

## Vector products

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \text{ and } \mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

## Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear more than twice in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \end{aligned}$$

and we have the following identities:

$$\mathbf{a} = \sum_{i=1}^3 a_i \mathbf{e}_i = a_i \mathbf{e}_i$$

$$A\mathbf{x} = a_{ij}x_j \mathbf{e}_i \text{ for } m \times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\dots]_j \delta_{jk} = [\dots]_k$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k$$

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \mathbf{e}_i$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk} \epsilon_{ijk} = 6.$$

## Transformations

Let matrix  $L$  relate basis  $\{\mathbf{e}_i\}$  to basis  $\{\mathbf{e}'_i\}$  with rule:

$$\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then  $L^T L = L L^T = I$ , and:

$$\ell_{ik} \ell_{jk} = \ell_{ki} \ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij} p_j \text{ for } \mathbf{p} = p_i \mathbf{e}_i = p'_i \mathbf{e}'_i.$$

## Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip} \ell_{jq} \ell_{kr} T_{pqr}$$

which relates frame  $S$  in  $\{\mathbf{e}_i\}$  to frame  $S'$  in  $\{\mathbf{e}'_i\}$  with rule  $\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j$ , etc.

Properties of tensors:

1. The addition of two rank  $n$  tensors is also a rank  $n$  tensor.
2. The multiplication of a rank  $m$  tensor with a rank  $n$  tensor yields a rank  $m + n$  tensor.
3. If  $T_{ijk\dots s}$  is a rank  $m$  tensor then  $T_{\mathbf{ii}k\dots s}$  is a rank  $m - 2$  tensor.
4. If  $T_{ij}$  is a tensor then  $T_{ji}$  is also a tensor. Explicitly:

$$\begin{aligned} T'_{ij} &= \ell_{ip} \ell_{jq} T_{pq} \implies T' = L T L^T \\ T'_{\mathbf{ji}} &= \ell_{jp} \ell_{iq} T_{pq}. \end{aligned}$$

## Symmetric tensors

$T_{ij}$  is a symmetric tensor when  $T_{ij} = T_{ji}$  in frame  $S$ . Then  $T'_{ij} = T'_{ji}$  in frame  $S'$ .

Similarly  $T_{ij}$  is an anti-symmetric tensor if  $T_{ij} = -T_{ji}$  and  $T'_{ij} = -T'_{ji}$ .

Finally **any tensor** can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

## Quotient theorem

Consider 9 entities  $T_{ij}$  in frame  $S$  and  $T'_{ij}$  in frame  $S'$ . Let  $b_i = T_{ij} a_j$  where  $a_j$  is a vector. If  $b_i$  always transforms as a vector then  $T_{ij}$  is a rank 2 tensor.

Generalising, let  $R_{ijk\dots r}$  be a rank  $m$  tensor and  $T_{ijk\dots s}$  a set of  $3^n$  numbers where  $n > m$ . If  $T_{ijk\dots s} R_{ijk\dots r}$  is a rank  $n - m$  tensor then  $T_{ijk\dots s}$  is a rank  $n$  tensor.

## Matrices

We define a  $m \times n$  matrix  $A$  as  $(A)_{ij} = a_{ij}$  where  $i = 1, \dots, m$  and  $j = 1, \dots, n$ .

- $\text{Tr } A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^T A^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a  $3 \times 3$  matrix  $A$  is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn} a_{1l} a_{2m} a_{3n} \\ &= \epsilon_{lmn} a_{l1} a_{m2} a_{n3}. \end{aligned}$$

Furthermore:

$$\begin{aligned} \epsilon_{ijk} \det A &= \epsilon_{lmn} a_{il} a_{jm} a_{kn} \\ \epsilon_{lmn} \det A &= \epsilon_{ijk} a_{il} a_{jm} a_{kn} \\ \det A &= \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} a_{il} a_{jm} a_{kn}. \end{aligned}$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3.  $\det A = \det A^T$
4.  $\det(AB) = \det A \cdot \det B$

These also apply to columns. Finally:

$$\epsilon_{ijk} \epsilon_{lmn} \det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting  $A = I$  yields:

$$\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

## Linear equations

Let  $\mathbf{y} = A\mathbf{x}$ . Then  $x_i = A_{ij}^{-1} y_j$  with:

$$\begin{aligned} A_{ij}^{-1} &= \frac{1}{2 \det A} \epsilon_{imn} \epsilon_{jpk} a_{pm} a_{qn} \\ &= \frac{1}{\det A} C_{ij}^T \end{aligned}$$

where  $C$  is the cofactor matrix of  $A$ .

## Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L) \ell_{ip} \ell_{jq} T_{pq}$$

where  $(L)_{ij} = \ell_{ij}$  and  $\det L = \pm 1$ .

Pseudovectors are rank 1 pseudotensors.

## Invariant tensors

Tensor  $T$  is invariant or isotropic if:

$$T_{ijk\dots} = \ell_{i\alpha} \ell_{j\beta} \ell_{k\gamma} \dots T_{\alpha\beta\gamma\dots}$$

for every orthogonal matrix  $L$ .

- If  $a_{ij}$  is a rank 2 invariant tensor then  $a_{ij} = \lambda \delta_{ij}$ .
- The most general rank 3 invariant pseudotensor is  $a_{ijk} = \lambda \epsilon_{ijk}$ . There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

## Rotation tensors

The clockwise rotation of position vector  $\mathbf{x}$  to  $\mathbf{y}$  about unit vector  $\hat{\mathbf{n}}$  is given by:

$$y_i = R_{ij}(\theta, \hat{\mathbf{n}})x_j$$

$$R_{ij}(\theta, \hat{\mathbf{n}}) = \delta_{ij} \cos \theta + (1 - \cos \theta)n_i n_j - \epsilon_{ijk} n_k \sin \theta$$

and is the rotation tensor.

## Reflections and inversions

The reflection of vector  $\mathbf{x}$  to  $\mathbf{y}$  in plane with unit vector  $\hat{\mathbf{n}}$  is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector  $\mathbf{x}$  to  $\mathbf{y}$  is given by  $\mathbf{y} = -\mathbf{x}$  and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

## Projections

We define  $P$  to be a parallel projection operator to vector  $\mathbf{u}$  if:

$$P\mathbf{u} = \mathbf{u} \text{ and } P\mathbf{v} = \mathbf{0}$$

where  $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ . Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define  $Q$  to be an orthogonal projection to vector  $\mathbf{u}$  if:

$$Q\mathbf{u} = \mathbf{0} \text{ and } Q\mathbf{v} = \mathbf{v}.$$

Here  $Q = I - P$ .

