

**T: Triangle inequalities**

Let  $\alpha, \beta \in \mathbb{R}$ . We then have that:

1.  $|\alpha| + |\beta| \geq |\alpha + \beta|$
2.  $||\alpha| - |\beta|| \leq |\alpha - \beta|$ .

**D: Supremum and infimum**

Let  $\alpha = \sup S$ . Then:

1.  $\forall s \in S; \alpha \geq s$
2.  $\forall a \in \mathbb{R} : \forall s \in S; a \geq s;$   
 $a \geq \alpha$

and similarly for infimum.

**T: Approximation property**

Consider bounded  $E \subset \mathbb{R}$ . Then:

$$\forall \epsilon > 0; \exists a \in E : \sup E - \epsilon < a \leq \sup E.$$

**D: Completeness of  $\mathbb{R}$** 

Every nonempty bounded subset of  $\mathbb{R}$  has an infimum and supremum.

**T: Archimedean property**

$$\forall a, b \in \mathbb{R}; a > 0; \exists n \in \mathbb{N} : na > b$$

**D1.1: Nested intervals**

A sequence of sets  $(I_n)_{n \in \mathbb{N}}$  is nested if  $I_1 \supset I_2 \supset I_3 \dots$ .

**T1.1: Nested interval property**

Let  $(I_n)_{n \in \mathbb{N}}$  be a sequence of nonempty, closed and bounded nested intervals. Then:

$$E = \bigcap_{n \in \mathbb{N}} I_n \neq \emptyset.$$

If  $\lambda(I_n) \rightarrow 0$  then  $E$  contains one number, where  $\lambda$  denotes length.

**T1.2**

Let  $E = [a, b]$  and that there exists an open collection of nested intervals  $(I_\alpha)_{\alpha \in A}$  such that:

$$E \subset \bigcup_{\alpha \in A} I_\alpha.$$

Then  $\exists \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset A$  such that:

$$E \subset I_{\alpha_1} \cup I_{\alpha_2} \cup \dots \cup I_{\alpha_n}.$$

**D1.2:  $\epsilon$ - $N$  convergence**

Let  $\lim_{n \rightarrow \infty} x_n = a$ . Then:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |x_n - a| < \epsilon.$$

**D1.3: Cauchy sequences**

The sequence  $(x_n)$  is Cauchy if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \implies |x_n - x_m| < \epsilon.$$

**T1.3 and T1.4**

Cauchy  $\iff \epsilon$ - $N$  convergent.

**T: Monotone convergence****D1.4: Subsequences**

The subsequence of  $(x_n)_{n \in \mathbb{N}}$  is a sequence of form  $(x_{n_k})_{k \in \mathbb{N}}$  and is a selection of the original sequence **taken in order**.

**T1.5: Bolzano-Weierstrass**

Every bounded real sequence has a convergent subsequence.

**D1.5: Limit inferior and superior**

Let  $(x_n)$  be a bounded real sequence. Then:

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right)$$

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right).$$

**T1.6**

The real sequence  $(x_n)$  is convergent if and only if:

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n.$$

**D1.6: Convergence of infinite series**

Series  $S = \sum_{k=1}^{\infty} a_k$  is convergent if:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_k < \infty.$$

Series  $S$  is **absolutely convergent** if  $\sum_{k=1}^{\infty} |a_k|$  is also convergent.

Otherwise  $S$  is conditionally convergent.

**T1.7: Cauchy criterion for series**

$S = \sum_{k=1}^{\infty} a_k$  is convergent **iff**:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall m \geq n \geq N \implies \left| \sum_{k=n+1}^m a_k \right| < \epsilon.$$

**T1.8**

Let  $S = \sum_{k=1}^{\infty} a_k$  be absolutely convergent. Let  $z : \mathbb{N} \rightarrow \mathbb{N}$  be a bijection. Then:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}.$$

**T1.9: Riemann rearrangement**

Let  $S = \sum_{k=1}^{\infty} a_k$  be conditionally convergent. Then there exists rearrangements such that  $S$  can take on any value.

