Honours Analysis Examples

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1 Lebesgue integrals

1.1 Characteristic and step functions

1. Let ϕ and ψ be step functions. Show that the following are also step functions:

   - $\alpha\phi + \beta\psi$
   - $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\}$
   - $\phi\psi$.

   Firstly define step function ϕ wrt finite set $\{x_0, x_1 \ldots, x_n\}$ and step function ψ wrt finite set $\{y_0, y_1 \ldots, y_n\}$. Their linear combination is a step function wrt the union of these two finite sets.

   The rest are proven similarly via construction.

2. Show that ϕ is a step function **if and only if**:

$$\phi(x) = \sum_{j=1}^{n} c_j \chi_{J_j}(x)$$

   where c_j is some constant and J_j some interval.

   From the definition of ϕ we can write:

$$\phi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1}, x_j)}(x) + \sum_{j=0}^{n} \phi(x_j)\chi_{\{x_j\}}$$

   and the opposite direction is shown by the first example and the fact that a characteristic function is a step function.

3. Write step function $\phi = \chi_{[0,1]}$ in two ways.

   We can write it as piecewise function. (def of chi)

   But also as def 4.1 constants.

   So integral either length of $[0, 1]$ or sum of constants times subintervals.

4. Show that the following function is Lebesgue integrable:

$$f(x) = \frac{1}{[x][x+1]}$$

where $[x]$ denotes the floor function.

So:

$$f = \sum_{j=1}^{\infty} |\frac{1}{j(j+1)}| \chi_{[j,j+1)}(x)$$

and integral value is 1.

5. Show that χ_E is integrable where $E \subset I$ and is bounded and <u>countable</u>.

$E = \{e_1, e_2, \dots\}$ and so:

$$\chi_E = \sum_{j=1}^{\infty} e_j \chi_{\{e_j\}}(x)$$

which has integral of zero.

6. Cantor set is Lebesgue integrable.

1.2 Riemann integrals

1. Show that step functions are Riemann-integrable.

   Because $\phi \leq \phi \leq \phi$ and using definition.

2. Show that if f is Riemann-integrable on $[a, b]$ then f is bounded and has bounded support.

   Further show that if $f(x) = 0$ when $x \notin [a, b]$ then our step functions are zero outside $[a, b]$.

3. Show that the following functions are **not** Riemann-integrable:

   - $f(x) = e^{-|x|}$
   - $g(x) = x^{-\frac{1}{2}}\chi_{(0,1)}(x)$

   Not quite sure about f yet, maybe because we cannot define a step function wrt to an infinite set?

   The function $g(x)$ has no upper step function.

4. Find infinitely countable $E \subset \mathbb{R}$ such that χ_E is Riemann-integrable.

   Is every such χ_E is Riemann-integrable?

   Consider $E = \{\frac{1}{n} : n \in \mathbb{N}\}$.

   Consider the Dirichlet function.

1.3 Lebesgue integrals

1.3.1 Basic properties

1.