Integrating factors

$$y' + P(x)y = Q(x)$$

$$I(x) = \exp\left(\int P(x)dx\right)$$

$$y(x) = \frac{1}{I(x)}\int Q(x)I(x)dx + \frac{\alpha}{I(x)}$$

Change of variables

$$y^{(n)} = F(y, y', \dots, y^{(n-1)}, t)$$

Let $x_{i+1} = y^{(i)}$ where $i \in \{0, 1, \dots, n-1\}$.

Picard-Lindelöf statement

Consider IVP: $x_i' = F_i(t, x_1, \dots, x_n)$ or that $\boldsymbol{x}' = \boldsymbol{F}(t, \boldsymbol{x})$ with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ and $\boldsymbol{x}$ is a vector with n entries.

It has **unique** solutions if:

$$F_i, \frac{\partial F_i}{\partial x_j} \text{ and } \frac{\partial F_i}{\partial t} \text{ are } \underline{\text{continuous}} \text{ in}$$

$R \subset \mathbb{R}^{n+1}$ where $(t, \boldsymbol{x}_0^T) \in R$.
Here $i, j \in \{1, \dots, n\}$.

Homogeneous systems

Consider $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x}$ where $\mathbf{A}$ is a $n \times n$ matrix. Substituting $\boldsymbol{x} = e^{rt}\boldsymbol{\xi}$ gives:

$$(\mathbf{A} - r_i\mathbf{I_n})\boldsymbol{\xi^{(i)}} = \mathbf{0}$$

where $i \in \{1, 2, \dots, n\}$.
Our general solution is then:

$$\begin{aligned}\boldsymbol{x}(t) &= \sum_{i=1}^{n} c_i e^{r_i t}\boldsymbol{\xi}^{(i)} \\ &= \sum_{i=1}^{n} c_i \boldsymbol{x}^{(i)} \\ &= \boldsymbol{\Psi}(t)\boldsymbol{c}.\end{aligned}$$

If initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ are given:

$$\boldsymbol{c} = \boldsymbol{\Psi}^{-1}(t_0)\boldsymbol{x}(t_0)$$

$$\boldsymbol{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\boldsymbol{x}_0.$$

Matrix exponentials

Given a $n \times n$ matrix $\boldsymbol{A}$:

$$\begin{aligned}e^{\boldsymbol{A}t} &= \sum_{n=0}^{\infty}\frac{(\boldsymbol{A}t)^n}{n!} \\ &= \boldsymbol{I}_n + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \dots.\end{aligned}$$

For system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

and that $e^{\boldsymbol{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$.

Diagonalisation

For $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ we have $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$.
If $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$ then $\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y}$.

Since our fundamental matrix with respect to $\boldsymbol{y}$ is a <u>diagonal</u> matrix $\boldsymbol{Q} = e^{\boldsymbol{D}t}$, the fundamental matrix with respect to $\boldsymbol{x}$ is $\boldsymbol{\Psi}(t) = \boldsymbol{T}e^{\boldsymbol{D}t}$ and:

$$e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{D}t}\boldsymbol{T}^{-1}.$$

Generalised eigenvectors

If eigenvalues do not generate enough eigenvectors to form n linearly independent solutions for a $n \times n$ matrix $\boldsymbol{A}$, then consider the following ansatz:

$$\boldsymbol{x} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

$$\therefore (\mathbf{A} - r_i\mathbf{I_n})\boldsymbol{\eta^{(i)}} = \boldsymbol{\xi^{(i)}}$$

Then this r_i produces <u>two</u> solutions:

$$\boldsymbol{x}^{(1)} = e^{rt}\boldsymbol{\xi}$$

$$\boldsymbol{x}^{(2)} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

Non-homogeneous systems

Consider non-homogeneous ODE system:

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}.$$

- **Change of basis**

  Let $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$ and since $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$:

  $$\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y} + \boldsymbol{T}^{-1}\boldsymbol{g}$$

  which is solved by integrating factors. Finally <u>revert back</u> to $\boldsymbol{x}$.

- **Variation of parameters**

  Find solution $\boldsymbol{x}_H = \boldsymbol{\Psi}\boldsymbol{c}$ to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Then let the non-homogeneous solution be $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{u}(t)$.

  $$\therefore \boldsymbol{\Psi}\boldsymbol{u}'(t) = \boldsymbol{g}(t)$$

  Row reduce before integrating.

- **Undetermined coefficients**

  Let non-homogeneous ODE system have solutions of form:

  $$\boldsymbol{x} = \boldsymbol{x}_H + \boldsymbol{x}_p$$

  where $\boldsymbol{x}_H$ is our homogeneous solution and $\boldsymbol{x}_p$ our <u>particular</u> solution.

Critical points

Consider non-linear ODE system

$$x' = F(x, y),$$

$$y' = G(x, y).$$

We define $\boldsymbol{x^0} = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$ as a **critical point** when $F(\boldsymbol{x^0}) = G(\boldsymbol{x^0}) = 0$.