**D1.7: Sequential continuity**

Let  $f : \text{dom}(f) \rightarrow \mathbb{R}$  where  $\text{dom}(f) \subset \mathbb{R}$ .  $f$  is continuous at  $\alpha \in \text{dom}(f)$  if:

$$\forall (x_n)_{n \in \mathbb{N}} \subset \text{dom}(f) : \lim_{n \rightarrow \infty} x_n = \alpha \implies \lim_{n \rightarrow \infty} f(x_n) = f(\alpha).$$

**T1.10**

Let  $\alpha \in \mathbb{R}$  and  $f, g$  continuous on  $D$ . Then  $\alpha f, f + g, fg$  are continuous on  $D$ .

**T1.11**

Let  $f$  be continuous at  $\alpha \in \mathbb{R}$  and  $g$  at  $f(\alpha)$ . Then  $g \circ f$  is continuous at  $\alpha$ .

**D1.12:  $\epsilon$ - $\delta$  continuity**

Let  $f : \text{dom}(f) \rightarrow \mathbb{R}$  where  $\text{dom}(f) \subset \mathbb{R}$ . Then  $f$  is continuous at  $\alpha \in \text{dom}(f)$  if:

$$\forall \epsilon > 0; \exists \delta > 0 : |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon.$$

**T1.13: Intermediate value theorem**

Let  $f$  be continuous on  $[a, b]$ . If  $f(a)f(b) < 0$  then:

$$\exists c \in (a, b) : f(c) = 0.$$

**T1.14: Extreme value theorem**

Let  $f$  be continuous on  $[a, b]$ . Then  $\exists c, d \in [a, b]$  such that:

$$f(c) = \inf\{f(x) : x \in [a, b]\}$$

$$f(d) = \sup\{f(x) : x \in [a, b]\}.$$

**T: Mean value theorem**

Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then:

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**D: Differentiability**

$f$  is differentiable at  $\alpha$  if:

$$f'(\alpha) = \lim_{h \rightarrow 0} \frac{f(\alpha + h) - f(\alpha)}{h}.$$

**T: Continuity test**

$f$  is continuous at  $\alpha$  if:

$$\lim_{x \rightarrow \alpha} f(x) = f(\alpha)$$

where the limit from left/right must both exist and be equal to each other.

**D2.1: Pointwise convergence**

$f_n \rightarrow f$  pointwise on  $E$  if:

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Here  $f_n : E \rightarrow \mathbb{R}$  and:

$$\begin{aligned} \forall x \in E; \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \\ \implies |f_n(x) - f(x)| < \epsilon. \end{aligned}$$

**D2.2: Uniform convergence**

$f_n \rightarrow f$  uniformly on  $E$  if:

$$\begin{aligned} \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \text{ and } \forall x \in E \\ \implies |f_n(x) - f(x)| < \epsilon. \end{aligned}$$

**P2.1**

The following statements are equivalent.

1.  $f_n \rightarrow f$  uniformly on  $E$
2.  $\lim_{n \rightarrow \infty} \sup_{x \in E} |f_n(x) - f(x)| = 0$
3.  $\exists a_n \rightarrow 0$  s.t.  $|f_n(x) - f(x)| \leq a_n$  for  $\forall x \in E$ .

**T2.1**

If  $f_n$  is continuous on  $E$  **and**  $f_n \rightarrow f$  uniformly on  $E$  then  $f$  is continuous on  $E$ .

**Remark**

If  $f$  is not continuous on  $E$  then  $f_n$  cannot be uniform on  $E$ .

