D: Functions

A function $f : X \to Y$ is an assignment of an element of Y to each element of X.

1. f is **injective** if:

$$\forall x_1, x_2 \in X; f(x_1) = f(x_2) \implies x_1 = x_2.$$

2. f is **surjective** if:

$$\forall y \in Y; \exists x \in X : y = f(x).$$

3. f is **bijective** if it is injective and surjective.

D: Groups

D: Abelian groups

D1.2.1(i): Fields

A field F is a set defined with:

1. Addition function $(+)$:

$$(+) : F \times F \to F; (\lambda, \mu) \mapsto \lambda + \mu$$

2. Multiplication function $(\cdot)$:

$$(\cdot) : F \times F \to F; (\lambda, \mu) \mapsto \lambda \cdot \mu$$

3. $\exists 0_F, 1_F \in F$ where $0_F \neq 1_F$ such that $(F, +)$ and $(F \backslash \{0_F\}, \cdot)$ form Abelian groups.
4. $\exists(-\lambda) \in F : \lambda + (-\lambda) = 0_F$
5. $\exists(\lambda^{-1}) \in F : \lambda \cdot (\lambda^{-1}) = 1_F$
6. $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F$

D1.2.1(ii): Vector spaces

A vector space V over a field F is an Abelian group $V := (V, +)$ with mapping:

$$F \times V \to V : (\lambda, \boldsymbol{v} \mapsto \lambda\boldsymbol{v})$$

where for $\forall \lambda, \mu \in F$ and $\forall \boldsymbol{v}, \boldsymbol{w} \in V$:

1. $\lambda(\boldsymbol{v} + \boldsymbol{w}) = (\lambda\boldsymbol{v}) + (\mu\boldsymbol{w})$
2. $(\lambda + \mu)\boldsymbol{v} = (\lambda\boldsymbol{v}) + (\mu\boldsymbol{w})$
3. $\lambda(\mu\boldsymbol{v}) = (\lambda\mu)\boldsymbol{v}$
4. $1_F\boldsymbol{v} = \boldsymbol{v}$

and is a F-vector space.

Remark

Let V be a F-vector space where $\boldsymbol{v} \in V$.

1. $0\boldsymbol{v} = 0$
2. $(-1)\boldsymbol{v} = -\boldsymbol{v}$
3. $\lambda\mathbf{0} = \mathbf{0}$ for $\forall \lambda \in F$.

D: Cartesian products

The Cartesian product of sets $X_1, \dots, X_n$ is defined as:

$$X_1 \times \cdots \times X_n := \{(x_1, \dots, x_n) : x_i \in X_i\}$$

where $1 \leq i \leq n$.

The projection of a Cartesian product is:

$$\text{pr}_i : X_1 \times \cdots \times X_n \to X_i; \quad (x_1, \dots, x_n) \mapsto x_i$$

D1.4.1: Vector subspaces

A vector subspace U of F-vector space V has the follwing properties:

1. $U \subset V$ and $\mathbf{0} \in U$.
2. Let $\boldsymbol{u}, \boldsymbol{v} \in U$ and $\lambda \in F$. Then $\boldsymbol{u} + \boldsymbol{v} \in U$ and $\lambda\boldsymbol{u} \in U$.

and is also a vector space.

P1.4.5

Let $T \subset V$ where V is a F-vector space. Then for all vector subspaces containing T, there exists a smallest vector subspace:

$$\text{span}(T) = \langle T \rangle_F \subset V$$

known as the vector subspace generated by T, or the span of T.

D1.4.7: Generating set

Let $T \subset V$ where V is a F-vector space. T is a generating set of V if:

$$\text{span}(T) = V$$

and is the linear combination of vectors in T over field F.

D1.4.9: Power sets

The power set of set X is:

$$\mathcal{P}(X) := \{U : U \subseteq X\}.$$

Let $\mathcal{U} \subseteq \mathcal{P}(X)$. Then:

$$\bigcup_{U \in \mathcal{U}} U := \{x \in X : (\exists U \in \mathcal{U} : x \in U)\}$$

$$\bigcap_{U \in \mathcal{U}} U := \{x \in X : \forall U \in \mathcal{U}; x \in U\}.$$

D1.5.1: Linear independence

Let V be a F-vector space and $L \subseteq V$. L is linearly independent if:

$$\alpha_1\boldsymbol{v}_1 + \cdots + \alpha_r\boldsymbol{v}_r = \mathbf{0} \implies \alpha_1 = \cdots = \alpha_r = 0$$

where $\boldsymbol{v}_i \in L$.

D1.5.8: Basis

A basis of a vector space V is a linearly independent generating set in V.

T1.5.11

Let V be a F-vector space.

Then $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ is a basis of V **iff**:

$$\Phi : F^r \to V; \quad (\alpha_1, \dots, \alpha_r) \mapsto \alpha_1\boldsymbol{v}_1 + \cdots + \alpha_r\boldsymbol{v}_r$$

is a bijection.

T1.5.12

Let V be a vector space and $E \subseteq V$. Then the following statements are equivalent:

1. E is a basis of V.
2. E is minimal among all generating sets, or that $E \setminus \{\boldsymbol{v}\}$ is not a basis for $\forall \boldsymbol{v} \in V$.
3. E is maximal among all linearly independent subsets. i.e. $E \cup \{\boldsymbol{v}\}$ is not linearly independent.

C1.5.13

Every finitely generated vector space has a finite basis.

T1.5.14

D1.5.15

T1.5.16

T1.6.1

T1.6.2

L1.6.3

L1.6.4

D1.6.5

C1.6.7

C1.6.8

T1.6.10