Vector products

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$$

$$\boldsymbol{a}\times\boldsymbol{b} = ab\sin\theta\,\hat{\boldsymbol{n}}$$

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a} \quad\text{and}\quad \boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear **more than twice** in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned}\epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{ijk} = -\epsilon_{ijk} = -\epsilon_{ijk}\end{aligned}$$

and we have the following identities:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i\boldsymbol{e}_i = a_i\boldsymbol{e}_i$$

$$A\boldsymbol{x} = a_{ij}x_j\boldsymbol{e}_i \text{ for } m\times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\ldots]_j\delta_{jk} = [\ldots]_k$$

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$$

$$\boldsymbol{e}_i\times\boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k$$

$$\boldsymbol{a}\times\boldsymbol{b} = \epsilon_{ijk}a_jb_k\boldsymbol{e}_i$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \epsilon_{ijk}a_ib_jc_k$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk}\epsilon_{ijk} = 6.$$

Transformations

Let matrix L relate basis $\{\boldsymbol{e}_i\}$ to basis $\{\boldsymbol{e}'_i\}$ with rule:

$$\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then $L^TL = LL^T = I$, and:

$$\ell_{ik}\ell_{jk} = \ell_{ki}\ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij}p_j \text{ for } \boldsymbol{p} = p_i\boldsymbol{e}_i = p'_i\boldsymbol{e}'_i.$$

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip}\ell_{jq}\ell_{kr}T_{pqr}$$

which relates frame S in $\{\boldsymbol{e}_i\}$ to frame S' in $\{\boldsymbol{e}'_i\}$ with rule $\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j$, etc.

Properties of tensors:

1. The addition of two rank n tensors is also a rank n tensor.
2. The multiplication of a rank m tensor with a rank n tensor yields a rank $m+n$ tensor.
3. If $T_{ijk\ldots s}$ is a rank m tensor then $T_{iik\ldots s}$ is a rank $m-2$ tensor.
4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} \implies T' = LTL^T$$

$$T'_{ji} = \ell_{jp}\ell_{iq}T_{pq}.$$

Symmetric tensors

T_{ij} is a symmetric tensor when $T_{ij} = T_{ji}$ in frame S. Then $T'_{ij} = T'_{ji}$ in frame S'.

Similarly T_{ij} is an anti-symmetric tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally any tensor can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij}+T_{ji}) + \frac{1}{2}(T_{ij}-T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S'. Let $b_i = T_{ij}a_j$ where a_j is a vector. If b_i always transforms as a vector then T_{ij} is a rank 2 tensor.

Generalising, let $R_{ijk\ldots r}$ be a rank m tensor and $T_{ijk\ldots s}$ a set of 3^n numbers where $n > m$. If $T_{ijk\ldots s}R_{ijk\ldots r}$ is a rank $n-m$ tensor then $T_{ijk\ldots s}$ is a rank n tensor.

Matrices

We define a $m\times n$ matrix A as $(A)_{ij} = a_{ij}$ where $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

- $\operatorname{Tr} A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^TA^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\begin{aligned}\det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn}a_{1l}a_{2m}a_{3n} \\ &= \epsilon_{lmn}a_{l1}a_{m2}a_{n3}.\end{aligned}$$

Furthermore:

$$\epsilon_{ijk}\det A = \epsilon_{lmn}a_{il}a_{jm}a_{kn}$$

$$\epsilon_{lmn}\det A = \epsilon_{ijk}a_{il}a_{jkm}a_{kn}$$

$$\det A = \frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}a_{il}a_{jm}a_{kn}.$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3. $\det A = \det A^T$
4. $\det(AB) = \det A\cdot\det B$

These also apply to columns. Finally:

$$\epsilon_{ijk}\epsilon_{lmn}\det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting $A = I$ yields:

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

Linear equations

Let $\boldsymbol{y} = A\boldsymbol{x}$. Then $x_i = A^{-1}_{ij}y_i$ with:

$$\begin{aligned}A^{-1}_{ij} &= \frac{1}{2}\frac{1}{\det A}\epsilon_{imn}\epsilon_{jpq}a_{pm}a_{qn} \\ &= \frac{1}{\det A}C^T_{ij}\end{aligned}$$

where C is the cofactor matrix of A.

Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L)\ell_{ip}\ell_{jq}T_{pq}$$

where $(L)_{ij} = \ell_{ij}$ and $\det L = \pm 1$.

