

## Integrating factors

$$y' + P(x)y = Q(x)$$

$$I(x) = \exp\left(\int P(x)dx\right)$$

$$y(x) = \frac{1}{I(x)} \int Q(x)I(x)dx + \frac{\alpha}{I(x)}$$

## Change of variables

$$y^{(n)} = F(y, y', \dots, y^{(n-1)}, t)$$

Let  $x_{i+1} = y^{(i)}$  where  $i \in \{0, 1, \dots, n-1\}$ .

## Picard-Lindelöf statement

Consider IVP:  $x'_i = F_i(t, x_1, \dots, x_n)$  or that  $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$  with  $\mathbf{x}(t_0) = \mathbf{x}_0$  and  $\mathbf{x}$  is a vector with  $n$  entries.

It has **unique** solutions if:

$$F_i, \frac{\partial F_i}{\partial x_j} \text{ and } \frac{\partial F_i}{\partial t} \text{ are } \underline{\text{continuous}} \text{ in}$$

$R \subset \mathbb{R}^{n+1}$  where  $(t, \mathbf{x}_0^T) \in R$ .

Here  $i, j \in \{1, \dots, n\}$ .

## Homogeneous systems

Consider  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  where  $\mathbf{A}$  is a  $n \times n$  matrix. Substituting  $\mathbf{x} = e^{rt}\boldsymbol{\xi}$  gives:

$$(\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\xi}^{(i)} = \mathbf{0}$$

where  $i \in \{1, 2, \dots, n\}$ .

Our general solution is then:

$$\begin{aligned} \mathbf{x}(t) &= \sum_{i=1}^n c_i e^{r_i t} \boldsymbol{\xi}^{(i)} \\ &= \sum_{i=1}^n c_i \mathbf{x}^{(i)} \\ &= \boldsymbol{\Psi}(t)\mathbf{c}. \end{aligned}$$

If initial conditions  $\mathbf{x}(t_0) = \mathbf{x}_0$  are given:

$$\mathbf{c} = \boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}(t_0)$$

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}_0.$$

## Matrix exponentials

Given a  $n \times n$  matrix  $\mathbf{A}$ :

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} \\ &= \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \dots \end{aligned}$$

For system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ :

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

and that  $e^{\mathbf{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$ .

## Diagonalisation

For  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  we have  $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$ .

If  $\mathbf{x} = \mathbf{T}\mathbf{y}$  then  $\mathbf{y}' = \mathbf{D}\mathbf{y}$ .

Since our fundamental matrix with respect to  $\mathbf{y}$  is a diagonal matrix  $\mathbf{Q} = e^{\mathbf{D}t}$ , the fundamental matrix with respect to  $\mathbf{x}$  is  $\boldsymbol{\Psi}(t) = \mathbf{T}e^{\mathbf{D}t}$  and:

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}.$$

## Generalised eigenvectors

If eigenvalues do not generate enough eigenvectors to form  $n$  linearly independent solutions for a  $n \times n$  matrix  $\mathbf{A}$ , then consider the following ansatz:

$$\mathbf{x} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

$$\therefore (\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\eta}^{(i)} = \boldsymbol{\xi}^{(i)}$$

Then this  $r_i$  produces two solutions:

$$\mathbf{x}^{(1)} = e^{rt}\boldsymbol{\xi}$$

$$\mathbf{x}^{(2)} = te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}.$$

## Non-homogeneous systems

Consider non-homogeneous ODE system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}.$$

### • Change of basis

Let  $\mathbf{x} = \mathbf{T}\mathbf{y}$  and since  $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$ :

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$$

which is solved by integrating factors. Finally revert back to  $\mathbf{x}$ .

### • Variation of parameters

Find solution  $\mathbf{x}_H = \boldsymbol{\Psi}\mathbf{c}$  to  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ . Then let the non-homogeneous solution be  $\mathbf{x} = \boldsymbol{\Psi}\mathbf{u}(t)$ .

$$\therefore \boldsymbol{\Psi}\mathbf{u}'(t) = \mathbf{g}(t)$$

Row reduce before integrating.

