Vector products

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

$$\mathbf{a} \times \mathbf{b} = ab\sin\theta\hat{\mathbf{n}}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$
 and $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$$

Suffix notation

- 1. A suffix that appears <u>twice</u> implies a summation.
- 2. Any suffix <u>cannot appear</u> more than twice in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right.$$

and the Levi-Civita as:

$$\epsilon_{ijk} = \left\{ \begin{array}{ll} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{array} \right.$$

Consequently:

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$
$$= -\epsilon_{ijk} = -\epsilon_{ijk} = -\epsilon_{ijk}$$

and we have the following identities:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i = a_i \boldsymbol{e}_i$$

 $A\mathbf{x} = a_{ij}x_j\mathbf{e}_i$ for $m \times n$ matrix A

$$\delta_{ii} = 3$$

$$[\ldots]_i \delta_{ik} = [\ldots]_k$$

$$e_i \cdot e_j = \delta_{ij}$$

$$e_i \times e_j = \epsilon_{ijk} e_k$$

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk} a_j b_k \boldsymbol{e}_i$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$$
 and $\epsilon_{ijk}\epsilon_{ijk} = 6$.

Transformations

Let matrix L relate basis $\{e_i\}$ to basis $\{e'_i\}$ with rule:

$$e'_i = \ell_{ij}e_j$$
 where $(L)_{ij} = \ell_{ij}$.

Then $L^T L = L L^T = I$, and:

$$\ell_{ik}\ell_{jk} = \ell_{ki}\ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij} p_j$$
 for $\boldsymbol{p} = p_i \boldsymbol{e}_i = p'_i \boldsymbol{e}'_i$.

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip}\ell_{jq}\ell_{kr}T_{pqr}$$

which relates frame S in $\{e_i\}$ to frame S' in $\{e'_i\}$ with rule $e'_i = \ell_{ij}e_j$, etc.

Properties of tensors:

- 1. The <u>addition</u> of two rank n tensors is also a rank n tensor.
- 2. The <u>multiplication</u> of a rank m tensor with a rank n tensor yields a rank m + n tensor.
- 3. If $T_{ijk...s}$ is a rank m tensor then $T_{iik...s}$ is a rank m-2 tensor.
- 4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} \implies T' = LTL^T$$
$$T'_{ii} = \ell_{ip}\ell_{iq}T_{pq}.$$

Symmetric tensors

 T_{ij} is a symmetric tensor when $T_{ij} = T_{ji}$ in frame S. Then $T'_{ij} = T'_{ji}$ in frame S'.

Similarly T_{ij} is an anti-symmetric tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally any tensor can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S'. Let $b_i = T_{ij}a_j$ where a_j is a vector. If b_i always transforms as a vector then T_{ij} is a rank 2 tensor.

Generalising, let $R_{ijk...r}$ be a rank m tensor and $T_{ijk...s}$ a set of 3^n numbers where n > m. If $T_{ijk...s}R_{ijk...r}$ is a rank n - m tensor then $T_{ijk...s}$ is a rank n tensor.

Matrices

We define a $m \times n$ matrix A as $(A)_{ij} = a_{ij}$ where i = 1, ..., m and j = 1, ..., n.

- $\operatorname{Tr} A = a_{ii}$
- \bullet $(A^T)_{ij} = a_{ii}$
- $\bullet \ (AB)^T = B^T A^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= \epsilon_{lmn} a_{1l} a_{2m} a_{3n}$$

 $=\epsilon_{lmn}a_{l1}a_{m2}a_{n3}.$

Furthermore:

$$\epsilon_{ijk} \det A = \epsilon_{lmn} a_{il} a_{jm} a_{kn}$$

$$\epsilon_{lmn} \det A = \epsilon_{ijk} a_{il} a_{jkm} a_{kn}$$

$$\det A = \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} a_{il} a_{jm} a_{kn}.$$

Properties of determinants:

