

Vector products

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \text{ and } \mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear more than twice in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \end{aligned}$$

and we have the following identities:

$$\mathbf{a} = \sum_{i=1}^3 a_i \mathbf{e}_i = a_i \mathbf{e}_i$$

$$A\mathbf{x} = a_{ij}x_j \mathbf{e}_i \text{ for } m \times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\dots]_j \delta_{jk} = [\dots]_k$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k$$

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \mathbf{e}_i$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk} \epsilon_{ijk} = 6.$$

Transformations

Let matrix L relate basis $\{\mathbf{e}_i\}$ to basis $\{\mathbf{e}'_i\}$ with rule:

$$\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then $L^T L = L L^T = I$, and:

$$\ell_{ik} \ell_{jk} = \ell_{ki} \ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij} p_j \text{ for } \mathbf{p} = p_i \mathbf{e}_i = p'_i \mathbf{e}'_i.$$

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip} \ell_{jq} \ell_{kr} T_{pqr}$$

which relates frame S in $\{\mathbf{e}_i\}$ to frame S' in $\{\mathbf{e}'_i\}$ with rule $\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j$, etc.

Properties of tensors:

1. The addition of two rank n tensors is also a rank n tensor.
2. The multiplication of a rank m tensor with a rank n tensor yields a rank $m + n$ tensor.
3. If $T_{ijk\dots s}$ is a rank m tensor then $T_{\mathbf{ii}k\dots s}$ is a rank $m - 2$ tensor.
4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$\begin{aligned} T'_{ij} &= \ell_{ip} \ell_{jq} T_{pq} \implies T' = L T L^T \\ T'_{\mathbf{ji}} &= \ell_{jp} \ell_{iq} T_{pq}. \end{aligned}$$

Symmetric tensors

T_{ij} is a symmetric tensor when $T_{ij} = T_{ji}$ in frame S . Then $T'_{ij} = T'_{ji}$ in frame S' .

Similarly T_{ij} is an anti-symmetric tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally **any tensor** can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S' . Let $b_i = T_{ij} a_j$ where a_j is a vector. If b_i always transforms as a vector then T_{ij} is a rank 2 tensor.

Generalising, let $R_{ijk\dots r}$ be a rank m tensor and $T_{ijk\dots s}$ a set of 3^n numbers where $n > m$. If $T_{ijk\dots s} R_{ijk\dots r}$ is a rank $n - m$ tensor then $T_{ijk\dots s}$ is a rank n tensor.

Matrices

We define a $m \times n$ matrix A as $(A)_{ij} = a_{ij}$ where $i = 1, \dots, m$ and $j = 1, \dots, n$.

- $\text{Tr } A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^T A^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn} a_{1l} a_{2m} a_{3n} \\ &= \epsilon_{lmn} a_{l1} a_{m2} a_{n3}. \end{aligned}$$

Furthermore:

$$\begin{aligned} \epsilon_{ijk} \det A &= \epsilon_{lmn} a_{il} a_{jm} a_{kn} \\ \epsilon_{lmn} \det A &= \epsilon_{ijk} a_{il} a_{jm} a_{kn} \\ \det A &= \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} a_{il} a_{jm} a_{kn}. \end{aligned}$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3. $\det A = \det A^T$
4. $\det(AB) = \det A \cdot \det B$

These also apply to columns. Finally:

$$\epsilon_{ijk} \epsilon_{lmn} \det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting $A = I$ yields:

$$\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

Linear equations

Let $\mathbf{y} = A\mathbf{x}$. Then $x_i = A_{ij}^{-1} y_j$ with:

$$\begin{aligned} A_{ij}^{-1} &= \frac{1}{2 \det A} \epsilon_{imn} \epsilon_{jpk} a_{pm} a_{qn} \\ &= \frac{1}{\det A} C_{ij}^T \end{aligned}$$

where C is the cofactor matrix of A .

Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L) \ell_{ip} \ell_{jq} T_{pq}$$

where $(L)_{ij} = \ell_{ij}$ and $\det L = \pm 1$.