### • Undetermined coefficients

Let non-homogeneous ODE system have solutions of form:

$$\mathbf{x} = \mathbf{x}_H + \mathbf{x}_p$$

where  $\mathbf{x}_H$  is our homogeneous solution and  $\mathbf{x}_p$  our particular solution.

## Critical points

Consider non-linear ODE system

$$\mathbf{x}' = \mathbf{F}(x, y),$$

$$\mathbf{y}' = \mathbf{G}(x, y).$$

We define  $\mathbf{x}^0 = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$  as a **critical point** when  $\mathbf{F}(\mathbf{x}^0) = \mathbf{G}(\mathbf{x}^0) = \mathbf{0}$ .

## Linearisation and stability

$$\text{Let } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x - x^0 \\ y - y^0 \end{bmatrix}.$$

$$\begin{aligned} \therefore u'_1 &\approx F(x^0, y^0) + \left(\frac{\partial F}{\partial x}\right)_{x^0} (x - x^0) \\ &\quad + \left(\frac{\partial F}{\partial y}\right)_{y^0} (y - y^0) \end{aligned}$$

$$\begin{aligned} \therefore u'_2 &\approx G(x^0, y^0) + \left(\frac{\partial G}{\partial x}\right)_{x^0} (x - x^0) \\ &\quad + \left(\frac{\partial G}{\partial y}\right)_{y^0} (y - y^0) \end{aligned}$$

$$\therefore \mathbf{u}' = \mathbf{A}\mathbf{u}$$

$$= \begin{bmatrix} \partial F / \partial x & \partial F / \partial y \\ \partial G / \partial x & \partial G / \partial y \end{bmatrix}_{\mathbf{x}=\mathbf{x}^0} \begin{bmatrix} x - x^0 \\ y - y^0 \end{bmatrix}$$

Critical points  $\mathbf{x}^0$  may also be classified:

Eigenvalues	Critical Points	Stability
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asympt. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asympt. stable
$r_1, r_2 = \lambda \pm i\mu$ ( $\lambda > 0$ )	focus	unstable
$r_1, r_2 = \lambda \pm i\mu$ ( $\lambda < 0$ )	focus	asympt. stable
$r_1 = i\mu, r_2 = -i\mu$	center	stable

Linearisation preserves critical point behaviour **except** when eigenvalues are of  $r = \pm i\mu$  form, for which then classification is unknown.

**Stable** critical points  $\mathbf{x}^0$ :

All solutions start and stay near  $\mathbf{x}^0$ .

$$\begin{aligned} \forall \epsilon > 0; \exists \delta > 0; \forall \mathbf{x}(t) = \boldsymbol{\phi}(t) : |\mathbf{x}(0) - \mathbf{x}^0| < \delta \\ \implies |\mathbf{x}(t) - \mathbf{x}^0| < \epsilon \text{ for } \forall t \geq 0 \end{aligned}$$

**Attracting** critical points  $\mathbf{x}^0$ :

All solutions tends to  $\mathbf{x}^0$ .

$$\begin{aligned} \forall \delta > 0 : |\mathbf{x}(0) - \mathbf{x}^0| < \delta \\ \implies \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^0 \end{aligned}$$

**Asymptotically stable** critical points  $\mathbf{x}^0$ : Attracting **and** stable.

## Lyapunov's theory and limit cycles

Consider  $\dot{x} = F(x, y)$  and  $\dot{y} = G(x, y)$  and let  $\mathbf{x}^0 \in D$  be a critical point. Let  $E : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined such that  $E(\mathbf{x}^0, y^0) = 0$ .

$$\therefore \frac{dE}{dt} = \frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G$$

- Let  $E > 0$  for  $\forall \mathbf{x} \neq \mathbf{x}^0$ .

If  $\frac{dE}{dt} \leq 0$  then  $\mathbf{x}^0$  is stable.

If  $\frac{dE}{dt} < 0$  then  $\mathbf{x}^0$  is asymptotically stable.

