **satisfy the Cauchy Riemann equations** at  $(x_0, y_0)$ .

Then  $f$  is differentiable at  $z_0$ .

**D1.4.9: Holomorphic functions**

$f$  is **holomorphic** at  $z_0$  if there exists a neighbourhood  $U$  of  $z_0$  such that  $f$  is defined and differentiable.

**D1.4.13: Harmonic equations**

$h(x, y)$  is harmonic if for  $\forall (x, y) \in \mathbb{R}^2$  it satisfies Laplace's equation:

$$\frac{\partial^2 h}{\partial x^2}(x, y) + \frac{\partial^2 h}{\partial y^2}(x, y) = 0.$$

**L1.4.14**

Let  $u(x, y), v(x, y)$  be twice continuously differentiable and that  $f(x + iy) = u + iv$  is holomorphic on  $\mathbb{C}$ .

Then  $u$  and  $v$  are harmonic.

**D1.4.15: Harmonic conjugates**

Let  $U \subseteq \mathbb{R}^2$  and  $u : U \rightarrow \mathbb{R}$  be harmonic. Then harmonic function  $v : U \rightarrow \mathbb{R}$  is a **harmonic conjugate** of  $u$  if complex function  $f = u + iv$  is holomorphic on  $U$ .

**D1.5.1: Polynomial degree**

Let  $P : \mathbb{C} \rightarrow \mathbb{C}$  be a polynomial. The **degree** of  $P$  is the highest power of the variable in  $P$ , denoted as  $\deg(P)$ .

**L1.5.2**

Let  $z_0 \in \mathbb{C}$ . Let complex functions  $f$  and  $g$  be holomorphic at  $z_0$ . Then  $f + g$ ,  $fg$  and  $f/g$  ( $g \neq 0$ ) are holomorphic at  $z_0$ .

**C1.5.3**

Let  $N \in \mathbb{N}$  and  $a_0, \dots, a_N \in \mathbb{C}$ .

Let  $P(z) = \sum_{n=0}^N a_n z^n$ .

Then  $P(z)$  is holomorphic on  $\mathbb{C}$  and:

$$P'(z_0) = \sum_{n=1}^N n a_n z_0^{n-1}.$$

**L1.5.4**

Let  $P(z) = \sum_{n=0}^N a_n z^n$  where  $a_i \in \mathbb{R}$  and  $P(z_0) = 0$  for  $z_0 \in \mathbb{C}$ . Then  $P(z_0^*) = 0$ .

**D1.5.5: Rational functions**

Let  $P, Q : \mathbb{C} \rightarrow \mathbb{C}$  be complex functions. Then  $R : \{z \in \mathbb{C} : Q(z) \neq 0\} \rightarrow \mathbb{C}$  with  $R(z) = P(z)/Q(z)$  is a rational function.

**L1.5.7**

The rational function  $R(z) = P(z)/Q(z)$  is holomorphic on  $\{z \in \mathbb{C} : Q(z) \neq 0\}$ .

**L1.5.8**

Let  $U \subseteq \mathbb{C}$  be open. Let  $g$  be holomorphic on  $U$  and  $f$  be holomorphic on  $g(U)$ .

Then  $f \circ g$  is holomorphic on  $U$ .

**L1.5.10**

Let  $U \subseteq \mathbb{R}^2$  be open and  $u, v : U \rightarrow \mathbb{R}$ .  $u$  and  $v$  satisfy the Cauchy-Riemann equations **iff**  $\bar{\partial}f = 0$ , where  $f = u + iv$  with map  $f : U \rightarrow \mathbb{C}$ .

**Remark**

$$\partial := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

$$\bar{\partial} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

**D1.6.1: Exponential function**

The complex exponential function is a function defined as  $\exp : \mathbb{C} \rightarrow \mathbb{C}$  and rule:

$$\exp(z) := e^x (\cos y + i \sin y)$$

for  $z = x + iy$  and  $|z| = e^x$ .

**P1.6.2**

Let  $z, w \in \mathbb{C}$ .

