EM S1 Handins

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Contents

Handin 1

1. Consider set T_{ijk} with 3^3 elements, satisfying:

$$v_i = T_{ijk} R_{jk}$$

where v_i is a vector and R_{jk} a rank 2 tensor.

Show that T_{ijk} is a rank 3 tensor.

Proof. Direct proof.

Firstly define T'_{ijk} in frame S' and T_{ijk} in frame S, where our frames are related by $\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j$. We then have that:

$$\begin{aligned} v'_i &= T'_{ijk} R'_{jk} \\ &= T'_{ijk} \ell_{jl} \ell_{km} R_{lm} \end{aligned}$$

and

$$\begin{aligned} v'_i &= \ell_{ij} v_j \\ &= \ell_{ij} T_{jkl} R_{kl}. \end{aligned}$$

$$\therefore T'_{ijk} \ell_{jl} \ell_{km} R_{lm} = \ell_{ij} T_{jkl} R_{kl}$$

Using the fact that R_{lm} is a tensor, we multiply both sides by vector a_m:

$$T'_{ijk} \ell_{jl} \ell_{km} R_{lm} a_m = \ell_{ij} T_{jkl} R_{kl} a_m.$$

The left hand side:

$$\begin{aligned} T'_{ijk} \ell_{jl} \ell_{km} R_{lm} a_m &= T'_{ijk} \ell_{jl} \ell_{km} a_l \\ &= T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} a_k. \end{aligned}$$

The right hand side:

$$\begin{aligned} \ell_{ij} T_{jkl} R_{kl} a_m &= \ell_{ij} T_{jkl} R_{kl} a_l \delta_{lm} \\ &= \ell_{ij} T_{jkl} \delta_{lm} a_k. \end{aligned}$$

Since equality still holds:

$$T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} a_k = \ell_{ij} T_{jkl} \delta_{lm} a_k.$$

$$\therefore \left(T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} - \ell_{ij} T_{jkl} \delta_{lm} \right) a_k = 0$$

Because a_k is a vector that is not always zero:

$$T'_{ijk} \ell_{jl} \ell_{km} \delta_{kl} = \ell_{ij} T_{jkl} \delta_{lm}.$$

$$\therefore T'_{ijk} \ell_{jk} \ell_{km} = \ell_{ij} T_{jkm}$$

Then multiply both sides by ℓ_{in}:

$$T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = \ell_{ij}\ell_{in}T_{jkm}.$$

$$\therefore T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = \delta_{jn}T_{jkm}$$

$$\therefore T'_{ijk}\ell_{jk}\ell_{km}\ell_{in} = T_{nkm}$$

$$\therefore T'_{ijk}\ell_{in}\ell_{jk}\ell_{km} = T_{nkm}$$

And this is by definition a third rank tensor. □

2. Define
$$\mathrm{d}\boldsymbol{r}_i = \frac{\partial \boldsymbol{r}}{\partial u_i}\mathrm{d}u_i$$
and the volume of the infinitesimal parallelepiped with $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$:
$$\mathrm{d}V = |\mathrm{d}\boldsymbol{r}_1 \cdot (\mathrm{d}\boldsymbol{r}_2 \times \mathrm{d}\boldsymbol{r}_3)|.$$
For part (i) show that:
$$\mathrm{d}V = |J|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$$
for $J = \det M$ where:
$$M_{ij} = \frac{\partial x_i}{\partial u_j}.$$

Using our definition of $\mathrm{d}\boldsymbol{r}_i$:
$$\begin{aligned}\mathrm{d}V &= |\mathrm{d}\boldsymbol{r}_1 \cdot (\mathrm{d}\boldsymbol{r}_2 \times \mathrm{d}\boldsymbol{r}_3)| \\ &= |\frac{\partial \boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2}\mathrm{d}u_2 \times \frac{\partial \boldsymbol{r}}{\partial u_3}\mathrm{d}u_3)| \\ &= |\frac{\partial \boldsymbol{r}}{\partial u_1} \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2} \times \frac{\partial \boldsymbol{r}}{\partial u_3})|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\end{aligned}$$
Since $\boldsymbol{r} = x_i\boldsymbol{e}_i$:
$$\frac{\partial \boldsymbol{r}}{\partial u_j} = \frac{\partial x_i}{\partial u_j}\boldsymbol{e}_i.$$
Now the triple scalar product of three vectors is equivalent to the determinant of a matrix consisting of these three vectors, as either rows or columns. Therefore:
$$\begin{aligned}|\frac{\partial \boldsymbol{r}}{\partial u_1} \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2} \times \frac{\partial \boldsymbol{r}}{\partial u_3})| &= \det \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \end{bmatrix} \\ &= \det M \\ &= J.\end{aligned}$$
Since volume is nonnegative:
$$\therefore \mathrm{d}V = |J|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3.$$

For part (ii) show:

- $(M^T M)_{ij} = g_{ij}$ for g_{ij} is the metric tensor.
- $\mathrm{d}V = \sqrt{g}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$ for $g_{ij} = (G)_{ij}$ and $g = \det G$.

