

**D1.1.1: Complex numbers**

Let  $z = x + iy$  and  $w = a + ib$  where  $x, y, a, b \in \mathbb{R}$ . Then  $z$  and  $w$  are complex numbers. Furthermore:

1.  $z = w$  **iff**  $x = a$  and  $y = b$ .
2.  $\operatorname{Re}(z) := x$  and  $\operatorname{Im}(z) := y$ .
3.  $|z| := \sqrt{x^2 + y^2}$
4. The **complex conjugate** of  $z$  is:

$$\bar{z} := x - iy.$$

5. Addition and multiplication:

$$(x + iy) + (a + ib) = (x + a) + i(y + b)$$

$$(x + iy)(a + ib) = (xa - yb) + i(xb + ya).$$

6.  $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$

with rule  $i^2 = -1$ .

**L1.1.3**

Let  $u, w, z \in \mathbb{C}$  where  $z = x + iy$ . Then:

1.  $z + w = w + z$  and  $zw = wz$ .
2.  $u + (z + w) = (u + z) + w$
3.  $u(zw) = (uz)w$
4.  $u(z + w) = uz + uw$
5.  $z + 0 = z$  and  $1z = z$ .
6.  $\exists(-z := -x + i(-y)) : z + (-z) = 0$ .
7.  $\exists z^{-1} : zz^{-1} = 1$  where:

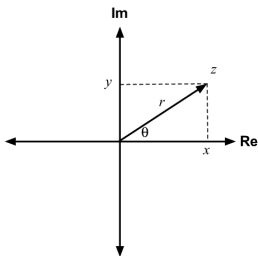
$$z^{-1} := \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}.$$

**D1.1.5 and D1.1.7: Polar form**

Let  $z \in \mathbb{C}$  and  $z = x + iy$ . Then:

$$z = r(\cos \theta + i \sin \theta) \\ = re^{i\theta}$$

for  $r = \sqrt{x^2 + y^2}$  in complex plane.

**L1.1.6**

Let  $\theta, \phi \in \mathbb{R}$  and  $n \in \mathbb{Z}$ . Then:

1.  $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$
2.  $e^{in\theta} = (e^{i\theta})^n$

due to de Moivre's formula:

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$$

**L1.1.9**

Let  $z, w \in \mathbb{C}$ . Then:

1.  $|z| = 0$  **iff**  $z = 0$ .
2.  $|\bar{z}| = |z|$
3.  $|zw| = |z||w|$
4.  $\bar{\bar{z}} = z$
5.  $|z|^2 = z\bar{z}$
6.  $\overline{z + w} = \bar{z} + \bar{w}$
7.  $\overline{zw} = \bar{z}\bar{w}$
8.  $|\operatorname{Re}(z)| \leq |z|$  and  $|\operatorname{Im}(z)| \leq |z|$ .
9.  $\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$
10.  $\operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$ .

**L1.1.10 – 11: Triangle inequalities**

Let  $z, w \in \mathbb{C}$ . Then:

1.  $|z + w| \leq |z| + |w|$
2.  $||z| - |w|| \leq |z - w|$ .

**D1.1.12: Argument of  $z$** 

Let  $z = |z|e^{i\theta}$ . Then:

$$\arg(z) := \theta \in (-\pi, \pi]$$

with period  $2\pi$ .

**P1.1.14**

Let  $z, w \in \mathbb{C}$ . Then:

1.  $\arg(zw) = \arg(z) + \arg(w)$
2.  $\arg(\bar{z}) = -\arg(z)$

and holds under modulo  $2\pi$ .

**D1.2.1: Open and closed  $\epsilon$ -discs**

Let  $z_0 \in \mathbb{C}$  and  $\epsilon > 0$ .

1. An **open**  $\epsilon$ -disc centred at  $z_0$  is:

$$D_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}.$$

2. A **closed**  $\epsilon$ -disc centred at  $z_0$  is:

$$\bar{D}_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}.$$

A **punctured**  $\epsilon$ -disc centred at  $z_0$  is:

$$D'_\epsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}.$$

**D1.2.2: Open sets**

Let  $U \subset \mathbb{C}$ . Set  $U$  is **open** if:

$$\forall z_0 \in U; \exists \epsilon > 0 : D_\epsilon(z_0) \subseteq U.$$

Subset  $F$  is **closed** if  $\mathbb{C} \setminus F$  is open.

A **neighbourhood** of point  $z_0 \in \mathbb{C}$  is an open set that contains  $z_0$ .

**L1.2.3**

Punctured disc  $D'_\epsilon(z_0)$  is open.

**D1.2.4: Limit points**

Let  $S \subseteq \mathbb{C}$ .  $z_0$  is a **limit point** of  $S$  if:

$$\forall \epsilon > 0; D'_\epsilon(z_0) \cap S \neq \emptyset.$$

The **closure** of  $S$  is set  $\bar{S}$  and contains  $S$  and **all** its limit points.

