

# Honours Differential Equations

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# Contents

<b>1</b>	<b>ODE systems</b>	<b>4</b>
1.1	Integrating factors . . . . .	4
1.2	Change of variables . . . . .	4
1.3	Existence and uniqueness for IVPs . . . . .	4
1.4	Homogeneous systems . . . . .	5
1.4.1	Unique eigenvalues . . . . .	5
1.4.2	Matrix exponentials . . . . .	5
1.4.3	Diagonalisation . . . . .	6
1.4.4	Generalised eigenvectors . . . . .	6
1.5	Non-homogeneous systems . . . . .	7
1.6	Critical points & linearisation . . . . .	8
1.7	Stability of critical points . . . . .	9
1.8	Lyapunov's theory and limit cycles . . . . .	9
<b>2</b>	<b>Fourier series</b>	<b>10</b>
2.1	Real Fourier series . . . . .	10
2.2	Complex Fourier series . . . . .	12
2.3	Parseval's theorem . . . . .	12
<b>3</b>	<b>PDEs</b>	<b>13</b>
3.1	Separation of variables . . . . .	13
3.2	Heat equation . . . . .	13
3.2.1	Standard boundary conditions . . . . .	13
3.2.2	Fixed boundary temperatures . . . . .	15
3.2.3	Insulated rod ends . . . . .	16
3.3	Wave equation . . . . .	17
3.3.1	Derivation . . . . .	17
3.3.2	Plucked string . . . . .	17
3.3.3	d'Alembert's solution . . . . .	18
3.3.4	General initial conditions . . . . .	19
3.4	Laplace's equation . . . . .	20
3.4.1	Rectangular boundary conditions . . . . .	20
3.4.2	Circular boundary conditions . . . . .	21
<b>4</b>	<b>Sturm-Liouville theory</b>	<b>23</b>
4.1	Regular S-L problems . . . . .	23
4.1.1	General 2nd order ODEs . . . . .	24
4.1.2	Lagrange's identity . . . . .	24
4.1.3	Series expansion . . . . .	25
4.1.4	General Parseval's identity for S-L problems . . . . .	25
4.2	Non-homogeneous S-L problems . . . . .	26
4.3	Non-homogeneous PDEs . . . . .	27
4.4	Singular S-L problems . . . . .	29

<b>5</b>	<b>Laplace transforms</b>	<b>30</b>
5.1	Properties . . . . .	30
5.1.1	Inversion formula . . . . .	30
5.1.2	Reduction of order . . . . .	30
5.1.3	Shifts, scaling and derivatives . . . . .	30
5.2	Applications . . . . .	31
5.2.1	Higher order ODEs . . . . .	31
5.2.2	Discontinuous source terms . . . . .	32
5.2.3	Impulse functions . . . . .	32
5.2.4	Convolutions . . . . .	33
5.3	Standard transforms . . . . .	35

# 1 ODE systems

## 1.1 Integrating factors

Consider linear DE of form

$$y' + P(x)y = Q(x)$$

The integrating factor for this DE is:

$$I(x) = \exp\left(\int P(x)dx\right)$$

and the solution to the linear DE is:

$$y(x) = \frac{1}{I(x)} \int Q(x)I(x)dx + \frac{\alpha}{I(x)}$$

where here  $\alpha$  is a constant.

## 1.2 Change of variables

For higher order differential equations of form

$$y^{(n)} = F(y, y', \dots, y^{(n-1)}, t),$$

consider **change of variables**  $x_{i+1} = y^{(i)}$  for  $i \in \{0, 1, \dots, n-1\}$ .

Taking derivatives with respect to time yields a first order matrix ODE system:

$$x'_j = F_j(t, x_1, \dots, x_n)$$

for  $j = 1, \dots, n$ . We either immediately write this as a matrix system or linearise near a critical point.

## 1.3 Existence and uniqueness for IVPs

An initial value problem (**IVP**) is defined as

$$\frac{dx}{dt} = f(x, t)$$

for **initial** condition  $x(t_0) = x_0$ . A solution  $x : I \rightarrow \mathbb{R}$  is a differentiable function that satisfies the IVP. Similarly for a first order system

$$x'_i = F_i(t, x_1, \dots, x_n)$$

to have a **unique** solution,  $F_i$  and  $\frac{\partial F_i}{\partial x_j}$  must be continuous in a region. Here  $i, j \in \{1, \dots, n\}$ . This is known as the Picard-Lindelöf theorem.

## 1.4 Homogeneous systems

### 1.4.1 Unique eigenvalues

Now consider  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is a  $n \times n$  matrix. Substituting  $\mathbf{x} = e^{rt}\boldsymbol{\xi}$  results in an eigenvalue problem:

$$(\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\xi}^{(i)} = \mathbf{0}$$

where  $i \in \{1, 2, \dots, n\}$ . Our general solution is then:

$$\begin{aligned}\mathbf{x}(t) &= \sum_{i=1}^n c_i e^{r_i t} \boldsymbol{\xi}^{(i)} \\ &= \sum_{i=1}^n c_i \mathbf{x}^{(i)} \\ &= \boldsymbol{\Psi}(t)\mathbf{c}\end{aligned}$$

where  $\boldsymbol{\Psi}(t)$  is our fundamental matrix satisfying  $\boldsymbol{\Psi}' = \mathbf{A}\boldsymbol{\Psi}$  and that:

$$\boldsymbol{\Psi}(t) = [\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}].$$

Furthermore if initial conditions  $\mathbf{x}(t_0) = \mathbf{x}_0$  are given we then have that:

$$\mathbf{c} = \boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}(t_0)$$

and

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}_0.$$

### 1.4.2 Matrix exponentials

We can also write our solutions as a matrix exponential, defined as such:

$$\begin{aligned}e^{\mathbf{A}t} &= \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} \\ &= \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \dots\end{aligned}$$

and since an exponential power series is infinitely differentiable:

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}.$$

Therefore it is then deduced that the solution to  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0).$$

and that  $e^{\mathbf{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$  where we are finding the coefficients to the general solution  $\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}$ .

### 1.4.3 Diagonalisation

Consider again  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  where  $\mathbf{A}$  is a  $n \times n$  matrix that is diagonalisable:

$$\mathbf{A}\boldsymbol{\xi}^{(i)} = r_i\boldsymbol{\xi}^{(i)}$$

$$\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$$

for here  $\mathbf{D}$  is our diagonal matrix containing our eigenvalues  $r_i$  and

$$\mathbf{T} = [\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}].$$

Then let  $\mathbf{x} = \mathbf{T}\mathbf{y}$ . After some algebra we have that:

$$\mathbf{y}' = \mathbf{D}\mathbf{y}$$

which have particular solutions  $\mathbf{y} = e^{r_i t} \mathbf{e}^{(i)}$  for  $i \in \{1, \dots, n\}$ .