- 1. Adding rows to each other does not change the determinant.
- 2. Interchanging two rows changes determinant signs.
- 3. $\det A = \det A^T$
- 4. $det(AB) = det A \cdot det B$

These also apply to columns. Finally:

$$\epsilon_{ijk}\epsilon_{lmn} \det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting A = I yields:

$$\epsilon_{ijk}\epsilon_{lmn} = \left| \begin{array}{ccc} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{array} \right|.$$

Linear equations

Let $\mathbf{y} = A\mathbf{x}$. Then $x_i = A_{ij}^{-1}y_i$ with:

$$\begin{split} A_{ij}^{-1} &= \frac{1}{2} \frac{1}{\det A} \epsilon_{imn} \epsilon_{jpq} a_{pm} a_{qn} \\ &= \frac{1}{\det A} C_{ij}^T \end{split}$$

where C is the cofactor matrix of A.

Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L)\ell_{ip}\ell_{jq}T_{pq}$$

where $(L)_{ij} = \ell_{ij}$ and $\det L = \pm 1$.

Pseudovectors are rank 1 pseudotensors.

Invariant tensors

Tensor T is invariant or isotropic if:

$$T_{ijk...} = \ell_{i\alpha}\ell_{j\beta}\ell_{k\gamma}\cdots T_{\alpha\beta\gamma...}$$

for every orthogonal matrix L.

- If a_{ij} is a rank 2 invariant tensor then $a_{ij} = \lambda \delta_{ij}$.
- The most general rank 3 invariant pseudotensor is $a_{ijk} = \lambda \epsilon_{ijk}$. There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

Rotation tensors

The clockwise <u>rotation</u> of position vector x to y about unit vector \hat{n} is given by:

$$y_i = R_{ij}(\theta, \hat{\boldsymbol{n}})x_j$$

$$R_{ij}(\theta, \hat{\boldsymbol{n}}) = \delta_{ij} \cos \theta + (1 - \cos \theta) n_i n_j - \epsilon_{ijk} n_k \sin \theta$$

and is the rotation tensor.

Reflections and inversions

The <u>reflection</u> of vector \boldsymbol{x} to \boldsymbol{y} in plane with unit vector $\hat{\boldsymbol{n}}$ is:

$$y_i = \sigma_{ij} x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The <u>inversion</u> of vector x to y is given by y = -x and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}$$
.

Projections

We define P to be a <u>parallel</u> projection operator to vector \mathbf{u} if:

$$Pu = u$$
 and $Pv = 0$

where $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{0}$. Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define Q to be an <u>orthogonal</u> projection to vector \boldsymbol{u} if:

$$Q\mathbf{u} = \mathbf{0}$$
 and $Q\mathbf{v} = \mathbf{v}$.

Here Q = I - P.

Inertia tensors

Let L denote the angular momentum of a rigid body about the origin of mass m, volume V and density ρ at position r with velocity v. Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_{V} \rho(r^2 \delta_{ij} - x_i x_j) dV$$

where $I_{ij}(O)$ is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2} I_{ij} \omega_i \omega_j = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega}.$$

Parallel axis theorem

Consider the same rigid body now with centre of mass G and let $\overrightarrow{OG} = \mathbf{R}$. Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2 \delta_{ij} - X_i X_j)$$
$$M = \int_V \rho'(\mathbf{r}') dV'.$$

Diagonalisation

Let $L = I_{ij}\omega_j$ where I_{ij} is a rank 2 tensor and let $L = \lambda \omega$. Then:

$$(I_{ij} - \lambda \delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda \delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for $P = \det I$, $Q = \frac{1}{2}[(\operatorname{tr} I)^2 - \operatorname{tr}(I^2)]$ and $R = \operatorname{tr} I$ given $\operatorname{\underline{tensor}} I$.

Real symmetric tensors

Let rank 2 real symmetric tensor T be diagonalisable with real eigenvalues $\lambda^{(i)}$ and orthonormal eigenvectors $\boldsymbol{\ell}^{(i)}$ where i=1,2,3. Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that $\det L = +1$ which transforms frame $S \to S'$.