**T2.5: Weierstrass M-test**

Let  $E \subset \mathbb{R}$  and  $f_k : E \rightarrow \mathbb{R}$ .

$$\exists M_k > 0 : \sum_{k=1}^{\infty} M_k < \infty.$$

If  $\forall k \in \mathbb{N}$  and  $\forall x \in E; |f_k(x)| \leq M_k$  then:

$$\sum_{k=1}^{\infty} f_k(x) \text{ converges uniformly on } E.$$

**D: Power series**

Let  $(a_n)$  be a real sequence and  $c \in \mathbb{R}$ . Then:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

is a power series centered at  $c$ , with **radius of convergence**:

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

where  $R = \infty$  implying that series converges everywhere.

**T3.1: Convergence of power series**

Let  $0 < R < \infty$ . If  $|x - c| < R$  then  $f_{PS}(x)$  converges absolutely.

If  $|x - c| > R$  then  $f_{PS}(x)$  diverges.

**T3.2: Continuity of power series**

Let  $0 < r < R$  where  $R$  is the radius of convergence of  $f_{PS}(x)$ .

Then for  $|x - c| \leq r$ ,  $f_{PS}(x)$  converges absolutely and uniformly to a continuous function  $f(x)$ .

**L3.1**

$\sum_{n=1}^{\infty} a_n(x-c)^n$  and  $\sum_{n=1}^{\infty} n a_n(x-c)^{n-1}$  have the same radius of convergence.

**T: Root and ratio tests**

Let  $S = \sum_{n=1}^{\infty} a_n$  and consider:

1. Ratio test:  $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$
2. Root test:  $\rho = \lim_{n \rightarrow \infty} |a_n|^{1/n}$ .

Then:

- $\rho < 1$ :  $S$  converges absolutely
- $\rho > 1$ :  $S$  diverges
- $\rho = 1$ : test is inconclusive.

**T3.3**

Let  $R$  be the radius of convergence of  $f_{PS}(x)$ . Then for  $\forall x : |x - c| < R$ ,  $f_{PS}(x)$  is **infinitely differentiable** and:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

**T: Taylor's theorem**

Let  $f$  be  $n$  times differentiable at  $\alpha \in \mathbb{R}$  where  $n \in \mathbb{N}$ . Then:

$$\begin{aligned} f(x) = \sum_{k=1}^n \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k \\ + h_n(x)(x-\alpha)^n \end{aligned}$$

where  $\lim_{x \rightarrow \alpha} h_n(x) = 0$ .

**Elementary expansions**

$$\bullet E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\begin{aligned} \bullet S(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \bullet C(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \end{aligned}$$

**D: Characteristic functions**

Let  $E \subset \mathbb{R}$ . The characteristic function is defined as a real function such that:

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise.} \end{cases}$$

**D4.1 and D4.2: Step functions**

The step function with respect to finite set  $\{x_0, \dots, x_n\}$  for some  $n \in \mathbb{N}$  is:

$$\phi(x) = \begin{cases} 0 & x < x_0 \text{ or } x > x_n \\ c_j & x \in (x_{j-1}, x_j); 1 \leq j \leq n \end{cases}$$

and its integral is defined as:

$$\int \phi = \sum_{j=1}^n c_j(x_j - x_{j-1}).$$

**D4.3: Lebesgue integrable**

$f : I \rightarrow \mathbb{R}$  is Lebesgue integrable on  $I$  if:

1.  $\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$
2.  $\forall x \in I; f(x) = \sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$

Here  $c_j \in \mathbb{R}$ ,  $J_i \subset I$  and is bounded for  $j \in \{1, 2, 3, \dots\}$ . Then:

$$\int_I f = \sum_{j=1}^{\infty} |c_j| \lambda(J_j).$$

**T4.1****T4.2: Basic properties**

Let  $f, g$  be integrable on  $I$  and  $\alpha, \beta \in \mathbb{R}$ .