Pseudovectors are rank 1 pseudotensors.

Invariant tensors

Tensor T is invariant or isotropic if:

$$T_{ijk\ldots} = \ell_{i\alpha}\ell_{j\beta}\ell_{k\gamma}\cdots T_{\alpha\beta\gamma\ldots}$$

for every orthogonal matrix L.

- If a_{ij} is a rank 2 invariant tensor then $a_{ij} = \lambda\delta_{ij}$.
- The most general rank 3 invariant pseudotensor is $a_{ijk} = \lambda\epsilon_{ijk}$. There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

Rotation tensors

The clockwise rotation of position vector $\boldsymbol{x}$ to $\boldsymbol{y}$ about unit vector $\hat{\boldsymbol{n}}$ is given by:

$$y_i = R_{ij}(\theta, \hat{\boldsymbol{n}})x_j$$

$$R_{ij}(\theta, \hat{\boldsymbol{n}}) = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk}n_k \sin\theta$$

and is the rotation tensor.

Reflections and inversions

The reflection of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ in plane with unit vector $\hat{\boldsymbol{n}}$ is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ is given by $\boldsymbol{y} = -\boldsymbol{x}$ and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

Projections

We define P to be a parallel projection operator to vector $\boldsymbol{u}$ if:

$$P\boldsymbol{u} = \boldsymbol{u} \text{ and } P\boldsymbol{v} = \boldsymbol{0}$$

where $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{0}$. Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define Q to be an orthogonal projection to vector $\boldsymbol{u}$ if:

$$Q\boldsymbol{u} = \boldsymbol{0} \text{ and } Q\boldsymbol{v} = \boldsymbol{v}.$$

Here $Q = I - P$.

Inertia tensors

Let $\boldsymbol{L}$ denote the angular momentum of a rigid body about the origin of mass m, volume V and density ρ at position $\boldsymbol{r}$ with velocity $\boldsymbol{v}$. Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2\delta_{ij} - x_i x_j)\mathrm{d}V$$

where $I_{ij}(O)$ is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2}I_{ij}\omega_i\omega_j = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{\omega}.$$

Parallel axis theorem

Consider the same rigid body now with centre of mass G and let $\overrightarrow{OG} = \boldsymbol{R}$. Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2\delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\boldsymbol{r}')\mathrm{d}V'.$$

Diagonalisation

Let $\boldsymbol{L} = I_{ij}\omega_j$ where I_{ij} is a rank 2 tensor and let $\boldsymbol{L} = \lambda\boldsymbol{\omega}$. Then:

$$(I_{ij} - \lambda\delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda\delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for $P = \det I$, $Q = \frac{1}{2}[(\operatorname{tr} I)^2 - \operatorname{tr}(I^2)]$ and $R = \operatorname{tr} I$ given tensor I.

Real symmetric tensors

Let rank 2 real symmetric tensor T be diagonalisable with real eigenvalues $\lambda^{(i)}$ and orthonormal eigenvectors $\boldsymbol{\ell}^{(i)}$ where $i = 1,2,3$. Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that $\det L = +1$ which transforms frame $S \to S'$.

Then since $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} = \lambda^{(i)}\delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ij}.$$

Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(x-a)^n$$

and is f expanded about $x = a$.

Trignometric expansions are in radians!

$$\therefore f(x+a) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x)a^n = \exp\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x)$$

Then for three dimensions:

$$\phi(\boldsymbol{r}+\boldsymbol{a}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)^n\phi(\boldsymbol{r}) = \exp(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)\phi(\boldsymbol{r}).$$

Curvilinear coordinates

Let x_i denote Cartesian coordinates and u_i denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \to (u_1, u_2, u_3)$$

where each $u_i = u_i(x_1, x_2, x_3)$ and:

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = u_1\boldsymbol{e}_{u_1} + u_2\boldsymbol{e}_{u_2} + u_3\boldsymbol{e}_{u_3}.$$

Scale factors

Let $u_1 \to u_1 + \mathrm{d}u_1$ in $\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, u_3)$. Then $\mathrm{d}\boldsymbol{r}$ in $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 := h_1\boldsymbol{e}_1\mathrm{d}u_1.$$

h_1 is the scale factor of unit vector $\boldsymbol{e}_1$:

$$h_1 = \left|\frac{\partial\boldsymbol{r}}{\partial u_1}\right| \text{ and } \boldsymbol{e}_1 = \frac{1}{h_1}\frac{\partial\boldsymbol{r}}{\partial u_1}.$$

If $\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$ then u_i is an **orthogonal** curvilinear coordinate system.

