D1.1.1: Complex numbers

Let $z = x + iy$ and $w = a + ib$ where $x, y, a, b \in \mathbb{R}$. Then z and w are complex numbers. Furthermore:

1. $z = w$ **iff** $x = a$ and $y = b$.
2. $\text{Re}(z) := x$ and $\text{Im}(z) := y$.
3. $|z| := \sqrt{x^2 + y^2}$
4. The **complex conjugate** of z is:
$$z^* := x - iy.$$
5. Addition and multiplication:
$$(x+iy)+(a+ib) = (x+a)+i(y+b)$$
$$(x+iy)(a+ib) = (xa-yb)+i(xb+ya).$$
6. $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$

with rule $i^2 = -1$.

L1.1.3

Let $u, w, z \in \mathbb{C}$ where $z = x + iy$. Then:

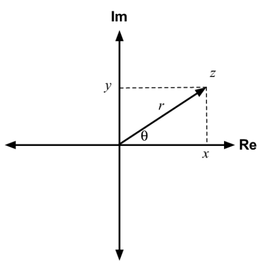
1. $z + w = w + z$ and $zw = wz$.
2. $u + (z + w) = (u + z) + w$
3. $u(zw) = (uz)w$
4. $u(z + w) = uz + uw$
5. $z + 0 = z$ and $1z = z$.
6. $\exists\big(-z := -x+i(-y)\big)$: $z+(-z) = 0$.
7. $\exists z^{-1} : zz^{-1} = 1$ where:
$$z^{-1} := \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

D1.1.5 and D1.1.7: Polar form

Let $z \in \mathbb{C}$ and $z = x + iy$. Then:

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

for $r = |z| = \sqrt{x^2 + y^2}$ and $\theta \in (-\pi, \pi]$ is given by $\tan\theta = y/x$.



L1.1.6

Let $\theta, \phi \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then:

1. $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$
2. $e^{in\theta} = (e^{i\theta})^n$

due to de Moivre's formula:

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

L1.1.9

Let $z, w \in \mathbb{C}$. Then:

1. $|z| = 0$ **iff** $z = 0$.
2. $|\overline{z}| = |z|$
3. $|zw| = |z||w|$
4. $(z^*)^* = z$
5. $|z|^2 = zz^*$ and $|z^2| = |z|^2$.
6. $(z + w)^* = z^* + w^*$
7. $(zw)^* = z^* w^*$
8. $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$.
9. $\text{Re}(z) = \frac{1}{2}(z + z^*)$
10. $\text{Im}(z) = \frac{1}{2i}(z - z^*)$.

L1.1.10 – 11: Triangle inequalities

Let $z, w \in \mathbb{C}$. Then:

1. $|z + w| \leq |z| + |w|$
2. $||z| - |w|| \leq |z - w|$.

D1.1.12: Argument of z

The set of all arguments of z is:

$$\begin{aligned} \arg(z) &:= \{\theta : z = |z|e^{i\theta}\} \\ &= \{\text{Arg}(z) + 2\pi k : k \in \mathbb{Z}\}. \end{aligned}$$

The principle argument of z satisfies $z = |z|e^{i\text{Arg}(z)}$ with $-\pi < \text{Arg}(z) \leq \pi$.

$$\therefore \text{Arg}(z) \equiv \arg(z) \mod 2\pi$$

$\text{Arg}(z)$ is discontinuous on the negative real axis since $\forall x, \epsilon > 0; -x \pm i\epsilon \to x$:

$$\lim_{\epsilon \to 0} \text{Arg}(-x \pm i\epsilon) = \pm\pi.$$

P1.1.14

Let $z, w \in \mathbb{C}$. Then:

1. $\arg(zw) = \arg(z) + \arg(w)$
2. $\arg(z^*) = -\arg(z)$

for these are set operations.

D1.2.1: Open and closed ϵ-discs

Let $z_0 \in \mathbb{C}$ and $\epsilon > 0$.

1. An **open** ϵ-disc centred at z_0 is:
$$D_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}.$$
2. A **closed** ϵ-disc centred at z_0 is:
$$\overline{D}_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}.$$

A **punctured** ϵ-disc centred at z_0 is:

$$D'_\epsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}.$$

D1.2.2: Open and closed sets

Let $U \subset \mathbb{C}$. Set U is **open** if:

$$\forall z_0 \in U; \exists \epsilon > 0 : D_\epsilon(z_0) \subseteq U.$$

Subset F is **closed** if $\mathbb{C} \setminus F$ is open.

A **neighbourhood** of point $z_0 \in \mathbb{C}$ is an open set that contains z_0.

Remark

$\emptyset$ is vacuously open. Therefore $\mathbb{C}$ is open **and** closed. A set like $D_2(0) \setminus D_1(0)$ is **neither closed nor open**.

The union and intersection of open sets is also an open set.

L1.2.3

Punctured disc $D'_\epsilon(z_0)$ is open.