- $E(\mathbf{x}^*) > 0$  and  $\frac{dE}{dt} > 0$

$\implies$  unstable  $\mathbf{x}^0$ . (flip both signs)

Postive definite:  $E(\mathbf{x}) > 0$  for  $\forall \mathbf{x} \neq \mathbf{x}^0$

Postive semidefinite:  
 $E(\mathbf{x}) \geq 0$  for  $\forall \mathbf{x} \neq \mathbf{x}^0$

**Limit cycles** are periodic solutions such that at least one other **non-closed trajectory** approaches it as  $t \rightarrow \infty$ .

Generally if our trajectory is enclosed by finite non-simple region and  $F, G$  have continuous partials then there is a limit cycle.

## Real Fourier series

The Fourier expansion of piecewise continuous  $f(x)$  on  $[-L, L]$  is:

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where  $f_{FS}(x) = f_{FS}(x + 2L)$ . If  $\alpha$  is a discontinuous point:

$$f_{FS}(\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}$$

for  $\alpha^+$  is the limit from the left.

Our Fourier coefficients are:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L \cos \frac{n\pi x}{L} f(x) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L \sin \frac{n\pi x}{L} f(x) dx.$$

## Orthogonality

Let  $S_n = \sin \frac{n\pi x}{L}$  and  $C_n = \cos \frac{n\pi x}{L}$ .

$$\langle S_n, S_m \rangle = \langle C_n, C_m \rangle = L \delta_{mn}$$

$$\langle S_n, C_m \rangle = 0.$$

Where:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$$\langle u(x), v(x) \rangle = \int_{-L}^L u(x)v(x) dx$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

**Even** functions:  $f(-x) = f(x)$

$$\therefore \int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$$

**Odd** functions:  $f(-x) = -f(x)$

$$\therefore \int_{-L}^L f(x) dx = 0$$

• **Even** function  $f(x)$  on  $[-L, L]$ :

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_0^L \cos \frac{n\pi x}{L} f(x) dx$$

• **Odd** function  $f(x)$  on  $[-L, L]$ :

$$f_{FS}(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} f(x) dx$$

## Extensions

Consider  $f(x)$  defined in  $[0, L]$  originally.

1. Define even function:

$$g(x) = \begin{cases} f(x) & x \in [0, L] \\ f(-x) & x \in (-L, 0) \end{cases}$$

with cosine series.

2. Define odd function:

$$h(x) = \begin{cases} f(x) & x \in (0, L) \\ 0 & x = 0, L \\ -f(-x) & x \in (-L, 0) \end{cases}$$

with sine series.

## Complex Fourier series

Similarly:

$$f_{FS}(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left( \frac{in\pi}{L} x \right)$$

$$e^{i\theta} = \sin \theta + i \cos \theta.$$

Then  $\forall n \in \mathbb{Z}$  we have that:

$$c_n = \frac{1}{2L} \int_{-L}^L \exp \left( -\frac{in\pi}{L} x \right) f(x) dx$$

$$c_n = \begin{cases} (a_n - ib_n)/2 & n > 0 \\ (a_0)/2 & n = 0 \\ (a_n + ib_n)/2 & n < 0. \end{cases}$$

The **inner product** is defined as:

$$\langle f, g \rangle = \int_{-L}^L f(x) g^*(x) dx.$$

$$\therefore \langle \exp \left( \frac{im\pi}{L} x \right), \exp \left( \frac{in\pi}{L} x \right) \rangle = 2L \delta_{mn}$$

## Parseval's theorem

$$\langle f, f \rangle = \int_{-L}^L |f(x)|^2 dx$$

$$= 2L \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$= L \left[ \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \right]$$

## Heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Let  $u(x, t) = X(x)T(t)$ .

$$\frac{1}{\alpha^2} \frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

$$\dot{T} + \alpha^2 \lambda T = 0$$

$$\lambda = \mu^2; X(x) = b_1 \cos \lambda^{1/2} x + b_2 \sin \lambda^{1/2} x$$

$$\lambda = -\mu^2; X(x) = b_1 \cosh \mu x + b_2 \sinh \mu x$$

$$T(t) = a_1 \exp(-\alpha^2 \lambda t)$$

## Standard boundary conditions

•  $u(x, 0) = f(x)$  for  $0 \leq x \leq L$

•  $u(0, t) = u(L, t) = 0$  for  $\forall t > 0$

$$X(0) = X(L) = 0$$

$$\therefore X_n = b_2 \sin \lambda_n^{1/2} x$$

$$\therefore \lambda_n = \left( \frac{n\pi}{L} \right)^2 \text{ for } \forall n \in \mathbb{N}$$