Linearisation and stability

Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x - x^0 \\ y - y^0 \end{bmatrix}$.

$$\begin{aligned}\therefore u_1' &\approx F(x^0, y^0) + \left(\frac{\partial F}{\partial x}\right)_{x^0}(x - x^0) \\ &\quad + \left(\frac{\partial F}{\partial y}\right)_{y^0}(y - y^0)\end{aligned}$$

$$\begin{aligned}\therefore u_2 &\approx G(x^0, y^0) + \left(\frac{\partial G}{\partial x}\right)_{x^0}(x - x^0) \\ &\quad + \left(\frac{\partial G}{\partial y}\right)_{y^0}(y - y^0)\end{aligned}$$

$$\begin{aligned}\therefore \boldsymbol{u}' &= \boldsymbol{A}\boldsymbol{u} \\ &= \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}_{\boldsymbol{x}=\boldsymbol{x^0}} \begin{bmatrix} x - x^0 \\ y - y^0 \end{bmatrix}\end{aligned}$$

Critical points $\boldsymbol{x^0}$ may also be classified:

Eigenvalues	Critical Points	Stability
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asymp. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asymp. stable
$r_1, r_2 = \lambda \pm i\mu\ (\lambda > 0)$	focus	unstable
$r_1, r_2 = \lambda \pm i\mu\ (\lambda < 0)$	focus	asymp. stable
$r_1 = i\mu, r_2 = -i\mu$	center	stable

Linearisation preserves critical point behaviour **except** when eigenvalues are of $r = \pm i\mu$ form, for which then classification is unknown.

Stable critical points $\boldsymbol{x^0}$:
All solutions <u>start</u> and <u>stay near</u> $\boldsymbol{x^0}$.

$$\forall\epsilon > 0; \exists\delta > 0; \forall\boldsymbol{x}(t) = \boldsymbol{\phi}(t) : |\boldsymbol{x}(0) - \boldsymbol{x^0}| < \delta$$
$$\implies |\boldsymbol{x}(t) - \boldsymbol{x^0}| < \epsilon \text{ for } \forall t \geq 0$$

Attracting critical points $\boldsymbol{x^0}$:
All solutions <u>tends</u> to $\boldsymbol{x^0}$.

$$\forall\delta > 0 : |\boldsymbol{x}(0) - \boldsymbol{x^0}| < \delta$$
$$\implies \lim_{t\to\infty}\boldsymbol{x}(t) = \boldsymbol{x^0}$$

Asymptotically stable critical points $\boldsymbol{x^0}$: Attracting **and** stable.

Lyapunov's theory and limit cycles

Consider $\dot{x} = F(x, y)$ and $\dot{y} = G(x, y)$ and let $\boldsymbol{x^0} \in D$ be a critical point. Let $E : D \subset \mathbb{R}^2 \to \mathbb{R}$ is defined such that $E(x^0, y^0) = 0$.

$$\therefore \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G$$

- Let $E > 0$ for $\forall\boldsymbol{x} \neq \boldsymbol{x^0}$.

  If $\frac{\mathrm{d}E}{\mathrm{d}t} \leq 0$ then $\boldsymbol{x^0}$ is stable.

  If $\frac{\mathrm{d}E}{\mathrm{d}t} < 0$ then $\boldsymbol{x^0}$ is asymptotically stable.

- $E(\boldsymbol{x}^*) > 0$ and $\frac{\mathrm{d}E}{\mathrm{d}t} > 0$

  $\implies$ unstable $\boldsymbol{x^0}$. (flip both signs)

Postive definite: $E(\boldsymbol{x}) > 0$ for $\forall \boldsymbol{x} \neq \boldsymbol{x^0}$

Postive semidefinite:
$E(\boldsymbol{x}) \geq 0$ for $\forall \boldsymbol{x} \neq \boldsymbol{x^0}$

Limit cycles are periodic solutions such that at least one other non-closed trajectory approaches it as $t \to \infty$.

Generally if our trajectory is enclosed by finite non-simple region and F, G have continuous partials then there is a limit cycle.

Real Fourier series

The Fourier expansion of piecewise continuous $f(x)$ on $[-L, L]$ is:

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

where $f_{FS}(x) = f_{FS}(x + \boldsymbol{2L})$. If α is a discontinuous point:

$$f_{FS}(\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}$$

for α^+ is the limit from the left.

