Vector products

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$$

$$\boldsymbol{a}\times\boldsymbol{b} = ab\sin\theta\hat{\boldsymbol{n}}$$

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a} \quad\text{and}\quad \boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear **more than twice** in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{ijk} = -\epsilon_{ijk} = -\epsilon_{ijk} \end{aligned}$$

and we have the following identities:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i\boldsymbol{e}_i = a_i\boldsymbol{e}_i$$

$$A\boldsymbol{x} = a_{ij}x_j\boldsymbol{e}_i \text{ for } m\times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\ldots]_j\delta_{jk} = [\ldots]_k$$

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$$

$$\boldsymbol{e}_i\times\boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k$$

$$\boldsymbol{a}\times\boldsymbol{b} = \epsilon_{ijk}a_jb_k\boldsymbol{e}_i$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \epsilon_{ijk}a_ib_jc_k$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk}\epsilon_{ijk} = 6.$$

Transformations

Let matrix L relate basis $\{\boldsymbol{e}_i\}$ to basis $\{\boldsymbol{e}'_i\}$ with rule:

$$\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then $L^TL = LL^T = I$, and:

$$\ell_{ik}\ell_{jk} = \ell_{ki}\ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij}p_j \text{ for } \boldsymbol{p} = p_i\boldsymbol{e}_i = p'_i\boldsymbol{e}'_i.$$

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip}\ell_{jq}\ell_{kr}T_{pqr}$$

which relates frame S in $\{\boldsymbol{e}_i\}$ to frame S' in $\{\boldsymbol{e}'_i\}$ with rule $\boldsymbol{e}'_i = \ell_{ij}\boldsymbol{e}_j$, etc.

Properties of tensors:

1. The addition of two rank n tensors is also a rank n tensor.
2. The multiplication of a rank m tensor with a rank n tensor yields a rank $m+n$ tensor.
3. If $T_{ijk\ldots s}$ is a rank m tensor then $T_{iik\ldots s}$ is a rank $m-2$ tensor.
4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} \implies T' = LTL^T$$
$$T'_{ji} = \ell_{jp}\ell_{iq}T_{pq}.$$

Symmetric tensors

T_{ij} is a symmetric tensor when $T_{ij} = T_{ji}$ in frame S. Then $T'_{ij} = T'_{ji}$ in frame S'.

Similarly T_{ij} is an anti-symmetric tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally any tensor can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S'. Let $b_i = T_{ij}a_j$ where a_j is a vector. If b_i always transforms as a vector then T_{ij} is a rank 2 tensor.

Generalising, let $R_{ijk\ldots r}$ be a rank m tensor and $T_{ijk\ldots s}$ a set of 3^n numbers where $n > m$. If $T_{ijk\ldots s}R_{ijk\ldots r}$ is a rank $n-m$ tensor then $T_{ijk\ldots s}$ is a rank n tensor.

Matrices

We define a $m\times n$ matrix A as $(A)_{ij} = a_{ij}$ where $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

- $\operatorname{Tr} A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^TA^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn}a_{1l}a_{2m}a_{3n} \\ &= \epsilon_{lmn}a_{l1}a_{m2}a_{n3}. \end{aligned}$$

Furthermore:

$$\epsilon_{ijk}\det A = \epsilon_{lmn}a_{il}a_{jm}a_{kn}$$

$$\epsilon_{lmn}\det A = \epsilon_{ijk}a_{il}a_{jkm}a_{kn}$$

$$\det A = \frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}a_{il}a_{jm}a_{kn}.$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3. $\det A = \det A^T$
4. $\det(AB) = \det A\cdot\det B$

These also apply to columns. Finally:

$$\epsilon_{ijk}\epsilon_{lmn}\det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting $A = I$ yields:

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

Linear equations

Let $\boldsymbol{y} = A\boldsymbol{x}$. Then $x_i = A^{-1}_{ij}y_i$ with:

$$\begin{aligned} A^{-1}_{ij} &= \frac{1}{2}\frac{1}{\det A}\epsilon_{imn}\epsilon_{jpq}a_{pm}a_{qn} \\ &= \frac{1}{\det A}C^T_{ij} \end{aligned}$$

where C is the cofactor matrix of A.

Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L)\ell_{ip}\ell_{jq}T_{pq}$$

where $(L)_{ij} = \ell_{ij}$ and $\det L = \pm 1$.

Pseudovectors are rank 1 pseudotensors.

Invariant tensors

Tensor T is invariant or isotropic if:

$$T_{ijk\ldots} = \ell_{i\alpha}\ell_{j\beta}\ell_{k\gamma}\cdots T_{\alpha\beta\gamma\ldots}$$

for every orthogonal matrix L.

- If a_{ij} is a rank 2 invariant tensor then $a_{ij} = \lambda\delta_{ij}$.
- The most general rank 3 invariant pseudotensor is $a_{ijk} = \lambda\epsilon_{ijk}$. There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

Rotation tensors

The clockwise rotation of position vector $\boldsymbol{x}$ to $\boldsymbol{y}$ about unit vector $\hat{\boldsymbol{n}}$ is given by:

$$y_i = R_{ij}(\theta, \hat{\boldsymbol{n}})x_j$$

$$R_{ij}(\theta, \hat{\boldsymbol{n}}) = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk}n_k\sin\theta$$

and is the rotation tensor.

Reflections and inversions

The reflection of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ in plane with unit vector $\hat{\boldsymbol{n}}$ is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ is given by $\boldsymbol{y} = -\boldsymbol{x}$ and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

Projections

We define P to be a parallel projection operator to vector $\boldsymbol{u}$ if:

$$P\boldsymbol{u} = \boldsymbol{u} \text{ and } P\boldsymbol{v} = \boldsymbol{0}$$

where $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{0}$. Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define Q to be an orthogonal projection to vector $\boldsymbol{u}$ if:

$$Q\boldsymbol{u} = \boldsymbol{0} \text{ and } Q\boldsymbol{v} = \boldsymbol{v}.$$

Here $Q = I - P$.

Inertia tensors

Let $\boldsymbol{L}$ denote the angular momentum of a rigid body about the origin of mass m, volume V and density ρ at position $\boldsymbol{r}$ with velocity $\boldsymbol{v}$. Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2\delta_{ij} - x_i x_j)\mathrm{d}V$$

where $I_{ij}(O)$ is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2}I_{ij}\omega_i\omega_j = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{\omega}.$$

Parallel axis theorem

Consider the same rigid body now with centre of mass G and let $\overrightarrow{OG} = \boldsymbol{R}$. Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2\delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\boldsymbol{r}')\mathrm{d}V'.$$

Diagonalisation

Let $\boldsymbol{L} = I_{ij}\omega_j$ where I_{ij} is a rank 2 tensor and let $\boldsymbol{L} = \lambda\boldsymbol{\omega}$. Then:

$$(I_{ij} - \lambda\delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda\delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for $P = \det I$, $Q = \frac{1}{2}[(\operatorname{tr} I)^2 - \operatorname{tr}(I^2)]$ and $R = \operatorname{tr} I$ given tensor I.

Real symmetric tensors

Let rank 2 real symmetric tensor T be diagonalisable with real eigenvalues $\lambda^{(i)}$ and orthonormal eigenvectors $\boldsymbol{\ell}^{(i)}$ where $i = 1, 2, 3$. Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that $\det L = +1$ which transforms frame $S \to S'$.

Then since $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$:

$$\begin{aligned} T'_{ij} &= \ell_{ip}\ell_{jq}T_{pq} \\ &= \lambda^{(i)}\delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ij}. \end{aligned}$$

Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(x-a)^n$$

and is f expanded about $x = a$.