By the definition of the metric tensor:

$$\begin{aligned} g_{ij} &= \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \\ &= M_{ki}M_{kj} \\ &= (M^T)_{ik}(M)_{kj} \\ &= (M^T M)_{ij}. \end{aligned}$$

Since $G = M^T M$ taking the determinants gives:

$$\begin{aligned} \det G &= \det\left(M^T M\right) \\ &= \det M^T \det M \\ &= (\det M)^2 \\ &= J^2. \end{aligned}$$

Then from part (i):

$$\therefore J = \pm\sqrt{g}$$

$$\begin{aligned} \therefore \mathrm{d}V &= |J|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 \\ &= \sqrt{g}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 \end{aligned}$$

For part (*iii*) show that given orthogonal curvilinear coordinates we have:

$$\mathrm{d}V = h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3$$

For OCCs, $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ and therefore:

$$\begin{aligned}
\mathrm{d}V &= |\mathrm{d}\boldsymbol{r}_1 \cdot (\mathrm{d}\boldsymbol{r}_2 \times \mathrm{d}\boldsymbol{r}_3)| \\
&= |\frac{\partial \boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 \cdot (\frac{\partial \boldsymbol{r}}{\partial u_2}\mathrm{d}u_2 \times \frac{\partial \boldsymbol{r}}{\partial u_3}\mathrm{d}u_3)| \\
&= |\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)|\, h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3 \\
&= |\boldsymbol{e}_1 \cdot \boldsymbol{e}_1|\, h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3 \\
&= h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3
\end{aligned}$$

where $h_i = |\frac{\partial \boldsymbol{r}}{\partial u_i}|$ and $\mathrm{d}u_i = \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial u_i}$.

3. For the first part of this problem we need to find the electric field $E(x)$ generated our rod of length $2a$ centered at $x = 0$ with total charge Q.

   The thin rod is also assumed to have <u>uniform</u> charge density of ρ.

   Then we can use the formula $F(x) = qE(x)$ to find force on our point charge q at $x = R$.

   By Coulomb's law:

$$\begin{aligned} E(x) &= \int_{-a}^{a} \frac{\rho}{4\pi\epsilon_0} \frac{x - x'}{(x - x')^3} \mathrm{d}x' \\ &= \frac{\rho}{4\pi\epsilon_0} \left[\frac{1}{x - x'} \right]_{x'=-a}^{x'=a} \\ &= \frac{\rho}{4\pi\epsilon_0} \frac{2a}{x^2 - a^2}. \end{aligned}$$

   Because we have a uniform charge density across $2a$:

$$\rho = \frac{Q}{2a}$$

   then the electric field generated by the thin rod becomes:

$$\begin{aligned} E(x) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{2a} \frac{2a}{x^2 - a^2} \\ &= \frac{Q}{4\pi\epsilon_0} \frac{1}{x^2 - a^2}. \end{aligned}$$

   The force on our point charge q at $x = R$ is then:

$$\begin{aligned} F(R) &= qE(R) \\ &= \frac{qQ}{4\pi\epsilon_0} \frac{1}{R^2 - a^2}. \end{aligned}$$

The force on charge q at $x = R$ by charge Q at $x_1 = 0$ is given by:

$$\begin{aligned} F(x) &= F(R) \\ &= \frac{qQ}{4\pi\epsilon_0}\frac{x - x_1}{|x - x_1|^3} \\ &= \frac{qQ}{4\pi\epsilon_0}\frac{R}{|R|^3} \\ &= \frac{qQ}{4\pi\epsilon_0}\frac{1}{R^2}. \end{aligned}$$

Comparing these two forces:

$$F_{rod} = \frac{qQ}{4\pi\epsilon_0}\frac{1}{R^2 - a^2}$$

$$F_{point} = \frac{qQ}{4\pi\epsilon_0}\frac{1}{R^2},$$

since

$$\frac{1}{R^2 - a^2} > \frac{1}{R^2}$$

therefore $F_{rod} > F_{point}$.

