Honours Analysis Workshops

Christopher Shen

Winter 2023

Contents

Workshop 6

1. Consider real function $f = x^2$. We know that f is continuous on $\mathbb{R}$, since it is a polynomial. So for $f : \mathbb{R} \to \mathbb{R}$ the $\epsilon - \delta$ definition states:

   $\forall \alpha \in \mathbb{R}; \ \forall \epsilon > 0; \ \exists \delta > 0; \ \forall x \in \mathbb{R};$
   $|x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon$

   Now we set $\alpha > 1$ and $\epsilon = 1$. Find **best** possible $\delta = \delta(\epsilon)$.

   So whilst we can choose $1 > \delta(\delta + 2\alpha)$, this is certainly not the best bound.

   Consider this approach instead:

   $$\alpha - \delta < x < \alpha + \delta \implies \alpha^2 - 1 < x^2 < \alpha^2 + 1$$
   $$\implies \sqrt{\alpha^2 - 1} < x < \sqrt{\alpha^2 + 1}$$

   i.e. since we have an implication:

   $$\sqrt{\alpha^2 - 1} < \alpha - \delta < x < \alpha + \delta < \sqrt{\alpha^2 + 1}$$

   This is really 4 inequalities, and we need to choose the best 2. So:

   $$\sqrt{\alpha^2 - 1} < \alpha - \delta \implies \delta < \alpha - \sqrt{\alpha^2 - 1}$$

   $$\alpha + \delta < \sqrt{\alpha^2 + 1} \implies \delta < -\alpha + \sqrt{\alpha^2 + 1}$$

   We can prove that $-\alpha + \sqrt{\alpha^2 + 1} > -\alpha + \sqrt{\alpha^2 + 1}$ by contradiction. Now for the lower bound:

   $$\sqrt{\alpha^2 - 1} < \alpha + \delta \implies -\alpha + \sqrt{\alpha^2 - 1} < \delta$$

   $$\alpha - \delta < \sqrt{\alpha^2 + 1} \implies \alpha - \sqrt{\alpha^2 + 1} < \delta$$

   By contradiction we have $\alpha - \sqrt{\alpha^2 + 1} > -\alpha + \sqrt{\alpha^2 - 1}$. Hence:

   $$\alpha - \sqrt{\alpha^2 + 1} < \delta < -\alpha + \sqrt{\alpha^2 + 1}$$

   ? our lower bound is wrong ?

Another approach

We begin from here:

$$\alpha - \delta < x < \alpha + \delta \implies \alpha^2 - 1 < x^2 < \alpha^2 + 1$$
$$\implies \sqrt{\alpha^2 - 1} < x < \sqrt{\alpha^2 + 1}.$$

It is clear from a graph that the distance from any x to α cannot exceed either $\alpha - \sqrt{\alpha^2 - 1}$ or $\sqrt{\alpha^2 + 1} - \alpha$ for our function to be continuous.

2. Now define $f : [0, 1] \to \mathbb{R}$ with rule $f(x) = x^2$.

   Show: $\forall \epsilon > 0; \exists \delta = \epsilon/2; \forall x, \alpha \in [0, 1]; |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon$

   Proof. We really should take the hint when given. Note that $\forall x, \alpha \in [0, 1]$:

   $$|x + \alpha| < |x| + |\alpha| \leq 2$$

   by the triangle inequality. (helpful to think of a triangle)

   Since polynomials are continuous we apply the $\epsilon - \delta$ continuity definition:

   $\forall \epsilon > 0; \exists \delta > 0; \forall x, \alpha \in [0, 1]; |x - \alpha| < \delta \implies |x^2 - \alpha^2| < \epsilon$

   Consider the final line:

   $$\begin{aligned} |x^2 - \alpha^2| &= |x + \alpha||x - \alpha| \\ &< 2|x - \alpha| \end{aligned}$$

   So if we choose $\delta = \frac{\epsilon}{2}$ given any ϵ then $|x - \alpha| < \epsilon$ and

   $$|x^2 - \alpha^2| < \epsilon.$$

   $\square$

3. Consider function $f : (0, \infty) \to \mathbb{R}$ with rule $f(x) = \frac{1}{x}$.

   Is this function **uniformly continuous**?

