

## Vector products

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \text{ and } \mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

## Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear more than twice in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \end{aligned}$$

and we have the following identities:

$$\mathbf{a} = \sum_{i=1}^3 a_i \mathbf{e}_i = a_i \mathbf{e}_i$$

$$A\mathbf{x} = a_{ij}x_j \mathbf{e}_i \text{ for } m \times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\dots]_j \delta_{jk} = [\dots]_k$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k$$

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \mathbf{e}_i$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk} \epsilon_{ijk} = 6.$$

## Transformations

Let matrix  $L$  relate basis  $\{\mathbf{e}_i\}$  to basis  $\{\mathbf{e}'_i\}$  with rule:

$$\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then  $L^T L = L L^T = I$ , and:

$$\ell_{ik} \ell_{jk} = \ell_{ki} \ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij} p_j \text{ for } \mathbf{p} = p_i \mathbf{e}_i = p'_i \mathbf{e}'_i.$$

## Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip} \ell_{jq} \ell_{kr} T_{pqr}$$

which relates frame  $S$  in  $\{\mathbf{e}_i\}$  to frame  $S'$  in  $\{\mathbf{e}'_i\}$  with rule  $\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j$ , etc.

Properties of tensors:

1. The addition of two rank  $n$  tensors is also a rank  $n$  tensor.
2. The multiplication of a rank  $m$  tensor with a rank  $n$  tensor yields a rank  $m + n$  tensor.
3. If  $T_{ijk\dots s}$  is a rank  $m$  tensor then  $T_{\mathbf{ii}k\dots s}$  is a rank  $m - 2$  tensor.
4. If  $T_{ij}$  is a tensor then  $T_{ji}$  is also a tensor. Explicitly:

$$\begin{aligned} T'_{ij} &= \ell_{ip} \ell_{jq} T_{pq} \implies T' = L T L^T \\ T'_{\mathbf{ji}} &= \ell_{jp} \ell_{iq} T_{pq}. \end{aligned}$$

## Symmetric tensors

$T_{ij}$  is a symmetric tensor when  $T_{ij} = T_{ji}$  in frame  $S$ . Then  $T'_{ij} = T'_{ji}$  in frame  $S'$ .

Similarly  $T_{ij}$  is an anti-symmetric tensor if  $T_{ij} = -T_{ji}$  and  $T'_{ij} = -T'_{ji}$ .

Finally **any tensor** can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

## Quotient theorem

Consider 9 entities  $T_{ij}$  in frame  $S$  and  $T'_{ij}$  in frame  $S'$ . Let  $b_i = T_{ij} a_j$  where  $a_j$  is a vector. If  $b_i$  always transforms as a vector then  $T_{ij}$  is a rank 2 tensor.

Generalising, let  $R_{ijk\dots r}$  be a rank  $m$  tensor and  $T_{ijk\dots s}$  a set of  $3^n$  numbers where  $n > m$ . If  $T_{ijk\dots s} R_{ijk\dots r}$  is a rank  $n - m$  tensor then  $T_{ijk\dots s}$  is a rank  $n$  tensor.

## Matrices

We define a  $m \times n$  matrix  $A$  as  $(A)_{ij} = a_{ij}$  where  $i = 1, \dots, m$  and  $j = 1, \dots, n$ .

- $\text{Tr } A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^T A^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a  $3 \times 3$  matrix  $A$  is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn} a_{1l} a_{2m} a_{3n} \\ &= \epsilon_{lmn} a_{l1} a_{m2} a_{n3}. \end{aligned}$$

Furthermore:

$$\begin{aligned} \epsilon_{ijk} \det A &= \epsilon_{lmn} a_{il} a_{jm} a_{kn} \\ \epsilon_{lmn} \det A &= \epsilon_{ijk} a_{il} a_{jm} a_{kn} \\ \det A &= \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} a_{il} a_{jm} a_{kn}. \end{aligned}$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3.  $\det A = \det A^T$
4.  $\det(AB) = \det A \cdot \det B$

These also apply to columns. Finally:

$$\epsilon_{ijk} \epsilon_{lmn} \det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting  $A = I$  yields:

$$\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

## Linear equations

Let  $\mathbf{y} = A\mathbf{x}$ . Then  $x_i = A_{ij}^{-1} y_j$  with:

$$\begin{aligned} A_{ij}^{-1} &= \frac{1}{2 \det A} \epsilon_{imn} \epsilon_{jpk} a_{pm} a_{qn} \\ &= \frac{1}{\det A} C_{ij}^T \end{aligned}$$

where  $C$  is the cofactor matrix of  $A$ .

## Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L) \ell_{ip} \ell_{jq} T_{pq}$$

where  $(L)_{ij} = \ell_{ij}$  and  $\det L = \pm 1$ .

Pseudovectors are rank 1 pseudotensors.

## Invariant tensors

Tensor  $T$  is invariant or isotropic if:

$$T_{ijk\dots} = \ell_{i\alpha} \ell_{j\beta} \ell_{k\gamma} \dots T_{\alpha\beta\gamma\dots}$$

for every orthogonal matrix  $L$ .

- If  $a_{ij}$  is a rank 2 invariant tensor then  $a_{ij} = \lambda \delta_{ij}$ .
- The most general rank 3 invariant pseudotensor is  $a_{ijk} = \lambda \epsilon_{ijk}$ . There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

## Rotation tensors

The clockwise rotation of position vector  $\mathbf{x}$  to  $\mathbf{y}$  about unit vector  $\hat{\mathbf{n}}$  is given by:

$$y_i = R_{ij}(\theta, \hat{\mathbf{n}})x_j$$

$$R_{ij}(\theta, \hat{\mathbf{n}}) = \delta_{ij} \cos \theta + (1 - \cos \theta)n_i n_j - \epsilon_{ijk} n_k \sin \theta$$

and is the rotation tensor.

## Reflections and inversions

The reflection of vector  $\mathbf{x}$  to  $\mathbf{y}$  in plane with unit vector  $\hat{\mathbf{n}}$  is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector  $\mathbf{x}$  to  $\mathbf{y}$  is given by  $\mathbf{y} = -\mathbf{x}$  and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

## Projections

We define  $P$  to be a parallel projection operator to vector  $\mathbf{u}$  if:

$$P\mathbf{u} = \mathbf{u} \text{ and } P\mathbf{v} = \mathbf{0}$$

where  $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ . Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define  $Q$  to be an orthogonal projection to vector  $\mathbf{u}$  if:

$$Q\mathbf{u} = \mathbf{0} \text{ and } Q\mathbf{v} = \mathbf{v}.$$

Here  $Q = I - P$ .

## Inertia tensors

Let  $\mathbf{L}$  denote the angular momentum of a rigid body about the origin of mass  $m$ , volume  $V$  and density  $\rho$  at position  $\mathbf{r}$  with velocity  $\mathbf{v}$ . Then:

$$\mathbf{L}_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2 \delta_{ij} - x_i x_j) dV$$

where  $I_{ij}(O)$  is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2} I_{ij} \omega_i \omega_j = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega}.$$

## Parallel axis theorem

Consider the same rigid body now with centre of mass  $G$  and let  $\overrightarrow{OG} = \mathbf{R}$ . Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2 \delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\mathbf{r}') dV'.$$

## Diagonalisation

Let  $\mathbf{L} = I_{ij}\omega_j$  where  $I_{ij}$  is a rank 2 tensor and let  $\mathbf{L} = \lambda\boldsymbol{\omega}$ . Then:

$$(I_{ij} - \lambda \delta_{ij})\omega_j = 0 \implies \det(I_{ij} - \lambda \delta_{ij}) = 0$$

where expanding this gives:

$$P - Q\lambda + R\lambda^2 - \lambda^3 = 0$$

for  $P = \det I$ ,  $Q = \frac{1}{2}[(\text{tr } I)^2 - \text{tr}(I^2)]$  and  $R = \text{tr } I$  given tensor  $I$ .

## Real symmetric tensors

Let rank 2 real symmetric tensor  $T$  be diagonalisable with real eigenvalues  $\lambda^{(i)}$  and orthonormal eigenvectors  $\boldsymbol{\ell}^{(i)}$  where  $i = 1, 2, 3$ . Let transformation matrix be:

$$L_{ij} = \ell_j^{(i)} = \begin{pmatrix} \ell_1^{(1)} & \ell_2^{(1)} & \ell_3^{(1)} \\ \ell_1^{(2)} & \ell_2^{(2)} & \ell_3^{(2)} \\ \ell_1^{(3)} & \ell_2^{(3)} & \ell_3^{(3)} \end{pmatrix}_{ij}$$

and always defined such that  $\det L = +1$  which transforms frame  $S \rightarrow S'$ .