4. For part (i) show that:
$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

The Lie brackets are defined:

$$[x, y] = xy - yx$$

and so
$$[T_i, T_j] = T_iT_j - T_jT_i.$$

Because
$$(T_k)_{ij} = -i\epsilon_{ijk}$$

then we have:
$$(T_i)_{lk} = -i\epsilon_{lki}$$
$$(T_j)_{km} = -i\epsilon_{kmj}$$

and swapping order gives:

$$(T_j)_{lk} = -i\epsilon_{lkj}$$
$$(T_i)_{km} = -i\epsilon_{kmi}.$$

It is important to use more indices here:

$$\begin{aligned}\big(T_iT_j - T_jT_i\big)_{lm} &= (T_i)_{lk}(T_j)_{km} - (T_j)_{lk}(T_i)_{km} \\ &= -\epsilon_{lki}\epsilon_{kmj} + \epsilon_{lkj}\epsilon_{kmi} \\ &= -\epsilon_{ilk}\epsilon_{kmj} + \epsilon_{jlk}\epsilon_{kmi}.\end{aligned}$$

We first consider $-\epsilon_{ilk}\epsilon_{kmj}$. Because we have

$$\begin{aligned}\epsilon_{ijk}\epsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\ &= \delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm} \\ &= \epsilon_{ilk}\epsilon_{kmj}\end{aligned}$$

where we swap $j \to l$, $l \to m$ and $m \to j$:

$$\therefore -\epsilon_{ilk}\epsilon_{kmj} = -\big(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\big)$$

$$\therefore \epsilon_{jlk}\epsilon_{kmi} = \delta_{jm}\delta_{li} - \delta_{ij}\delta_{lm}$$

Then:

$$\begin{aligned}\big(T_iT_j - T_jT_i\big)_{lm} &= -\epsilon_{ilk}\epsilon_{kmj} + \epsilon_{jlk}\epsilon_{kmi}\\ &= -\big(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\big) + \delta_{jm}\delta_{li} - \delta_{ij}\delta_{lm}\\ &= -\delta_{im}\delta_{lj} + \delta_{jm}\delta_{li}.\end{aligned}$$

Due to the symmetry of the Kronecker delta:

$$\begin{aligned}\big(T_iT_j - T_jT_i\big)_{lm} &= -\delta_{im}\delta_{lj} + \delta_{jm}\delta_{li}\\ &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\\ &= \epsilon_{ijk}\epsilon_{klm}\\ &= \epsilon_{ijk}\epsilon_{lmk}\\ &= -i\epsilon_{ijk} \cdot -i\epsilon_{lmk}\\ &= -i\epsilon_{ijk} \cdot (T_k)_{lm}.\end{aligned}$$

$$\therefore [T_i, T_j]_{lm} = -i\epsilon_{ijk} \cdot (T_k)_{lm}$$
$$\therefore [T_i, T_j] = -i\epsilon_{ijk} T_k$$

For part (ii) we want to show:

- T_i is hermitian
- $\mathrm{Tr}(T_iT_j) = 2\delta_{ij}$

The definition of a hermitian matrix is as such:

$$H = (H^T)^*$$

$$H_{ij} = (H^T)^*_{ij}$$

where $*$ denotes the complex conjugate. So:

$$(T_i)_{jk} = -i\epsilon_{ijk}.$$

$$\begin{aligned} \therefore (T_i^T)_{jk} &= (T_i)_{kj} \\ &= -i\epsilon_{kji} \\ &= -i\epsilon_{ikj} \\ &= i\epsilon_{ijk} \end{aligned}$$

Then by the definition of the complex conjugate:

$$\therefore (T_i^T)^*_{jk} = -i\epsilon_{ijk}.$$

Since $(T_i)_{jk} = (T_i^T)^*_{jk}$ our matrix T_i is hermitian.

For the second part we firstly define:

$$(T_i)_{lk} = -i\epsilon_{lki}$$

and

$$(T_j)_{km} = -i\epsilon_{kmj}.$$

$$\begin{aligned} \therefore (T_i)_{lk}(T_j)_{km} &= (T_iT_j)_{lm} \\ &= -\epsilon_{lki}\epsilon_{kmj} \\ &= -\epsilon_{ilk}\epsilon_{kmj} \end{aligned}$$

Now:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\ &= \delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm} \\ &= \epsilon_{ilk}\epsilon_{kmj} \end{aligned}$$

if we swap $j \to l$, $l \to m$ and $m \to j$.

$$
\begin{aligned}
\therefore (T_i T_j)_{lm} &= -\epsilon_{ilk}\epsilon_{kmj} \\
&= -\big(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm}\big)
\end{aligned}
$$

Taking the trace of a matrix is summing up its diagonals:

$$
\begin{aligned}
\mathrm{Tr}(T_i T_j) &= (T_i T_j)_{ll} \\
&= -\big(\delta_{il}\delta_{lj} - \delta_{ij}\delta_{ll}\big) \\
&= -\big(\delta_{ij} - 3\delta_{ij}\big) \\
&= 2\delta_{ij}.
\end{aligned}
$$

Finally for part (*iii*) consider:

$$R(\alpha, \boldsymbol{n}) = \exp\bigl(-i\alpha \boldsymbol{n} \cdot \boldsymbol{T}\bigr)$$

where α is our rotation angle about <u>unit</u> axis vector $\boldsymbol{n}$.