D1.2.4: Limit points

Let $S \subseteq \mathbb{C}$. z_0 is a **limit point** of S if:

$$\forall \epsilon > 0; D'_\epsilon(z_0) \cap S \neq \emptyset.$$

The **closure** of S is set $\overline{S}$ and contains S and **all** its limit points.

L1.2.6

Let $S \subseteq \mathbb{C}$. S is closed **iff** $S = \overline{S}$.

D1.2.7: Bounded sets

Let $S \subseteq \mathbb{C}$. Set S is **bounded** if:

$$\forall z \in S; \exists M > 0 : |z| \leq S.$$

D1.2.8: ϵ-N convergence

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$ be a sequence and $z \in \mathbb{C}$. Then $\lim\limits_{n \to \infty} z_n = z$ if:

$$\begin{aligned} &\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \\ &\implies |z_n - z| < \epsilon. \end{aligned}$$

L1.2.9

Let $z_n, z \in \mathbb{C}$ where $z_n = a_n + ib_n$.

Then $\lim\limits_{n \to \infty} z_n = z$ **iff**:

$$\text{Re}(z) = \lim_{n \to \infty} a_n \text{ and } \text{Im}(z) = \lim_{n \to \infty} b_n.$$

L1.2.10

Let $S \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. Then $z \in \overline{S}$ **iff**:

$$\exists z_n \in S : z = \lim_{n \to \infty} z_n.$$

D1.2.11: Cauchy sequences

z_n is a Cauchy sequence if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \implies |z_n - z_m| < \epsilon.$$

L1.2.12

z_n is convergent **iff** z_n is Cauchy.

D1.2.14: Bounded sequences

z_n is bounded if:

$$\forall n \in \mathbb{N}; \exists M > 0 : |z_n| \leq M.$$

L1.2.15: Bolzano-Weierstrass

Let z_n be a bounded sequence. Then:

$$\exists (z_{n_k})_{k, n_k \in \mathbb{N}} : \lim_{k \to \infty} z_{n_k} = z \in \mathbb{C}$$

or that z_n has a convergent subsequence.

A selection of a sequence is a subsequence.

D1.3.1: Bounded functions

Let $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$. Then f is a bounded function if:

$$\forall z \in S; \exists M > 0 : |f(z)| \leq M.$$

D1.3.2: ϵ-δ convergence

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}$ and $a_0 \in \mathbb{C}$. Then $\lim_{z \to z_0} f(z) = a_0$ if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : 0 < |z - z_0| < \delta \implies |f(z) - a_0| < \epsilon.$$

L1.3.3

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}$ and $a_0 \in \mathbb{C}$ where $z_0 = x_0 + iy_0$ and $f = u + iv$.

Then $\lim_{z \to z_0} f(z) = a_0$ **iff**:

$$\mathrm{Re}(a_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y)$$

and

$$\mathrm{Im}(a_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y).$$

L1.3.4

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}, f : S \to \mathbb{C}, a_0 \in \mathbb{C}$ and sequence $w_n \in S \setminus \{z_0\}$.

If $\lim_{z \to z_0} f(z) = a_0$ and $\lim_{n \to \infty} w_n = z_0$ then:

$$\lim_{n \to \infty} f(w_n) = a_0.$$

L1.3.5: Limit identities

Let $S \subseteq \mathbb{C}, z_0 \in \overline{S}$ and $a_0, b_0 \in \mathbb{C}$.
Let $f, g : S \to \mathbb{C}$.

If $\lim_{z \to z_0} f(z) = a_0$ and $\lim_{z \to z_0} g(z) = b_0$ then:

1. $\lim_{z \to z_0} (f(z) + g(z)) = a_0 + b_0$
2. $\lim_{z \to z_0} (f(z)g(z)) = a_0 b_0$
3. $\lim_{z \to z_0} \left(\frac{f(z)}{g(z)} \right) = \frac{a_0}{b_0}$ if $b_0 \neq 0$.

D1.3.6: ϵ-δ continuity

Let $S \subseteq \mathbb{C}, f : S \to \mathbb{C}$ and $z_0 \in S$. Then f is continuous at z_0 if:

$$\forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

L1.3.7

Let $f : \mathbb{C} \to \mathbb{C}$ with rule $f = u + iv$ and $z_0 = x_0 + iy_0 \in \mathbb{C}$.

Then f is continuous at z_0 **iff** u and v are continuous at (x_0, y_0).

L1.3.8

If $f, g : \mathbb{C} \to \mathbb{C}$ are continuous at z_0 then:

1. $f + g$ is continuous at z_0.
2. fg is continuous at z_0.
3. f/g is continuous at z_0. $(g \neq 0)$

D: Image and preimage

Let $f : X \to Y$ where $A \subseteq X$ and $B \subseteq Y$. The image of A is:

$$f(A) = \{f(x) : x \in A\}$$

and the preimage of B is:

$$f^{-1}(B) = \{x : f(x) \in B\}.$$

L1.3.9

Let $U \subseteq \mathbb{C}$ be an open set. $f : \mathbb{C} \to \mathbb{C}$ is continuous **iff** $\forall U \subseteq \mathbb{C}; f^{-1}(U)$ is open for $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\}$.

