Vector products

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta$$

$$\boldsymbol{a} \times \boldsymbol{b} = ab\sin\theta\,\hat{\boldsymbol{n}}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \quad \text{and} \quad \boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear **more than twice** in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{ijk} = -\epsilon_{ijk} = -\epsilon_{ijk} \end{aligned}$$

and we have the following identities:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i = a_i \boldsymbol{e}_i$$

$$A\boldsymbol{x} = a_{ij} x_j \boldsymbol{e}_i \text{ for } m \times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\ldots]_j \delta_{jk} = [\ldots]_k$$

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk} \boldsymbol{e}_k$$

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk} a_j b_k \boldsymbol{e}_i$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk}\epsilon_{ijk} = 6.$$

Transformations

Let matrix L relate basis $\{\boldsymbol{e}_i\}$ to basis $\{\boldsymbol{e}'_i\}$ with rule:

$$\boldsymbol{e}'_i = \ell_{ij} \boldsymbol{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then $L^T L = L L^T = I$, and:

$$\ell_{ik}\ell_{jk} = \ell_{ki}\ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij} p_j \text{ for } \boldsymbol{p} = p_i \boldsymbol{e}_i = p'_i \boldsymbol{e}'_i.$$

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip}\ell_{jq}\ell_{kr} T_{pqr}$$

which relates frame S in $\{\boldsymbol{e}_i\}$ to frame S' in $\{\boldsymbol{e}'_i\}$ with rule $\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j$, etc.

Properties of tensors:

1. The addition of two rank n tensors is also a rank n tensor.
2. The multiplication of a rank m tensor with a rank n tensor yields a rank $m+n$ tensor.
3. If $T_{ijk\ldots s}$ is a rank m tensor then $T_{iik\ldots s}$ is a rank $m-2$ tensor.
4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$T'_{ij} = \ell_{ip}\ell_{jq} T_{pq} \implies T' = L T L^T$$
$$T'_{ji} = \ell_{jp}\ell_{iq} T_{pq}.$$

Symmetric tensors

T_{ij} is a symmetric tensor when $T_{ij} = T_{ji}$ in frame S. Then $T'_{ij} = T'_{ji}$ in frame S'.

Similarly T_{ij} is an anti-symmetric tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally any tensor can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S'. Let $b_i = T_{ij} a_j$ where a_j is a vector. If b_i always transforms as a vector then T_{ij} is a rank 2 tensor.

Generalising, let $R_{ijk\ldots r}$ be a rank m tensor and $T_{ijk\ldots s}$ a set of 3^n numbers where $n > m$. If $T_{ijk\ldots s} R_{ijk\ldots r}$ is a rank $n-m$ tensor then $T_{ijk\ldots s}$ is a rank n tensor.

Matrices

We define a $m \times n$ matrix A as $(A)_{ij} = a_{ij}$ where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- $\operatorname{Tr} A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^T A^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn} a_{1l} a_{2m} a_{3n} \\ &= \epsilon_{lmn} a_{l1} a_{m2} a_{n3}. \end{aligned}$$

Furthermore:

$$\epsilon_{ijk} \det A = \epsilon_{lmn} a_{il} a_{jm} a_{kn}$$

$$\epsilon_{lmn} \det A = \epsilon_{ijk} a_{il} a_{jkm} a_{kn}$$

$$\det A = \frac{1}{3!} \epsilon_{ijk}\epsilon_{lmn} a_{il} a_{jm} a_{kn}.$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3. $\det A = \det A^T$
4. $\det(AB) = \det A \cdot \det B$

These also apply to columns. Finally:

$$\epsilon_{ijk}\epsilon_{lmn} \det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting $A = I$ yields:

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

Linear equations

Let $\boldsymbol{y} = A\boldsymbol{x}$. Then $x_i = A^{-1}_{ij} y_i$ with:

$$\begin{aligned} A^{-1}_{ij} &= \frac{1}{2} \frac{1}{\det A} \epsilon_{imn}\epsilon_{jpq} a_{pm} a_{qn} \\ &= \frac{1}{\det A} C^T_{ij} \end{aligned}$$

where C is the cofactor matrix of A.

Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L)\ell_{ip}\ell_{jq} T_{pq}$$

where $(L)_{ij} = \ell_{ij}$ and $\det L = \pm 1$.

Pseudovectors are rank 1 pseudotensors.

Invariant tensors

Tensor T is invariant or isotropic if:

$$T_{ijk\ldots} = \ell_{i\alpha}\ell_{j\beta}\ell_{k\gamma} \cdots T_{\alpha\beta\gamma\ldots}$$

for every orthogonal matrix L.

- If a_{ij} is a rank 2 invariant tensor then $a_{ij} = \lambda\delta_{ij}$.
- The most general rank 3 invariant pseudotensor is $a_{ijk} = \lambda\epsilon_{ijk}$. There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

Rotation tensors

The clockwise rotation of position vector $\boldsymbol{x}$ to $\boldsymbol{y}$ about unit vector $\hat{\boldsymbol{n}}$ is given by:

$$y_i = R_{ij}(\theta, \hat{\boldsymbol{n}}) x_j$$

$$R_{ij}(\theta, \hat{\boldsymbol{n}}) = \delta_{ij}\cos\theta + (1-\cos\theta) n_i n_j - \epsilon_{ijk} n_k \sin\theta$$

and is the rotation tensor.

Reflections and inversions

The reflection of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ in plane with unit vector $\hat{\boldsymbol{n}}$ is:

$$y_i = \sigma_{ij} x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ is given by $\boldsymbol{y} = -\boldsymbol{x}$ and is defined as:

$$y_i = P_{ij} x_j$$

$$P_{ij} = \delta_{ij}.$$

Projections

We define P to be a parallel projection operator to vector $\boldsymbol{u}$ if:

$$P\boldsymbol{u} = \boldsymbol{u} \text{ and } P\boldsymbol{v} = \boldsymbol{0}$$

where $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{0}$. Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define Q to be an orthogonal projection to vector $\boldsymbol{u}$ if:

$$Q\boldsymbol{u} = \boldsymbol{0} \text{ and } Q\boldsymbol{v} = \boldsymbol{v}.$$

Here $Q = I - P$.

