

T: Triangle inequalities

Let $\alpha, \beta \in \mathbb{R}$. We then have that:

1. $|\alpha| + |\beta| \geq |\alpha + \beta|$
2. $||\alpha| - |\beta|| \leq |\alpha - \beta|$.

D: Supremum and infimum

Let $\alpha = \sup S$. Then:

1. $\forall s \in S; \alpha \geq s$
2. $\forall a \in \mathbb{R} : \forall s \in S; a \geq s;$
 $\quad \quad \quad \color{red}{a \geq \alpha}$

and similarly for infimum.

T: Approximation property

Consider bounded $E \subset \mathbb{R}$. Then:

$$\forall \epsilon > 0; \exists a \in E : \sup E - \epsilon < a \leq \sup E.$$

D: Completeness of \mathbb{R}

Every nonempty bounded subset of \mathbb{R} has an infimum and supremum.

T: Archimedean property

$$\forall a, b \in \mathbb{R}; a > 0; \exists n \in \mathbb{N} : na > b$$

D1.1: Nested intervals

A sequence of sets $(I_n)_{n \in \mathbb{N}}$ is nested if $I_1 \supset I_2 \supset I_3 \dots$.

T1.1: Nested interval property

Let $(I_n)_{n \in \mathbb{N}}$ be a sequence of nonempty, closed and bounded nested intervals. Then:

$$E = \bigcap_{n \in \mathbb{N}} I_n \neq \emptyset.$$

If $\lambda(I_n) \rightarrow 0$ then E contains one number, where λ denotes length.

T1.2

Let $E = [a, b]$ and that there exists an open collection of nested intervals $(I_\alpha)_{\alpha \in A}$ such that:

$$E \subset \bigcup_{\alpha \in A} I_\alpha.$$

Then $\exists \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset A$ such that:

$$E \subset I_{\alpha_1} \cup I_{\alpha_2} \cup \dots \cup I_{\alpha_n}.$$

D1.2: ϵ - N convergence

Let $\lim_{n \rightarrow \infty} x_n = a$. Then:

$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |x_n - a| < \epsilon$. Otherwise S is conditionally convergent.

D1.3: Cauchy sequences

The sequence (x_n) is Cauchy if:

$$\begin{aligned} \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \\ \implies |x_n - x_m| < \epsilon. \end{aligned}$$

T1.3 and T1.4

Cauchy $\iff \epsilon$ - N convergent.

T: Monotone convergence

Let $(x_n)_{n \in \mathbb{N}}$ be increasing and bounded above. Then:

$$\lim_{n \rightarrow \infty} x_n = \sup_{n \in \mathbb{N}} \{x_n\}$$

and similarly for sequences that are decreasing and bounded below.

D1.4: Subsequences

The subsequence of $(x_n)_{n \in \mathbb{N}}$ is a sequence of form $(x_{n_k})_{k \in \mathbb{N}}$ and is a selection of the original sequence **taken in order**.

T1.5: Bolzano-Weierstrass

Every bounded real sequence has a convergent subsequence.

D1.5: Limit inferior and superior

Let (x_n) be a bounded real sequence. Then:

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sup_{k \geq n} x_k \right)$$

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} x_k \right).$$

T1.6

The real sequence (x_n) is convergent if and only if:

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n.$$

D1.6: Convergence of infinite series

Series $S = \sum_{k=1}^{\infty} a_k$ is convergent if:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_k < \infty.$$

Series S is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ is also convergent.

T1.7: Cauchy criterion for series

$S = \sum_{k=1}^{\infty} a_k$ is convergent **iff**:

$$\begin{aligned} \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall m \geq n \geq N \\ \implies \left| \sum_{k=n+1}^m a_k \right| < \epsilon. \end{aligned}$$

T1.8

Let $S = \sum_{k=1}^{\infty} a_k$ be absolutely convergent. Let $z : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Then:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}.$$

T1.9: Riemann rearrangement

Let $S = \sum_{k=1}^{\infty} a_k$ be conditionally convergent. Then there exists rearrangements such that S can take on any value.