L1.3.10

Let $f : S \to \mathbb{C}$ be continuous. Let $S \subseteq \mathbb{C}$ be closed and bounded.

Then $f(S)$ is closed and bounded.

D1.4.1: Differentiability

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $f : U \to \mathbb{C}$. Then f is differentiable at z_0 if the following limit exists:

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

L1.4.3

Differentiability $\implies$ continuity.

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. If $f : U \to \mathbb{C}$ is differentiable at z_0 then f is continuous at z_0.

L1.4.4

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $f, g : U \to \mathbb{C}$ be differentiable at z_0. Then $f + g$, fg and f/g (where $g(z_0) \neq 0$) are all differentiable at z_0.

L1.4.5: Chain rule

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0. Let $g : U \to \mathbb{C}$ be such that $g(U)$ is a neighbourhood of $g(z_0)$. Assume that g is differentiable at z_0 and f is differentiable at $g(z_0)$. Then $f \circ g$ is differentiable at z_0:

$$(f \circ g)'(z_0) = f(g(z_0))g'(z_0).$$

T1.4.6: Cauchy-Riemann equations

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0.
Let $f : U \to \mathbb{C}$ be differentiable at z_0.
Let $z_0 = x_0 + iy_0$ and $f = u + iv$. Then:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$
$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

and are the Cauchy-Riemann equations.

T1.4.8

Let $z_0 \in \mathbb{C}$ and U a neighbourhood of z_0 for $z_0 = x_0 + iy_0$. Let $f : U \to \mathbb{C}$ where $f = u + iv$.

Assume that u and v have continuous first derivatives on a neighbourhood of (x_0, y_0) **and** also that they satisfy the Cauchy Riemann equations at (x_0, y_0).

Then f is differentiable at z_0.

D1.4.9: Holomorphic functions

f is **holomorphic** at z_0 if there exists a neighbourhood U of z_0 such that f is defined and differentiable.

D1.4.13: Harmonic equations

$h(x, y)$ is harmonic if for $\forall (x, y) \in \mathbb{R}^2$ it satisfies Laplace's equation:

$$\frac{\partial^2 h}{\partial x^2}(x, y) + \frac{\partial^2 h}{\partial y^2}(x, y) = 0.$$

L1.4.14

Let $u(x, y), v(x, y)$ be twice continuously differentiable and that $f(x + iy) = u + iy$ is holomorphic on $\mathbb{C}$.

Then u and v are harmonic.

D1.4.15: Harmonic conjugates

Let $U \subseteq \mathbb{R}^2$ and $u : U \to \mathbb{R}$ be harmonic. Then harmonic function $v : U \to \mathbb{R}$ is a **harmonic conjugate** of u if complex function $f = u + iv$ is holomorphic on U.

D1.5.1: Polynomial degree

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. The **degree** of P is the highest power of the variable in P, denoted as $\deg(P)$.

L1.5.2

Let $z_0 \in \mathbb{C}$. Let complex functions f and g be holomorphic at z_0. Then $f+g$, fg and f/g $(g \neq 0)$ are holomorphic at z_0.

C1.5.3

Let $N \in \mathbb{N}$ and $a_0, \dots, a_N \in \mathbb{C}$.

Let $P(z) = \sum_{n=0}^{N} a_n z^n$.

Then $P(z)$ is holomorphic on $\mathbb{C}$ and:

$$P'(z_0) = \sum_{n=1}^{N} n a_n z_0^{n-1}.$$

L1.5.4

Let $P(z) = \sum_{n=0}^{N} a_n z^n$ where $a_i \in \mathbb{R}$ and $P(z_0) = 0$ for $z_0 \in \mathbb{C}$. Then $P(z_0^*) = 0$.

D1.5.5: Rational functions

Let $P, Q : \mathbb{C} \to \mathbb{C}$ be complex functions. Then $R : \{z \in \mathbb{C} : Q(z) \neq 0\} \to \mathbb{C}$ with $R(z) = P(z)/Q(z)$ is a rational function.

L1.5.7

The rational function $R(z) = P(z)/Q(z)$ is holomorphic on $\{z \in \mathbb{C} : Q(z) \neq 0\}$.

L1.5.8

Let $U \subseteq \mathbb{C}$ be open. Let g be holomorphic on U and f be holomorphic on $g(U)$.

Then $f \circ g$ is holomorphic on U.

L1.5.10

Let $U \subseteq \mathbb{R}^2$ be open and $u, v : U \to \mathbb{R}$. u and v satisfy the Cauchy-Riemann equations **iff** $\bar{\partial} f = 0$, where $f = u + iv$ with map $f : U \to \mathbb{C}$.