Trignometric expansions are in radians!

$$\begin{aligned} \therefore f(x+a) &= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x)a^n \\ &= \exp\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x) \end{aligned}$$

Then for three dimensions:

$$\begin{aligned} \phi(\boldsymbol{r}+\boldsymbol{a}) &= \sum_{n=0}^{\infty}\frac{1}{n!}(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)^n\phi(\boldsymbol{r}) \\ &= \exp(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)\phi(\boldsymbol{r}). \end{aligned}$$

Curvilinear coordinates

Let x_i denote Cartesian coordinates and u_i denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \to (u_1, u_2, u_3)$$

where each $u_i = u_i(x_1, x_2, x_3)$ and:

$$\begin{aligned} \boldsymbol{r} &= x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \\ &= u_1\boldsymbol{e}_{u_1} + u_2\boldsymbol{e}_{u_2} + u_3\boldsymbol{e}_{u_3}. \end{aligned}$$

Scale factors

Let $u_1 \to u_1 + \mathrm{d}u_1$ in $\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, u_3)$. Then $\mathrm{d}\boldsymbol{r}$ in $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 := h_1\boldsymbol{e}_1\mathrm{d}u_1.$$

h_1 is the scale factor of unit vector $\boldsymbol{e}_1$:

$$h_1 = \left|\frac{\partial\boldsymbol{r}}{\partial u_1}\right| \text{ and } \boldsymbol{e}_1 = \frac{1}{h_1}\frac{\partial\boldsymbol{r}}{\partial u_1}.$$

If $\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$ then u_i is an **orthogonal** curvilinear coordinate system.

Vector and arc length

The vector length $\mathrm{d}\boldsymbol{r}$ of $\boldsymbol{r}$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3}h_i\mathrm{d}u_i\boldsymbol{e}_i$$

where $u_i \to u_i + \mathrm{d}u_i$ for $\forall i = 1, 2, 3$.

Then the arc length $\mathrm{d}s$ is defined as:

$$\begin{aligned} (\mathrm{d}s)^2 &= \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} \\ &= g_{ij}\mathrm{d}u_i\mathrm{d}u_j \end{aligned}$$

where g_{ij} is the metric tensor:

$$\begin{aligned} g_{ij} = g_{ji} &= \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \\ &= h_i h_j(\boldsymbol{e}_i\cdot\boldsymbol{e}_j). \end{aligned}$$

Area and volume

Let $\mathrm{d}\boldsymbol{r}_i = h_i\boldsymbol{e}_i\mathrm{d}u_i$ denote vector length when $u_i \to u_i + \mathrm{d}u_i$. (**No** sum!)

The infinitesimal vector area formed by $\mathrm{d}\boldsymbol{r}_1$ and $\mathrm{d}\boldsymbol{r}_2$ is:

$$\mathrm{d}\boldsymbol{S} = (h_1\mathrm{d}\boldsymbol{u}_1\boldsymbol{e}_1)\times(h_2\mathrm{d}\boldsymbol{u}_2\boldsymbol{e}_2).$$

Similarly the infinitesimal volume formed by edges $\mathrm{d}\boldsymbol{r}_1$, $\mathrm{d}\boldsymbol{r}_2$ and $\mathrm{d}\boldsymbol{r}_3$ is:

$$\begin{aligned} \mathrm{d}V &= |(\mathrm{d}\boldsymbol{r}_1\times\mathrm{d}\boldsymbol{r}_2)\cdot\mathrm{d}\boldsymbol{r}_3| \\ &= \sqrt{g}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 \end{aligned}$$

where $g = \det(g_{ij})$.

Cylindrical coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$ where ρ represents the radial distance from the origin and ϕ is the anticlockwise rotation angle on the x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_1 + \rho\sin\phi\boldsymbol{e}_2 + z\boldsymbol{e}_3$$

$$h_\rho = 1, \quad \boldsymbol{e}_\rho = \cos\phi\boldsymbol{e}_1 + \sin\phi\boldsymbol{e}_2$$

$$h_\phi = \rho, \quad \boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

$$h_z = 1, \quad \boldsymbol{e}_z = \boldsymbol{e}_3$$

and forms an orthogonal set.