Our Fourier coefficients are:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} f(x)\mathrm{d}x$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L} f(x)\mathrm{d}x.$$

Orthogonality

Let $S_n = \sin\frac{n\pi x}{L}$ and $C_n = \cos\frac{n\pi x}{L}$.

$$\langle S_n, S_m\rangle = \langle C_n, C_m\rangle = L\delta_{mn}$$

$$\langle S_n, C_m\rangle = 0.$$

Where:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$$\langle u(x), v(x)\rangle = \int_{-L}^{L} u(x)v(x)\mathrm{d}x$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)].$$

Even functions: $f(-x) = f(x)$

$$\therefore \int_{-L}^{L} f(x)\mathrm{d}x = 2\int_{0}^{L} f(x)\mathrm{d}x$$

Odd functions: $f(-x) = -f(x)$

$$\therefore \int_{-L}^{L} f(x)\mathrm{d}x = 0$$

- **Even** function $f(x)$ on $[-L, L]$:

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_{0}^{L} \cos\frac{n\pi x}{L} f(x)\mathrm{d}x$$

- **Odd** function $f(x)$ on $[-L, L]$:

$$f_{FS}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_{0}^{L} \sin\frac{n\pi x}{L} f(x)\mathrm{d}x$$

Extensions

Consider $f(x)$ defined in $[0, L]$ originally.

1. Define even function:

$$g(x) = \begin{cases} f(x) & x \in [0, L] \\ f(-x) & x \in (-L, 0) \end{cases}$$

with cosine series.

2. Define odd function:

$$h(x) = \begin{cases} f(x) & x \in (0, L) \\ 0 & x = 0, L \\ -f(-x) & x \in (-L, 0) \end{cases}$$

with sine series.

Complex Fourier series

Similarly:

$$f_{FS}(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi}{L}x\right)$$

$$e^{i\theta} = \sin\theta + i\cos\theta.$$

Then $\forall n \in \mathbb{Z}$ we have that:

$$c_n = \frac{1}{2L}\int_{-L}^{L} \exp\left(-\frac{in\pi}{L}x\right) f(x)\mathrm{d}x$$

$$c_n = \begin{cases} (a_n - ib_n)/2 & n > 0 \\ (a_0)/2 & n = 0 \\ (a_n + ib_n)/2 & n < 0. \end{cases}$$

The **inner product** is defined as:

$$\langle f, g\rangle = \int_{-L}^{L} f(x)g^*(x)\mathrm{d}x.$$

$$\therefore \langle \exp\left(\frac{\boldsymbol{im}\pi}{L}x\right), \exp\left(\frac{\boldsymbol{in}\pi}{L}x\right)\rangle = 2L\delta_{mn}$$

Parseval's theorem

$$\begin{aligned}\langle f, f\rangle &= \int_{-L}^{L} |f(x)|^2\mathrm{d}x \\ &= 2L\sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= L\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)\right]\end{aligned}$$

Heat equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$$

Let $u(x, t) = X(x)T(t)$.

$$\frac{1}{\alpha^2}\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

$$\dot{T} + \alpha^2\lambda T = 0$$

$$\lambda = \mu^2; X(x) = b_1\cos\lambda^{1/2}x + b_2\sin\lambda^{1/2}x$$

$$\lambda = -\mu^2; X(x) = b_1\cosh\mu x + b_2\sinh\mu x$$

$$T(t) = a_1\exp\left(-\alpha^2\lambda t\right)$$

Standard boundary conditions

- $u(x, 0) = f(x)$ for $0 \leq x \leq L$
- $u(0, t) = u(L, t) = 0$ for $\forall t > 0$

$$X(0) = X(L) = 0$$

$$\therefore X_n = b_2\sin\lambda_n^{1/2}x$$

$$\therefore \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for } \forall n \in \mathbb{N}$$

Our general solution must then be:

$$u(x, t) = \sum_{n=1}^{\infty} c_n\exp\left(-\alpha^2\lambda_n t\right)\sin\lambda_n^{1/2}x$$

$$c_n = \frac{2}{L}\int_{0}^{L}\sin\left(\lambda_n^{1/2}x\right)f(x)\mathrm{d}x.$$

Fixed boundary temperatures

- $u(0, t) = T_1$
- $u(L, t) = T_2$
- $u(x, 0) = f(x)$

$$v(x) = \lim_{t\to\infty} u(x, t)$$

Since $v'' = 0$, $v(0) = T_1$ and $v(L) = T_2$:

$$v(x) = \frac{T_2 - T_1}{L}x + T_1.$$

We then deduce that:

$$u(x, t) = v(x) + \omega(x, t)$$

where $\omega(x, t)$ satisfies conditions:

- $\omega(0, t) = \omega(L, t) = 0$
- $\omega(x, 0) = f(x) - v(x)$

$$\therefore \omega(x, t) = \sum_{n=1}^{\infty} c_n\exp\left(-\alpha^2\lambda_n t\right)\sin\lambda_n^{1/2}x$$

For $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and

$$c_n = \frac{2}{L}\int_{0}^{L}\sin\left(\lambda_n^{1/2}x\right)\left(f(x) - v(x)\right)\mathrm{d}x.$$

Insulated rod ends

- $\dfrac{\partial}{\partial x}u(0,t) = \dfrac{\partial}{\partial x}u(L,t) = 0$
- $u(x,0) = f(x)$

$$X'(0) = X'(L) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\alpha^2 \lambda_n t\right) \cos \lambda_n^{1/2} x$$

$$c_n = \frac{2}{L}\int_0^L \cos\left(\lambda_n^{1/2} x\right) f(x) \mathrm{d}x$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t) = X(x)T(t)$.