Remark

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

D1.6.1: Exponential function

The complex exponential function is a function defined as $\exp : \mathbb{C} \to \mathbb{C}$ and rule:

$$\exp(z) := e^x(\cos y + i \sin y)$$

for $z = x + iy$ and $|z| = e^x$.

P1.6.2

Let $z, w \in \mathbb{C}$.

1. $\exp(z)$ is holomorphic on $\mathbb{C}$.
2. $\exp(z) = \exp'(z)$
3. $\exp(z + w) = \exp(z)\exp(w)$
4. $\exp(z + 2\pi i) = \exp(z)$

D1.6.6: Cosine and sine functions

$$\cos(z) := \frac{1}{2}\Big(\exp(iz) + \exp(-iz)\Big)$$

$$\sin(z) := \frac{1}{2i}\Big(\exp(iz) - \exp(-iz)\Big)$$

L1.6.7

Let $z \in \mathbb{C}$ where $z = x + iy$. Then:

1. $\cos(z)$ and $\sin(z)$ are holomorphic at z, with $\cos'(z) = -\sin(z)$ and $\sin'(z) = \cos(z)$.
2. $\cos^2(z) + \sin^2(z) = 1$
3. $\cos(z + 2\pi) = \cos(z)$
   $\sin(z + 2\pi) = \sin(z)$

L1.6.8

Let $z, w \in \mathbb{C}$. Then:

1. $\sin(z + \pi/2) = \cos(z)$
2. $\sin(z + w)$
   $= \sin(z)\cos(w) + \sin(w)\cos(z)$
3. $\cos(z + w)$
   $= \cos(z)\cos(w) - \sin(z)\sin(w)$.

L1.6.9

Let $z \in \mathbb{C}$ where $z = x + iy$. Then:

$\sin(x + iy)$
$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

$\cos(x + iy)$
$= \cos(x)\cosh(y) - i\sin(x)\sinh(y)$.

D1.6.11: Hyperbolic functions

$$\cosh(z) := \frac{1}{2}\Big(\exp(z) + \exp(-z)\Big)$$

$$\sinh(z) := \frac{1}{2}\Big(\exp(z) - \exp(-z)\Big)$$

L1.6.12

Let $z \in \mathbb{C}$. Then $\sinh(iz) = i\sin(z)$ and $\cosh(iz) = \cos(z)$.

D1.7.1: Logarithm function

Let $z \neq 0 \in \mathbb{C}$. Then:

$$\log(z) := \{w \in \mathbb{C} : z = \exp(w)\}$$

and is the complex **natural** logarithm.

L1.7.3

Let $z, w \in \mathbb{C}$ be nonzero. Then:

1. $\log(z) = \{\ln|z| + i\mathrm{Arg}(z) + i2\pi k\}$
2. $\log(zw) = \log(z) + \log(w)$
3. $\log(1/z) = -\log(z)$

where $k \in \mathbb{Z}$ and $\ln(x)$ denotes the real valued natural logarithm of x.

D1.7.5: Principle branch of $\log z$

The principle branch of the logarithm function is defined as:

$$\mathrm{Log} : \mathbb{C}\backslash\{0\} \to \mathbb{C};$$

$$\mathrm{Log}(z) := \ln|z| + i\mathrm{Arg}(z)$$

and is discontinuous on the negative real axis since $\forall x, \epsilon > 0; -x \pm i\epsilon \to x$ yet:

$$\lim_{\epsilon \to 0} \mathrm{Log}(-x \pm i\epsilon) = \ln|z| \pm i\pi.$$

i.e. the limit on the axis does not exist.

D1.7.7: Branch cuts

A branch cut $L \subset \mathbb{C}$ is removed so that we may define a holomorphic branch of a multivalued function on $\mathbb{C}\backslash L$.

The half-line from z_0 at angle ϕ is:

$$L_{z_0,\phi} = \{z \in \mathbb{C} : z = z_0 + re^{i\phi}; r \geq 0\}$$

and $D_{z_0,\phi} = \mathbb{C}\backslash L_{z_0,\phi}$.

D1.7.9

Let $\phi \in \mathbb{R}$. Then:

$$\phi < \mathrm{Arg}_\phi(z) \leq \phi + 2\pi$$

$$\mathrm{Log}_\phi(z) := \ln|z| + i\mathrm{Arg}_\phi(z).$$

L1.7.10

Branch $\mathrm{Log}_\phi(z)$ is holomorphic on $D_{0,\phi}$:

$$\forall z \in D_{0,\phi}; \frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{Log}_\phi(z)\right] = \frac{1}{z}.$$

L1.7.11

Let $\phi \in \mathbb{R}$, $U \subseteq \mathbb{C}$ be open and $g : U \to \mathbb{C}$ be holomorphic on U. Then $\mathrm{Log}_\phi\big(g(z)\big)$ is holomorphic on $U \cap g^{-1}(D_\phi)$.

