Honours Differential Equations Assignments

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1. Consider $f(x) = \frac{x^2}{2}$ where $0 \leq x < L$.

   Part (a) wants us to find its Fourier series with period $2L$.

   Define an extension of our function f as g:

$$g(x) = \begin{cases} f(x) & x \in [0, L) \\ f(-x) & x \in [-L, 0) \end{cases}$$

   and therefore g has period $2L$. Since g is also an even function its Fourier series can only consist of cosine terms.

$$\therefore g_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

   Its coefficients are:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} g(x)dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)dx + \frac{1}{L}\int_{-L}^{0} f(-x)dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)dx - \frac{1}{L}\int_{0}^{-L} f(-x)dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)dx + \frac{1}{L}\int_{0}^{L} f(x^*)dx^* \\ &= \frac{2}{L}\int_{0}^{L} f(x)dx \end{aligned}$$

   where we define $x = -x^*$ and similarly

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} g(x)dx \\ &= \frac{1}{L}\int_{0}^{L} \cos\frac{n\pi x}{L} f(x)dx + \frac{1}{L}\int_{-L}^{0} \cos\frac{n\pi x}{L} f(-x)dx \\ &= \frac{2}{L}\int_{0}^{L} \cos\frac{n\pi x}{L} f(x)dx. \end{aligned}$$

Calculating this explicitly:

$$\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L f(x)dx \\
&= \frac{2}{L}\int_0^L \frac{1}{2}x^2 dx \\
&= \frac{1}{L}\left[\frac{x^3}{3}\right]_0^L \\
&= \frac{1}{3}L^2
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L} f(x)dx \\
&= \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}\frac{1}{2}x^2 dx \\
&= \frac{1}{L}\int_0^L x^2\cos\frac{n\pi x}{L}dx \\
&= \frac{1}{n\pi}\left[x^2\sin\frac{n\pi x}{L}\right]_0^L - \frac{2}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}dx \\
&= \frac{L^2}{n\pi}\sin n\pi - \frac{2}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}dx \\
&= \frac{L^2}{n\pi}\sin n\pi - \frac{2}{n\pi}\left(-\frac{L^2}{n\pi}\cos n\pi + \left(\frac{L}{n\pi}\right)^2\sin n\pi\right) \\
&= \left(\frac{L^2}{n\pi} - \frac{2L^2}{(n\pi)^3}\right)\sin n\pi + \left(\frac{2L^2}{(n\pi)^2}\right)\cos n\pi \\
&= (-1)^n\frac{2L^2}{(n\pi)^2}
\end{aligned}$$

where we have integrated by parts, and $\sin n\pi = 0$ for $\forall n \in \mathbb{N}$.

Putting all this together we get our Fourier series for f:

$$\therefore f_{FS}(x) = \frac{L^2}{6} + \sum_{n=1}^{\infty}\left((-1)^n\frac{2L^2}{(n\pi)^2}\cos\frac{n\pi x}{L}\right)$$

and is valid for $\forall x \in [0, L)$.

For part (b) we want $f_{FS}(0)$. By Fourier's convergence theorem:

$$f_{FS}(0) = \frac{L^2}{6} + \sum_{n=1}^{\infty} \left((-1)^n \frac{2L^2}{(n\pi)^2} \right).$$

For part (c):

$$\begin{aligned} f(0) &= \frac{L^2}{6} + \sum_{n=1}^{\infty} \left((-1)^n \frac{2L^2}{(n\pi)^2} \cos \frac{n\pi x}{L} \right) \\ &= 0 \end{aligned}$$

$$\therefore \frac{L^2}{6} + \frac{2L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

2. For part (a) we given the solution to Laplace's equation:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

find expressions for coefficients a_0, a_n and b_n.

Because our solution has period 2π, using the Euler-Fourier formulae:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)f(\theta)d\theta$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta)f(\theta)d\theta.$$

For part (b) we begin with the following expression:

$$\begin{aligned} u(r,\theta) = &\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)d\psi \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \left(\cos(n\theta)\int_{-\pi}^{\pi}\cos(n\psi)f(\psi)d\psi + \sin(n\theta)\int_{-\pi}^{\pi}\sin(n\psi)f(\psi)d\psi\right)\end{aligned}$$