Our general solution must then be:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx.$$

## Fixed boundary temperatures

•  $u(0, t) = T_1$

•  $u(L, t) = T_2$

•  $u(x, 0) = f(x)$

$$v(x) = \lim_{t \rightarrow \infty} u(x, t)$$

Since  $v'' = 0$ ,  $v(0) = T_1$  and  $v(L) = T_2$ :

$$v(x) = \frac{T_2 - T_1}{L} x + T_1.$$

We then deduce that:

$$u(x, t) = v(x) + \omega(x, t)$$

where  $\omega(x, t)$  satisfies conditions:

•  $\omega(0, t) = \omega(L, t) = 0$

•  $\omega(x, 0) = f(x) - v(x)$

$$\therefore \omega(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

For  $\lambda_n = \left( \frac{n\pi}{L} \right)^2$  and

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) (f(x) - v(x)) dx.$$

### Insulated rod ends

- $\frac{\partial}{\partial x}u(0, t) = \frac{\partial}{\partial x}u(L, t) = 0$
- $u(x, 0) = f(x)$

$$X'(0) = X'(L) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \cos \lambda_n^{1/2} x$$

$$c_n = \frac{2}{L} \int_0^L \cos(\lambda_n^{1/2} x) f(x) dx$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

### Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Let  $u(x, t) = X(x)T(t)$ .

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = -\lambda$$

$$X'' + \lambda X = 0$$

$$\ddot{T} + c^2 \lambda T = 0$$

### Plucked string

- $u(0, t) = u(L, t) = 0$
- $\frac{\partial}{\partial t}u(x, 0) = 0$
- $u(x, 0) = f(x)$

$$X(0) = X(L) = 0 \text{ and } \dot{T}(0) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x \cos c \lambda_n^{1/2} t$$

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

### General initial conditions

- $u(0, t) = u(L, t) = 0$
- $\frac{\partial}{\partial t}u(x, 0) = g(x)$
- $u(x, 0) = f(x)$

$$u(x, t) = \sum_{n=1}^{\infty} \sin \lambda_n^{1/2} x \times \left( a_n \cos c \lambda_n^{1/2} t + b_n \sin c \lambda_n^{1/2} t \right)$$

$$a_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx$$

$$b_n = \frac{1}{c \lambda_n^{1/2}} \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) g(x) dx$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

### Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let  $u(x, y) = X(x)Y(y)$ .

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$X'' - \lambda X = 0$$

$$Y'' + \lambda Y = 0$$

### Rectangular boundary conditions

- $u(x, 0) = u(x, b) = 0$
- $u(0, y) = 0$  and  $u(a, y) = f(y)$

Here  $x \in [0, a]$  and  $y \in [0, b]$ .

$$X(0) = 0 \text{ and } Y(0) = Y(b) = 0$$

$$\therefore Y_n = a_1 \sin(\lambda_n^{1/2} y) \text{ for } \lambda_n = \frac{n^2 \pi^2}{b^2}$$

$$\therefore X_n = a_3 \sinh(\lambda_n^{1/2} x)$$

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} x) \sin(\lambda_n^{1/2} y)$$

$$c_n = \frac{2}{b \sinh(\lambda_n^{1/2} a)} \int_0^b \sin(\lambda_n^{1/2} y) f(y) dy$$

### Circular boundary conditions

- $u(a, \theta) = f(\theta)$
- $u(r, \theta)$  is bounded

Here  $r \in [0, a]$  and  $\theta \in [0, 2\pi]$ .