$\boldsymbol{T}$ is a vector of generator matrices.

Our aims are:

- Show $\bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)^2_{ij} = \delta_{ij} - n_i n_j$.
- Show $\bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)^3_{ij} = \bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)_{ij}$.
- General formula for $\bigl(\boldsymbol{n} \cdot \boldsymbol{T}\bigr)^m_{ij}$ where $m > 3$.
- Expand $\exp\bigl(-i\alpha \boldsymbol{n} \cdot \boldsymbol{T}\bigr)$ as a power series.
- Recover standard rotation tensor of form:

$$R_{ij}(\alpha, \boldsymbol{n}) = \delta_{ij} \cos\alpha + n_i n_j \bigl(1 - \cos\alpha\bigr) - \epsilon_{ijk} n_k \sin\alpha.$$

Firstly define:

$$\begin{aligned}
(\boldsymbol{n} \cdot \boldsymbol{T})_{ik} &= n_\alpha (T_\alpha)_{ik} = n_\alpha \cdot -i\epsilon_{ik\alpha} \\
(\boldsymbol{n} \cdot \boldsymbol{T})_{kj} &= n_\beta (T_\beta)_{kj} = n_\beta \cdot -i\epsilon_{kj\beta}
\end{aligned}$$

then we have that

$$\begin{aligned}
(\boldsymbol{n} \cdot \boldsymbol{T})^2_{ij} &= n_\alpha (T_\alpha)_{ik} \cdot n_\beta (T_\beta)_{kj} \\
&= n_\alpha n_\beta \cdot -1 \cdot \epsilon_{ik\alpha} \epsilon_{kj\beta}.
\end{aligned}$$

Using the standard identity we get:

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{klm} &= \epsilon_{jki}\epsilon_{klm} \\
&= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\
&= \delta_{\alpha j}\delta_{i\beta} - \delta_{\alpha\beta}\delta_{ij} \\
&= \epsilon_{ik\alpha}\epsilon_{kj\beta}
\end{aligned}$$

where $j \to i$, $i \to \alpha$, $l \to j$ and $m \to \beta$.

Then substituting back into our equation:

$$\begin{aligned}
(\boldsymbol{n} \cdot \boldsymbol{T})^2_{ij} &= n_\alpha n_\beta \cdot -1 \cdot \epsilon_{ik\alpha}\epsilon_{kj\beta} \\
&= n_\alpha n_\beta \bigl(\delta_{\alpha\beta}\delta_{ij} - \delta_{\alpha j}\delta_{i\beta}\bigr) \\
&= \delta_{ij} - n_i n_j
\end{aligned}$$

Now we show that $\big(\boldsymbol{n}\cdot\boldsymbol{T}\big)^3_{ij} = \big(\boldsymbol{n}\cdot\boldsymbol{T}\big)_{ij}$. Define:

$$(\boldsymbol{n}\cdot\boldsymbol{T})^2_{ik} = \delta_{ik} - n_i n_k$$

$$(\boldsymbol{n}\cdot\boldsymbol{T})_{kj} = n_l (T_l)_{kj} = n_l \cdot -i\epsilon_{kjl}$$

and so multiplying them together gives:

$$\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{T})^3_{ij} &= (\boldsymbol{n}\cdot\boldsymbol{T})^2_{ik}(\boldsymbol{n}\cdot\boldsymbol{T})_{kj} \\
&= (\delta_{ik} - n_i n_k) n_l \cdot -i\epsilon_{kjl} \\
&= -in_l\big(\epsilon_{ijl} - n_i n_k \epsilon_{kjl}\big) \\
&= -in_l\epsilon_{ijl} + in_l n_i n_k \epsilon_{kjl} \\
&= -in_l\epsilon_{ijl} + in_i \delta_{lk}\epsilon_{kjl} \\
&= -in_l\epsilon_{ijl} \\
&= (\boldsymbol{n}\cdot\boldsymbol{T})_{ij}.
\end{aligned}$$

The general formula for $\big(\boldsymbol{n}\cdot\boldsymbol{T}\big)^m_{ij}$ takes the form:

$$\big(\boldsymbol{n}\cdot\boldsymbol{T}\big)^m_{ij} = \begin{cases} \delta_{ij} - n_i n_j & m \text{ even} \\ \big(\boldsymbol{n}\cdot\boldsymbol{T}\big)_{ij} & m \text{ odd.} \end{cases}$$