## Inertia tensors

Let  $\mathbf{L}$  denote the angular momentum of a rigid body about the origin of mass  $m$ , volume  $V$  and density  $\rho$  at position  $\mathbf{r}$  with velocity  $\mathbf{v}$ . Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2 \delta_{ij} - x_i x_j) dV$$

where  $I_{ij}(O)$  is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2} I_{ij} \omega_i \omega_j = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega}.$$

## Parallel axis theorem

Consider the same rigid body now with centre of mass  $G$  and let  $\overrightarrow{OG} = \mathbf{R}$ . Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2 \delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\mathbf{r}') dV'.$$

## Diagonalisation

Let  $\mathbf{L} = I_{ij}\omega_j$  where  $I_{ij}$  is a rank 2 tensor and let  $\mathbf{L} = \lambda\boldsymbol{\omega}$ . Then:

$$(I_{ij} - \lambda \delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda \delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for  $P = \det I$ ,  $Q = \frac{1}{2}[(\text{tr } I)^2 - \text{tr}(I^2)]$  and  $R = \text{tr } I$  given tensor  $I$ .

## Real symmetric tensors

Let rank 2 real symmetric tensor  $T$  be diagonalisable with real eigenvalues  $\lambda^{(i)}$  and orthonormal eigenvectors  $\ell^{(i)}$  where  $i = 1, 2, 3$ . Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that  $\det L = +1$  which transforms frame  $S \rightarrow S'$ .

Then since  $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$ :

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} = \lambda^{(i)}\delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ij}.$$

## Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

and is  $f$  expanded about  $x = a$ .

Trigonometric expansions are in radians!

$$\begin{aligned} \therefore f(x+a) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)a^n \\ &= \exp\left(a \frac{d}{dx}\right) f(x) \end{aligned}$$

Then for three dimensions:

$$\begin{aligned} \phi(\mathbf{r} + \mathbf{a}) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{a} \cdot \nabla_{\mathbf{r}})^n \phi(\mathbf{r}) \\ &= \exp(\mathbf{a} \cdot \nabla_{\mathbf{r}}) \phi(\mathbf{r}). \end{aligned}$$

## Curvilinear coordinates

Let  $x_i$  denote Cartesian coordinates and  $u_i$  denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \rightarrow (u_1, u_2, u_3)$$

where each  $u_i = u_i(x_1, x_2, x_3)$  and:

$$\begin{aligned} \mathbf{r} &= x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \\ &= u_1 \mathbf{e}_{u_1} + u_2 \mathbf{e}_{u_2} + u_3 \mathbf{e}_{u_3}. \end{aligned}$$

## Scale factors

Let  $u_1 \rightarrow u_1 + du_1$  in  $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$ . Then  $d\mathbf{r}$  in  $\mathbf{r} \rightarrow \mathbf{r} + d\mathbf{r}$  is defined as:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 := h_1 \mathbf{e}_1 du_1.$$

$h_1$  is the scale factor of unit vector  $\mathbf{e}_1$ :

$$h_1 = \left| \frac{\partial \mathbf{r}}{\partial u_1} \right| \text{ and } \mathbf{e}_1 = \frac{1}{h_1} \frac{\partial \mathbf{r}}{\partial u_1}.$$

If  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$  then  $u_i$  is an **orthogonal** curvilinear coordinate system.

## Vector and arc length

The vector length  $d\mathbf{r}$  of  $\mathbf{r}$  is defined as:

$$d\mathbf{r} = \sum_{i=1}^3 h_i du_i \mathbf{e}_i$$

where  $u_i \rightarrow u_i + du_i$  for  $\forall i = 1, 2, 3$ .

Then the arc length  $ds$  is defined as:

$$\begin{aligned} (ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= g_{ij} du_i du_j \end{aligned}$$

where  $g_{ij}$  is the metric tensor:

$$\begin{aligned} g_{ij} &= g_{ji} = \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \\ &= h_i h_j (\mathbf{e}_i \cdot \mathbf{e}_j). \end{aligned}$$

## Area and volume

Let  $d\mathbf{r}_i = h_i \mathbf{e}_i du_i$  denote vector length when  $u_i \rightarrow u_i + du_i$ . (**No** sum!)

The infinitesimal vector area or **surface element** formed by  $d\mathbf{r}_1$  and  $d\mathbf{r}_2$  is:

$$d\mathbf{S} = (h_1 du_1 \mathbf{e}_1) \times (h_2 du_2 \mathbf{e}_2).$$

Similarly the infinitesimal volume formed by edges  $d\mathbf{r}_1$ ,  $d\mathbf{r}_2$  and  $d\mathbf{r}_3$  is:

$$\begin{aligned} dV &= |(d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3| \\ &= \sqrt{g} du_1 du_2 du_3 \end{aligned}$$

where  $g = \det(g_{ij})$ .

## Cylindrical coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$  where  $\rho$  represents the radial distance from the origin and  $\phi$  is the anticlockwise rotation angle on the  $x$ - $y$  plane. In Cartesian unit vectors:

$$\begin{aligned} \mathbf{r} &= \rho \cos \phi \mathbf{e}_1 + \rho \sin \phi \mathbf{e}_2 + z \mathbf{e}_3 \\ h_\rho &= 1, \quad \mathbf{e}_\rho = \cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2 \\ h_\phi &= \rho, \quad \mathbf{e}_\phi = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2 \\ h_z &= 1, \quad \mathbf{e}_z = \mathbf{e}_3 \end{aligned}$$

and forms an orthogonal set.

## Spherical coordinates

$(u_1, u_2, u_3) = (r, \theta, \phi)$  where  $\theta$  represents the clockwise rotation angle in  $y$ - $z$  plane and  $\phi$  the anticlockwise rotation angle in  $x$ - $y$  plane. In Cartesian unit vectors:

$$\mathbf{r} = r \sin \theta \cos \phi \mathbf{e}_1 + r \sin \theta \sin \phi \mathbf{e}_2 + r \cos \theta \mathbf{e}_3$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin \theta$$

$$\mathbf{e}_r = \sin \theta \cos \phi \mathbf{e}_1 + \sin \theta \sin \phi \mathbf{e}_2 + \cos \theta \mathbf{e}_3$$

$$\mathbf{e}_\theta = \cos \theta \cos \phi \mathbf{e}_1 + \cos \theta \sin \phi \mathbf{e}_2 - \sin \theta \mathbf{e}_3$$

$$\mathbf{e}_\phi = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2$$

and also forms an orthogonal set.



The inverse relations are given by:

$$\mathbf{e}_1 = \sin \theta \cos \phi \mathbf{e}_r + \cos \theta \cos \phi \mathbf{e}_\theta - \sin \phi \mathbf{e}_\phi$$

$$\mathbf{e}_2 = \sin \theta \sin \phi \mathbf{e}_r + \cos \theta \sin \phi \mathbf{e}_\theta + \cos \phi \mathbf{e}_\phi$$

$$\mathbf{e}_3 = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta.$$

We notice that the transformation is an orthogonal matrix – its inverse is simply its transpose.

