Honours Differential Equations Assignments

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Contents

Assignment 1 **3**

Assignment 2 **4**

Assignment 3 **5**

Assignment 4 **11**

Assignment 1

1.

Assignment 2

1.

Assignment 3

1. Consider $f(x) = \frac{x^2}{2}$ where $0 \le x < L$.

   Part (a) wants us to find its Fourier series with period $2L$.

   Define an extension of our function f as g:

$$g(x) = \begin{cases} f(x) & x \in [0, L) \\ f(-x) & x \in [-L, 0) \end{cases}$$

   and therefore g has period $2L$. Since g is also an even function its Fourier series can only consist of cosine terms.

$$\therefore g_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

   Its coefficients are:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} g(x)dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)dx + \frac{1}{L}\int_{-L}^{0} f(-x)dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)dx - \frac{1}{L}\int_{0}^{-L} f(-x)dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)dx + \frac{1}{L}\int_{0}^{L} f(x^*)dx^* \\ &= \frac{2}{L}\int_{0}^{L} f(x)dx \end{aligned}$$

   where we define $x = -x^*$ and similarly

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} g(x)dx \\ &= \frac{1}{L}\int_{0}^{L} \cos\frac{n\pi x}{L} f(x)dx + \frac{1}{L}\int_{-L}^{0} \cos\frac{n\pi x}{L} f(-x)dx \\ &= \frac{2}{L}\int_{0}^{L} \cos\frac{n\pi x}{L} f(x)dx. \end{aligned}$$

Calculating this explicitly:

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{L}\int_0^L \frac{1}{2}x^2 dx \\ &= \frac{1}{L}\left[\frac{x^3}{3}\right]_0^L \\ &= \frac{1}{3}L^2 \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L} f(x)dx \\ &= \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\frac{1}{2}x^2 dx \\ &= \frac{1}{L}\int_0^L x^2\cos\frac{n\pi x}{L} dx \\ &= \frac{1}{n\pi}\left[x^2\sin\frac{n\pi x}{L}\right]_0^L - \frac{2}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}dx \\ &= \frac{L^2}{n\pi}\sin n\pi - \frac{2}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}dx \\ &= \frac{L^2}{n\pi}\sin n\pi - \frac{2}{n\pi}\left(-\frac{L^2}{n\pi}\cos n\pi + \left(\frac{L}{n\pi}\right)^2\sin n\pi\right) \\ &= \left(\frac{L^2}{n\pi} - \frac{2L^2}{(n\pi)^3}\right)\sin n\pi + \left(\frac{2L^2}{(n\pi)^2}\right)\cos n\pi \\ &= (-1)^n\frac{2L^2}{(n\pi)^2} \end{aligned}$$

where we have integrated by parts, and $\sin n\pi = 0$ for $\forall n \in \mathbb{N}$.

Putting all this together we get our Fourier series for f:

$$\therefore f_{FS}(x) = \frac{L^2}{6} + \sum_{n=1}^{\infty}\left((-1)^n\frac{2L^2}{(n\pi)^2}\cos\frac{n\pi x}{L}\right)$$

and is valid for $\forall x \in [0, L)$.

For part (b) we want $f_{FS}(0)$. By Fourier's convergence theorem:

$$f_{FS}(0) = \frac{L^2}{6} + \sum_{n=1}^{\infty} \left((-1)^n \frac{2L^2}{(n\pi)^2} \right).$$

For part (c):

$$\begin{aligned} f(0) &= \frac{L^2}{6} + \sum_{n=1}^{\infty} \left((-1)^n \frac{2L^2}{(n\pi)^2} \cos \frac{n\pi x}{L} \right) \\ &= 0 \end{aligned}$$

$$\therefore \frac{L^2}{6} + \frac{2L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

2. For part (a) we given the solution to Laplace's equation:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

find expressions for coefficients a_0, a_n and b_n.