Vector and arc length

The vector length $\mathrm{d}\boldsymbol{r}$ of $\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3}h_i\mathrm{d}u_i\boldsymbol{e}_i$$

where $u_i \to u_i + \mathrm{d}u_i$ for $\forall i = 1,2,3$.

Then the arc length $\mathrm{d}s$ is defined as:

$$(\mathrm{d}s)^2 = \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} = g_{ij}\mathrm{d}u_i\mathrm{d}u_j$$

where g_{ij} is the metric tensor:

$$g_{ij} = g_{ji} = \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} = h_i h_j(\boldsymbol{e}_i\cdot\boldsymbol{e}_j).$$

Area and volume

Let $\mathrm{d}\boldsymbol{r}_i = h_i\boldsymbol{e}_i\mathrm{d}u_i$ denote vector length when $u_i \to u_i + \mathrm{d}u_i$. (**No** sum!)

The infinitesimal vector area formed by $\mathrm{d}\boldsymbol{r}_1$ and $\mathrm{d}\boldsymbol{r}_2$ is:

$$\mathrm{d}\boldsymbol{S} = (h_1\mathrm{d}\boldsymbol{u}_1\boldsymbol{e}_1)\times(h_2\mathrm{d}\boldsymbol{u}_2\boldsymbol{e}_2).$$

Similarly the infinitesimal volume formed by edges $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$ is:

$$\mathrm{d}V = |(\mathrm{d}\boldsymbol{r}_1\times\mathrm{d}\boldsymbol{r}_2)\cdot\mathrm{d}\boldsymbol{r}_3| = \sqrt{g}\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3$$

where $g = \det(g_{ij})$.

Cylindrical coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$ where ρ represents the radial distance from the origin and ϕ is the anticlockwise rotation angle on the x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_1 + \rho\sin\phi\boldsymbol{e}_2 + z\boldsymbol{e}_3$$

$$h_\rho = 1, \quad \boldsymbol{e}_\rho = \cos\phi\boldsymbol{e}_1 + \sin\phi\boldsymbol{e}_2$$

$$h_\phi = \rho, \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

$$h_z = 1, \quad \boldsymbol{e}_z = \boldsymbol{e}_3$$

and forms an orthogonal set.

Spherical coordinates

$(u_1, u_2, u_3) = (r, \theta, \phi)$ where θ represents the clockwise rotation angle in y-z plane and ϕ the anticlockwise rotation angle in x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = r\sin\theta\cos\phi\boldsymbol{e}_1 + r\sin\theta\sin\phi\boldsymbol{e}_2 + r\cos\theta\boldsymbol{e}_3$$

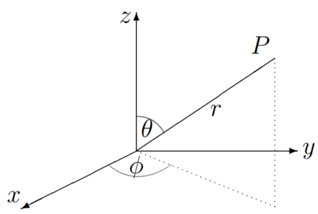
$$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$$

$$\boldsymbol{e}_r = \sin\theta\cos\phi\boldsymbol{e}_1 + \sin\theta\sin\phi\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\boldsymbol{e}_1 + \cos\theta\sin\phi\boldsymbol{e}_2 - \sin\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

and also forms an orthogonal set.



Gradient

The gradient of a scalar field $f(\boldsymbol{r})$ is:

$$\mathrm{d}f(\boldsymbol{r}) := \boldsymbol{\nabla} f(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}$$

when $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r} \implies f \to f + \mathrm{d}f$. Taking the total differential of f yields:

$$\boldsymbol{\nabla} f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i$$

where $\{\boldsymbol{e}_i\}$ is orthogonal.

Divergence

The divergence of a vector field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} := \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

for surface δS bounds infinitesimal δV. In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 F_2 h_3) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right\}.$$

Curl

The curl of a vector field $\boldsymbol{F}$ in the direction of unit vector $\hat{\boldsymbol{n}}$ is:

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) := \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

where curve δC encloses plane δS. In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{vmatrix}.$$

Laplacian

The Laplacian of a scalar field f is:

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f)$$

and in orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla}^2 f = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right\}.$$

The Laplacian of a vector field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla}^2 \boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}).$$

Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x_i}\boldsymbol{e}_i$$

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{F} = u_j \frac{\partial}{\partial x_j} F_i$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_i}{\partial x_i}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}\boldsymbol{e}_i$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