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0.$$

Let  $u(r, \theta) = R(r)\Theta(\theta)$ .

$$\therefore r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \lambda$$

$$\therefore \ddot{\Theta} + \lambda \Theta = 0$$

$$\therefore r^2 R'' + r R' = \lambda R$$

For the first ODE if  $\lambda \leq 0$  then we get at best constant solutions. If  $\lambda > 0$ :

$$\Theta(\theta) = a_1 \cos \lambda^{1/2} \theta + a_2 \sin \lambda^{1/2} \theta$$

and since periodicity must be preserved:

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\therefore \lambda_n^{1/2} = n \text{ or } \lambda_n^{1/2} = 0$$

where  $n \in \mathbb{N}$ . So when  $\lambda_n = 0$ :

$$r^2 R'' + r R' = 0$$

and since  $u(r, \theta)$  is bounded we get only constant solutions. If  $\lambda_n = n^2$  then:

$$r^2 R'' + r R' - n^2 R = 0$$

with solutions of form  $R(r) = r^\alpha$  which yields  $R_n(r) = c_n r^n$ . Then:

$$u(r, \theta) = \frac{p_0}{2} + \sum_{n=1}^{\infty} r^n \left( q_n \cos \lambda_n^{1/2} \theta + r_n \sin \lambda_n^{1/2} \theta \right)$$

$$p_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$q_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} \cos(\lambda_n^{1/2} \theta) f(\theta) d\theta$$

$$r_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} \sin(\lambda_n^{1/2} \theta) f(\theta) d\theta.$$

### Regular S-L problems

A regular S-L problem defined on  $[0, 1]$  is:

$$-\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

$$\bullet a_1 y(0) + a_2 y'(0) = 0$$

$$\bullet b_1 y(1) + b_2 y'(1) = 0$$

$$\bullet p(x), p'(x), q(x), r(x) \text{ are } \underline{\text{continuous}}$$

$$\bullet p(x), r(x) \text{ are } \underline{\text{strictly positive}}$$

which yields **real** eigenvalues  $\lambda_n$  and **orthonormal** eigenfunctions  $\phi_n(x)$ :

$$\langle \phi_m, \phi_n \rangle = \int_0^1 r(x) \phi_m(x) \phi_n(x) dx = \delta_{mn}$$

in Hilbert space  $L^2([0, 1], r(x)dx)$ .

Let  $\phi_n(x) = k_n y_n(x)$ .

$$\therefore k_n = \frac{1}{\sqrt{\langle y_n, y_n \rangle}} = \left( \int_0^1 r(x) y_n^2(x) dx \right)^{-1/2}$$

### General 2nd order ODEs

$$-P(x) \frac{d^2 y}{dx^2} - \omega(x) \frac{dy}{dx} + q(x)y = \lambda r(x)y$$

$$F(x) = \exp \left[ \int_0^x \frac{\omega(s) - p'(s)}{p(s)} ds \right]$$

$$-\frac{d}{dx} \left[ F(x) P(x) \frac{dy}{dx} \right] + F(x) q(x) y = \lambda F(x) r(x) y$$

## Lagrange's identity

Let functions  $u$  and  $v$  satisfy regular S-L boundary conditions. Then:

$$\langle \mathcal{L}[u], v \rangle = \langle u, \mathcal{L}[v] \rangle$$

where:

$$\langle u, v \rangle = \int_0^1 uv^* dx$$

$$\mathcal{L}[u] = -\frac{d}{dx} \left[ p(x) \frac{du}{dx} \right] + q(x)u.$$

$$\begin{aligned} \therefore \langle \mathcal{L}[u], v \rangle - \langle u, \mathcal{L}[v] \rangle &= - \left[ p \left( u'v^* - u(v^*)' \right) \right]_0^1 \\ &= - \left[ p(x) \left( \frac{du}{dx} \cdot v^* - u \cdot \frac{dv^*}{dx} \right) \right]_0^1 \\ [pu'v^*]' &= (pu')'v^* + pu'(v^*)' \\ [pu(v^*)']' &= (p(v^*)')'u + pu'(v^*)' \end{aligned}$$

## S-L series expansion

The set of orthonormal eigenfunctions  $\{\phi_n(x)\}$  from a S-L problem defined on  $[0, 1]$  may be used to expand function  $f(x)$ :

$$f_\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$c_n = \int_0^1 r(x) \phi_n(x) f(x) dx.$$

in  $L^2([0, 1], r(x)dx)$ . We also have the general Parseval's identity:

$$\int_0^1 r(x) [f(x)]^2 dx = \sum_{n=1}^{\infty} c_n^2.$$

## Non-homogeneous S-L problems

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

$$\mathcal{L}[y] = -\frac{d}{dx} \left[ P(x) \frac{dy}{dx} \right] + q(x)y$$

Firstly solve corresponding homogeneous problem:

$$\mathcal{L}[y] = \lambda r(x)y$$

for non-trivial  $\lambda_n$  and  $\phi_n(x)$ .