## Gradient

The gradient of a scalar field  $f(\mathbf{r})$  is:

$$df(\mathbf{r}) := \nabla f(\mathbf{r}) \cdot d\mathbf{r}$$

when  $\mathbf{r} \rightarrow \mathbf{r} + d\mathbf{r} \implies f \rightarrow f + df$ . Taking the total differential of  $f$  yields:

$$\nabla f = \sum_{i=1}^3 \frac{1}{h_i} \frac{\partial f}{\partial u_i} \mathbf{e}_i$$

where  $\{\mathbf{e}_i\}$  is orthogonal.

## Divergence

The divergence of a vector field  $\mathbf{F}$  is:

$$\nabla \cdot \mathbf{F} := \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \int_{\delta S} \mathbf{F} \cdot d\mathbf{S}$$

for surface  $\delta S$  bounds infinitesimal  $\delta V$ .

In orthogonal curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (h_1 F_2 h_3) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right\}.$$

## Curl

The curl of a vector field  $\mathbf{F}$  in the direction of unit vector  $\hat{\mathbf{n}}$  is:

$$\hat{\mathbf{n}} \cdot (\nabla \times \mathbf{F}) := \lim_{\delta S \rightarrow 0} \frac{1}{\delta S} \oint_{\delta C} \mathbf{F} \cdot d\mathbf{r}$$

where curve  $\delta C$  encloses plane  $\delta S$ .

In orthogonal curvilinear coordinates:

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

## Laplacian

The Laplacian of a scalar field  $f$  is:

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

and in orthogonal curvilinear coordinates:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right\}.$$

The Laplacian of a vector field  $\mathbf{F}$  is:

$$\nabla^2 \mathbf{F} = \nabla (\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

## Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\nabla f = \frac{\partial f}{\partial x_i} \mathbf{e}_i$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

$$\nabla \times \mathbf{F} = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \mathbf{e}_i$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

The following operator acts on **both** scalar and vector fields:

$$(\mathbf{u} \cdot \nabla) \mathbf{F} = u_j \frac{\partial}{\partial x_j} F_i.$$

If  $\psi$  is a scalar field and  $\mathbf{v}$  a vector field:

$$\nabla \times (\nabla \psi) = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = \mathbf{0}$$

$$\nabla \cdot (\psi \mathbf{v}) = \nabla \psi \cdot \mathbf{v} + \psi \nabla \cdot \mathbf{v}$$

$$\nabla \times (\psi \mathbf{v}) = \nabla \psi \times \mathbf{v} + \psi \nabla \times \mathbf{v}.$$

Let  $\mathbf{r} = x_i \mathbf{e}_i$  and  $r = (x_i^2)^{1/2}$ . Then:

$$\bullet \nabla r = \frac{\mathbf{r}}{r} \text{ and } \nabla \left( \frac{1}{r} \right) = -\frac{\mathbf{r}}{r^3}$$

$$\bullet \nabla r^n = n r^{n-2} \mathbf{r}$$

$$\bullet \nabla \cdot \mathbf{r} = 3 \text{ and } \nabla \times \mathbf{r} = \mathbf{0}$$

$$\bullet \nabla \times (\mathbf{c} \times \mathbf{r}) = 2\mathbf{c}$$

$$\bullet \nabla \cdot (\mathbf{c} \times \mathbf{r}) = \mathbf{0} \text{ for constant } \mathbf{c}.$$

## Divergence theorem

Let surface  $S$  **enclose** volume  $V$ . Then:

$$\int_V \nabla \cdot \mathbf{F} dV = \oint_S \mathbf{F} \cdot d\mathbf{S}$$

where  $\mathbf{F}$  is a vector field.

## Stokes' theorem

Let closed curve  $C$  bound open surface  $S$  and let  $\mathbf{F}$  be a vector field. Then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

for  $C$  is traversed in anticlockwise sense.

## Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x - a) := \begin{cases} \infty & x = a \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions this becomes:

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) := \delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0)$$

where  $(x, y, z)$  are Cartesian coordinates.

In orthogonal curvilinear coordinates:

$$\delta(\mathbf{r} - \mathbf{a}) = \frac{1}{h_1 h_2 h_3} \delta(u_1 - a_1) \cdot \delta(u_2 - a_2) \delta(u_3 - a_3).$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

which yields:

$$x \delta(x) = 0 \text{ and } \delta(cx) = \frac{1}{|c|} \delta(x).$$

If simple solutions of  $g(x) = 0$  are  $x_i$ :

$$\int_{-\infty}^{\infty} f(x) \delta(g(x)) dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}.$$

## Coulomb's law

Consider charges  $q$  and  $q_1$  at positions  $\mathbf{r}$  and  $\mathbf{r}_1$ . The force on charge  $q$  at  $\mathbf{r}$  due to charge  $q_1$  at  $\mathbf{r}_1$  is:

$$\mathbf{F}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{qq_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$

where  $qq_1 > 0$  denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419 \times 10^{-12} \text{C}^2 \text{N}^{-1} \text{m}^{-2}.$$

Charge has units Coulombs (C) and an electron has charge  $-1.60218 \times 10^{-19} \text{C}$ .

## Electric fields

The electric field is induced by a charge distribution and defined in terms of the force on a small positive test charge  $q$ :

$$\mathbf{E}(\mathbf{r}) := \lim_{q \rightarrow 0} \frac{1}{q} \mathbf{F}.$$

Then for our two charges  $q$  and  $q_1$ :

$$\mathbf{F}_1(\mathbf{r}) = q\mathbf{E}_1(\mathbf{r})$$

where  $q_1$  produces electric field  $\mathbf{E}_1$ .

$$\therefore \mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$

$$\therefore \phi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\mathbf{r} - \mathbf{r}_1|}$$

## Principle of superposition

For a set of charges  $q_i$  at position  $\mathbf{r}_i$  the total electric field at  $\mathbf{r}$  is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}.$$

For object with **charge density**  $\rho(\mathbf{r}')$  its overall electric field at  $\mathbf{r}$  is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

where  $\rho(\mathbf{r}')$  is charge divided by volume. The **type** of integral (surface or line) is dependent on the object in consideration.