D1.7: Sequential continuity

Let $f : \text{dom}(f) \rightarrow \mathbb{R}$ where $\text{dom}(f) \subset \mathbb{R}$. f is continuous at $\alpha \in \text{dom}(f)$ if:

$$\begin{aligned} \forall (x_n)_{n \in \mathbb{N}} \subset \text{dom}(f) : \lim_{n \rightarrow \infty} x_n = \alpha \\ \implies \lim_{n \rightarrow \infty} f(x_n) = f(\alpha). \end{aligned}$$

T1.10

Let $\alpha \in \mathbb{R}$ and f, g continuous on D . Then $\alpha f, f + g, fg$ are continuous on D .

T1.11

Let f be continuous at $\alpha \in \mathbb{R}$ and g at $f(\alpha)$. Then $g \circ f$ is continuous at α .

D1.12: ϵ - δ continuity

Let $f : \text{dom}(f) \rightarrow \mathbb{R}$ where $\text{dom}(f) \subset \mathbb{R}$. Then f is continuous at $\alpha \in \text{dom}(f)$ if:

$$\begin{aligned} \forall \epsilon > 0; \exists \delta > 0 : |x - \alpha| < \delta \\ \implies |f(x) - f(\alpha)| < \epsilon. \end{aligned}$$

T1.13: Intermediate value theorem

Let f be continuous on $[a, b]$. If $f(a)f(b) < 0$ then:

$$\exists c \in (a, b) : f(c) = 0.$$

T1.14: Extreme value theorem

Let f be continuous on $[a, b]$. Then $\exists c, d \in [a, b]$ such that:

$$f(c) = \inf\{f(x) : x \in [a, b]\}$$

$$f(d) = \sup\{f(x) : x \in [a, b]\}.$$

T: Mean value theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b) . Then:

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

D: Differentiability

f is differentiable at α if:

$$f'(\alpha) = \lim_{h \rightarrow 0} \frac{f(\alpha + h) - f(\alpha)}{h}.$$

T: Continuity test

f is continuous at α if:

$$\lim_{x \rightarrow \alpha} f(x) = f(\alpha)$$

where the limit from left/right must both exist and be equal to each other.

D2.1: Pointwise convergence

$f_n \rightarrow f$ pointwise on E if:

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Here $f_n : E \rightarrow \mathbb{R}$ and:

$$\begin{aligned} \forall x \in E; \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \\ \implies |f_n(x) - f(x)| < \epsilon. \end{aligned}$$

D2.2: Uniform convergence

$f_n \rightarrow f$ uniformly on E if:

$$\begin{aligned} \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \text{ and } \forall x \in E \\ \implies |f_n(x) - f(x)| < \epsilon. \end{aligned}$$

P2.1

The following statements are equivalent.

1. $f_n \rightarrow f$ uniformly on E
2. $\lim_{n \rightarrow \infty} \sup_{x \in E} |f_n(x) - f(x)| = 0$
3. $\exists a_n \rightarrow 0$ s.t. $|f_n(x) - f(x)| \leq a_n$ for $\forall x \in E$.

T2.1

If f_n is continuous on E and $f_n \rightarrow f$ uniformly on E then f is continuous on E .

Remark

If f is not continuous on E then f_n cannot be uniform on E .

T2.5: Weierstrass M-test

Let $E \subset \mathbb{R}$ and $f_k : E \rightarrow \mathbb{R}$.

$$\exists M_k > 0 : \sum_{k=1}^{\infty} M_k < \infty.$$

If $\forall k \in \mathbb{N}$ and $\forall x \in E; |f_k(x)| \leq M_k$ then:

$$\sum_{k=1}^{\infty} f_k(x) \text{ converges uniformly on } E.$$

D: Power series

Let (a_n) be a real sequence and $c \in \mathbb{R}$. Then:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

is a power series centered at c , with **radius of convergence**:

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

where $R = \infty$ implying that series converges everywhere.

T3.1: Convergence of power series

Let $0 < R < \infty$. If $|x - c| < R$ then $f_{PS}(x)$ converges absolutely.

If $|x - c| > R$ then $f_{PS}(x)$ diverges.

T3.2: Continuity of power series

Let $0 < r < R$ where R is the radius of convergence of $f_{PS}(x)$.

Then for $|x - c| \leq r$, $f_{PS}(x)$ converges absolutely and uniformly to a continuous function $f(x)$.

