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2.1 True tensors

tensor algebra

2.1.1 Rank 2 quotient theorem

The **quotient theorem** is as an alternative definition for tensors. In the context of <u>rank 2</u> tensors it states that if b_i always transforms as a <u>vector</u> in

$$b_i = T_{ij} a_j$$

and that a_j is also a vector then T_{ij} is a rank 2 tensor.

Proof. We egregiously define entity T_{ij} in frame S and T'_{ij} in frame S'.

The usual transformation laws apply, namely $\boldsymbol{e}'_i = \ell_{ij} \boldsymbol{e}_j$. By definition:

$$\begin{aligned} b'_i &= T'_{ij} a'_j \\ &= T'_{ij} \ell_{jk} a_k \end{aligned}$$

Also directly from transformation laws:

$$\begin{aligned} b'_i &= \ell_{ij} b_j \\ &= \ell_{ij} T_{jk} a_k \end{aligned}$$

$$\therefore \left(T'_{ij} \ell_{jk} - \ell_{ij} T_{jk}\right) a_k = 0$$

Since a_k are constants of our vector it must then be that:

$$T'_{ij} \ell_{jk} = \ell_{ij} T_{jk}$$

$$\therefore T'_{ij} \ell_{jk} \ell_{mk} = \ell_{ij} \ell_{mk} T_{jk}$$

Where here we aim to eliminate the first two ℓs. Finally:

$$T'_{im} = \ell_{ij} \ell_{mk} T_{jk}$$

□

2.1.2 General quotient theorem

Let $R_{ij\ldots r}$ be a rank m tensor, and $T_{ij\ldots s}$ be a set of 3^n numbers where $n > m$.

If $R_{ij\ldots r} T_{ij\ldots s}$ is a rank $n - m$ tensor then $T_{ij\ldots s}$ is a rank n tensor.

symmetric and anti symmetric tensors

2.2 Matrices as tensors

2.3 Pseudotensors

Firstly note that $\det L = +1$ for rotations, and $\det L = -1$ for reflections and inversions. Recall the transformation law $\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j$.

A second rank **pseudotensor** is defined:

$$T'_{ij} = (\det L)\ell_{ip}\ell_{jq}T_{pq}.$$

Furthermore a rank 1 pseudotensor is a **pseudovector** and is defined as:

$$T'_i = (\det L)\ell_{ip}T_p.$$

Finally a **pseudoscalar** is a rank 0 pseudotensor:

$$a' = (\det L) \cdot a,$$

and changes sign under transformation.

2.4 Invariant tensors

2.5 Rotation tensors

2.6 Reflections, inversions and projections

active and passive transformations

maybe merge with rotations?

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4.1.2 Divergence

4.1.3 Curl

chain rules, important identities

4.2 Integrals theorems

4.2.1 Line, volume and surface integrals

4.2.2 Divergence theorem

4.2.3 Stokes's theorem

Consider surface S enclosed by line C. We then have that:

$$\int_S \boldsymbol{\nabla} \times \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}.$$

5 Curvilinear coordinates

5.1 Orthogonal curvilinear coordinates

5.1.1 Scale factors and basis vectors

Consider change of variables:

$$(x_1, x_2, x_3) \leftrightarrow (u_1, u_2, u_3)$$

where u_i are our curvilinear coordinates, and

$$u_i = u_i(x_1, x_2, x_3)$$
$$x_i = x_i(u_1, u_2, u_3).$$

Then we define:

$$\begin{aligned} \mathrm{d}\boldsymbol{r}_i &= \frac{\partial \boldsymbol{r}}{\partial u_i}\mathrm{d}u_i \\ &= h_i \boldsymbol{e}_i \mathrm{d}u_i \end{aligned}$$

where $h_i = |\frac{\partial \boldsymbol{r}}{\partial u_i}|$ is our **scale factor** and

$$\boldsymbol{e}_i = \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial u_i}$$

is our **basis vector** of unit length for a specific set of curvilinear coordinates.

Now if the basis vectors satisfy

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

we have an orthogonal set of curvilinear coordinates.

5.1.2 Cylindrical coordinates

We define cylindrical coordinates as

$$(u_1, u_2, u_3) = (\rho, \phi, z)$$

and with the following relation to Cartesian coordinates:

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_x + \rho\sin\phi\boldsymbol{e}_y + z\boldsymbol{e}_z.$$

Furthermore:

$$h_\rho = 1 \quad \text{and} \quad \boldsymbol{e}_\rho = \cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y$$
$$h_\phi = \rho \quad \text{and} \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y$$
$$h_z = 1 \quad \text{and} \quad \boldsymbol{e}_z = \boldsymbol{e}_z.$$

Here ϕ is the anticlockwise rotation of the xy-plane.