## Electrostatic Maxwell's equations

Because  $\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$ :

$$\mathbf{E}(\mathbf{r}) = -\nabla \left( \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \right)$$

and therefore for all static electric fields:

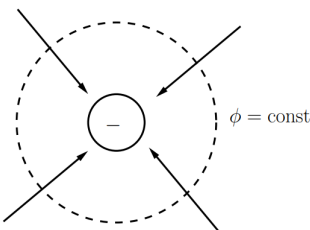
$$\nabla \times \mathbf{E} = \mathbf{0}.$$

$\mathbf{E}$  is a **conservative** vector field where its line integral is **independent** of path. Furthermore it may be written as:

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$$

for  $\phi(\mathbf{r})$  is the potential of  $\mathbf{E}$ .

$$\therefore \phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$



The **potential difference** between two points  $A$  and  $B$  is the energy per unit charge needed to move a small charge  $q$  from  $A$  to  $B$ :

$$\begin{aligned} V_{A \rightarrow B} &= \lim_{q \rightarrow 0} \frac{1}{q} W_{A \rightarrow B} \\ &= -\frac{1}{q} \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \phi_B - \phi_A. \end{aligned}$$

A charge distribution  $\rho(\mathbf{r}')$  in an external electric field has potential energy:

$$W = \int_V \rho(\mathbf{r}') \phi_{ext}(\mathbf{r}') dV'.$$

Because  $\nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}')$ :

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}.$$

## Electric dipoles

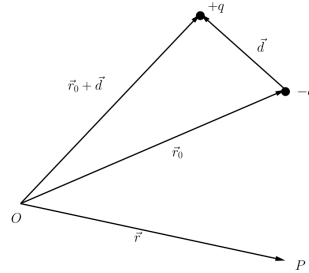
An electric dipole at  $\mathbf{r}_0$  is defined as two charges  $-q$  at  $\mathbf{r}_0$  and  $+q$  at  $\mathbf{r}_0 + \mathbf{d}$  which generates **dipole moment**:

$$\mathbf{p} = q\mathbf{d}$$

and in the dipole limit this is defined as:

$$\mathbf{p} := \lim_{\substack{q \rightarrow \infty \\ d \rightarrow 0}} q\mathbf{d}$$

known as an ideal dipole.



The electrostatic potential generated by this ideal dipole at  $\mathbf{r}_0$  is given by:

$$\begin{aligned} \phi(\mathbf{r}) &= \phi_q + \phi_{-q} \\ &= \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|\mathbf{r} - \mathbf{r}_0 - \mathbf{d}|} - \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) \\ &\approx \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3} \end{aligned}$$

for the first term is expanded in powers of  $-\mathbf{d}$  about  $\mathbf{r} - \mathbf{r}_0$ .

The electric field generated is:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \left[ -\frac{\mathbf{p}}{|\mathbf{r} - \mathbf{r}_0|^3} \right. \\ &\quad \left. + \frac{3\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} (\mathbf{r} - \mathbf{r}_0) \right]. \end{aligned}$$

If the ideal dipole is at the origin:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left( \frac{3\mathbf{p} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{p}}{r^3} \right).$$

Let ideal dipole moment  $\mathbf{p}$  be parallel to the  $z$ -axis. Then in spherical coordinates  $(r, \theta, \chi)$ ,  $\mathbf{r} = r\mathbf{e}_r$ ,  $\mathbf{p} = p\mathbf{e}_z$  and:

$$\phi(\mathbf{r}) = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0} \left( \frac{2\cos\theta}{r^3} \mathbf{e}_r + \frac{\sin\theta}{r^3} \mathbf{e}_\theta \right).$$

## Force, torque and energy

The **force** on a **dipole** at  $\mathbf{r}$  from external electric field  $\mathbf{E}_{ext}(\mathbf{r})$  is:

$$\begin{aligned} \mathbf{F} &= -q\mathbf{E}_{ext}(\mathbf{r}) + q\mathbf{E}_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx (\mathbf{p} \cdot \nabla) \mathbf{E}_{ext}(\mathbf{r}). \end{aligned}$$

The **torque** on a dipole at  $\mathbf{r}$  about the axis  $\mathbf{r}$  due to  $\mathbf{E}_{ext}(\mathbf{r})$  is:

$$\begin{aligned} \mathbf{G} &= \boldsymbol{\tau}_{-q} + \boldsymbol{\tau}_q \\ &= -q\mathbf{0} \times \mathbf{E}_{ext}(\mathbf{r}) + q\mathbf{d} \times \mathbf{E}_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx \mathbf{p} \times \mathbf{E}_{ext}(\mathbf{r}). \end{aligned}$$

The **energy** of a dipole at  $\mathbf{r}$  from external electric field  $\mathbf{E}_{ext}(\mathbf{r}) = -\nabla\phi_{ext}(\mathbf{r})$  is:

$$\begin{aligned} W &= -q\phi_{ext}(\mathbf{r}) + q\phi_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx -\mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{r}) \end{aligned}$$

and  $\mathbf{F} = -\nabla W$ .

## Multipole expansion

Consider object with volume  $V$  and charge distribution  $\rho(\mathbf{r}')$ . Let origin be in the object. Then the potential at  $\mathbf{r}$  is:

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \\ &\approx \frac{1}{4\pi\epsilon_0} \left( \frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{Q_{ij}x_ix_j}{2r^5} \right) \end{aligned}$$

where  $Q$  is the **total charge** in  $V$ :

$$Q = \int_V \rho(\mathbf{r}') dV'$$

$\mathbf{p}$  the **dipole moment** about the origin:

$$\mathbf{p} = \int_V \mathbf{r}' \rho(\mathbf{r}') dV'$$

and  $Q_{ij}$  the **quadrupole tensor**:

$$Q_{ij} = \int_V \rho(\mathbf{r}') \left[ 3x'_i x'_j - (r')^2 \delta_{ij} \right] dV'.$$

If  $Q \neq 0$  then in the far zone ( $r \gg r_0$ ) the first term (monopole term) dominates.

If  $Q = 0$  and  $\mathbf{p} = \mathbf{0}$  then the third term (quadrupole term) dominates in the far zone and etc.

## Interaction energy

By expanding  $\phi_{ext}(\mathbf{r})$  about  $\mathbf{r} = \mathbf{0}$ :

$$\begin{aligned} W &= \int_V \rho(\mathbf{r}') \phi_{ext}(\mathbf{r}') dV' \\ &= Q \phi_{ext}(\mathbf{0}) - \mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{0}) \\ &\quad - \frac{1}{6} Q_{ij} \frac{\partial (\mathbf{E}_{ext}(\mathbf{0}))_i}{\partial x_j} + \dots \end{aligned}$$

and is the potential energy of a charge distribution  $\rho(\mathbf{r})$  in  $\mathbf{E}_{ext}$ .

## Gauss' law

For object with charge distribution  $\rho(\mathbf{r})$  and volume  $V$  enclosed by surface  $S$ :

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\epsilon_0}$$

where  $Q_{enc}$  is total charge enclosed by  $V$ :

$$Q_{enc} = \int_V \rho(\mathbf{r}') dV'$$

and is useful for symmetric problems.

## Boundaries in electrostatics

Let  $\sigma$  be the charge density of a surface separating electric fields  $\mathbf{E}_1$  and  $\mathbf{E}_2$ .

1. Normal component of electric field is discontinuous across surface by:

$$\hat{\mathbf{n}} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \frac{\sigma}{\epsilon_0}.$$

2. Tangential component of electric field is continuous across surface:

$$\mathbf{E}_{||} := \hat{\mathbf{n}} \times \mathbf{E}_1 = \hat{\mathbf{n}} \times \mathbf{E}_2.$$

## Conductors

Conductors have surplus electrons that can move freely when an electric field is applied. **In electrostatics:**

1. Conductors are in equilibrium, all charges are at rest and reside on the surface of the conductor.

Hence inside a conductor  $\rho(\mathbf{r}) = 0$ ,  $\mathbf{E}(\mathbf{r}) = \mathbf{0}$  and  $\phi = \text{constant}$ .