1.  $\exp(z)$  is holomorphic on  $\mathbb{C}$ .
2.  $\exp(z) = \exp'(z)$
3.  $\exp(z + w) = \exp(z) \exp(w)$
4.  $\exp(z + 2\pi i) = \exp(z)$

**D1.6.6: Cosine and sine functions**

$$\cos(z) := \frac{1}{2} (\exp(iz) + \exp(-iz))$$

$$\sin(z) := \frac{1}{2i} (\exp(iz) - \exp(-iz))$$

**L1.6.7**

Let  $z \in \mathbb{C}$  where  $z = x + iy$ . Then:

1.  $\cos(z)$  and  $\sin(z)$  are holomorphic at  $z$ , with  $\cos'(z) = -\sin(z)$  and  $\sin'(z) = \cos(z)$ .
2.  $\cos^2(z) + \sin^2(z) = 1$
3.  $\cos(z + 2\pi) = \cos(z)$   
 $\sin(z + 2\pi) = \sin(z)$

**L1.6.8**

Let  $z, w \in \mathbb{C}$ . Then:

1.  $\sin(z + \pi/2) = \cos(z)$
2.  $\sin(z + w)$   
 $= \sin(z) \cos(w) + \sin(w) \cos(z)$
3.  $\cos(z + w)$   
 $= \cos(z) \cos(w) - \sin(z) \sin(w)$ .

**L1.6.9**

Let  $z \in \mathbb{C}$  where  $z = x + iy$ . Then:

$$\begin{aligned} \sin(x + iy) &= \sin(x) \cosh(y) + i \cos(x) \sinh(y) \\ \cos(x + iy) &= \cos(x) \cosh(y) - i \sin(x) \sinh(y). \end{aligned}$$

**D1.6.11: Hyperbolic functions**

$$\cosh(z) := \frac{1}{2} (\exp(z) + \exp(-z))$$

$$\sinh(z) := \frac{1}{2} (\exp(z) - \exp(-z))$$

**L1.6.12**

Let  $z \in \mathbb{C}$ . Then  $\sinh(iz) = i \sin(z)$  and  $\cosh(iz) = \cos(z)$ .

**D1.7.1: Logarithm function**

Let  $z \neq 0 \in \mathbb{C}$ . Then:

$$\log(z) := \{w \in \mathbb{C} : z = \exp(w)\}$$

and is the complex **natural** logarithm.

**L1.7.3**

Let  $z, w \in \mathbb{C}$  be nonzero. Then:

1.  $\log(z) = \{\ln|z| + i\text{Arg}(z) + i2\pi k\}$
2.  $\log(zw) = \log(z) + \log(w)$
3.  $\log(1/z) = -\log(z)$

where  $k \in \mathbb{Z}$  and  $\ln(x)$  denotes the real valued natural logarithm of  $x$ .

**D1.7.5: Principle branch of  $\log z$** 

The principle branch of the logarithm function is defined as:

$$\text{Log} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C};$$

$$\text{Log}(z) := \ln|z| + i\text{Arg}(z)$$

and is **discontinuous on the negative real axis** since  $\forall x, \epsilon > 0; -x \pm i\epsilon \rightarrow x$  yet:

$$\lim_{\epsilon \rightarrow 0} \text{Log}(-x \pm i\epsilon) = \ln|x| \pm i\pi.$$

i.e. the limit on the axis does not exist.

**D1.7.7: Branch cuts**

A branch cut  $L \subset \mathbb{C}$  is removed so that we may define a holomorphic branch of a multivalued function on  $\mathbb{C} \setminus L$ .

The half-line from  $z_0$  at angle  $\phi$  is:

$$L_{z_0, \phi} = \{z \in \mathbb{C} : z = z_0 + re^{i\phi}; r \geq 0\}$$

and  $D_{z_0, \phi} = \mathbb{C} \setminus L_{z_0, \phi}$ .

