

Integrating factors

$$y' + P(x)y = Q(x)$$

$$I(x) = \exp\left(\int P(x)dx\right)$$

$$y(x) = \frac{1}{I(x)} \int Q(x)I(x)dx + \frac{\alpha}{I(x)}$$

Change of variables

$$y^{(n)} = F(y, y', \dots, y^{(n-1)}, t)$$

Let $x_{i+1} = y^{(i)}$ where $i \in \{0, 1, \dots, n-1\}$.

Picard-Lindelöf statement

Consider IVP: $x'_i = F_i(t, x_1, \dots, x_n)$ or that $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ and \mathbf{x} is a vector with n entries.

It has **unique** solutions if:

$$F_i, \frac{\partial F_i}{\partial x_j} \text{ and } \frac{\partial F_i}{\partial t} \text{ are continuous in}$$

$R \subset \mathbb{R}^{n+1}$ where $(t, \mathbf{x}_0^T) \in R$. Here $i, j \in \{1, \dots, n\}$.

Homogeneous systems

Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where \mathbf{A} is a $n \times n$ matrix. Substituting $\mathbf{x} = e^{rt}\boldsymbol{\xi}$ gives:

$$(\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\xi}^{(i)} = \mathbf{0}$$

where $i \in \{1, 2, \dots, n\}$.

Our general solution is then:

$$\begin{aligned} \mathbf{x}(t) &= \sum_{i=1}^n c_i e^{r_i t} \boldsymbol{\xi}^{(i)} \\ &= \sum_{i=1}^n c_i \mathbf{x}^{(i)} \\ &= \boldsymbol{\Psi}(t)\mathbf{c}. \end{aligned}$$

If initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ are given:

$$\mathbf{c} = \boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}(t_0)$$

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}_0.$$

Matrix exponentials

Given a $n \times n$ matrix \mathbf{A} :

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} \\ &= \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \dots \end{aligned}$$

For system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

and that $e^{\mathbf{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$.

Diagonalisation

For $\mathbf{x}' = \mathbf{A}\mathbf{x}$ we have $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$.

If $\mathbf{x} = \mathbf{T}\mathbf{y}$ then $\mathbf{y}' = \mathbf{D}\mathbf{y}$.

Since our fundamental matrix with respect to \mathbf{y} is a diagonal matrix $\mathbf{Q} = e^{\mathbf{D}t}$, the fundamental matrix with respect to \mathbf{x} is $\boldsymbol{\Psi}(t) = \mathbf{T}e^{\mathbf{D}t}$ and:

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}.$$

Generalised eigenvectors

If eigenvalues do not generate enough eigenvectors to form n linearly independent solutions for a $n \times n$ matrix \mathbf{A} , then consider the following ansatz:

$$\mathbf{x} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

$$\therefore (\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\eta}^{(i)} = \boldsymbol{\xi}^{(i)}$$

Then this r_i produces two solutions:

$$\mathbf{x}^{(1)} = e^{rt}\boldsymbol{\xi}$$

$$\mathbf{x}^{(2)} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

Non-homogeneous systems

Consider non-homogeneous ODE system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}.$$

- Change of basis**

Let $\mathbf{x} = \mathbf{T}\mathbf{y}$ and since $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$:

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$$

which is solved by integrating factors. Finally revert back to \mathbf{x} .

- Variation of parameters**

Find solution $\mathbf{x}_H = \boldsymbol{\Psi}\mathbf{c}$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then let the non-homogeneous solution be $\mathbf{x} = \boldsymbol{\Psi}\mathbf{u}(t)$.

$$\therefore \boldsymbol{\Psi}\mathbf{u}'(t) = \mathbf{g}(t)$$

Row reduce before integrating.

- Undetermined coefficients**

Let non-homogeneous ODE system have solutions of form:

$$\mathbf{x} = \mathbf{x}_H + \mathbf{x}_p$$

where \mathbf{x}_H is our homogeneous solution and \mathbf{x}_p our particular solution.

Critical points

Consider non-linear ODE system

$$\mathbf{x}' = \mathbf{F}(x, y),$$

$$\mathbf{y}' = \mathbf{G}(x, y).$$

We define $\mathbf{x}^0 = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$ as a **critical point** when $\mathbf{F}(\mathbf{x}^0) = \mathbf{G}(\mathbf{x}^0) = \mathbf{0}$.