   So firstly uniform continuity only makes sense if our function is already continuous. Since $x = 0$ is removed, our function is continuous and we may consider uniform continuity.

   Here we claim that f is **not** uniformly continuous.

   Note that the following two notions of uniform continuity is equivalent:

   - $\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in I; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$

   - $\forall s_n, t_n \in I; \lim_{n \to \infty} |s_n - t_n| = 0 \implies \lim_{n \to \infty} |f(s_n) - f(t_n)| = 0$

   given function $f : I \to \mathbb{R}$. This makes our life easy. To disprove uniform continuity we just need to negate the second condition:

   $$\exists s_n, t_n \in I; \lim_{n \to \infty} |s_n - t_n| = 0 \text{ and } \lim_{n \to \infty} |f(s_n) - f(t_n)| \neq 0.$$

   Choose $s_n = \frac{1}{n}$ and $t_n = \frac{2}{n}$ and we are finished.

4. Consider function $f : [a, \infty) \to \mathbb{R}$ for $\boldsymbol{a > 0}$ and $f(x) = \frac{1}{x}$.

   Is this function uniformly continuous?

   Proof. We claim that f is uniformly continuous.

   So we need:

   $$\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in [a, \infty) : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

   Substituting f we have that $|x - y| < \delta \implies |\frac{1}{x} - \frac{1}{y}| < \epsilon$.

   Firstly without loss of generality define $x > y \geq a > 0$.

   $$\therefore \frac{1}{a} > \frac{1}{y} > \frac{1}{x} \implies \frac{1}{a^2} > \frac{1}{xy}.$$

   Now consider:

   $$|\frac{1}{x} - \frac{1}{y}| = \frac{|x - y|}{xy}$$
   $$< \frac{|x - y|}{a^2}$$
   $$< \epsilon$$

   if we choose $\delta = a^2 \epsilon$. Therefore:

   $$\forall \epsilon > 0; \exists \delta = a^2 \epsilon; \forall x, y \in [a, \infty) : |x - y| < \delta \implies |\frac{1}{x} - \frac{1}{y}| < \epsilon,$$

   or that f is uniformly continuous on $[a, \infty)$ where $a > 0$. $\qquad \square$

Mean value theorem approach
Not finished!
Firstly $f(x) = \frac{1}{x}$ is differentiable on $[a, \infty)$ as we have the following limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= -\frac{1}{x^2} \quad \text{if} \quad x \neq 0.$$

Define $x > y \geq a > 0$. By the mean value theorem we have:

$$\forall x, y \in [a, \infty); \exists c \in [y, x]; f'(c) = \frac{f(x) - f(y)}{x - y},$$

or that

$$|\frac{1}{x} - \frac{1}{y}| = \frac{1}{x} - \frac{1}{y}$$
$$= -\frac{1}{c^2}(x - y).$$

Now for fixed $[y, x]$, the mean value theorem states that $c > \min\{x, y\}$ and hence $c > a$. (this holds $\forall x, y$) $\therefore \frac{1}{a^2} > \frac{1}{c^2} \implies -\frac{1}{a^2} > -\frac{1}{c^2}$ and:

$$|\frac{1}{x} - \frac{1}{y}| = \frac{1}{x} - \frac{1}{y}$$
$$= -\frac{1}{c^2}(x - y)$$
$$< -\frac{1}{a^2}(x - y).$$

Since x and y are non-negative $x - y = |x - y|$ and:

$$|\frac{1}{x} - \frac{1}{y}| < -\frac{1}{a^2}|x - y| = -\frac{\delta}{a^2} = \epsilon.$$

Now pick $\delta = -a^2\epsilon$ and we are finished.