D1.8.1: α-th power of z

Let $z, \alpha \in \mathbb{C}$. Then the α-th power of z is: $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\}$ for $z \neq 0$.

T1.8.4

Let $\alpha, z \neq 0 \in \mathbb{C}$.

1. If $\alpha \in \mathbb{Z}$ there is one value of z^α.
2. If $\alpha = p/q \in \mathbb{Q}$ for p, q are coprime then there are q values of z^α.
3. If α is irrational or complex then there are infinite values of z^α.

D1.8.5: Roots of unity

Let q be a positive integer. Then:

$$1^{1/q} = 1, \omega, \dots, \omega^{q-1}; \omega := \exp(2\pi i/q)$$

are the q roots of unity.

D1.8.7: Principle branch of z^α

Let $z \in D$ such that $\mathrm{Log}(z)$ is defined. Then the principle branch of z^α is:

$$z^\alpha := \exp\big(\alpha \mathrm{Log}(z)\big).$$

L1.8.8

Let $\alpha, \beta, z \in \mathbb{C}$ for $z \neq 0$. Then:

$$z^\alpha z^\beta = z^{\alpha+\beta}.$$

L1.8.9

A branch of z^α is holomorphic on D_ϕ and:

$$\forall z \in D_\phi; (z^\alpha)' = \alpha z^{\alpha-1}.$$

D2.1.1: Conformal maps

Let $U \subseteq \mathbb{C}$ be open and let $f : U \to \mathbb{C}$. f is **conformal** if it preserves angles.

i.e. that the angle between tangent lines must remain invariant under mapping.

T2.1.2

Let $U \subseteq \mathbb{C}$ be open and let $f : U \to \mathbb{C}$ be a holomorphic function. Define $T \subseteq U$:

$$T = \{z_0 \in U : f'(z_0) \neq 0\}.$$

Then f preserves angles at every $z_0 \in T$.

i.e. f is a conformal mapping on T.

D2.2.1: Möbius transformations

f is a Möbius transformation if:

$$f(z) = \frac{az+b}{cz+d} \text{ where } a, b, c, d \in \mathbb{C},$$

$ad \neq bc$ and normalisation $ad - bc = 1$.

L2.2.3

Define $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = 1$ to be associated with the following:

$$f_M(z) = \frac{az+b}{cz+d}.$$

Then $f_{M^{-1}} = f_M^{-1}$ and:

$$f_{M_1 M_2} = f_{M_1} \circ f_{M_2}.$$

D2.3.1: Extended complex plane

The extended complex plane is the set:

$$\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

such that for all $a, b \neq 0 \in \mathbb{C}$:

$$a + \infty = \infty, \ \ b \cdot \infty = \infty,$$
$$\frac{b}{0} = \infty \text{ and } \frac{b}{\infty} = 0.$$

D2.3.2.1: Riemann spheres

The Riemann sphere is the unit sphere S^2 in $\mathbb{R}^3$ defined by:

$$S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$$

with north pole $N := (0, 0, 1)$.

D2.3.2.2: Stereographic projections

Let $\phi : \widetilde{\mathbb{C}} \to S^2$ be a bijective mapping such that points $z \in \widetilde{\mathbb{C}}$ and $\phi(z), N \in S^2$ are **colinear**. Then from calculation:

$$\phi(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$
$$\lim_{|z|\to\infty} \phi(z) = N$$

where we denote $z = x + iy = (x, y, 0)$.

The **stereographic projection** is the inverse mapping $\psi : S^2 \to \widetilde{\mathbb{C}}$ of ϕ where:

$$\psi(X, Y, Z) = \begin{cases} \frac{X+iY}{1-Z} & (X, Y, Z) \neq N \\ \infty & (X, Y, Z) = N \end{cases}$$

since we define $\phi(\infty) := N$.

L2.3.4

Stereographic projections maps a circle to a **circline**. (i.e. circle or line)

D2.4.1

1. **Translations**: $f(z) = z + b$ where $b \in \mathbb{C}$.
2. **Rotations**: $f(z) = az$ where $a = e^{i\theta}$ and $a \in \mathbb{C}$.
3. **Dilations**: $f(z) = rz$ where $r > 0 \in \mathbb{R}$.
4. **Inversions**: $f(z) = 1/z$.

T2.4.2

Let f be a Möbius transformation.

Then f consists of a **finite composition** of translations, rotations, dilations and inversions **iff**:

$$f(\infty) \neq \infty$$

i.e. f does not fix the point at infinity.

C2.4.3

If f is a Möbius transformation then it maps circlines to circlines.

L2.5.1

Let f be a Möbius transformation and let $z_2, z_3, z_4 \in \widetilde{\mathbb{C}}$ be distinct points such that $f(z_2) = z_2$, $f(z_3) = z_3$ and $f(z_4) = z_4$.