Then since  $T_{pq}\ell_q^{(i)} = \lambda^{(i)}\ell_p^{(i)}$ :

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} = \lambda^{(i)}\delta_{ij} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}_{ij}.$$

## Taylor expansions

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

and is  $f$  expanded about  $x = a$ .

Trigonometric expansions are in radians!

$$\begin{aligned} \therefore f(x+a) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)a^n \\ &= \exp\left(a \frac{d}{dx}\right) f(x) \end{aligned}$$

Then for three dimensions:

$$\begin{aligned} \phi(\mathbf{r} + \mathbf{a}) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{a} \cdot \nabla_{\mathbf{r}})^n \phi(\mathbf{r}) \\ &= \exp(\mathbf{a} \cdot \nabla_{\mathbf{r}}) \phi(\mathbf{r}). \end{aligned}$$

## Curvilinear coordinates

Let  $x_i$  denote Cartesian coordinates and  $u_i$  denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \rightarrow (u_1, u_2, u_3)$$

where each  $u_i = u_i(x_1, x_2, x_3)$  and:

$$\begin{aligned} \mathbf{r} &= x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \\ &= u_1 \mathbf{e}_{u_1} + u_2 \mathbf{e}_{u_2} + u_3 \mathbf{e}_{u_3}. \end{aligned}$$

## Scale factors

Let  $u_1 \rightarrow u_1 + du_1$  in  $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$ . Then  $d\mathbf{r}$  in  $\mathbf{r} \rightarrow \mathbf{r} + d\mathbf{r}$  is defined as:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 := h_1 \mathbf{e}_1 du_1.$$

$h_1$  is the scale factor of unit vector  $\mathbf{e}_1$ :

$$h_1 = \left| \frac{\partial \mathbf{r}}{\partial u_1} \right| \text{ and } \mathbf{e}_1 = \frac{1}{h_1} \frac{\partial \mathbf{r}}{\partial u_1}.$$

If  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$  then  $u_i$  is an **orthogonal** curvilinear coordinate system.

## Vector and arc length

The vector length  $d\mathbf{r}$  of  $\mathbf{r}$  is defined as:

$$d\mathbf{r} = \sum_{i=1}^3 h_i du_i \mathbf{e}_i$$

where  $u_i \rightarrow u_i + du_i$  for  $\forall i = 1, 2, 3$ .

Then the arc length  $ds$  is defined as:

$$\begin{aligned} (ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= g_{ij} du_i du_j \end{aligned}$$

where  $g_{ij}$  is the metric tensor:

$$\begin{aligned} g_{ij} &= g_{ji} = \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \\ &= h_i h_j (\mathbf{e}_i \cdot \mathbf{e}_j). \end{aligned}$$

## Area and volume

Let  $d\mathbf{r}_i = h_i \mathbf{e}_i du_i$  denote vector length when  $u_i \rightarrow u_i + du_i$ . (**No** sum!)

The infinitesimal vector area formed by  $d\mathbf{r}_1$  and  $d\mathbf{r}_2$  is:

$$d\mathbf{S} = (h_1 d\mathbf{u}_1 \mathbf{e}_1) \times (h_2 d\mathbf{u}_2 \mathbf{e}_2).$$

Similarly the infinitesimal volume formed by edges  $d\mathbf{r}_1$ ,  $d\mathbf{r}_2$  and  $d\mathbf{r}_3$  is:

$$\begin{aligned} dV &= |(d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3| \\ &= \sqrt{g} du_1 du_2 du_3 \end{aligned}$$

where  $g = \det(g_{ij})$ .

## Cylindrical coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$  where  $\rho$  represents the radial distance from the origin and  $\phi$  is the anticlockwise rotation angle on the  $x$ - $y$  plane. In Cartesian unit vectors:

$$\begin{aligned} \mathbf{r} &= \rho \cos \phi \mathbf{e}_1 + \rho \sin \phi \mathbf{e}_2 + z \mathbf{e}_3 \\ h_\rho &= 1, \quad \mathbf{e}_\rho = \cos \phi \mathbf{e}_1 + \sin \phi \mathbf{e}_2 \\ h_\phi &= \rho, \quad \mathbf{e}_\phi = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2 \\ h_z &= 1, \quad \mathbf{e}_z = \mathbf{e}_3 \end{aligned}$$

and forms an orthogonal set.