Since our fundamental matrix with respect to  $\mathbf{y}$  is  $\mathbf{Q} = e^{\mathbf{D}t}$ , the fundamental matrix with respect to  $\mathbf{x}$  is:

$$\boldsymbol{\Psi}(t) = \mathbf{T}e^{\mathbf{D}t}$$

and we get an expression for the matrix exponential of  $\mathbf{A}$ :

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$$

where  $e^{\mathbf{D}t}$  is a diagonal matrix with entries  $e^{r_i t}$ .

### 1.4.4 Generalised eigenvectors

If eigenvalues do not generate enough eigenvectors to form  $n$  linearly independent solutions for a  $n \times n$  matrix  $\mathbf{A}$  we try the following ansatz:

$$\mathbf{x} = te^{r_i t}\boldsymbol{\xi} + e^{r_i t}\boldsymbol{\eta}$$

which gives

$$(\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\eta}^{(i)} = \boldsymbol{\xi}^{(i)}.$$

Therefore we end up with:

$$\mathbf{x}^{(1)} = e^{r_i t}\boldsymbol{\xi}$$

and

$$\mathbf{x}^{(2)} = te^{r_i t}\boldsymbol{\xi} + e^{r_i t}\boldsymbol{\eta}.$$

The process is algorithmic — we try again until it works.

## 1.5 Non-homogeneous systems

Consider non-homogeneous ODE system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}.$$

There are a couple of different approaches we can take to solve such a system.

- **Change of basis**

Let  $\mathbf{x} = \mathbf{T}\mathbf{y}$ , where  $\mathbf{T}$  is our eigenvector matrix from diagonalisation. So  $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$ , and after some algebra we obtain:

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$$

which can be solved by integrating factors. Finally revert back to  $\mathbf{x}$ .

- **Variation of parameters**

So  $\mathbf{x}_H = \mathbf{\Psi}\mathbf{c}$  solves the  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ , where  $\mathbf{c}$  is a constant vector.

We then assume that the solution to our non-homogeneous system takes the form:

$$\mathbf{x} = \mathbf{\Psi}\mathbf{u}$$

for here  $\mathbf{u} = \mathbf{u}(t)$ . We then get  $\mathbf{\Psi}\mathbf{u}' = \mathbf{g}$ , which can be solved by eliminating variables and integrating.

- **Method of undetermined coefficients**

Our non-homogeneous ODE system has solutions of form:

$$\mathbf{x} = \mathbf{x}_H + \mathbf{x}_p$$

Solving the homogeneous ODE gives us  $\mathbf{x}_H$ .

On the other hand we just need to find a **particular solution**  $\mathbf{x}_p$  that satisfies our non-homogeneous ODE. Then our solution is complete.

Whilst the fastest, this method is not guaranteed to work.

## 1.6 Critical points & linearisation

Consider non-linear ODE system

$$x' = F(x, y),$$

$$y' = G(x, y).$$

We define  $\mathbf{x}^0 = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$  as a **critical point** when  $F(\mathbf{x}^0) = G(\mathbf{x}^0) = 0$ .

Non-linear systems may then be linearised by Taylor expanding them around a critical point  $\mathbf{x}^0$ , and discarding higher order terms.

i.e. let  $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  where  $u_1 = x - x^0$  and  $u_2 = y - y^0$ .

$$\therefore u'_1 = x'$$

$$\approx F(x^0, y^0) + \left( \frac{\partial F}{\partial x} \right)_{x^0} (x - x^0) + \left( \frac{\partial F}{\partial y} \right)_{y^0} (y - y^0)$$

$$\therefore u'_2 = y'$$

$$\approx G(x^0, y^0) + \left( \frac{\partial G}{\partial x} \right)_{x^0} (x - x^0) + \left( \frac{\partial G}{\partial y} \right)_{y^0} (y - y^0)$$

Then we end up with the following linear system:

$$\mathbf{u}' = \mathbf{A}\mathbf{u}$$

where  $\mathbf{A} = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}_{\mathbf{x}=\mathbf{x}^0}$  and is a  $2 \times 2$  Jacobian matrix.

Our critical points  $\mathbf{x}^0$  may also be classified:

Eigenvalues	Critical Points	Stability
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asympt. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asympt. stable
$r_1, r_2 = \lambda \pm i\mu$ ( $\lambda > 0$ )	focus	unstable
$r_1, r_2 = \lambda \pm i\mu$ ( $\lambda < 0$ )	focus	asympt. stable
$r_1 = i\mu, r_2 = -i\mu$	center	stable

Linearisation preserves critical point behaviour **except** when eigenvalues are of  $r = \pm i\mu$  form, for which then classification is unknown.



## 1.7 Stability of critical points

**Stable** critical points  $\mathbf{x}^0$ : All solutions start and stay near  $\mathbf{x}^0$ .

$$\forall \epsilon > 0; \exists \delta > 0; \forall \mathbf{x}_{\text{solution}} \text{ to } \mathbf{x}' = \mathbf{F}(\mathbf{x}, t): \\ |\mathbf{x}(0) - \mathbf{x}^0| < \delta \implies |\mathbf{x}(t) - \mathbf{x}^0| < \epsilon \text{ for } \forall t \geq 0$$

**Attracting** critical points  $\mathbf{x}^0$ : All solutions tends to  $\mathbf{x}^0$ .

$$\forall \delta > 0 : |\mathbf{x}(0) - \mathbf{x}^0| < \delta \implies \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^0$$

**Asymptotically stable** critical points  $\mathbf{x}^0$ : Attracting **and** stable

## 1.8 Lyapunov's theory and limit cycles

In this section  $\dot{\mathbf{x}}$  means its first time derivative. So consider:

$$\dot{x} = F(x, y),$$

$$\dot{y} = G(x, y)$$

defined in  $\mathbb{R}^2$ . Let  $\mathbf{x}^0 \in D$  be a critical point.