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -\lambda$$

$$X'' + \lambda X = 0$$

$$\ddot{T} + c^2 \lambda T = 0$$

Plucked string

- $u(0,t) = u(L,t) = 0$
- $\dfrac{\partial}{\partial t}u(x,0) = 0$
- $u(x,0) = f(x)$

$$X(0) = X(L) = 0 \text{ and } \dot{T}(0) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x \cos c\lambda_n^{1/2} t$$

$$c_n = \frac{2}{L}\int_0^L \sin\left(\lambda_n^{1/2} x\right) f(x) \mathrm{d}x$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

General initial conditions

- $u(0,t) = u(L,t) = 0$
- $\dfrac{\partial}{\partial t}u(x,0) = g(x)$
- $u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin \lambda_n^{1/2} x \times \left(a_n \cos c\lambda_n^{1/2} t + b_n \sin c\lambda_n^{1/2} t\right)$$

$$a_n = \frac{2}{L}\int_0^L \sin\left(\lambda_n^{1/2} x\right) f(x) \mathrm{d}x$$

$$b_n = \frac{1}{c\lambda_n^{1/2}} \frac{2}{L}\int_0^L \sin\left(\lambda_n^{1/2} x\right) g(x) \mathrm{d}x$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u(x,y) = X(x)Y(y)$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$X'' - \lambda X = 0$$

$$Y'' + \lambda Y = 0$$

Rectangular boundary conditions

- $u(x,0) = u(x,b) = 0$
- $u(0,y) = 0$ and $u(a,y) = f(y)$

Here $x \in [0,a]$ and $y \in [0,b]$.

$$X(0) = 0 \text{ and } Y(0) = Y(b) = 0$$

$$\therefore Y_n = a_1 \sin\left(\lambda_n^{1/2} y\right) \text{ for } \lambda_n = \frac{n^2\pi^2}{b^2}$$

$$\therefore X_n = a_3 \sinh\left(\lambda_n^{1/2} x\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\lambda_n^{1/2} x\right) \sin\left(\lambda_n^{1/2} y\right)$$

$$c_n = \frac{2}{b \sinh\left(\lambda_n^{1/2} a\right)} \int_0^b \sin\left(\lambda_n^{1/2} y\right) f(y) dy$$

Circular boundary conditions

- $u(a,\theta) = f(\theta)$
- $u(r,\theta)$ is bounded

Here $r \in [0,a]$ and $\theta \in [0, 2\pi]$.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0.$$

Let $u(r,\theta) = R(r)\Theta(\theta)$.

$$\therefore r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \lambda$$

$$\therefore \ddot{\Theta} + \lambda\Theta = 0$$

$$\therefore r^2 R'' + rR' = \lambda R$$

For the first ODE if $\lambda \leq 0$ then we get at best constant solutions. If $\lambda > 0$:

$$\Theta(\theta) = a_1 \cos \lambda^{1/2}\theta + a_2 \sin \lambda^{1/2}\theta$$

and since <u>periodicity</u> must be preserved:

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\therefore \lambda_n^{1/2} = n \text{ or } \lambda_n^{1/2} = 0$$

where $n \in \mathbb{N}$. So when $\lambda_n = 0$:

$$r^2 R'' + rR' = 0$$

and since $u(r,\theta)$ is bounded we get only constant solutions. If $\lambda_n = n^2$ then:

$$r^2 R'' + rR' - n^2 R = 0$$

with solutions of form $R(r) = r^\alpha$ which yields $R_n(r) = c_n r^n$. Then:

$$u(r,\theta) = \frac{p_0}{2} + \sum_{n=1}^{\infty} r^n \left(q_n \cos \lambda_n^{1/2}\theta + r_n \sin \lambda_n^{1/2}\theta\right)$$

$$p_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta$$

$$q_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} \cos\left(\lambda_n^{1/2}\theta\right) f(\theta)\mathrm{d}\theta$$

$$r_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} \sin\left(\lambda_n^{1/2}\theta\right) f(\theta)\mathrm{d}\theta.$$

Regular S-L problems

Consider the following eigenvalue ODE:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \lambda r(x) y$$

where $r(x)$ is our <u>weight function</u>. We define the following <u>boundary conditions</u>:

1. $a_1 y(0) + a_2 y'(0) = 0$
2. $b_1 y(1) + b_2 y'(1) = 0.$

This is a **regular Sturm-Liouville** problem, where $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are <u>continuous</u> functions and $p(x)$, $r(x)$ are <u>strictly positive</u> functions for $\forall x \in [0,1]$.