If ψ is a scalar field and $\boldsymbol{v}$ a vector field:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi) = \boldsymbol{0}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{0}$$

$$\boldsymbol{\nabla} \cdot (\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi \cdot \boldsymbol{v} + \psi\boldsymbol{\nabla} \cdot \boldsymbol{v}$$

$$\boldsymbol{\nabla} \times (\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi \times \boldsymbol{v} + \psi\boldsymbol{\nabla} \times \boldsymbol{v}.$$

Let $\boldsymbol{r} = x_i\boldsymbol{e}_i$ and $r = (x_i^2)^{1/2}$. Then:

- $\boldsymbol{\nabla} r = \frac{\boldsymbol{r}}{r}$ and $\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\boldsymbol{r}}{r^3}$
- $\boldsymbol{\nabla} r^n = n r^{n-2}\boldsymbol{r}$
- $\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3$ and $\boldsymbol{\nabla} \times \boldsymbol{r} = \boldsymbol{0}$
- $\boldsymbol{\nabla} \times (\boldsymbol{c} \times \boldsymbol{r}) = 2\boldsymbol{c}$
- $\boldsymbol{\nabla} \cdot (\boldsymbol{c} \times \boldsymbol{r}) = \boldsymbol{0}$ for constant $\boldsymbol{c}$.

Divergence theorem

Let surface S enclose volume V. Then:

$$\iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{F}\,\mathrm{d}V = \oint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

where $\boldsymbol{F}$ is a vector field.

Stokes' theorem

Let closed curve C bound open surface S and let $\boldsymbol{F}$ be a vector field. Then:

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

for C is traversed in anticlockwise sense.

Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x-a) := \begin{cases} \infty & x = a \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions this becomes:

$$\begin{aligned} \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0) &:= \delta(\boldsymbol{r} - \boldsymbol{r}_0) \\ &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \end{aligned}$$

where (x, y, z) are Cartesian coordinates. In orthogonal curvilinear coordinates:

$$\begin{aligned} \delta(\boldsymbol{r} - \boldsymbol{a}) = &\frac{1}{h_1 h_2 h_3}\delta(u_1 - a_1) \\ &\cdot \delta(u_2 - a_2)\delta(u_3 - a_3). \end{aligned}$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

which yields:

$$x\delta(x) = 0 \text{ and } \delta(cx) = \frac{1}{|c|}\delta(x).$$

If simple solutions of $g(x) = 0$ are x_i:

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))\mathrm{d}x = \sum_i \frac{f(x_i)}{|g'(x_i)|}.$$

Coulomb's law

Consider charges q and q_1 at positions $\boldsymbol{r}$ and $\boldsymbol{r}_1$. The force on charge q at $\boldsymbol{r}$ due to charge q_1 at $\boldsymbol{r}_1$ is:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q q_1(\boldsymbol{r} - \boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}$$

where $qq_1 > 0$ denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419 \times 10^{-12}\mathrm{C^2N^{-1}m^{-2}}.$$

Charge has units Coulombs (C) and an electron has charge -1.60218×10^{-19}C.

Electric fields

The electric field is generated by a charge configuration and defined in terms of the force on a small positive test charge q:

$$\boldsymbol{E}(\boldsymbol{r}) := \lim_{q \to 0}\frac{1}{q}\boldsymbol{F}.$$

Then for our two charges q and q_1:

$$\boldsymbol{F}_1(\boldsymbol{r}) = q\boldsymbol{E}_1(\boldsymbol{r})$$

where q_1 produces electric field $\boldsymbol{E}_1$.

$$\therefore \boldsymbol{E}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1(\boldsymbol{r} - \boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}$$

Principle of superposition

For a set of charges q_i at position $\boldsymbol{r}_i$ the total electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}.$$

For object with **charge density** $\rho(\boldsymbol{r}')$ its overall electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'$$

where $\rho(\boldsymbol{r}')$ is charge divided by volume.