5. 5

6. 6

7. Prove that the following statements are equivalent:

   - $f : I \to \mathbb{R}$ is uniformly continuous.

     i.e. $\forall \epsilon > 0;\ \exists \delta > 0;\ \forall x, y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$

   - $\forall s_n, t_n \in I : \lim\limits_{n \to \infty} |s_n - t_n| = 0 \implies \lim\limits_{n \to \infty} |f(s_n) - f(t_n)| = 0$

Proof. $\to$ direction

Direct proof. Assume that f is uniformly continuous:

$$\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Also assume that:

$$\forall s_n, t_n \in I : \lim_{n \to \infty} |s_n - t_n| = 0.$$

But this may also be written as:

$$\forall \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N \implies |s_n - t_n| < \delta,$$

and since the definition of uniform continuity holds $\forall x, y \in I$ we may set $x = s_n$ and $y = t_n$. Combining our assumptions we get:

$$\begin{aligned} \forall \epsilon > 0; \exists \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N &\implies |s_n - t_n| < \delta \\ &\implies |f(s_n) - f(t_n)| < \epsilon \end{aligned}$$

But really what we want is:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |f(s_n) - f(t_n)| < \epsilon$$

Or that:

$$\lim_{n \to \infty} |f(s_n) - f(t_n)| = 0.$$

$\square$

Proof. ← direction

Proof by contradiction. Assume that if:

$$\forall s_n, t_n \in I : \lim_{n \to \infty} |s_n - t_n| = 0 \implies \lim_{n \to \infty} |f(s_n) - f(t_n)| = 0,$$

then f is **not** uniformly continuous. i.e. that:

$$\exists \epsilon > 0; \forall \delta > 0; \exists x, y \in I : |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon.$$

So our first assumption gives us:

$$\forall \epsilon > 0; \exists N_1 \in \mathbb{N} : \forall n \geq N_1 \implies |f(s_n) - f(t_n)| < \epsilon$$

and holds true $\forall s_n, t_n \in I$ with condition:

$$\forall \delta > 0; \exists N_2 \in \mathbb{N} : \forall n \geq N_2 \implies |s_n - t_n| < \delta.$$

So taking $N = \max\{N_1, N_2\}$ and combining the previous two statements:

$$\forall \epsilon > 0; \forall \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N \implies |s_n - t_n| < \delta$$
$$\implies |f(s_n) - f(t_n)| < \epsilon$$

The definition for **not** uniformly continuous only makes sense if x and y are sequences, since if they are real numbers then the following implies that they must be equal:

$$\forall \delta > 0; \exists x, y \in I : |x - y| < \delta,$$

and we reach a contradiction from the implication $\exists! \epsilon > 0 : 0 \geq \epsilon$. So for sequences x_n and y_n the previous statement implies:

$$\lim_{n \to \infty} |x_n - y_n| = 0$$

But we also assumed that $\forall s_n, t_n \in I : \lim_{n \to \infty} |s_n - t_n| = 0$.

This justifies setting $x_n = s_n$ and $y_n = t_n$ with condition $\forall n \geq N$.

$$\therefore \exists \epsilon > 0; \forall \delta > 0; \exists s_n, t_n \in I; |s_n - t_n| < \delta \text{ and } |f(s_n) - f(t_n)| \geq \epsilon.$$

But clearly this means that:

$$\lim_{n \to \infty} |f(s_n) - f(t_n)| \neq 0.$$

Then by truth tables f must be uniformly continuous. $\square$

Workshop 7

1. Let $f(x) = [x]$ for $\forall x \in \mathbb{R}$. Find the following integrals:

$$\int_{(0,5)} f$$

and

$$\int_{(-\frac{7}{3}, \frac{12}{5}]} f.$$

Note that here we denote $[x]$ as the **floor function**. The floor function **rounds down** its input to the closest integer. So for example we have that $[3.5] = 3$ and $[-2.5] = -3$.

Let's first consider the open interval $(0, 5)$. Notice that we can write f as a sum of characteristic functions each with interval of length 1:

$$f(x) = \sum_{j=1}^{5} (j-1) \chi_{[j-1,j)}(x).$$

Recall the characteristic function definition:

$$\chi_{[j-1,j)} = \begin{cases} 1 & x \in [j-1, j) \\ 0 & \text{otherwise.} \end{cases}$$

Integrating this gives us:

$$\int_{(0,5)} f = \int \sum_{j=1}^{5} (j-1) \chi_{[j-1,j)}(x)$$

$$= \sum_{j=1}^{5} (j-1) \int \chi_{[j-1,j)}(x)$$

$$= \sum_{j=1}^{5} (j-1)$$

$$= 10.$$

So now consider the semi-open interval $(-\frac{7}{3}, \frac{12}{5}]$, for $f(x) = [x]$.