Then the general solution is:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

$$c_n = \int_0^1 \phi_n(x) f(x) dx.$$

## Non-homogeneous PDEs

Consider the following PDE:

$$r(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ p(x) \frac{\partial u}{\partial x} \right] - q(x)u(x, t) + F(x, t)$$

with conditions:

- $\frac{\partial}{\partial x} u(0, t) - h_1 u(0, t) = 0$
- $\frac{\partial}{\partial x} u(1, t) - h_2 u(1, t) = 0$
- $u(x, 0) = f(x).$

Firstly solve the homogeneous case.

Let  $u(x, t) = X(x)T(t)$ .

$$\therefore \frac{1}{rX} [p'X' + pX'' - qX] = \frac{\dot{T}}{T} = -\lambda$$

This yields two ODEs:

$$\dot{T} + \lambda T = 0$$

$$-[pX']' + qX = \lambda rX$$

with boundary conditions:

$$X'(0) - h_1 X(0) = 0$$

$$X'(1) - h_2 X(1) = 0$$

Clearly our second ODE is of S-L form. We are now going to assume that this regular S-L problem has non-trivial  $\lambda_n$  and orthonormal eigenfunctions  $\phi_n(x)$ . Then:

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x).$$

Since we have a S-L problem:

$$\begin{aligned} \sum_{n=1}^{\infty} \dot{b}_n(t) \phi_n(x) &= \sum_{n=1}^{\infty} b_n(t) [-\lambda_n \phi_n(x)] \\ &+ \frac{F(x, t)}{r(x)} \end{aligned}$$

$$\frac{F(x, t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t) \phi_n(x)$$

$$\gamma_n(t) = \int_0^1 F(x, t) \phi_n(x) dx$$

$$\sum_{n=1}^{\infty} [\dot{b}_n(t) + \lambda_n b_n(t) - \gamma_n(t)] \phi_n(x) = 0$$

$$b_n(t) = e^{-\lambda_n t} \int_0^t \gamma_n(s) e^{\lambda_n s} ds + e^{-\lambda_n t} b_n(0)$$

finally using initial conditions:

$$b_n(0) = \int_0^1 r(x) f(x) \phi_n(x) dx$$

and all this is in  $L^2([0, 1], r(x)dx)$ .

## Singular S-L problems

Consider the following ODE:

$$-\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

but now some of  $p(x)$ ,  $q(x)$  and  $r(x)$  are discontinuous at  $x = 0$  and/or  $x = 1$ . This is a **singular** S-L problem with boundary condition:

$$b_1 y(1) + b_2 y'(1) = 0$$

or

$$a_1 y(0) + a_2 y'(0) = 0.$$

Now singular S-L problems at  $x = 0$  may be self-adjoint or that they yield:

- $\lambda_n \in \mathbb{R}$  (Real eigenvalues)
- $\langle \phi_m, \phi_n \rangle = \delta_{mn}$  (Orthogonal eigenfunctions)

if they satisfy Lagrange's identity.

Consider singular S-L problem at  $x = 0$ :

$$\begin{aligned} \int_{\epsilon}^1 (\mathcal{L}[u]v - u\mathcal{L}[v]) dx &= \left[ -p(x) (u'(x)v(x) - u(x)v'(x)) \right]_{\epsilon}^1 \\ &= p(\epsilon) (u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)) \end{aligned}$$

and tends to zero if and only if:

$$\lim_{\epsilon \rightarrow \infty} [p(\epsilon) (u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon))] = 0,$$

where we define:

$$\mathcal{L}[u] = -\frac{d}{dx} \left[ p(x) \frac{du}{dx} \right] + q(x)u.$$

Therefore this is the condition for our singular S-L problem to have real eigenvalues and orthogonal eigenfunctions.