The power series for an exponential is:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and so:

$$\begin{aligned}
\Big(\exp\big(-i\alpha\boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= \sum_{k=0}^{\infty}\Big(\frac{1}{k!}[-i\alpha]^k(\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij}\Big) \\
&= 1 + [-i\alpha](\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij} \\
&\quad + \sum_{k=2,4,\ldots} \frac{1}{k!}[-i\alpha]^k(\delta_{ij} - n_i n_j) \\
&\quad + \sum_{k=3,5,\ldots} \frac{1}{k!}[-i\alpha]^k(\boldsymbol{n}\cdot\boldsymbol{T})_{ij}.
\end{aligned}$$

Set $k = 2n$ for the first sum and $k = 2m+1$ for the second. Here $n, m \in \mathbb{N}$.

$$\begin{aligned}\therefore \Big(\exp\big(-i\alpha \boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= 1 + [-i\alpha](\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij} \\ &\quad + (\delta_{ij} - n_i n_j)\Big(\big(\sum_{n=0}^{\infty}(-1)^n \frac{\alpha^{2n}}{(2n)!}\big) - 1\Big) \\ &\quad + (\boldsymbol{n}\cdot\boldsymbol{T})_{ij} \cdot -i\Big(\big(\sum_{m=0}^{\infty}(-1)^m \frac{\alpha^{2m+1}}{(2m+1)!}\big) - \alpha\Big)\end{aligned}$$

Recognising this as the series expansion for cosine and sine:

$$\begin{aligned}\therefore \Big(\exp\big(-i\alpha \boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= 1 + [-i\alpha](\boldsymbol{n}\cdot\boldsymbol{T})^k_{ij} \\ &\quad + (\delta_{ij} - n_i n_j)\Big(\cos\alpha - 1\Big) \\ &\quad + (\boldsymbol{n}\cdot\boldsymbol{T})_{ij} \cdot -i\Big(\sin\alpha - \alpha\Big)\end{aligned}$$

Since $(\boldsymbol{n}\cdot\boldsymbol{T})_{ij} = n_k (T_k)_{ij} = -i n_k \epsilon_{ijk}$:

$$\begin{aligned}\Big(\exp\big(-i\alpha \boldsymbol{n}\cdot\boldsymbol{T}\big)\Big)_{ij} &= 1 - \alpha n_k \epsilon_{ijk} + (\delta_{ij} - n_i n_j)\Big(\cos\alpha - 1\Big) - n_k\epsilon_{ijk}\Big(\sin\alpha - \alpha\Big) \\ &= \delta_{ij}\cos\alpha + n_i n_j\Big(1 - \cos\alpha\Big) - \epsilon_{ijk} n_k \sin\alpha\end{aligned}$$

where we use the bogus argument:

$$\begin{aligned}1 - \delta_{ij} &= \frac{1}{3}\delta_{ii} - \frac{1}{3}\delta_{jj}\delta_{ij} \\ &= \frac{1}{3}\delta_{ii} - \frac{1}{3}\delta_{ii} \\ &= 0.\end{aligned}$$

Handin 2

1. For part (i) consider uniformly charged region bounded by two spheres, of radius b and a where $b > a$. Let the charge density of this region be ρ. Find its potential $\phi(\boldsymbol{r})$.

   Let $\boldsymbol{r} = r\boldsymbol{e}_r + \theta\boldsymbol{e}_\theta + \phi\boldsymbol{e}_\phi$ where $r \in (a, b)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ be our parametrisation of the region. We have Gauss's law:

$$\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}.$$

   Because of spherical symmetry, our electric field is of only radial form:

$$\boldsymbol{E}(\boldsymbol{r}) = E_r(r)\boldsymbol{e}_r.$$

   Evaluating our surface integral:

$$\begin{aligned}
\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} &= \iint \boldsymbol{E} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \phi}\right) \mathrm{d}\theta\mathrm{d}\phi \\
&= \iint E_r(r)\boldsymbol{e}_r \cdot (r^2 \sin\theta \boldsymbol{e}_r)\mathrm{d}\theta\mathrm{d}\phi \\
&= r^2 E_r(r) \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta \mathrm{d}\theta\mathrm{d}\phi \\
&= 4\pi r^2 E_r(r).
\end{aligned}$$

   Now the total charge enclosed is $Q_{enc} = \rho V$ where V is enclosed volume.