We write this function as a sum of characteristic functions:

$$f(x) = -3\chi_{(-\frac{7}{3}, -2)} + -2\chi_{[-2,-1)} + -1\chi_{[-1,0)} + 0 + 1\chi_{[1,2)} + 2\chi_{[2, \frac{12}{5}]}$$

$$= -3\chi_{(-\frac{7}{3}, -2)} + \sum_{j=-2}^{1} j\chi_{[j,j+1)} + 2\chi_{[2, \frac{12}{5}]}.$$

Then integrating gives:

$$\int_{(-\frac{7}{3}, \frac{12}{5}]} f = -1 + -2 + \frac{4}{5}$$

$$= -\frac{11}{5}.$$

2. Let $f(x) = [nx]^2$ for $\forall x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Show that:

$$\int_{(0,1)} f = \frac{1}{n} \sum_{j=1}^{n-1} j^2.$$

So first consider:

$$x \in [\frac{j}{n}, \frac{j+1}{n})$$

for $j \in \{1, \ldots, n-1\}$. Multiplying each of these intervals by n gives:

$$nx \in [j, j+1),$$

and this also works for negative n values. Taking the floor for each interval:

$$[nx] = j \quad \text{for} \quad \forall nx \in [j, j+1)$$

and squaring this gives $[nx]^2 = j^2$ for $\forall j \in \{1, \ldots, n-1\}$.

So we can now write our function f as the sum of characteristic functions:

$$f(x) = 0 \cdot \chi_{(0,\frac{j}{n})} + \sum_{j=1}^{n-1} j^2 \chi_{[\frac{j}{n}, \frac{j+1}{n})}(x)$$

and integrating this gives:

$$\int_{(0,1)} f = \sum_{j=1}^{n-1} j^2 \int \chi_{[\frac{j}{n}, \frac{j+1}{n})}(x)$$

$$= \sum_{j=1}^{n-1} j^2 \cdot \frac{1}{n},$$

since we have defined n intervals each of length $\frac{1}{n}$. Finally:

$$\int_{(0,1)} f = \frac{1}{n} \sum_{j=1}^{n-1} j^2$$

$$= \frac{1}{n} \cdot \frac{n(n-1)(2n-1)}{6}$$

$$= \frac{1}{6}(n-1)(2n-1).$$

3. Let $f(x) = \frac{1}{[x]^2}$ for $\forall x \geq 1$. Show that f is integrable on $[1, \infty)$ and:

$$\int_{[1,\infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

Choose $J_j = [j, j+1)$ for $j \in \mathbb{N}$. Firstly we verify that:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) = \sum_{j=1}^{\infty} \frac{1}{j^2}$$
$$= \frac{\pi^2}{6} < \infty$$

with $c_j = \frac{1}{j^2}$ and our interval of choice being of length 1.

Now $\forall x \in J_i$ where $i \in \mathbb{N}$, we have that:

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} \chi_{J_j}(x)$$
$$= \frac{1}{i^2} < \infty$$

Hence we have proven that f is Lebesgue integrable on $[1, \infty)$, and:

$$\int_{[1,\infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2} \int \chi_{J_j}(x)$$
$$= \sum_{j=1}^{\infty} \frac{1}{j^2} \lambda(J_j)$$
$$= \sum_{j=1}^{\infty} \frac{1}{j^2},$$

and we are finished.

4. So now consider the function:

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}.$$

Show that f is integrable on every **bounded** interval I and:

$$\int_I f = \lambda(I).$$

Proof. Firstly choose:

$$c_j = \begin{cases} 1 & j = 1 \\ -1 & j > 1 \end{cases},$$

and

$$J_j = \begin{cases} I & j = 1 \\ q_{j-1} & j > 1 \end{cases},$$

where we define $I \cap \mathbb{Q} = \{q_1, q_2, \dots\}$ and $j \in \mathbb{N}$.

Then:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) = \lambda(I) + \sum_{j=2}^{\infty} -\lambda(\{q_{j-1}\})$$
$$= \lambda(I)$$
$$< \infty$$

since interval I is bounded and hence of finite length.

Finally $\forall x \in I$:

$$f(x) = \sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x)$$
$$= \chi_I(x) + \sum_{j=1}^{\infty} -\chi_{\{q_j\}}(x)$$
$$< \infty$$

So if $x \in \mathbb{Q}$ then $f(x) = 0$, and vice versa.

