

D: Functions

A function $f : X \rightarrow Y$ is an assignment of an element of Y to each element of X .

1. f is **injective** if:

$$\forall x_1, x_2 \in X; f(x_1) = f(x_2) \\ \implies x_1 = x_2.$$

2. f is **surjective** if:

$$\forall y \in Y; \exists x \in X : y = f(x).$$

3. f is **bijective** if it is injective and surjective.

D: Groups

A group G is a set defined with:

1. Composition operator (\cdot) such that $x \cdot y = xy$.
2. $\forall x, y, z \in G; (xy)z = x(yz)$
3. $\exists e \in G : ex = xe = x$ for $\forall x \in G$.
4. $\exists x^{-1} \in G : xx^{-1} = x^{-1}x = e$ for $\forall x \in G$.

G is **Abelian** if $\forall x, y \in G; xy = yx$.

D1.2.1(i): Fields

A field F is a set defined with:

1. Addition function $(+)$:

$$(+) : F \times F \rightarrow F; (\lambda, \mu) \mapsto \lambda + \mu$$

2. Multiplication function (\cdot) :

$$(\cdot) : F \times F \rightarrow F; (\lambda, \mu) \mapsto \lambda \cdot \mu$$

3. $\exists 0_F, 1_F \in F$ where $0_F \neq 1_F$ such that $(F, +)$ and $(F \setminus \{0_F\}, \cdot)$ form Abelian groups.
4. $\exists (-\lambda) \in F : \lambda + (-\lambda) = 0_F$
5. $\exists (\lambda^{-1}) \in F : \lambda \cdot (\lambda^{-1}) = 1_F$
6. $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F$

D1.2.1(ii): Vector spaces

A vector space V over a field F is an Abelian group $V := (V, +)$ with mapping:

$$F \times V \rightarrow V : (\lambda, \mathbf{v}) \mapsto \lambda \mathbf{v}$$

where for $\forall \lambda, \mu \in F$ and $\forall \mathbf{v}, \mathbf{w} \in V$:

1. $\lambda(\mathbf{v} + \mathbf{w}) = (\lambda\mathbf{v}) + (\lambda\mathbf{w})$
2. $(\lambda + \mu)\mathbf{v} = (\lambda\mathbf{v}) + (\mu\mathbf{v})$
3. $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$
4. $1_F\mathbf{v} = \mathbf{v}$

and is a F -vector space.

Remark

Let V be a F -vector space where $\mathbf{v} \in V$.

1. $0\mathbf{v} = \mathbf{0}$
2. $(-1)\mathbf{v} = -\mathbf{v}$
3. $\lambda\mathbf{0} = \mathbf{0}$ for $\forall \lambda \in F$.

D: Cartesian products

The Cartesian product of sets X_1, \dots, X_n is defined as:

$$X_1 \times \dots \times X_n := \{(x_1, \dots, x_n) : x_i \in X_i\}$$

where $1 \leq i \leq n$.

The projection of a Cartesian product is:

$$\text{pr}_i : X_1 \times \dots \times X_n \rightarrow X_i; \\ (x_1, \dots, x_n) \mapsto x_i$$

D1.4.1: Vector subspaces

A vector subspace U of F -vector space V has the following properties:

1. $U \subset V$ and $\mathbf{0} \in U$.
2. Let $\mathbf{u}, \mathbf{v} \in U$ and $\lambda \in F$. Then $\mathbf{u} + \mathbf{v} \in U$ and $\lambda\mathbf{u} \in U$.

and is also a vector space.

P1.4.5

Let $T \subset V$ where V is a F -vector space. Then for all vector subspaces containing T , there exists a smallest vector subspace:

$$\text{span}(T) = \langle T \rangle_F \subset V$$

known as the vector subspace generated by T , or the span of T .

D1.4.7: Generating set

Let $T \subset V$ where V is a F -vector space. T is a generating set of V if:

$$\text{span}(T) = V$$

and is the linear combination of vectors in T over field F .

D1.4.9: Power sets

The power set of set X is:

$$\mathcal{P}(X) := \{U : U \subseteq X\}.$$

Let $\mathcal{U} \subseteq \mathcal{P}(X)$. Then:

$$\bigcup_{U \in \mathcal{U}} U := \{x \in X : (\exists U \in \mathcal{U} : x \in U)\}$$

$$\bigcap_{U \in \mathcal{U}} U := \{x \in X : \forall U \in \mathcal{U}; x \in U\}.$$

D1.5.1: Linear independence

Let V be a F -vector space and $L \subseteq V$. L is linearly independent if:

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_r \mathbf{v}_r = \mathbf{0} \\ \implies \alpha_1 = \dots = \alpha_r = 0$$

where $\mathbf{v}_i \in L$.