$$\therefore 4\pi r^2 E_r(r) = \begin{cases} \dfrac{\rho}{\epsilon_0}\dfrac{4}{3}\pi(b^3 - a^3) & r > b \\[2ex] \dfrac{\rho}{\epsilon_0}\dfrac{4}{3}\pi(r^3 - a^3) & r \in [a, b] \\[2ex] 0 & r < a \end{cases}$$

   Then rearranging:

$$E_r(r) = \begin{cases} \dfrac{1}{3}\dfrac{\rho}{\epsilon_0}(b^3 - a^3)\dfrac{1}{r^2} & r > b \\[2ex] \dfrac{1}{3}\dfrac{\rho}{\epsilon_0}\left(r - \dfrac{a^3}{r^2}\right) & r \in [a, b] \\[2ex] 0 & r < a. \end{cases}$$

Since $\boldsymbol{E} = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$ we then have that $E_r(r) = -\dfrac{\partial\phi}{\partial r}$ and:

$$\phi(r) = -\int E_r(r)\mathrm{d}r.$$

So when $r \in [a, b]$:

$$\begin{aligned}\phi(r) &= -\int \frac{\rho}{\epsilon_0}\frac{1}{3}(r - a^3 r^{-2})\mathrm{d}r \\ &= -\frac{1}{3}\frac{\rho}{\epsilon_0}\left[\frac{r^2}{2} + \frac{a^3}{r}\right] + C_1\end{aligned}$$

and when $r > b$:

$$\begin{aligned}\phi(r) &= -\frac{1}{3}\frac{\rho}{\epsilon_0}(b^3 - a^3)\int r^{-2}\mathrm{d}r \\ &= \frac{1}{3}\frac{\rho}{\epsilon_0}(b^3 - a^3)\frac{1}{r} + C_2\end{aligned}$$

where $C_2 = 0$ so that $\phi \to 0$. For continuity, $C_1 = \dfrac{1}{2}\dfrac{\rho}{\epsilon_0}b^2$.

$$\therefore \phi(r) = \begin{cases} \dfrac{1}{3}\dfrac{\rho}{\epsilon_0}(b^3 - a^3)\dfrac{1}{r} & r > b \\ \\ -\dfrac{1}{3}\dfrac{\rho}{\epsilon_0}\left[\dfrac{r^2}{2} + \dfrac{a^3}{r}\right] + \dfrac{1}{2}\dfrac{\rho}{\epsilon_0}b^2 & r \in [a, b] \\ \\ 0 & r < a. \end{cases}$$

For part (ii):

$$\begin{aligned}E_r(r) &= \frac{1}{3}\frac{\rho}{\epsilon_0}\left(r - \frac{a^3}{r^2}\right) \\ &= \frac{1}{3}\frac{\rho}{\epsilon_0}\frac{1}{r^2}(r^3 - a^3) \\ &= \frac{\rho}{\epsilon_0}(r - a)\frac{1}{3r^2}(r^2 + a^2 + ra) \\ &= \frac{\rho}{\epsilon_0}(b - a)\frac{1}{3b^2}(b^2 + b^2 + rb)\end{aligned}$$

but since $b \to a$ we then have that:

$$E_r(r) = \frac{\rho}{\epsilon_0}(b - a) = \frac{\sigma}{\epsilon_0}.$$

2. For part (i) a dipole at $\boldsymbol{r}_1$ with moment $\boldsymbol{p}_1$ generates an electric field:

$$\boldsymbol{E}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}_1 \cdot (\boldsymbol{r}-\boldsymbol{r}_1))(\boldsymbol{r}-\boldsymbol{r}_1) - |\boldsymbol{r}-\boldsymbol{r}_1|^2\boldsymbol{p}_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^5}$$

and so at point $\boldsymbol{r}_2$ we have:

$$\boldsymbol{E}_1(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}_1 \cdot (\boldsymbol{r}_2-\boldsymbol{r}_1))(\boldsymbol{r}_2-\boldsymbol{r}_1) - |\boldsymbol{r}_2-\boldsymbol{r}_1|^2\boldsymbol{p}_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^5}.$$

For part (ii) the force induced by our dipole at $\boldsymbol{r}_1$ with moment $\boldsymbol{p}_1$ on <u>another</u> dipole at $\boldsymbol{r}_2$ with moment $\boldsymbol{p}_2$ is:

$$\begin{aligned}\boldsymbol{F}_2(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0}\Big[&-\frac{15}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^7}\big(\boldsymbol{p}_1\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\big)\big(\boldsymbol{p}_2\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\big)(\boldsymbol{r}_2-\boldsymbol{r}_1)\\ &+\frac{3}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^5}\Big((\boldsymbol{p}_1\cdot\boldsymbol{p}_2)(\boldsymbol{r}_2-\boldsymbol{r}_1)+\Big(\boldsymbol{p}_1\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\Big)\boldsymbol{p}_2\\ &+\Big(\boldsymbol{p}_2\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\Big)\boldsymbol{p}_1\Big)\Big].\end{aligned}$$

For part (iii) find the torque $\boldsymbol{G}_1$ on dipole $\boldsymbol{p}_1$ at $\boldsymbol{r}_1$ due to the electric field $\boldsymbol{E}_2$ generated by dipole $\boldsymbol{p}_2$ at $\boldsymbol{r}_2$.