5.1.3 Spherical coordinates

We define the spherical coordinates as

$$(u_1, u_2, u_3) = (r, \theta, \phi)$$

$$\boldsymbol{r} = r\sin\theta\cos\phi\boldsymbol{e}_x + r\sin\theta\sin\phi\boldsymbol{e}_y + r\cos\theta\boldsymbol{e}_z$$

where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ represent the Cartesian unit vectors.

Now $\phi \in [0, 2\pi]$ is the rotation angle in xy-plane, and $\theta \in [0, \pi]$ in z-plane. We also have that:

$$h_r = 1 \quad \text{and} \quad \boldsymbol{e}_r = \sin\theta\cos\phi\boldsymbol{e}_x + \sin\theta\sin\phi\boldsymbol{e}_y + \cos\theta\boldsymbol{e}_z$$

$$h_\theta = r \quad \text{and} \quad \boldsymbol{e}_\theta = \cos\theta\cos\phi\boldsymbol{e}_x + \cos\theta\sin\phi\boldsymbol{e}_y - \sin\theta\boldsymbol{e}_z$$

$$h_\phi = r\sin\theta \quad \text{and} \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y.$$

5.2 Length, area and volume

5.2.1 Vector and arc length

Firstly the **vector length** due to infinitesimal change in all directions is

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3} h_i \mathrm{d}u_i \boldsymbol{e}_i.$$

It is important to note that summation notation does not work here.

Now the **arc length** of $\mathrm{d}\boldsymbol{r}$ is:

$$\begin{aligned} \mathrm{d}s &= |\mathrm{d}\boldsymbol{r}| \\ &= \sqrt{\mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r}} \end{aligned}$$

and we define the **metric tensor** as

$$\begin{aligned} g_{ij} &= \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \\ &= \frac{\partial \boldsymbol{r}}{\partial u_i} \cdot \frac{\partial \boldsymbol{r}}{\partial u_j}. \end{aligned}$$

Since $\mathrm{d}\boldsymbol{r} = \mathrm{d}x_k$ we then the following relation:

$$(\mathrm{d}s)^2 = g_{ij} \mathrm{d}u_i \mathrm{d}u_j.$$

5.2.2 Vector area

5.2.3 Volume

The volume of the infinitesimal parallelepiped defined by $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$ is:

$$\begin{aligned}
\mathrm{d}V &= |(\mathrm{d}\boldsymbol{r}_1 \times \mathrm{d}\boldsymbol{r}_2) \cdot \mathrm{d}\boldsymbol{r}_3| \\
&= h_1\, h_2\, h_3\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3\, |(\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot \boldsymbol{e}_3| \\
&= \sqrt{g}\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u_3
\end{aligned}$$

where g is the determinant of the metric tensor.

6 Electrostatics

6.1 Dirac delta function

The one dimensional **Dirac delta** is defined:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0, \end{cases}$$

and can be thought of as infinitely sharp at $x = 0$ and zero elsewhere.

It satisfies some useful properties:

- $\delta(x-a) = \lim_{\sigma \to 0} \left[\frac{1}{|\sigma|\sqrt{\pi}} \exp\left(-\frac{(x-a)^2}{\sigma^2} \right) \right]$

  i.e. an infinitely sharp Gaussian. (generalised functions)
- **Sift property**
  $$\int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a)$$
- Let x_i be the solutions to $g(x_i) = 0$. Then:
  $$\int_{\mathbb{R}} f(x)\delta[g(x)]dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

Now we consider the **3D Dirac delta**, which is defined as follows:

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

given Cartesian coordinates (x_1, x_2, x_3). It also satisfies the **sift** property:

$$\int_{\mathbb{R}^3} f(\boldsymbol{r})\delta(\boldsymbol{r} - \boldsymbol{r}_0) = f(\boldsymbol{r}_0).$$

The three dimensional Dirac delta defined in a orthogonal curvilinear coordinate system (u_1, u_2, u_3) is as follows:

$$\delta(\boldsymbol{r} - \boldsymbol{a}) = \frac{1}{h_1 h_2 h_3}\delta(u_1 - a_1)\delta(u_2 - a_2)\delta(u_3 - a_3)$$

for h_1, h_2 and h_3 are the scale factors.

