# Contents

# 1 Suffix notation

# 2 Cartesian tensors

## 2.1 True tensors

tensor algebra

### 2.1.1 Rank 2 quotient theorem

The **quotient theorem** is as an alternative definition for tensors. In the context of rank 2 tensors it states that if $b_i$ always transforms as a vector in

$$b_i = T_{ij} a_j$$

and that $a_j$ is also a vector then $T_{ij}$ is a rank 2 tensor.

*Proof.* We egregiously define entity $T_{ij}$ in frame $S$ and $T'_{ij}$ in frame $S'$.

The usual transformation laws apply, namely $\boldsymbol{e}'_i = \ell_{ij} \boldsymbol{e}_j$. By definition:

$$\begin{aligned} b'_i &= T'_{ij} a'_j \\ &= T'_{ij} \ell_{jk} a_k \end{aligned}$$

Also directly from transformation laws:

$$\begin{aligned} b'_i &= \ell_{ij} b_j \\ &= \ell_{ij} T_{jk} a_k \end{aligned}$$

$$\therefore \left( T'_{ij} \ell_{jk} - \ell_{ij} T_{jk} \right) a_k = 0$$

Since $a_k$ are constants of our vector it must then be that:

$$T'_{ij} \ell_{jk} = \ell_{ij} T_{jk}$$

$$\therefore T'_{ij} \ell_{jk} \ell_{mk} = \ell_{ij} \ell_{mk} T_{jk}$$

Where here we aim to eliminate the first two $\ell$s. Finally:

$$T'_{im} = \ell_{ij} \ell_{mk} T_{jk}$$

$\square$

### 2.1.2 General quotient theorem

Let $R_{ij\ldots r}$ be a rank $m$ tensor, and $T_{ij\ldots s}$ be a set of $3^n$ numbers where $n > m$.

If $R_{ij\ldots r} T_{ij\ldots s}$ is a rank $n - m$ tensor then $T_{ij\ldots s}$ is a rank $n$ tensor.

symmetric and anti symmetric tensors

### 2.2 Matrices as tensors

### 2.3 Pseudotensors

Firstly note that $\det L = +1$ for rotations, and $\det L = -1$ for reflections and inversions. Recall the transformation law $\boldsymbol{e}_i' = \ell_{ij}\boldsymbol{e}_j$.

A second rank **pseudotensor** is defined:

$$T'_{ij} = (\det L)\ell_{ip}\ell_{jq}T_{pq}.$$

Furthermore a rank 1 pseudotensor is a **pseudovector** and is defined as:

$$T'_i = (\det L)\ell_{ip}T_p.$$

Finally a **pseudoscalar** is a rank 0 pseudotensor:

$$a' = (\det L) \cdot a,$$

and changes sign under transformation.

### 2.4 Invariant tensors

### 2.5 Rotation tensors

### 2.6 Reflections, inversions and projections

active and passive transformations

maybe merge with rotations?

### 2.7 Inertia tensors

## 3 Taylor expansions

# 4 Vector calculus

## 4.1 Vector operators

### 4.1.1 Gradient

### 4.1.2 Divergence

### 4.1.3 Curl

chain rules, important identities

## 4.2 Integrals theorems

### 4.2.1 Line, volume and surface integrals

### 4.2.2 Divergence theorem

### 4.2.3 Stokes's theorem

# 5 Curvilinear coordinates

## 5.1 Orthogonal curvilinear coordinates

### 5.1.1 Scale factors and basis vectors

Consider change of variables:

$$(x_1, x_2, x_3) \leftrightarrow (u_1, u_2, u_3)$$

where $u_i$ are our curvilinear coordinates, and

$$u_i = u_i(x_1, x_2, x_3)$$
$$x_i = x_i(u_1, u_2, u_3).$$

Then we define:

$$\begin{aligned} \mathrm{d}\boldsymbol{r}_i &= \frac{\partial \boldsymbol{r}}{\partial u_i}\mathrm{d}u_i \\ &= h_i \boldsymbol{e}_i \mathrm{d}u_i \end{aligned}$$

where $h_i = |\frac{\partial \boldsymbol{r}}{\partial u_i}|$ is our **scale factor** and

$$\boldsymbol{e}_i = \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial u_i}$$

is our **basis vector** of unit length for a specific set of curvilinear coordinates.