Then $f(z) = z$. (identity transformation)

T2.5.2

Given distinct points $z_2, z_3, z_4 \in \widetilde{\mathbb{C}}$ there exists a <u>unique</u> Möbius transformation f:

$$f(z_2) = 1, \ \ f(z_3) = 0 \text{ and } f(z_4) = \infty.$$

Explicitly this mapping is given by:

$$f(z) = \frac{z_2 - z_4}{z_2 - z_3} \frac{z - z_3}{z - z_4}.$$

C2.5.3

Let $z_2, z_3, z_4, w_2, w_3, w_4 \in \widetilde{\mathbb{C}}$ be distinct points. Then there is a unique Möbius transformation f such that:

$$f(z_2) = w_3, \ \ f(z_3) = w_3 \text{ and } f(z_4) = w_4.$$

D2.5.4: Cross ratios

Let $z_1, z_2, z_3, z_4 \in \widetilde{\mathbb{C}}$ be distinct points and let f be a Möbius transformation that maps $(z_2, z_3, z_4) \mapsto (1, 0, \infty)$.

Then the **cross ratio** is defined as:

$$[z_1, z_2, z_3, z_4] := f(z_1).$$

T2.5.6

Let $z_1, z_2, z_3, z_4 \in \widetilde{\mathbb{C}}$ be distinct and let f be a Möbius transformation. Then:

$$[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4].$$

D3.1.1: Integrable functions

Let $f : [a, b] \to \mathbb{C}; f = u + iv$. Then f is **integrable** if $u(t)$ and $v(t)$ are integrable.

$$\therefore \int_a^b f := \int_a^b u + i \int_a^b v \in \mathbb{C}.$$

f is integrable if it is continuous.

L3.1.2

Let $f, g : [a, b] \to \mathbb{C}$ be integrable and $\alpha, \beta \in \mathbb{C}$. Then:

1. $\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$
2. Let $f = F'$ be continuous and that $F : [a, b] \to \mathbb{C}$. Then:
$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a).$$
3. $\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\mathrm{d}t.$

D3.2.1: Contours

A contour $\Gamma \subset \mathbb{C}$ is a curve that connects z_0 **to** $z_1 \in \mathbb{C}$. We define $\Gamma = \mathrm{im}(\gamma)$ where:

$\gamma : [t_0, t_1] \to \mathbb{C}; \gamma(t_0) = z_0$ and $\gamma(t_1) = z_1$.

Contour Γ is **regular** if its first derivative is continuous and $\gamma'(t) \neq 0$ for $\forall t$.

D3.2.3: Contour integrals

Let Γ be a regular curve connecting points z_0 and z_1. Let $f : \Gamma \to \mathbb{C}$ be continuous. Then the integral of f along Γ is:

$$\int_\Gamma f(z)\mathrm{d}z = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

Let there exist $\gamma_i : [t_0^i, t_1^i] \to \mathbb{C}$ such that $\gamma_i(t_0^1) = z_0$, $\gamma_i(t_1^i) = \gamma_{i+1}(t_0^{i+1})$ and $\gamma_n(t_1^n) = z_1$ where $\Gamma_i = \mathrm{im}(\gamma_i)$. Then:

$$\int_\Gamma f(z)\mathrm{d}z = \sum_{i=1}^{n} \int_{\Gamma_i} f(z)\mathrm{d}z.$$

D3.2.7: Contour arclengths

The **arclength** of a regular curve Γ is:

$$\ell(\Gamma) = \int_{t_0}^{t_1} |\gamma'(t)|\mathrm{d}t = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t.$$

If Γ is the arc of a circle with radius r traced by an angle θ then $\ell(\Gamma) = r\theta$.

L3.2.9: M-L lemma

Let Γ be regular and let $f : \Gamma \to \mathbb{C}$ be a continuous function. Then:

$$\left|\int_\Gamma f(z)\mathrm{d}z\right| \leq \max_{z\in\Gamma} |f(z)|\ell(\Gamma).$$

D3.3.1: Domains

Set D is a **domain** if it is **open** and every two points in D is connected by a contour that is fully contained in D.

L3.3.2

Let D be a domain and let $u : D \to \mathbb{R}$ be differentiable, where $u'_x = u'_y = 0$ on D. Then $u(x, y)$ is constant on D.

D3.3.3: Antiderivatives

Let D be a domain and let $f : D \to \mathbb{C}$ be continuous. f has an antiderivative on D if $\exists F : D \to \mathbb{C} : \forall z \in D; F'(z) = f(z)$.

T3.3.5: FTC

Let D be a domain and let continuous $f : D \to \mathbb{C}$ have an antiderivative F on D. If contour $\Gamma \subset D$ connects z_0 to z_1:

$$\int_\Gamma f(z)\mathrm{d}z = F(z_1) - F(z_0).$$

C3.3.6

Let f be holomorphic on domain D and $f'(z) = 0$ for $\forall z \in D$. Then f is constant.