Then since $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq}$$

$$= \lambda^{(i)} \delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ii}.$$

Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a) (x - a)^n$$

and is f expanded about x = a.

Trignometric expansions are in radians!

$$\therefore f(x+a) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) a^n$$
$$= \exp\left(a \frac{d}{dx}\right) f(x)$$

Then for three dimensions:

$$\phi(\mathbf{r} + \mathbf{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{a} \cdot \nabla_r)^n \phi(\mathbf{r})$$
$$= \exp(\mathbf{a} \cdot \nabla_r) \phi(\mathbf{r}).$$

Curvilinear coordinates

Let x_i denote Cartesian coordinates and u_i denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \rightarrow (u_1, u_2, u_3)$$

where each $u_i = u_i(x_1, x_2, x_3)$ and:

$$r = x_1 e_1 + x_2 e_2 + x_3 e_3$$

= $u_1 e_{u_1} + u_2 e_{u_2} + u_3 e_{u_3}$.

Scale factors

Let $u_1 \to u_1 + du_1$ in $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$. Then $d\mathbf{r}$ in $\mathbf{r} \to \mathbf{r} + d\mathbf{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial u_1} \mathrm{d}u_1 := h_1 \boldsymbol{e}_1 \mathrm{d}u_1.$$

 h_1 is the scale factor of unit vector e_1 :

$$h_1 = \left| \frac{\partial \boldsymbol{r}}{\partial u_1} \right| \text{ and } \boldsymbol{e}_1 = \frac{1}{h_1} \frac{\partial \boldsymbol{r}}{\partial u_1}.$$

If $e_i \cdot e_j = \delta_{ij}$ then u_i is an **orthogonal** curvilinear coordinate system.

Vector and arc length

The vector length $d\mathbf{r}$ of \mathbf{r} is defined as:

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3} h_i \mathrm{d}u_i \boldsymbol{e}_i$$

where $u_i \to u_i + du_i$ for $\forall i = 1, 2, 3$.

Then the arc length $\mathrm{d}s$ is defined as:

$$(\mathrm{d}s)^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r}$$
$$= g_{ij} \, \mathrm{d}u_i \, \mathrm{d}u_j$$

where g_{ij} is the metric tensor:

$$g_{ij} = g_{ji} = \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j}$$
$$= h_i h_j (\mathbf{e}_i \cdot \mathbf{e}_j).$$

Area and volume

Let $d\mathbf{r}_i = h_i \mathbf{e}_i du_i$ denote vector length when $u_i \to u_i + du_i$. (No sum!)

The infinitesimal vector area formed by $d\mathbf{r}_1$ and $d\mathbf{r}_2$ is:

$$d\mathbf{S} = (h_1 d\mathbf{u}_1 \mathbf{e}_1) \times (h_2 d\mathbf{u}_2 \mathbf{e}_2).$$

Similarly the infinitesimal volume formed by edges $d\mathbf{r}_1$, $d\mathbf{r}_2$ and $d\mathbf{r}_3$ is:

$$dV = |(d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3|$$
$$= \sqrt{g} du_1 du_2 du_3$$

where $g = \det(g_{ij})$.

Cylindrical coordinates

 $(u_1, u_2, u_3) = (\rho, \phi, z)$ where ρ represents the radial distance from the origin and ϕ is the anticlockwise rotation angle on the x-y plane. In Cartesian unit vectors:

$$\mathbf{r} = \rho \cos \phi \mathbf{e}_1 + \rho \sin \phi \mathbf{e}_2 + z \mathbf{e}_3$$

$$h_{\rho} = 1$$
, $e_{\rho} = \cos \phi e_1 + \sin \phi e_2$

$$h_{\phi} = \rho$$
, $e_{\phi} = -\sin\phi e_1 + \cos\phi e_2$

$$h_z = 1$$
, $e_z = e_3$

and forms an orthogonal set.