2. An electric field is always normal to the surface of a conductor:

$$E_{\perp} = \frac{\sigma}{\epsilon_0} \text{ and } E_{||} = 0.$$

The presence of an external electric field induces a charge distribution  $\sigma$  on the surface of our conductor. This changes the external electric field as it needs to be normal to the surface of the conductor.

## Poisson's equation

Because  $\mathbf{E} = -\nabla\phi$  and  $\nabla \cdot \mathbf{E} = \rho(\mathbf{r})/\epsilon_0$ :

$$\nabla^2 \phi = -\frac{\rho(\mathbf{r})}{\epsilon_0}.$$

We can solve this by direct integration or using the **method of images**.

Given volume under consideration place fictitious charge outside the volume such that the system still satisfies Poisson's equation with boundary conditions.

This potential is our solution.

## Electrostatic energy

The work needed to move point charge  $q$  from  $\mathbf{r}_A$  to  $\mathbf{r}_B$  in  $\mathbf{E}(\mathbf{r})$  is:

$$W_{A \rightarrow B} = qV_{A \rightarrow B}.$$

Then  $W_{\infty \rightarrow B} = q\phi(\mathbf{r}_B)$  since potential  $\phi$  vanishes at infinity.

Generalising, the work needed to move a system of  $n$  charges  $q_i$  from infinity to  $\mathbf{r}$  is a double sum with overcounting as each charge contributes to the electric field:

$$W_e = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i,j (i \neq j)}^n \frac{q_i q_j}{|\mathbf{r}_j - \mathbf{r}_i|}.$$

Furthermore the energy needed to move a continuous charge distribution  $\rho(\mathbf{r}')$  from infinity to position  $\mathbf{r}$  is:

$$\begin{aligned} W_e &= \frac{1}{2} \int_V \rho(\mathbf{r}) \phi(\mathbf{r}) dV \\ &= \frac{\epsilon_0}{2} \int_V |\mathbf{E}(\mathbf{r})|^2 dV. \end{aligned}$$

## Capacitors

A capacitor is formed by two conductors 1 and 2 with equal and opposite charges  $Q$  and  $-Q$ . The **capacitance** ( $1\text{CV}^{-1}$ ) of a capacitor is defined as:

$$C := \frac{Q}{V}$$

where  $Q = \sigma A$  for  $A$  is the surface area of **one** conductor and potential difference  $V = \phi_1 - \phi_2$  from the conductors.

The energy stored in a capacitor is the amount of work done to move charge across the two conductors. So to move charge  $dq$  from conductor with  $+q$ :

$$dW = \left(\frac{q}{C}\right) dq$$

and integrating this up to  $Q$  gives:

$$W = \frac{1}{2} \frac{Q^2}{C}.$$

## Currents

An elementary current is generated by a charge  $q$  moving at velocity  $\mathbf{v}$ .

The **bulk current density** is:

$$\mathbf{J}(\mathbf{r}) := \rho(\mathbf{r})\mathbf{v}$$

for  $\rho(\mathbf{r})$  is the volume charge density.

The **surface current density** is:

$$\mathbf{K}(\mathbf{r}) := \sigma(\mathbf{r})\mathbf{v}$$

for  $\sigma(\mathbf{r})$  is the surface charge density.

The **line charge density** is:

$$\mathbf{I}(\mathbf{r}) := \lambda(\mathbf{r})\mathbf{v}$$

for  $\lambda(\mathbf{r})$  is the line charge density.

The infinitesimal **current element** is:

$$d\mathbf{I}(\mathbf{r}) := \begin{cases} \mathbf{J}(\mathbf{r})dV & \text{volume current} \\ \mathbf{K}(\mathbf{r})dS & \text{surface current} \\ \mathbf{I}(\mathbf{r})dr & \text{line current} \end{cases}$$

Current  $I$  has units Ampères (A) but the infinitesimal current element  $d\mathbf{I}(\mathbf{r})$  has units of current/volume, etc.

Note:  $1\text{A} := 1\text{Cs}^{-1}$

Consider volume  $V$  bounded by surface  $S$  with total charge  $Q$ . Because the **total charge is conserved**:

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0$$

$$Q = \int_V \rho(\mathbf{r}, t) dV.$$

An electric field can induce a current in a conductor, via **Ohm's law**:

$$\mathbf{J} \approx \sigma_{ij} E_j$$

where  $\sigma_{ij}$  is the conductivity tensor.

1. Perfect conductors:  
 $\sigma \rightarrow \infty$  and  $\mathbf{E} = \mathbf{0}$ .

2. Insulators:  $\sigma = 0$ .

The **electromotive force** (emf) is:

$$\begin{aligned} \mathcal{E}_{1 \rightarrow 2} &:= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{r} \\ &= \phi(\mathbf{r}_1) - \phi(\mathbf{r}_2) \end{aligned}$$

since  $\mathbf{E} = -\nabla\phi$ . (static case)

### Biot-Savart law

The magnetic field at  $\mathbf{r}$  generated by a static current loop carrying current  $I$  is:

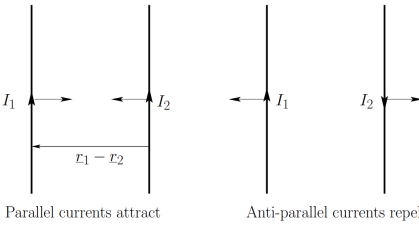
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

and is the right grip rule.



Magnetic fields have units Teslas (T), where  $\text{NC}^{-1}\text{m}^{-1}\text{s} = \text{T}$  and:

$$\mu_0 = 1.25664 \dots \times 10^{-6} \text{NA}^{-2}.$$



No force is induced from perpendicular currents.

### Lozentz force

Through physical experiments the **force density** on a charge distribution  $\rho(\mathbf{r})$  in electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$  is:

$$\mathbf{f} := \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

where  $\mathbf{J}$  is the **current density** and integrating yields the **right hand rule**.



Current is the movement of charge.

For point charge  $q$  at  $\mathbf{r}'$  with velocity  $\mathbf{v}$  in  $\mathbf{E}$  and  $\mathbf{B}$  its net force is the integral of the force density over volume  $V$ :

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

since  $\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}')$  and  $\mathbf{J} = \rho(\mathbf{r})\mathbf{v}$ .

### Magnetostatic Maxwell's equations

Because  $\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$ :

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \times \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV'$$

therefore  $\nabla \cdot \mathbf{B} = 0$  due to:

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - (\nabla \times \mathbf{v}) \cdot \mathbf{u}.$$

This implies that magnetic fields always form **closed loops** — there are no sources or sinks for magnetic fields.