Therefore our function f is integrable on bounded I with formula:

$$\int_I f = \lambda(I).$$

$\square$

5. Let $f : [a, b] \to \mathbb{R}$ be continuous.

   Let $M = \sup\limits_{x \in [a,b]} |f(x)|$ and $p > 0$. For part (a) show that:

   $\forall \epsilon : (0 < \epsilon < M/2); \exists (\alpha, \beta) \subset [a, b] :$

   $$(M - \epsilon)^p (\beta - \alpha) \leq \int_a^b |f(x)|^p dx \leq M^p (b - a).$$

   Proof. Direct proof.

   Using the approximation property for suprema, $\exists x_0 \in [a, b]$:

   $$\sup_{x \in [a,b]} |f(x)| - \epsilon < |f(x_0)|.$$

   Then choose a $(\alpha, \beta) \subset [a, b]$ such that $\forall \epsilon > 0; \forall x \in (\alpha, \beta) :$

   $$\sup_{x \in [a,b]} |f(x)| - \epsilon < |f(x)|.$$

   Since these are strictly positive values taking the power of p preserves signs:
   $$(M - \epsilon)^p < |f(x)|^p.$$

   Also by the definition of supremum, for $\forall x \in [a, b] :$

   $$|f(x)| \leq \sup_{x \in [a,b]} |f(x)|$$

   and taking the pth gives:
   $$|f(x)|^p \leq M^p.$$

   Assuming the integrability of f we use the integral comparison test:

   $$\int_a^b |f(x)|^p \leq M^p (b - a).$$

   Similarly:
   $$(M - \epsilon)^p (\beta - \alpha) < \int_\alpha^\beta |f(x)|^p dx.$$

   But because $(\alpha, \beta) \subset [a, b]$:

   $$\therefore (M - \epsilon)^p (\beta - \alpha) < \int_\alpha^\beta |f(x)|^p dx \leq \int_a^b |f(x)|^p dx \leq M^p (b - a).$$

   $\square$

For part (b) we want:

$$\lim_{p \to \infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} = M$$

Proof. From part (a) we have that $\forall \epsilon : 0 < \epsilon < M/2$:

$$(M - \epsilon)^p (\beta - \alpha) \leq \int_a^b |f(x)|^p dx \leq M^p (b - a)$$

where $a < \alpha < \beta < b$. Taking the pth root gives:

$$(M - \epsilon)(\beta - \alpha)^{1/p} \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \leq M(b - a)^{1/p}.$$

Now by definition $\beta - \alpha > 0$ and $b - a > 0$. So taking $p \to \infty$:

$$M - \epsilon \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \leq M < M + \epsilon$$

for all $0 < \epsilon < M/2$, by monotone convergence theorem. Then:

$$\left| \left(\int_a^b |f(x)|^p dx \right)^{1/p} - M \right| < \epsilon,$$

or that

$$\lim_{p \to \infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} = M.$$

$\square$

6. Let $f(x) = n$ for $\forall x \in ((n+1)^{-2}, n^{-2}]$ and $n \in \mathbb{N}$. Show that:

$$\int_{(0,1]} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

We write our function as the following sum:

$$f(x) = \sum_{j=1}^{\infty} \chi_{(0, \frac{1}{j^2}]}(x).$$

This expression is clearly finite. We then check that:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) = \sum_{j=1}^{\infty} \lambda((0, \frac{1}{j^2}])$$
$$= \sum_{j=1}^{\infty} \frac{1}{j^2}$$
$$= \frac{\pi^2}{6} < \infty.$$

Finally:

$$\therefore \int_{(0,1)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

But why not:

$$f(x) = \sum_{j=1}^{\infty} j \cdot \chi_{(\frac{1}{(j+1)^2}, \frac{1}{j^2}]}(x)?$$

Workshop 8

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6

7. Define $L(x) = \displaystyle\int_1^x \frac{\mathrm{d}t}{t}$ for $\forall x > 0$. Show:

   - $L(xy) = L(x) + L(y)$

   - $L'(x) = \dfrac{1}{x}$

   - $L_{inv}(x) = E(x)$, where we define $E(x) = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

