Honours Analysis Examples

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Contents

1 Lebesgue integrals **3**
1.1 Characteristic and step functions 3

1 Lebesgue integrals

1.1 Characteristic and step functions

1. Let ϕ and ψ be step functions. Show that the following are also step functions:

   - $\alpha\phi + \beta\psi$
   - $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\}$
   - $\phi\psi$.

   Firstly define step function ϕ wrt finite set $\{x_0, x_1 \dots, x_n\}$ and step function ψ wrt finite set $\{y_0, y_1 \dots, y_n\}$. Their linear combination is a step function wrt the union of these two finite sets.

   The rest are proven similarly via construction.

2. Show that ϕ is a step function **if and only if**:

   $$\phi(x) = \sum_{j=1}^{n} c_j \chi_{J_j}(x)$$

   where c_j is some constant and J_j some interval.

   From the definition of ϕ we can write:

   $$\phi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1}, x_j)}(x) + \sum_{j=0}^{n} \phi(x_j) \chi_{\{x_j\}}$$

   and the opposite direction is shown by the first example and the fact that a characteristic function is a step function.