D2.1: Pointwise convergence

$f_n \to f$ pointwise on E if:

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Here $f_n : E \to \mathbb{R}$.

$$\forall x \in E; \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N$$
$$\implies |f_n(x) - f(x)| < \epsilon$$

D2.2: Uniform convergence

$f_n \to f$ uniformly on E if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \text{ and } \forall x \in E$$
$$\implies |f_n(x) - f(x)| < \epsilon$$

P2.1

The following statements are equivalent.

1. $f_n \to f$ uniformly on E
2. $\lim\limits_{n\to\infty} \sup\limits_{x\in E} |f_n(x) - f(x)| = 0$
3. $\exists a_n \to 0$ s.t. $|f_n(x) - f(x)| \leq a_n$ for $\forall x \in E$.

T2.1

If f_n is continuous on E **and** $f_n \to f$ uniformly on E then f is continuous on E.

Remark

If f is <u>not continuous</u> on E then f_n <u>cannot</u> be uniform on E.

T2.5: Weierstrass M-test

Let $E \subset \mathbb{R}$ and $f_k : E \to \mathbb{R}$.
$\exists M_k > 0 : \sum\limits_{k=1}^{\infty} M_k < \infty$.
If $\forall k \in \mathbb{N}$ and $\forall x \in E$; $|f_k(x)| \leq M_k$ then:
$\sum\limits_{k=1}^{\infty} f_k(x)$ converges uniformly on E.