Now if the basis vectors satisfy

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

we have an orthogonal set of curvilinear coordinates.

### 5.1.2 Cylindrical coordinates

We define cylindrical coordinates as

$$(u_1, u_2, u_3) = (\rho, \phi, z)$$

and with the following relation to Cartesian coordinates:

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_x + \rho\sin\phi\boldsymbol{e}_y + z\boldsymbol{e}_z.$$

Furthermore:

$$\begin{aligned} h_\rho = 1 \quad &\text{and} \quad \boldsymbol{e}_\rho = \cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y \\ h_\phi = \rho \quad &\text{and} \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y \\ h_z = 1 \quad &\text{and} \quad \boldsymbol{e}_z = \boldsymbol{e}_z. \end{aligned}$$

Here $\phi$ is the anticlockwise rotation of the $xy$-plane.

#### 5.1.3 Spherical coordinates

## 5.2 Length, area and volume

### 5.2.1 Vector and arc length

Firstly the **vector length** due to infinitesimal change in all directions is

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3} h_i \mathrm{d}u_i \boldsymbol{e}_i.$$

It is important to note that summation notation does not work here.

Now the **arc length** of $\mathrm{d}\boldsymbol{r}$ is:

$$\begin{aligned} \mathrm{d}s &= |\mathrm{d}\boldsymbol{r}| \\ &= \sqrt{\mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r}} \end{aligned}$$

and we define the **metric tensor** as

$$\begin{aligned} g_{ij} &= \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \\ &= \frac{\partial \boldsymbol{r}}{\partial u_i} \cdot \frac{\partial \boldsymbol{r}}{\partial u_j}. \end{aligned}$$

Since $\mathrm{d}\boldsymbol{r} = \mathrm{d}x_k$ we then the following relation:

$$(\mathrm{d}s)^2 = g_{ij}\mathrm{d}u_i\mathrm{d}u_j.$$

### 5.2.2 Vector area

### 5.2.3 Volume

The volume of the infinitesimal parallelepiped defined by $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$ is:

$$\begin{aligned}\mathrm{d}V &= |(\mathrm{d}\boldsymbol{r}_1 \times \mathrm{d}\boldsymbol{r}_2) \cdot \mathrm{d}\boldsymbol{r}_3| \\ &= h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3\, |(\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot \boldsymbol{e}_3| \\ &= \sqrt{g}\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3\end{aligned}$$

where $g$ is the <u>determinant</u> of the metric tensor.

# 6 Electrostatics

## 6.1 Dirac delta function

The one dimensional **Dirac delta** is defined:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0, \end{cases}$$

and can be thought of as infinitely sharp at $x = 0$ and zero elsewhere.

It satisfies some useful properties:

- $\delta(x-a) = \lim_{\sigma \to 0} \left[ \frac{1}{|\sigma|\sqrt{\pi}} \exp\left( -\frac{(x-a)^2}{\sigma^2} \right) \right]$

  i.e. an infinitely sharp Gaussian. (generalised functions)
- **Sift property**
$$\int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a)$$
- Let $x_i$ be the solutions to $g(x_i) = 0$. Then:
$$\int_{\mathbb{R}} f(x)\delta[g(x)]dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

Now we consider the **3D Dirac delta**, which is defined as follows:

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

given Cartesian coordinates $(x_1, x_2, x_3)$. It also satisfies the **sift** property:

$$\int_{\mathbb{R}^3} f(\boldsymbol{r})\delta(\boldsymbol{r} - \boldsymbol{r}_0) = f(\boldsymbol{r}_0).$$

The three dimensional Dirac delta defined in a orthogonal curvilinear coordinate system $(u_1, u_2, u_3)$ is as follows:

$$\delta(\boldsymbol{r} - \boldsymbol{a}) = \frac{1}{h_1 h_2 h_3}\delta(u_1 - a_1)\delta(u_2 - a_2)\delta(u_3 - a_3)$$

for $h_1, h_2$ and $h_3$ are the scale factors.