Inertia tensors

Let $\boldsymbol{L}$ denote the angular momentum of a rigid body about the origin of mass m, volume V and density ρ at position $\boldsymbol{r}$ with velocity $\boldsymbol{v}$. Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2\delta_{ij} - x_i x_j)\mathrm{d}V$$

where $I_{ij}(O)$ is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2} I_{ij}\omega_i\omega_j = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{\omega}.$$

Parallel axis theorem

Consider the same rigid body now with centre of mass G and let $\overrightarrow{OG} = \boldsymbol{R}$. Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2\delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\boldsymbol{r}')\mathrm{d}V'.$$

Diagonalisation

Let $\boldsymbol{L} = I_{ij}\omega_j$ where I_{ij} is a rank 2 tensor and let $\boldsymbol{L} = \lambda\boldsymbol{\omega}$. Then:

$$(I_{ij} - \lambda\delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda\delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for $P = \det I$, $Q = \frac{1}{2}[(\operatorname{tr} I)^2 - \operatorname{tr}(I^2)]$ and $R = \operatorname{tr} I$ given tensor I.

Real symmetric tensors

Let rank 2 real symmetric tensor T be diagonalisable with real eigenvalues $\lambda^{(i)}$ and orthonormal eigenvectors $\boldsymbol{\ell}^{(i)}$ where $i = 1, 2, 3$. Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that $\det L = +1$ which transforms frame $S \to S'$.

Then since $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$:

$$\begin{aligned} T'_{ij} &= \ell_{ip}\ell_{jq}T_{pq} \\ &= \lambda^{(i)}\delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ij}. \end{aligned}$$

Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

and is f expanded about $x = a$.

Trignometric expansions are in radians!

$$\begin{aligned} \therefore f(x+a) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) a^n \\ &= \exp\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) \end{aligned}$$

Then for three dimensions:

$$\begin{aligned} \phi(\boldsymbol{r}+\boldsymbol{a}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)^n \phi(\boldsymbol{r}) \\ &= \exp(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)\phi(\boldsymbol{r}). \end{aligned}$$

Curvilinear coordinates

Let x_i denote Cartesian coordinates and u_i denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \to (u_1, u_2, u_3)$$

where each $u_i = u_i(x_1, x_2, x_3)$ and:

$$\begin{aligned} \boldsymbol{r} &= x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \\ &= u_1\boldsymbol{e}_{u_1} + u_2\boldsymbol{e}_{u_2} + u_3\boldsymbol{e}_{u_3}. \end{aligned}$$

Scale factors

Let $u_1 \to u_1 + \mathrm{d}u_1$ in $\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, u_3)$. Then $\mathrm{d}\boldsymbol{r}$ in $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 := h_1\boldsymbol{e}_1\mathrm{d}u_1.$$

h_1 is the scale factor of unit vector $\boldsymbol{e}_1$:

$$h_1 = \left|\frac{\partial\boldsymbol{r}}{\partial u_1}\right| \text{ and } \boldsymbol{e}_1 = \frac{1}{h_1}\frac{\partial\boldsymbol{r}}{\partial u_1}.$$

If $\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$ then u_i is an **orthogonal** curvilinear coordinate system.

Vector and arc length

The vector length $\mathrm{d}\boldsymbol{r}$ of $\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3} h_i \mathrm{d}u_i \boldsymbol{e}_i$$

where $u_i \to u_i + \mathrm{d}u_i$ for $\forall i = 1, 2, 3$.

Then the arc length $\mathrm{d}s$ is defined as:

$$\begin{aligned} (\mathrm{d}s)^2 &= \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} \\ &= g_{ij}\mathrm{d}u_i\mathrm{d}u_j \end{aligned}$$

where g_{ij} is the metric tensor:

$$\begin{aligned} g_{ij} = g_{ji} &= \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \\ &= h_i h_j(\boldsymbol{e}_i\cdot\boldsymbol{e}_j). \end{aligned}$$

Area and volume

Let $\mathrm{d}\boldsymbol{r}_i = h_i\boldsymbol{e}_i\mathrm{d}u_i$ denote vector length when $u_i \to u_i + \mathrm{d}u_i$. (**No** sum!)

The infinitesimal vector area or **surface element** formed by $\mathrm{d}\boldsymbol{r}_1$ and $\mathrm{d}\boldsymbol{r}_2$ is:

$$\mathrm{d}\boldsymbol{S} = (h_1\mathrm{d}\boldsymbol{u}_1\boldsymbol{e}_1)\times(h_2\mathrm{d}\boldsymbol{u}_2\boldsymbol{e}_2).$$

Similarly the infinitesimal volume formed by edges $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$ is:

$$\begin{aligned} \mathrm{d}V &= |(\mathrm{d}\boldsymbol{r}_1\times\mathrm{d}\boldsymbol{r}_2)\cdot\mathrm{d}\boldsymbol{r}_3| \\ &= \sqrt{g}\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 \end{aligned}$$

where $g = \det(g_{ij})$.

Cylindrical coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$ where ρ represents the radial distance from the origin and ϕ is the anticlockwise rotation angle on the x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_1 + \rho\sin\phi\boldsymbol{e}_2 + z\boldsymbol{e}_3$$

$$h_\rho = 1, \quad \boldsymbol{e}_\rho = \cos\phi\boldsymbol{e}_1 + \sin\phi\boldsymbol{e}_2$$

$$h_\phi = \rho, \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

$$h_z = 1, \quad \boldsymbol{e}_z = \boldsymbol{e}_3$$

and forms an orthogonal set.

Spherical coordinates

$(u_1, u_2, u_3) = (r, \theta, \phi)$ where θ represents the clockwise rotation angle in y-z plane and ϕ the anticlockwise rotation angle in x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = r\sin\theta\cos\phi\boldsymbol{e}_1 + r\sin\theta\sin\phi\boldsymbol{e}_2 + r\cos\theta\boldsymbol{e}_3$$

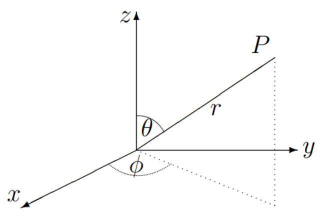
$$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$$

$$\boldsymbol{e}_r = \sin\theta\cos\phi\boldsymbol{e}_1 + \sin\theta\sin\phi\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\boldsymbol{e}_1 + \cos\theta\sin\phi\boldsymbol{e}_2 - \sin\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

and also forms an orthogonal set.