## Spherical coordinates

$(u_1, u_2, u_3) = (r, \theta, \phi)$  where  $\theta$  represents the clockwise rotation angle in  $y$ - $z$  plane and  $\phi$  the anticlockwise rotation angle in  $x$ - $y$  plane. In Cartesian unit vectors:

$$\mathbf{r} = r \sin \theta \cos \phi \mathbf{e}_1 + r \sin \theta \sin \phi \mathbf{e}_2 + r \cos \theta \mathbf{e}_3$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin \theta$$

$$\mathbf{e}_r = \sin \theta \cos \phi \mathbf{e}_1 + \sin \theta \sin \phi \mathbf{e}_2 + \cos \theta \mathbf{e}_3$$

$$\mathbf{e}_\theta = \cos \theta \cos \phi \mathbf{e}_1 + \cos \theta \sin \phi \mathbf{e}_2 - \sin \theta \mathbf{e}_3$$

$$\mathbf{e}_\phi = -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2$$

and also forms an orthogonal set.



## Gradient

The gradient of a scalar field  $f(\mathbf{r})$  is:

$$df(\mathbf{r}) := \nabla f(\mathbf{r}) \cdot d\mathbf{r}$$

when  $\mathbf{r} \rightarrow \mathbf{r} + d\mathbf{r} \implies f \rightarrow f + df$ .  
Taking the total differential of  $f$  yields:

$$\nabla f = \sum_{i=1}^3 \frac{1}{h_i} \frac{\partial f}{\partial u_i} \mathbf{e}_i$$

where  $\{\mathbf{e}_i\}$  is orthogonal.

## Divergence

The divergence of a vector field  $\mathbf{F}$  is:

$$\nabla \cdot \mathbf{F} := \lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \int_{\delta S} \mathbf{F} \cdot d\mathbf{S}$$

for surface  $\delta S$  bounds infinitesimal  $\delta V$ .  
In orthogonal curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (h_1 F_2 h_3) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right\}.$$

## Curl

The curl of a vector field  $\mathbf{F}$  in the direction of unit vector  $\hat{\mathbf{n}}$  is:

$$\hat{\mathbf{n}} \cdot (\nabla \times \mathbf{F}) := \lim_{\delta S \rightarrow 0} \frac{1}{\delta S} \oint_{\delta C} \mathbf{F} \cdot d\mathbf{r}$$

where curve  $\delta C$  encloses plane  $\delta S$ .  
In orthogonal curvilinear coordinates:

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{vmatrix}.$$

## Laplacian

The Laplacian of a scalar field  $f$  is:

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

and in orthogonal curvilinear coordinates:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right\}.$$

The Laplacian of a vector field  $\mathbf{F}$  is:

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

## Vector calculus identities

Particularly for Cartesian coordinates we can apply the suffix notation:

$$\nabla f = \frac{\partial f}{\partial x_i} \mathbf{e}_i$$

$$(\mathbf{u} \cdot \nabla) \mathbf{F} = u_j \frac{\partial}{\partial x_j} F_i$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

$$\nabla \times \mathbf{F} = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \mathbf{e}_i$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \text{ and } \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

If  $\psi$  is a scalar field and  $\mathbf{v}$  a vector field:

$$\nabla \times (\nabla \psi) = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = \mathbf{0}$$

$$\nabla \cdot (\psi \mathbf{v}) = \nabla \psi \cdot \mathbf{v} + \psi \nabla \cdot \mathbf{v}$$

$$\nabla \times (\psi \mathbf{v}) = \nabla \psi \times \mathbf{v} + \psi \nabla \times \mathbf{v}.$$

Let  $\mathbf{r} = x_i \mathbf{e}_i$  and  $r = (x_i^2)^{1/2}$ . Then:

- $\nabla r = \frac{\mathbf{r}}{r}$  and  $\nabla \left( \frac{1}{r} \right) = -\frac{\mathbf{r}}{r^3}$
- $\nabla r^n = n r^{n-2} \mathbf{r}$
- $\nabla \cdot \mathbf{r} = 3$  and  $\nabla \times \mathbf{r} = \mathbf{0}$
- $\nabla \times (\mathbf{c} \times \mathbf{r}) = 2\mathbf{c}$
- $\nabla \cdot (\mathbf{c} \times \mathbf{r}) = \mathbf{0}$  for constant  $\mathbf{c}$ .