**L1.2.6**

Let  $S \subseteq \mathbb{C}$ .  $S$  is **closed** **iff**  $S = \bar{S}$ .

**D1.2.7: Bounded sets**

Let  $S \subseteq \mathbb{C}$ . Set  $S$  is **bounded** if:

$$\forall z \in S; \exists M > 0 : |z| \leq M.$$

**D1.2.8:  $\epsilon$ -N convergence**

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

Let  $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$  be a sequence and  $z \in \mathbb{C}$ . Then  $\lim_{n \rightarrow \infty} z_n = z$  if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \\ \implies |z_n - z| < \epsilon.$$

**L1.2.9**

Let  $z_n, z \in \mathbb{C}$  where  $z_n = a_n + ib_n$ .

Then  $\lim_{n \rightarrow \infty} z_n = z$  **iff**:

$$\operatorname{Re}(z) = \lim_{n \rightarrow \infty} a_n \text{ and } \operatorname{Im}(z) = \lim_{n \rightarrow \infty} b_n.$$

**L1.2.10**

Let  $S \subseteq \mathbb{C}$  and  $z \in \mathbb{C}$ . Then  $z \in \bar{S}$  **iff**:

$$\exists z_n \in S : z = \lim_{n \rightarrow \infty} z_n.$$

**D1.2.11: Cauchy sequences**

$z_n$  is a Cauchy sequence if:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n, m \geq N \\ \implies |z_n - z_m| < \epsilon.$$

**L1.2.12**

$z_n$  is convergent **iff**  $z_n$  is Cauchy.

**D1.2.14: Bounded sequences**

$z_n$  is bounded if:

$$\forall n \in \mathbb{N}; \exists M > 0 : |z_n| \leq M.$$

**L1.2.15: Bolzano-Weierstrass**

Let  $z_n$  be a bounded sequence. Then:

$$\exists (z_{n_k})_{k, n_k \in \mathbb{N}} : \lim_{k \rightarrow \infty} z_{n_k} = z \in \mathbb{C}$$

or that  $z_n$  has a convergent subsequence.

A selection of a sequence is a subsequence.

**D1.3.1: Bounded functions**

Let  $S \subseteq \mathbb{C}$  and  $f : S \rightarrow \mathbb{C}$ . Then  $f$  is a bounded function if:

$$\forall z \in S; \exists M > 0 : |f(z)| \leq M.$$

**D1.3.2:  $\epsilon$ - $\delta$  convergence**

Let  $S \subseteq \mathbb{C}$ ,  $z_0 \in \overline{S}$ ,  $f : S \rightarrow \mathbb{C}$  and  $a_0 \in \mathbb{C}$ . Then  $\lim_{z \rightarrow z_0} f(z) = a_0$  if:

$$\begin{aligned} \forall z \in S; \forall \epsilon > 0; \exists \delta > 0 : 0 < |z - z_0| < \delta \\ \implies |f(z) - a_0| < \epsilon. \end{aligned}$$

**L1.3.3**

Let  $S \subseteq \mathbb{C}$ ,  $z_0 \in \overline{S}$ ,  $f : S \rightarrow \mathbb{C}$  and  $a_0 \in \mathbb{C}$  where  $z_0 = x_0 + iy_0$  and  $f = u + iv$ .

Then  $\lim_{z \rightarrow z_0} f(z) = a_0$  iff:

$$\operatorname{Re}(a_0) = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} u(x, y)$$

and

$$\operatorname{Im}(a_0) = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} v(x, y).$$

**L1.3.4**

Let  $S \subseteq \mathbb{C}$ ,  $z_0 \in \overline{S}$ ,  $f : S \rightarrow \mathbb{C}$ ,  $a_0 \in \mathbb{C}$  and sequence  $w_n \in S \setminus \{z_0\}$ .

If  $\lim_{z \rightarrow z_0} f(z) = a_0$  and  $\lim_{n \rightarrow \infty} w_n = z_0$  then:

$$\lim_{n \rightarrow \infty} f(w_n) = a_0.$$

**L1.3.5**

limit algebra

**D1.3.6:  $\epsilon$ - $\delta$  continuity****L1.3.7****L1.3.8**

composition of functions are also continuous

**L1.3.9****L1.3.10****D1.4.1: Differentiability****L1.4.3**

differentiability implies continuity

**L1.4.4**

derivative rules

**L1.4.5: Chain rule****T1.4.6: Cauchy-Riemann equations****T1.4.8 ?****D1.4.9: Holomorphic functions****D1.4.13: Harmonic equations****L1.4.14****D1.4.15: Harmonic conjugates****D1.5.1: Complex polynomials****L1.5.2**