Linearisation and stability

$$\text{Let } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x - x^0 \\ y - y^0 \end{bmatrix}.$$

$$\therefore u'_1 \approx F(x^0, y^0) + \left(\frac{\partial F}{\partial x}\right)_{x^0} (x - x^0)$$

$$+ \left(\frac{\partial F}{\partial y}\right)_{y^0} (y - y^0)$$

$$\therefore u'_2 \approx G(x^0, y^0) + \left(\frac{\partial G}{\partial x}\right)_{x^0} (x - x^0)$$

$$+ \left(\frac{\partial G}{\partial y}\right)_{y^0} (y - y^0)$$

$$\therefore \mathbf{u}' = \mathbf{A}\mathbf{u}$$

$$= \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}_{\mathbf{x}=\mathbf{x}^0} \begin{bmatrix} x - x^0 \\ y - y^0 \end{bmatrix}$$

Critical points \mathbf{x}^0 may also be classified:

Eigenvalues	Critical Points	Stability
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asympt. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asympt. stable
$r_1, r_2 = \lambda \pm i\mu$ ($\lambda > 0$)	focus	unstable
$r_1, r_2 = \lambda \pm i\mu$ ($\lambda < 0$)	focus	asympt. stable
$r_1 = i\mu, r_2 = -i\mu$	center	stable

Linearisation preserves critical point behaviour **except** when eigenvalues are of $r = \pm i\mu$ form, for which then classification is unknown.

Stable critical points \mathbf{x}^0 :

All solutions start and stay near \mathbf{x}^0 .

$$\begin{aligned} \forall \epsilon > 0; \exists \delta > 0; \forall \mathbf{x}(t) = \boldsymbol{\phi}(t) : |\mathbf{x}(0) - \mathbf{x}^0| < \delta \\ \implies |\mathbf{x}(t) - \mathbf{x}^0| < \epsilon \text{ for } \forall t \geq 0 \end{aligned}$$

Attracting critical points \mathbf{x}^0 :

All solutions tends to \mathbf{x}^0 .

$$\begin{aligned} \forall \delta > 0 : |\mathbf{x}(0) - \mathbf{x}^0| < \delta \\ \implies \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^0 \end{aligned}$$

Asymptotically stable critical points \mathbf{x}^0 : Attracting **and** stable.

Lyapunov's theory and limit cycles

Consider $\dot{x} = F(x, y)$ and $\dot{y} = G(x, y)$ and let $\mathbf{x}^0 \in D$ be a critical point. Let $E : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined such that $E(\mathbf{x}^0, y^0) = 0$.

$$\therefore \frac{dE}{dt} = \frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G$$

- Let $E > 0$ for $\forall \mathbf{x} \neq \mathbf{x}^0$.**

If $\frac{dE}{dt} \leq 0$ then \mathbf{x}^0 is stable.

If $\frac{dE}{dt} < 0$ then \mathbf{x}^0 is asymptotically stable.

- $E(\mathbf{x}^*) > 0$ and $\frac{dE}{dt} > 0$**

\implies unstable \mathbf{x}^0 . (flip both signs)

Postive definite: $E(x) > 0$ for $\forall x \neq x^0$

Postive semidefinite:

$E(x) \geq 0$ for $\forall x \neq x^0$

Limit cycles are periodic solutions such that at least one other **non-closed trajectory** approaches it as $t \rightarrow \infty$.

Generally if our trajectory is enclosed by finite non-simple region and F, G have continuous partials then there is a limit cycle.

Real Fourier series

The Fourier expansion of piecewise continuous $f(x)$ on $[-L, L]$ is:

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where $f_{FS}(x) = f_{FS}(x + 2L)$. If α is a discontinuous point:

$$f_{FS}(\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}$$

for α^+ is the limit from the left.

Our Fourier coefficients are:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L \cos \frac{n\pi x}{L} f(x) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L \sin \frac{n\pi x}{L} f(x) dx.$$

Orthogonality

Let $S_n = \sin \frac{n\pi x}{L}$ and $C_n = \cos \frac{n\pi x}{L}$.

$$\langle S_n, S_m \rangle = \langle C_n, C_m \rangle = L\delta_{mn}$$

$$\langle S_n, C_m \rangle = 0.$$

Where:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$$\langle u(x), v(x) \rangle = \int_{-L}^L u(x)v(x) dx$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)].$$

Even functions: $f(-x) = f(x)$

$$\therefore \int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$$

Odd functions: $f(-x) = -f(x)$

$$\therefore \int_{-L}^L f(x) dx = 0$$

• **Even function** $f(x)$ on $[-L, L]$:

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_0^L \cos \frac{n\pi x}{L} f(x) dx$$

• **Odd function** $f(x)$ on $[-L, L]$:

$$f_{FS}(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} f(x) dx$$

Extensions

Consider $f(x)$ defined in $[0, L]$ originally.