## Divergence theorem

Let surface  $S$  enclose volume  $V$ . Then:

$$\iiint_V \nabla \cdot \mathbf{F} dV = \oint_S \mathbf{F} \cdot d\mathbf{S}$$

where  $\mathbf{F}$  is a vector field.

## Stokes' theorem

Let closed curve  $C$  bound open surface  $S$  and let  $\mathbf{F}$  be a vector field. Then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

for  $C$  is traversed in anticlockwise sense.

## Dirac delta function

The Dirac delta in 1D is defined as:

$$\delta(x - a) := \begin{cases} \infty & x = a \\ 0 & \text{otherwise.} \end{cases}$$

In three dimensions this becomes:

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) := \delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0)$$

where  $(x, y, z)$  are Cartesian coordinates.

In orthogonal curvilinear coordinates:

$$\delta(\mathbf{r} - \mathbf{a}) = \frac{1}{h_1 h_2 h_3} \delta(u_1 - a_1) \cdot \delta(u_2 - a_2) \delta(u_3 - a_3).$$

Importantly we have the **sift** property:

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

which yields:

$$x \delta(x) = 0 \text{ and } \delta(cx) = \frac{1}{|c|} \delta(x).$$

If simple solutions of  $g(x) = 0$  are  $x_i$ :

$$\int_{-\infty}^{\infty} f(x) \delta(g(x)) dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}.$$

## Coulomb's law

Consider charges  $q$  and  $q_1$  at positions  $\mathbf{r}$  and  $\mathbf{r}_1$ . The force on charge  $q$  at  $\mathbf{r}$  due to charge  $q_1$  at  $\mathbf{r}_1$  is:

$$\mathbf{F}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{qq_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$

where  $qq_1 > 0$  denotes repulsion.

The permittivity of free space is given by:

$$\epsilon_0 = 8.85419 \times 10^{-12} \text{C}^2 \text{N}^{-1} \text{m}^{-2}.$$

Charge has units Coulombs (C) and an electron has charge  $-1.60218 \times 10^{-19} \text{C}$ .

## Electric fields

The electric field is induced by a charge distribution and defined in terms of the force on a small positive test charge  $q$ :

$$\mathbf{E}(\mathbf{r}) := \lim_{q \rightarrow 0} \frac{1}{q} \mathbf{F}.$$

Then for our two charges  $q$  and  $q_1$ :

$$\mathbf{F}_1(\mathbf{r}) = q\mathbf{E}_1(\mathbf{r})$$

where  $q_1$  produces electric field  $\mathbf{E}_1$ .

$$\therefore \mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}$$

## Principle of superposition

For a set of charges  $q_i$  at position  $\mathbf{r}_i$  the total electric field at  $\mathbf{r}$  is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}.$$

For object with **charge density**  $\rho(\mathbf{r}')$  its overall electric field at  $\mathbf{r}$  is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

where  $\rho(\mathbf{r}')$  is charge divided by volume.

## Electrostatic Maxwell's equations

Because  $\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$ :

$$\mathbf{E}(\mathbf{r}) = -\nabla \left( \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \right)$$

and therefore for all static electric fields:

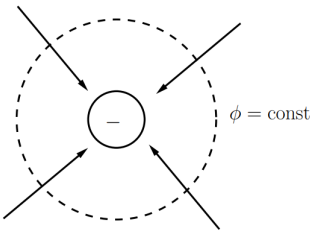
$$\nabla \times \mathbf{E} = \mathbf{0}.$$

$\mathbf{E}$  is a **conservative** vector field where its line integral is **independent** of path. Furthermore it may be written as:

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$$

for  $\phi(\mathbf{r})$  is the potential of  $\mathbf{E}$ .

$$\therefore \phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$



The **potential difference** between two points  $A$  and  $B$  is the energy per unit charge needed to move a small charge  $q$  from  $A$  to  $B$ :

$$\begin{aligned} V_{A \rightarrow B} &= \lim_{q \rightarrow 0} \frac{1}{q} W_{A \rightarrow B} \\ &= -\frac{1}{q} \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \phi_B - \phi_A. \end{aligned}$$

A charge distribution  $\rho(\mathbf{r}')$  in an external electric field has potential energy:

$$W = \int_V \rho(\mathbf{r}') \phi_{ext}(\mathbf{r}') dV'.$$

Because  $\nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}')$ :

