Honours Analysis

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7.1 Characteristic and step functions

Definition 7.1 (Characteristic functions)**.**
The characteristic function is defined as a real function such that:

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & \textit{otherwise} \end{cases}$$

where $E \subset \mathbb{R}$.

Remark. *Therefore the integral of a characteristic function is:*

$$\int \chi_E = \lambda(E)$$

for $\lambda(E)$ is the length of an internal E.

Definition 7.2 (Step functions)**.**
The step function with respect to finite set $\{x_0, \ldots, x_n\}$ for some $n \in \mathbb{N}$ is:

$$\phi(x) = \begin{cases} 0 & x < x_0 \textit{ or } x > x_n \\ c_j & \textit{if } x \in (x_{j-1}, x_j);\ 1 \leq j \leq n. \end{cases}$$

Remark. *Step functions are a sum of characteristic functions:*

$$\phi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1}, x_j)}(x)$$

and its integral is

$$\int \phi = \sum_{j=1}^{n} c_j (x_{j-1} - x_j).$$

Importantly the <u>sum</u> of two step functions is another step function.

7.2 Lebesgue integrals

Consider function $f : I \to \mathbb{R}$. This function is **Lebesgue integrable** on our interval I if:

1. $\sum_{j=1}^{\infty} |c_j|\lambda(J_j) < \infty$

2. $\forall x \in I; f(x) = \sum_{j=1}^{\infty} |c_j|\chi_{J_j}(x) < \infty$

Here $c_j \in \mathbb{R}$, $J_i \subset I$ and is bounded for $j \in \{1, 2, 3, \dots\}$.

i.e. that our function's area and height are defined. Therefore:

$$\int_I f = \sum_{j=1}^{\infty} |c_j|\lambda(J_j)$$

and integral value is invariant of interval type. (open, semi-open or closed)

7.2.1 Properties of Lebesgue integrals

Let functions f, g be Lebesgue integrable on I and $\alpha, \beta \in \mathbb{R}$. Then:

1. $\alpha f + \beta g$ is Lebesgue integrable on I, and:
$$\int_I \alpha f + \beta g = \alpha \int_I f + \beta \int_I g.$$

2. If $f \geq g$ on I then:
$$\int_I f \geq \int_I g.$$

3. $$\int_I |f| \geq |\int_I f|$$

4. $\max\{f, g\}$ and $\min\{f, g\}$ are integrable on I. Furthermore:
$$\max\{f, g\} = \frac{f + g}{2} + \frac{|f - g|}{2}$$
and
$$\min\{f, g\} = \frac{f + g}{2} - \frac{|f - g|}{2}.$$

5. fg is integrable on I if one of the functions is bounded.

6. Let $f \geq 0$ where $\int_I f = 0$.

   The function h is integrable on I if $0 \leq h \leq f$.

7.2.2 Integration on subintervals

Let $J \subset I$. We then have the following statements.

1. If f is integrable on I then f is integrable on J.
2. Let $f(x) = 0$ for $\forall x \in I \setminus J$ and f integrable on J. Then:
$$\int_J f = \int_I f.$$
3. Assume that $\forall x \in I; f(x) \geq 0$. If f is integrable on I then:
$$\int_I f \geq \int_J f.$$
4. Let $I = \bigcup_{n=1}^{\infty} I_n$ where I_n are all disjoint sets.

   Let f be integrable on each I_n. We have that:
$$f \text{ is integrable on } I \iff \sum_{n=1}^{\infty} \int_{I_n} f$$
   and that the following equality holds:
$$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} f.$$

The regular integral calculus properties hold:

1. $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
2. $$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

7.2.3 Maclaurin-Cauchy integral test

Now let f be a non-negative, **monotone decreasing** function on $[p, \infty)$. Then:
$$\sum_{n=p}^{\infty} f(n) < \infty \iff f \text{ is integrable on } [p, \infty)$$
where $p \in \mathbb{Z}$. Furthermore:
$$\sum_{n=p}^{\infty} f(n) < \infty \iff \int_p^{\infty} f(x)dx < \infty.$$

7.3 Riemann integrals

A real function f is **Riemann-integrable** if it has bounded support. i.e:

$$\forall \epsilon > 0; \exists \phi, \psi : \phi \leq f \leq \psi \ \ \underline{\text{and}} \quad \int \psi - \int \phi < \epsilon,$$

where ψ and ϕ are step functions.