Spherical coordinates

 $(u_1, u_2, u_3) = (r, \theta, \phi)$ where θ represents the clockwise rotation angle in y-z plane and ϕ the anticlockwise rotation angle in x-y plane. In Cartesian unit vectors:

 $r = r \sin \theta \cos \phi e_1 + r \sin \theta \sin \phi e_2 + r \cos \theta e_3$

$$h_r = 1, \ h_\theta = r, \ h_\phi = r \sin \theta$$

 $e_r = \sin \theta \cos \phi e_1 + \sin \theta \sin \phi e_2 + \cos \theta e_3$ $e_{\theta} = \cos \theta \cos \phi e_1 + \cos \theta \sin \phi e_2 - \sin \theta e_3$

$$e_{\phi} = -\sin\phi e_1 + \cos\phi e_2$$

and also forms an orthogonal set.



Gradient

The gradient of a scalar field $f(\mathbf{r})$ is:

$$df(\boldsymbol{r}) := \boldsymbol{\nabla} f(\boldsymbol{r}) \cdot d\boldsymbol{r}$$

when $r \to r + dr \implies f \to f + df$. Taking the total differential of f yields:

$$\nabla f = \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial u_i} e_i$$

where $\{e_i\}$ is orthogonal.

Divergence

The divergence of a vector field \mathbf{F} is:

$$\nabla \cdot \boldsymbol{F} := \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \boldsymbol{F} \cdot \mathrm{d} \boldsymbol{S}$$

for surface δS bounds infinitesimal δV . In orthogonal curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (h_1 F_2 h_3) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right\}.$$

Curl

The curl of a vector field F in the direction of unit vector \hat{n} is:

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) := \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \boldsymbol{F} \cdot d\boldsymbol{r}$$

where curve δC encloses plane δS . In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla}\times\boldsymbol{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3\\\frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3}\\h_1a_1 & h_2a_2 & h_3a_3\end{vmatrix}$$

Laplacian

The Laplacian of a scalar field f is:

$$\mathbf{\nabla}^2 f = \mathbf{\nabla} \cdot (\mathbf{\nabla} f)$$

and in orthogonal curvilinear coordinates:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right\}.$$

The Laplacian of a vector field F is:

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\nabla f = \frac{\partial f}{\partial x_i} e_i$$

$$(\boldsymbol{u} \cdot \nabla) \boldsymbol{F} = u_j \frac{\partial}{\partial x_j} F_i$$

$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_i}{\partial x_i}$$

$$\nabla \times \boldsymbol{F} = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} e_i$$

$$\frac{\partial x_i}{\partial x_i} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

If ψ is a scalar field and \boldsymbol{v} a vector field:

$$egin{aligned} oldsymbol{
abla} imes (oldsymbol{
abla} \psi) &= \mathbf{0} \ oldsymbol{
abla} \cdot (oldsymbol{v} oldsymbol{v}) &= oldsymbol{
abla} \psi \cdot oldsymbol{v} + \psi oldsymbol{
abla} \cdot oldsymbol{v} \ oldsymbol{
abla} imes (\psi oldsymbol{v}) &= oldsymbol{
abla} \psi imes oldsymbol{v} + \psi oldsymbol{
abla} imes oldsymbol{v}. \end{aligned}$$

Let $\mathbf{r} = x_i \mathbf{e}_i$ and $r = (x_i^2)^{1/2}$. Then:

•
$$\nabla r = \frac{\mathbf{r}}{r}$$
 and $\nabla \left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$

- $\bullet \ \nabla r^n = nr^{n-2}r$
- $\nabla \cdot \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = \mathbf{0}$
- $\nabla \times (\boldsymbol{c} \times \boldsymbol{r}) = 2\boldsymbol{c}$
- $\nabla \cdot (c \times r) = 0$ for constant c.

Divergence theorem

Let surface S enclose volume V. Then:

$$\iiint_{V} \nabla \cdot \boldsymbol{F} dV = \oint_{S} \boldsymbol{F} \cdot d\boldsymbol{S}$$

where \boldsymbol{F} is a vector field.