Electrostatic Maxwell's equations

Because $\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right)$$

and therefore for all static electric fields:

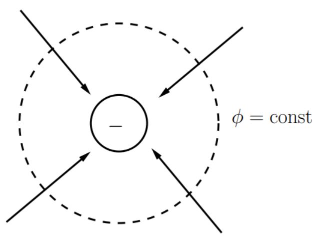
$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}.$$

$\boldsymbol{E}$ is a **conservative** vector field where its line integral is independent of path. Furthermore it may be written as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$$

for $\phi(\boldsymbol{r})$ is the potential of $\boldsymbol{E}$.

$$\therefore \phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'$$



The **potential difference** between two points A and B is the energy per unit charge needed to move a small charge q from A to B:

$$\begin{aligned} V_{A\to B} &= \lim_{q\to 0}\frac{1}{q}W_{A\to B} \\ &= -\frac{1}{q}\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} \\ &= \phi_B - \phi_A. \end{aligned}$$

A charge distribution $\rho(\boldsymbol{r}')$ in an external electric field has potential energy:

$$W = \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}')\mathrm{d}V'.$$

Because $\boldsymbol{\nabla}^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

Electric dipoles

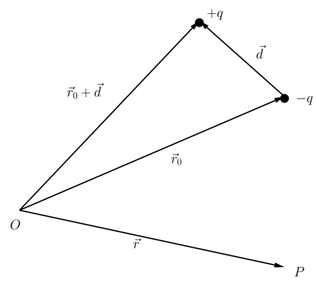
An electric dipole at $\boldsymbol{r}_0$ is defined as two charges $-q$ at $\boldsymbol{r}_0$ and $+q$ at $\boldsymbol{r}_0+\boldsymbol{d}$ which generates **dipole moment**:

$$\boldsymbol{p} = q\boldsymbol{d}$$

and in the dipole limit this is defined as:

$$\boldsymbol{p} := \lim_{\substack{q\to\infty \\ \boldsymbol{d}\to\boldsymbol{0}}} q\boldsymbol{d}$$

known as an ideal dipole.



The electrostatic potential generated by this ideal dipole at $\boldsymbol{r}_0$ is given by:

$$\begin{aligned} \phi(\boldsymbol{r}) &= \phi_q + \phi_{-q} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{d}|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\right) \\ &\approx \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3} \end{aligned}$$

for the first term is expanded in powers of $-\boldsymbol{d}$ about $\boldsymbol{r}-\boldsymbol{r}_0$.

The electric field generated is:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) =& \frac{1}{4\pi\epsilon_0}\Bigg[-\frac{\boldsymbol{p}}{|\boldsymbol{r}-\boldsymbol{r}_0|^3} \\ &+ \frac{3\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^5}(\boldsymbol{r}-\boldsymbol{r}_0)\Bigg]. \end{aligned}$$

If the ideal dipole is at the origin:

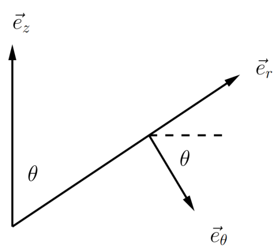
$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\boldsymbol{p}\cdot\boldsymbol{r}}{r^5}\boldsymbol{r} - \frac{\boldsymbol{p}}{r^3}\right).$$

Let ideal dipole moment $\boldsymbol{p}$ be parallel to the z-axis. Then in spherical coordinates (r,θ,χ), $\boldsymbol{r} = r\boldsymbol{e}_r$, $\boldsymbol{p} = p\boldsymbol{e}_z$ and:

$$\phi(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0}\left(\frac{2\cos\theta}{r^3}\boldsymbol{e}_r + \frac{\sin\theta}{r^3}\boldsymbol{e}_\theta\right).$$



Force, torque and energy

The **force** on a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} \boldsymbol{F} &= -q\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{E}_{ext}(\boldsymbol{r}+\boldsymbol{d}) \\ &\approx (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E}_{ext}(\boldsymbol{r}). \end{aligned}$$

The **torque** on a dipole at $\boldsymbol{r}$ about the axis $\boldsymbol{r}$ due to $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} \boldsymbol{G} &= \boldsymbol{\tau}_{-q} + \boldsymbol{\tau}_q \\ &= -q\boldsymbol{0}\times\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{d}\times\boldsymbol{E}_{ext}(\boldsymbol{r}+\boldsymbol{d}) \\ &\approx \boldsymbol{p}\times\boldsymbol{E}_{ext}(\boldsymbol{r}). \end{aligned}$$

The **energy** of a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} W &= -q\phi_{ext}(\boldsymbol{r}) + q\phi_{ext}(\boldsymbol{r}+\boldsymbol{d}) \\ &\approx -\boldsymbol{p}\cdot\boldsymbol{E}_{ext}(\boldsymbol{r}) \end{aligned}$$

and $\boldsymbol{F} = -\boldsymbol{\nabla}W$.

