

D: Functions

A function $f : X \rightarrow Y$ is an assignment of an element of Y to each element of X .

1. f is **injective** if:

$$\forall x_1, x_2 \in X; f(x_1) = f(x_2) \\ \implies x_1 = x_2.$$

2. f is **surjective** if:

$$\forall y \in Y; \exists x \in X : y = f(x).$$

3. f is **bijective** if it is injective and surjective.

D: Groups

A group G is a set defined with:

1. Composition operator (\cdot) such that $x \cdot y = xy$.
2. $\forall x, y, z \in G; (xy)z = x(yz)$
3. $\exists e \in G : ex = xe = x$ for $\forall x \in G$.
4. $\exists x^{-1} \in G : xx^{-1} = x^{-1}x = e$ for $\forall x \in G$.

G is **Abelian** if $\forall x, y \in G; xy = yx$.

D1.2.1(i): Fields

A field F is a set defined with:

1. Addition function $(+)$:
 $(+) : F \times F \rightarrow F; (\lambda, \mu) \mapsto \lambda + \mu$
2. Multiplication function (\cdot) :
 $(\cdot) : F \times F \rightarrow F; (\lambda, \mu) \mapsto \lambda \cdot \mu$
3. $\exists 0_F, 1_F \in F$ where $0_F \neq 1_F$ such that $(F, +)$ and $(F \setminus \{0_F\}, \cdot)$ form Abelian groups.
4. $\exists (-\lambda) \in F : \lambda + (-\lambda) = 0_F$
5. $\exists (\lambda^{-1}) \in F : \lambda \cdot (\lambda^{-1}) = 1_F$
6. $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F$

D1.2.1(ii): Vector spaces

A vector space V over a field F is an Abelian group $V := (V, +)$ with mapping:

$$F \times V \rightarrow V : (\lambda, \mathbf{v}) \mapsto \lambda \mathbf{v}$$

where for $\forall \lambda, \mu \in F$ and $\forall \mathbf{v}, \mathbf{w} \in V$:

1. $\lambda(\mathbf{v} + \mathbf{w}) = (\lambda\mathbf{v}) + (\lambda\mathbf{w})$
2. $(\lambda + \mu)\mathbf{v} = (\lambda\mathbf{v}) + (\mu\mathbf{v})$
3. $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$
4. $1_F\mathbf{v} = \mathbf{v}$

and is a F -vector space.

Remark

Let V be a F -vector space where $\mathbf{v} \in V$.

1. $0\mathbf{v} = \mathbf{0}$
2. $(-1)\mathbf{v} = -\mathbf{v}$
3. $\lambda\mathbf{0} = \mathbf{0}$ for $\forall \lambda \in F$.

D: Cartesian products

The Cartesian product of sets X_1, \dots, X_n is defined as:

$$X_1 \times \dots \times X_n := \{(x_1, \dots, x_n) : x_i \in X_i\}$$

where $1 \leq i \leq n$.

The projection of a Cartesian product is:

$$\text{pr}_i : X_1 \times \dots \times X_n \rightarrow X_i; \\ (x_1, \dots, x_n) \mapsto x_i$$

D1.4.1: Vector subspaces

A vector subspace U of F -vector space V has the following properties:

1. $U \subset V$ and $\mathbf{0} \in U$.
2. Let $\mathbf{u}, \mathbf{v} \in U$ and $\lambda \in F$. Then $\mathbf{u} + \mathbf{v} \in U$ and $\lambda\mathbf{u} \in U$.

and is also a vector space.

P1.4.5

Let $T \subset V$ where V is a F -vector space. Then for all vector subspaces containing T , there exists a smallest vector subspace:

$$\text{span}(T) = \langle T \rangle_F \subset V$$

known as the vector subspace generated by T , or the span of T .

D1.4.7: Generating set

Let $T \subset V$ where V is a F -vector space. T is a generating set of V if:

$$\text{span}(T) = V$$

and is the linear combination of vectors in T over field F .