1. Define even function:

$$g(x) = \begin{cases} f(x) & x \in [0, L] \\ f(-x) & x \in (-L, 0) \end{cases}$$

with cosine series.

2. Define odd function:

$$h(x) = \begin{cases} f(x) & x \in (0, L) \\ 0 & x = 0, L \\ -f(-x) & x \in (-L, 0) \end{cases}$$

with sine series.

Complex Fourier series

Similarly:

$$f_{FS}(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left(\frac{in\pi}{L} x \right)$$

$$e^{i\theta} = \sin \theta + i \cos \theta.$$

Then $\forall n \in \mathbb{Z}$ we have that:

$$c_n = \frac{1}{2L} \int_{-L}^L \exp \left(-\frac{in\pi}{L} x \right) f(x) dx$$

$$c_n = \begin{cases} (a_n - ib_n)/2 & n > 0 \\ (a_0)/2 & n = 0 \\ (a_n + ib_n)/2 & n < 0. \end{cases}$$

The **inner product** is defined as:

$$\langle f, g \rangle = \int_{-L}^L f(x) g^*(x) dx.$$

$$\therefore \langle \exp \left(\frac{im\pi}{L} x \right), \exp \left(\frac{in\pi}{L} x \right) \rangle = 2L\delta_{mn}$$

Parseval's theorem

$$\langle f, f \rangle = \int_{-L}^L |f(x)|^2 dx$$

$$= 2L \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$= L \left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \right]$$

Heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Let $u(x, t) = X(x)T(t)$.

$$\frac{1}{\alpha^2} \frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

$$\dot{T} + \alpha^2 \lambda T = 0$$

$$\lambda = \mu^2; X(x) = b_1 \cos \lambda^{1/2} x + b_2 \sin \lambda^{1/2} x$$

$$\lambda = -\mu^2; X(x) = b_1 \cosh \mu x + b_2 \sinh \mu x$$

$$T(t) = a_1 \exp(-\alpha^2 \lambda t)$$

Standard boundary conditions

$$\bullet u(x, 0) = f(x) \text{ for } 0 \leq x \leq L$$

$$\bullet u(0, t) = u(L, t) = 0 \text{ for } \forall t > 0$$

$$X(0) = X(L) = 0$$

$$\therefore X_n = b_2 \sin \lambda_n^{1/2} x$$

$$\therefore \lambda_n = \left(\frac{n\pi}{L} \right)^2 \text{ for } \forall n \in \mathbb{N}$$

Our general solution must then be:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx.$$

Fixed boundary temperatures

$$\bullet u(0, t) = T_1$$

$$\bullet u(L, t) = T_2$$

$$\bullet u(x, 0) = f(x)$$

$$v(x) = \lim_{t \rightarrow \infty} u(x, t)$$

Since $v'' = 0$, $v(0) = T_1$ and $v(L) = T_2$:

$$v(x) = \frac{T_2 - T_1}{L} x + T_1.$$

We then deduce that:

$$u(x, t) = v(x) + \omega(x, t)$$

where $\omega(x, t)$ satisfies conditions:

$$\bullet \omega(0, t) = \omega(L, t) = 0$$

$$\bullet \omega(x, 0) = f(x) - v(x)$$

$$\therefore \omega(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

For $\lambda_n = \left(\frac{n\pi}{L} \right)^2$ and

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) (f(x) - v(x)) dx.$$

Insulated rod ends

- $\frac{\partial}{\partial x}u(0, t) = \frac{\partial}{\partial x}u(L, t) = 0$
- $u(x, 0) = f(x)$

$$X'(0) = X'(L) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \cos \lambda_n^{1/2} x$$

$$c_n = \frac{2}{L} \int_0^L \cos(\lambda_n^{1/2} x) f(x) dx$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Let $u(x, t) = X(x)T(t)$.

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -\lambda$$

$$X'' + \lambda X = 0$$

$$\ddot{T} + c^2 \lambda T = 0$$

Plucked string

- $u(0, t) = u(L, t) = 0$
- $\frac{\partial}{\partial t}u(x, 0) = 0$
- $u(x, 0) = f(x)$

$$X(0) = X(L) = 0 \text{ and } \dot{T}(0) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x \cos c \lambda_n^{1/2} t$$