Similarly for singular S-L problem at  $x = 1$  it is self-adjoint if:

$$\lim_{\epsilon \rightarrow \infty} [p(1-\epsilon) (u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon))] = 0$$

Finally whilst eigenvalues may be real they are not necessarily discrete!

## Laplace transforms

So let  $f(t)$  be defined for  $t \in [0, \infty)$ . Its Laplace transform is:

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.$$

Let **functions of exponential order** be defined as:

$$\forall t \in [0, \infty); \exists A > 0 \text{ and } B \in \mathbb{R} : |f(t)| \leq Ae^{Bt}$$

where  $f$  is piecewise continuous and we denote such functions as  $f \in E$ .

For sufficiently large  $s$ , the Laplace transform  $f \in E$  converges.

## Reduction of order

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

For  $\forall f, f' \in E$  and generalising:

$$\begin{aligned}\mathcal{L}[f^{(n)}(t)] &= s^n \mathcal{L}[f(t)] - s^{(n-1)} f^{(0)}(0) \\ &\quad - s^{(n-2)} f^{(1)}(0) \\ &\quad - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0).\end{aligned}$$

## Shifts, scaling and derivatives

Let  $F(s) = \mathcal{L}[f(t)](s)$ . We then have that:

### 1. s-shift:

$$\mathcal{L}[e^{-ct} f(t)](s) = F(s + c)$$

where  $s + c > \gamma$ .

### 2. t-shift:

Let  $c \geq 0$  and  $f(t) = 0$  if  $t < 0$ .

$$\therefore \mathcal{L}[f(t - c)](s) = e^{-sc} F(s)$$

In terms of the unit step function:

$$\mathcal{L}[g(t - c)u_c(t)](s) = e^{-sc} G(s)$$

where  $G(s) = \mathcal{L}[g(t)](s)$  and  $g(t)$  any normal function.

### 3. s-derivative:

$$\mathcal{L}[t f(t)](s) = -\frac{d}{ds} F(s)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

### 4. scaling:

$$\mathcal{L}[f(ct)] = \frac{1}{c} F\left(\frac{s}{c}\right)$$

$$\frac{1}{c} \mathcal{L}\left[f\left(\frac{t}{c}\right)\right] = F(cs)$$

where  $c > 0$ .

## Higher order ODEs

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) = f(t)$$

$$Z(s)\mathcal{L}[y(t)] = \mathcal{L}[f(t)] + Z_0(s)$$

Where  $Z(s)$  is a degree  $n$  polynomial and  $Z_0(s)$  a degree  $n-1$  polynomial dependent on our initial conditions.

If the source term is of the following form:

$$f(t) = t^n e^{at} (A \cos bt + B \sin bt)$$

then  $\mathcal{L}[f(t)]$  is rational and therefore:

$$\mathcal{L}[y(t)] = \frac{\mathcal{L}[f(t)]}{Z(s)} + \frac{Z_0(s)}{Z(s)}$$

where we can solve this via standard transforms.

## Discontinuous source terms

Consider the following ODE:

$$Ay''(t) + By'(t) + Cy(t) = g(t)$$

where  $g(t)$  is piecewise continuous:

$$g(t) = f(t)[u_a(t) - u_b(t)] = \begin{cases} f(t) & t \in [a, b) \\ 0 & t \notin [a, b) \end{cases}$$

for  $b > a$ . This is the **unit step function**:

$$u_c(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq c. \end{cases}$$

$$\therefore \mathcal{L}[u_c(t)](s) = \frac{e^{-sc}}{s}$$

Furthermore we can define a shift of  $f(t)$  by  $c > 0$  to the right by:

$$f(t - c)u_c(t)$$

$$\mathcal{L}[f(t - c)u_c(t)] = e^{-sc} F(s)$$

where  $F(s) = \mathcal{L}[f(t)]$ .