Spherical coordinates

$(u_1, u_2, u_3) = (r, \theta, \phi)$ where θ represents the clockwise rotation angle in y-z plane and ϕ the anticlockwise rotation angle in x-y plane. In Cartesian unit vectors:

$$\boldsymbol{r} = r\sin\theta\cos\phi\boldsymbol{e}_1 + r\sin\theta\sin\phi\boldsymbol{e}_2 + r\cos\theta\boldsymbol{e}_3$$

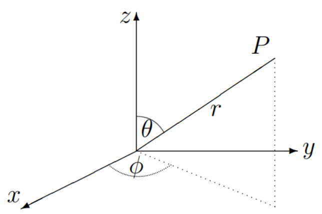
$$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$$

$$\boldsymbol{e}_r = \sin\theta\cos\phi\boldsymbol{e}_1 + \sin\theta\sin\phi\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\boldsymbol{e}_1 + \cos\theta\sin\phi\boldsymbol{e}_2 - \sin\theta\boldsymbol{e}_3$$

$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

and also forms an orthogonal set.



Gradient

The gradient of a scalar field $f(\boldsymbol{r})$ is:

$$\mathrm{d}f(\boldsymbol{r}) := \boldsymbol{\nabla} f(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r}$$

when $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r} \implies f \to f + \mathrm{d}f$. Taking the total differential of f yields:

$$\boldsymbol{\nabla} f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i$$

where $\{\boldsymbol{e}_i\}$ is orthogonal.

Divergence

The divergence of a vector field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} := \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta S}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$$

for surface δS bounds infinitesimal δV. In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(h_1F_2h_3) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right\}.$$

Curl

The curl of a vector field $\boldsymbol{F}$ in the direction of unit vector $\hat{\boldsymbol{n}}$ is:

$$\hat{\boldsymbol{n}}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) := \lim_{\delta S\to 0}\frac{1}{\delta S}\oint_{\delta C}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$$

where curve δC encloses plane δS. In orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix}.$$

Laplacian

The Laplacian of a scalar field f is:

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f)$$

and in orthogonal curvilinear coordinates:

$$\boldsymbol{\nabla}^2 f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right\}.$$

The Laplacian of a vector field $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla}^2\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}).$$

Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x_i}\boldsymbol{e}_i$$

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{F} = u_j\frac{\partial}{\partial x_j}F_i$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_i}{\partial x_i}$$

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}\boldsymbol{e}_i$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

If ψ is a scalar field and $\boldsymbol{v}$ a vector field:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) = \boldsymbol{0}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) = \boldsymbol{0}$$

$$\boldsymbol{\nabla}\cdot(\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi\cdot\boldsymbol{v} + \psi\boldsymbol{\nabla}\cdot\boldsymbol{v}$$

$$\boldsymbol{\nabla}\times(\psi\boldsymbol{v}) = \boldsymbol{\nabla}\psi\times\boldsymbol{v} + \psi\boldsymbol{\nabla}\times\boldsymbol{v}.$$

Let $\boldsymbol{r} = x_i\boldsymbol{e}_i$ and $r = (x_i^2)^{1/2}$. Then:

- $\boldsymbol{\nabla} r = \dfrac{\boldsymbol{r}}{r}$ and $\boldsymbol{\nabla}\left(\dfrac{1}{r}\right) = -\dfrac{\boldsymbol{r}}{r^3}$
- $\boldsymbol{\nabla} r^n = nr^{n-2}\boldsymbol{r}$
- $\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3$ and $\boldsymbol{\nabla}\times\boldsymbol{r} = \boldsymbol{0}$
- $\boldsymbol{\nabla}\times(\boldsymbol{c}\times\boldsymbol{r}) = 2\boldsymbol{c}$
- $\boldsymbol{\nabla}\cdot(\boldsymbol{c}\times\boldsymbol{r}) = \boldsymbol{0}$ for constant $\boldsymbol{c}$.

Divergence theorem

Let surface S enclose volume V. Then:

$$\iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{F}\mathrm{d}V = \oint_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}$$

where $\boldsymbol{F}$ is a vector field.

Stokes' theorem

Let closed curve C bound open surface S and let $\boldsymbol{F}$ be a vector field. Then:

$$\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \iint_S(\boldsymbol{\nabla}\times\boldsymbol{F})\cdot\mathrm{d}\boldsymbol{S}$$

for C is traversed in anticlockwise sense.

Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x-a) := \begin{cases} \infty & x = a \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions this becomes:

$$\begin{aligned}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}_0) &:= \delta(\boldsymbol{r}-\boldsymbol{r}_0) \\ &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\end{aligned}$$

where (x, y, z) are Cartesian coordinates.

In orthogonal curvilinear coordinates:

$$\begin{aligned}\delta(\boldsymbol{r}-\boldsymbol{a}) =& \frac{1}{h_1h_2h_3}\delta(u_1-a_1) \\ &\cdot\delta(u_2-a_2)\delta(u_3-a_3).\end{aligned}$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

which yields:

$$x\delta(x) = 0 \text{ and } \delta(cx) = \frac{1}{|c|}\delta(x).$$

If simple solutions of $g(x) = 0$ are x_i:

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))\mathrm{d}x = \sum_i \frac{f(x_i)}{|g'(x_i)|}.$$

Coulomb's law

Consider charges q and q_1 at positions $\boldsymbol{r}$ and $\boldsymbol{r}_1$. The force on charge q at $\boldsymbol{r}$ due to charge q_1 at $\boldsymbol{r}_1$ is:

$$\boldsymbol{F}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{qq_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}$$

where $qq_1 > 0$ denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419\times 10^{-12}\mathrm{C^2N^{-1}m^{-2}}.$$

Charge has units Coulombs (C) and an electron has charge -1.60218×10^{-19}C.

Electric fields

The electric field is generated by a charge configuration and defined in terms of the force on a small positive test charge q:

$$\boldsymbol{E}(\boldsymbol{r}) := \lim_{q\to 0}\frac{1}{q}\boldsymbol{F}.$$

Then for our two charges q and q_1:

$$\boldsymbol{F}_1(\boldsymbol{r}) = q\boldsymbol{E}_1(\boldsymbol{r})$$

where q_1 produces electric field $\boldsymbol{E}_1$.

$$\therefore \boldsymbol{E}_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}$$

Principle of superposition

For a set of charges q_i at position $\boldsymbol{r}_i$ the total electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i(\boldsymbol{r}-\boldsymbol{r}_i)}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}.$$

For object with **charge density** $\rho(\boldsymbol{r}')$ its overall electric field at $\boldsymbol{r}$ is:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'$$

where $\rho(\boldsymbol{r}')$ is charge divided by volume.

Electrostatic Maxwell's equations

Because $\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right)$$

and therefore for all <u>static</u> electric fields:

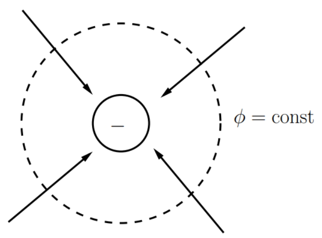
$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}.$$

$\boldsymbol{E}$ is a **conservative** vector field where its line integral is independent of path. Furthermore it may be written as:

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$$

for $\phi(\boldsymbol{r})$ is the potential of $\boldsymbol{E}$.

$$\therefore \phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'$$



The **potential difference** between two points A and B is the energy per unit charge needed to move a small charge q from A to B:

$$\begin{aligned}V_{A\to B} &= \lim_{q\to 0}\frac{1}{q}W_{A\to B}\\ &= -\frac{1}{q}\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}\\ &= \phi_B - \phi_A.\end{aligned}$$

A charge distribution $\rho(\boldsymbol{r}')$ in an <u>external</u> electric field has potential energy:

$$W = \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}')\mathrm{d}V'.$$

Because $\boldsymbol{\nabla}^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')$:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

Electric dipoles

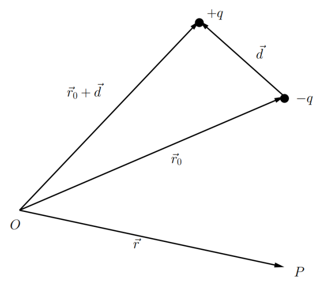
An electric dipole at $\boldsymbol{r}_0$ is defined as two charges $-q$ at $\boldsymbol{r}_0$ and $+q$ at $\boldsymbol{r}_0+\boldsymbol{d}$ which generates **dipole moment**:

$$\boldsymbol{p} = q\boldsymbol{d}$$

and in the <u>dipole limit</u> this is defined as:

$$\boldsymbol{p} := \lim_{\substack{q\to\infty\\ \boldsymbol{d}\to\boldsymbol{0}}} q\boldsymbol{d}$$

known as an ideal dipole.



