

Honours Analysis

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1 Real numbers

1.1 Properties of real numbers

1.2 Nested interval property and compactness

1.3 Triangle inequalities

2 Real sequences

3 Infinite series

4 Continuity and differentiability

5 Pointwise and uniform convergence

definition for pointwise and uniform convergence

uniform convergence supremum

limits and integration applications

weierstrass m test

uniform continuity - if δ is purely in ϵ form

6 Power series

7 Lebesgue integration

7.1 Characteristic and step functions

Definition 7.1 (Characteristic functions).

The characteristic function is defined as a real function such that:

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

where $E \subset \mathbb{R}$.

Remark. Therefore the integral of a characteristic function is:

$$\int \chi_E = \lambda(E)$$

for $\lambda(E)$ is the length of an interval E .

Definition 7.2 (Step functions).

The step function with respect to finite set $\{x_0, \dots, x_n\}$ for some $n \in \mathbb{N}$ is:

$$\phi(x) = \begin{cases} 0 & x < x_0 \text{ or } x > x_n \\ c_j & \text{if } x \in (x_{j-1}, x_j); \ 1 \leq j \leq n. \end{cases}$$

Remark. Step functions are a sum of characteristic functions:

$$\phi(x) = \sum_{j=1}^n c_j \chi_{(x_{j-1}, x_j)}(x)$$

and its integral is

$$\int \phi = \sum_{j=1}^n c_j (x_{j-1} - x_j).$$

Importantly the sum of two step functions is another step function.

7.2 Lebesgue integrals

Consider function $f : I \rightarrow \mathbb{R}$. This function is **Lebesgue integrable** on our interval I if:

1. $\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$
2. $\forall x \in I; f(x) = \sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$

Here $c_j \in \mathbb{R}$, $J_j \subset I$ and is bounded for $j \in \{1, 2, 3, \dots\}$.

i.e. that our function's area and height are defined. Therefore:

$$\int_I f = \sum_{j=1}^{\infty} |c_j| \lambda(J_j)$$

and integral value is invariant of interval type. (open, semi-open or closed)

7.2.1 Properties of Lebesgue integrals

Let functions f, g be Lebesgue integrable on I and $\alpha, \beta \in \mathbb{R}$. Then:

1. $\alpha f + \beta g$ is Lebesgue integrable on I , and:

$$\int_I \alpha f + \beta g = \alpha \int_I f + \beta \int_I g.$$

2. If $f \geq g$ on I then:

$$\int_I f \geq \int_I g.$$

- 3.

$$\int_I |f| \geq \left| \int_I f \right|$$

4. $\max\{f, g\}$ and $\min\{f, g\}$ are integrable on I . Furthermore:

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$$

and

$$\min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

5. fg is integrable on I if one of the functions is bounded.

6. Let $f \geq 0$ where $\int_I f = 0$.

The function h is integrable on I if $0 \leq h \leq f$.

7.2.2 Integration on subintervals

Let $J \subset I$. We then have the following statements.

1. If f is integrable on I then f is integrable on J .
2. Let $f(x) = 0$ for $\forall x \in I \setminus J$ and f integrable on J . Then:

$$\int_J f = \int_I f.$$

3. Assume that $\forall x \in I; f(x) \geq 0$. If f is integrable on I then:

$$\int_I f \geq \int_J f.$$

4. Let $I = \bigcup_{n=1}^{\infty} I_n$ where I_n are all disjoint sets.

Let f be integrable on each I_n . We have that:

$$f \text{ is integrable on } I \iff \sum_{n=1}^{\infty} \int_{I_n} f$$

and that the following equality holds:

$$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} f.$$

The regular integral calculus properties hold:

- 1.

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

- 2.

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

7.2.3 Maclaurin-Cauchy integral test

Now let f be a non-negative, **monotone decreasing** function on $[p, \infty)$. Then:

$$\sum_{n=p}^{\infty} f(n) < \infty \iff f \text{ is integrable on } [p, \infty)$$

where $p \in \mathbb{Z}$. Furthermore:

$$\sum_{n=p}^{\infty} f(n) < \infty \iff \int_p^{\infty} f(x)dx < \infty.$$

7.3 Riemann integrals

A real function f is **Riemann-integrable** if it has bounded support. i.e:

$$\forall \epsilon > 0; \exists \phi, \psi : \phi \leq f \leq \psi \quad \text{and} \quad \int \psi - \int \phi < \epsilon,$$

where ψ and ϕ are step functions.