## 6.2 Coulomb's law

Consider the force on charge $q$ at $\boldsymbol{r}$ due to charge $q_1$ at $\boldsymbol{r}_1$:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{qq_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3},$$

for here $\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$ in vacuum.

Physically, like charges ($qq_1 > 0$) repel while opposite charges ($qq_1 < 0$) attract.

We then define an **electric field** as the force on a small positive test charge:

$$\boldsymbol{E}(\boldsymbol{r}) = \lim_{q\to 0}\left(\frac{1}{q}\boldsymbol{F}(\boldsymbol{r})\right).$$

The force on a charge $q$ at $\boldsymbol{r}$ from the origin in this electric field is:

$$\boldsymbol{F}(\boldsymbol{r}) = q\boldsymbol{E}(\boldsymbol{r}).$$

A negative point charge is a sink whereas a positive point charge is a source.

Consider a collection of charges $q_i$ at position $\boldsymbol{r}_i$. The **principle of superposition** tells us that:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i\left(\frac{q_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right).$$

Now consider a continuous charged object with volume $V$ and **charge density** $\rho(\boldsymbol{r}')$. It generates the following electric field:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}dV'.$$

Returning to the electric field generated by a point charge $q_1$ at position $\boldsymbol{r}_1$:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^3},$$

this is a **conservative field**, and we may write it as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r}),$$

where:

$$\phi(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1|}.$$

Conservative fields have zero curl, and their line integrals are path independent. This namely applies to finding work done.

### 6.3 Electrostatic Maxwell's equations

## 6.4 Electric dipoles

**Dipoles** consist of two equal and **opposite point charges** that are $\boldsymbol{d}$ apart.

An **ideal dipole** is defined as when the following **dipole limit** is finite and constant:

$$\boldsymbol{p} = \lim_{\substack{q\to\infty \\ \boldsymbol{d}\to 0}} q\boldsymbol{d}.$$

A **dipole moment** is simply $\boldsymbol{p} = q\boldsymbol{d}$. The **dipole potential** at $\boldsymbol{r}_0$ is:

$$\begin{aligned}\phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{d}|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3},\end{aligned}$$

where we have Taylor expanded the first term about $|\boldsymbol{r}-\boldsymbol{r}_0|$. For simplicity we set $\boldsymbol{r}_0 = \boldsymbol{0}$. Then the **electric field** generated by our dipole at the origin is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\boldsymbol{p}\cdot\boldsymbol{r}}{r^5}\boldsymbol{r} - \frac{1}{r^3}\boldsymbol{p}\right),$$

since $\boldsymbol{E} = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$. Note that these formulae are in Cartesian coordinates.

Now we repeat this in spherical.

Force, torque and energy.

### 6.4.1 Multidipole expansion

potential

work done

## 6.5 Gauss's law

Gauss's law is the integral form of Maxwell's first equation:

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}$$

where $Q_{enc}$ is the total charge enclosed by volume $V$. This result follows from the application of the divergence theorem and is useful in problems with symmetry.

### 6.5.1 Boundaries

### 6.5.2 Conductors

special case for electrostatics

## 6.6 Poisson's equation

In electrostatics we have:

$$\boldsymbol{\nabla}^2\phi = \frac{\rho}{\epsilon_0}$$

where $\rho$ is our charge density. This is the **Poisson's equation** and is a consequence of the fact that $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$.

### 6.6.1 Existence and uniqueness of solutions

The existence of solutions is given by the fact that:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi.$$

Poisson's equation has **unique** solution $\phi$ if we have volume $V$ bounded by surface $S$ and one of the following boundary conditions:

1.