Similarly using the following identity:

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v}(\nabla \cdot \mathbf{u})$$

$$\text{and } \nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

### Ampère's law

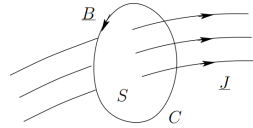
Using the divergence theorem, there is no magnetic flux through a **closed** surface:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0.$$

From Stokes' theorem:

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{enc}$$

which implies that the circulation of a magnetic field  $\mathbf{B}$  around a closed loop  $C$  is proportional to the total current  $I$  that passes through the enclosed surface.



### Boundaries in magnetostatics

Let conductor with current density  $\mathbf{K}$  separate magnetic fields  $\mathbf{B}_1$  and  $\mathbf{B}_2$ .

1. Normal component of magnetic field is continuous across surface:

$$\mathbf{B}_\perp := \mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}.$$

2. Tangential component of magnetic field is **discontinuous** across surface:

$$\hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0 \mathbf{K}.$$

### Magnetic vector potentials

Because  $\nabla \cdot (\nabla \times \mathbf{v}) = 0$  and  $\nabla \cdot \mathbf{B} = 0$ :

$$\exists \mathbf{A} : \mathbf{B} = \nabla \times \mathbf{A}$$

known as the **vector potential**. Then:

$$\begin{aligned} \nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ &= \mu_0 \mathbf{J} \end{aligned}$$

$$\therefore \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

where we set  $\nabla \cdot \mathbf{A} = 0$  and solution:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

and boundary condition  $\lim_{r \rightarrow \infty} \mathbf{A}(\mathbf{r}) = \mathbf{0}$ .

### Magnetic dipoles

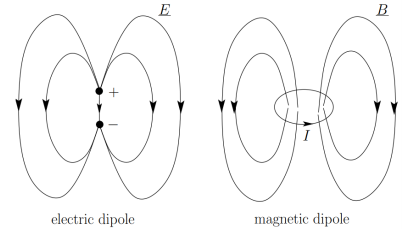
The vector potential for a current loop positioned at the origin in the far zone is:

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \oint_C \frac{I d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{\mu_0}{4\pi} \oint_C \frac{1}{r} \left( 1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \dots \right) d\mathbf{r}' \\ &\approx \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \end{aligned}$$

where  $\mathbf{m}$  is the magnetic dipole moment:

$$\begin{aligned} \mathbf{m} &= I \int_S d\mathbf{S} \\ &= \frac{1}{2} \oint_C \mathbf{r} \times d\mathbf{I} \end{aligned}$$

via Stokes' theorem.



### Force and torque

The Lorentz force on a current loop due to an external magnetic field  $\mathbf{B}$  is:

$$\begin{aligned} \mathbf{F} &= I \oint_C d\mathbf{r}' \times \mathbf{B}(\mathbf{r}') \\ &= \nabla(\mathbf{m} \cdot \mathbf{B}). \end{aligned}$$

The torque on a current loop due to an external magnetic field  $\mathbf{B}$  is:

$$\begin{aligned} \mathbf{G} &= \oint_C \mathbf{r}' \times [I d\mathbf{r}' \times \mathbf{B}(\mathbf{r}')] \\ &= \mathbf{m} \times \mathbf{B}. \end{aligned}$$

### Motional electromotive force

The electromotive force (emf) is the **work needed** for unit point charge to circulate around a conductor loop:

$$\begin{aligned} \mathcal{E} &= \oint_C \mathbf{f} \cdot d\mathbf{r} \\ &= \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} \end{aligned}$$

for  $\mathbf{v}$  is the velocity of the charge and  $\mathbf{f}$  is the force density on point charge.



## Magnetic induction

Faraday's law of induction states that a **change** in magnetic flux  $\Phi$  induces an emf in a conductor loop:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where **magnetic flux** is defined as the **total magnetic field passing through the region** bounded by the conductor loop:

$$\begin{aligned}\Phi &= \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= \mathbf{B} \cdot \mathbf{S} \text{ if constant } \mathbf{B}\end{aligned}$$

and any surface  $S$  enclosing the region.



For static charges in a conductor loop  $C$  with time dependent magnetic field:

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{r} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

and using Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

In words, a time dependent magnetic field **always accompanies** a spatial and time dependent electric field.

## Galilean relativity

If the velocity of frame  $S'$  in  $S$  is  $\mathbf{v}$ :

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$$

$$dt' = dt$$

for point  $P$  has position  $\mathbf{r}$  in frame  $S$  and position  $\mathbf{r}'$  in frame  $S'$ .

Let circuit  $C$  be in motion in frame  $S$  with velocity  $\mathbf{v}$  with respect to  $\mathbf{B}(\mathbf{r}, t)$ . Then let  $C$  be stationary in frame  $S'$ . Since the electromotive force generated is the same regardless of frames, in frame  $S$ :

$$\mathcal{E} = \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$$

and as  $\mathbf{v} = \mathbf{0}$  in frame  $S'$ :

$$\mathcal{E} = \oint_C \mathbf{E}' \cdot d\mathbf{r}.$$

Equating the two statements:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

but only applies at  $v \ll c$ .

## Mutual and self inductance

Consider conductor loops 1 and 2 with current  $I_1$  and  $I_2$ . The magnetic vector potential generated by loop 1 is:

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 d\mathbf{r}'_1}{|\mathbf{r} - \mathbf{r}'_1|}$$

and the magnetic flux in loop 2 is:

$$\begin{aligned}\Phi_{2 \leftarrow 1} &= \frac{\mu_0 I_1}{4\pi} \oint_{C_2} d\mathbf{r}'_2 \cdot \oint_{C_1} \frac{d\mathbf{r}'_1}{|\mathbf{r}'_2 - \mathbf{r}'_1|} \\ &= M_{21} I_1\end{aligned}$$

where  $M_{21}$  is the Neumann's formula for mutual induction. Then for two loops:

$$\begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix} = - \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \begin{pmatrix} \frac{dI_1}{dt} \\ \frac{dI_2}{dt} \end{pmatrix}$$

where  $M = M_{21} = M_{12}$ .  $L_1$  and  $L_2$  are the self-inductance for each loop.

formula for self inductance

$M_{21}$  only equal when we have equal areas

## Magnetic field energy

Consider an inductor generating field  $\mathbf{B}$  with current  $I$  and self-inductance  $L$ :

$$\begin{aligned}dW_m &= idt \cdot -\mathcal{E} \\ &= idt \cdot \frac{d\Phi_i}{dt} \\ &= iL di\end{aligned}$$

and since  $\Phi_{1 \leftarrow 1} = IL$  the energy stored our inductor is an integral over current:

$$W_m = \int_0^I dW_m = \frac{\Phi_{1 \leftarrow 1}^2}{2L} \text{ or } \frac{1}{2} \Phi_{1 \leftarrow 1} I.$$