Pseudovectors are rank 1 pseudotensors.

Invariant tensors

Tensor T is invariant or isotropic if:

$$T_{ijk\dots} = \ell_{i\alpha} \ell_{j\beta} \ell_{k\gamma} \dots T_{\alpha\beta\gamma\dots}$$

for every orthogonal matrix L .

- If a_{ij} is a rank 2 invariant tensor then $a_{ij} = \lambda \delta_{ij}$.
- The most general rank 3 invariant pseudotensor is $a_{ijk} = \lambda \epsilon_{ijk}$. There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

Rotation tensors

The clockwise rotation of position vector \mathbf{x} to \mathbf{y} about unit vector $\hat{\mathbf{n}}$ is given by:

$$y_i = R_{ij}(\theta, \hat{\mathbf{n}})x_j$$

$$R_{ij}(\theta, \hat{\mathbf{n}}) = \delta_{ij} \cos \theta + (1 - \cos \theta)n_i n_j - \epsilon_{ijk} n_k \sin \theta$$

and is the rotation tensor.

Reflections and inversions

The reflection of vector \mathbf{x} to \mathbf{y} in plane with unit vector $\hat{\mathbf{n}}$ is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector \mathbf{x} to \mathbf{y} is given by $\mathbf{y} = -\mathbf{x}$ and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

Projections

We define P to be a parallel projection operator to vector \mathbf{u} if:

$$P\mathbf{u} = \mathbf{u} \text{ and } P\mathbf{v} = \mathbf{0}$$

where $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$. Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define Q to be an orthogonal projection to vector \mathbf{u} if:

$$Q\mathbf{u} = \mathbf{0} \text{ and } Q\mathbf{v} = \mathbf{v}.$$

Here $Q = I - P$.

Inertia tensors

Let \mathbf{L} denote the angular momentum of a rigid body about the origin of mass m , volume V and density ρ at position \mathbf{r} with velocity \mathbf{v} . Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2 \delta_{ij} - x_i x_j) dV$$

where $I_{ij}(O)$ is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2} I_{ij} \omega_i \omega_j = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega}.$$

Parallel axis theorem

Consider the same rigid body now with centre of mass G and let $\overrightarrow{OG} = \mathbf{R}$. Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2 \delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\mathbf{r}') dV'.$$

Diagonalisation

Let $\mathbf{L} = I_{ij}\omega_j$ where I_{ij} is a rank 2 tensor and let $\mathbf{L} = \lambda\boldsymbol{\omega}$. Then:

$$(I_{ij} - \lambda \delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda \delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for $P = \det I$, $Q = \frac{1}{2}[(\text{tr } I)^2 - \text{tr}(I^2)]$ and $R = \text{tr } I$ given tensor I .

Real symmetric tensors

Let rank 2 real symmetric tensor T be diagonalisable with real eigenvalues $\lambda^{(i)}$ and orthonormal eigenvectors $\ell^{(i)}$ where $i = 1, 2, 3$. Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that $\det L = +1$ which transforms frame $S \rightarrow S'$.

Then since $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} = \lambda^{(i)}\delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ij}.$$

Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

and is f expanded about $x = a$.

Trigonometric expansions are in radians!

$$\begin{aligned} \therefore f(x+a) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)a^n \\ &= \exp\left(a \frac{d}{dx}\right) f(x) \end{aligned}$$

Then for three dimensions:

$$\begin{aligned} \phi(\mathbf{r} + \mathbf{a}) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{a} \cdot \nabla_{\mathbf{r}})^n \phi(\mathbf{r}) \\ &= \exp(\mathbf{a} \cdot \nabla_{\mathbf{r}}) \phi(\mathbf{r}). \end{aligned}$$

Curvilinear coordinates

Let x_i denote Cartesian coordinates and u_i denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \rightarrow (u_1, u_2, u_3)$$

where each $u_i = u_i(x_1, x_2, x_3)$ and:

$$\begin{aligned} \mathbf{r} &= x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \\ &= u_1 \mathbf{e}_{u_1} + u_2 \mathbf{e}_{u_2} + u_3 \mathbf{e}_{u_3}. \end{aligned}$$