The function  $E : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is a Lyapunov function where  $E(x^0, y^0) = 0$ , whenever it exists. Note that the time derivative of E is:

$$\frac{dE}{dt} = \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G$$

- Let  $E > 0$  for  $\forall \mathbf{x} \neq \mathbf{x}^0$ .  
If  $\frac{dE}{dt} \leq 0$  then  $\mathbf{x}^0$  is stable.  
If  $\frac{dE}{dt} < 0$  then  $\mathbf{x}^0$  is asymptotically stable.
- If every neighbourhood of  $\mathbf{x}^0$  contains  $\mathbf{x}^*$  such that  $E(\mathbf{x}^*) > 0$   
**and** if  $\frac{dE}{dt} > 0$  then  $\mathbf{x}^0$  is unstable.

Now **limit cycles** are defined as periodic solutions such that at least one other **non-closed trajectory** approaches the limit cycle as  $t \rightarrow \infty$ .

## 2 Fourier series

### 2.1 Real Fourier series

Let  $f(x)$  and  $f'(x)$  be **piecewise continuous** in  $[-L, L]$  with **period**  $2L$ .  
i.e.  $f(x) = f(x + 2L)$  for  $\forall x$ . Then the Fourier series for  $f(x)$  is

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

The **convergence** of our Fourier series depends on the continuity of  $f(x)$ :

- If  $f(x)$  is continuous then  $f_{FS}(x) = f(x)$ .
- If  $f(\alpha)$  is discontinuous then at point  $\alpha$  we have

$$f_{FS}(\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}.$$

Note that  $f(x)$  is continuous at  $\alpha$  if  $f(\alpha) = \lim_{x \rightarrow \alpha} f(x)$  and we define:

$$f(\alpha^-) = \lim_{x \rightarrow \alpha^-} f(x)$$

and

$$f(\alpha^+) = \lim_{x \rightarrow \alpha^+} f(x),$$

i.e. limits from left and right respectively. It is important to also note that the derivative of a Fourier series is **not necessarily convergent**.

Now consider  $S_n = \sin \frac{n\pi x}{L}$  and  $C_n = \cos \frac{n\pi x}{L}$ . We then have the following **orthogonality relations**:

$$\langle S_n, S_m \rangle = \langle C_n, C_m \rangle = L\delta_{mn}$$

$$\langle S_n, C_m \rangle = 0$$

where we define the inner product as:

$$\langle u(x), v(x) \rangle = \int_{-L}^L u(x)v(x)dx$$

and use the following trigonometric identities:

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

Now integrating the following expression:

$$\int_{-L}^L \left( \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right) dx = \int_{-L}^L f(x) dx$$

gives:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx.$$

Similarly:

$$a_n = \frac{1}{L} \int_{-L}^L \cos \frac{n\pi x}{L} f(x) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L \sin \frac{n\pi x}{L} f(x) dx.$$

Note that  $\delta_{mn}$  is the **Kronecker delta** and is defined as:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Furthermore notice that:

- The Fourier series of **even** functions contains only **cosines**.
- The Fourier series of **odd** functions contains only **sines**.

**Even** functions are defined  $f(-x) = f(x)$ , and:

$$\int_{-L}^L f_{\text{even}} dx = 2 \int_0^L f_{\text{even}} dx.$$

Similarly **odd** functions are defined  $f(-x) = -f(x)$ , and:

$$\int_{-L}^L f_{\text{odd}} dx = 0.$$

We can also extend a function defined in  $[0, L]$  in several ways:

1. Define even function:

$$g(x) = \begin{cases} f(x) & x \in [0, L] \\ f(-x) & x \in (-L, 0) \end{cases}$$

where its Fourier expansion is a cosine series.

2. Define odd function:

$$h(x) = \begin{cases} f(x) & x \in (0, L) \\ 0 & x = 0, L \\ -f(-x) & x \in (-L, 0) \end{cases}$$

where its Fourier expansion is a sine series.

## 2.2 Complex Fourier series

Expanding  $f(x)$  defined in  $[-L, L]$  with period  $2L$ :

$$f_{FS}(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi}{L}x\right)$$

using Euler's formula  $e^{i\theta} = \cos \theta + i \sin \theta$ . Its coefficients are:

$$c_n = \frac{1}{2L} \int_{-L}^L \exp\left(-\frac{in\pi}{L}x\right) f(x) dx$$

for  $\forall n \in \mathbb{Z}$  and:

$$c_n = \begin{cases} (a_n - ib_n)/2 & n > 0 \\ (a_0)/2 & n = 0 \\ (a_n + ib_n)/2 & n < 0. \end{cases}$$

Here we define the **inner product** for complex functions as

$$\langle f, g \rangle = \int_{\alpha}^{\beta} f^*(x)g(x)dx$$

where  $f^*(x)$  is the complex conjugate of  $f(x)$ . Then:

$$\begin{aligned} \left\langle \exp\left(\frac{im\pi}{L}x\right), \exp\left(\frac{in\pi}{L}x\right) \right\rangle &= \int_{-L}^L \exp\left(-\frac{im\pi}{L}x\right) \exp\left(\frac{in\pi}{L}x\right) dx \\ &= 2L\delta_{mn} \end{aligned}$$

and since  $f(x) = f_{FS}(x)$  we obtain our formula.

## 2.3 Parseval's theorem

Parseval's theorem states that given a periodic  $f(x)$  with convergent Fourier series we have that

$$\begin{aligned} \langle f, f \rangle &= \int_{-L}^L |f(x)|^2 dx \\ &= 2L \sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= L \left[ \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \right] \end{aligned}$$

and is derived by orthogonality.

### 3 PDEs

#### 3.1 Separation of variables

The only methodology considered is separation of variables. So for PDE:

$$\hat{D}[u(x_1, \dots, x_n)] = 0$$

where  $\hat{D}$  is our differential operator, we look for solutions of form:

$$u(x_1, \dots, x_n) = X_1(x_1) \cdots X_n(x_n)$$

subject to **initial** and **boundary** conditions.

#### 3.2 Heat equation

The heat equation is an equation of the following form:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

where  $\alpha^2$  is the thermal diffusivity constant.

##### 3.2.1 Standard boundary conditions

We firstly define:

- **Initial condition:**  $u(x, 0) = f(x)$  for  $0 \leq x \leq L$
- **Boundary condition:**  $u(0, t) = u(L, t) = 0$  for  $\forall t > 0$

Let solutions be of form:

$$\begin{aligned} u(x, t) &= X(x) \cdot T(t) \\ \therefore X(x) \cdot \dot{T}(t) &= \alpha^2 X''(x) \cdot T(t) \end{aligned}$$

Only a constant function may satisfy the first equality:

$$\frac{1}{\alpha^2} \frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda.$$

Writing this as two ODEs:

$$\begin{aligned} \dot{T} + \alpha^2 \lambda T &= 0 \\ X'' + \lambda X &= 0. \end{aligned}$$

The first one we can directly integrate, yielding:

$$T(t) = a_1 \exp(-\alpha^2 \lambda t).$$

The second ODE is a spring system, hence it has solution of form:

$$X(x) = b_1 \cos \lambda^{1/2} x + b_2 \sin \lambda^{1/2} x$$

where  $\lambda > 0$  and all other cases yields at best constant solutions.

