Rearrangement of series

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Example 1: An incorrect argument

Let $S = 1 + 2 + 3 + 4 + \dots$ and let:

$$S_1 = 1 - 1 + 1 - 1 + 1 - 1 + \dots \stackrel{?}{=} \frac{1}{2}.$$

Example 1: An incorrect argument

Consider $S_2 = 1 - 2 + 3 - 4 + 5 - 6 + \dots$.

$$\begin{aligned} 2S_2 &= 1 - 2 + 3 - 4 + 5 - 6 + \dots \\ &\qquad + 1 - 2 + 3 - 4 + 5 - 6 + \dots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \dots \\ &\overset{?}{=} \frac{1}{2} \end{aligned}$$

$$\therefore S_2 = 1 - 2 + 3 - 4 + 5 - 6 + \dots = \frac{1}{4}$$

Example 1: An incorrect argument

And finally:

$$\begin{aligned} S - S_2 &= 1 + 2 + 3 + 4 + 5 + \dots \\ &\quad - (1 - 2 + 3 - 4 + 5 - 6 + \dots) \\ &= 4(1 + 2 + 3 + 4 + \dots) \\ &= 4S \end{aligned}$$

$$\therefore S = 1 + 2 + 3 + 4 + 5 + \dots = -\frac{1}{12}$$

Example 1: An incorrect argument

This is completely wrong!

$$S = \sum_{k=1}^{\infty} a_k \text{ exists } \textbf{if and only if } \lim_{n\to\infty} \sum_{k=1}^{n} a_k < \infty.$$

In our case:

$$\lim_{n\to\infty} \left[1 - 1 + 1 - 1 + 1 - 1 + \dots\right] \text{ DNE}$$

and

$$\begin{aligned} S &= 1 + 2 + 3 + 4 + \dots \\ &= \infty \\ &\neq -\frac{1}{12}. \end{aligned}$$

Example 2: A conditionally convergent series

$$\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{1}{n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
$$= \ln 2$$

This is the alternating harmonic series.

Example 2: A conditionally convergent series

$$\text{Know: } 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln 2$$

$$\therefore \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\ln 2$$

$$\begin{aligned} &1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\ &\quad + \frac{1}{2} \qquad\quad - \frac{1}{4} \qquad\quad + \frac{1}{6} + \dots \\ &= \frac{3}{2}\ln 2 \end{aligned}$$

$$\therefore 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2$$

Example 2: A conditionally convergent series

Our previous argument was only valid because:

$$\sum_{n=1}^{\infty}\left(|(-1)^{n+1}\frac{1}{n}|\right) = \sum_{n=1}^{\infty}\frac{1}{n}$$
$$= \infty$$

Definition (Conditional convergence)

Let $S = \sum_{k=1}^{\infty} a_k$. Series S is conditionally convergent if $\sum_{k=1}^{\infty} a_k < \infty$ yet $\sum_{k=1}^{\infty} |a_k| = \infty$.

Theorem (Riemann rearrangement)

Let $S = \sum_{k=1}^{\infty} a_k$ be a conditionally convergent series. Then there exists rearrangements $z : \mathbb{N} \to \mathbb{N}$ such that:

$$\sum_{k=1}^{\infty} a_{z(k)}$$

may take on any value.