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

General initial conditions

- $u(0, t) = u(L, t) = 0$
- $\frac{\partial}{\partial t}u(x, 0) = g(x)$
- $u(x, 0) = f(x)$

$$u(x, t) = \sum_{n=1}^{\infty} \sin \lambda_n^{1/2} x \times \left(a_n \cos c \lambda_n^{1/2} t + b_n \sin c \lambda_n^{1/2} t \right)$$

$$a_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx$$

$$b_n = \frac{1}{c \lambda_n^{1/2}} \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) g(x) dx$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u(x, y) = X(x)Y(y)$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$X'' - \lambda X = 0$$

$$Y'' + \lambda Y = 0$$

Rectangular boundary conditions

- $u(x, 0) = u(x, b) = 0$
- $u(0, y) = 0$ and $u(a, y) = f(y)$

Here $x \in [0, a]$ and $y \in [0, b]$.

$$X(0) = 0 \text{ and } Y(0) = Y(b) = 0$$

$$\therefore Y_n = a_1 \sin(\lambda_n^{1/2} y) \text{ for } \lambda_n = \frac{n^2 \pi^2}{b^2}$$

$$\therefore X_n = a_3 \sinh(\lambda_n^{1/2} x)$$

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} x) \sin(\lambda_n^{1/2} y)$$

$$c_n = \frac{2}{b \sinh(\lambda_n^{1/2} a)} \int_0^b \sin(\lambda_n^{1/2} y) f(y) dy$$

Circular boundary conditions

- $u(a, \theta) = f(\theta)$
- $u(r, \theta)$ is bounded

Here $r \in [0, a]$ and $\theta \in [0, 2\pi]$.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0.$$

Let $u(r, \theta) = R(r)\Theta(\theta)$.

$$\therefore r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \lambda$$

$$\therefore \ddot{\Theta} + \lambda \Theta = 0$$

$$\therefore r^2 R'' + r R' = \lambda R$$

For the first ODE if $\lambda \leq 0$ then we get at best constant solutions. If $\lambda > 0$:

$$\Theta(\theta) = a_1 \cos \lambda^{1/2} \theta + a_2 \sin \lambda^{1/2} \theta$$

and since periodicity must be preserved:

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\therefore \lambda_n^{1/2} = n \text{ or } \lambda_n^{1/2} = 0$$

where $n \in \mathbb{N}$. So when $\lambda_n = 0$:

$$r^2 R'' + r R' = 0$$

and since $u(r, \theta)$ is bounded we get only constant solutions. If $\lambda_n = n^2$ then:

$$r^2 R'' + r R' - n^2 R = 0$$

with solutions of form $R(r) = r^\alpha$ which yields $R_n(r) = c_n r^n$. Then:

$$u(r, \theta) = \frac{p_0}{2} + \sum_{n=1}^{\infty} r^n \left(q_n \cos \lambda_n^{1/2} \theta + r_n \sin \lambda_n^{1/2} \theta \right)$$

$$p_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$q_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} \cos(\lambda_n^{1/2} \theta) f(\theta) d\theta$$

$$r_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} \sin(\lambda_n^{1/2} \theta) f(\theta) d\theta.$$

Regular S-L problems

A regular S-L problem defined on $[0, 1]$ is:

$$-\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

$$\bullet a_1 y(0) + a_2 y'(0) = 0$$

$$\bullet b_1 y(1) + b_2 y'(1) = 0$$

$$\bullet p(x), \quad p'(x), \quad q(x), \quad r(x) \quad \text{are} \quad \underline{\text{continuous}}$$

$$\bullet p(x), r(x) \text{ are } \underline{\text{strictly positive}}$$

which yields **real** eigenvalues λ_n and **orthonormal** eigenfunctions $\phi_n(x)$:

$$\langle \phi_m, \phi_n \rangle = \int_0^1 r(x) \phi_m(x) \phi_n(x) dx = \delta_{mn}$$

in Hilbert space $L^2([0, 1], r(x)dx)$.

Let $\phi_n(x) = k_n y_n(x)$.

$$\begin{aligned} \therefore k_n &= \frac{1}{\sqrt{\langle y_n, y_n \rangle}} \\ &= \left(\int_0^1 r(x) y_n^2(x) dx \right)^{-1/2} \end{aligned}$$

General 2nd order ODEs

$$-p(x) \frac{d^2 y}{dx^2} - \omega(x) \frac{dy}{dx} + q(x)y = \lambda r(x)y$$

$$F(x) = \exp \left[\int_0^x \frac{\omega(s) - p'(s)}{p(s)} ds \right]$$

$$-\frac{d}{dx} \left[F(x)p(x) \frac{dy}{dx} \right] + F(x)q(x)y = \lambda F(x)r(x)y$$

Lagrange's identity

Let functions u and v satisfy regular S-L boundary conditions. Then:

$$\langle \mathcal{L}[u], v \rangle = \langle u, \mathcal{L}[v] \rangle$$

where:

$$\langle u, v \rangle = \int_0^1 uv^* dx$$

$$\mathcal{L}[u] = -\frac{d}{dx} \left[p(x) \frac{du}{dx} \right] + q(x)u.$$

$$\begin{aligned} \therefore \langle \mathcal{L}[u], v \rangle - \langle u, \mathcal{L}[v] \rangle &= - \left[p(u'v^* - u(v^*)') \right]_0^1 \\ &= - \left[p(x) \left(\frac{du}{dx} \cdot v^* - u \cdot \frac{dv^*}{dx} \right) \right]_0^1 \end{aligned}$$

$$[pu'v^*]' = (pu')'v^* + pu'(v^*)'$$

$$[pu(v^*)]' = (p(v^*)')'u + pu'(v^*)'$$

S-L series expansion

The set of orthonormal eigenfunctions $\{\phi_n(x)\}$ from a S-L problem defined on $[0, 1]$ may be used to expand function $f(x)$:

$$f_\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$c_n = \int_0^1 r(x) \phi_n(x) f(x) dx.$$

in $L^2([0, 1], r(x)dx)$. We also have the general Parseval's identity:

$$\int_0^1 r(x) [f(x)]^2 dx = \sum_{n=1}^{\infty} c_n^2.$$

Non-homogeneous S-L problems

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

$$\mathcal{L}[y] = -\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + q(x)y$$

Firstly solve corresponding homogeneous problem:

$$\mathcal{L}[y] = \lambda r(x)y$$

for non-trivial λ_n and $\phi_n(x)$.

Then the general solution is:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

$$c_n = \int_0^1 \phi_n(x) f(x) dx.$$

Non-homogeneous PDEs

Consider the following PDE:

$$\begin{aligned} r(x) \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left[p(x) \frac{\partial u}{\partial x} \right] \\ &\quad - q(x)u(x, t) + F(x, t) \end{aligned}$$

with conditions:

- $\frac{\partial}{\partial x} u(0, t) - h_1 u(0, t) = 0$
- $\frac{\partial}{\partial x} u(1, t) - h_2 u(1, t) = 0$
- $u(x, 0) = f(x)$.

Firstly solve the homogeneous case.

Let $u(x, t) = X(x)T(t)$.

$$\therefore \frac{1}{rX} [p'X' + pX'' - qX] = \frac{\dot{T}}{T} = -\lambda$$

This yields two ODEs:

$$\dot{T} + \lambda T = 0$$

$$-[pX']' + qX = \lambda rX$$

with boundary conditions:

$$X'(0) - h_1 X(0) = 0$$

$$X'(1) - h_2 X(1) = 0$$

Clearly our second ODE is of S-L form. We are now going to assume that this regular S-L problem has non-trivial λ_n and orthonormal eigenfunctions $\phi_n(x)$. Then:

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x).$$

Since we have a S-L problem:

$$\begin{aligned} \sum_{n=1}^{\infty} \dot{b}_n(t) \phi_n(x) &= \sum_{n=1}^{\infty} b_n(t) [-\lambda_n \phi_n(x)] \\ &\quad + \frac{F(x, t)}{r(x)} \end{aligned}$$

$$\frac{F(x, t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t) \phi_n(x)$$

$$\gamma_n(t) = \int_0^1 F(x, t) \phi_n(x) dx$$

$$\sum_{n=1}^{\infty} [\dot{b}_n(t) + \lambda_n b_n(t) - \gamma_n(t)] \phi_n(x) = 0$$

$$b_n(t) = e^{-\lambda_n t} \int_0^t \gamma_n(s) e^{\lambda_n s} ds + e^{-\lambda_n t} b_n(0)$$

finally using initial conditions:

$$b_n(0) = \int_0^1 r(x) f(x) \phi_n(x) dx$$

and all this is in $L^2([0, 1], r(x)dx)$.

Singular S-L problems

Consider the following ODE:

$$-\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

but now some of $p(x)$, $q(x)$ and $r(x)$ are discontinuous at $x = 0$ and/or $x = 1$. This is a **singular** S-L problem with boundary condition:

$$b_1 y(1) + b_2 y'(1) = 0$$

or

$$a_1 y(0) + a_2 y'(0) = 0.$$

Now singular S-L problems at $x = 0$ may be self-adjoint or that they yield:

- $\lambda_n \in \mathbb{R}$ (Real eigenvalues)
- $\langle \phi_m, \phi_n \rangle = \delta_{mn}$ (Orthogonal eigenfunctions)

if they satisfy Lagrange's identity.

