D: Functions

injections, surjections and bijections

D: Groups

D: Abelian groups

D1.2.1(i): Fields

A field F is a set defined with:

1. Addition function $(+)$:

$$(+) : F \times F \to F; (\lambda, \mu) \mapsto \lambda + \mu$$

2. Multiplication function $(\cdot)$:

$$(\cdot) : F \times F \to F; (\lambda, \mu) \mapsto \lambda \cdot \mu$$

3. $\exists 0_F, 1_F \in F$ where $0_F \neq 1_F$ such that $(F, +)$ and $(F \backslash \{0_F\}, \cdot)$ form Abelian groups.
4. $\exists (-\lambda) \in F : \lambda + (-\lambda) = 0_F$
5. $\exists (\lambda^{-1}) \in F : \lambda \cdot (\lambda^{-1}) = 1_F$
6. $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F$

D1.2.1(ii): Vector spaces

A vector space V over a field F is an Abelian group $V := (V, +)$ with mapping:

$$F \times V \to V : (\lambda, \boldsymbol{v} \mapsto \lambda\boldsymbol{v})$$

where for $\forall \lambda, \mu \in F$ and $\forall \boldsymbol{v}, \boldsymbol{w} \in V$:

1. $\lambda(\boldsymbol{v} + \boldsymbol{w}) = (\lambda\boldsymbol{v}) + (\mu\boldsymbol{w})$
2. $(\lambda + \mu)\boldsymbol{v} = (\lambda\boldsymbol{v}) + (\mu\boldsymbol{w})$
3. $\lambda(\mu\boldsymbol{v}) = (\lambda\mu)\boldsymbol{v}$
4. $1_F\boldsymbol{v} = \boldsymbol{v}$

and is a F-vector space.

Remark

Let V be a F-vector space where $\boldsymbol{v} \in V$.

1. $0\boldsymbol{v} = 0$
2. $(-1)\boldsymbol{v} = -\boldsymbol{v}$
3. $\lambda\mathbf{0} = \mathbf{0}$ for $\forall \lambda \in F$.

D: Cartesian products

The Cartesian product of sets $X_1, \ldots, X_n$ is defined as:

$$X_1 \times \cdots \times X_n := \{(x_1, \ldots, x_n) : x_i \in X_i\}$$

where $1 \leq i \leq n$.

The projection of a Cartesian product is:

$$\begin{aligned} \mathrm{pr}_i : X_1 \times \cdots \times X_n \to X_i; \\ (x_1, \ldots, x_n) \mapsto x_i \end{aligned}$$

D1.4.1: Vector subspaces

A vector subspace U of F-vector space V has the follwing properties:

1. $U \subset V$ and $\mathbf{0} \in U$.
2. Let $\boldsymbol{u}, \boldsymbol{v} \in U$ and $\lambda \in F$.
   Then $\boldsymbol{u} + \boldsymbol{v} \in U$ and $\lambda\boldsymbol{u} \in U$.

and is also a vector space.

P1.4.5

Let $T \subset V$ where V is a F-vector space. Then for all vector subspaces containing T, there exists a smallest vector subspace:

$$\mathrm{span}(T) = \langle T \rangle_F \subset V$$

known as the vector subspace generated by T, or the span of T.

D1.4.7: Generating set

Let $T \subset V$ where V is a F-vector space. T is a generating set of V if:

$$\mathrm{span}(T) = V$$

and is the linear combination of vectors in T over field F.

D1.4.9: Power sets

D1.5.1: Linear independence

D1.5.8: Basis

T1.5.11: ???

T1.5.12: ???

C1.5.13