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}.$$

## Electric dipoles

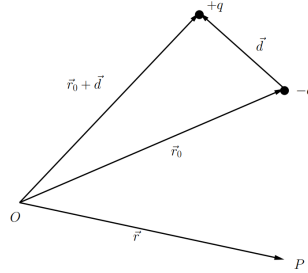
An electric dipole at  $\mathbf{r}_0$  is defined as two charges  $-q$  at  $\mathbf{r}_0$  and  $+q$  at  $\mathbf{r}_0 + \mathbf{d}$  which generates **dipole moment**:

$$\mathbf{p} = q\mathbf{d}$$

and in the dipole limit this is defined as:

$$\mathbf{p} := \lim_{\substack{q \rightarrow \infty \\ d \rightarrow 0}} q\mathbf{d}$$

known as an ideal dipole.



The electrostatic potential generated by this ideal dipole at  $\mathbf{r}_0$  is given by:

$$\begin{aligned} \phi(\mathbf{r}) &= \phi_q + \phi_{-q} \\ &= \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|\mathbf{r} - \mathbf{r}_0 - \mathbf{d}|} - \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) \\ &\approx \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3} \end{aligned}$$

for the first term is expanded in powers of  $-\mathbf{d}$  about  $\mathbf{r} - \mathbf{r}_0$ .

The electric field generated is:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \left[ -\frac{\mathbf{p}}{|\mathbf{r} - \mathbf{r}_0|^3} \right. \\ &\quad \left. + \frac{3\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^5} (\mathbf{r} - \mathbf{r}_0) \right]. \end{aligned}$$

If the ideal dipole is at the origin:

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \\ \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \left( \frac{3\mathbf{p} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{p}}{r^3} \right). \end{aligned}$$

Let ideal dipole moment  $\mathbf{p}$  be parallel to the  $z$ -axis. Then in spherical coordinates  $(r, \theta, \chi)$ ,  $\mathbf{r} = r\mathbf{e}_r$ ,  $\mathbf{p} = p\mathbf{e}_z$  and:

$$\phi(\mathbf{r}) = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0} \left( \frac{2\cos\theta}{r^3} \mathbf{e}_r + \frac{\sin\theta}{r^3} \mathbf{e}_\theta \right).$$



## Force, torque and energy

The **force** on a **dipole** at  $\mathbf{r}$  from external electric field  $\mathbf{E}_{ext}(\mathbf{r})$  is:

$$\begin{aligned} \mathbf{F} &= -q\mathbf{E}_{ext}(\mathbf{r}) + q\mathbf{E}_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx (\mathbf{p} \cdot \nabla) \mathbf{E}_{ext}(\mathbf{r}). \end{aligned}$$

The **torque** on a dipole at  $\mathbf{r}$  about the axis  $\mathbf{r}$  due to  $\mathbf{E}_{ext}(\mathbf{r})$  is:

$$\begin{aligned} \mathbf{G} &= \boldsymbol{\tau}_{-q} + \boldsymbol{\tau}_q \\ &= -q\mathbf{0} \times \mathbf{E}_{ext}(\mathbf{r}) + q\mathbf{d} \times \mathbf{E}_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx \mathbf{p} \times \mathbf{E}_{ext}(\mathbf{r}). \end{aligned}$$

The **energy** of a dipole at  $\mathbf{r}$  from external electric field  $\mathbf{E}_{ext}(\mathbf{r}) = -\nabla\phi_{ext}(\mathbf{r})$  is:

$$\begin{aligned} W &= -q\phi_{ext}(\mathbf{r}) + q\phi_{ext}(\mathbf{r} + \mathbf{d}) \\ &\approx -\mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{r}) \end{aligned}$$

and  $\mathbf{F} = -\nabla W$ .

## Multipole expansion

Consider object with volume  $V$  and charge distribution  $\rho(\mathbf{r}')$ . Let origin be in the object. Then the potential at  $\mathbf{r}$  is:

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \\ &\approx \frac{1}{4\pi\epsilon_0} \left( \frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{Q_{ij}x_i x_j}{2r^5} \right) \end{aligned}$$

where  $Q$  is the **total charge** in  $V$ :

$$Q = \int_V \rho(\mathbf{r}') dV'$$

$\mathbf{p}$  the **dipole moment** about the origin:

$$\mathbf{p} = \int_V \mathbf{r}' \rho(\mathbf{r}') dV'$$

and  $Q_{ij}$  the **quadrupole tensor**:

$$Q_{ij} = \int_V \rho(\mathbf{r}') \left[ 3x'_i x'_j - (r')^2 \delta_{ij} \right] dV'.$$

If  $Q \neq 0$  then in the far zone ( $r \gg r_0$ ) the first term (monopole term) dominates.