However this time before proceeding we need to consider boundary conditions:

$$X(0) = X(L) = 0.$$

We find  $X(0) = b_1 = 0$  and  $X(L) = b_2 \sin \lambda^{1/2} L = 0$ .

The second equation implies that  $\lambda$  must of the following form:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for } \forall n \in \mathbb{N}$$

and so

$$X'' + \lambda X = 0 \implies X_n = b_2 \sin \lambda_n^{1/2} x.$$

Since  $\lambda$  is discretised:

$$\therefore T_n = a_1 \exp(-\alpha^2 \lambda_n t).$$

Our general solution must then be:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

where  $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ . Using initial condition  $u(x, 0) = f(x)$ :

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x$$

and we recognise this as an odd Fourier series with period  $2L$ .

$$\begin{aligned} \therefore \int_{-L}^L \sin(\lambda_n^{1/2} x) f(x) dx &= \sum_{n=1}^{\infty} c_n \int_{-L}^L \left(\sin(\lambda_n^{1/2} x)\right)^2 dx \\ \therefore 2 \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx &= c_n L \end{aligned}$$

The final step we split the integration range and use  $x = -x^*$ .

$$\therefore c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx$$

This is fine because we can extend  $u(x, t)$  via reflection for negative  $x$ .

### 3.2.2 Fixed boundary temperatures

We reconsider the heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

but now with the following **non-homogeneous boundary conditions**:

- $u(0, t) = T_1$
- $u(L, t) = T_2$
- $u(x, 0) = f(x)$

Physically our rod has fixed boundary temperatures, namely  $T_1$  and  $T_2$ .

We approach this problem with a change of variables:

$$v(x) = \lim_{t \rightarrow \infty} u(x, t).$$

Using our boundary conditions  $v$  must be linear:

$$\therefore v(x) = \frac{T_2 - T_1}{L}x + T_1$$

since  $v'' = 0$ ,  $v(0) = T_1$  and  $v(L) = T_2$ . We then deduce that:

$$u(x, t) = v(x) + \omega(x, t)$$

for  $\omega(x, t)$  satisfies the same heat equation with initial conditions:

- $\omega(0, t) = \omega(L, t) = 0$
- $\omega(x, 0) = f(x) - v(x)$

Recognising this as our initial example:

$$\omega(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

where again  $\lambda_n = \left(\frac{n\pi}{L}\right)^2$  and because  $\omega(x, t)$  is a Fourier series with period  $2L$ :

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) (f(x) - v(x)) dx.$$

### 3.2.3 Insulated rod ends

For the final example we consider:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

and define the following conditions:

- $\frac{\partial}{\partial x} u(0, t) = \frac{\partial}{\partial x} u(L, t) = 0$
- $u(x, 0) = f(x)$ .

Let  $u(x, t) = X(x)T(t)$  and we get:

$$X'' + \lambda X = 0$$

where  $X'(0) = X'(L) = 0$  and

$$\dot{T} + \alpha^2 \lambda T = 0.$$

If  $\lambda \leq 0$  we get at best constant solutions. So let  $\lambda > 0$  and we get solutions:

$$X(x) = c_1 \cos \lambda^{1/2} x + c_2 \sin \lambda^{1/2} x.$$

After substituting for boundary conditions we have that  $c_2 = 0$  and:

$$\lambda_n^2 = \frac{n\pi}{L}$$

for  $n \in \mathbb{N}$ . Solving the time component ODE yields:

$$T_n(t) = k \exp(-\alpha^2 \lambda_n t)$$

and hence we have general solutions of form:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \cos \lambda_n^{1/2} x.$$

Finally using our initial conditions:

$$c_n = \frac{2}{L} \int_0^L \cos(\lambda_n^{1/2} x) f(x) dx$$

where we define an even extension of  $u(x, t)$  in negative  $x$  to obtain this formula.



### 3.3 Wave equation

#### 3.3.1 Derivation

Consider string with length  $x$  and time dependent vertical displacement  $u(x, t)$ . Let  $\theta(x, t)$  denote the angle formed by the string. By Newton's second law:

$$\begin{aligned}\rho \Delta x \frac{\partial^2 u}{\partial t^2} &= T \left[ \sin \theta(x + \Delta x, t) - \sin \theta(x, t) \right] \\ &\approx T \left[ \theta(x + \Delta x, t) - \theta(x, t) \right] \\ &= T \frac{\partial \theta}{\partial x} \Delta x + \mathcal{O}(\Delta x^2) \\ &\approx T \frac{\partial^2 u}{\partial x^2} \Delta x\end{aligned}$$

where  $\rho$  is mass per unit length and  $T$  our string tension. If  $\Delta x \rightarrow 0$  then:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for  $c^2 = T/\rho$  and is our wave speed.

#### 3.3.2 Plucked string

So consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for  $t \geq 0$  and  $x \in [0, L]$  subject to the following conditions:

- $u(0, t) = u(L, t) = 0$
- $\frac{\partial}{\partial t} u(x, 0) = 0$
- $u(x, 0) = f(x)$ .

Let  $u(x, t) = X(x)T(t)$  and substituting gives:

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -\lambda$$

where this is two ODEs:

$$X'' + \lambda X = 0 \text{ with } X(0) = X(L) = 0$$

and

$$\ddot{T} + c^2 \lambda T = 0 \text{ with } \dot{T}(0) = 0.$$

The first spatial ODE we have solved before:

$$X_n(x) = b_2 \sin \lambda_n^{1/2} x \text{ with } \lambda_n = \frac{n^2 \pi^2}{L^2}.$$

The time component ODE is yet another SHM system:

$$T_n(t) = a_n \cos c\lambda_n^{1/2}t + b_n \sin c\lambda_n^{1/2}t$$

and hence by initial conditions we obtain:

$$T_n(t) = a_n \cos c\lambda_n^{1/2}t.$$

Combining our results the general solution takes the following form:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2}x \cos c\lambda_n^{1/2}t$$

where  $\lambda_n^{1/2} = \frac{n\pi}{L}$ . Since the initial string shape is  $u(x, 0) = f(x)$ :

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2}x) f(x) dx.$$

Our displacement  $u(x, t)$  may also be written as a sum of travelling waves:

$$\begin{aligned} u(x, t) &= \frac{1}{2} \sum_{n=1}^{\infty} c_n \left[ \sin \lambda_n^{1/2}(x - at) + \sin \lambda_n^{1/2}(x + at) \right] \\ &= \frac{1}{2} \left[ \underbrace{h(x - at)}_{+x \text{ direction}} + \underbrace{h(x + at)}_{-x \text{ direction}} \right]. \end{aligned}$$

### 3.3.3 d'Alembert's solution

Consider the 1D wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with change of variables  $\xi = x - ct$  and  $\eta = x + ct$ . Using the chain rule:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$u(\xi, \eta) = p(\eta) + q(\xi)$$

or that we have solutions:

$$u(x, t) = p(x + ct) + q(x - ct)$$

where functions  $p$  and  $q$  represent travelling waves in  $\pm x$  directions.