So we have that:

$$\boldsymbol{G}_1(\boldsymbol{r}_1) = \boldsymbol{p}_1 \times \boldsymbol{E}_2(\boldsymbol{r}_1)$$

where

$$\boldsymbol{E}_2(\boldsymbol{r}_1) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}_2 \cdot (\boldsymbol{r}_1-\boldsymbol{r}_2))(\boldsymbol{r}_1-\boldsymbol{r}_2) - |\boldsymbol{r}_1-\boldsymbol{r}_2|^2\boldsymbol{p}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^5}$$

and evaluating this expression:

$$\boldsymbol{G}_1(\boldsymbol{r}_1) = \frac{1}{4\pi\epsilon_0}\frac{3\big(\boldsymbol{p}_2 \cdot (\boldsymbol{r}_1-\boldsymbol{r}_2)\big)\boldsymbol{p}_1 \times (\boldsymbol{r}_1-\boldsymbol{r}_2) - |\boldsymbol{r}_1-\boldsymbol{r}_2|^2\ \boldsymbol{p}_1\times\boldsymbol{p}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^5}.$$

For part (iv) find the torque $\boldsymbol{G}_2$ on dipole $\boldsymbol{p}_2$ at $\boldsymbol{r}_1$ from the electric field $\boldsymbol{E}_1$ generated by $\boldsymbol{p}_1$ at $\boldsymbol{r}_1$.

Now the torque on $\boldsymbol{p}_2$ at $\boldsymbol{r}_2$ is then:

$$\begin{aligned}
\boldsymbol{G}_2(\boldsymbol{r}_2) &= \tau_{+q} + \tau_{-q} \\
&= q(\boldsymbol{0} + \boldsymbol{d}) \times \boldsymbol{E}_1(\boldsymbol{r}_2 + \boldsymbol{d}) - q\boldsymbol{0} \times \boldsymbol{E}_1(\boldsymbol{r}_2) \\
&= q\boldsymbol{d} \times \Big[\boldsymbol{E}_1(\boldsymbol{r}_2) + (\boldsymbol{d} \cdot \boldsymbol{\nabla})\boldsymbol{E}_1(\boldsymbol{r}_2) + \dots\Big] \\
&\approx q\boldsymbol{d} \times \boldsymbol{E}_1(\boldsymbol{r}_2) + \boldsymbol{d} \times (q\boldsymbol{d} \cdot \boldsymbol{\nabla})\boldsymbol{E}_1(\boldsymbol{r}_2) \\
&= \boldsymbol{p}_2 \times \boldsymbol{E}_1(\boldsymbol{r}_2) + \boldsymbol{d} \times (\boldsymbol{p}_2 \cdot \boldsymbol{\nabla})\boldsymbol{E}_1(\boldsymbol{r}_2) \\
&= \boldsymbol{p}_2 \times \boldsymbol{E}_1(\boldsymbol{r}_2) + \boldsymbol{d} \times \boldsymbol{F}_2(\boldsymbol{r}_2)
\end{aligned}$$

where dipole $\boldsymbol{p}_2 = q\boldsymbol{d}$ has $-q$ charge at $\boldsymbol{r}_2$ and $+q$ charge at $\boldsymbol{r}_2 + \boldsymbol{d}$ with the limit $\boldsymbol{d} \to \boldsymbol{0}$.

In the dipole limit our two dipoles $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are infinitely close.
There probably exists dipole interactions between these two dipoles.
So let $\boldsymbol{r}_1 = \boldsymbol{r}_2 - \boldsymbol{d}$, and in the limit $\boldsymbol{d} \to \boldsymbol{0}$ we get:

$$\boldsymbol{G}_2(\boldsymbol{r}_1) = \boldsymbol{p}_2 \times \boldsymbol{E}_1(\boldsymbol{r}_2) + (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{F}_2(\boldsymbol{r}_2)$$

where $\boldsymbol{r}_2 \to \boldsymbol{r}_1$.