D1.4.9: Power sets

The power set of set X is:

$$\mathcal{P}(X) := \{U : U \subseteq X\}.$$

Let $\mathcal{U} \subseteq \mathcal{P}(X)$. Then:

$$\bigcup_{U \in \mathcal{U}} U := \{x \in X : (\exists U \in \mathcal{U} : x \in U)\}$$

$$\bigcap_{U \in \mathcal{U}} U := \{x \in X : \forall U \in \mathcal{U}; x \in U\}.$$

D1.5.1: Linear independence

Let V be a F -vector space and $L \subseteq V$. L is linearly independent if:

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_r \mathbf{v}_r = \mathbf{0} \\ \implies \alpha_1 = \dots = \alpha_r = 0$$

where $\mathbf{v}_i \in L$.

D1.5.8: Basis

A basis of a vector space V is a linearly independent generating set in V .

T1.5.11

Let V be a F -vector space.

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a basis of V **iff**:

$$\Phi : F^r \rightarrow V;$$

$$(\alpha_1, \dots, \alpha_r) \mapsto \alpha_1 \mathbf{v}_1 + \dots + \alpha_r \mathbf{v}_r$$

is a bijection.

T1.5.12

Let V be a vector space and $E \subseteq V$. Then the following statements are equivalent:

1. E is a basis of V .
2. E is minimal among all generating sets, or that $E \setminus \{\mathbf{v}\}$ is not a basis for $\forall \mathbf{v} \in V$.
3. E is maximal amongst all linearly independent subsets. i.e. $E \cup \{\mathbf{v}\}$ is not linearly independent.

C1.5.13

Every finitely generated vector space has a finite basis. (any vector space too!)

T1.5.14

Let V be a vector space.

1. Let $L \subseteq V$ be linearly independent and set E be minimal amongst all generating sets of V . Let $L \subseteq E$. Then E is a basis of V .
2. Let $E \subseteq V$ be a generating set and L be maximal amongst all linearly independent subsets of V .

Let $L \subseteq E$. Then E is a basis of V .

D1.5.15

Let X be a set and F be a field. Then:

$$\text{maps}(X, F) := \{f : (\forall f : X \rightarrow F)\}$$

and is a F -vector space under pointwise addition and multiplication via scalars.

Remark

The subset of all mappings which sends almost all elements of X to 0 is defined: (all but finitely many)

$$F\langle X \rangle \subseteq \text{maps}(X, F)$$

and is a vector subspace.

T1.5.16

Let V be a F -vector space.

Then $(v_i)_{i \in I}$ is a basis for V **iff**:

$$\forall v \in V; \exists! (a_i)_{i \in I} \subseteq F : v = \sum_{i \in I} a_i v_i.$$

T1.6.1

Let V be a vector space. Let $L \subset V$ be a linearly independent subset and $E \subseteq V$ a generating set. Then $|L| \leq |E|$.

T1.6.2: Steinitz exchange theorem

Let V be a vector space, $L \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set.

Then there exists an **injective** function $\phi : L \rightarrow E$ such that:

$$(E \setminus \phi(L)) \cup L$$

is also a generating set for V .

L1.6.3: Exchange lemma

Let V be a vector space. Let $M \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set where $M \subseteq E$.

If $\exists w \in V \setminus M$ such that set $M \cup \{w\}$ is linearly independent then:

$$\exists e \in E \setminus M : (E \setminus e) \cup \{w\} \text{ is generating.}$$

C1.6.4

Let V be a finitely generated vector space.

1. V has finite basis.
2. V cannot have infinite basis.
3. Any two basis of V have the same number of elements.

D1.6.5: Dimension

The dimension of finite F -vector space V is the cardinality of one its basis.

For infinite vector spaces: $\dim(V) = \infty$.

C1.6.7

Let V be a finitely generated vector space.

1. Every linearly independent $L \subseteq V$ has **at most** $\dim(V)$ elements and if $|L| = \dim(V)$ then L is a basis.
2. Every generating set $E \subseteq V$ has **at least** $\dim(V)$ elements and if $|E| = \dim(V)$ then E is a basis.

C1.6.8

A proper vector subspace of a vector space with finite dimension has itself a strictly smaller dimension.

T1.6.10

Let V be a vector space and $U, W \subseteq V$ be vector subspaces. Then:

$$\begin{aligned} \dim(U + W) + \dim(U \cap W) \\ = \dim(U) + \dim(W). \end{aligned}$$