D3.3.7: Closed contours

Γ is **closed** if its endpoints are the same.

L3.3.9: Path independence

Let $f : D \to \mathbb{C}$ be continuous where D is a domain. The following are equivalent:

1. f has an antiderivative on D.
2. For all **closed** contours $\Gamma \subset D$:
$$\oint_\Gamma f(z)\mathrm{d}z = 0.$$
3. Integrals are independent of path, regardless of contour chosen in D.

D3.4.1: Loops

Γ is **simple** if it has no self intersections except at the endpoints.

Loops are simple and closed contours.

T3.4.2: Jordan curve theorem

Let Γ be a loop in $\mathbb{C}$. Then Γ defines the following two regions:

1. bounded interior: $\mathrm{Int}(\Gamma)$
2. unbounded exterior: $\mathrm{Ext}(\Gamma)$

where $\mathbb{C} = \mathrm{Int}(\Gamma) \cup \Gamma \cup \mathrm{Ext}(\Gamma)$.

D3.4.4: Positively oriented loops

Loop Γ is **positively oriented** if $\mathrm{Int}(\Gamma)$ is always remain on the left hand side when traversing its parametrisation.

D3.4.6: Simply connected domains

Domain D is **simply connected** if:

for all loops $\Gamma \subset D; \mathrm{Int}(\Gamma) \subseteq D$.

T3.4.8: Cauchy integral theorem

Let Γ be a **loop**. Let f be holomorphic inside and on contour Γ. Then:

$$\int_\Gamma f(z)\mathrm{d}z = 0.$$

C3.4.9

Let D be a simply connected domain and let f be holomorphic on D. Then f has an antiderivative on D.

T3.4.11

Consider loop Γ and point $z_0 \notin \Gamma$. Then:

$$\int_\Gamma \frac{1}{z - z_0}\mathrm{d}z = \begin{cases} 2\pi i & z_0 \in \mathrm{Int}(\Gamma) \\ 0 & \text{otherwise.} \end{cases}$$

T3.4.12: Deformation theorem

Let f be holomorphic on loops Γ_1, Γ_2 and $\big(\mathrm{Int}(\Gamma_1) \setminus \mathrm{Int}(\Gamma_2)\big) \cup \big(\mathrm{Int}(\Gamma_2) \setminus \mathrm{Int}(\Gamma_1)\big)$.

$$\text{Then } \int_{\Gamma_1} f(z)\mathrm{d}z = \int_{\Gamma_2} f(z)\mathrm{d}z.$$

T3.5.1: Cauchy integral formula

Let Γ be a loop. Let $z_0 \in \mathrm{Int}(\Gamma)$ and let f be holomorphic on $\Gamma \cup \mathrm{Int}(\Gamma)$. Then:

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\mathrm{d}z.$$

T3.5.3

Let Γ be a contour on domain D. Let $g : D \to \mathbb{C}$ be continuous on Γ. Then the following $G : D \setminus \Gamma \to \mathbb{C}$ is holomorphic:

$$G(z) = \int_\Gamma \frac{g(w)}{(w - z)^n}\mathrm{d}w$$

$$G'(z) = n \int_\Gamma \frac{g(w)}{(w - z)^{n+1}}\mathrm{d}w$$

given $n \in \{1, 2, \ldots\}$.

C3.5.5: Infinite differentiability

Let f be holomorphic on domain D. Then f is infinitely differentiable on D and all its derivatives are holomorphic on D.

T3.5.6

Consider loop Γ. Let f be holomorphic on $\Gamma \cup \mathrm{Int}(\Gamma)$ and let $z \in \Gamma$. Then f is infinitely differentiable at z and:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z_0)^{n+1}}\mathrm{d}w$$

where $n \in \mathbb{N}$.

T3.5.12

Let D be a domain. Let $f : D \to \mathbb{C}$ be continuous and that for all loops $\Gamma \subset D$:

$$\int_\Gamma f(z)\mathrm{d}z = 0.$$

Then f is holomorphic on D.

T3.6.1

Let D be a domain. Let $z_0 \in D$, $R > 0$ and $\overline{D_R}(z_0) \subseteq D$. Consider holomorphic function f on D such that:

$$\exists M > 0; \forall z \in D : |f(z)| \leq M.$$

Then for all $n \in \mathbb{N}$:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

T3.6.2: Liouville's theorem

f is constant **iff** f is holomorphic **and** bounded on $\mathbb{C}$.

T3.6.3: FTA

Every complex polynomial has a root.

T3.7.1

Let f be holomorphic on domain D. Let $z_0 \in D$, $R > 0$ and $\overline{D_R}(z_0) \subseteq D$. Then:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R\exp(it))\mathrm{d}t.$$

T3.7.5: Maximum modulus

Let function f be holomorphic on domain D. Let $\exists M > 0 : \forall z \in D; |f(z)| \leq M$, or that f is also bounded.