6.2 Coulomb's law

Consider the force on charge q at $\boldsymbol{r}$ due to charge q_1 at $\boldsymbol{r}_1$:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{qq_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3},$$

for here $\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$ in vacuum.

Physically, like charges ($qq_1 > 0$) repel while opposite charges ($qq_1 < 0$) attract.

We then define an **electric field** as the force on a small positive test charge:

$$\boldsymbol{E}(\boldsymbol{r}) = \lim_{q\to 0}\left(\frac{1}{q}\boldsymbol{F}(\boldsymbol{r})\right).$$

The force on a charge q at $\boldsymbol{r}$ from the origin in this electric field is:

$$\boldsymbol{F}(\boldsymbol{r}) = q\boldsymbol{E}(\boldsymbol{r}).$$

A negative point charge is a sink whereas a positive point charge is a source.

Consider a collection of charges q_i at position $\boldsymbol{r}_i$. The **principle of superposition** tells us that:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i\left(\frac{q_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}\right).$$

Now consider a continuous charged object with volume V and **charge density** $\rho(\boldsymbol{r}')$. It generates the following electric field:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}dV'.$$

Returning to the electric field generated by a point charge q_1 at position $\boldsymbol{r}_1$:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^3},$$

this is a **conservative field**, and we may write it as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r}),$$

where:

$$\phi(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1|}.$$

Conservative fields have zero curl, and their line integrals are path independent. This namely applies to finding work done.

6.3 Electrostatic Maxwell's equations

Firstly for a continuous charge distribution:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= -\frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V' \\ &= -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right)\end{aligned}$$

and therefore $\boldsymbol{\nabla}\times\boldsymbol{E}=\boldsymbol{0}$ for <u>static electric fields</u>.

Hence electrostatic fields are <u>conversative fields</u>:

$$\int_{C_1}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r} = \int_{C_2}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r}$$

and that

$$-\int_a^b \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r} = \phi(b)-\phi(a)$$

where $\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$. Therefore:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'.$$

6.4 Electric dipoles

Dipoles consist of two equal and **opposite point charges** that are $\boldsymbol{d}$ apart.

An **ideal dipole** is defined as when the following **dipole limit** is finite and constant:

$$\boldsymbol{p} = \lim_{\substack{q\to\infty \\ \boldsymbol{d}\to 0}} q\boldsymbol{d}.$$

A **dipole moment** is simply $\boldsymbol{p} = q\boldsymbol{d}$. The **dipole potential** at $\boldsymbol{r}_0$ is:

$$\begin{aligned}\phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{d}|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3},\end{aligned}$$

where we have Taylor expanded the first term about $|\boldsymbol{r}-\boldsymbol{r}_0|$. For simplicity we set $\boldsymbol{r}_0 = \boldsymbol{0}$. Then the **electric field** generated by our dipole at the origin is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\boldsymbol{p}\cdot\boldsymbol{r}}{r^5}\boldsymbol{r} - \frac{1}{r^3}\boldsymbol{p}\right),$$

since $\boldsymbol{E} = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$. Note that these formulae are in Cartesian coordinates.

Now we repeat this in spherical.

Force, torque and energy.

6.4.1 Multidipole expansion

potential

work done

6.5 Gauss's law

Gauss's law is the integral form of Maxwell's first equation:

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is the total charge enclosed by volume V. This result follows from the application of the divergence theorem and is useful in problems with symmetry.

6.5.1 Boundaries

6.5.2 Conductors

special case for electrostatics

6.6 Poisson's equation

In electrostatics we have:

$$\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

where ρ is our charge density. This is the **Poisson's equation** and is a consequence of the fact that $\nabla \times \boldsymbol{E} = \mathbf{0}$ and $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$.

6.6.1 Existence and uniqueness of solutions

The existence of solutions is given by the fact that:

$$\boldsymbol{E} = -\nabla \phi.$$

Poisson's equation has **unique** solution ϕ if we have volume V bounded by surface S and one of the following boundary conditions:

1.

method of images

6.7 Capacitors

7 Magnetostatics

charge distribution $\implies$ electric field

current $\implies$ magnetic field

7.1 Currents

Elementary current

Bulk current density

Surface current density

Line current

units!

Infinitesimal current element (dependent on material)
units: $Cs^{-1}m = Am$

Note that $J = Am^{-2}$.

Current flowing through surface and line.