Because  $\Phi_{1 \leftarrow 1} = \int_S \mathbf{B} \cdot d\mathbf{S}$ :

$$W_m = \frac{1}{2} \oint_C \mathbf{A} \cdot I d\mathbf{r}$$

and generalising this to volume integrals:

$$W_m = \frac{1}{2} \int_V \mathbf{J} \cdot \mathbf{A} dV$$

where  $\mathbf{J}$  is the current density. Since:

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - (\nabla \times \mathbf{v}) \cdot \mathbf{u}$$

and assuming that the field vanish at  $\infty$ :

$$W_m = \frac{1}{2\mu_0} \int_V B^2 dV$$

with **magnetic field density**:

$$w_m = \frac{1}{2\mu_0} B^2.$$

## Circuits

Consider an LRC circuit with alternating current source and electromotive force:

$$\begin{aligned}\mathcal{E}_S &= V_S \cos(\phi + \omega t) \\ &= \text{Re}[V_0 e^{i\omega t}]\end{aligned}$$

where  $V_0 = V_S e^{i\phi}$ . Equating emfs:

$$\begin{aligned}\mathcal{E}_S &= V_L + V_C + V_R \\ &= L \frac{dI}{dt} + V_C(t) + IR \\ I &= C \frac{dV_C}{dt}\end{aligned}$$

and due to linearity we assume that:

$$\begin{aligned}I(t) &= \text{Re}[I_0 e^{i\omega t}] \\ V_C(t) &= \text{Re}[V_{C_0} e^{i\omega t}].\end{aligned}$$

After substituting into our equations:

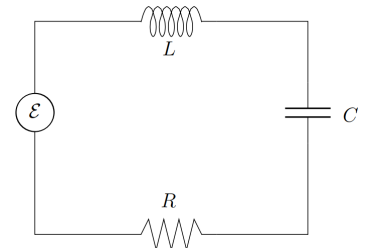
$$V_0 = i\omega L I_0 + V_{C_0} + R I_0$$

$$I_0 = i\omega C V_0 \text{ and } Z := \frac{V_0}{I_0}$$

which yields solutions:

$$\begin{aligned}I(t) &= \text{Re} \left[ \frac{V_0}{Z} e^{i\omega t} \right] \\ &= \frac{V_S \cos(\omega t + \phi - \psi)}{[R^2 + (\omega L - \frac{1}{\omega C})^2]^{1/2}}\end{aligned}$$

and  $\psi = \arctan \left[ \frac{\omega L - \frac{1}{\omega C}}{R} \right]$ .



current/voltage in series/parallel circuits

voltage of specific components

## Electromagnetic waves

From considering the capacitor paradox:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Since in vacuum  $\rho = 0$  and  $\mathbf{J} = \mathbf{0}$ :

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$$

and using  $\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$ :

$$\begin{aligned}\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \\ \Rightarrow \nabla^2 \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.\end{aligned}$$

Similarly for our electric field  $\mathbf{E}$ :

$$\begin{aligned}\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} \\ \Rightarrow \nabla^2 \mathbf{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.\end{aligned}$$

Let  $F$  be any component of fields  $\mathbf{E}(\mathbf{r}, t)$  or  $\mathbf{B}(\mathbf{r}, t)$ . Then it satisfies the following:

$$\nabla^2 F = \mu_0 \epsilon_0 \frac{\partial^2 F}{\partial t^2}.$$

Substituting solutions of form:

$$\begin{aligned}F(t, x, y, z) &= f(\mathbf{k} \cdot \mathbf{x} - \omega t) \\ \Rightarrow |\mathbf{k}|^2 &= \mu_0 \epsilon_0 \omega^2\end{aligned}$$

and with phase velocity:

$$\begin{aligned}v_{\text{phase}} &= \frac{\omega}{|\mathbf{k}|} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \\ &= 3.10 \times 10^8 \text{ms}^{-1} = c\end{aligned}$$

which is the speed of light and implies that light is also an electromagnetic wave!

## Lorentz transformations

It is then postulated that:

- The **speed of light is universal**.  $c$  is **frame invariant** and classically only propagate forwards in time.

In Minkowsky spacetime we have that:

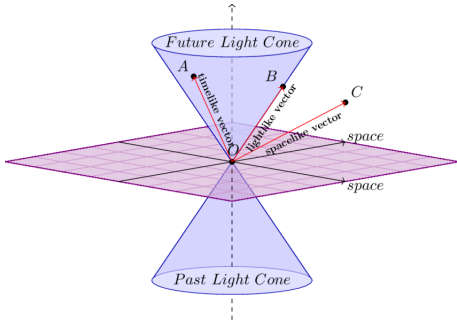
$$(\Delta S)^2 = (\Delta S')^2$$

between two frames  $S, S'$  and:

$$(\Delta S)^2 := (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

for events  $A$  and  $B$ ,  $\Delta x = x_B - x_A$ , etc.

- $(\Delta S)^2 = 0$ : light-like separated
- $(\Delta S)^2 > 0$ : time-like separated
- $(\Delta S)^2 < 0$ : space-like separated



A **boost**  $B_x$  in the  $\mathbf{e}_x$  direction is defined:

$$\mathbf{x}' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x}$$

where  $\mathbf{x} = (ct, x, y, z)^T$ .

Generally we can relate reference frames  $S$  and  $S'$  by a composition of rotations and boosts. This forms a group, denoted by  $\text{SO}(3) = \{R_x, R_y, R_z, B_x, B_y, B_z\}$ .

Practically if frame  $S'$  is moving at  $v\mathbf{e}_x$  with respect to frame  $S$  then:

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(-\beta ct + x)$$

$$y' = y \text{ and } z' = z$$

where  $\gamma = (1 - \beta^2)^{-1/2}$  and  $\beta = v/c$ .

## Time and length

Consider object in rest in frame  $S'$ , which is moving with respect to frame  $S$ . Then:

- lifetime in  $S$ :  $\gamma\tau$
- length in  $S$ :  $\ell_0/\gamma$

for time  $\tau$  and length  $\ell_0$  are its physical quantities in frame  $S'$ .

## Electromagnetic energy

By considering the Lorentz force with:

$$dW = \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot v d\mathbf{t}$$

and generalising to a charge distribution:

$$\frac{dW}{dt} = \int_V \mathbf{J} \cdot \mathbf{E} d^3\mathbf{r}.$$

Using Maxwell's equations:

$$\underbrace{\frac{dW}{dt}}_{\text{power}} + \int_{\partial V} \mathbf{S} \cdot d\mathbf{a} + \frac{dU_{em}}{dt} = 0$$

where  $U_{em}$  represents the total energy stored in the electric and magnetic fields:

$$U_{em} = \int_V \left( \frac{1}{\mu_0} |\mathbf{B}|^2 + \epsilon_0 |\mathbf{E}|^2 \right) d^3\mathbf{r}$$

and we define the **Poynting vector**:

$$\mathbf{S} := \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

Likewise its surface integral denotes the **rate** at which energy flows.