If there exists $z_0 \in D$ such that $|f(z_0)|$ is maximised then f is constant on D.

D4.1.1: Infinite series

L4.1.4

L4.1.8: Comparison test

L4.1.9

L4.1.11: Ratio test

D4.1.12: Pointwise convergence

D4.1.14: Uniform convergence

L4.1.17

L4.1.19: Weierstrass M-test

L4.1.21

L4.1.22

T4.1.23

D4.2.1: Power series

T4.2.2: Radius of convergence

include t4.2.4

T4.2.6

T4.3.2: Taylor series

Let f be holomorphic on $D_R(z_0)$ where $R > 0$. Then the **Taylor series** for f *centred at* z_0 defined as:

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

converges to $f(z)$ for all $z \in D_R(z_0)$ and *uniformly* on $\overline{D}_r(z_0)$ for all $r \in [0, R)$.

D4.3.4: Analytic functions

P4.3.8

L4.3.9

T4.3.11

uniqueness of taylor series

D4.4.3: Annuluses

Let $r, R \in [0, \infty) \cup \{\infty\}$. Then we define the **annulus** centred at z_0 as:

$$A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

$$\overline{A}_{r,R}(z_0) = \{z \in \mathbb{C} : r \leq |z - z_0| \leq R\}.$$

T4.4.4: Laurent series

Let f be holomorphic on $A_{r,R}(z_0)$ where $0 \leq r < R \leq \infty$. Then f can be written as a **Laurent series** centred at z_0:

$$\sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

which converges to $f(z)$ on $A_{r,R}(z_0)$ and *uniformly* on $\overline{A}_{r_1,R_1}(z_0); \forall r_1, R_1 \in (r, R)$.

Given loop $\Gamma \subset A_{r,R}(z_0)$ and $z_0 \in \operatorname{Int}(\Gamma)$:

$$a_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{j+1}} \mathrm{d}z.$$

The Laurent development is *unique*.

D4.5.1: Singularities of f

Let D be a domain, let $z_0 \in \mathbb{C}$ and let $f : D \to \mathbb{C}$. If f is *not* holomorphic at point z_0 then z_0 is a **singularity** of f.

z_0 is an **isolated singularity** if $\exists R > 0$ such that f is holomorphic on $D'_R(z_0)$.

D4.5.3: Zeros of f

Let U be a neighbourhood of z_0 and let f be holomorphic on U. z_0 is a **zero** of f if $f(z_0) = 0$. z_0 is a *zero of order m* if:

$$\exists m \in \mathbb{Z}_{>0} : f(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

but $f^{(m)}(z_0) \neq 0$. A **simple zero** is a zero of order 1. An **isolated zero** z_0 is if $\exists R > 0 : f(z) \neq 0$ for all $z \in D'_R(z_0)$.

P4.5.4

Let U be a neighbourhood of z_0 and let f be holomorphic on U. Let z_0 be a zero of *finite* order. Then z_0 is isolated.

C4.5.5

Let U be a neighbourhood of z_0 and let f be holomorphic on U. Let there exist sequence $(z_n)_{n \in \mathbb{N}} \subset U$ such that $z_n \to z_0$ and $f(z_n) = 0$.

Then f is zero on a disc centred at z_0.

C4.5.6

Let z_0 be a singularity of rational function $f = P/Q$. Then z_0 is isolated.

D4.5.7

Let f be holomorphic on $D'_R(z_0)$ where $R > 0$ and z_0 an isolated singularity. Then f has a Laurent expansion centred at z_0 which is valid on $A_{0,R}(z_0)$:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j.$$

Furthermore we define that:

1. z_0 is a **removable singularity** if $a_j = 0$ for all $j < 0$.
2. z_0 is a **pole** *of order m* if $a_j = 0$ for $j < -m$ and $a_{-m} \neq 0$.
3. z_0 is an **essential singularity** if $a_j \neq 0$ for infinitely many $j < 0$.

T4.5.8

Let f be holomorphic on $D'_R(z_0)$ where $R > 0$ and z_0 a removable singularity. Then $f(z_0)$ can be *redefined* so that f is holomorphic at z_0.

L4.5.12

D4.6.1: Analytic continuations

T4.6.4: Identity theorem

C4.6.5

C4.6.7

C4.6.8

T5.1.1

D5.1.2: Residues of f

L5.1.4

L5.1.5

L5.1.7

T5.1.10: Cauchy residue theorem

D5.2.1: Meromorphic functions

L5.2.2

D5.2.3

T5.2.5: Argument principle

C5.2.6

T5.2.7: Rouché's theorem

T5.2.14: Open mapping theorem

L5.4.5: Jordan lemma

D5.5.1: Improper integrals

D5.5.2: Principle value of integrals

L5.5.3

L5.6.3