Furthermore the following statements are equivalent:

1. f is Riemann-integrable, where f is a real bounded function with bounded support $[a, b]$.
2. $\sup \left\{ \int \phi \right\} = \inf \left\{ \int \psi \right\}$, and is the integral value.
3. $\forall \epsilon > 0; \exists \{a = x_0 < \dots < x_n = b\} :$

$$\sum_{j=1}^n \left(\sup_{x \in (x_{j-1}, x_j)} f(x) - \inf_{x \in (x_{j-1}, x_j)} f(x) \right) (x_j - x_{j-1}) < \epsilon$$

and

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| \cdot \lambda(I_j) < \epsilon$$

where we define $I_j = (x_{j-1}, x_j)$ and $j \in \{1, \dots, n\}$.

Now let:

$$m_j = \inf_{x \in I_j} f(x)$$

$$M_j = \sup_{x \in I_j} f(x)$$

and it makes sense to define step functions

$$\phi_* \leq f \leq \phi^*(x)$$

with respect to $\{x_0, \dots, x_n\}$ where:

$$\phi_*(x) = \sum_{j=1}^n m_j \chi_{I_j}(x) + \sum_{j=1}^n f(x_j) \chi_{\{x_j\}}(x)$$

and

$$\phi^*(x) = \sum_{j=1}^n M_j \chi_{I_j}(x) + \sum_{j=1}^n f(x_j) \chi_{\{x_j\}}(x).$$

If f is Riemann-integrable then it is automatically Lebesgue-integrable, but not necessarily the opposite way. So Lebesgue-integrals are a superset of Riemann-integrals.

Note that closed intervals are **uniformly continuous**.

Let $g : [a, b] \rightarrow \mathbb{R}$ and that:

$$f(x) = \begin{cases} g(x) & x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

We then have that:

1. If g is continuous on $[a, b]$ then f is Riemann-integrable.
2. If g is a montone function then f is Riemann-integrable.

7.4 Fundamental theorem of calculus

Let $g : I \rightarrow \mathbb{R}$ be integrable on I and that

$$G(x) = \int_{x_0}^x g(x) dx$$

for $\forall x \in I$ and fixed $x_0 \in I$.

If $g(x)$ is continuous at $x \in I$ then:

$$\frac{d}{dx} G(x) = g(x).$$

Furthermore if $G(x)$ and $g(x)$ are continuous on the interval I :

$$\int_a^b g(x) dx = G(b) - G(a)$$

for $\forall a, b \in I$.

7.5 Integration of sequences

Consider $(f_n)_{n \in \mathbb{N}}$ that are integrable on I . Assume the following:

- $\sum_{n=1}^{\infty} \int_I |f_n| < \infty$
- $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ for $\forall x \in I$.

Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then f is integrable on I and

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n.$$

The following result is a useful test for integrability.

Let $f_n \geq 0$ on I and that $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then:

$$f \text{ is integrable on } I \iff \sum_{n=1}^{\infty} \int_I f_n < \infty.$$

7.5.1 Monotone convergence for integration

Now consider a monotone increasing sequence of functions $(f_n)_{n \in \mathbb{N}}$:

$$f_1 \leq f_2 \leq f_3 \leq \dots$$

Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then:

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

and furthermore:

$$\sup_{n \in \mathbb{N}} \int_I f_n = \lim_{n \rightarrow \infty} \int_I f_n < \infty.$$

7.5.2 Fatoux's lemma

Let $f_n > 0$ be integrable functions on I and that:

$$f(x) = \liminf_{n \rightarrow \infty} f_n(x)$$

for $\forall x \in I$. If

$$\liminf_{n \rightarrow \infty} \int_I f_n(x) < \infty$$

then f is integrable on I and:

$$\int_I f \leq \liminf_{n \rightarrow \infty} \int_I f_n(x).$$

An immediate result is the following.

Let f_n be integrable on the interval I and that:

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

If $|f_n(x)| \leq g(x)$ where $\int_I g < \infty$ then:

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

A final result is that if $f_n : (a, b) \rightarrow \mathbb{R}$ are integrable functions, and that:

$$f_n \rightarrow f \text{ uniformly on } (a, b),$$

we then have that:

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

8 Fourier analysis

8.1 L^2 space

l_2 norm of a function

inner product

cauchy schwarz inequalities

minkowski inequalities

convergence in l_2

orthonormal systems

T5.2

bessel's inequality

riemann lemma

complete orthonormal systems

T5.4

8.2 Fourier series

trigonometric polynomial (fs)

complex fourier series

fourier coefficients

euler formula

lemma 5.1: orthgonality of FS

convolution of fs

dirichlet kernel

8.3 Convergence of Fourier series

8.3.1 Approximations

8.3.2 L^2 convergence

8.3.3 Pointwise convergence