The inverse relations are given by:

$$\boldsymbol{e}_1 = \sin\theta\cos\phi\boldsymbol{e}_r + \cos\theta\cos\phi\boldsymbol{e}_\theta - \sin\phi\boldsymbol{e}_\phi$$

$$\boldsymbol{e}_2 = \sin\theta\sin\phi\boldsymbol{e}_r + \cos\theta\sin\phi\boldsymbol{e}_\theta + \cos\phi\boldsymbol{e}_\phi$$

$$\boldsymbol{e}_3 = \cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta.$$

We notice that the transformation is an orthogonal matrix – its inverse is simply its transpose.

Gradient

The gradient of a scalar field $f(\boldsymbol{r})$ is:

$$\mathrm{d}f(\boldsymbol{r}) := \boldsymbol{\nabla} f(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}$$

when $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r} \implies f \to f + \mathrm{d}f$. Taking the total differential of f yields:

$$\boldsymbol{\nabla} f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i$$

where $\{\boldsymbol{e}_i\}$ is orthogonal.

Divergence

The divergence of a vector field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} := \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

for surface δS bounds infinitesimal δV.
In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 F_2 h_3) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right\}.$$

Curl

The curl of a vector field $\boldsymbol{F}$ in the direction of unit vector $\hat{\boldsymbol{n}}$ is:

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) := \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

where curve δC encloses plane δS.
In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

Laplacian

The Laplacian of a scalar field f is:

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f)$$

and in orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla}^2 f = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right\}.$$

The Laplacian of a vector field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla}^2 \boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}).$$

Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x_i}\boldsymbol{e}_i$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_i}{\partial x_i}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}\boldsymbol{e}_i$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

The following operator acts on **both** scalar and vector fields:

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{F} = u_j \frac{\partial}{\partial x_j} F_i.$$

If ψ is a scalar field and $\boldsymbol{v}$ a vector field:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi) = \boldsymbol{0}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{0}$$

$$\boldsymbol{\nabla} \cdot (\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi \cdot \boldsymbol{v} + \psi\boldsymbol{\nabla} \cdot \boldsymbol{v}$$

$$\boldsymbol{\nabla} \times (\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi \times \boldsymbol{v} + \psi\boldsymbol{\nabla} \times \boldsymbol{v}.$$

Let $\boldsymbol{r} = x_i\boldsymbol{e}_i$ and $r = (x_i^2)^{1/2}$. Then:

- $\boldsymbol{\nabla} r = \dfrac{\boldsymbol{r}}{r}$ and $\boldsymbol{\nabla}\left(\dfrac{1}{r}\right) = -\dfrac{\boldsymbol{r}}{r^3}$
- $\boldsymbol{\nabla} r^n = n r^{n-2}\boldsymbol{r}$
- $\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3$ and $\boldsymbol{\nabla} \times \boldsymbol{r} = \boldsymbol{0}$
- $\boldsymbol{\nabla} \times (\boldsymbol{c} \times \boldsymbol{r}) = 2\boldsymbol{c}$
- $\boldsymbol{\nabla} \cdot (\boldsymbol{c} \times \boldsymbol{r}) = \boldsymbol{0}$ for constant $\boldsymbol{c}$.

Divergence theorem

Let surface S enclose volume V. Then:

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{F}\mathrm{d}V = \oint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

where $\boldsymbol{F}$ is a vector field.

Stokes' theorem

Let closed curve C bound open surface S and let $\boldsymbol{F}$ be a vector field. Then:

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

for C is traversed in anticlockwise sense.

Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x-a) := \begin{cases} \infty & x = a \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions this becomes:

$$\begin{aligned} \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0) &:= \delta(\boldsymbol{r} - \boldsymbol{r}_0) \\ &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \end{aligned}$$

where (x, y, z) are Cartesian coordinates.

In orthogonal curvilinear coordinates:

$$\begin{aligned} \delta(\boldsymbol{r} - \boldsymbol{a}) = &\frac{1}{h_1 h_2 h_3}\delta(u_1 - a_1) \\ &\cdot \delta(u_2 - a_2)\delta(u_3 - a_3). \end{aligned}$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

which yields:

$$x\delta(x) = 0 \text{ and } \delta(cx) = \frac{1}{|c|}\delta(x).$$

If simple solutions of $g(x) = 0$ are x_i:

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))\mathrm{d}x = \sum_i \frac{f(x_i)}{|g'(x_i)|}.$$

Coulomb's law

Consider charges q and q_1 at positions $\boldsymbol{r}$ and $\boldsymbol{r}_1$. The force on charge q at $\boldsymbol{r}$ due to charge q_1 at $\boldsymbol{r}_1$ is:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q q_1(\boldsymbol{r} - \boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|^3}$$

where $qq_1 > 0$ denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419 \times 10^{-12}\mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}.$$

Charge has units Coulombs (C) and an electron has charge -1.60218×10^{-19}C.

Electric fields

The electric field is induced by a charge distribution and defined in terms of the force on a small positive test charge q:

$$\boldsymbol{E}(\boldsymbol{r}) := \lim_{q\to 0}\frac{1}{q}\boldsymbol{F}.$$

Then for our two charges q and q_1:

$$\boldsymbol{F}_1(\boldsymbol{r}) = q\boldsymbol{E}_1(\boldsymbol{r})$$

where q_1 produces electric field $\boldsymbol{E}_1$.

$$\therefore \boldsymbol{E}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}$$

$$\therefore \phi_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\boldsymbol{r}-\boldsymbol{r}_1|}$$

Principle of superposition

For a set of charges q_i at position $\boldsymbol{r}_i$ the total electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}.$$

For object with **charge density** $\rho(\boldsymbol{r}')$ its overall electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'$$

where $\rho(\boldsymbol{r}')$ is charge divided by volume. The **type** of integral (surface or line) is dependent on the object in consideration.