Multipole expansion

Consider object with volume V and charge distribution $\rho(\boldsymbol{r}')$. Let origin be in the object. Then the potential at $\boldsymbol{r}$ is:

$$\begin{aligned} \phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' \\ &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{Q_{ij}x_i x_j}{2r^5}\right) \end{aligned}$$

where Q is the **total charge** in V:

$$Q = \int_V \rho(\boldsymbol{r}')\mathrm{d}V'$$

$\boldsymbol{p}$ the **dipole moment** about the origin:

$$\boldsymbol{p} = \int_V \boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V'$$

and Q_{ij} the **quadrupole tensor**:

$$Q_{ij} = \int_V \rho(\boldsymbol{r}')\Big[3x_i'x_j' - (r')^2\delta_{ij}\Big]\mathrm{d}V'.$$

If $Q \neq 0$ then in the far zone $(r \gg r_0)$ the first term (monopole term) dominates.

If $Q = 0$ and $\boldsymbol{p} = \boldsymbol{0}$ then the third term (quadruple term) dominates in the far zone and etc.

Interaction energy

By expanding $\phi_{ext}(\boldsymbol{r})$ about $\boldsymbol{r} = \boldsymbol{0}$:

$$\begin{aligned} W &= \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}')\mathrm{d}V' \\ &= Q\phi_{ext}(\boldsymbol{0}) - \boldsymbol{p}\cdot\boldsymbol{E}_{ext}(\boldsymbol{0}) \\ &\quad - \frac{1}{6}Q_{ij}\frac{\partial\big(\boldsymbol{E}_{ext}(\boldsymbol{0})\big)_i}{\partial x_j} + \ldots \end{aligned}$$

and is the potential energy of a charge distribution $\rho(\boldsymbol{r})$ in $\boldsymbol{E}_{ext}$.

Gauss' law

For object with charge distribution $\rho(\boldsymbol{r}')$ and volume V enclosed by surface S:

$$\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is total charge enclosed by V:

$$Q_{enc} = \int_V \rho(\boldsymbol{r}')\mathrm{d}V'$$

and is useful for symmetric problems.

Boundaries

Let σ be the charge density of a surface separating electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$.

1. Normal component of electric field is <u>discontinuous</u> across surface by:

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \frac{\sigma}{\epsilon_0}.$$

2. Tangential component of electric field is continuous across surface:

$$\boldsymbol{E}_\parallel := \hat{\boldsymbol{n}} \times \boldsymbol{E}_1 = \hat{\boldsymbol{n}} \times \boldsymbol{E}_2.$$

Conductors

Conductors have surplus electrons that can move freely when an electric field is applied. **In electrostatics**:

1. Conductors are in equilibrium, all charges are at rest and reside on the <u>surface</u> of the conductor.

   Hence inside a conductor $\rho(\boldsymbol{r}) = 0$, $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}$ and $\phi = \text{constant}$.

2. An electric field is always <u>normal</u> to the surface of a conductor:

$$E_\perp = \frac{\sigma}{\epsilon_0} \text{ and } E_\parallel = 0.$$

The presence of an external electric field induces a charge distribution σ on the surface of our conductor. This changes the external electric field as it needs to be normal to the surface of the conductor.

Poisson's equation

Because $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho(\boldsymbol{r})/\epsilon_0$:

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

We can solve this by direct integration or using the **method of images**.

Given volume under consideration place fictitious charge <u>outside</u> the volume such that the system still satisfies Poisson's equation with boundary conditions.

This potential is our solution.

Electrostatic energy

The work needed to move point charge q from $\boldsymbol{r}_A$ to $\boldsymbol{r}_B$ in $\boldsymbol{E}(\boldsymbol{r})$ is:

$$W_{A\to B} = qV_{A\to B}.$$

Then $W_{\infty\to B} = q\phi(\boldsymbol{r}_B)$ since potential ϕ vanishes in the far zone.

Furthermore the energy needed to move a continuous charge distribution $\rho(\boldsymbol{r}')$ from infinity to position $\boldsymbol{r}$ is:

$$\begin{aligned} W_e &= \frac{1}{2}\int_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V \\ &= \frac{\epsilon_0}{2}\int_V |\boldsymbol{E}(\boldsymbol{r})|^2\mathrm{d}V. \end{aligned}$$

Capacitors

Currents

Lozentz force

Biot-Savart law

Magnetostatic Maxwell's equations

Ampère's law

normal and tangent components of conducting surfaces

Magnetic dipoles