Because our solution has period 2π, using the Euler-Fourier formulae:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)f(\theta)d\theta$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)f(\theta)d\theta.$$

For part (b) we begin with the following expression:

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)d\psi \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \left(\cos(n\theta)\int_{-\pi}^{\pi}\cos(n\psi)f(\psi)d\psi + \sin(n\theta)\int_{-\pi}^{\pi}\sin(n\psi)f(\psi)d\psi\right) \end{aligned}$$

Now since $e^{in(\theta-\psi)} = \cos n(\theta-\psi) + i\sin n(\theta-\psi)$:

$$\begin{aligned} Re(e^{in(\theta-\psi)}) &= \cos n(\theta-\psi) \\ &= \cos(n\theta)\cos(n\psi) + \sin(n\theta)\sin(n\psi) \end{aligned}$$

$$\begin{aligned} \therefore \int_{-\pi}^{\pi} Re(e^{in(\theta-\psi)})f(\psi)d\psi &= \int_{-\pi}^{\pi} (\cos(n\theta)\cos(n\psi) + \sin(n\theta)\sin(n\psi))\, f(\psi)d\psi \\ &= \cos(n\theta)\int_{-\pi}^{\pi}\cos(n\psi)f(\psi)d\psi \\ &\quad + \sin(n\theta)\int_{-\pi}^{\pi}\sin(n\psi)f(\psi)d\psi \end{aligned}$$

Therefore our original expression becomes:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)d\psi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} r^n Re(e^{in(\theta-\psi)})f(\psi)d\psi$$

Taking the real component of each element in a sum is equivalent to taking the real component of the overall sum:

$$\therefore u(r,\theta) = \frac{1}{\pi} Re\left(\frac{1}{2}\int_{-\pi}^{\pi} f(\psi)d\psi + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} r^n e^{in(\theta-\psi)} f(\psi)d\psi\right)$$

Then by the linearity of integrals:

$$\therefore u(r,\theta) = \frac{1}{\pi} Re\left(\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n e^{in(\theta-\psi)}\right] f(\psi)d\psi\right)$$

Finally for part (c) we have that

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

and let:

$$\begin{aligned} q &= re^{i(\theta-\psi)} \\ &= r\left(\cos(\theta-\psi) + i\sin(\theta-\psi)\right). \end{aligned}$$

Now $|q| < 1$ since we defined $r < 1$ and $e^{i2\pi} = 1$. Returning to our sum:

$$\begin{aligned} \sum_{n=1}^{\infty} q^n &= \frac{1}{1-q} - 1 \\ &= \frac{q}{1-q}. \end{aligned}$$

Furthermore:

$$\begin{aligned} \frac{1}{2} + \sum_{n=1}^{\infty} q^n &= \frac{1}{2}\frac{1+q}{1-q} \\ &= \frac{1}{2}\frac{1 + r(\cos(\theta-\psi) + i\sin(\theta-\psi))}{1 - r(\cos(\theta-\psi) + i\sin(\theta-\psi))} \\ &= \frac{1}{2}\left[\frac{(1+rc) + i(rs)}{(1-rc) + i(-rs)}\right] \times \frac{1-rc+i(rs)}{1-rc+i(rs)} \\ &= \frac{1}{2}\frac{1-r^2+i(2rs)}{1+r^2-2rc} \end{aligned}$$

$$\therefore Re(\frac{1}{2} + \sum_{n=1}^{\infty} q^n) = \frac{1-r^2}{1+r^2-2rc}$$

Finally we have that:

$$\begin{aligned}
u(r,\theta) &= \frac{1}{\pi} Re \left(\int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n e^{in(\theta-\psi)} \right] f(\psi) d\psi \right) \\
&= \frac{1}{\pi} \left(\int_{-\pi}^{\pi} Re \left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n e^{in(\theta-\psi)} \right] f(\psi) d\psi \right) \\
&= \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\psi)} f(\psi) d\psi \right).
\end{aligned}$$

Assignment 4

1. For part (a), $\phi_n(x)$ are the orthonormal eigenfunctions and λ_n the real eigenvalues of the corresponding homogeneous regular S-L problem:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \lambda r(x)y$$

with initial conditions:

- $\alpha_1 y(0) + \alpha_2 y'(0) = 0$
- $\beta_1 y(1) + \beta_2 y'(1) = 0.$

for $x \in [0,1]$. The solution to the following:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \mu r(x)y + f(x)$$

is then:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

where:

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

and

$$c_n = \int_0^1 \phi_n(x) f(x)\mathrm{d}x.$$

For part (b) solve the following:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + 7y(x) = 2\sin 5x + 3\sin 7x$$

with boundary conditions $y(0) = y(\pi) = 0$ for $\forall x \in [0, \pi]$.