Scale factors

Let $u_1 \rightarrow u_1 + du_1$ in $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$. Then $d\mathbf{r}$ in $\mathbf{r} \rightarrow \mathbf{r} + d\mathbf{r}$ is defined as:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 := h_1 \mathbf{e}_1 du_1.$$

h_1 is the scale factor of unit vector \mathbf{e}_1 :

$$h_1 = \left| \frac{\partial \mathbf{r}}{\partial u_1} \right| \text{ and } \mathbf{e}_1 = \frac{1}{h_1} \frac{\partial \mathbf{r}}{\partial u_1}.$$

If $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ then u_i is an **orthogonal** curvilinear coordinate system.

Vector and arc length

The vector length $d\mathbf{r}$ of \mathbf{r} is defined as:

$$d\mathbf{r} = \sum_{i=1}^3 h_i du_i \mathbf{e}_i$$

where $u_i \rightarrow u_i + du_i$ for $\forall i = 1, 2, 3$.

Then the arc length ds is defined as:

$$\begin{aligned} (ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= g_{ij} du_i du_j \end{aligned}$$

where g_{ij} is the metric tensor:

$$\begin{aligned} g_{ij} &= g_{ji} = \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \\ &= h_i h_j (\mathbf{e}_i \cdot \mathbf{e}_j). \end{aligned}$$

Area and volume

Let $d\mathbf{r}_i = h_i \mathbf{e}_i du_i$ denote vector length when $u_i \rightarrow u_i + du_i$. (**No** sum!)

The infinitesimal vector area formed by $d\mathbf{r}_1$ and $d\mathbf{r}_2$ is:

$$d\mathbf{S} = (h_1 d\mathbf{r}_1 \times h_2 d\mathbf{r}_2).$$

Similarly the infinitesimal volume formed by edges $d\mathbf{r}_1$, $d\mathbf{r}_2$ and $d\mathbf{r}_3$ is:

$$\begin{aligned} dV &= |(d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3| \\ &= \sqrt{g} du_1 du_2 du_3 \end{aligned}$$

where $g = \det(g_{ij})$.

Cylindrical coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$ where ρ represents the radial distance from the origin and ϕ is the anticlockwise rotation angle on the x - y plane. In Cartesian unit vectors:

$$\begin{aligned} \mathbf{r} &= \rho \cos \phi \mathbf{e}_1 + \rho \sin \phi \mathbf{e}_2 + z \mathbf{e}_3 \\ h_\rho &= 1, \quad \mathbf{e}_\rho = \cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2 \\ h_\phi &= \rho, \quad \mathbf{e}_\phi = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2 \\ h_z &= 1, \quad \mathbf{e}_z = \mathbf{e}_3 \end{aligned}$$

and forms an orthogonal set.

Spherical coordinates

$(u_1, u_2, u_3) = (r, \theta, \phi)$ where θ represents the clockwise rotation angle in y - z plane and ϕ the anticlockwise rotation angle in x - y plane. In Cartesian unit vectors:

$$\mathbf{r} = r \sin \theta \cos \phi \mathbf{e}_1 + r \sin \theta \sin \phi \mathbf{e}_2 + r \cos \theta \mathbf{e}_3$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin \theta$$

$$\mathbf{e}_r = \sin \theta \cos \phi \mathbf{e}_1 + \sin \theta \sin \phi \mathbf{e}_2 + \cos \theta \mathbf{e}_3$$

$$\mathbf{e}_\theta = \cos \theta \cos \phi \mathbf{e}_1 + \cos \theta \sin \phi \mathbf{e}_2 - \sin \theta \mathbf{e}_3$$

$$\mathbf{e}_\phi = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2$$

and also forms an orthogonal set.