Eigenvalues λ_n yield **eigenfunctions** $\phi_n(x)$ which are <u>nontrivial solutions</u> to our S-L problem. Important consequences include:

- Eigenvalues λ_n of a S-L problem are **real**.

  Furthermore each eigenvalue corresponds to one eigenfunction.

- Eigenfunctions $\phi_n(x)$ are <u>orthogonal</u>:

  $$\langle \phi_m, \phi_n \rangle = \int_0^1 r(x)\phi_m(x)\phi_n(x)\mathrm{d}x = \delta_{mn}$$

  in Hilbert space $L^2([0,1], r(x)\mathrm{d}x)$.

Note that our eigenfunctions are orthonormal and so we define:

$$\phi_n(x) = k_n y_n(x)$$

where k_n is our scale factor. Since $\langle \phi_n, \phi_n \rangle = 1$:

$$\therefore \int_0^1 r(x) k_n^2 y_n^2(x)\mathrm{d}x = 1$$

and so we have that:

$$k_n = \frac{1}{\sqrt{\langle y_n, y_n\rangle}} = \Big(\int_0^1 r(x) y_n^2(x)\mathrm{d}x\Big)^{-1/2}.$$

0.0.1 General 2nd order ODEs

Consider the following general 2nd order eigenvalue ODE:

$$-P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \omega(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = \lambda r(x)y.$$

Multiply this by the following integrating factor:

$$F(x) = \exp\Big[\int_0^x \frac{\omega(s) - p'(s)}{p(s)}\mathrm{d}s\Big]$$

yields an ODE of S-L form:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big[F(x)P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\Big] + F(x)q(x)y = \lambda F(x)r(x)y.$$

0.0.2 Lagrange's identity

Our previous definition is motivated by the **Lagrange's identity**:

$$\langle \mathcal{L}[u], v\rangle - \langle u, \mathcal{L}[v]\rangle = -\Big[p\Big(u'v^* - u(v^*)'\Big)\Big]_0^1 = -\Big[p(x)(\frac{\mathrm{d}u}{\mathrm{d}x}\cdot v^* - u\cdot\frac{\mathrm{d}v^*}{\mathrm{d}x})\Big]_0^1$$

where $u = u(x)$, $v = v(x)$ are complex functions and

$$\mathcal{L}[u] = -\frac{\mathrm{d}}{\mathrm{d}x}\Big[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\Big] + q(x)u.$$

Here the inner product is defined as

$$\langle u, v\rangle = \int_0^1 uv^*\mathrm{d}x$$

and we have integrated by parts using the following identities:

$$\big[pu'v^*\big]' = \big(pu'\big)'v^* + pu'(v^*)'$$

$$\big[pu(v^*)'\big]' = \Big(p(v^*)'\Big)'u + pu'(v^*)'.$$

For a S-L problem its properties follow from:

$$\langle \mathcal{L}[u], v\rangle = \langle u, \mathcal{L}[v]\rangle$$

where functions u and v satisfy its boundary conditions.

0.0.3 Series expansion

Now the set of orthonormal eigenfunctions $\{\phi_n(x)\}$ from a S-L problem with boundary conditions may be used to expand function $f(x)$:

$$f_\phi(x) = \sum_{n=1}^\infty c_n\phi_n(x)$$

for $\forall x \in [0,1]$. Integrating this on both sides:

$$\int_0^1 r(x)\phi_m(x)f(x)\mathrm{d}x = \int_0^1 r(x)\phi_m(x)\sum_{n=1}^\infty c_n\phi_n(x)\mathrm{d}x = \sum_{n=1}^\infty c_n\int_0^1 r(x)\phi_m(x)\phi_n(x)\mathrm{d}x = \sum_{n=1}^\infty c_n\delta_{mn} = c_m$$

and so we get a general formula for our coefficients:

$$c_n = \int_0^1 r(x)\phi_n(x)f(x)\mathrm{d}x.$$

If $f(x)$ and $f'(x)$ are piecewise continuous on $x \in [0,1]$ then:

$$\forall x \in (0,1); \sum_{n=1}^\infty c_n\phi_n(x) = \frac{f(x^-) + f(x^+)}{2}$$

where these are the left and right handed limits.

0.0.4 General Parseval's identity for S-L problems

We have that:

$$\int_0^1 r(x)\big[f(x)\big]^2\mathrm{d}x = \sum_{n=1}^\infty c_n^2$$

where

$$f(x) = \sum_{n=1}^\infty c_n\phi_n(x)$$

and

$$c_n = \int_0^1 r(x)\phi_n(x)f(x)\mathrm{d}x.$$

0.1 Non-homogeneous S-L problems

Consider:

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

where $f(x)$ is our non-homogeneous term, and that we define

$$\mathcal{L}[y] = -\frac{\mathrm{d}}{\mathrm{d}x}\Big[P(x)\frac{\mathrm{d}y}{\mathrm{d}x}\Big] + q(x)y.$$

Assume that this ODE satisfies regular S-L boundary conditions.