L3.1

$\sum_{n=1}^{\infty} a_n(x - c)^n$ and $\sum_{n=1}^{\infty} n a_n(x - c)^{n-1}$ have the same radius of convergence.

T: Root and ratio tests

Let $S = \sum_{n=1}^{\infty} a_n$ and consider:

1. Ratio test: $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$
2. Root test: $\rho = \lim_{n \rightarrow \infty} |a_n|^{1/n}$.

Then:

- $\rho < 1$: S converges absolutely
- $\rho > 1$: S diverges
- $\rho = 1$: test is inconclusive.

T3.3

Let R be the radius of convergence of $f_{PS}(x)$. Then for $\forall x : |x - c| < R$, $f_{PS}(x)$ is **infinitely differentiable** and:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

T: Taylor's theorem

Let f be n times differentiable at $\alpha \in \mathbb{R}$ where $n \in \mathbb{N}$. Then:

$$\begin{aligned} f(x) = \sum_{k=1}^n \frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k \\ + h_n(x)(x - \alpha)^n \end{aligned}$$

where $\lim_{x \rightarrow \alpha} h_n(x) = 0$.

Elementary expansions

- $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

D: Characteristic functions

Let $E \subset \mathbb{R}$. The characteristic function is defined as a real function such that:

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise.} \end{cases}$$

D4.1 and D4.2: Step functions

The step function with respect to finite set $\{x_0, \dots, x_n\}$ for some $n \in \mathbb{N}$ is:

$$\phi(x) = \begin{cases} 0 & x < x_0 \text{ or } x > x_n \\ c_j & x \in (x_{j-1}, x_j); 1 \leq j \leq n \end{cases}$$

and its integral is defined as:

$$\int \phi = \sum_{j=1}^n c_j (x_j - x_{j-1}).$$

D4.3: Lebesgue integrable

$f : I \rightarrow \mathbb{R}$ is Lebesgue integrable on I if:

1. $\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$
2. $\forall x \in I; f(x) = \sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$

Here $c_j \in \mathbb{R}$, $J_i \subset I$ and is bounded for $j \in \{1, 2, 3, \dots\}$. Then:

$$\int_I f = \sum_{j=1}^{\infty} |c_j| \lambda(J_j).$$

T4.1

Let $c_j, d_j \in \mathbb{R}$ and J_j, K_j be bounded intervals where $j \in \{1, 2, \dots\}$. Let:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

$$\sum_{j=1}^{\infty} |d_j| \lambda(K_j) < \infty.$$

If:

$$\forall x; \sum_{j=1}^{\infty} c_j \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \chi_{K_j}(x) :$$

$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty \text{ and}$$

$$\sum_{j=1}^{\infty} |d_j| \chi_{K_j}(x) < \infty$$

$$\text{then } \sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{\infty} d_j \lambda(K_j).$$

T4.2: Basic properties

Let f, g be integrable on I and $\alpha, \beta \in \mathbb{R}$.

1. $\alpha f + \beta g$ is integrable on I and:

$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g.$$

2. If $f \geq g$ on I then $\int_I f \geq \int_I g$.

3. $|f|$ is integrable on I and:

$$\int_I |f| \geq \left| \int_I f \right|.$$

4. If f or g is bounded on I then fg is integrable on I .

5. If $f \geq 0$ and $\int_I f = 0$,
then $\forall h$ such that $0 \leq h \leq f$ is also
integrable on I .

T4.3

Let f_n be integrable on I where:

$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty.$$

1. Let f be defined as:

$$\forall x \in I; f(x) = \sum_{n=1}^{\infty} f_n(x) :$$

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Then f is integrable on I and:

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n.$$

2. Let each $f_n \geq 0$ and:

$$\forall x \in I; f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Then f is integrable on I iff:

$$\sum_{n=1}^{\infty} \int_I f_n < \infty.$$

T4.4: MCT for integrals

Let f_n be monotone increasing sequence of functions on I and that:

$$\forall x \in I; f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Then f is integrable on I iff:

$$\sup_{n \in \mathbb{N}} \int_I f_n = \lim_{n \rightarrow \infty} \int_I f_n < \infty.$$

Furthermore:

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

D4.4: Riemann integrable

f is Riemann-integrable if:

$$\forall \epsilon > 0; \exists \phi, \psi : \phi \leq f \leq \psi$$

$$\text{and } \int \psi - \int \phi < \epsilon$$

where ϕ and ψ are step functions,
i.e. the bounded support of f .