If  $Q = 0$  and  $\mathbf{p} = \mathbf{0}$  then the third term (quadrupole term) dominates in the far zone and etc.

## Interaction energy

By expanding  $\phi_{ext}(\mathbf{r})$  about  $\mathbf{r} = \mathbf{0}$ :

$$\begin{aligned} W &= \int_V \rho(\mathbf{r}') \phi_{ext}(\mathbf{r}') dV' \\ &= Q\phi_{ext}(\mathbf{0}) - \mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{0}) \\ &\quad - \frac{1}{6} Q_{ij} \frac{\partial(\mathbf{E}_{ext}(\mathbf{0}))_i}{\partial x_j} + \dots \end{aligned}$$

and is the potential energy of a charge distribution  $\rho(\mathbf{r})$  in  $\mathbf{E}_{ext}$ .

## Gauss' law

For object with charge distribution  $\rho(\mathbf{r}')$  and volume  $V$  enclosed by surface  $S$ :

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\epsilon_0}$$

where  $Q_{enc}$  is total charge enclosed by  $V$ :

$$Q_{enc} = \int_V \rho(\mathbf{r}') dV'$$

and is useful for symmetric problems.

## Boundaries

Let  $\sigma$  be the charge density of a surface separating electric fields  $\mathbf{E}_1$  and  $\mathbf{E}_2$ .

1. Normal component of electric field is discontinuous across surface by:

$$\hat{\mathbf{n}} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \frac{\sigma}{\epsilon_0}.$$

2. Tangential component of electric field is continuous across surface:

$$\mathbf{E}_{\parallel} := \hat{\mathbf{n}} \times \mathbf{E}_1 = \hat{\mathbf{n}} \times \mathbf{E}_2.$$

## Conductors

Conductors have surplus electrons that can move freely when an electric field is applied. **In electrostatics:**

1. Conductors are in equilibrium, all charges are at rest and reside on the surface of the conductor.

Hence inside a conductor  $\rho(\mathbf{r}) = 0$ ,  $\mathbf{E}(\mathbf{r}) = \mathbf{0}$  and  $\phi = \text{constant}$ .

2. An electric field is always normal to the surface of a conductor:

$$E_{\perp} = \frac{\sigma}{\epsilon_0} \text{ and } E_{\parallel} = 0.$$

The presence of an external electric field induces a charge distribution  $\sigma$  on the surface of our conductor. This changes the external electric field as it needs to be normal to the surface of the conductor.

## Poisson's equation

Because  $\mathbf{E} = -\nabla\phi$  and  $\nabla \cdot \mathbf{E} = \rho(\mathbf{r})/\epsilon_0$ :

$$\nabla^2 \phi = -\frac{\rho(\mathbf{r})}{\epsilon_0}.$$

We can solve this by direct integration or using the **method of images**.

Given volume under consideration place fictitious charge outside the volume such that the system still satisfies Poisson's equation with boundary conditions.

This potential is our solution.

## Electrostatic energy

The work needed to move point charge  $q$  from  $\mathbf{r}_A$  to  $\mathbf{r}_B$  in  $\mathbf{E}(\mathbf{r})$  is:

$$W_{A \rightarrow B} = qV_{A \rightarrow B}.$$

Then  $W_{\infty \rightarrow B} = q\phi(\mathbf{r}_B)$  since potential  $\phi$  vanishes at infinity.

Generalising, the work needed to move a system of  $n$  charges  $q_i$  from infinity to  $\mathbf{r}$  is a double sum with overcounting as each charge contributes to the electric field:

$$W_e = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i,j(i \neq j)}^n \frac{q_i q_j}{|\mathbf{r}_j - \mathbf{r}_i|}.$$

Furthermore the energy needed to move a continuous charge distribution  $\rho(\mathbf{r}')$  from infinity to position  $\mathbf{r}$  is:

$$\begin{aligned} W_e &= \frac{1}{2} \int_V \rho(\mathbf{r}) \phi(\mathbf{r}) dV \\ &= \frac{\epsilon_0}{2} \int_V |\mathbf{E}(\mathbf{r})|^2 dV. \end{aligned}$$

## Capacitors

A capacitor is formed by two conductors 1 and 2 with equal and opposite charges  $Q$  and  $-Q$ . The **capacitance** ( $1\text{CV}^{-1}$ ) of a capacitor is defined as:

$$C := \frac{Q}{V}$$

where  $Q = \sigma A$  for  $A$  is the surface area of **one** conductor and potential difference  $V = \phi_1 - \phi_2$  from the conductors.