Firstly we solve the corresponding homogeneous S-L problem:

$$\mathcal{L}[y] = \lambda r(x)y$$

for non-trivial λ_n and $\phi_n(x)$.

Now let the general solution to our non-homogeneous S-L problem be:

$$y(x) = \sum_{n=1}^\infty b_n\phi_n(x)$$

and substituting this yields:

$$\sum_{n=1}^\infty b_n\mathcal{L}[\phi_n(x)] = r(x)\sum_{n=1}^\infty b_n\lambda_n\phi_n(x) = \mu r(x)\sum_{n=1}^\infty b_n\phi_n(x) + f(x).$$

Rearranging then gives:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^\infty b_n(\lambda_n - \mu)\phi_n(x).$$

Now we can also expand $\dfrac{f(x)}{r(x)}$ in terms of our eigenfunctions:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^\infty c_n\phi_n(x)$$

with

$$c_n = \int_0^1 r(x)\phi_n(x)\frac{f(x)}{r(x)}\mathrm{d}x = \int_0^1 \phi_n(x)f(x)\mathrm{d}x$$

and so equating yields the following relation

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

0.2 Non-homogeneous PDEs

Consider the following non-homogeneous PDE:

$$r(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big[p(x)\frac{\partial u}{\partial x}\Big] - q(x)u(x,t) + F(x,t)$$

with boundary and initial conditions:

- $\frac{\partial}{\partial x}u(0,t) - h_1u(0,t) = 0$
- $\frac{\partial}{\partial x}u(1,t) - h_2u(1,t) = 0$
- $u(x,0) = f(x)$.

Firstly we solve the homogenous case via separation of variables. Let:

$$u(x,t) = X(x)T(t)$$

and after some algebra:

$$\frac{1}{rX}\Big[p'X' + pX'' - qX\Big] = \frac{\dot{T}}{T} = -\lambda.$$

This yields two ODEs:

$$\dot{T} + \lambda T = 0$$

$$-\left[pX'\right]' + qX = \lambda r X$$

with boundary conditions:

$$X'(0) - h_1 X(0) = 0$$

$$X'(1) - h_2 X(1) = 0$$

Clearly our second ODE is of S-L form. We are now going to assume that this regular S-L problem has non-trivial λ_n and orthonormal eigenfunctions $\phi_n(x)$.

Let the general solution to our PDE be:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x).$$

Substituting this into our PDE yields:

$$r(x)\sum_{n=1}^{\infty} \dot{b_n}(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Big[\Big(p(x)\phi_n'(x)\Big)' - q(x)\phi_n(x)\Big] + F(x,t).$$

Now since we have a S-L problem:

$$\Big(p(x)\phi_n'(x)\Big)' - q(x)\phi_n(x) = -\lambda_n \phi_n(x) r(x)$$

and after dividing through our PDE by $r(x)$ we get:

$$\sum_{n=1}^{\infty} \dot{b_n}(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\Big[-\lambda_n\phi_n(x)\Big] + \frac{F(x,t)}{r(x)}$$

But we can also expand our non-homogeneous term as a sum:

$$\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)$$

and its coefficients are:

$$\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)\mathrm{d}x$$

in $L^2([0,1], r(x))$. Therefore after some algebra:

$$\sum_{n=1}^{\infty}\Big[\dot{b_n}(t) + \lambda_n b_n(t) - \gamma_n(t)\Big]\phi_n(x) = 0$$

and yields the following ODE:

$$\dot{b_n}(t) + \lambda_n b_n(t) - \gamma_n(t) = 0.$$

Solution is via integrating factors:

$$b_n(t) = e^{-\lambda_n t}\int_0^t \gamma_n(s)e^{\lambda_n s}\mathrm{d}s + e^{-\lambda_n t}b_n(0).$$

Finally using $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) = f(x)$$

and we get

$$b_n(0) = \int_0^1 r(x)f(x)\phi_n(x)\mathrm{d}x.$$

0.3 Singular S-L problems

Consider the following ODE:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \lambda r(x)y$$

but now $p(x)$, $q(x)$ and $r(x)$ are discontinuous at $x = 0$ and/or $x = 1$. This is a **singular** S-L problem with boundary condition:

$$b_1 y(1) + b_2 y'(1) = 0$$

or

$$a_1 y(0) + a_2 y'(0) = 0.$$

Now singular S-L problems at $x = 0$ may be self-adjoint or that they yield:

- $\lambda_n \in \mathbb{R}$ (Real eigenvalues)
- $\langle\phi_m, \phi_n\rangle = \delta_{mn}$ (Orthogonal eigenfunctions)

if they satisfy Lagrange's identity. Consider singular S-L problem at $x = 0$:

$$\int_\epsilon^1 \Big(\mathcal{L}[u]v - u\mathcal{L}[v]\Big)\mathrm{d}x = \Big[-p(x)\Big(u'(x)v(x) - u(x)v'(x)\Big)\Big]_\epsilon^1 = p(\epsilon)\Big(u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\Big)$$

and tends to zero if and only if:

$$\lim_{\epsilon\to\infty}\Big[p(\epsilon)\Big(u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\Big)\Big] = 0,$$

where we define:

$$\mathcal{L}[u] = -\frac{\mathrm{d}}{\mathrm{d}x}\Big[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\Big] + q(x)u.$$

Therefore this is the condition for our singular S-L problem to have real eigenvalues and orthogonal eigenfunctions.