**D1.7.9**

Let  $\phi \in \mathbb{R}$ . Then:

$$\phi < \text{Arg}_\phi(z) \leq \phi + 2\pi$$

$$\text{Log}_\phi(z) := \ln|z| + i\text{Arg}_\phi(z).$$

**L1.7.10**

Branch  $\text{Log}_\phi(z)$  is holomorphic on  $D_{0, \phi}$ :

$$\forall z \in D_{0, \phi}; \frac{d}{dz} [\text{Log}_\phi(z)] = \frac{1}{z}.$$

**L1.7.11**

Let  $\phi \in \mathbb{R}$ ,  $U \subseteq \mathbb{C}$  be open and  $g : U \rightarrow \mathbb{C}$  be holomorphic on  $U$ . Then  $\text{Log}_\phi(g(z))$  is holomorphic on  $U \cap g^{-1}(D_\phi)$ .

**D1.8.1:  $\alpha$ -th power of  $z$** 

Let  $z, \alpha \in \mathbb{C}$ . Then the  $\alpha$ -th power of  $z$  is:  $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\}$  for  $z \neq 0$ .

**T1.8.4**

Let  $\alpha, z \neq 0 \in \mathbb{C}$ .

1. If  $\alpha \in \mathbb{Z}$  there is one value of  $z^\alpha$ .
2. If  $\alpha = p/q \in \mathbb{Q}$  for  $p, q$  are coprime then there are  $q$  values of  $z^\alpha$ .
3. If  $\alpha$  is irrational or complex then there are infinite values of  $z^\alpha$ .

**D1.8.5: Roots of unity**

Let  $q$  be a positive integer. Then:

$$1^{1/q} = 1, \omega, \dots, \omega^{q-1}, \omega := \exp(2\pi i/q)$$

are the  $q$  roots of unity.

**D1.8.7: Principle branch of  $z^\alpha$** 

Let  $z \in D$  such that  $\text{Log}(z)$  is defined. Then the principle branch of  $z^\alpha$  is:

$$z^\alpha := \exp(\alpha \text{Log}(z)).$$

**L1.8.8**

Let  $\alpha, \beta, z \in \mathbb{C}$  for  $z \neq 0$ . Then:

$$z^\alpha z^\beta = z^{\alpha+\beta}.$$

**L1.8.9**

A branch of  $z^\alpha$  is holomorphic on  $D_\phi$  and:

$$\forall z \in D_\phi; (z^\alpha)' = \alpha z^{\alpha-1}.$$

**D2.1.1: Conformal maps**

Let  $U \subseteq \mathbb{C}$  be open and let  $f : U \rightarrow \mathbb{C}$ .  $f$  is **conformal** if it preserves angles.

i.e. that the angle between **tangent** lines must remain invariant under mapping.

**T2.1.2**

Let  $U \subseteq \mathbb{C}$  be open and let  $f : U \rightarrow \mathbb{C}$  be a holomorphic function. Define  $T \subseteq U$ :

$$T = \{z_0 \in U : f'(z_0) \neq 0\}.$$

Then  $f$  preserves angles at every  $z_0 \in T$ .

i.e.  $f$  is a conformal mapping on  $T$ .

**D2.2.1: Möbius transformations**

$f$  is a Möbius transformation if:

$$f(z) = \frac{az + b}{cz + d} \text{ where } a, b, c, d \in \mathbb{C},$$

$ad \neq bc$  and normalisation  $ad - bc = 1$ .

**L2.2.3**

Define  $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $\det(M) = 1$  to be associated with the following:

$$f_M(z) = \frac{az + b}{cz + d}.$$

Then  $f_{M^{-1}} = f_M^{-1}$  and:

$$f_{M_1 M_2} = f_{M_1} \circ f_{M_2}.$$

**D2.3.1: Extended complex plane**

The extended complex plane is the set:

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

such that for all  $a, b \neq 0 \in \mathbb{C}$ :

$$a + \infty = \infty, \quad b \cdot \infty = \infty,$$

$$\frac{b}{0} = \infty \text{ and } \frac{b}{\infty} = 0.$$

**D2.3.2.1: Riemann spheres**

The Riemann sphere is the unit sphere  $S^2$  in  $\mathbb{R}^3$  defined by:

$$S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$$

with north pole  $N := (0, 0, 1)$ .