Electrostatic Maxwell's equations

Because $\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right)$$

and therefore for all static electric fields:

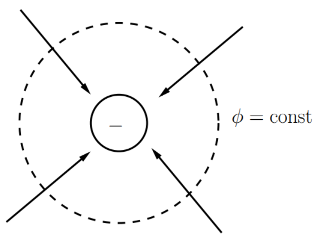
$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}.$$

$\boldsymbol{E}$ is a **conservative** vector field where its line integral is independent of path. Furthermore it may be written as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$$

for $\phi(\boldsymbol{r})$ is the potential of $\boldsymbol{E}$.

$$\therefore \phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'$$



The **potential difference** between two points A and B is the energy per unit charge needed to move a small charge q from A to B:

$$\begin{aligned} V_{A\to B} &= \lim_{q\to 0}\frac{1}{q}W_{A\to B} \\ &= -\frac{1}{q}\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} \\ &= \phi_B - \phi_A. \end{aligned}$$

A charge distribution $\rho(\boldsymbol{r}')$ in an external electric field has potential energy:

$$W = \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}')\mathrm{d}V'.$$

Because $\boldsymbol{\nabla}^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

Electric dipoles

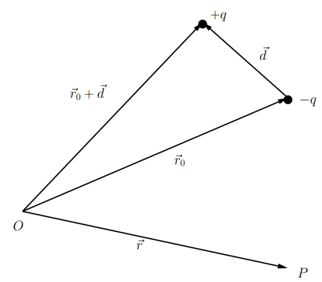
An electric dipole at $\boldsymbol{r}_0$ is defined as two charges $-q$ at $\boldsymbol{r}_0$ and $+q$ at $\boldsymbol{r}_0+\boldsymbol{d}$ which generates **dipole moment**:

$$\boldsymbol{p} = q\boldsymbol{d}$$

and in the dipole limit this is defined as:

$$\boldsymbol{p} := \lim_{\substack{q\to\infty \\ \boldsymbol{d}\to\boldsymbol{0}}} q\boldsymbol{d}$$

known as an ideal dipole.



The electrostatic potential generated by this ideal dipole at $\boldsymbol{r}_0$ is given by:

$$\begin{aligned} \phi(\boldsymbol{r}) &= \phi_q + \phi_{-q} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{d}|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\right) \\ &\approx \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3} \end{aligned}$$

for the first term is expanded in powers of $-\boldsymbol{d}$ about $\boldsymbol{r}-\boldsymbol{r}_0$.

The electric field generated is:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) = &\frac{1}{4\pi\epsilon_0}\left[-\frac{\boldsymbol{p}}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}\right. \\ &\left. + \frac{3\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^5}(\boldsymbol{r}-\boldsymbol{r}_0)\right]. \end{aligned}$$

If the ideal dipole is at the origin:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\boldsymbol{p}\cdot\boldsymbol{r}}{r^5}\boldsymbol{r} - \frac{\boldsymbol{p}}{r^3}\right).$$

Let ideal dipole moment $\boldsymbol{p}$ be parallel to the z-axis. Then in spherical coordinates (r,θ,χ), $\boldsymbol{r} = r\boldsymbol{e}_r$, $\boldsymbol{p} = p\boldsymbol{e}_z$ and:

$$\phi(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0}\left(\frac{2\cos\theta}{r^3}\boldsymbol{e}_r + \frac{\sin\theta}{r^3}\boldsymbol{e}_\theta\right).$$

Force, torque and energy

The **force** on a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} \boldsymbol{F} &= -q\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{E}_{ext}(\boldsymbol{r}+\boldsymbol{d}) \\ &\approx (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E}_{ext}(\boldsymbol{r}). \end{aligned}$$

The **torque** on a dipole at $\boldsymbol{r}$ about the axis $\boldsymbol{r}$ due to $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} \boldsymbol{G} &= \boldsymbol{\tau}_{-q} + \boldsymbol{\tau}_q \\ &= -q\boldsymbol{0}\times\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{d}\times\boldsymbol{E}_{ext}(\boldsymbol{r}+\boldsymbol{d}) \\ &\approx \boldsymbol{p}\times\boldsymbol{E}_{ext}(\boldsymbol{r}). \end{aligned}$$

The **energy** of a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned} W &= -q\phi_{ext}(\boldsymbol{r}) + q\phi_{ext}(\boldsymbol{r}+\boldsymbol{d}) \\ &\approx -\boldsymbol{p}\cdot\boldsymbol{E}_{ext}(\boldsymbol{r}) \end{aligned}$$

and $\boldsymbol{F} = -\boldsymbol{\nabla}W$.

Multipole expansion

Consider object with volume V and charge distribution $\rho(\boldsymbol{r}')$. Let origin be in the object. Then the potential at $\boldsymbol{r}$ is:

$$\begin{aligned} \phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' \\ &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{Q_{ij}x_ix_j}{2r^5}\right) \end{aligned}$$

where Q is the **total charge** in V:

$$Q = \int_V \rho(\boldsymbol{r}')\mathrm{d}V'$$

$\boldsymbol{p}$ the **dipole moment** about the origin:

$$\boldsymbol{p} = \int_V \boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V'$$

and Q_{ij} the **quadrupole tensor**:

$$Q_{ij} = \int_V \rho(\boldsymbol{r}')\left[3x_i'x_j' - (r')^2\delta_{ij}\right]\mathrm{d}V'.$$

If $Q \neq 0$ then in the far zone ($r \gg r_0$) the first term (monopole term) dominates.

If $Q = 0$ and $\boldsymbol{p} = \boldsymbol{0}$ then the third term (quadruple term) dominates in the far zone and etc.

Interaction energy

By expanding $\phi_{ext}(\boldsymbol{r})$ about $\boldsymbol{r}=\boldsymbol{0}$:

$$\begin{aligned} W &= \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}')\mathrm{d}V' \\ &= Q\phi_{ext}(\boldsymbol{0}) - \boldsymbol{p}\cdot\boldsymbol{E}_{ext}(\boldsymbol{0}) \\ &\quad - \frac{1}{6}Q_{ij}\frac{\partial\big(\boldsymbol{E}_{ext}(\boldsymbol{0})\big)_i}{\partial x_j} + \dots \end{aligned}$$

and is the potential energy of a charge distribution $\rho(\boldsymbol{r})$ in $\boldsymbol{E}_{ext}$.

Gauss' law

For object with charge distribution $\rho(\boldsymbol{r}')$ and volume V enclosed by surface S:

$$\int_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is total charge enclosed by V:

$$Q_{enc} = \int_V \rho(\boldsymbol{r}')\mathrm{d}V'$$

and is useful for symmetric problems.

Boundaries in electrostatics

Let σ be the charge density of a surface separating electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$.