First define change of variables $x = \pi t$.

$$\therefore y(x) \iff y(t)$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}y(t = \frac{x}{\pi}) = \frac{1}{\pi}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}y(t = \frac{x}{\pi})\right) = \frac{1}{\pi^2}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$$

Then our ODE becomes:

$$\frac{1}{\pi^2}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 7y(t) = 2\sin 5\pi t + 3\sin 7\pi t$$

with boundary conditions $y(0) = y(1) = 0$ for $\forall t \in [0,1]$. This is now of S-L form, and its corresponding homogeneous S-L system is:

$$y'' + \pi^2\lambda y = 0.$$

We know that S-L problems have real valued eigenvalues, and so we consider the sign of λ separately. Now if $\lambda = 0$ we have linear solutions:

$$y = a_1 t + a_2$$

but since $y(0) = a_2 = 0$ and $y = a_1 = 0$, only trivial solutions remain.

If $\lambda < 0$ we have solutions:

$$y = b_1 \cosh \pi\sqrt{\lambda}t + b_2 \sinh \pi\sqrt{\lambda}t$$

and using boundary conditions, $y(0) = b_1 = 0$ and $y(1) = b_2 \sinh \pi\sqrt{\lambda} = 0$. This also yields only trivial solutions since:

$$\sinh \pi\sqrt{\lambda} = \frac{1}{2}\left(e^{\pi\sqrt{\lambda}} - e^{-\pi\sqrt{\lambda}}\right) = 0$$

implies $e^{2\pi\sqrt{\lambda}} = 1$ and the only eigenvalue satifying this is $\lambda = 0$.

Finally if $\lambda > 0$ we have solutions of form:

$$y = c_1 \sin \pi\sqrt{\lambda}t + c_2 \cos \pi\sqrt{\lambda}t$$

and our boundary conditions yields $y(0) = c_2 = 0$ and:

$$y(1) = c_1 \sin \pi\sqrt{\lambda} = 0$$

implies $\lambda_n = n^2$ where $n \in \mathbb{N}$. So our eigenfunctions are:

$$\begin{aligned}\phi_n(t) &= k_n \sin \pi\sqrt{\lambda_n}t \\ &= k_n \sin n\pi t.\end{aligned}$$

Since $\langle \phi_n, \phi_n \rangle = 1$ and $r(x) = \pi^2$ we then have that:

$$\int_0^1 \pi^2 k_n^2 (\sin^2 n\pi t)\mathrm{d}t = 1.$$

$$\therefore k_n = \frac{1}{\pi}\left[1 - \frac{1}{2\pi n}\sin 2\pi n\right]^{-1/2}$$

Returning to our original ODE and rearranging it into S-L form:

$$-y'' = 7\pi^2 y - \pi^2\big(2\sin 5\pi t + 3\sin 7\pi t\big)$$

where we have $\mu = 7$, $r(t) = \pi^2$ and

$$f(t) = -\pi^2\big(2\sin 5\pi t + 3\sin 7\pi t\big).$$

Let solutions be of the following form:

$$y(t) = \sum_{n=1}^{\infty} b_n \phi_n(t)$$

where

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

and

$$\begin{aligned}
c_n &= \int_0^1 \phi_n(x) f(x)\mathrm{d}x \\
&= -\pi^2 k_n\Big[\int_0^1 \sin n\pi t\big(2\sin 5\pi t + 3\sin 7\pi t\big)\mathrm{d}t\Big] \\
&= -\pi^2 k_n\Big[2\int_0^1 \sin(n\pi t)\sin(5\pi t)\mathrm{d}t + 3\int_0^1 \sin(n\pi t)\sin(7\pi t)\mathrm{d}t\Big] \\
&= -\pi^2 k_n\Big[\frac{\sin(n-5)\pi}{(n-5)\pi} - \frac{\sin(n+5)\pi}{(n+5)\pi}\Big] \\
&\quad - \frac{3}{2}\pi^2 k_n\Big[\frac{\sin(n-7)\pi}{(n-7)\pi} - \frac{\sin(n+7)\pi}{(n+7)\pi}\Big]
\end{aligned}$$

for $n \neq 5, 7$. We then have that:

$$b_n = \frac{c_n}{n^2 - 7}$$

and

$$k_n = \frac{1}{\pi}\Big[1 - \frac{1}{2\pi n}\sin 2\pi n\Big]^{-1/2}.$$

These coefficients form the solutions to our ODE:

$$y(t) = \sum_{n=1}^{\infty} b_n \phi_n(t)$$

where

$$\phi_n(t) = k_n \sin n\pi t.$$

2. For part (a) find the Laplace transform of:

$$f(t) = \sinh at.$$

Firstly we have that:

$$\begin{aligned} e^{-st}\sinh at &= \frac{1}{2}e^{-st}\Big[e^{at} + e^{-at}\Big] \\ &= \frac{1}{2}\Big[e^{-(s-a)t} - e^{-(s+a)t}\Big]. \end{aligned}$$