### 3.3.4 General initial conditions

Again consider:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

but now with the following conditions:

- $u(0, t) = u(L, t) = 0$
- $\frac{\partial}{\partial t} u(x, 0) = g(x)$
- $u(x, 0) = f(x)$ .

Using  $u(x, t) = X(x)T(t)$  we get two ODEs:

$$X'' + \lambda X = 0 \text{ with } X(0) = X(L) = 0$$

$$\ddot{T} + c^2 \lambda T = 0.$$

We have solved the spatial component ODE:

$$X_n(x) = b_n \sin \lambda_n^{1/2} x \text{ with } \lambda_n = \frac{n^2 \pi^2}{L^2}.$$

However now the time component ODE yields:

$$T_n(t) = a_n \cos c \lambda_n^{1/2} t + b_n \sin c \lambda_n^{1/2} t$$

which gives us the following general solution:

$$u(x, t) = \sum_{n=1}^{\infty} \sin \lambda_n^{1/2} x \left( a_n \cos c \lambda_n^{1/2} t + b_n \sin c \lambda_n^{1/2} t \right).$$

To find our coefficients we use our initial conditions:

$$a_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx$$

and

$$b_n = \frac{1}{c \lambda_n^{1/2}} \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) g(x) dx.$$

### 3.4 Laplace's equation

Laplace's equation takes the form  $\nabla^2 u = 0$ . In two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and we only consider boundary conditions. (Dirichlet conditions)

#### 3.4.1 Rectangular boundary conditions

We open with the following example:

- **Boundary for y:**  $u(x, 0) = u(x, b) = 0$
- **Boundary for x:**  $u(0, y) = 0$  and  $u(a, y) = f(y)$

where  $x \in [0, a]$  and  $y \in [0, b]$ . Begin by separation of variables:

$$\begin{aligned} u(x, y) &= X(x) \cdot Y(y) \\ \therefore \frac{X''}{X} &= -\frac{Y''}{Y} = \lambda. \end{aligned}$$

Recognising the previous statement as two ODEs:

$$X'' - \lambda X = 0 \text{ for } X(0) = 0$$

$$Y'' + \lambda Y = 0 \text{ for } Y(0) = Y(b) = 0$$

The second ODE we have already solved in the heat equation. It has solution:

$$Y_n = a_1 \sin(\lambda_n^{1/2} y) \text{ for } \lambda_n = \frac{n^2 \pi^2}{b^2}.$$

The first ODE has solutions of form:

$$X_n = a_2 \cosh(\lambda_n^{1/2} x) + a_3 \sinh(\lambda_n^{1/2} x)$$

where these are the hyperbolic functions:

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}).$$

Using our boundary condition  $X(0) = 0$  gives:

$$X_n = a_3 \sinh(\lambda_n^{1/2} x).$$

Now putting all of this together we get:

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} x) \sin(\lambda_n^{1/2} y)$$

To find coefficients  $c_n$  we use  $u(a, y) = f(y)$ .

$$\therefore f(y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} a) \sin(\lambda_n^{1/2} y)$$

Since we have a Fourier series with period  $2b$ :

$$\begin{aligned} \int_{-b}^b \sin(\lambda_n^{1/2} y) f(y) dy &= \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} a) \int_{-b}^b \sin(\lambda_n^{1/2} y) dy \\ &= c_n \sinh(\lambda_n^{1/2} a) \cdot b \end{aligned}$$

We can split the first integral to give us:

$$c_n = \frac{2}{b \sinh(\lambda_n^{1/2} a)} \int_0^b \sin(\lambda_n^{1/2} y) f(y) dy$$

where  $\lambda_n = \left(\frac{n\pi}{b}\right)^2$  and our solution is complete.

### 3.4.2 Circular boundary conditions

Now we solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

but with a circular boundary. In polar coordinates  $(r, \theta)$ :

- $u(a, \theta) = f(\theta)$
- $u(r, \theta)$  is bounded

where  $a$  is the radius of our circle and  $\theta \in [0, 2\pi]$ . Since  $u = u(x, y)$ :

$$u'_\theta = u'_x x'_\theta + u'_y y'_\theta$$

$$u''_{\theta\theta} = (u''_{xx} x'_\theta + u''_{xy} y'_\theta) x'_\theta + u'_x x''_{\theta\theta} + (u''_{yy} y'_\theta + u''_{xy} x'_\theta) y'_\theta + u'_y y''_{\theta\theta}$$

$$u'_r = u'_x x'_r + u'_y y'_r$$

$$u''_{rr} = (u''_{xx} x'_r + u''_{xy} y'_r) x'_r + u'_x x''_{rr} + (u''_{xy} x'_r + u''_{yy} y'_r) y'_r + u'_y y''_{rr}$$

and here we have used the chain rule.

Applying these derivatives we obtain the following equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0.$$

Using separation of variables:

$$u(r, \theta) = R(r)\Theta(\theta)$$

$$\therefore r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \lambda$$

where  $\lambda$  is our separation constant.

$$\therefore \ddot{\Theta} + \lambda\Theta = 0$$

$$\therefore r^2 R'' + rR' = \lambda R$$

## 4 Sturm-Liouville theory

### 4.1 Regular S-L problems

Sturm-Liouville theory is a general theory for 2nd order ODEs.

Consider the following eigenvalue ODE:

$$-\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

where  $r(x)$  is our weight function. We define the following boundary conditions:

1.  $a_1 y(0) + a_2 y'(0) = 0$
2.  $b_1 y(1) + b_2 y'(1) = 0$ .