1. Normal component of electric field is discontinuous across surface by:

$$\hat{\boldsymbol{n}}\cdot(\boldsymbol{E}_2-\boldsymbol{E}_1) = \frac{\sigma}{\epsilon_0}.$$

2. Tangential component of electric field is continuous across surface:

$$\boldsymbol{E}_{\parallel} := \hat{\boldsymbol{n}}\times\boldsymbol{E}_1 = \hat{\boldsymbol{n}}\times\boldsymbol{E}_2.$$

Conductors

Conductors have surplus electrons that can move freely when an electric field is applied. **In electrostatics**:

1. Conductors are in equilibrium, all charges are at rest and reside on the surface of the conductor.

   Hence inside a conductor $\rho(\boldsymbol{r})=0$, $\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{0}$ and $\phi=$ constant.

2. An electric field is always normal to the surface of a conductor:

$$E_\perp = \frac{\sigma}{\epsilon_0} \text{ and } E_\parallel = 0.$$

The presence of an external electric field induces a charge distribution σ on the surface of our conductor. This changes the external electric field as it needs to be normal to the surface of the conductor.

Poisson's equation

Because $\boldsymbol{E}=-\boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho(\boldsymbol{r})/\epsilon_0$:

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

We can solve this by direct integration or using the **method of images**.

Given volume under consideration place fictitious charge outside the volume such that the system still satisfies Poisson's equation with boundary conditions.

This potential is our solution.

Electrostatic energy

The work needed to move point charge q from $\boldsymbol{r}_A$ to $\boldsymbol{r}_B$ in $\boldsymbol{E}(\boldsymbol{r})$ is:

$$W_{A\to B} = qV_{A\to B}.$$

Then $W_{\infty\to B} = q\phi(\boldsymbol{r}_B)$ since potential ϕ vanishes at infinity.

Generalising, the work needed to move a system of n charges q_i from infinity to $\boldsymbol{r}$ is a double sum with overcounting as each charge contributes to the electric field:

$$W_e = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i,j(i\neq j)}^{n}\frac{q_iq_j}{|\boldsymbol{r}_j-\boldsymbol{r}_i|}.$$

Furthermore the energy needed to move a continuous charge distribution $\rho(\boldsymbol{r}')$ from infinity to position $\boldsymbol{r}$ is:

$$\begin{aligned} W_e &= \frac{1}{2}\int_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V \\ &= \frac{\epsilon_0}{2}\int_V |\boldsymbol{E}(\boldsymbol{r})|^2\mathrm{d}V. \end{aligned}$$

Capacitors

A capacitor is formed by two conductors 1 and 2 with equal and opposite charges Q and $-Q$. The **capacitance** ($1\mathrm{CV}^{-1}$) of a capacitor is defined as:

$$C := \frac{Q}{V}$$

where $Q=\sigma A$ for A is the surface area of one conductor and potential difference $V=\phi_1-\phi_2$ from the conductors.

The energy stored in a capacitor is the amount of work done to move charge across the two conductors. So to move charge $\mathrm{d}q$ from conductor with $+q$:

$$\mathrm{d}W = \left(\frac{q}{C}\right)\mathrm{d}q$$

and integrating this up to Q gives:

$$W = \frac{1}{2}\frac{Q^2}{C}.$$

Currents

An elementary current is generated by a charge q moving at velocity $\boldsymbol{v}$.

The **bulk current density** is:

$$\boldsymbol{J}(\boldsymbol{r}) := \rho(\boldsymbol{r})\boldsymbol{v}$$

for $\rho(\boldsymbol{r})$ is the volume charge density.

The **surface current density** is:

$$\boldsymbol{K}(\boldsymbol{r}) := \sigma(\boldsymbol{r})\boldsymbol{v}$$

for $\sigma(\boldsymbol{r})$ is the surface charge density.

The **line charge density** is:

$$\boldsymbol{I}(\boldsymbol{r}) := \lambda(\boldsymbol{r})\boldsymbol{v}$$

for $\lambda(\boldsymbol{r})$ is the line charge density.

The infinitesimal **current element** is:

$$\mathrm{d}\boldsymbol{I}(\boldsymbol{r}) := \begin{cases} \boldsymbol{J}(\boldsymbol{r})\mathrm{d}V & \text{volume current} \\ \boldsymbol{K}(\boldsymbol{r})\mathrm{d}S & \text{surface current} \\ \boldsymbol{I}(\boldsymbol{r})\mathrm{d}r & \text{line current} \end{cases}$$

Current I has units Ampères (A) but the infinitesimal current element $\mathrm{d}\boldsymbol{I}(\boldsymbol{r})$ has units of current/volume, etc.

Note: $1\mathrm{A} := 1\mathrm{Cs}^{-1}$

Consider volume V bounded by surface S with total charge Q. Because the total charge is conserved:

$$\frac{\partial\rho(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r},t) = 0$$

$$Q = \int_V \rho(\boldsymbol{r},t)\mathrm{d}V.$$

An electric field can induce a current in a conductor, via **Ohm's law**:

$$\boldsymbol{J} \approx \sigma_{ij}E_j$$

where σ_{ij} is the conductivity tensor.

1. Perfect conductors:
   $\sigma\to\infty$ and $\boldsymbol{E}=\boldsymbol{0}$.

2. Insulators: $\sigma=0$.

The **electromotive force** (emf) is:

$$\begin{aligned} \mathcal{E}_{1\to2} &:= \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r} \\ &= \phi(\boldsymbol{r}_1)-\phi(\boldsymbol{r}_2) \end{aligned}$$

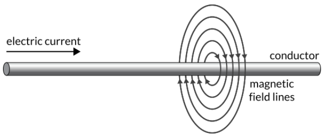
since $\boldsymbol{E}=-\boldsymbol{\nabla}\phi$. (static case)

Biot-Savart law

The magnetic field at $\boldsymbol{r}$ generated by a static current loop carrying current I is:

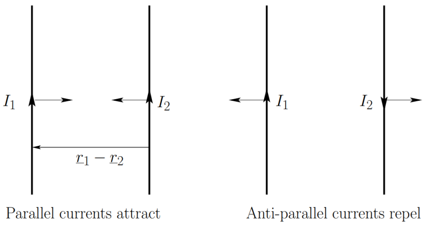
$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\oint_C \frac{I\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

and is the right grip rule.



Magnetic fields have units Teslas (T), where $\mathrm{NC}^{-1}\mathrm{m}^{-1}\mathrm{s} = \mathrm{T}$ and:

$$\mu_0 = 1.25664\cdots \times 10^{-6}\mathrm{NA}^{-2}.$$



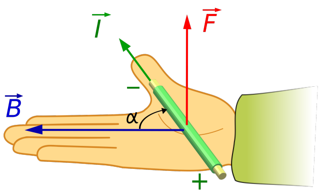
No force is induced from perpendicular currents.