Furthermore the following statements are equivalent:

1. f is Riemann-integrable, where f is a real bounded function with bounded support $[a, b]$.
2. $\sup\left\{\int \phi\right\} = \inf\left\{\int \psi\right\}$, and is the integral value.
3. $\forall \epsilon > 0; \exists \{a = x_0 < \cdots < x_n = b\}$:

$$\sum_{j=1}^{n} \left(\sup_{x \in (x_{j-1} - x_j)} f(x) - \inf_{x \in (x_{j-1} - x_j)} f(x) \right) (x_j - x_{j-1}) < \epsilon$$

and

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \cdot \lambda(I_j) < \epsilon$$

where we define $I_j = (x_{j-1}, x_j)$ and $j \in \{1, \ldots, n\}$.

Now let:

$$m_j = \inf_{x \in I_j} f(x)$$

$$M_j = \sup_{x \in I_j} f(x)$$

and it makes sense to define step functions

$$\phi_* \leq f \leq \phi^*(x)$$

with respect to $\{x_0, \ldots, x_n\}$ where:

$$\phi_*(x) = \sum_{j=1}^{n} m_j \chi_{I_j}(x) + \sum_{j=1}^{n} f(x_j) \chi_{\{x_j\}}(x)$$

and

$$\phi^*(x) = \sum_{j=1}^{n} M_j \chi_{I_j}(x) + \sum_{j=1}^{n} f(x_j) \chi_{\{x_j\}}(x).$$

If f is Riemann-integrable then it is automatically Lebesgue-integrable, but not necessarily the opposite way. So Lebesgue-integrals are a superset of Riemann-integrals.

Note that <u>closed</u> intervals are **uniformly continuous**.

Let $g : [a, b] \to \mathbb{R}$ and that:

$$f(x) = \begin{cases} g(x) & x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

We then have that:

1. If g is continuous on $[a, b]$ then f is Riemann-integrable.
2. If g is a <u>montone</u> function then f is Riemann-integrable.

7.4 Fundamental theorem of calculus

Let $g : I \to \mathbb{R}$ be integrable on I and that

$$G(x) = \int_{x_0}^{x} g(x)\mathrm{d}x$$

for $\forall x \in I$ and <u>fixed</u> $x_0 \in I$.

If $g(x)$ is continuous at $x \in I$ then:

$$\frac{\mathrm{d}}{\mathrm{d}x} G(x) = g(x).$$

Furthermore if $G(x)$ and $g(x)$ are continuous on the interval I:

$$\int_a^b g(x)\mathrm{d}x = G(b) - G(a)$$

for $\forall a, b \in I$.

7.5 Integration of sequences

Consider $(f_n)_{n\in\mathbb{N}}$ that are integrable on I. Assume the following:

- $\displaystyle\sum_{n=1}^{\infty} \int_I |f_n| < \infty$
- $\displaystyle\sum_{n=1}^{\infty} |f_n(x)| < \infty$ for $\forall x \in I$.

Let $f(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$. Then f is integrable on I and

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n.$$

The following result is a useful test for integrability.

Let $f_n \geq 0$ on I and that $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then:

$$f \text{ is integrable on } I \iff \sum_{n=1}^{\infty} \int_I f_n < \infty.$$

7.5.1 Monotone convergence for integration

Now consider a monotone increasing sequence of functions $(f_n)_{n\in\mathbb{N}}$:

$$f_1 \leq f_2 \leq f_3 \leq \dots .$$

Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then:

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

and furthermore:

$$\sup_{n\in\mathbb{N}} \int_I f_n = \lim_{n\to\infty} \int_I f_n < \infty.$$

7.5.2 Fatoux's lemma

Let $f_n > 0$ be integrable functions on I and that:

$$f(x) = \liminf_{n\to\infty} f_n(x)$$

for $\forall x \in I$. If

$$\liminf_{n\to\infty} \int_I f_n(x) < \infty$$

then f is integrable on I and:

$$\int_I f \leq \liminf_{n\to\infty} \int_I f_n(x).$$

An immediate result is the following.

Let f_n be integrable on the interval I and that:

$$f(x) = \lim_{n\to\infty} f_n(x).$$

If $|f_n(x)| \leq g(x)$ where $\int_I g < \infty$ then:

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

A final result is that if $f_n : (a, b) \to \mathbb{R}$ are integrable functions, and that:

$$f_n \to f \text{ uniformly on } (a, b),$$

we then have that:

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

8 Fourier analysis

8.1 L^2 space

l2 norm of a function

inner product

cauchy schwarz inequalities

minkowski inequalities

convergence in l2

orthonormal systems

T5.2

bessel's inequality

riemann lemma

complete orthonormal systems

T5.4

8.2 Fourier series

trigonometric polynomial (fs)

complex fourier series

fourier coefficients

euler formula

lemma 5.1: orthgonality of FS

convolution of fs

dirichlet kernel

8.3 Convergence of Fourier series

8.3.1 Approximations

8.3.2 L^2 convergence

8.3.3 Pointwise convergence