The energy stored in a capacitor is the amount of work done to move charge across the two conductors. So to move charge  $dq$  from conductor with  $+q$ :

$$dW = \left(\frac{q}{C}\right) dq$$

and integrating this up to  $Q$  gives:

$$W = \frac{1}{2} \frac{Q^2}{C}.$$

## Currents

An elementary current is generated by a charge  $q$  moving at velocity  $\mathbf{v}$ .

The **bulk current density** is:

$$\mathbf{J}(\mathbf{r}) := \rho(\mathbf{r})\mathbf{v}$$

for  $\rho(\mathbf{r})$  is the volume charge density.

The **surface current density** is:

$$\mathbf{K}(\mathbf{r}) := \sigma(\mathbf{r})\mathbf{v}$$

for  $\sigma(\mathbf{r})$  is the surface charge density.

The **line charge density** is:

$$\mathbf{I}(\mathbf{r}) := \lambda(\mathbf{r})\mathbf{v}$$

for  $\lambda(\mathbf{r})$  is the line charge density.

The infinitesimal **current element** is:

$$d\mathbf{I}(\mathbf{r}) := \begin{cases} \mathbf{J}(\mathbf{r})dV & \text{volume current} \\ \mathbf{K}(\mathbf{r})dS & \text{surface current} \\ \mathbf{I}(\mathbf{r})dr & \text{line current} \end{cases}$$

Current  $I$  has units Ampères (A) but the infinitesimal current element  $d\mathbf{I}(\mathbf{r})$  has units of current/volume, etc.

Note:  $1\text{A} := 1\text{Cs}^{-1}$

Consider volume  $V$  bounded by surface  $S$  with total charge  $Q$ . Because the **total charge is conserved**:

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0$$

$$Q = \int_V \rho(\mathbf{r}, t) dV.$$

An electric field can induce a current in a conductor, via **Ohm's law**:

$$\mathbf{J} \approx \sigma_{ij} E_j$$

where  $\sigma_{ij}$  is the conductivity tensor.

1. Perfect conductors:  
 $\sigma \rightarrow \infty$  and  $\mathbf{E} = \mathbf{0}$ .

2. Insulators:  $\sigma = 0$ .

The **electromotive force** (emf) is:

$$\begin{aligned} \mathcal{E}_{1 \rightarrow 2} &:= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{r} \\ &= \phi(\mathbf{r}_1) - \phi(\mathbf{r}_2) \end{aligned}$$

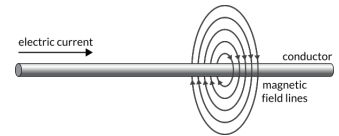
since  $\mathbf{E} = -\nabla\phi$ .

## Biot-Savart law

The magnetic field at  $\mathbf{r}$  generated by a static current loop carrying current  $I$  is:

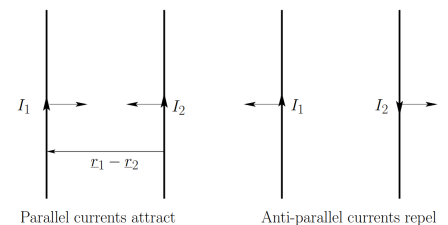
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

and is the right grip rule.



Magnetic fields have units Teslas (T), where  $\text{NC}^{-1}\text{m}^{-1}\text{s} = \text{T}$  and:

$$\mu_0 = 1.25664 \dots \times 10^{-6} \text{NA}^{-2}.$$



However perpendicular currents do not induce force.

**Lozentz force**

Through physical experiments the **force density** on a charge distribution at  $\mathbf{r}$  in electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$  is:

$$\mathbf{f} := \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

Let point charge  $q$  at  $\mathbf{r}'$  be moving at  $\mathbf{v}$ . The net force on  $q$  is the integral of the force density over volume  $V$ :

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

since  $\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}')$  and  $\mathbf{J} = \rho(\mathbf{r})\mathbf{v}$ .

**Magnetostatic Maxwell's equations****Ampère's law**

normal and tangent components of conducting surfaces

**Magnetic dipoles**