Stokes' theorem

Let closed curve C bound open surface S and let F be a vector field. Then:

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

for C is traversed in anticlockwise sense.

Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x-a) := \left\{ \begin{array}{ll} \infty & x = a \\ 0 & \text{otherwise.} \end{array} \right.$$

In three dimensions this becomes:

$$\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0) := \delta(\boldsymbol{r} - \boldsymbol{r}_0)$$
$$= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

where (x, y, z) are Cartesian coordinates.

In orthogonal curvilinear coordinates:

$$\delta(\mathbf{r} - \mathbf{a}) = \frac{1}{h_1 h_2 h_3} \delta(u_1 - a_1)$$
$$\cdot \delta(u_2 - a_2) \delta(u_3 - a_3).$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

which yields:

$$x\delta(x) = 0$$
 and $\delta(cx) = \frac{1}{|c|}\delta(x)$.

If simple solutions of g(x) = 0 are x_i :

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))dx = \sum_{i} \frac{f(x_i)}{|g'(x_i)|}.$$

Coulomb's law

Consider charges q and q_1 at positions r and r_1 . The force on charge q at r due to charge q_1 at r_1 is:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{qq_1(\boldsymbol{r} - \boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}$$

where $qq_1 > 0$ denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419 \times 10^{-12} \text{C}^2 \text{N}^{-1} \text{m}^{-2}$$
.

Charge has units Coulombs (C) and an electron has charge -1.60218×10^{-19} C.

Electric fields

The electric field is induced by a charge distribution and defined in terms of the force on a small positive test charge q:

$$\boldsymbol{E}(\boldsymbol{r}) := \lim_{q \to 0} \frac{1}{q} \boldsymbol{F}.$$

Then for our two charges q and q_1 :

$$\boldsymbol{F}_1(\boldsymbol{r}) = q\boldsymbol{E}_1(\boldsymbol{r})$$

where q_1 produces electric field E_1 .

$$\therefore \boldsymbol{E}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1(\boldsymbol{r} - \boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}$$

Principle of superposition

For a set of charges q_i at position r_i the total electric field at r is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i} \frac{q_i(\boldsymbol{r} - \boldsymbol{r}_i)}{|\boldsymbol{r} - \boldsymbol{r}_i|^3}.$$

For object with **charge density** $\rho(\mathbf{r}')$ its overall electric field at \mathbf{r} is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{V} \rho(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}V'$$

where $\rho(\mathbf{r}')$ is charge divided by volume.

Electrostatic Maxwell's equations

Because
$$\nabla \left(\frac{1}{|r-r'|} \right) = -\frac{r-r'}{|r-r'|^3}$$
:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla} \left(\frac{1}{4\pi\epsilon_0} \int_{V} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} dV' \right)$$

and therefore for all static electric fields:

$$\nabla \times E = 0.$$

E is a **conservative** vector field where its line integral is independent of path. Furthermore it may be written as:

$$E(r) = -\nabla \phi(r)$$

for $\phi(\mathbf{r})$ is the potential of \mathbf{E} .

$$\therefore \phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$



The **potential difference** between two points A and B is the energy per unit charge needed to move a small charge q from A to B:

$$V_{A \to B} = \lim_{q \to 0} \frac{1}{q} W_{A \to B}$$
$$= -\frac{1}{q} \int_{C} \mathbf{F} \cdot d\mathbf{r}$$
$$= \phi_{B} - \phi_{A}.$$

A charge distribution $\rho(\mathbf{r}')$ in an <u>external</u> electric field has potential energy:

$$W = \int_{V} \rho(\mathbf{r'}) \phi_{ext}(\mathbf{r'}) dV'.$$
Because $\nabla^{2} \left(\frac{1}{|\mathbf{r} - \mathbf{r'}|} \right) = -4\pi \delta(\mathbf{r} - \mathbf{r'})$:
$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon}.$$

Electric dipoles

An electric dipole at r_0 is defined as two charges -q at r_0 and +q at $r_0 + d$ which generates **dipole moment**:

$$\boldsymbol{p} = q\boldsymbol{d}$$

and in the dipole limit this is defined as:

$$\boldsymbol{p} \coloneqq \lim_{\substack{q \to \infty \\ \boldsymbol{d} \to \boldsymbol{0}}} q \boldsymbol{d}$$

known as an ideal dipole.