This is a **regular Sturm-Liouville** problem, where  $p(x)$ ,  $p'(x)$ ,  $q(x)$ ,  $r(x)$  are continuous functions and  $p(x)$ ,  $r(x)$  are strictly positive functions for  $\forall x \in [0, 1]$ .

**Eigenvalues**  $\lambda_n$  yield **eigenfunctions**  $\phi_n(x)$  which are nontrivial solutions to our S-L problem. Important consequences include:

- Eigenvalues  $\lambda_n$  of a S-L problem are **real**.  
Furthermore each eigenvalue corresponds to one eigenfunction.
- Eigenfunctions  $\phi_n(x)$  are orthogonal:

$$\langle \phi_m, \phi_n \rangle = \int_0^1 r(x) \phi_m(x) \phi_n(x) dx = \delta_{mn}$$

in Hilbert space  $L^2([0, 1], r(x)dx)$ .

Note that our eigenfunctions are orthonormal and so we define:

$$\phi_n(x) = k_n y_n(x)$$

where  $k_n$  is our scale factor. Since  $\langle \phi_n, \phi_n \rangle = 1$ :

$$\therefore \int_0^1 r(x) k_n^2 y_n^2(x) dx = 1$$

and so we have that:

$$\begin{aligned} k_n &= \frac{1}{\sqrt{\langle y_n, y_n \rangle}} \\ &= \left( \int_0^1 r(x) y_n^2(x) dx \right)^{-1/2}. \end{aligned}$$

#### 4.1.1 General 2nd order ODEs

Consider the following general 2nd order eigenvalue ODE:

$$-P(x)\frac{d^2y}{dx^2} - \omega(x)\frac{dy}{dx} + q(x)y = \lambda r(x)y.$$

Multiply this by the following integrating factor:

$$F(x) = \exp\left[\int_0^x \frac{\omega(s) - p'(s)}{p(s)} ds\right]$$

yields an ODE of S-L form:

$$-\frac{d}{dx}\left[F(x)P(x)\frac{dy}{dx}\right] + F(x)q(x)y = \lambda F(x)r(x)y.$$

#### 4.1.2 Lagrange's identity

Our previous definition is motivated by the **Lagrange's identity**:

$$\begin{aligned}\langle \mathcal{L}[u], v \rangle - \langle u, \mathcal{L}[v] \rangle &= -\left[p(u'v^* - u(v^*)')\right]_0^1 \\ &= -\left[p(x)\left(\frac{du}{dx} \cdot v^* - u \cdot \frac{dv^*}{dx}\right)\right]_0^1\end{aligned}$$

where  $u = u(x)$ ,  $v = v(x)$  are complex functions and

$$\mathcal{L}[u] = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u.$$

Here the inner product is defined as

$$\langle u, v \rangle = \int_0^1 uv^* dx$$

and we have integrated by parts using the following identities:

$$[pu'v^*]' = (pu')'v^* + pu'(v^*)'$$

$$[pu(v^*)']' = (p(v^*)')'u + pu'(v^*)'.$$

For a S-L problem its properties follow from:

$$\langle \mathcal{L}[u], v \rangle = \langle u, \mathcal{L}[v] \rangle$$

where functions  $u$  and  $v$  satisfy its boundary conditions.



### 4.1.3 Series expansion

Now the set of orthonormal eigenfunctions  $\{\phi_n(x)\}$  from a S-L problem with boundary conditions may be used to expand function  $f(x)$ :

$$f_\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

for  $\forall x \in [0, 1]$ . Integrating this on both sides:

$$\begin{aligned} \int_0^1 r(x) \phi_m(x) f(x) dx &= \int_0^1 r(x) \phi_m(x) \sum_{n=1}^{\infty} c_n \phi_n(x) dx \\ &= \sum_{n=1}^{\infty} c_n \int_0^1 r(x) \phi_m(x) \phi_n(x) dx \\ &= \sum_{n=1}^{\infty} c_n \delta_{mn} \\ &= c_m \end{aligned}$$

and so we get a general formula for our coefficients:

$$c_n = \int_0^1 r(x) \phi_n(x) f(x) dx.$$

If  $f(x)$  and  $f'(x)$  are piecewise continuous on  $x \in [0, 1]$  then:

$$\forall x \in (0, 1); \sum_{n=1}^{\infty} c_n \phi_n(x) = \frac{f(x^-) + f(x^+)}{2}$$

where these are the left and right handed limits.

### 4.1.4 General Parseval's identity for S-L problems

We have that:

$$\int_0^1 r(x) [f(x)]^2 dx = \sum_{n=1}^{\infty} c_n^2$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

and

$$c_n = \int_0^1 r(x) \phi_n(x) f(x) dx.$$

## 4.2 Non-homogeneous S-L problems

Consider:

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

where  $f(x)$  is our non-homogeneous term, and that we define

$$\mathcal{L}[y] = -\frac{d}{dx} \left[ P(x) \frac{dy}{dx} \right] + q(x)y.$$

Assume that this ODE satisfies regular S-L boundary conditions.

Firstly we solve the corresponding homogeneous S-L problem:

$$\mathcal{L}[y] = \lambda r(x)y$$

for non-trivial  $\lambda_n$  and  $\phi_n(x)$ .

Now let the general solution to our non-homogeneous S-L problem be:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

and substituting this yields:

$$\begin{aligned} \sum_{n=1}^{\infty} b_n \mathcal{L}[\phi_n(x)] &= r(x) \sum_{n=1}^{\infty} b_n \lambda_n \phi_n(x) \\ &= \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x). \end{aligned}$$

Rearranging then gives:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} b_n (\lambda_n - \mu) \phi_n(x).$$

Now we can also expand  $\frac{f(x)}{r(x)}$  in terms of our eigenfunctions:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

with

$$\begin{aligned} c_n &= \int_0^1 r(x) \phi_n(x) \frac{f(x)}{r(x)} dx \\ &= \int_0^1 \phi_n(x) f(x) dx \end{aligned}$$

and so equating yields the following relation

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

### 4.3 Non-homogeneous PDEs

Consider the following non-homogeneous PDE:

$$r(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ p(x) \frac{\partial u}{\partial x} \right] - q(x)u(x, t) + F(x, t)$$

with boundary and initial conditions:

- $\frac{\partial}{\partial x} u(0, t) - h_1 u(0, t) = 0$
- $\frac{\partial}{\partial x} u(1, t) - h_2 u(1, t) = 0$
- $u(x, 0) = f(x).$

Firstly we solve the homogenous case via separation of variables. Let:

$$u(x, t) = X(x)T(t)$$

and after some algebra:

$$\frac{1}{rX} \left[ p'X' + pX'' - qX \right] = \frac{\dot{T}}{T} = -\lambda.$$

This yields two ODEs:

$$\begin{aligned} \dot{T} + \lambda T &= 0 \\ -[pX']' + qX &= \lambda rX \end{aligned}$$

with boundary conditions:

$$\begin{aligned} X'(0) - h_1 X(0) &= 0 \\ X'(1) - h_2 X(1) &= 0 \end{aligned}$$

Clearly our second ODE is of S-L form. We are now going to assume that this regular S-L problem has non-trivial  $\lambda_n$  and orthonormal eigenfunctions  $\phi_n(x)$ .