## Impulse functions

The Dirac delta is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

and has the following properties:

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0)$$

$$\frac{d}{dt} [u_{t_0}(t)] = \delta(t - t_0)$$

$$\mathcal{L}[\delta(t - t_0)] = e^{-st_0}.$$

If  $t_0 = 0$  then  $\mathcal{L}[\delta(t)] = \lim_{t_0 \rightarrow 0} (e^{-st_0}) = 1$ .

Consider the following impulse ODE:

$$y''(t) + y(t) = \delta(t)$$

with initial conditions  $y(0) = y'(0) = 0$ :

$$\mathcal{L}[y(t)] = \frac{1}{s^2 + 1} \lim_{t_0 \rightarrow 0} (e^{-st_0}).$$

It is important that we do not evaluate the limit here! Then by inspection:

$$\begin{aligned}y(t) &= \lim_{t_0 \rightarrow 0} (\sin(t - t_0)u_{t_0}(t)) \\ &= \sin(t)u_0(t).\end{aligned}$$

## Convolutions

Let functions  $f, g : [0, \infty) \rightarrow \mathbb{R}$ .

$$\begin{aligned}f(t) * g(t) &= \int_0^t f(s)g(t - s)ds \\ &= \int_0^t g(s)f(t - s)ds\end{aligned}$$

- $f * (g + h) = f * g + f * h$
- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * 1 \neq f$
- $f * f \neq f^2$ .

## Convolution theorem

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

For if  $f, g \in E$  then  $f * g \in E$ . Note:

$$\int_0^t f(s)ds = \int_0^t f(s)u_0(t - s)ds$$

$$\mathcal{L}\left[\int_0^t f(s)ds\right] = \frac{\mathcal{L}[f(t)]}{s}.$$

Now consider the following ODE:

$$ay''(t) + by'(t) + cy(t) = g(t)$$

with  $y(0) = \alpha$  and  $y'(0) = \beta$ .

Here  $a, b, c, \alpha, \beta \in \mathbb{R}$ .

$$\begin{aligned}\therefore \mathcal{L}[y(t)] &= \Phi(s) + \Psi(s) \\ &= \frac{(as + b)\alpha + a\beta}{as^2 + bs + c} \\ &\quad + \frac{1}{as^2 + bs + c} \mathcal{L}[g(t)]\end{aligned}$$

$$\therefore y(t) = \phi(t) + \psi(t)$$

$$ah''(t) + bh'(t) + ch(t) = \delta(t)$$

$$h(0) = h'(0) = 0$$

$$H(s) = \mathcal{L}[h(t)] = \frac{1}{as^2 + bs + c}$$

$$\begin{aligned}\therefore \Psi(s) &= \frac{1}{as^2 + bs + c} \mathcal{L}[g(t)] \\ &= H(s)\mathcal{L}[g(t)]\end{aligned}$$

$$\begin{aligned}\mathcal{L}[\psi(t)] &= \mathcal{L}[h(t)]\mathcal{L}[g(t)] \\ &= \mathcal{L}[h(t) * g(t)].\end{aligned}$$

$$\psi(t) = h(t) * g(t)$$

$$= \int_0^t h(s)g(t - s)ds.$$

## Standard transforms

- $\mathcal{L}[t^n e^{at}] = \frac{n!}{(s - a)^{n+1}}$  where  $n \in \mathbb{N}$  and  $s > a$ .
- $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$ ,
- $\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}$  where  $s > 0$ .
- $\mathcal{L}[\sinh at] = \frac{a}{s^2 - a^2}$ ,
- $\mathcal{L}[\cosh at] = \frac{s}{s^2 - a^2}$  where  $s > |a|$ .
- $\mathcal{L}[e^{at} \sin bt] = \frac{b}{(s - a)^2 + b^2}$ ,
- $\mathcal{L}[e^{at} \cos bt] = \frac{s - a}{(s - a)^2 + b^2}$  where  $s > a$ .
- $\mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}$  where  $s > 0$ .
- $\mathcal{L}[\delta(t - c)] = e^{-cs}$ .

# Poincaré Diagram: Classification of Phase Portraits in the $(\det A, \text{Tr } A)$ -plane

$$\Delta = (\text{Tr } A)^2 - 4 \det A$$