Consider singular S-L problem at $x = 0$:

$$\begin{aligned} &\int_{\epsilon}^1 (\mathcal{L}[u]v - u\mathcal{L}[v]) dx \\ &= \left[-p(x) (u'(x)v(x) - u(x)v'(x)) \right]_{\epsilon}^1 \\ &= p(\epsilon) (u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)) \end{aligned}$$

and tends to zero if and only if:

$$\lim_{\epsilon \rightarrow 0} [p(\epsilon) (u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon))] = 0,$$

where we define:

$$\mathcal{L}[u] = -\frac{d}{dx} \left[p(x) \frac{du}{dx} \right] + q(x)u.$$

Therefore this is the condition for our singular S-L problem to have real eigenvalues and orthogonal eigenfunctions.

Similarly for singular S-L problem at $x = 1$ it is self-adjoint if:

$$\lim_{\epsilon \rightarrow 0} \left[p(1 - \epsilon) (u'(1 - \epsilon)v(1 - \epsilon) - u(1 - \epsilon)v'(1 - \epsilon)) \right] = 0.$$

Finally whilst eigenvalues may be real they are not necessarily discrete!

Laplace transforms

So let $f(t)$ be defined for $t \in [0, \infty)$. Its Laplace transform is:

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.$$

Let **functions of exponential order** be defined as:

$$\forall t \in [0, \infty); \exists A > 0 \text{ and } B \in \mathbb{R} : |f(t)| \leq Ae^{Bt}$$

where f is piecewise continuous and we denote such functions as $f \in E$.

For sufficiently large s , the Laplace transform $f \in E$ converges.

Reduction of order

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

For $\forall f, f' \in E$ and generalising:

$$\begin{aligned}\mathcal{L}[f^{(n)}(t)] &= s^n \mathcal{L}[f(t)] - s^{(n-1)} f^{(0)}(0) \\ &\quad - s^{(n-2)} f^{(1)}(0) \\ &\quad - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0).\end{aligned}$$

Shifts, scaling and derivatives

Let $F(s) = \mathcal{L}[f(t)](s)$. We then have that:

1. s-shift:

$$\mathcal{L}[e^{-ct} f(t)](s) = F(s+c)$$

where $s+c > \gamma$.

2. t-shift:

Let $c \geq 0$ and $f(t) = 0$ if $t < 0$.

$$\therefore \mathcal{L}[f(t-c)](s) = e^{-sc} F(s)$$

In terms of the unit step function:

$$\mathcal{L}[g(t-c)u_c(t)](s) = e^{-sc} G(s)$$

where $G(s) = \mathcal{L}[g(t)](s)$ and $g(t)$ any normal function.

3. s-derivative:

$$\mathcal{L}[tf(t)](s) = -\frac{d}{ds} F(s)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

4. scaling:

$$\mathcal{L}[f(ct)] = \frac{1}{c} F\left(\frac{s}{c}\right)$$

$$\frac{1}{c} \mathcal{L}\left[f\left(\frac{t}{c}\right)\right] = F(cs)$$

where $c > 0$.

Higher order ODEs

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) = f(t)$$

$$Z(s)\mathcal{L}[y(t)] = \mathcal{L}[f(t)] + Z_0(s)$$

Where $Z(s)$ is a degree n polynomial and $Z_0(s)$ a degree $n-1$ polynomial dependent on our initial conditions.

If the source term is of the following form:

$$f(t) = t^n e^{at} (A \cos bt + B \sin bt)$$

then $\mathcal{L}[f(t)]$ is rational and therefore:

$$\mathcal{L}[y(t)] = \frac{\mathcal{L}[f(t)]}{Z(s)} + \frac{Z_0(s)}{Z(s)}$$

where we can solve this via standard transforms.

Discontinuous source terms

Consider the following ODE:

$$Ay''(t) + By'(t) + Cy(t) = g(t)$$

where $g(t)$ is piecewise continuous:

$$g(t) = f(t)[u_a(t) - u_b(t)] = \begin{cases} f(t) & t \in [a, b) \\ 0 & t \notin [a, b) \end{cases}$$

for $b > a$. This is the **unit step function**:

$$u_c(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq c. \end{cases}$$

$$\therefore \mathcal{L}[u_c(t)](s) = \frac{e^{-sc}}{s}$$

Furthermore we can define a shift of $f(t)$ by $c > 0$ to the right by:

$$f(t-c)u_c(t)$$

$$\mathcal{L}[f(t-c)u_c(t)] = e^{-sc} F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

Impulse functions

The Dirac delta is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

and has the following properties:

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0)$$

$$\frac{d}{dt} [u_{t_0}(t)] = \delta(t-t_0)$$

$$\mathcal{L}[\delta(t-t_0)] = e^{-st_0}.$$

If $t_0 = 0$ then $\mathcal{L}[\delta(t)] = \lim_{t_0 \rightarrow 0} (e^{-st_0}) = 1$.