Gradient

The gradient of a scalar field $f(\mathbf{r})$ is:

$$df(\mathbf{r}) := \nabla f(\mathbf{r}) \cdot d\mathbf{r}$$

when $\mathbf{r} \rightarrow \mathbf{r} + d\mathbf{r} \implies f \rightarrow f + df$. Taking the total differential of f yields:

$$\nabla f = \sum_{i=1}^3 \frac{1}{h_i} \frac{\partial f}{\partial u_i} \mathbf{e}_i$$

where $\{\mathbf{e}_i\}$ is orthogonal.

Divergence

The divergence of a vector field \mathbf{F} is:

$$\nabla \cdot \mathbf{F} := \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \int_{\delta S} \mathbf{F} \cdot d\mathbf{S}$$

for surface δS bounds infinitesimal δV . In orthogonal curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (h_1 F_2 h_3) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right\}.$$

Curl

The curl of a vector field \mathbf{F} in the direction of unit vector $\hat{\mathbf{n}}$ is:

$$\hat{\mathbf{n}} \cdot (\nabla \times \mathbf{F}) := \lim_{\delta S \rightarrow 0} \frac{1}{\delta S} \oint_{\delta C} \mathbf{F} \cdot d\mathbf{r}$$

where curve δC encloses plane δS .

In orthogonal curvilinear coordinates:

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{vmatrix}.$$

Laplacian

The Laplacian of a scalar field f is:

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

and in orthogonal curvilinear coordinates:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right\}.$$

The Laplacian of a vector field \mathbf{F} is:

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\nabla f = \frac{\partial f}{\partial x_i} \mathbf{e}_i$$

$$(\mathbf{u} \cdot \nabla) \mathbf{F} = u_j \frac{\partial}{\partial x_j} F_i$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

$$\nabla \times \mathbf{F} = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \mathbf{e}_i$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

If ψ is a scalar field and \mathbf{v} a vector field:

$$\nabla \times (\nabla \psi) = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = \mathbf{0}$$

$$\nabla \cdot (\psi \mathbf{v}) = \nabla \psi \cdot \mathbf{v} + \psi \nabla \cdot \mathbf{v}$$

$$\nabla \times (\psi \mathbf{v}) = \nabla \psi \times \mathbf{v} + \psi \nabla \times \mathbf{v}.$$

Let $\mathbf{r} = x_i \mathbf{e}_i$ and $r = (x_i^2)^{1/2}$. Then:

- $\nabla r = \frac{\mathbf{r}}{r}$ and $\nabla \left(\frac{1}{r} \right) = -\frac{\mathbf{r}}{r^3}$
- $\nabla r^n = n r^{n-2} \mathbf{r}$
- $\nabla \cdot \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = \mathbf{0}$
- $\nabla \times (\mathbf{c} \times \mathbf{r}) = 2\mathbf{c}$
- $\nabla \cdot (\mathbf{c} \times \mathbf{r}) = \mathbf{0}$ for constant \mathbf{c} .

Divergence theorem

Let surface S enclose volume V . Then:

$$\iiint_V \nabla \cdot \mathbf{F} dV = \oint_S \mathbf{F} \cdot d\mathbf{S}$$

where \mathbf{F} is a vector field.

Stokes' theorem

Let closed curve C bound open surface S and let \mathbf{F} be a vector field. Then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

for C is traversed in anticlockwise sense.

Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x - a) := \begin{cases} \infty & x = a \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions this becomes:

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) := \delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0)$$

where (x, y, z) are Cartesian coordinates.

In orthogonal curvilinear coordinates:

$$\delta(\mathbf{r} - \mathbf{a}) = \frac{1}{h_1 h_2 h_3} \delta(u_1 - a_1) \cdot \delta(u_2 - a_2) \delta(u_3 - a_3).$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

which yields:

$$x \delta(x) = 0 \text{ and } \delta(cx) = \frac{1}{|c|} \delta(x).$$

If simple solutions of $g(x) = 0$ are x_i :

$$\int_{-\infty}^{\infty} f(x) \delta(g(x)) dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}.$$

Coulomb's law

Consider charges q and q_1 at positions \mathbf{r} and \mathbf{r}_1 . The force on charge q at \mathbf{r} due to charge q_1 at \mathbf{r}_1 is:

$$\mathbf{F}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{qq_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$

where $qq_1 > 0$ denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419 \times 10^{-12} \text{C}^2 \text{N}^{-1} \text{m}^{-2}.$$

Charge has units Coulombs (C) and an electron has charge $-1.60218 \times 10^{-19} \text{C}$.