Similarly for singular S-L problem at $x = 1$ it is self-adjoint if:

$$\lim_{\epsilon\to\infty}\Big[p(1-\epsilon)\Big(u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon)\Big)\Big] = 0.$$

Finally whilst eigenvalues may be real they are not necessarily discrete!

1 Laplace transforms

So let $f(t)$ be defined for $t \in [0,\infty)$. Its Laplace transform is:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\mathrm{d}t.$$

Let **functions of exponential order** be defined as:

$$\forall t \in [0,\infty); \exists A > 0 \text{ and } B \in \mathbb{R} : |f(t)| \le Ae^{Bt}$$

where f is piecewise continous and we denote such functions as $f \in E$.

For sufficiently large s, the Laplace transform $f \in E$ converges.

1.1 Properties

1.1.1 Inversion formula

Now let $F(s) = \mathcal{L}[f(t)]$. We have the following inversion formula:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}F(s)\mathrm{d}s.$$

1.1.2 Reduction of order

Applying the Laplace transform to derivatives:

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

for $\forall f, f' \in E$ and generalising this via induction gives:

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{(n-1)}f^{(0)}(0) - s^{(n-2)}f \\ - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

1.1.3 Shifts, scaling and derivatives

Let $F(s) = \mathcal{L}[f(t)](s)$. We then have that:

1. **s-shift**:
$$\mathcal{L}[e^{-ct}f(t)](s) = F(s+c)$$
where $s + c > \gamma$.

2. **t-shift**:
Let $c \ge 0$ and $f(t) = 0$ if $t < 0$. Then:
$$\mathcal{L}[f(t-c)](s) = e^{-sc}F(s).$$
Furthermore utilising the unit step function:
$$\mathcal{L}[g(t-c)u_c(t)](s) = e^{-sc}G(s)$$
where $G(s) = \mathcal{L}[g(t)](s)$ and $g(t)$ any normal function.

3. **s-derivative**:
$$\mathcal{L}[tf(t)](s) = -\frac{\mathrm{d}}{\mathrm{d}s}F(s).$$
We can also extend this to the nth derivative:
$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s).$$

4. **scaling**:
$$\mathcal{L}[f(ct)] = \frac{1}{c}F(\frac{s}{c})$$
and
$$\frac{1}{c}\mathcal{L}[f(\frac{t}{c})] = F(cs)$$
where $c > 0$.

1.2 Applications

1.2.1 Higher order ODEs

So consider the following nth order ODE:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = f(t)$$

for $f \in E$. After taking the Laplace transform of both sides we get:

$$Z(s)\mathcal{L}[y(t)] = \mathcal{L}[f(t)] + Z_0(s)$$

where $Z(s)$ is a degree n polynomial and $Z_0(s)$ a degree $n-1$ polynomial dependent on our initial conditions.

Now if our source term is of the following form:

$$f(t) = t^n e^{at}\big(A\cos bt + B\sin bt\big)$$

then $\mathcal{L}[f(t)]$ is rational and therefore:

$$\mathcal{L}[y(t)] = \frac{\mathcal{L}[f(t)]}{Z(s)} + \frac{Z_0(s)}{Z(s)}$$

where we can solve this via standard transforms.

1.2.2 Discontinuous source terms

Since the Laplace transform is convergent for all <u>piecewise</u> continuous functions, we can use it to solve ODEs with discontinous source terms:

$$Ay''(t) + By'(t) + Cy(t) = g(t)$$

where $g(t)$ is piecewise continuous and of the following form:

$$g(t) = f(t)[u_a(t) - u_b(t)] = \begin{cases} f(t) & t \in [a,b) \\ 0 & t \notin [a,b) \end{cases}$$

for $b > a$. This is the **unit step function**:

$$u_c(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq c \end{cases}$$

and is also known as the Heaviside function. Its Laplace transform is:

$$\mathcal{L}[u_c(t)](s) = \frac{e^{-sc}}{s}.$$

Furthermore we can define a <u>shift</u> of $f(t)$ by $c > 0$ to the <u>right</u> by:

$$f(t-c)u_c(t)$$

$$\mathcal{L}[f(t-c)u_c(t)] = e^{-sc}F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

1.2.3 Impulse functions

The **Dirac delta** is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

and physically a “strike” on a system in infinitely short time. We have that:

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\mathrm{d}t = f(t_0)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[u_{t_0}(t)\Big] = \delta(t-t_0)$$

and

$$\mathcal{L}[\delta(t-t_0)] = e^{-st_0}.$$

If $t_0 = 0$ then $\mathcal{L}[\delta(t)] = \lim_{t_0\to 0}\big(e^{-st_0}\big) = 1$.