D1.5.8: Basis

A basis of a vector space V is a linearly independent generating set in V .

T1.5.11

Let V be a F -vector space.

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a basis of V **iff**:

$$\Phi : F^r \rightarrow V;$$

$$(\alpha_1, \dots, \alpha_r) \mapsto \alpha_1 \mathbf{v}_1 + \dots + \alpha_r \mathbf{v}_r$$

is a bijection.

T1.5.12

Let V be a vector space and $E \subseteq V$. Then the following statements are equivalent:

1. E is a basis of V .
2. E is minimal among all generating sets, or that $E \setminus \{\mathbf{v}\}$ is not a basis for $\forall \mathbf{v} \in V$.
3. E is maximal amongst all linearly independent subsets. i.e. $E \cup \{\mathbf{v}\}$ is not linearly independent.

C1.5.13

Every finitely generated vector space has a finite basis. (any vector space too!)

T1.5.14

Let V be a vector space.

1. Let $L \subseteq V$ be linearly independent and set E be minimal amongst all generating sets of V . Let $L \subseteq E$. Then E is a basis of V .
2. Let $E \subseteq V$ be a generating set and L be maximal amongst all linearly independent subsets of V .

Let $L \subseteq E$. Then E is a basis of V .

D1.5.15

Let X be a set and F be a field. Then:

$$\text{maps}(X, F) := \{f : (\forall f : X \rightarrow F)\}$$

and is a F -vector space under pointwise addition and multiplication via scalars.

Remark

The subset of all mappings which sends almost all elements of X to 0 is defined: (all but finitely many)

$$F\langle X \rangle \subseteq \text{maps}(X, F)$$

and is a vector subspace.

T1.5.16

Let V be a F -vector space.

Then $(\mathbf{v}_i)_{i \in I}$ is a basis for V iff:

$$\forall \mathbf{v} \in V; \exists! (a_i)_{i \in I} \subseteq F : \mathbf{v} = \sum_{i \in I} a_i \mathbf{v}_i.$$

T1.6.1

Let V be a vector space. Let $L \subset V$ be a linearly independent subset and $E \subseteq V$ a generating set. Then $|L| \leq |E|$.

T1.6.2: Steinitz exchange theorem

Let V be a vector space, $L \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set.

Then there exists an **injective** function $\phi : L \rightarrow E$ such that:

$$(E \setminus \phi(L)) \cup L$$

is also a generating set for V .

L1.6.3: Exchange lemma

Let V be a vector space. Let $M \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set where $M \subseteq E$.

If $\exists \mathbf{w} \in V \setminus M$ such that set $M \cup \{\mathbf{w}\}$ is linearly independent then:

$\exists \mathbf{e} \in E \setminus M : (E \setminus \mathbf{e}) \cup \{\mathbf{w}\}$ is generating.

C1.6.4

Let V be a finitely generated vector space.

1. V has finite basis.
2. V cannot have infinite basis.
3. Any two basis of V have the same number of elements.

D1.6.5: Dimension

The dimension of finite F -vector space V is the cardinality of one its basis.

For infinite vector spaces: $\dim(V) = \infty$.

C1.6.7

Let V be a finitely generated vector space.

1. Every linearly independent $L \subseteq V$ has **at most** $\dim(V)$ elements and if $|L| = \dim(V)$ then L is a basis.
2. Every generating set $E \subseteq V$ has **at least** $\dim(V)$ elements and if $|E| = \dim(V)$ then E is a basis.

C1.6.8

A proper vector subspace of a vector space with finite dimension has itself a strictly smaller dimension.

T1.6.10

Let V be a vector space and $U, W \subseteq V$ be vector subspaces. Then:

$$\begin{aligned} \dim(U + W) + \dim(U \cap W) \\ = \dim(U) + \dim(W). \end{aligned}$$

D1.7.1: Linear mappings

Let V and W be F -vector spaces. A mapping $f : V \rightarrow W$ is F -linear or a **homomorphism** of vector spaces if for $\forall \mathbf{v}_1, \mathbf{v}_2 \in V$ and $\forall \lambda \in F$:

1. $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$
2. $f(\lambda \mathbf{v}_1) = \lambda f(\mathbf{v}_1)$.

Furthermore bijective linear mappings are an **isomorphism** of vector spaces.

A homomorphism from a vector space to itself is an **endomorphism**.

An isomorphism of a vector space to itself is an **automorphism**.

D1.7.5: Fixed points

In a linear mapping a fixed point is sent to itself. For mapping $f : X \rightarrow X$ the **set of fixed points** is:

$$X^f = \{x \in X : f(x) = x\}.$$

D1.7.6: Complementary subspaces?

Vector subspaces V_1, V_2 of vector space V are **complementary** if the mapping:

$$V_1 \times V_2 \rightarrow V$$

is a bijection.