Let the general solution to our PDE be:

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x).$$

Substituting this into our PDE yields:

$$r(x) \sum_{n=1}^{\infty} \dot{b}_n(t) \phi_n(x) = \sum_{n=1}^{\infty} b_n(t) \left[ \left( p(x) \phi_n'(x) \right)' - q(x) \phi_n(x) \right] + F(x, t).$$

Now since we have a S-L problem:

$$\left(p(x)\phi'_n(x)\right)' - q(x)\phi_n(x) = -\lambda_n\phi_n(x)r(x)$$

and after dividing through our PDE by  $r(x)$  we get:

$$\sum_{n=1}^{\infty} \dot{b}_n(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t) \left[-\lambda_n\phi_n(x)\right] + \frac{F(x,t)}{r(x)}.$$

But we can also expand our non-homogeneous term as a sum:

$$\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x)$$

and its coefficients are:

$$\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)dx$$

in  $L^2([0,1], r(x))$ . Therefore after some algebra:

$$\sum_{n=1}^{\infty} \left[\dot{b}_n(t) + \lambda_n b_n(t) - \gamma_n(t)\right]\phi_n(x) = 0$$

and yields the following ODE:

$$\dot{b}_n(t) + \lambda_n b_n(t) - \gamma_n(t) = 0.$$

Solution is via integrating factors:

$$b_n(t) = e^{-\lambda_n t} \int_0^t \gamma_n(s)e^{\lambda_n s} ds + e^{-\lambda_n t} b_n(0).$$

Finally using  $u(x,0) = f(x)$ :

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) = f(x)$$

and we get

$$b_n(0) = \int_0^1 r(x)f(x)\phi_n(x)dx.$$

#### 4.4 Singular S-L problems

Consider the following ODE:

$$-\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

but now  $p(x)$ ,  $q(x)$  and  $r(x)$  are discontinuous at  $x = 0$  and/or  $x = 1$ .  
This is a **singular** S-L problem with boundary condition:

$$b_1 y(1) + b_2 y'(1) = 0$$

or

$$a_1 y(0) + a_2 y'(0) = 0.$$

Now singular S-L problems at  $x = 0$  may be self-adjoint or that they yield:

- $\lambda_n \in \mathbb{R}$  (Real eigenvalues)
- $\langle \phi_m, \phi_n \rangle = \delta_{mn}$  (Orthogonal eigenfunctions)

if they satisfy Lagrange's identity. Consider singular S-L problem at  $x = 0$ :

$$\begin{aligned} \int_{\epsilon}^1 (\mathcal{L}[u]v - u\mathcal{L}[v]) dx &= \left[ -p(x) \left( u'(x)v(x) - u(x)v'(x) \right) \right]_{\epsilon}^1 \\ &= p(\epsilon) \left( u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon) \right) \end{aligned}$$

and tends to zero if and only if:

$$\lim_{\epsilon \rightarrow \infty} \left[ p(\epsilon) \left( u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon) \right) \right] = 0,$$

where we define:

$$\mathcal{L}[u] = -\frac{d}{dx} \left[ p(x) \frac{du}{dx} \right] + q(x)u.$$

Therefore this is the condition for our singular S-L problem to have real eigenvalues and orthogonal eigenfunctions.

Similarly for singular S-L problem at  $x = 1$  it is self-adjoint if:

$$\lim_{\epsilon \rightarrow \infty} \left[ p(1-\epsilon) \left( u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon) \right) \right] = 0.$$

Finally whilst eigenvalues may be real they are not necessarily discrete!

## 5 Laplace transforms

So let  $f(t)$  be defined for  $t \in [0, \infty)$ . Its Laplace transform is:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt.$$

Let **functions of exponential order** be defined as:

$$\forall t \in [0, \infty); \exists A > 0 \text{ and } B \in \mathbb{R} : |f(t)| \leq Ae^{Bt}$$

where  $f$  is piecewise continuous and we denote such functions as  $f \in E$ .

For sufficiently large  $s$ , the Laplace transform  $f \in E$  converges.

### 5.1 Properties

#### 5.1.1 Inversion formula

Now let  $F(s) = \mathcal{L}[f(t)]$ . We have the following inversion formula:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds. \end{aligned}$$

#### 5.1.2 Reduction of order

Applying the Laplace transform to derivatives:

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

for  $\forall f, f' \in E$  and generalising this via induction gives:

$$\begin{aligned} \mathcal{L}[f^{(n)}(t)] &= s^n \mathcal{L}[f(t)] - s^{(n-1)} f^{(0)}(0) - s^{(n-2)} f^{(1)}(0) \\ &\quad - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0). \end{aligned}$$

#### 5.1.3 Shifts, scaling and derivatives

Let  $F(s) = \mathcal{L}[f(t)](s)$ . We then have that:

1. **s-shift:**

$$\mathcal{L}[e^{-ct} f(t)](s) = F(s + c)$$

where  $s + c > \gamma$ .

2. **t-shift:**

Let  $c \geq 0$  and  $f(t) = 0$  if  $t < 0$ . Then:

$$\mathcal{L}[f(t - c)](s) = e^{-sc} F(s).$$

Furthermore utilising the unit step function:

$$\mathcal{L}[g(t - c)u_c(t)](s) = e^{-sc} G(s)$$

where  $G(s) = \mathcal{L}[g(t)](s)$  and  $g(t)$  any normal function.

**3. s-derivative:**

$$\mathcal{L}[tf(t)](s) = -\frac{d}{ds}F(s).$$

We can also extend this to the  $n$ th derivative:

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s).$$

**4. scaling:**

$$\mathcal{L}[f(ct)] = \frac{1}{c}F\left(\frac{s}{c}\right)$$

and

$$\frac{1}{c}\mathcal{L}\left[f\left(\frac{t}{c}\right)\right] = F(cs)$$

where  $c > 0$ .