**D2.3.2.2: Stereographic projections**

Let  $\phi : \tilde{\mathbb{C}} \rightarrow S^2$  be a bijective mapping such that points  $z \in \tilde{\mathbb{C}}$  and  $\phi(z), N \in S^2$  are **colinear**. Then from calculation:

$$\phi(z) = \left( \frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

$$\lim_{|z| \rightarrow \infty} \phi(z) = N$$

where we denote  $z = x + iy = (x, y, 0)$ .

The **stereographic projection** is the inverse mapping  $\psi : S^2 \rightarrow \tilde{\mathbb{C}}$  of  $\phi$  where:

$$\psi(X, Y, Z) = \begin{cases} \frac{X+iY}{1-Z} & (X, Y, Z) \neq N \\ \infty & (X, Y, Z) = N \end{cases}$$

since we define  $\phi(\infty) := N$ .

**L2.3.4**

Stereographic projections maps a circle to a **circline**. (i.e. circle or line)

**D2.4.1**

1. **Translations:**  $f(z) = z + b$  where  $b \in \mathbb{C}$ .
2. **Rotations:**  $f(z) = az$  where  $a = e^{i\theta}$  and  $a \in \mathbb{C}$ .
3. **Dilations:**  $f(z) = rz$  where  $r > 0 \in \mathbb{R}$ .
4. **Inversions:**  $f(z) = 1/z$ .

**T2.4.2**

Let  $f$  be a Möbius transformation.

Then  $f$  consists of a **finite composition** of translations, rotations, dilations and inversions **iff**:

$$f(\infty) \neq \infty$$

i.e.  $f$  does not fix the point at infinity.

**C2.4.3**

If  $f$  is a Möbius transformation then it maps circlines to circlines.

**L2.5.1**

Let  $f$  be a Möbius transformation and let  $z_2, z_3, z_4 \in \tilde{\mathbb{C}}$  be distinct points such that  $f(z_2) = z_2, f(z_3) = z_3$  and  $f(z_4) = z_4$ .

Then  $f(z) = z$ . (identity transformation)

**T2.5.2**

Given distinct points  $z_2, z_3, z_4 \in \tilde{\mathbb{C}}$  there exists a unique Möbius transformation  $f$ :

$$f(z_2) = 1, \quad f(z_3) = 0 \text{ and } f(z_4) = \infty.$$

Explicitly this mapping is given by:

$$f(z) = \frac{z_2 - z_4}{z_2 - z_3} \frac{z - z_3}{z - z_4}.$$

**C2.5.3**

Let  $z_2, z_3, z_4, w_2, w_3, w_4 \in \tilde{\mathbb{C}}$  be distinct points. Then there is a unique Möbius transformation  $f$  such that:

$$f(z_2) = w_3, \quad f(z_3) = w_3 \text{ and } f(z_4) = w_4.$$

**D2.5.4: Cross ratios**

Let  $z_1, z_2, z_3, z_4 \in \tilde{\mathbb{C}}$  be distinct points and let  $f$  be a Möbius transformation that maps  $(z_2, z_3, z_4) \mapsto (1, 0, \infty)$ .

Then the **cross ratio** is defined as:

$$[z_1, z_2, z_3, z_4] := f(z_1).$$

**T2.5.6**

Let  $z_1, z_2, z_3, z_4 \in \tilde{\mathbb{C}}$  be distinct and let  $f$  be a Möbius transformation. Then:

$$[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4].$$

**D3.1.1: Integrable functions**

Let  $f : [a, b] \rightarrow \mathbb{C}; f = u + iv$ . Then  $f$  is **integrable** if  $u(t)$  and  $v(t)$  are integrable.

$$\therefore \int_a^b f := \int_a^b u + i \int_a^b v \in \mathbb{C}.$$

$f$  is integrable if it is continuous.