Now since $e^{in(\theta-\psi)} = \cos n(\theta-\psi) + i\sin n(\theta-\psi)$:

$$\begin{aligned} Re(e^{in(\theta-\psi)}) &= \cos n(\theta-\psi) \\ &= \cos(n\theta)\cos(n\psi) + \sin(n\theta)\sin(n\psi)\end{aligned}$$

$$\begin{aligned} \therefore \int_{-\pi}^{\pi} Re(e^{in(\theta-\psi)})f(\psi)d\psi &= \int_{-\pi}^{\pi}\left(\cos(n\theta)\cos(n\psi) + \sin(n\theta)\sin(n\psi)\right)f(\psi)d\psi \\ &= \cos(n\theta)\int_{-\pi}^{\pi}\cos(n\psi)f(\psi)d\psi \\ &\quad + \sin(n\theta)\int_{-\pi}^{\pi}\sin(n\psi)f(\psi)d\psi\end{aligned}$$

Therefore our original expression becomes:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)d\psi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} r^n Re(e^{in(\theta-\psi)})f(\psi)d\psi$$

Taking the real component of each element in a sum is equivalent to taking the real component of the overall sum:

$$\therefore u(r,\theta) = \frac{1}{\pi} Re\left(\frac{1}{2}\int_{-\pi}^{\pi} f(\psi)d\psi + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} r^n e^{in(\theta-\psi)} f(\psi)d\psi\right)$$

Then by the linearity of integrals:

$$\therefore u(r,\theta) = \frac{1}{\pi} Re\left(\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n e^{in(\theta-\psi)}\right] f(\psi)d\psi\right)$$

Finally for part (c) we have that

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

and let:

$$\begin{aligned} q &= re^{i(\theta-\psi)} \\ &= r\left(\cos(\theta-\psi) + i\sin(\theta-\psi)\right). \end{aligned}$$

Now $|q| < 1$ since we defined $r < 1$ and $e^{i2\pi} = 1$. Returning to our sum:

$$\begin{aligned} \sum_{n=1}^{\infty} q^n &= \frac{1}{1-q} - 1 \\ &= \frac{q}{1-q}. \end{aligned}$$

Furthermore:

$$\begin{aligned} \frac{1}{2} + \sum_{n=1}^{\infty} q^n &= \frac{1}{2}\frac{1+q}{1-q} \\ &= \frac{1}{2}\frac{1 + r(\cos(\theta-\psi) + i\sin(\theta-\psi))}{1 - r(\cos(\theta-\psi) + i\sin(\theta-\psi))} \\ &= \frac{1}{2}\left[\frac{(1+rc) + i(rs)}{(1-rc) + i(-rs)}\right] \times \frac{1-rc+i(rs)}{1-rc+i(rs)} \\ &= \frac{1}{2}\frac{1-r^2+i(2rs)}{1+r^2-2rc} \end{aligned}$$

$$\therefore Re(\frac{1}{2} + \sum_{n=1}^{\infty} q^n) = \frac{1-r^2}{1+r^2-2rc}$$

Finally we have that:

$$\begin{aligned}
u(r,\theta) &= \frac{1}{\pi} Re\left(\int_{-\pi}^{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty} r^n e^{in(\theta-\psi)}\right] f(\psi)d\psi\right) \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} Re\left[\frac{1}{2}+\sum_{n=1}^{\infty} r^n e^{in(\theta-\psi)}\right] f(\psi)d\psi\right) \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\psi)} f(\psi)d\psi\right).
\end{aligned}$$

Assignment 4

1. For part (a), $\phi_n(x)$ are the orthonormal eigenfunctions and λ_n the real eigenvalues of the corresponding homogeneous regular S-L problem:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+q(x)y=\lambda r(x)y$$

with initial conditions:

- $\alpha_1 y(0)+\alpha_2 y'(0)=0$
- $\beta_1 y(1)+\beta_2 y'(1)=0.$

for $x\in[0,1]$. The solution to the following:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+q(x)y=\mu r(x)y+f(x)$$

is then:

$$y(x)=\sum_{n=1}^{\infty}b_n\phi_n(x)$$

where:

$$b_n=\frac{c_n}{\lambda_n-\mu}$$

and

$$c_n=\int_0^1\phi_n(x)f(x)\mathrm{d}x.$$

For part (b) solve the following:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x)+7y(x)=2\sin 5x+3\sin 7x$$

with boundary conditions $y(0)=y(\pi)=0$ for $\forall x\in[0,\pi]$.

First define change of variables $x=\pi t$.

$$\therefore y(x)\iff y(t)$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}y(t=\frac{x}{\pi})=\frac{1}{\pi}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}y(t=\frac{x}{\pi})\right)=\frac{1}{\pi^2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

Then our ODE becomes:

$$\frac{1}{\pi^2}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 7y(t) = 2\sin 5\pi t + 3\sin 7\pi t$$

with boundary conditions $y(0) = y(1) = 0$ for $\forall t \in [0, 1]$. This is now of S-L form, and its corresponding homogeneous S-L system is: