Vector products

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$$

$$\boldsymbol{a}\times\boldsymbol{b} = ab\sin\theta\hat{\boldsymbol{n}}$$

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear **more than twice** in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{ijk} = -\epsilon_{ijk} = -\epsilon_{ijk} \end{aligned}$$

and we have the following identities:

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i\boldsymbol{e}_i = a_i\boldsymbol{e}_i$$

$$A\boldsymbol{x} = a_{ij}x_j\boldsymbol{e}_i \text{ for } m\times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\dots]_j\delta_{jk} = [\dots]_k$$

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$$

$$\boldsymbol{e}_i\times\boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k$$

$$\boldsymbol{a}\times\boldsymbol{b} = \epsilon_{ijk}a_jb_k\boldsymbol{e}_i$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \epsilon_{ijk}a_ib_jc_k$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk}\epsilon_{ijk} = 6.$$

Transformations

Let matrix L relate basis $\{\boldsymbol{e}_i\}$ to basis $\{\boldsymbol{e}_i'\}$ with rule:

$$\boldsymbol{e}_i' = \ell_{ij}\boldsymbol{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then $L^TL = LL^T = I$, and:

$$\ell_{ik}\ell_{jk} = \ell_{ki}\ell_{kj} = \delta_{ij}$$

$$p_i' = \ell_{ij}p_j \text{ for } \boldsymbol{p} = p_i\boldsymbol{e}_i = p_i'\boldsymbol{e}_i'.$$

Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip}\ell_{jq}\ell_{kr}T_{pqr}$$

which relates frame S in $\{\boldsymbol{e}_i\}$ to frame S' in $\{\boldsymbol{e}_i'\}$ with rule $\boldsymbol{e}_i' = \ell_{ij}\boldsymbol{e}_j$, etc.

Properties of tensors:

1. The addition of two rank n tensors is also a rank n tensor.
2. The multiplication of a rank m tensor with a rank n tensor yields a rank $m+n$ tensor.
3. If $T_{ijk\dots s}$ is a rank m tensor then $T_{iik\dots s}$ is a rank $m-2$ tensor.
4. If T_{ij} is a tensor then T_{ji} is also a tensor. Explicitly:

$$T'_{ij} = \ell_{ip}\ell_{jq}T_{pq} \implies T' = LTL^T$$
$$T'_{ji} = \ell_{jp}\ell_{iq}T_{pq}.$$

Symmetric tensors

T_{ij} is a symmetric tensor when $T_{ij} = T_{ji}$ in frame S. Then $T'_{ij} = T'_{ji}$ in frame S'.

Similarly T_{ij} is an anti-symmetric tensor if $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$.

Finally any tensor can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij}+T_{ji}) + \frac{1}{2}(T_{ij}-T_{ji}).$$

Quotient theorem

Consider 9 entities T_{ij} in frame S and T'_{ij} in frame S'. Let $b_i = T_{ij}a_j$ where a_j is a vector. If b_i always transforms as a vector then T_{ij} is a rank 2 tensor.

Generalising, let $R_{ijk\dots r}$ be a rank m tensor and $T_{ijk\dots s}$ a set of 3^n numbers where $n>m$. If $T_{ijk\dots s}R_{ijk\dots r}$ is a rank $n-m$ tensor then $T_{ijk\dots s}$ is a rank n tensor.

Matrices

We define a $m\times n$ matrix A as $(A)_{ij} = a_{ij}$ where $i = 1,\dots,m$ and $j = 1,\dots,n$.

- $\mathrm{Tr}\,A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^TA^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a 3×3 matrix A is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn}a_{1l}a_{2m}a_{3n} \\ &= \epsilon_{lmn}a_{l1}a_{m2}a_{n3}. \end{aligned}$$

Furthermore:

$$\epsilon_{ijk}\det A = \epsilon_{lmn}a_{il}a_{jm}a_{kn}$$

$$\epsilon_{lmn}\det A = \epsilon_{ijk}a_{il}a_{jkm}a_{kn}$$

$$\det A = \frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}a_{il}a_{jm}a_{kn}.$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3. $\det A = \det A^T$
4. $\det(AB) = \det A\cdot\det B$

These also apply to columns. Finally:

$$\epsilon_{ijk}\epsilon_{lmn}\det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting $A = I$ yields:

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

Linear equations

Let $\boldsymbol{y} = A\boldsymbol{x}$. Then $x_i = A^{-1}_{ij}y_i$ with:

$$\begin{aligned} A^{-1}_{ij} &= \frac{1}{2}\frac{1}{\det A}\epsilon_{imn}\epsilon_{jpq}a_{pm}a_{qn} \\ &= \frac{1}{\det A}C^T_{ij} \end{aligned}$$

where C is the cofactor matrix of A.

Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L)\ell_{ip}\ell_{jq}T_{pq}$$

where $(L)_{ij} = \ell_{ij}$ and $\det L = \pm 1$.

Pseudovectors are rank 1 pseudotensors.

Invariant tensors

Tensor T is invariant or isotropic if:

$$T_{ijk\dots} = \ell_{i\alpha}\ell_{j\beta}\ell_{k\gamma}\cdots T_{\alpha\beta\gamma\dots}$$

for every orthogonal matrix L.

- If a_{ij} is a rank 2 invariant tensor then $a_{ij} = \lambda\delta_{ij}$.
- The most general rank 3 invariant pseudotensor is $a_{ijk} = \lambda\epsilon_{ijk}$. There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

Rotation tensors

The clockwise rotation of position vector $\boldsymbol{x}$ to $\boldsymbol{y}$ about unit vector $\hat{\boldsymbol{n}}$ is given by:

$$y_i = R_{ij}(\theta, \hat{\boldsymbol{n}})x_j$$

$$R_{ij}(\theta, \hat{\boldsymbol{n}}) = \delta_{ij}\cos\theta + (1-\cos\theta)n_i n_j - \epsilon_{ijk}n_k\sin\theta$$

and is the rotation tensor.

Reflections and inversions

The reflection of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ in plane with unit vector $\hat{\boldsymbol{n}}$ is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector $\boldsymbol{x}$ to $\boldsymbol{y}$ is given by $\boldsymbol{y} = -\boldsymbol{x}$ and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

Projections

We define P to be a parallel projection operator to vector $\boldsymbol{u}$ if:

$$P\boldsymbol{u} = \boldsymbol{u} \text{ and } P\boldsymbol{v} = \boldsymbol{0}$$

where $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{0}$. Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define Q to be an orthogonal projection to vector $\boldsymbol{u}$ if:

$$Q\boldsymbol{u} = \boldsymbol{0} \text{ and } Q\boldsymbol{v} = \boldsymbol{v}.$$

Here $Q = I - P$.

Inertia tensors

Let $\boldsymbol{L}$ denote the angular momentum of a rigid body about the origin of mass m, volume V and density ρ at position $\boldsymbol{r}$ with velocity $\boldsymbol{v}$. Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2\delta_{ij} - x_i x_j)\mathrm{d}V$$

where $I_{ij}(O)$ is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2}I_{ij}\omega_i\omega_j = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{\omega}.$$

Parallel axis theorem

Consider the same rigid body now with centre of mass G and let $\overrightarrow{OG} = \boldsymbol{R}$. Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2\delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\boldsymbol{r}')\mathrm{d}V'.$$

Diagonalisation

Taylor series

In the one-dimensional case we have:

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(x-a)^n$$

and is f expanded about $x = a$.

$$\therefore f(x+a) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x)a^n = \exp\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x)$$

Then for three dimensions:

$$\phi(\boldsymbol{r}+\boldsymbol{a}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)^n\phi(\boldsymbol{r}) = \exp(\boldsymbol{a}\cdot\boldsymbol{\nabla}_r)\phi(\boldsymbol{r}).$$

Curvilinear coordinates

Let x_i denote Cartesian coordinates and u_i denote curvilinear coordinates. Then:

$$(x_1, x_2, x_3) \to (u_1, u_2, u_3)$$

where each $u_i = u_i(x_1, x_2, x_3)$.

Scale factors

If component u_1 of $\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, u_3)$ is changed by $\mathrm{d}u_1$ then $\boldsymbol{r} \to \boldsymbol{r} + \mathrm{d}\boldsymbol{r}$, where:

$$\mathrm{d}\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial u_1}\mathrm{d}u_1 := h_1\boldsymbol{e}_1\mathrm{d}u_1.$$

h_1 is the scale factor of unit vector $\boldsymbol{e}_1$:

$$h_1 = \left|\frac{\partial\boldsymbol{r}}{\partial u_1}\right| \text{ and } \boldsymbol{e}_1 = \frac{1}{h_1}\frac{\partial\boldsymbol{r}}{\partial u_1}.$$

If $\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$ then u_i is an **orthogonal** curvilinear coordinate system.

Length, area and volume

The vector length of $\boldsymbol{r}$ when $u_i \to u_i + \mathrm{d}u_i$ is changed for $\forall i = 1, 2, 3$ is defined as:

$$\mathrm{d}\boldsymbol{r} = \sum_{i=1}^{3}h_i\mathrm{d}u_i\boldsymbol{e}_i.$$

Cylindrical coordinates

Spherical coordinates