EM S1 Tutorials

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1 Tutorial 3

1. Let T_{ij} be a rank 2 tensor, and S_{ij} a rank 2 pseudotensor.

   Show that:

   - $T_{ij}S_{ik}$ is a pseudotensor.
   - $T_{ij}S_{ij}$ is a pseudoscalar.

   So by definition we have the following:

   $$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq}$$

   $$S'_{ij} = (\det L)\ell_{ip}\ell_{jq}S_{pq}.$$

   Changing indices for the pseudotensor:

   $$S'_{ik} = (\det L)\ell_{ir}\ell_{ks}S_{rs}$$

   We <u>must</u> swap for r and s here! Then:

   $$\begin{aligned} T'_{ij}S'_{ik} &= (\det L)\ell_{ip}\ell_{jq}\ell_{ir}\ell_{ks}T_{pq}S_{rs} \\ &= (\det L)\delta_{pr}\ell_{jq}\ell_{ks}T_{pq}S_{rs} \\ &= (\det L)\ell_{jq}\ell_{ks}T_{rq}S_{rs} \end{aligned}$$

   Then set $\alpha_{qs} = T_{rq}S_{rs}$, and therefore $T_{ij}S_{ik}$ is a pseudotensor. i.e:

   $$\alpha'_{jk} = (\det L)\ell_{jq}\ell_{ks}\alpha_{qs}.$$

   For the second part set $j = k$. Therefore:

   $$\begin{aligned} T'_{ij}S'_{ij} &= (\det L)\ell_{jq}\ell_{js}T_{rq}S_{rs} \\ &= (\det L)\delta_{qs}T_{rq}S_{rs} \\ &= (\det L)T_{rq}S_{rq}, \end{aligned}$$

   which is the transformation law for pseudoscalars.

   Notice how the equation is similar to a dot product, yielding a scalar quantity.

2 Tutorial 5

1. Current in crystal from applied electric field is explored. Consider:

$$j_i = \sigma_{ij} E_j,$$

where j_i is the current density, σ_{ij} is the conductivity of our crystal and E_j the electric field we subject our crystal to.

Now by the **quotient theorem** clearly if j_i and E_j are vectors then σ_{ij} is a rank 2 tensor.

The conductivity tensor of our crystal is defined as:

$$\sigma_{ij} = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{ij}$$

and to find electric field directions where no current flows we need to solve $\sigma_{ij} E_j = 0$. This can be done by writing our equation as an augmented matrix and then row reducing, which yields:

$$\boldsymbol{E} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}.$$

3 Tutorial 7

1. For part (i), we are asked to find $\boldsymbol{\nabla}(\boldsymbol{c}\cdot\boldsymbol{r})^n$.

   This is the **gradient** operator and acts only on scalar fields $\phi(\boldsymbol{r})$. Note:

   $$\boldsymbol{\nabla}\phi(\boldsymbol{r}) = \frac{\partial\phi}{\partial x_i}\boldsymbol{e}_i,$$

   where $\boldsymbol{r} = x_i\boldsymbol{e}_i$. Using the chain rule we get:

   $$\begin{aligned}\boldsymbol{\nabla}(\boldsymbol{c}\cdot\boldsymbol{r})^n &= n(\boldsymbol{c}\cdot\boldsymbol{r})^{n-1}\boldsymbol{\nabla}(\boldsymbol{c}\cdot\boldsymbol{r})\\ &= n\boldsymbol{c}(\boldsymbol{c}\cdot\boldsymbol{r})^{n-1}.\end{aligned}$$

   Our solution makes sense since we obtain a vector quantity. Now however our question asks us to show this using suffix notation:

   $$\begin{aligned}\boldsymbol{\nabla}(\boldsymbol{c}\cdot\boldsymbol{r})^n &= \boldsymbol{e}_i\frac{\partial}{\partial x_i}(\boldsymbol{c}\cdot\boldsymbol{r})^n\\ &= n(\boldsymbol{c}\cdot\boldsymbol{r})^{n-1}\boldsymbol{e}_i\frac{\partial}{\partial x_i}(c_jx_j)\\ &= n(\boldsymbol{c}\cdot\boldsymbol{r})^{n-1}\boldsymbol{e}_ic_j\delta_{ij}\\ &= n(\boldsymbol{c}\cdot\boldsymbol{r})^{n-1}\boldsymbol{c}.\end{aligned}$$

   For part (ii) we want $\boldsymbol{\nabla}r^n$.