The electrostatic potential generated by this ideal dipole at r_0 is given by:

$$\begin{split} \phi(\boldsymbol{r}) &= \phi_q + \phi_{-q} \\ &= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0 - \boldsymbol{d}|} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} \right) \\ &\approx \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}_0|^3} \end{split}$$

for the first term is expanded in powers of $-\mathbf{d}$ about $\mathbf{r} - \mathbf{r}_0$.

The electric field generated is:

$$E(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[-\frac{\mathbf{p}}{|\mathbf{r} - \mathbf{r}_0|^3} + \frac{3\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} (\mathbf{r} - \mathbf{r}_0) \right].$$

If the ideal dipole is at the origin:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

$$m{E}(m{r}) = rac{1}{4\pi\epsilon_0} \left(rac{3m{p}\cdotm{r}}{r^5}m{r} - rac{m{p}}{r^3}
ight).$$

Let ideal dipole moment \boldsymbol{p} be parallel to the z-axis. Then in spherical coordinates $(r, \theta, \chi), \boldsymbol{r} = r\boldsymbol{e}_r, \boldsymbol{p} = p\boldsymbol{e}_z$ and:

$$\phi(\mathbf{r}) = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0} \left(\frac{2\cos\theta}{r^3} \boldsymbol{e}_r + \frac{\sin\theta}{r^3} \boldsymbol{e}_\theta \right).$$



Force, torque and energy

The force on a dipole at r from external electric field $E_{ext}(r)$ is:

$$F = -qE_{ext}(r) + qE_{ext}(r + d)$$

$$\approx (p \cdot \nabla)E_{ext}(r).$$

The **torque** on a dipole at r about the axis r due to $E_{ext}(r)$ is:

$$egin{aligned} m{G} &= m{ au}_{-q} + m{ au}_q \ &= -q m{0} imes m{E}_{ext}(m{r}) + q m{d} imes m{E}_{ext}(m{r} + m{d}) \ &pprox m{p} imes m{E}_{ext}(m{r}). \end{aligned}$$

The **energy** of a dipole at r from external electric field $E_{ext}(r) = -\nabla \phi_{ext}(r)$ is:

$$W = -q\phi_{ext}(\mathbf{r}) + q\phi_{ext}(\mathbf{r} + \mathbf{d})$$

$$\approx -\mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{r})$$

and $\boldsymbol{F} = -\boldsymbol{\nabla}W$.

Multipole expansion

Consider object with volume V and charge distribution $\rho(\mathbf{r}')$. Let origin be in the object. Then the potential at \mathbf{r} is:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

$$\approx \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{Q_{ij} x_i x_j}{2r^5} \right)$$

where Q is the **total charge** in V:

$$Q = \int_{V} \rho(\mathbf{r}') \mathrm{d}V'$$

p the **dipole moment** about the origin:

$$p = \int_{V} r' \rho(r') dV'$$

and Q_{ij} the quadrupole tensor:

$$Q_{ij} = \int_{V} \rho(\mathbf{r}') \left[3x_i' x_j' - (r')^2 \delta_{ij} \right] dV'.$$

If $Q \neq 0$ then in the far zone $(r \gg r_0)$ the first term (monopole term) dominates.

If Q = 0 and $\mathbf{p} = \mathbf{0}$ then the third term (quadruple term) dominates in the far zone and etc.