Taking the Laplace transform of the following ODE

$$y''(t) + y(t) = \delta(t)$$

with initial conditions $y(0) = y'(0) = 0$ gives:

$$\mathcal{L}[y(t)] = \frac{1}{s^2+1}\lim_{t_0\to 0}\big(e^{-st_0}\big).$$

It is important that we do not evaluate the limit here!

Finally by inspection we have that:

$$\begin{aligned} y(t) &= \lim_{t_0\to 0}\Big(\sin(t-t_0)u_{t_0}(t)\Big) \\ &= \sin(t)u_0(t). \end{aligned}$$

1.2.4 Convolutions

Convolutions can also be helpful in solving ODEs. They are defined:

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(s)g(t-s)\mathrm{d}s \\ &= \int_0^t g(s)f(t-s)\mathrm{d}s \end{aligned}$$

for functions $f, g : [0,\infty) \to \mathbb{R}$ and has the following properties:

- $f * (g+h) = f*g + f*h$
- $f * g = g * f$
- $f * (g*h) = (f*g)*h$
- $f * 1 \neq f$
- $f * f \neq f^2$.

Importantly we have the **convolution theorem**:

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

where if $f, g \in E$ then $f * g \in E$, or that they are functions of exponential order so that our Laplace transforms converge. Note that:

$$\int_0^t f(s)\mathrm{d}s = \int_0^t f(s)u_0(t-s)\mathrm{d}s$$

and therefore

$$\mathcal{L}\Big[\int_0^t f(s)\mathrm{d}s\Big] = \frac{\mathcal{L}[f(t)]}{s}.$$

Now consider the following ODE:

$$ay''(t) + by'(t) + cy(t) = g(t)$$

with initial conditions $y(0) = \alpha$ and $y'(0) = \beta$. Here $a, b, c, \alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} \therefore \mathcal{L}[y(t)] &= \Phi(s) + \Psi(s) \\ &= \frac{(as+b)\alpha + a\beta}{as^2+bs+c} + \frac{1}{as^2+bs+c}\mathcal{L}[g(\end{aligned}$$

$$\begin{aligned} \therefore y(t) &= \mathcal{L}^{-1}[\Phi(s)] + \mathcal{L}^{-1}[\Psi(s)] \\ &= \phi(t) + \psi(t) \end{aligned}$$

The first expression $\phi(t)$ can be found via standard transforms. For the second expression we define the **transfer function**[1]:

$$H(s) = \frac{1}{as^2+bs+c}$$

where it is the Laplace transform of the following corresponding ODE:

$$ah''(t) + bh'(t) + ch(t) = \delta(t)$$

with initial conditions $h(0) = h'(0) = 0$. $\therefore H(s) = \mathcal{L}[h(t)]$

This is helpful since:

$$\begin{aligned} \Psi(s) &= \frac{1}{as^2+bs+c}\mathcal{L}[g(t)] \\ &= H(s)\mathcal{L}[g(t)] \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}[\psi(t)] &= \mathcal{L}[h(t)]\mathcal{L}[g(t)] \\ &= \mathcal{L}[h(t) * g(t)]. \end{aligned}$$

Applying the convolution theorem we get:

$$\begin{aligned} \psi(t) &= h(t) * g(t) \\ &= \int_0^t h(s)g(t-s)\mathrm{d}s. \end{aligned}$$

[1] Also known as a Green's function.

1.3 Standard transforms

- $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ where $n \in \mathbb{N}$ and $s > 0$.
- $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$ where $s > a$.
- $\mathcal{L}[t^n e^{at}] = \dfrac{n!}{(s-a)^{n+1}}$ where $n \in \mathbb{N}$ and $s > a$.
- $\mathcal{L}[\sin at] = \dfrac{a}{s^2+a^2}$,

  $\mathcal{L}[\cos at] = \dfrac{s}{s^2+a^2}$ where $s > 0$.
- $\mathcal{L}[\sinh at] = \dfrac{a}{s^2-a^2}$,

  $\mathcal{L}[\cosh at] = \dfrac{s}{s^2-a^2}$ where $s > |a|$.
- $\mathcal{L}[e^{at}\sin bt] = \dfrac{b}{(s-a)^2+b^2}$,

  $\mathcal{L}[e^{at}\cos bt] = \dfrac{s-a}{(s-a)^2+b^2}$ where $s > a$.
- $\mathcal{L}[u_c(t)] = \dfrac{e^{-cs}}{s}$ where $s > 0$.
- $\mathcal{L}[\delta(t-c)] = e^{-cs}$.