If  $\int_{\partial V} \mathbf{S} \cdot d\mathbf{a} < 0$  then power flows **into** the bounded surface and vice versa.

## Maxwell's stress tensor

Since Newton's second law states that:

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}$$

using the Lorentz force we have that:

$$\frac{d\mathbf{P}}{dt} = -\frac{d}{dt} \int_V \epsilon_0 \mu_0 \mathbf{S} d^3\mathbf{r} + \int_{\partial V} \mathbf{T} \cdot d\mathbf{a}$$

where  $\mathbf{T}$  is the **stress tensor**:

$$\begin{aligned}T_{ij} &= \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} |\mathbf{E}|^2 \right) \\ &+ \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} |\mathbf{B}|^2 \right).\end{aligned}$$

## Relativistic kinematics

Now we interpret particles of *constant* motion under the context of inertial frames. Define frame  $S'$  to be moving at  $v\mathbf{e}_x$  with respect to frame  $S$ .

If a particle is moving at  $\mathbf{u}' = (u'_x, u'_y, u'_z)$  with respect to frame  $S'$  then:

$$cdt' = \gamma(cdt - \beta dx)$$

$$dx' = \gamma(-\beta cdt + dx) \text{ and so on...}$$

which yields the following relation:

$$\frac{1}{c} u'_x := \frac{dx'}{cdt'} \Rightarrow \frac{-\beta c + u_x}{c - \beta u_x}$$

where  $\mathbf{u} = (u_x, u_y, u_z)$  is the velocity of the particle with respect to frame  $S$ .

## Instantaneous rest frames

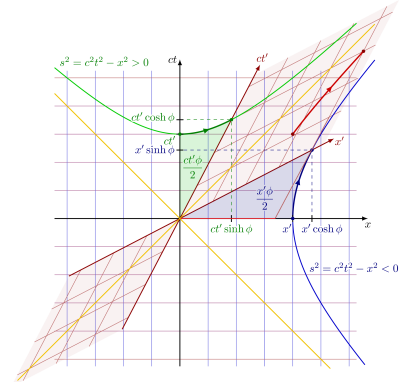
Generally, for every object, there exists a *unique rest frame* such that it has no velocity with respect to this frame.

Let a particle have *instantaneous* velocity  $\mathbf{u}(t) = \frac{d\mathbf{x}}{dt}$  with respect to frame  $S$ . Then the frame  $S'$  with boost  $\beta = u/c$  is its rest frame. The time element in frame  $S'$  is the **proper time element**:

$$dt' := d\tau = \frac{1}{c} \left( 1 - \frac{u^2}{c^2} \right)^{1/2} dt.$$

## Invariant $(\Delta S)^2$ trajectories

Good time to work through tutorial sheet 17 question 3.





## Plane wave solutions

In *vacuum*, the Maxwell's equations may be recast into a classical wave equation:

$$\nabla^2 \{\mathbf{E}, \mathbf{B}\} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \{\mathbf{E}, \mathbf{B}\}$$

$$\mu_0 \epsilon_0 = c^{-2}.$$

We then look for solutions of linear form:

$$\mathbf{E} = \text{Re} \left\{ \mathbf{E}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \right\}$$

$$\mathbf{B} = \text{Re} \left\{ \mathbf{B}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \right\}$$

known as **plane wave** solutions:

$$|\mathbf{k}| = \frac{\omega}{c}$$

from substituting into our wave equation. The *frequency*  $\omega$  can correspond to a color in the visible electromagnetic spectrum. Since  $\rho = 0$  and  $\mathbf{J} = \mathbf{0}$  we have that:

$$\mathbf{k} \cdot \mathbf{E} = 0, \quad \mathbf{k} \cdot \mathbf{B} = 0,$$

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B} \quad \text{and} \quad -\mathbf{k} \times \mathbf{B} = \frac{\omega}{c^2} \mathbf{E}.$$

i.e. that  $(\hat{\mathbf{k}}, \hat{\mathbf{E}}, \hat{\mathbf{B}})$  forms a right-handed *orthonormal basis* and:

$$\mathbf{E} = \left( a_1 e^{i\delta_1} \hat{\mathbf{e}}_1 + a_2 e^{i\delta_2} \hat{\mathbf{e}}_2 \right) e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})}$$

$$\implies a_1 \hat{\mathbf{e}}_1 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_1)$$

$$+ a_2 \hat{\mathbf{e}}_2 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_2)$$

where  $a_1, a_2, \delta_1, \delta_2 \in \mathbb{R}$ . The unit vectors  $\hat{\mathbf{e}}_1$  and  $\hat{\mathbf{e}}_2$  are perpendicular to each other as well as the **wave vector**  $\mathbf{k}$ , which defines the direction of wave propagation.

Fixing  $\mathbf{E}$  also fixes the magnetic field:

$$\mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}.$$

## Polarisation

Every plane wave solution with fixed  $\omega$  is **monochromatic**. We then define the *polarisation* vector  $\hat{\mathbf{n}}$  of a plane wave as:

$$\mathbf{E} = |\mathbf{E}| \hat{\mathbf{n}} \quad \text{where} \quad \hat{\mathbf{n}} \cdot \mathbf{k} = 0.$$

Hence every electromagnetic wave can be thought of as a superposition of polarised monochromatic transverse waves.

1. **Linear** polarisation:  $\delta_1 = \delta_2$

$$\mathbf{E}_{linear} = (a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2)$$

$$\cdot \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_1)$$

2. **Circular** polarisation:  $\delta_1 \pm \frac{\pi}{2} = \delta_2$

$$\mathbf{E}_{circ} = a_1 \left[ \hat{\mathbf{e}}_1 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_1) \right.$$

$$\mp \hat{\mathbf{e}}_2 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_1) \left. \right]$$

## Energy in electromagnetic waves

Given an arbitrary plane wave solution:

$$\mathbf{E} = \mathbf{E}_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)$$

the corresponding electromagnetic wave energy density enclosed by a volume is:

$$u_{em} = u_e + u_m$$

$$:= \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2$$

$$= \epsilon_0 |\mathbf{E}_0|^2 \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)$$

where  $u_e = u_m$  for all space and time.

$$\therefore \langle u_{em} \rangle = \epsilon_0 |\mathbf{E}_0|^2$$

The Poynting vector can be interpreted as the *flux of energy* in an unit area:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

$$= \frac{1}{\mu_0 \omega} \mathbf{E} \times (\mathbf{k} \times \mathbf{E})$$

$$= \frac{\mathbf{k}}{\omega \mu_0} |\mathbf{E}|^2$$

$$\implies (c u_{em}) \hat{\mathbf{k}}.$$

## Gauge freedoms

coulomb gauge

lorenz gauge

## Retarded potentials

magnetic and electric potentials for the  
lorenz gauge

## Electric dipole radiation

compute electric field potential

magnetic field vector potential

poynting vector

## 4-vectors

## Electric fields in matter

## Magnetic materials