Finally for part (v) verify that:

$$\boldsymbol{G}_2(\boldsymbol{r}_1) = -\boldsymbol{G}_1(\boldsymbol{r}_1).$$

Using previous parts we have that:

$$\boldsymbol{p}_2 \times \boldsymbol{E}_1(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{3\big(\boldsymbol{p}_1 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)\big)\boldsymbol{p}_2 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1) - |\boldsymbol{r}_2 - \boldsymbol{r}_1|^2\ \boldsymbol{p}_2 \times \boldsymbol{p}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^5}$$

$$\begin{aligned}
(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{F}_2(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{3}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^5}\Big[&-\big(\boldsymbol{p}_1 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)\big)\boldsymbol{p}_2 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1) \\
&+ -\big(\boldsymbol{p}_2 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)\big)\boldsymbol{p}_1 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)\Big]
\end{aligned}$$

Adding these two up:

$$\begin{aligned}
\boldsymbol{G}_2(\boldsymbol{r}_1) &= -\frac{1}{4\pi\epsilon_0}\frac{3\big(\boldsymbol{p}_2 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)\big)\boldsymbol{p}_1 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1) + |\boldsymbol{r}_2 - \boldsymbol{r}_1|^2\ \boldsymbol{p}_2 \times \boldsymbol{p}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^5} \\
&= -\boldsymbol{G}_1(\boldsymbol{r}_1).
\end{aligned}$$

3. For part (i) using Gauss's law:

$$\begin{aligned}
\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} &= \iint E_\rho(\rho) \cdot \Big(\frac{\partial \boldsymbol{r}}{\partial \phi} \times \frac{\partial \boldsymbol{r}}{\partial z}\Big) \mathrm{d}\phi \mathrm{d}z \\
&= E_\rho(\rho) \int_z \int_0^{2\pi} \rho \mathrm{d}\phi \mathrm{d}z \\
&= z E_\rho(\rho) \rho 2\pi
\end{aligned}$$

and when $\rho \in (a, b)$ this is equivalent to:

$$z E_\rho(\rho) \rho 2\pi = -\frac{\lambda}{\epsilon_0} z.$$

When $\rho \notin (a, b)$ choose spherical shell with radius ρ. By definition it has no charge and hence:

$$E_\rho(\rho) = 0.$$

So we have that:

$$E_\rho(\rho) = \begin{cases} -\dfrac{\lambda}{2\pi\epsilon_0} \dfrac{1}{\rho} & \rho \in (a, b) \\ \\ 0 & \rho \notin (a, b). \end{cases}$$

For part (ii) since $\boldsymbol{E} = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$ integrating gives:

$$\phi(\rho) = \begin{cases} \dfrac{\lambda}{2\pi\epsilon_0} \ln \rho & \rho \in (a, b) \\ \\ 0 & \rho \notin (a, b) \end{cases}$$

The potential difference between the two plates is

$$\begin{aligned}
V &= \phi(b) - \phi(a) \\
&= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a}
\end{aligned}$$

and since $Q = \lambda$:

$$\begin{aligned}
C &= \frac{Q}{V} \\
&= \frac{2\pi\epsilon_0}{\ln b/a}.
\end{aligned}$$

For part (iii) the energy per unit length is:

$$\begin{aligned} W &= \frac{Q}{2}(\phi_b - \phi_a) \\ &= \frac{\lambda^2}{4\pi\epsilon_0}\ln\frac{b}{a}. \end{aligned}$$

For part (iv):

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int \mathrm{d}V|\boldsymbol{E}(\boldsymbol{r})|^2 \\ &= \frac{\epsilon_0}{2}\frac{\lambda^2}{4\pi^2\epsilon_0}\int_z\int_0^{2\pi}\int_a^b \frac{1}{\rho^2}|\rho|\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z \\ &= \frac{\lambda^2}{4\pi\epsilon_0}\ln\frac{b}{a}. \end{aligned}$$

For part (v) show that when b is slightly larger than a we have that:

$$C = \frac{2\pi\epsilon_0}{\ln b/a} \approx \frac{2\pi a\epsilon_0}{d}$$

where a is the radius of a cylinder and d is plate separation distance.

Here capacitance in per unit length. Let $b - a = \epsilon$ where ϵ is a small quantity. Then consider the following:

$$\begin{aligned} \ln\frac{b}{a} &= \ln\left(1 + \frac{\epsilon}{a}\right) \\ &\approx \frac{\epsilon}{a} \end{aligned}$$

and

$$\begin{aligned} \frac{2\pi\epsilon_0}{\ln b/a} &\approx \frac{2\pi\epsilon_0}{\frac{\epsilon}{a}} \\ &= \frac{2\pi a\epsilon_0}{b-a} \end{aligned}$$

which is an expression for the capacitance of a thinly separated parallel-plate capacitor.