The electrostatic potential generated by this ideal dipole at $\boldsymbol{r}_0$ is given by:

$$\begin{aligned}\phi(\boldsymbol{r}) &= \phi_q + \phi_{-q}\\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{d}|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}\right)\\ &\approx \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}\end{aligned}$$

for the first term is expanded in powers of $-\boldsymbol{d}$ about $\boldsymbol{r}-\boldsymbol{r}_0$.

The electric field generated is:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) =& \frac{1}{4\pi\epsilon_0}\left[-\frac{\boldsymbol{p}}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}\right.\\ &\left.+\frac{3\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^5}(\boldsymbol{r}-\boldsymbol{r}_0)\right].\end{aligned}$$

If the ideal dipole is at the origin:

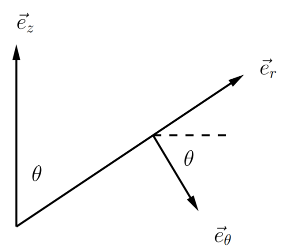
$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{3\boldsymbol{p}\cdot\boldsymbol{r}}{r^5}\boldsymbol{r} - \frac{\boldsymbol{p}}{r^3}\right).$$

Let ideal dipole moment $\boldsymbol{p}$ be parallel to the z-axis. Then in spherical coordinates (r,θ,χ), $\boldsymbol{r} = r\boldsymbol{e}_r$, $\boldsymbol{p} = p\boldsymbol{e}_z$ and:

$$\phi(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0}\left(\frac{2\cos\theta}{r^3}\boldsymbol{e}_r + \frac{\sin\theta}{r^3}\boldsymbol{e}_\theta\right).$$



Force, torque and energy

The **force** <u>on</u> a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned}\boldsymbol{F} &= -q\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{E}_{ext}(\boldsymbol{r}+\boldsymbol{d})\\ &\approx (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E}_{ext}(\boldsymbol{r}).\end{aligned}$$

The **torque** <u>on</u> a dipole at $\boldsymbol{r}$ about the axis $\boldsymbol{r}$ due to $\boldsymbol{E}_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned}\boldsymbol{G} &= \boldsymbol{\tau}_{-q} + \boldsymbol{\tau}_q\\ &= -q\boldsymbol{0}\times\boldsymbol{E}_{ext}(\boldsymbol{r}) + q\boldsymbol{d}\times\boldsymbol{E}_{ext}(\boldsymbol{r}+\boldsymbol{d})\\ &\approx \boldsymbol{p}\times\boldsymbol{E}_{ext}(\boldsymbol{r}).\end{aligned}$$

The **energy** of a dipole at $\boldsymbol{r}$ from external electric field $\boldsymbol{E}_{ext}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi_{ext}(\boldsymbol{r})$ is:

$$\begin{aligned}W &= -q\phi_{ext}(\boldsymbol{r}) + q\phi_{ext}(\boldsymbol{r}+\boldsymbol{d})\\ &\approx -\boldsymbol{p}\cdot\boldsymbol{E}_{ext}(\boldsymbol{r})\end{aligned}$$

and $\boldsymbol{F} = -\boldsymbol{\nabla}W$.