Electric fields

The electric field is generated by a charge configuration and defined in terms of the force on a small positive test charge q :

$$\mathbf{E}(\mathbf{r}) := \lim_{q \rightarrow 0} \frac{1}{q} \mathbf{F}.$$

Then for our two charges q and q_1 :

$$\mathbf{F}_1(\mathbf{r}) = q\mathbf{E}_1(\mathbf{r})$$

where q_1 produces electric field \mathbf{E}_1 .

$$\therefore \mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$

Principle of superposition

For a set of charges q_i at position \mathbf{r}_i the total electric field at \mathbf{r} is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}.$$

For object with **charge density** $\rho(\mathbf{r}')$ its overall electric field at \mathbf{r} is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

where $\rho(\mathbf{r}')$ is charge divided by volume.

Electrostatic Maxwell's equations

Because $\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$:

$$\mathbf{E}(\mathbf{r}) = -\nabla \left(\frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \right)$$

and therefore for all static electric fields:

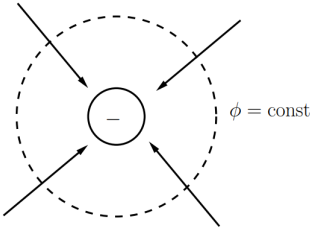
$$\nabla \times \mathbf{E} = \mathbf{0}.$$

\mathbf{E} is a **conservative** vector field where its line integral is **independent** of path. Furthermore it may be written as:

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$$

for $\phi(\mathbf{r})$ is the potential of \mathbf{E} .

$$\therefore \phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$



The **potential difference** between two points A and B is the energy per unit charge needed to move a small charge q from A to B :

$$\begin{aligned} V_{A \rightarrow B} &= \lim_{q \rightarrow 0} \frac{1}{q} W_{A \rightarrow B} \\ &= -\frac{1}{q} \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \phi_B - \phi_A. \end{aligned}$$

A charge distribution $\rho(\mathbf{r}')$ in an external electric field has potential energy:

$$W = \int_V \rho(\mathbf{r}') \phi_{ext}(\mathbf{r}') dV'.$$

Because $\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}')$:

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}.$$

Electric dipoles

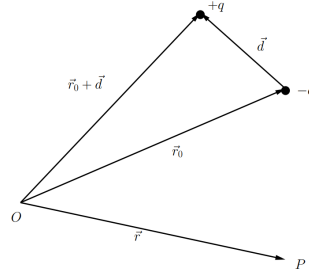
An electric dipole at \mathbf{r}_0 is defined as two charges $-q$ at \mathbf{r}_0 and $+q$ at $\mathbf{r}_0 + \mathbf{d}$ which generates **dipole moment**:

$$\mathbf{p} = q\mathbf{d}$$

and in the dipole limit this is defined as:

$$\mathbf{p} := \lim_{\substack{q \rightarrow \infty \\ d \rightarrow 0}} q\mathbf{d}$$

known as an ideal dipole.



The electrostatic potential generated by this ideal dipole at \mathbf{r}_0 is given by:

$$\begin{aligned} \phi(\mathbf{r}) &= \phi_q + \phi_{-q} \\ &= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0 - \mathbf{d}|} - \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) \\ &\approx \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3} \end{aligned}$$

for the first term is expanded in powers of $-\mathbf{d}$ about $\mathbf{r} - \mathbf{r}_0$.