Lozentz force

Through physical experiments the **force density** on a charge distribution $\rho(\boldsymbol{r})$ in electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ is:

$$\boldsymbol{f} := \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}$$

where $\boldsymbol{J}$ is the **current density** and integrating yields the **right hand rule**.



Current is the movement of charge.

For point charge q at $\boldsymbol{r}'$ with velocity $\boldsymbol{v}$ in $\boldsymbol{E}$ and $\boldsymbol{B}$ its net force is the integral of the force density over volume V:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

since $\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}-\boldsymbol{r}')$ and $\boldsymbol{J} = \rho(\boldsymbol{r})\boldsymbol{v}$.

Magnetostatic Maxwell's equations

Because $\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$:

$$\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi}\int_V \boldsymbol{J}(\boldsymbol{r}')\times\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\mathrm{d}V'$$

therefore $\boldsymbol{\nabla}\cdot\boldsymbol{B} = \boldsymbol{0}$ due to:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{\nabla}\times\boldsymbol{u})\cdot\boldsymbol{v} - (\boldsymbol{\nabla}\times\boldsymbol{v})\cdot\boldsymbol{u}.$$

This implies that magnetic fields always form **closed loops** — there are no sources or sinks for magnetic fields.

Similarly using the following identity:

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{u}\\ - \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\end{aligned}$$

and $\boldsymbol{\nabla}^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')$:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J}.$$

Ampère's law

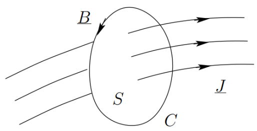
Using the divergence theorem, there is no magnetic flux through a **closed** surface:

$$\oint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0.$$

From Stokes' theorem:

$$\oint_C \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{r} = \mu_0 I_{enc}$$

which implies that the circulation of a magnetic field $\boldsymbol{B}$ around a closed loop C is proportional to the total current I that passes through the enclosed surface.



Boundaries in magnetostatics

Let conductor with current density $\boldsymbol{K}$ separate magnetic fields $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.

1. Normal component of magnetic field is continuous across surface:

$$\boldsymbol{B}_\perp := \boldsymbol{B}_1\cdot\hat{\boldsymbol{n}} = \boldsymbol{B}_2\cdot\hat{\boldsymbol{n}}.$$

2. Tangential component of magnetic field is discontinuous across surface:

$$\hat{\boldsymbol{n}}\times(\boldsymbol{B}_2 - \boldsymbol{B}_1) = \mu_0\boldsymbol{K}.$$

Magnetic vector potentials

Because $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$:

$$\exists\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$$

known as the **vector potential**. Then:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{B} &= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A}\\ &= \mu_0\boldsymbol{J}\end{aligned}$$

$$\therefore \boldsymbol{\nabla}^2\boldsymbol{A} = -\mu_0\boldsymbol{J}$$

where we set $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ and solution:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'$$

and boundary condition $\lim_{r\to\infty}\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{0}$.

Magnetic dipoles

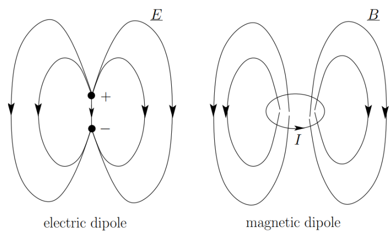
The vector potential for a current loop positioned at the origin in the far zone is:

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\oint_C \frac{I\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\\ &= \frac{\mu_0}{4\pi}\oint_C \frac{1}{r}\left(1 + \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2} + \dots\right)\mathrm{d}\boldsymbol{r}'\\ &\approx \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}\end{aligned}$$

where $\boldsymbol{m}$ is the magnetic dipole moment:

$$\begin{aligned}\boldsymbol{m} &= I\int_S \mathrm{d}\boldsymbol{S}\\ &= \frac{1}{2}\oint_C \boldsymbol{r}\times\mathrm{d}\boldsymbol{I}\end{aligned}$$

via Stokes' theorem.



Force and torque

The Lorentz force on a current loop due to an external magnetic field $\boldsymbol{B}$ is:

$$\begin{aligned}\boldsymbol{F} &= I\oint_C \mathrm{d}\boldsymbol{r}'\times\boldsymbol{B}(\boldsymbol{r}')\\ &= \boldsymbol{\nabla}(\boldsymbol{m}\cdot\boldsymbol{B}).\end{aligned}$$

The torque on a current loop due to an external magnetic field $\boldsymbol{B}$ is:

$$\begin{aligned}\boldsymbol{G} &= \oint_C \boldsymbol{r}'\times[I\mathrm{d}\boldsymbol{r}'\times\boldsymbol{B}(\boldsymbol{r}')]\\ &= \boldsymbol{m}\times\boldsymbol{B}.\end{aligned}$$

Motional electromotive force

The electromotive force (emf) is the work needed for unit point charge to circulate around a conductor loop:

$$\begin{aligned}\mathcal{E} &= \oint_C \boldsymbol{f}\cdot\mathrm{d}\boldsymbol{r}\\ &= \oint_C (\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{r}\end{aligned}$$

for $\boldsymbol{v}$ is the velocity of the charge and $\boldsymbol{f}$ is the force density on point charge.

Magnetic induction

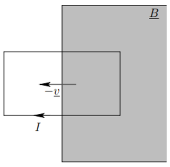
Faraday's law of induction states that a **change** in magnetic flux Φ induces an emf in a conductor loop:

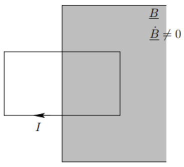
$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

where **magnetic flux** is defined as the total magnetic field passing through the region bounded by the conductor loop:

$$\begin{aligned}\Phi &= \int_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}\\ &= \boldsymbol{B}\cdot\boldsymbol{S} \text{ if constant } \boldsymbol{B}\end{aligned}$$

and any surface S enclosing the region.





For static charges in a conductor loop C with time dependent magnetic field:

$$\mathcal{E} = \oint_C \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S}$$

and using Stokes' theorem:

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

In words, a time dependent magnetic field **always accompanies** a spatial and time dependent electric field.

Galilean relativity

If the velocity of frame S' in S is $\boldsymbol{v}$:

$$\boldsymbol{r'} = \boldsymbol{r} - \boldsymbol{v}t$$

$$\mathrm{d}t' = \mathrm{d}t$$

for point P has position $\boldsymbol{r}$ in frame S and position $\boldsymbol{r'}$ in frame S'.