Multipole expansion

Consider object with volume V and charge distribution $\rho(\boldsymbol{r}')$. Let origin be in the object. Then the potential at $\boldsymbol{r}$ is:

$$\begin{aligned}\phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\\ &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{Q_{ij}x_ix_j}{2r^5}\right)\end{aligned}$$

where Q is the **total charge** in V:

$$Q = \int_V \rho(\boldsymbol{r}')\mathrm{d}V'$$

$\boldsymbol{p}$ the **dipole moment** about the origin:

$$\boldsymbol{p} = \int_V \boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V'$$

and Q_{ij} the **quadrupole tensor**:

$$Q_{ij} = \int_V \rho(\boldsymbol{r}')\left[3x_i'x_j' - (r')^2\delta_{ij}\right]\mathrm{d}V'.$$

If $Q \neq 0$ then in the far zone ($r \gg r_0$) the first term (monopole term) dominates.

If $Q = 0$ and $\boldsymbol{p} = \boldsymbol{0}$ then the third term (quadruple term) dominates in the far zone and etc.

Interaction energy

By expanding $\phi_{ext}(\boldsymbol{r})$ about $\boldsymbol{r} = \boldsymbol{0}$:

$$\begin{aligned}W &= \int_V \rho(\boldsymbol{r}')\phi_{ext}(\boldsymbol{r}')\mathrm{d}V'\\ &= Q\phi_{ext}(\boldsymbol{0}) - \boldsymbol{p}\cdot\boldsymbol{E}_{ext}(\boldsymbol{0})\\ &\quad - \frac{1}{6}Q_{ij}\frac{\partial\left(\boldsymbol{E}_{ext}(\boldsymbol{0})\right)_i}{\partial x_j} + \ldots\end{aligned}$$

and is the potential energy of a charge distribution $\rho(\boldsymbol{r})$ in $\boldsymbol{E}_{ext}$.

Gauss' law

For object with charge distribution $\rho(\boldsymbol{r}')$ and volume V enclosed by surface S:

$$\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is total charge enclosed by V:

$$Q_{enc} = \int_V \rho(\boldsymbol{r}')\mathrm{d}V'$$

useful for symmetric problems.

Boundaries and conductors

Let σ be the charge density of a surface separating electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$.

1. Normal component of electric field is discontinuous across surface by:

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{E}_2 - \boldsymbol{E}_1) = \frac{\sigma}{\epsilon_0}.$$

2. Tangential component of electric field is continuous across surface:

$$\boldsymbol{E}_\parallel := \hat{\boldsymbol{n}} \times \boldsymbol{E}_1 = \hat{\boldsymbol{n}} \times \boldsymbol{E}_2.$$

Conductors (Electrostatics)

Conductors have surplus electrons that can move freely when an electric field is applied. In electrostatics:

1. For conductors in equilibrium, all charges are at rest and reside on the surface of the conductor.

   Hence inside a conductor $\rho(\boldsymbol{r}) = 0$, $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}$ and $\phi =$ constant.

2. An electric field is always normal to the surface of a conductor:

$$E_\perp = \frac{\sigma}{\epsilon_0} \text{ and } E_\parallel = 0.$$

The presence of an external electric field induces a charge distribution σ on the surface of our conductor. This changes the external electric field as it need to be normal to the surface of the conductor.

Poisson's equation

Because $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho(\boldsymbol{r})/\epsilon_0$:

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

We can solve this by direct integration or using the **method of images**.

Given volume under consideration place fictitious charge outside the volume such that the system still satisfies Poisson's equation with boundary conditions.

This potential is our solution.

Electrostatic energy

Because the work done to move point charge q from $\boldsymbol{r}_A$ to $\boldsymbol{r}_B$ in $\boldsymbol{E}(\boldsymbol{r})$ is:

$$W_{A\to B} = qV_{A\to B}$$

then $W_{\infty\to B} = q\phi(\boldsymbol{r}_B)$.

Furthermore the energy needed to move a continuous charge distribution $\rho(\boldsymbol{r}')$ from infinity to position $\boldsymbol{r}$ is:

$$\begin{aligned} W_e &= \frac{1}{2}\int_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V \\ &= \frac{\epsilon_0}{2}\int_V |\boldsymbol{E}(\boldsymbol{r})|^2\mathrm{d}V. \end{aligned}$$

Capacitors

Currents

Lozentz force

Biot-Savart law

Magnetostatic Maxwell's equations

Ampère's law

normal and tangent components of conducting surfaces

Magnetic dipoles