D: Supremum

T: Approximation lemma

D: Completeness of $\mathbb{R}$

Every nonempty bounded subset of $\mathbb{R}$ has an infimum and supremum.

T: Archimedean property

$\forall a, b \in \mathbb{R}; a > 0; \exists n \in \mathbb{N} : na > b$

D1.1: Nested intervals

T1.1: Nested interval property

D2.1: Pointwise convergence

$f_n \to f$ pointwise on E if:

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Here $f_n : E \to \mathbb{R}$.

$$\forall x \in E; \forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

D2.2: Uniform convergence

$f_n \to f$ uniformly on E if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \text{ and } \forall x \in E \implies |f_n(x) - f(x)| < \epsilon$$

P2.1

The following statements are equivalent.

1. $f_n \to f$ uniformly on E
2. $\lim\limits_{n\to\infty} \sup\limits_{x\in E} |f_n(x) - f(x)| = 0$
3. $\exists a_n \to 0$ s.t. $|f_n(x) - f(x)| \leq a_n$ for $\forall x \in E$.

T2.1

If f_n is continuous on E **and** $f_n \to f$ uniformly on E then f is continuous on E.

Remark

If f is not continuous on E then f_n cannot be uniform on E.

T2.5: Weierstrass M-test

Let $E \subset \mathbb{R}$ and $f_k : E \to \mathbb{R}$.

$\exists M_k > 0 : \sum\limits_{k=1}^{\infty} M_k < \infty.$

If $\forall k \in \mathbb{N}$ and $\forall x \in E$; $|f_k(x)| \leq M_k$ then: $\sum\limits_{k=1}^{\infty} f_k(x)$ converges uniformly on E.

D: Power series

Let (a_n) be a real sequence and $c \in \mathbb{R}$. Then:

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

is a power series centered at c, with **radius of convergence**:

$$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$

where $R = \infty$ implying that series converges everywhere.

T3.1: Convergence of power series

Let $0 < R < \infty$. If $|x-c| < R$ then $f_{PS}(x)$ converges absolutely.

If $|x - c| > R$ then $f_{PS}(x)$ diverges.

T3.2: Continuity of power series

Let $0 < r < R$ where R is the radius of convergence of $f_{PS}(x)$.

Then for $|x - c| \leq r$, $f_{PS}(x)$ converges absolutely and uniformly to a continuous function $f(x)$.

L3.1

$\sum\limits_{n=1}^{\infty} a_n(x-c)^n$ and $\sum\limits_{n=1}^{\infty} n a_n (x-c)^{n-1}$ have the same radius of convergence.

T3.3

Let R be the radius of convegence of $f_{PS}(x)$. Then for $\forall x : |x - c| < R$, $f_{PS}(x)$ is **infinitely differentiable**.

$$f_{PS}(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$\therefore a_n = \frac{f^{(n)}(c)}{n!}$$

Elementary expansions

- $E(x) = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$