   Firstly we need to prove an important result, namely that:

   $$\boldsymbol{\nabla}r = \frac{1}{r}\boldsymbol{r},$$

   where $r = |\boldsymbol{r}|^2 = \sqrt{x_j^2}$ for $\boldsymbol{r} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.

   So consider the following:

   $$\begin{aligned}\boldsymbol{\nabla}r &= \frac{\partial r}{\partial x_i}\boldsymbol{e}_i\\ &= \boldsymbol{e}_i\frac{\partial}{\partial x_i}\sqrt{x_j^2}\\ &= \boldsymbol{e}_i\cdot\frac{1}{2}\frac{1}{\sqrt{x_j^2}}\cdot 2x_j\frac{\partial x_j}{\partial x_i}\\ &= \boldsymbol{e}_i\cdot\frac{1}{r}\cdot x_j\delta_{ij}\\ &\implies \frac{1}{r}\boldsymbol{r}.\end{aligned}$$

Now that this is established we may use it:

$$\begin{aligned}\boldsymbol{\nabla} r^n &= \frac{\partial}{\partial r}(r^n)\cdot\boldsymbol{\nabla} r\\ &= n\cdot r^{n-2}\boldsymbol{r},\end{aligned}$$

and we are finished. We can also do this with suffix notation:

$$\begin{aligned}\boldsymbol{\nabla} r^n &= \boldsymbol{e}_i\frac{\partial}{\partial x_i}[x_j^2]^{n/2}\\ &= \frac{n}{2}[x_j^2]^{(n-2)/2}\boldsymbol{e}_i\frac{\partial}{\partial x_i}[x_k^2]\\ &= \frac{n}{2}[x_j^2]^{(n-2)/2}\boldsymbol{e}_i\cdot 2x_k\delta_{ik}\\ &= nr^{n-2}\boldsymbol{r}.\end{aligned}$$

For part (iii), we need to find $\boldsymbol{\nabla}\cdot(r^n\boldsymbol{r})$.

This is the **divergence** operator and is defined:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\partial E_i}{\partial x_i},$$

where $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r})$ and is a <u>vector field</u>. Therefore:

$$\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3.$$

We also need the <u>chain rule for divergence</u>:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\phi\boldsymbol{a}) &= \frac{\partial}{\partial x_i}(\phi\boldsymbol{a})\\ &\implies (\boldsymbol{\nabla}\phi)\cdot\boldsymbol{a} + \phi(\boldsymbol{\nabla}\cdot\boldsymbol{a}).\end{aligned}$$

Putting all this together we get:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(r^n\boldsymbol{r}) &= (\boldsymbol{\nabla} r^n)\cdot\boldsymbol{r} + r^n(\boldsymbol{\nabla}\cdot\boldsymbol{r})\\ &= (n+3)r^n.\end{aligned}$$

We can also do this in suffix notation:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(r^n\boldsymbol{r}) &= \frac{\partial}{\partial x_i}\left[r^n x_i\right]\\ &= \left(\frac{\partial}{\partial x_i}r^n\right)x_i + r^n\left(\frac{\partial}{\partial x_i}x_i\right)\\ &= \left(nr^{n-1}\frac{\partial}{\partial x_i}[x_k^2]^{1/2}\right)x_i + r^n\delta_{ii}\\ &= (n+3)\boldsymbol{r}^n.\end{aligned}$$

For part (iv) we want $\boldsymbol{\nabla} \times (r^n \boldsymbol{r})$.