**5.2 Applications****5.2.1 Higher order ODEs**

So consider the following  $n$ th order ODE:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = f(t)$$

for  $f \in E$ . After taking the Laplace transform of both sides we get:

$$Z(s)\mathcal{L}[y(t)] = \mathcal{L}[f(t)] + Z_0(s)$$

where  $Z(s)$  is a degree  $n$  polynomial and  $Z_0(s)$  a degree  $n - 1$  polynomial dependent on our initial conditions.

Now if our source term is of the following form:

$$f(t) = t^n e^{at} (A \cos bt + B \sin bt)$$

then  $\mathcal{L}[f(t)]$  is rational and therefore:

$$\mathcal{L}[y(t)] = \frac{\mathcal{L}[f(t)]}{Z(s)} + \frac{Z_0(s)}{Z(s)}$$

where we can solve this via standard transforms.

### 5.2.2 Discontinuous source terms

Since the Laplace transform is convergent for all piecewise continuous functions, we can use it to solve ODEs with discontinuous source terms:

$$Ay''(t) + By'(t) + Cy(t) = g(t)$$

where  $g(t)$  is piecewise continuous and of the following form:

$$g(t) = f(t)[u_a(t) - u_b(t)] = \begin{cases} f(t) & t \in [a, b) \\ 0 & t \notin [a, b) \end{cases}$$

for  $b > a$ . This is the **unit step function**:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

and is also known as the Heaviside function. Its Laplace transform is:

$$\mathcal{L}[u_c(t)](s) = \frac{e^{-sc}}{s}.$$

Furthermore we can define a shift of  $f(t)$  by  $c > 0$  to the right by:

$$f(t - c)u_c(t)$$

$$\mathcal{L}[f(t - c)u_c(t)] = e^{-sc}F(s)$$

where  $F(s) = \mathcal{L}[f(t)]$ .

### 5.2.3 Impulse functions

The **Dirac delta** is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

and physically a “strike” on a system in infinitely short time. We have that:

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = f(t_0)$$

$$\frac{d}{dt}[u_{t_0}(t)] = \delta(t - t_0)$$

and

$$\mathcal{L}[\delta(t - t_0)] = e^{-st_0}.$$

If  $t_0 = 0$  then  $\mathcal{L}[\delta(t)] = \lim_{t_0 \rightarrow 0} (e^{-st_0}) = 1$ .



Taking the Laplace transform of the following ODE

$$y''(t) + y(t) = \delta(t)$$

with initial conditions  $y(0) = y'(0) = 0$  gives:

$$\mathcal{L}[y(t)] = \frac{1}{s^2 + 1} \lim_{t_0 \rightarrow 0} (e^{-st_0}).$$

It is important that we do not evaluate the limit here!

Finally by inspection we have that:

$$\begin{aligned} y(t) &= \lim_{t_0 \rightarrow 0} (\sin(t - t_0)u_{t_0}(t)) \\ &= \sin(t)u_0(t). \end{aligned}$$

#### 5.2.4 Convolutions

Convolutions can also be helpful in solving ODEs. They are defined:

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(s)g(t-s)ds \\ &= \int_0^t g(s)f(t-s)ds \end{aligned}$$

for functions  $f, g : [0, \infty) \rightarrow \mathbb{R}$  and has the following properties:

- $f * (g + h) = f * g + f * h$
- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * 1 \neq f$
- $f * f \neq f^2$ .

Importantly we have the **convolution theorem**:

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

where if  $f, g \in E$  then  $f * g \in E$ , or that they are functions of exponential order so that our Laplace transforms converge. Note that:

$$\int_0^t f(s)ds = \int_0^t f(s)u_0(t-s)ds$$

and therefore

$$\mathcal{L}\left[\int_0^t f(s)ds\right] = \frac{\mathcal{L}[f(t)]}{s}.$$

Now consider the following ODE:

$$ay''(t) + by'(t) + cy(t) = g(t)$$

with initial conditions  $y(0) = \alpha$  and  $y'(0) = \beta$ . Here  $a, b, c, \alpha, \beta \in \mathbb{R}$ .

$$\begin{aligned}\therefore \mathcal{L}[y(t)] &= \Phi(s) + \Psi(s) \\ &= \frac{(as + b)\alpha + a\beta}{as^2 + bs + c} + \frac{1}{as^2 + bs + c} \mathcal{L}[g(t)]\end{aligned}$$

$$\begin{aligned}\therefore y(t) &= \mathcal{L}^{-1}[\Phi(s)] + \mathcal{L}^{-1}[\Psi(s)] \\ &= \phi(t) + \psi(t)\end{aligned}$$

The first expression  $\phi(t)$  can be found via standard transforms. For the second expression we define the **transfer function**<sup>1</sup>:

$$H(s) = \frac{1}{as^2 + bs + c}$$

where it is the Laplace transform of the following corresponding ODE:

$$ah''(t) + bh'(t) + ch(t) = \delta(t)$$

with initial conditions  $h(0) = h'(0) = 0$ .  $\therefore H(s) = \mathcal{L}[h(t)]$

This is helpful since:

$$\begin{aligned}\Psi(s) &= \frac{1}{as^2 + bs + c} \mathcal{L}[g(t)] \\ &= H(s) \mathcal{L}[g(t)]\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}[\psi(t)] &= \mathcal{L}[h(t)] \mathcal{L}[g(t)] \\ &= \mathcal{L}[h(t) * g(t)].\end{aligned}$$

Applying the convolution theorem we get:

$$\begin{aligned}\psi(t) &= h(t) * g(t) \\ &= \int_0^t h(s)g(t-s)ds.\end{aligned}$$

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<sup>1</sup>Also known as a Green's function.

### 5.3 Standard transforms

- $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$  where  $n \in \mathbb{N}$  and  $s > 0$ .
- $\mathcal{L}[e^{at}] = \frac{1}{s-a}$  where  $s > a$ .
- $\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$  where  $n \in \mathbb{N}$  and  $s > a$ .
- $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$ ,
- $\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}$  where  $s > 0$ .
- $\mathcal{L}[\sinh at] = \frac{a}{s^2 - a^2}$ ,
- $\mathcal{L}[\cosh at] = \frac{s}{s^2 - a^2}$  where  $s > |a|$ .
- $\mathcal{L}[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2}$ ,
- $\mathcal{L}[e^{at} \cos bt] = \frac{s-a}{(s-a)^2 + b^2}$  where  $s > a$ .
- $\mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}$  where  $s > 0$ .
- $\mathcal{L}[\delta(t-c)] = e^{-cs}$ .