T1.7.7

A F -vector space V is isomorphic to F^n iff $\dim(V) = n$, for $n \in \mathbb{N}$ and F a field.

L1.7.8

Let V, W be F -vector spaces and let B be a basis of V . Then the following mapping:

$$\text{hom}_F(V, W) \rightarrow \text{maps}(B, W); f \mapsto f_B$$

is a bijection.

Remark

Let V, W be F -vector spaces. The set of all homomorphisms from V to W is:

$$\text{hom}_F(V, W) \subseteq \text{maps}(B, W).$$

P1.7.9

Let $f : V \rightarrow W$ be a linear mapping, where V, W are vector spaces.

1. If f is injective, there exists map $g : W \rightarrow V$ such that $g \circ f = \text{id}_V$. i.e. it has a **left inverse**.
2. If f is surjective, there exists map $g : W \rightarrow V$ such that $f \circ g = \text{id}_W$. i.e. it has a **right inverse**.

D1.8.1: Image and kernel

Let $f : V \rightarrow W$ be a linear mapping. The **image** of this linear mapping f is:

$$\text{im}(f) := f(V) \subseteq W$$

and is a vector subspace of W .

The **kernel** of this linear mapping f is:

$$\ker(f) := f^{-1}(\mathbf{0}) = \{\mathbf{v} \in V : f(\mathbf{v}) = \mathbf{0}\}$$

and is the preimage of the zero vector in linear mapping f .

L1.8.2

A linear mapping $f : V \rightarrow W$ is injective iff $\ker(f) = \{\mathbf{0}\}$.

T1.8.4: Rank-nullity theorem

Let $f : V \rightarrow W$ be a linear mapping and V, W are vector spaces. Then:

$$\dim(V) = \dim(\ker(f)) + \dim(\text{im}(f)).$$

T2.1.1

Let F be a field and $m, n \in \mathbb{N}$.

Then there exists a bijection:

$$M : \text{hom}_F(F^m, F^n) \rightarrow \text{mat}(n \times m; F);$$

$$f \mapsto [f]$$

and attaches each linear mapping f with its **representing matrix** $M(f) := [f]$.

Remark

The set of matrices with n rows and m columns with entries in field F is:

$$\text{mat}(n \times m; F).$$

D2.1.6: Matrix products

The product $A \circ B = AB$ is defined:

$$(AB)_{ik} = \sum_{j=1}^m A_{ij}B_{jk}$$

where $A \in \text{mat}(n \times m; F)$, F a field, $B \in \text{mat}(m \times \ell; F)$ and $m, n, \ell \in \mathbb{N}$. This is matrix multiplication, with mapping:

$$\begin{aligned} &\text{mat}(n \times m; F) \times \text{mat}(m \times \ell; F) \\ &\rightarrow \text{mat}(n \times \ell; F); \\ &(A, B) \mapsto AB. \end{aligned}$$

T2.1.8

Let $g : F^\ell \rightarrow F^m$ and $f : F^m \rightarrow F^n$ be linear mappings. Then $[f \circ g] = [f] \circ [g]$.

P2.1.9

Let $A, A' \in \text{mat}(n \times m; F)$.

Let $B, B' \in \text{mat}(m \times \ell; F)$.

Let $C, C' \in \text{mat}(\ell \times k; F)$.

Let $k, \ell, m, n \in \mathbb{N}$ and denote $I = I_m$ as the $(m \times m)$ identity matrix. Then:

1. $(A + A')B = AB + A'B$
2. $A(B + B') = AB + AB'$
3. $IB = B$
4. $AI = A$
5. $(AB)C = A(BC)$.

D2.2.1: Invertible matrices

A matrix A is **invertible** if:

$$\exists B, C : BA = I \text{ and } AC = I.$$

D2.2.2: Elementary matrices

Elementary matrices are square matrices that differs from the identity matrix by at most one entry.

T2.2.3

Every square matrix with entries in a field can be written as a product of elementary matrices.

D2.2.4: Smith normal form

Matrices with non-zero entries along the diagonal are in Smith normal form. e.g:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

T2.2.5

For every $A \in \text{mat}(n \times m; F)$, there exists invertible matrices P and Q such that PAQ is of Smith normal form.

D2.2.7: Column and row rank

Let matrix $A \in \text{mat}(n \times m; F)$.

The column rank of A is the dimension of the subspace of F^n generated by the columns of A .

Similarly the row rank of A is the dimension of the subspace of F^m generated by the rows of A .

T2.2.8

Column and row ranks are equal.

D2.2.9: Full rank matrices

Let matrix $A \in \text{mat}(n \times m; F)$.

A is full rank if $\text{rank}(A) = \min(m, n)$.