Honours Differential Equations Examples

Christopher Shen

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5 Laplace transforms

1. Find the Laplace transform of the following function:

$$y = \exp(at).$$

So we have that:

$$\begin{aligned}\mathcal{L}[y] &= \int_0^\infty \exp\Big((a-s)t\Big)\mathrm{d}t \\ &= \frac{1}{s-a}\end{aligned}$$

and is only defined if $s > a$.

2. Solve:

$$y'' + y = \sin 2t$$

with $y(0) = 2$ and $y'(0) = 1$.

Now taking the Laplace transforms on both sides:

$$\begin{aligned}\mathcal{L}[y'' + y] &= \mathcal{L}[y''] + \mathcal{L}[y] \\ &= s^2\mathcal{L}[y] - sy(0) - y'(0) + \mathcal{L}[y] \\ &= (s^2+1)\mathcal{L}[y] - sy(0) - y'(0)\end{aligned}$$

and for the RHS:

$$\begin{aligned}\mathcal{L}[\sin 2t] &= \int_0^\infty e^{-st}\sin 2t\mathrm{d}t \\ &= \frac{2}{s^2+4}.\end{aligned}$$

Equating terms and arranging gives:

$$\mathcal{L}[y] = \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

and by inspection we have that:

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$