T4.5

f is Riemann-integrable if and only if:

$$\sup \left\{ \int \phi : \phi \leq f \right\} = \inf \left\{ \int \psi : f \leq \psi \right\}$$

where ϕ and ψ are step functions.

T4.6

If f is Riemann-integrable on I then f is also Lebesgue-integrable on I .

Remark

The converse of T4.6 is not true.

L4.1

Let f be a bounded function with bounded support on $[a, b]$. The following statements are equivalent:

1. f is Riemann-integrable.
2. $\forall \epsilon > 0; \exists \{a = x_0 < \dots < x_n = b\} :$

$$\sum_{j=1}^n (M_j - m_j)(x_j - x_{j-1}) < \epsilon$$

where we define:

$$M_j = \sup_{x \in (x_{j-1}, x_j)} \{f(x)\}$$

$$m_j = \inf_{x \in (x_{j-1}, x_j)} \{f(x)\}.$$

3. $\forall \epsilon > 0; \exists \{a = x_0 < \dots < x_n = b\} :$

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

where $I_j = (x_{j-1}, x_j)$.

T4.7

Let $g : [a, b] \rightarrow \mathbb{R}$ and f be such that $f(x) = g(x)$ if $x \in [a, b]$ and $f(x) = 0$ otherwise.

1. If g is continuous on $[a, b]$ then f is Riemann-integrable.
2. If g is a monotone function then f is Riemann-integrable.

T4.8

Let $J \subset I$.

1. If f is integrable on I
then f is integrable on J .
2. If f is integrable on J and
for $\forall x \in I \setminus J; f(x) = 0$
then f is integrable on I .

$$\text{Furthermore: } \int_J f = \int_I f.$$

3. If f is integrable on I and $f(x) \geq 0$
for $\forall x \in I$ then:

$$\int_I f \geq \int_J f.$$

4. Assume that I can be written as the union of disjoint intervals I_n and that f is integrable on each I_n .

Then f is integrable on I iff:

$$\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty.$$

If this is true then:

$$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} |f|.$$

T4.9

If any two of the following integrals exists:

$$\int_a^b f, \quad \int_b^c f, \quad \int_a^c f$$

then so does the third and:

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

T4.10: FTC I

Let g be integrable on I and let:

$$G(x) = \int_{x_0}^x g(s) ds$$

where $x, x_0 \in I$.

If g is continuous at x then:

$$\frac{d}{dx} G(x) = g(x).$$

T4.11: FTC II

Let $f'(x)$ be continuous on I . Then:

$$\int_a^b f'(x) dx = f(b) - f(a)$$

where $a, b \in I$.

L4.2: Fatoux's lemma

Let $f_n \geq 0$ be integrable on I and:

$$\forall x \in I; f(x) = \liminf_{n \rightarrow \infty} f_n(x).$$

If $\liminf_{n \rightarrow \infty} \int_I f_n < \infty$ then:

$$\int_I f \leq \liminf_{n \rightarrow \infty} \int_I f_n.$$

T4.12: Dominated convergence

Let f_n, g be integrable on I and:

$$\forall x \in I; f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

If $|f_n(x)| \leq g(x)$ for $\forall x \in I$
then f is integrable on I and:

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

T4.13

Let f_n be integrable on (a, b) and that
 $f_n \rightarrow f$ uniformly on (a, b) .

Then f is integrable on (a, b) and:

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n.$$

D5.1: L^2 space**D5.2: Inner products****T5.1: Cauchy-Schwarz inequality****C: Minkowski's inequality****D5.3: L^2 convergence****D5.4: Orthonormal systems****T5.2****T5.3: Bessel's inequality****C: Riemann-Lebesgue lemma****D5.5: Completeness****T5.4****D5.6: Trigonometric polynomial****L5.1****D5.7: Fourier series****D5.8: Fourier partial sums****D5.9: Convolutions****L5.2****L5.3: Dirichlet kernel****L5.4: Fejér kernel****T5.5: Fejér's theorem****D5.10: Approximation of unity****T5.6****C****L5.5****T5.7****C: Parseval's theorem**