**L3.1.2**

Let  $f, g : [a, b] \rightarrow \mathbb{C}$  be integrable and  $\alpha, \beta \in \mathbb{C}$ . Then:

- $\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$
- Let  $f = F'$  be continuous and that  $F : [a, b] \rightarrow \mathbb{C}$ . Then:

$$\int_a^b f(t)dt = F(b) - F(a).$$

$$3. \left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt.$$

**D3.2.1: Contours**

A contour  $\Gamma \subset \mathbb{C}$  is a curve that connects  $z_0$  to  $z_1 \in \mathbb{C}$ . We define  $\Gamma = \text{im}(\gamma)$  where:

$$\gamma : [t_0, t_1] \rightarrow \mathbb{C}; \gamma(t_0) = z_0 \text{ and } \gamma(t_1) = z_1.$$

Contour  $\Gamma$  is **regular** if its first derivative is continuous and  $\gamma'(t) \neq 0$  for  $\forall t$ .

**D3.2.3: Contour integrals**

Let  $\Gamma$  be a regular curve connecting points  $z_0$  and  $z_1$ . Let  $f : \Gamma \rightarrow \mathbb{C}$  be continuous. Then the integral of  $f$  along  $\Gamma$  is:

$$\int_{\Gamma} f(z)dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt.$$

Let there exist  $\gamma_i : [t_0^i, t_1^i] \rightarrow \mathbb{C}$  such that  $\gamma_i(t_0^i) = z_0$ ,  $\gamma_i(t_1^i) = \gamma_{i+1}(t_0^{i+1})$  and  $\gamma_n(t_1^n) = z_1$  where  $\Gamma_i = \text{im}(\gamma_i)$ . Then:

$$\int_{\Gamma} f(z)dz = \sum_{i=1}^n \int_{\Gamma_i} f(z)dz.$$

**D3.2.7: Contour arclengths**

The **arclength** of a regular curve  $\Gamma$  is:

$$\begin{aligned} \ell(\Gamma) &= \int_{t_0}^{t_1} |\gamma'(t)|dt \\ &= \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}dt. \end{aligned}$$

If  $\Gamma$  is the arc of a circle with radius  $r$  traced by an angle  $\theta$  then  $\ell(\Gamma) = r\theta$ .

**L3.2.9: M-L lemma**

Let  $\Gamma$  be regular and let  $f : \Gamma \rightarrow \mathbb{C}$  be a continuous function. Then:

$$\left| \int_{\Gamma} f(z)dz \right| \leq \max_{z \in \Gamma} |f(z)|\ell(\Gamma).$$

**D3.3.1: Domains**

Set  $D$  is a **domain** if it is **open** and every two points in  $D$  is connected by a contour that is fully contained in  $D$ .

**L3.3.2**

Let  $D$  be a domain and let  $u : D \rightarrow \mathbb{R}$  be differentiable, where  $u'_x = u'_y = 0$  on  $D$ . Then  $u(x, y)$  is constant on  $D$ .

**D3.3.3: Antiderivatives**

Let  $D$  be a domain and let  $f : D \rightarrow \mathbb{C}$  be continuous.  $f$  has an antiderivative **on  $D$**  if  $\exists F : D \rightarrow \mathbb{C} : \forall z \in D; F'(z) = f(z)$ .

**T3.3.5: FTC**

Let  $D$  be a domain and let continuous  $f : D \rightarrow \mathbb{C}$  have an antiderivative  $F$  on  $D$ . If contour  $\Gamma \subset D$  **connects  $z_0$  to  $z_1$** :

$$\int_{\Gamma} f(z)dz = F(z_1) - F(z_0).$$

**C3.3.6**

Let  $f$  be holomorphic on domain  $D$  and  $f'(z) = 0$  for  $\forall z \in D$ . Then  $f$  is constant.

**D3.3.7: Closed contours**

$\Gamma$  is **closed** if its endpoints are the same.