The electric field generated is:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \left[-\frac{\mathbf{p}}{|\mathbf{r} - \mathbf{r}_0|^3} \right. \\ &\quad \left. + \frac{3\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} (\mathbf{r} - \mathbf{r}_0) \right]. \end{aligned}$$

If the ideal dipole is at the origin:

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \\ \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{p} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{p}}{r^3} \right). \end{aligned}$$

Let ideal dipole moment \mathbf{p} be parallel to the z -axis. Then in spherical coordinates (r, θ, χ) , $\mathbf{r} = r\mathbf{e}_r$, $\mathbf{p} = p\mathbf{e}_z$ and:

$$\phi(\mathbf{r}) = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0} \left(\frac{2\cos\theta}{r^3} \mathbf{e}_r + \frac{\sin\theta}{r^3} \mathbf{e}_\theta \right).$$



Force, torque and energy

The **force** on a **dipole at \mathbf{r}** from external electric field $\mathbf{E}_{ext}(\mathbf{r})$ is:

$$\begin{aligned} \mathbf{F} &= -q\mathbf{E}_{ext}(\mathbf{r}) + q\mathbf{E}_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx (\mathbf{p} \cdot \nabla) \mathbf{E}_{ext}(\mathbf{r}). \end{aligned}$$

The **torque** on a dipole at \mathbf{r} about the axis \mathbf{r} due to $\mathbf{E}_{ext}(\mathbf{r})$ is:

$$\begin{aligned} \mathbf{G} &= \boldsymbol{\tau}_{-q} + \boldsymbol{\tau}_q \\ &= -q\mathbf{0} \times \mathbf{E}_{ext}(\mathbf{r}) + q\mathbf{d} \times \mathbf{E}_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx \mathbf{p} \times \mathbf{E}_{ext}(\mathbf{r}). \end{aligned}$$

The **energy** of a dipole at \mathbf{r} from external electric field $\mathbf{E}_{ext}(\mathbf{r}) = -\nabla\phi_{ext}(\mathbf{r})$ is:

$$\begin{aligned} W &= -q\phi_{ext}(\mathbf{r}) + q\phi_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx -\mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{r}) \end{aligned}$$

and $\mathbf{F} = -\nabla W$.

Multipole expansion

Consider object with volume V and charge distribution $\rho(\mathbf{r}')$. Let origin be in the object. Then the potential at \mathbf{r} is:

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \\ &\approx \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{Q_{ij}x_i x_j}{2r^5} \right) \end{aligned}$$

where Q is the **total charge** in V :

$$Q = \int_V \rho(\mathbf{r}') dV'$$

\mathbf{p} the **dipole moment** about the origin:

$$\mathbf{p} = \int_V \mathbf{r}' \rho(\mathbf{r}') dV'$$

and Q_{ij} the **quadrupole tensor**:

$$Q_{ij} = \int_V \rho(\mathbf{r}') \left[3x'_i x'_j - (r')^2 \delta_{ij} \right] dV'$$

If $Q \neq 0$ then in the far zone ($r \gg r_0$) the first term (monopole term) dominates.

If $Q = 0$ and $\mathbf{p} = \mathbf{0}$ then the third term (quadrupole term) dominates in the far zone and etc.

Interaction energy

By expanding $\phi_{ext}(\mathbf{r})$ about $\mathbf{r} = \mathbf{0}$:

$$\begin{aligned} W &= \int_V \rho(\mathbf{r}') \phi_{ext}(\mathbf{r}') dV' \\ &= Q\phi_{ext}(\mathbf{0}) - \mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{0}) \\ &\quad - \frac{1}{6} Q_{ij} \frac{\partial(\mathbf{E}_{ext}(\mathbf{0}))_i}{\partial x_j} + \dots \end{aligned}$$

and is the potential energy of a charge distribution $\rho(\mathbf{r})$ in \mathbf{E}_{ext} .

Gauss' law

For object with charge distribution $\rho(\mathbf{r}')$
and volume V enclosed by surface S :

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is total charge enclosed by V .

boundaries and conductors

Poisson's equation

method of images

Electrostatic energy**Capacitors****Currents****Lozentz force****Biot-Savart law****Magnetostatic Maxwell's equations****Ampère's law**

normal and tangent components of conducting surfaces

Magnetic dipoles