Interaction energy

By expanding $\phi_{ext}(\mathbf{r})$ about $\mathbf{r} = \mathbf{0}$:

$$W = \int_{V} \rho(\mathbf{r}') \phi_{ext}(\mathbf{r}') dV'$$

$$= Q \phi_{ext}(\mathbf{0}) - \mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{0})$$

$$- \frac{1}{6} Q_{ij} \frac{\partial (\mathbf{E}_{ext}(\mathbf{0}))_{i}}{\partial x_{i}} + \dots$$

and is the potential energy of a charge distribution $\rho(\mathbf{r})$ in \mathbf{E}_{ext} .

Gauss' law

For object with charge distribution $\rho(\mathbf{r}')$ and volume V enclosed by surface S:

$$\int_{S} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is total charge enclosed by V:

$$Q_{enc} = \int_{V} \rho(\mathbf{r}') \mathrm{d}V'$$

and is useful for symmetric problems.

Boundaries

Let σ be the charge density of a surface separating electric fields E_1 and E_2 .

1. Normal component of electric field is <u>discontinuous</u> across surface by:

$$\hat{m{n}}\cdot(m{E}_2-m{E}_1)=rac{\sigma}{\epsilon_0}.$$

2. Tangential component of electric field is continuous across surface:

$$E_{\parallel} := \hat{\boldsymbol{n}} \times \boldsymbol{E}_1 = \hat{\boldsymbol{n}} \times \boldsymbol{E}_2.$$

Conductors

Conductors have surplus electrons that can move freely when an electric field is applied. **In electrostatics**:

1. Conductors are in equilibrium, all charges are at rest and reside on the surface of the conductor.

Hence inside a conductor $\rho(\mathbf{r}) = 0$, $\mathbf{E}(\mathbf{r}) = \mathbf{0}$ and $\phi = \text{constant}$.

2. An electric field is always <u>normal</u> to the surface of a conductor:

$$E_{\perp} = \frac{\sigma}{\epsilon_0} \text{ and } E_{\parallel} = 0.$$

The presence of an external electric field induces a charge distribution σ on the surface of our conductor. This changes the external electric field as it needs to be normal to the surface of the conductor.

Poisson's equation

Because $\mathbf{E} = -\nabla \phi$ and $\nabla \cdot \mathbf{E} = \rho(\mathbf{r})/\epsilon_0$:

$$\nabla^2 \phi = -\frac{\rho(\mathbf{r})}{\epsilon_0}.$$

We can solve this by direct integration or using the **method of images**.

Given volume under consideration place fictitious charge <u>outside</u> the volume such that the system still satisfies Poisson's equation with boundary conditions.

This potential is our solution.

Electrostatic energy

The work needed to move point charge q from \mathbf{r}_A to \mathbf{r}_B in $\mathbf{E}(\mathbf{r})$ is:

$$W_{A\to B} = qV_{A\to B}.$$

Then $W_{\infty \to B} = q\phi(\mathbf{r}_B)$ since potential ϕ vanishes at infinity.

Generalising, the work needed to move a system of n charges q_i from infinity to r is a double sum with overcounting as each charge contributes to the electric field:

$$W_e = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i,j(i\neq j)}^n \frac{q_i q_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}.$$

Furthermore the energy needed to move a continuous charge distribution $\rho(\mathbf{r}')$ from infinity to position \mathbf{r} is:

$$W_e = rac{1}{2} \int_V
ho(\mathbf{r}) \phi(\mathbf{r}) \mathrm{d}V$$

= $rac{\epsilon_0}{2} \int_V |\mathbf{E}(\mathbf{r})|^2 \mathrm{d}V.$

Capacitors

A capacitor is formed by two conductors 1 and 2 with equal and opposite charges Q and -Q. The **capacitance** (1CV⁻¹) of a capacitor is defined as:

$$C := \frac{Q}{V}$$

where $Q = \sigma A$ for A is the surface area of one conductor and potential difference $V = \phi_1 - \phi_2$ from the conductors.