Let circuit C be in motion in frame S with velocity $\boldsymbol{v}$ with respect to $\boldsymbol{B}(\boldsymbol{r},t)$. Then let C be stationary in frame S'. Since the electromotive force generated is the same regardless of frames, in frame S:

$$\mathcal{E} = \oint_C (\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{r}$$

and as $\boldsymbol{v} = \boldsymbol{0}$ in frame S':

$$\mathcal{E} = \oint_C \boldsymbol{E}'\cdot\mathrm{d}\boldsymbol{r}.$$

Equating the two statements:

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$$

but only applies at $v \ll c$.

Mutual and self inductance

Consider conductor loops 1 and 2 with current I_1 and I_2. The magnetic vector potential generated by loop 1 is:

$$\boldsymbol{A}_1(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\oint_{C_1}\frac{I_1\mathrm{d}\boldsymbol{r}_1'}{|\boldsymbol{r}-\boldsymbol{r}_1'|}$$

and the magnetic flux in loop 2 is:

$$\begin{aligned}\Phi_{2\leftarrow 1} &= \frac{\mu_0 I_1}{4\pi}\oint_{C_2}\mathrm{d}\boldsymbol{r}_2'\cdot\oint_{C_1}\frac{\mathrm{d}\boldsymbol{r}_1'}{|\boldsymbol{r}_2'-\boldsymbol{r}_1'|}\\ &= M_{21}I_1\end{aligned}$$

where M_{21} is the Neumann's formula for mutual induction. Then for two loops:

$$\begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2\end{pmatrix} = -\begin{pmatrix}L_1 & M\\ M & L_2\end{pmatrix}\begin{pmatrix}\frac{\mathrm{d}I_1}{\mathrm{d}t}\\ \frac{\mathrm{d}I_2}{\mathrm{d}t}\end{pmatrix}$$

where $M = M_{21} = M_{12}$. L_1 and L_2 are the self-inductance for each loop.

formula for self inductance

M_{21} only equal when we have equal areas

Magnetic field energy

Consider an inductor generating field $\boldsymbol{B}$ with current I and self-inductance L:

$$\begin{aligned}\mathrm{d}W_m &= i\mathrm{d}t\cdot -\mathcal{E}\\ &= i\mathrm{d}t\cdot\frac{\mathrm{d}\Phi_i}{\mathrm{d}t}\\ &= iL\mathrm{d}i\end{aligned}$$

and since $\Phi_{1\leftarrow 1} = IL$ the energy stored our inductor is an integral over current:

$$W_m = \int_0^I \mathrm{d}W_m = \frac{\Phi_{1\leftarrow 1}^2}{2L} \text{ or } \frac{1}{2}\Phi_{1\leftarrow 1}I.$$

Because $\Phi_{1\leftarrow 1} = \int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$:

$$W_m = \frac{1}{2}\oint_C \boldsymbol{A}\cdot I\mathrm{d}\boldsymbol{r}$$

and generalising this to volume integrals:

$$W_m = \frac{1}{2}\int_V \boldsymbol{J}\cdot\boldsymbol{A}\mathrm{d}V$$

where $\boldsymbol{J}$ is the current density. Since:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{\nabla}\times\boldsymbol{u})\cdot\boldsymbol{v} - (\boldsymbol{\nabla}\times\boldsymbol{v})\cdot\boldsymbol{u}$$

and assuming that the field vanish at ∞:

$$W_m = \frac{1}{2\mu_0}\int_V B^2\mathrm{d}V$$

with **magnetic field density**:

$$w_m = \frac{1}{2\mu_0}B^2.$$

Circuits

Consider an LRC circuit with alternating current source and electromotive force:

$$\begin{aligned}\mathcal{E}_S &= V_S\cos(\phi+\omega t)\\ &= \mathrm{Re}[V_0 e^{i\omega t}]\end{aligned}$$

where $V_0 = V_S e^{i\phi}$. Equating emfs:

$$\begin{aligned}\mathcal{E}_S &= V_L + V_C + V_R\\ &= L\frac{\mathrm{d}I}{\mathrm{d}t} + V_C(t) + IR\end{aligned}$$

$$I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}$$

and due to linearity we assume that:

$$I(t) = \mathrm{Re}[I_0 e^{i\omega t}]$$

$$V_C(t) = \mathrm{Re}[V_{C_0}e^{i\omega t}].$$

After subsituting into our equations:

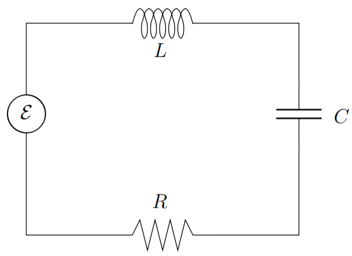
$$V_0 = i\omega L I_0 + V_{C_0} + RI_0$$

$$I_0 = i\omega C V_0 \text{ and } Z := \frac{V_0}{I_0}$$

which yields solutions:

$$\begin{aligned}I(t) &= \mathrm{Re}\left[\frac{V_0}{Z}e^{i\omega t}\right]\\ &= \frac{V_S\cos(\omega t+\phi-\psi)}{\left[R^2 + (\omega L - \frac{1}{\omega C})^2\right]^{1/2}}\end{aligned}$$

and $\psi = \arctan\left[\frac{\omega L - \frac{1}{\omega C}}{R}\right]$.



current/voltage in series/parallel circuits

voltage of specific components

Electromagnetic waves

From considering the capacitor paradox:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}.$$

Since in vacuum $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \mu_0\epsilon_0\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{E})$$

and using $\boldsymbol{\nabla}\times\boldsymbol{E} = \frac{\partial\boldsymbol{B}}{\partial t}$:

$$\begin{aligned}&\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{\nabla}^2\boldsymbol{B} = -\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\boldsymbol{B}\\ \implies &\boldsymbol{\nabla}^2\boldsymbol{B} = \mu_0\epsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2}.\end{aligned}$$

Similarly for our electric field $\boldsymbol{E}$:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \boldsymbol{\nabla}^2\boldsymbol{E} = -\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\boldsymbol{E}$$
$$\implies \boldsymbol{\nabla}^2\boldsymbol{E} = \mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}.$$

Let F be any component of fields $\boldsymbol{E}(\boldsymbol{r},t)$ or $\boldsymbol{B}(\boldsymbol{r},t)$. Then it satisfies the following:

$$\boldsymbol{\nabla}^2 F = \mu_0\epsilon_0\frac{\partial^2 F}{\partial t^2}.$$

Substituting solutions of form:

$$F(t,x,y,z) = f(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)$$
$$\implies |\boldsymbol{k}|^2 = \mu_0\epsilon_0\omega^2$$

and with phase velocity:

$$v_{phase} = \frac{\omega}{|\boldsymbol{k}|} = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$
$$= 3.10\times10^8\text{ms}^{-1} = c$$

which is the speed of light and implies that light is also an electromagnetic wave!