Consider the following impulse ODE:

$$y''(t) + y(t) = \delta(t)$$

with initial conditions $y(0) = y'(0) = 0$:

$$\mathcal{L}[y(t)] = \frac{1}{s^2 + 1} \lim_{t_0 \rightarrow 0} (e^{-st_0}).$$

It is important that we do not evaluate the limit here! Then by inspection:

$$\begin{aligned}y(t) &= \lim_{t_0 \rightarrow 0} (\sin(t-t_0)u_{t_0}(t)) \\ &= \sin(t)u_0(t).\end{aligned}$$

Convolutions

Let functions $f, g: [0, \infty) \rightarrow \mathbb{R}$.

$$\begin{aligned}f(t) * g(t) &= \int_0^t f(s)g(t-s)ds \\ &= \int_0^t g(s)f(t-s)ds\end{aligned}$$

- $f * (g+h) = f * g + f * h$
- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * 1 \neq f$
- $f * f \neq f^2$.

Convolution theorem

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

For if $f, g \in E$ then $f * g \in E$. Note:

$$\int_0^t f(s)ds = \int_0^t f(s)u_0(t-s)ds$$

$$\mathcal{L}\left[\int_0^t f(s)ds\right] = \frac{\mathcal{L}[f(t)]}{s}.$$

Now consider the following ODE:

$$ay''(t) + by'(t) + cy(t) = g(t)$$

with $y(0) = \alpha$ and $y'(0) = \beta$.

Here $a, b, c, \alpha, \beta \in \mathbb{R}$.

$$\begin{aligned}\therefore \mathcal{L}[y(t)] &= \Phi(s) + \Psi(s) \\ &= \frac{(as+b)\alpha + a\beta}{as^2 + bs + c} \\ &\quad + \frac{1}{as^2 + bs + c} \mathcal{L}[g(t)]\end{aligned}$$

$$\therefore y(t) = \phi(t) + \psi(t)$$

$$ah''(t) + bh'(t) + ch(t) = \delta(t)$$

$$h(0) = h'(0) = 0$$

$$H(s) = \mathcal{L}[h(t)] = \frac{1}{as^2 + bs + c}$$

$$\begin{aligned}\therefore \Psi(s) &= \frac{1}{as^2 + bs + c} \mathcal{L}[g(t)] \\ &= H(s)\mathcal{L}[g(t)]\end{aligned}$$

$$\begin{aligned}\mathcal{L}[\psi(t)] &= \mathcal{L}[h(t)]\mathcal{L}[g(t)] \\ &= \mathcal{L}[h(t) * g(t)].\end{aligned}$$

$$\psi(t) = h(t) * g(t)$$

$$= \int_0^t h(s)g(t-s)ds.$$

Standard transforms

$$\bullet \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \text{ where } n \in \mathbb{N}.$$

$$\bullet \mathcal{L}[e^{at}] = \frac{1}{s-a} \text{ where } s > a.$$

$$\bullet \mathcal{L}[\sin at] = \frac{a}{s^2 + a^2},$$

$$\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2} \text{ where } s > 0.$$