The energy stored in a capacitor is the amount of work done to move charge across the two conductors. So to move charge dq from conductor with +q:

$$dW = \left(\frac{q}{C}\right) dq$$

and integrating this up to Q gives:

$$W = \frac{1}{2} \frac{Q^2}{C}.$$

Currents

An elementary current is generated by a charge q moving at velocity $\boldsymbol{v}.$

The bulk current density is:

$$\boldsymbol{J}(\boldsymbol{r}) := \rho(\boldsymbol{r})\boldsymbol{v}$$

for $\rho(\mathbf{r})$ is the volume charge density.

The surface current density is:

$$K(r) := \sigma(r)v$$

for $\sigma(\mathbf{r})$ is the surface charge density.

The line charge density is:

$$\boldsymbol{I}(\boldsymbol{r}) := \lambda(\boldsymbol{r})\boldsymbol{v}$$

for $\lambda(\mathbf{r})$ is the line charge density.

The infinitesimal current element is:

$$\mathrm{d} oldsymbol{I}(oldsymbol{r}) \coloneqq \left\{ egin{array}{ll} oldsymbol{J}(oldsymbol{r}) \mathrm{d} oldsymbol{V} & \mathrm{volume\ current} \\ oldsymbol{K}(oldsymbol{r}) \mathrm{d} oldsymbol{S} & \mathrm{surface\ current} \\ oldsymbol{I}(oldsymbol{r}) \mathrm{d} oldsymbol{r} & \mathrm{line\ current} \end{array}
ight.$$

Current I has units Ampères (A) but the infinitesimal current element dI(r) has units of current/volume, etc.

Note:
$$1A := 1Cs^{-1}$$

Consider volume V bounded by surface S with total charge Q. Because the total charge is conserved:

$$\frac{\partial \rho(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}(\boldsymbol{r},t) = 0$$

$$Q = \int_{V} \rho(\boldsymbol{r}, t) dV.$$

An electric field can induce a current in a conductor, via **Ohm's law**:

$$\boldsymbol{J} \approx \sigma_{ij} E_j$$

where σ_{ij} is the conductivity tensor.

- 1. Perfect conductors: $\sigma \to \infty$ and E = 0.
- 2. Insulators: $\sigma = 0$.

The **electromotive force** (emf) is:

$$\mathcal{E}_{1 o 2} := \int_{m{r}_1}^{m{r}_2} m{E} \cdot \mathrm{d}m{r}
onumber \ = \phi(m{r}_1) - \phi(m{r}_2)$$

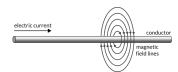
since $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$.

Biot-Savart law

The magnetic field at r generated by a static current loop carrying current I is:

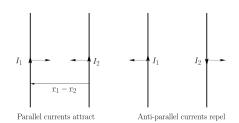
$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I \mathrm{d} \boldsymbol{r}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

and is the right grip rule.



Magnetic fields have units Teslas (T), where $NC^{-1}m^{-1}s=T$ and:

$$\mu_0 = 1.25664 \cdots \times 10^{-6} \text{NA}^{-2}.$$



However perpendicular currents do not induce force.

Lozentz force

Through physical experiments the force density on a charge distribution at r in electric field E and magnetic field B is:

$$f := \rho E + J \times B$$
.

Let point charge q at r' be moving at v. The net force on q is the integral of the force density over volume V:

$$F = q(E + v \times B)$$

since $\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}')$ and $\mathbf{J} = \rho(\mathbf{r})\mathbf{v}$.

Magnetostatic Maxwell's equations

Because
$$\nabla \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) = -\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$
:

$$\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi} \int_{V} \boldsymbol{J}(\boldsymbol{r}') \times \boldsymbol{\nabla} \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \mathrm{d}V'$$

therefore $\nabla \cdot \boldsymbol{B} = \boldsymbol{0}$ due to:

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v} - (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{u}.$$

This implies that magnetic fields always form **closed loops** — there are no sources or sinks for magnetic fields.

Similarly using the following identity:

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}(\nabla \cdot \boldsymbol{v})$$

Ampère's law

normal and tangent components of conducting surfaces

Magnetic dipoles