Lorentz transformations

It is then postulated that:

- The **speed of light is universal**. c is frame invariant and classically only propagate forwards in time.

In Minkowsky spacetime we have that:

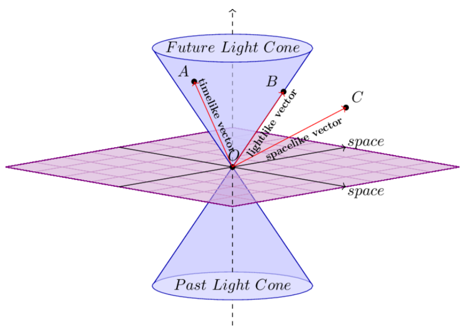
$$(\Delta S)^2 = (\Delta S')^2$$

between two frames S, S' and:

$$(\Delta S)^2 := (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

for events A and B, $\Delta x = x_B - x_A$, etc.

- $(\Delta S)^2 = 0$: light-like separated
- $(\Delta S)^2 > 0$: time-like separated
- $(\Delta S)^2 < 0$: space-like separated



A **boost** B_x in the $\boldsymbol{e}_x$ direction is defined:

$$\boldsymbol{x}' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\boldsymbol{x}$$

where $\boldsymbol{x} = (ct, x, y, z)^T$.

Generally we can relate reference frames S and S' by a composition of rotations and boosts. This forms a group, denoted by $\text{SO}(3) = \{R_x, R_y, R_z, B_x, B_y, B_z\}$.

Practically if frame S' is moving at $v\boldsymbol{e}_x$ with respect to frame S then:

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(-\beta ct + x)$$
$$y' = y \text{ and } z' = z$$

where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = v/c$.

Time and length

Consider object in rest in frame S', which is moving with respect to frame S. Then:

- lifetime in S: $\gamma\tau$
- length in S: ℓ_0/γ

for time τ and length ℓ_0 are its physical quantities in frame S'.

Electromagnetic energy

By considering the Lorentz force with:

$$\text{d}W = \boldsymbol{F}\cdot\text{d}\boldsymbol{r} = \boldsymbol{F}\cdot\boldsymbol{v}\text{d}t$$

and generalising to a charge distribution:

$$\frac{\text{d}W}{\text{d}t} = \int_V \boldsymbol{J}\cdot\boldsymbol{E}\,\text{d}^3\boldsymbol{r}.$$

Using Maxwell's equations:

$$\underbrace{\frac{\text{d}W}{\text{d}t}}_{\text{power}} + \int_{\partial V}\boldsymbol{S}\cdot\text{d}\boldsymbol{a} + \frac{\text{d}U_{em}}{\text{d}t} = 0$$

where U_{em} represents the total energy stored in the electric and magnetic fields:

$$U_{em} = \int_V \left(\frac{1}{\mu_0}|\boldsymbol{B}|^2 + \epsilon_0|\boldsymbol{E}|^2\right)\text{d}^3\boldsymbol{r}$$

and we define the **Poynting vector**:

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}.$$

Likewise its surface integral denotes the *rate* at which energy flows.

If $\int_{\partial V}\boldsymbol{S}\cdot\text{d}\boldsymbol{a} < 0$ then power flows **into** the bounded surface and vice versa.

Maxwell's stress tensor

Since Newton's second law states that:

$$\boldsymbol{F} = \frac{\text{d}}{\text{d}t}\boldsymbol{P}_{mech}$$

using the Lorentz force we have that:

$$\frac{\text{d}\boldsymbol{P}_{mech}}{\text{d}t} = -\frac{\text{d}}{\text{d}t}\int_V \epsilon_0\mu_0\boldsymbol{S}\,\text{d}^3\boldsymbol{r} + \int_{\partial V}\boldsymbol{T}\cdot\text{d}\boldsymbol{a}$$

where $\boldsymbol{T}$ is the **stress tensor**:

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}|\boldsymbol{E}|^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}|\boldsymbol{B}|^2\right).$$

Relativistic kinematics

Monochromatic plane waves

In *vacuum*, the Maxwell's equations may be recast into a classical wave equation:

$$\boldsymbol{\nabla}^2\{\boldsymbol{E},\boldsymbol{B}\} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\{\boldsymbol{E},\boldsymbol{B}\}$$

$$\mu_0\epsilon_0 = c^{-2}.$$

We then look for solutions of linear form:

$$\boldsymbol{E} = \text{Re}\left\{\boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}\right\}$$

$$\boldsymbol{B} = \text{Re}\left\{\boldsymbol{B}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}\right\}$$

known as **plane wave** solutions:

$$|\boldsymbol{k}| = \frac{\omega}{c}$$

from subsituting into our wave equation. Since $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$ we have that:

$$\boldsymbol{k}\cdot\boldsymbol{E} = 0, \qquad \boldsymbol{k}\cdot\boldsymbol{B} = 0,$$

$$\boldsymbol{k}\times\boldsymbol{E} = \omega\boldsymbol{B} \quad \text{and} \quad -\boldsymbol{k}\times\boldsymbol{B} = \frac{\omega}{c^2}\boldsymbol{E}.$$

i.e. that $(\hat{\boldsymbol{k}}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{B}})$ forms a right-handed *orthonormal basis* and:

$$\boldsymbol{E} = \left(a_1 e^{i\delta_1}\hat{\boldsymbol{\epsilon}}_1 + a_2 e^{i\delta_2}\hat{\boldsymbol{\epsilon}}_2\right)e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$$

where $a_1, a_2, \delta_1, \delta_2 \in \mathbb{R}$. The unit vectors $\hat{\boldsymbol{\epsilon}}_1$ and $\hat{\boldsymbol{\epsilon}}_2$ are perpendicular to each other.

Radiation

compute electric field potential

magnetic field vector potential

4-vectors