$$\bullet \mathcal{L}[\sinh at] = \frac{a}{s^2 - a^2},$$

$$\mathcal{L}[\cosh at] = \frac{s}{s^2 - a^2} \text{ where } s > |a|.$$

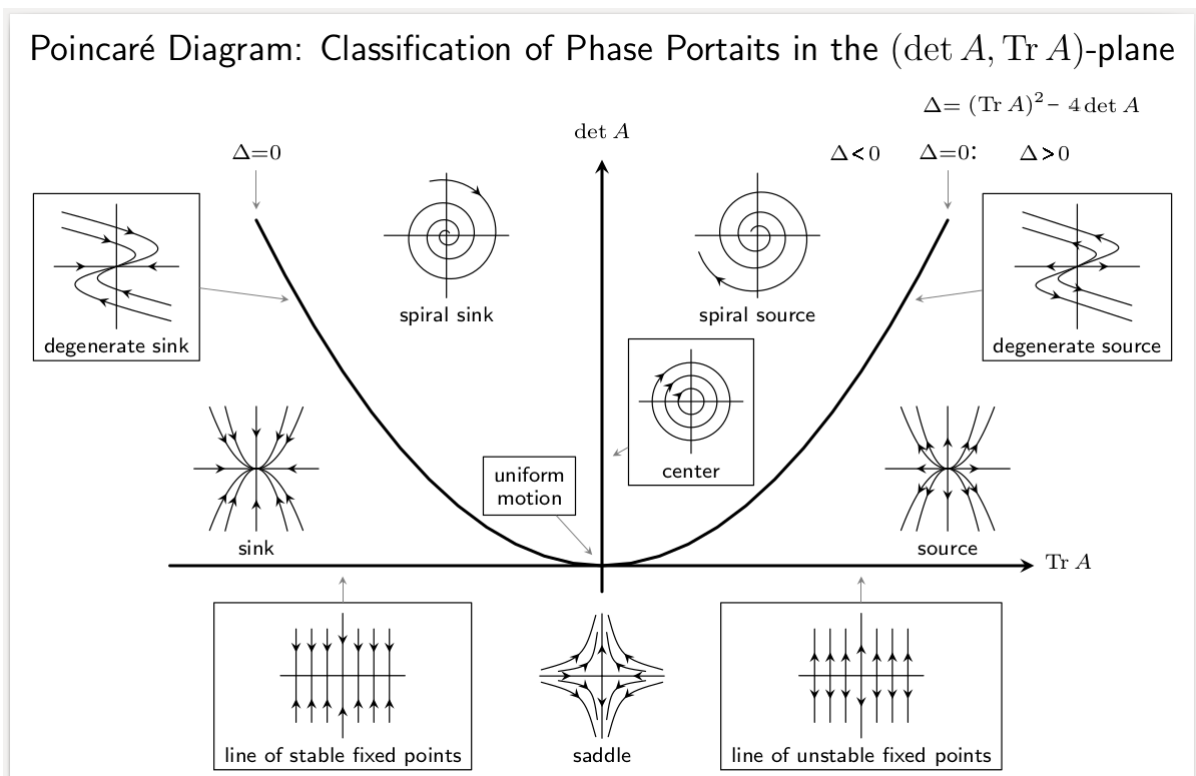
$$\bullet \mathcal{L}[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2},$$

$$\mathcal{L}[e^{at} \cos bt] = \frac{s-a}{(s-a)^2 + b^2}$$

where $s > a$.

$$\bullet \mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s} \text{ where } s > 0.$$

$$\bullet \mathcal{L}[\delta(t-c)] = e^{-cs}.$$



Miscellaneous

- Polar coordinates**

$$x = r \cos \theta$$

$$y = r \sin \theta$$

- Taylor's formula**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

- Partial fractions**

$$\frac{N(x)}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$

$$\frac{N(x)}{(ax+b)(x^2+c)} = \frac{A}{ax+b} + \frac{Bx+C}{x^2+c}$$

- Hyperbolic functions**

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

Periodicity: $2\pi i$

- Integrals**

$$\int \frac{dx}{\sin x} = \ln |\csc x - \cot x| + k$$

$$\int \frac{dx}{\cos x} = \ln |\tan x - \sec x| + k$$

- Bessel's equation**

Let $t = \sqrt{\lambda}x$. Then:

$$-(xy')' = \lambda xy$$

$$t \frac{d^2 y}{dt^2} + \frac{dy}{dt} + ty = 0.$$

This is Bessel's equation of order zero. Its general solution is:

$$y = c_1 J_0(t) + c_2 Y_0(t)$$

where J_0 and Y_0 are Bessel functions. Since Y_0 is unbounded we choose $c_2 = 0$ so that the solution is physical and set our orthonormal eigenfunctions to be:

$$\phi_n(x) = k_n J_0(\sqrt{\lambda_n}x)$$

$$J_0(\sqrt{\lambda}x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^m x^{2m}}{2^{2m} (m!)^2}.$$

- Inverse formula**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Further transforms**

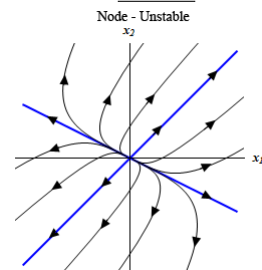
$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$$

where $n \in \mathbb{N}$ and $s > a$.

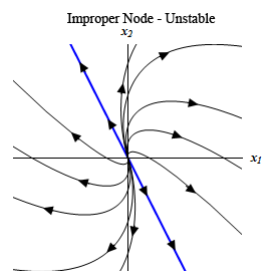
$$\mathcal{L}[\delta(t-c)f(t)] = e^{-cs}f(c).$$

Phase diagrams

- Real and unique eigenvalues:



- Repeat eigenvalues (ξ and η):



- Repeat eigenvalues (**two** eigenvectors, $\xi^{(1)}$ and $\xi^{(2)}$):