**L3.3.9: Path independence**

Let  $f : D \rightarrow \mathbb{C}$  be continuous where  $D$  is a domain. The following are **equivalent**:

- $f$  has an antiderivative on  $D$ .
- For all **closed** contours  $\Gamma \subset D$ :

$$\oint_{\Gamma} f(z)dz = 0.$$

- Integrals are independent of path, regardless of contour chosen in  $D$ .

**D3.4.1: Loops**

$\Gamma$  is **simple** if it has no self intersections except at the endpoints.

**Loops** are **simple** and **closed** contours.

**T3.4.2: Jordan curve theorem**

Let  $\Gamma$  be a loop in  $\mathbb{C}$ . Then  $\Gamma$  defines the following two regions:

- bounded interior:  $\text{Int}(\Gamma)$
- unbounded exterior:  $\text{Ext}(\Gamma)$

where  $\mathbb{C} = \text{Int}(\Gamma) \cup \Gamma \cup \text{Ext}(\Gamma)$ .

**D3.4.4: Positively oriented loops**

Loop  $\Gamma$  is **positively oriented** if  $\text{Int}(\Gamma)$  is always remain on the **left** hand side when traversing its parametrisation.

**D3.4.6: Simply connected domains**

Domain  $D$  is **simply connected** if:

for all **loops**  $\Gamma \subset D; \text{Int}(\Gamma) \subseteq D$ .

**T3.4.8: Cauchy integral theorem**

Let  $\Gamma$  be a **loop**. Let  $f$  be holomorphic inside and on contour  $\Gamma$ . Then:

$$\int_{\Gamma} f(z)dz = 0.$$

**C3.4.9**

Let  $D$  be a simply connected domain and let  $f$  be holomorphic on  $D$ . Then  $f$  has an antiderivative on  $D$ .

**T3.4.11**

Consider loop  $\Gamma$  and point  $z_0 \notin \Gamma$ . Then:

$$\int_{\Gamma} \frac{1}{z - z_0} dz = \begin{cases} 2\pi i & z_0 \in \text{Int}(\Gamma) \\ 0 & \text{otherwise.} \end{cases}$$

**T3.4.12: Deformation theorem**

Let  $f$  be holomorphic on loops  $\Gamma_1, \Gamma_2$  and  $(\text{Int}(\Gamma_1) \setminus \text{Int}(\Gamma_2)) \cup (\text{Int}(\Gamma_2) \setminus \text{Int}(\Gamma_1))$ .

$$\text{Then } \int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz.$$

**T3.5.1: Cauchy integral formula**

Let  $\Gamma$  be a loop. Let  $z_0 \in \text{Int}(\Gamma)$  and let  $f$  be holomorphic on  $\Gamma \cup \text{Int}(\Gamma)$ . Then:

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz.$$

**T3.5.3**

Let  $\Gamma$  be a contour on domain  $D$ . Let  $g : D \rightarrow \mathbb{C}$  be continuous on  $\Gamma$ . Then the following  $G : D \setminus \Gamma \rightarrow \mathbb{C}$  is holomorphic:

$$G(z) = \int_{\Gamma} \frac{g(w)}{(w - z)^n} dw$$

$$G'(z) = n \int_{\Gamma} \frac{g(w)}{(w - z)^{n+1}} dw$$

given  $n \in \{1, 2, \dots\}$ .

**C3.5.5: Infinite differentiability**

Let  $f$  be holomorphic on domain  $D$ . Then  $f$  is infinitely differentiable on  $D$  and all its derivatives are holomorphic on  $D$ .

**T3.5.6**

Consider loop  $\Gamma$ . Let  $f$  be holomorphic on  $\Gamma \cup \text{Int}(\Gamma)$  and let  $z \in \Gamma$ . Then  $f$  is infinitely differentiable at  $z$  and:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

where  $n \in \mathbb{N}$ .

**T3.5.12**

Let  $D$  be a domain. Let  $f : D \rightarrow \mathbb{C}$  be continuous and that for all loops  $\Gamma \subset D$ :

$$\int_{\Gamma} f(z)dz = 0.$$

Then  $f$  is holomorphic on  $D$ .