So note the chain rule for curl:

$$\boldsymbol{\nabla} \times (\phi \boldsymbol{a}) = \boldsymbol{\nabla}\phi \times \boldsymbol{a} + \phi \boldsymbol{\nabla} \times \boldsymbol{a}.$$

Therefore:

$$\begin{aligned}
\boldsymbol{\nabla} \times (r^n \boldsymbol{r}) &= \boldsymbol{\nabla} r^n \times \boldsymbol{r} + r^n \boldsymbol{\nabla} \times \boldsymbol{r} \\
&= n \cdot r^{n-2} \cdot \boldsymbol{r} \times \boldsymbol{r} + r^n \cdot \boldsymbol{0} \\
&= \boldsymbol{0}.
\end{aligned}$$

We can also do this with suffix notation:

$$\begin{aligned}
\boldsymbol{\nabla} \times (r^n \boldsymbol{r}) &= \boldsymbol{e}_i \epsilon_{ijk} \frac{\partial}{\partial x_j} [r^n \boldsymbol{r}]_k \\
&= \boldsymbol{e}_i \epsilon_{ijk} \frac{\partial}{\partial x_j} [r^n x_k] \\
&= \boldsymbol{e}_i \epsilon_{ijk} \left(\frac{\partial}{\partial x_j} [r^n] x_k + r^n \frac{\partial}{\partial x_j} x_k \right) \\
&= \boldsymbol{e}_i \epsilon_{ijk} \left(n r^{n-2} x_i x_k + r^n \delta_{jk} \right) \\
&= 0.
\end{aligned}$$

For part (v) we want $\boldsymbol{\nabla} \cdot (\boldsymbol{c} \times \boldsymbol{r})$.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{c} \times \boldsymbol{r}) &= \frac{\partial}{\partial x_i} [\epsilon_{ijk} c_j x_k] \\
&= \epsilon_{ijk} c_j \cdot \frac{\partial}{\partial x_i} x_k \\
&= \epsilon_{ijk} c_j \delta_{ik} = 0.
\end{aligned}$$

For part (vi) we want $\boldsymbol{\nabla} \times (\boldsymbol{c} \times \boldsymbol{r})$.

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{c} \times \boldsymbol{r}) &= \epsilon_{ijk} \cdot \frac{\partial}{\partial x_j} [\epsilon_{klm} c_l x_m] \\
&= \epsilon_{ijk} \epsilon_{klm} \cdot c_l \delta_{mj} \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) c_l \delta_{mj} \\
&= 2c_i \\
&= \boldsymbol{c}.
\end{aligned}$$

For part (vii) we want $(\boldsymbol{c} \cdot \boldsymbol{\nabla}) \boldsymbol{r}$.

$$\begin{aligned}
(\boldsymbol{c} \cdot \boldsymbol{\nabla}) \boldsymbol{r} &= c_i \frac{\partial}{\partial x_i} x_j \boldsymbol{e}_j \\
&= c_i \delta_{ij} \boldsymbol{e}_j \\
&= \boldsymbol{c}.
\end{aligned}$$

2.

3.

4. For (i) we are asked to show the following:
   - $x\delta(x) = 0$
   - $\delta(cx) = \frac{1}{|c|}\delta(x)$

   So we have the **sift** property:

   $$\int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a).$$

   Taking $f(x) = x$ and $a = 0$ gives:

   $$\int_{\mathbb{R}} x\delta(x)dx = 0.$$

   Recall the **fundamental theorem of calculus**:

   $$\frac{d}{dx}\int_{x_0}^{x} g(t)dt = g(x).$$

   Let $g(t) = t\delta(t)$ and integrate over $\mathbb{R}$. $\therefore x\delta(x) = 0$.

   Now consider the following identity:

   $$\int_{\mathbb{R}} \delta(x)dx = 1.$$

   We aim to write $\delta(cx)$ into this form. First assume that $c > 0$:

   $$\begin{aligned}\int_{-\infty}^{\infty} \delta(cx)dx &= \int_{-\infty}^{\infty} \frac{1}{c}\delta(cx)d(cx) \\ &= \frac{1}{c}.\end{aligned}$$

   If $c = 0$ then our equality holds trivially. Now assume that $c < 0$. We can write c as $c = -|c|$. or that:

   $$\int_{\mathbb{R}} |c|\delta(cx)dx = 1.$$

   It must then be that:

   $$|c|\delta(cx) = \delta(x).$$