For the first part we want to show:

$$\int_1^{yx} \frac{\mathrm{d}t}{t} = \int_1^x \frac{\mathrm{d}t}{t} + \int_1^y \frac{\mathrm{d}t}{t}.$$

Beginning from the left hand side let $t = x\alpha$.

$$\therefore \int_{t=1}^{t=xy} \implies \int_{\alpha=\frac{1}{x}}^{\alpha=y}$$

$$\therefore \mathrm{d}t = x\mathrm{d}\alpha$$

$$\therefore \frac{1}{t} = \frac{1}{x\alpha}$$

Now splitting this integral via T4.9 gives:

$$\int_{t=1}^{t=yx} \frac{\mathrm{d}t}{t} = \int_{\alpha=\frac{1}{x}}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha}$$

$$= \int_{\alpha=1}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} + \int_{\alpha=\frac{1}{x}}^{\alpha=1} \frac{\mathrm{d}\alpha}{\alpha}$$

$$= \int_{\alpha=1}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} + \int_{\beta=1}^{\beta=x} \frac{\mathrm{d}\beta}{\beta}$$

where we set $\alpha = \frac{1}{x}\beta$ in the second integral.

$$\therefore L(xy) = L(x) + L(y)$$

Using the fundamental theorem of calculus:

$$L(x) = \int_1^x \frac{\mathrm{d}t}{t} \implies \frac{\mathrm{d}}{\mathrm{d}x} L(x) = \frac{1}{x}$$

since $\forall t > 0$, $\frac{1}{t}$ is continuous.

For the final part let's first define our functions:

$$E : \mathbb{R} \to \mathbb{R}$$

$$L : \mathbb{R}^+ \to \mathbb{R}$$

where $\mathbb{R}^+ = \mathbb{R}\backslash\{0,\dots\}$ represents the positive reals. Then define:

$$E(x) = z$$

for $x, z \in \mathbb{R}$ and:

$$L(y) = x$$

for $y \in \mathbb{R}^+$.

For these two functions to be inverses of each other we must show that:

$$E(L(y)) = y$$

and

$$L(E(x)) = x.$$

Consider

$$\frac{\mathrm{d}}{\mathrm{d}y}E(L(y)) = E(L(y))\frac{1}{y}.$$

Rearranging this and taking integrals:

$$\int_1^{E(L(y))} \frac{1}{E(L(y))}\mathrm{d}E(L(y)) = \int_1^y \frac{1}{y}\mathrm{d}y.$$

This gives:

$$\Big[L(E(L(y)))\Big]_{E(L(y))=1}^{E(L(y))=E(L(y))} = [L(y)]_1^y$$

or that:

$$L(E(L(y))) = L(y).$$

$$\therefore E(L(y)) = y$$

This is fine since $y \in \mathbb{R}^+ \subset \mathbb{R}$. Similarly consider the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}L(E(x)) = \frac{1}{E(x)}E(x) = 1.$$

Here $L(E(x))$ is defined as $\forall x \in \mathbb{R}; E(x) > 0$.

Integrating our expression as an indefinite integral:

$$L(E(x)) = x + k$$

and we find that $k = 0$ by setting $x = 0$.

$$\therefore L(E(x)) = x$$

8. Let $g : [a, b] \to \mathbb{R}$ be continuous, and that $g \geq 0$ for $\forall x \in [a, b]$. Then let:

$$\int_a^b g(x)\mathrm{d}x = 0.$$

Show that $\forall x \in [a, b]$ we have $g(x) = 0$.

Firstly because $g \geq 0$ splitting the integral using T4.9:

$$\int_a^b g(x)\mathrm{d}x = \int_a^c g(x)\mathrm{d}x + \int_c^b g(x)\mathrm{d}x = 0$$

implies that $\forall c \in [a, b]$:

$$\int_a^c g(x)\mathrm{d}x = 0$$

as areas of positive functions are always positive.

Since $g(x)$ is continuous we can use the fundamental theorem of calculus.

Let:

$$G(x) = \int_a^x g(t)\mathrm{d}t = 0$$

for $\forall x \in [a, b]$ as shown above. We then have that:

$$g(x) = \frac{\mathrm{d}}{\mathrm{d}x}G(x) = 0$$

for $\forall x \in [a, b]$.