1.  $\alpha f + \beta g$  is integrable on  $I$  and:

$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g.$$

2. If  $f \geq g$  on  $I$  then  $\int_I f \geq \int_I g$ .

3.  $|f|$  is integrable on  $I$  and:

$$\int_I |f| \geq \left| \int_I f \right|.$$

4. If  $f$  or  $g$  is bounded on  $I$  then  $fg$  is integrable on  $I$ .

5. If  $f \geq 0$  and  $\int_I f = 0$ , then  $\forall h$  such that  $0 \leq h \leq f$  is also integrable on  $I$ .

**T4.3****T4.4: MCT for integrals**

Let  $f_n$  be monotone increasing sequence of functions on  $I$  and that:

$$\forall x \in I; f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Then  $f$  is integrable on  $I$  **iff**:

$$\sup_{n \in \mathbb{N}} \int_I f_n = \lim_{n \rightarrow \infty} \int_I f_n < \infty.$$

Furthermore:

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

**D4.4: Riemann integrable**

$f$  is Riemann-integrable if:

$$\forall \epsilon > 0; \exists \phi, \psi : \phi \leq f \leq \psi$$

$$\text{and } \int \psi - \int \phi < \epsilon$$

where  $\phi$  and  $\psi$  are step functions,  
i.e. the bounded support of  $f$ .

**T4.5**

$f$  is Riemann-integrable if and only if:

$$\sup \left\{ \int \phi : \phi \leq f \right\} = \inf \left\{ \int \psi : f \leq \psi \right\}$$

where  $\phi$  and  $\psi$  are step functions.

**T4.6**

If  $f$  is Riemann-integrable on  $I$  then  $f$  is also Lebesgue-integrable on  $I$ .

**Remark**

The converse of T4.6 is not true.

**L4.1**

Let  $f$  be a bounded function with bounded support on  $[a, b]$ . The following statements are equivalent:

1.  $f$  is Riemann-integrable.
2.  $\forall \epsilon > 0; \exists \{a = x_0 < \dots < x_n = b\} :$

$$\sum_{j=1}^n (M_j - m_j)(x_j - x_{j-1}) < \epsilon$$

where we define:

$$M_j = \sup_{x \in (x_{j-1}, x_j)} \{f(x)\}$$

$$m_j = \inf_{x \in (x_{j-1}, x_j)} \{f(x)\}.$$

3.  $\forall \epsilon > 0; \exists \{a = x_0 < \dots < x_n = b\} :$

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

where  $I_j = (x_{j-1}, x_j)$ .

**T4.7**

Let  $g : [a, b] \rightarrow \mathbb{R}$  and  $f$  be such that  $f(x) = g(x)$  if  $x \in [a, b]$  and  $f(x) = 0$  otherwise.

1. If  $g$  is continuous on  $[a, b]$  then  $f$  is Riemann-integrable.

2. If  $g$  is a monotone function then  $f$  is Riemann-integrable.

**T4.8**

Let  $J \subset I$ .

1. If  $f$  is integrable on  $I$  then  $f$  is integrable on  $J$ .

2. If  $f$  is integrable on  $J$  and for  $\forall x \in I \setminus J; f(x) = 0$  then  $f$  is integrable on  $I$ .

Furthermore:  $\int_J f = \int_I f$ .

**T4.9****T4.10: FTC I****T4.11: FTC II****L4.2: Fatoux's lemma****T4.12: Dominated convergence****T4.13****D5.1:  $L^2$  space****D5.2: Inner products****T5.1: Cauchy-Schwarz inequality****C: Minkowski's inequality****D5.3:  $L^2$  convergence****D5.4: Orthonormal systems****T5.2****T5.3: Bessel's inequality****C: Riemann-Lebesgue lemma****D5.5: Completeness****T5.4****D5.6: Trigonometric polynomial****L5.1****D5.7: Fourier series****D5.8: Fourier partial sums****D5.9: Convolutions****L5.2****L5.3: Dirichlet kernel****L5.4: Fejér kernel****T5.5: Fejér's theorem****D5.10: Approximation of unity****T5.6****C****L5.5****T5.7****C: Parseval's theorem**