Therefore:

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^\infty e^{-st}\sinh(at)\mathrm{d}t \\ &= \int_0^\infty \frac{1}{2}\Big[e^{-(s-a)t} - e^{-(s+a)t}\Big]\mathrm{d}t \\ &= \lim_{T\to\infty}\Big[-\frac{1}{s-a}e^{-(s-a)t} + \frac{1}{s+a}e^{-(s+a)t}\Big]_0^T \\ &= \frac{a}{s^2-a^2}. \end{aligned}$$

For part (b) show that:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

for all non-negative n.

We proceed via induction. Let $n = 1$:

$$\begin{aligned} \mathcal{L}[t] &= \int_0^\infty te^{-st}\mathrm{d}t \\ &= \Big[-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\Big]_0^\infty \\ &= \frac{1}{s^2} \end{aligned}$$

since an exponential grows faster than a linear one.

We then assume that the following is true:

$$\mathcal{L}[t^k] = \frac{k!}{s^{k+1}}$$

and we want to show:

$$\mathcal{L}[t^{k+1}] = \frac{(k+1)!}{s^{k+2}}.$$

So integrating this by parts we get:

$$\begin{aligned}
\mathcal{L}[t^{k+1}] &= \int_0^\infty t^{k+1}e^{-st}\mathrm{d}t \\
&= \left[-\frac{1}{s}t^{k+1}e^{-st}\right]_0^\infty + \frac{k+1}{s}\int_0^\infty t^k e^{-st}\mathrm{d}t \\
&= \frac{k+1}{s}\mathcal{L}[t^k] \\
&= \frac{(k+1)!}{s^{k+2}}.
\end{aligned}$$

Now verifying this for $k = 0$:

$$\begin{aligned}
\mathcal{L}[t^0] &= \int_0^\infty e^{-st}\mathrm{d}t \\
&= \left[-\frac{1}{s}e^{-st}\right]_0^\infty \\
&= \frac{1}{s}.
\end{aligned}$$

For part (c) solve the following ODE:

$$y^{(4)}(t) = 3\sinh 2t$$

with initial conditions:

$$y(0) = y^{(1)}(0) = y^{(2)}(0) = y^{(3)}(0) = 1.$$

First begin by taking the Laplace transforms of both sides:

$$\mathcal{L}[y^{(4)}] = \mathcal{L}[3\sinh 2t] = \frac{6}{s^2-4}.$$

And our left hand side becomes:

$$\begin{aligned}
\mathcal{L}[y^{(4)}] &= s^4\mathcal{L}[y(t)] - s^3y^{(0)}(0) - s^2y^{(1)}(0) - sy^{(2)}(0) - y^{(3)}(0) \\
&= s^4\mathcal{L}[y(t)] - s^3 - s^2 - s - 1.
\end{aligned}$$

Equating and rearranging:

$$\begin{aligned}\mathcal{L}[y(t)] &= \frac{6}{s^4(s^2-4)} + \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} \\ &= \frac{1}{s} + \frac{5}{8}\frac{1}{s^2} + \frac{1}{s^3} + -\frac{1}{2}\frac{1}{s^4} + \frac{3}{16}\frac{2}{s^2-4}.\end{aligned}$$

Since we have the following standard transforms:

$$\frac{1}{s} = \mathcal{L}[1], \quad \frac{1}{s^2} = \mathcal{L}[t], \quad \frac{1}{s^3} = \frac{1}{2}\mathcal{L}[t^2], \quad \frac{1}{s^4} = \frac{1}{6}\mathcal{L}[t^3]$$

and

$$\frac{2}{s^2-4} = \mathcal{L}[\sinh 2t]$$

we have that by inspection:

$$y(t) = 1 + \frac{5}{8}t + \frac{1}{2}t^2 - \frac{1}{12}t^3 + \frac{3}{16}\sinh 2t.$$