D: Functions

A function $f : X \to Y$ is an assignment of an element of Y to each element of X.

1. f is **injective** if:
$$\forall x_1, x_2 \in X; f(x_1) = f(x_2) \implies x_1 = x_2.$$
2. f is **surjective** if:
$$\forall y \in Y; \exists x \in X : y = f(x).$$
3. f is **bijective** if it is injective and surjective.

D: Groups

A group G is a set defined with:

1. Composition operator $(\cdot)$ such that $x \cdot y = xy$.
2. $\forall x, y, z \in G$; $(xy)z = x(yz)$
3. $\exists e \in G : ex = xe = x$ for $\forall x \in G$.
4. $\exists x^{-1} \in G : xx^{-1} = x^{-1}x = e$ for $\forall x \in G$.

G is **Abelian** if $\forall x, y \in G; xy = yx$.

D1.2.1(i): Fields

A field F is a set defined with:

1. Addition function $(+)$:
$$(+) : F \times F \to F; (\lambda, \mu) \mapsto \lambda + \mu$$
2. Multiplication function $(\cdot)$:
$$(\cdot) : F \times F \to F; (\lambda, \mu) \mapsto \lambda \cdot \mu$$
3. $\exists 0_F, 1_F \in F$ where $0_F \neq 1_F$ such that $(F, +)$ and $(F \backslash \{0_F\}, \cdot)$ form Abelian groups.
4. $\exists(-\lambda) \in F : \lambda + (-\lambda) = 0_F$
5. $\exists(\lambda^{-1}) \in F : \lambda \cdot (\lambda^{-1}) = 1_F$
6. $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu \in F$

D1.2.1(ii): Vector spaces

A vector space V over a field F is an Abelian group $V := (V, +)$ with mapping:
$$F \times V \to V : (\lambda, \boldsymbol{v} \mapsto \lambda\boldsymbol{v})$$
where for $\forall \lambda, \mu \in F$ and $\forall \boldsymbol{v}, \boldsymbol{w} \in V$:

1. $\lambda(\boldsymbol{v} + \boldsymbol{w}) = (\lambda\boldsymbol{v}) + (\mu\boldsymbol{w})$
2. $(\lambda + \mu)\boldsymbol{v} = (\lambda\boldsymbol{v}) + (\mu\boldsymbol{w})$
3. $\lambda(\mu\boldsymbol{v}) = (\lambda\mu)\boldsymbol{v}$
4. $1_F\boldsymbol{v} = \boldsymbol{v}$

and is a F-vector space.

Remark

Let V be a F-vector space where $\boldsymbol{v} \in V$.

1. $0\boldsymbol{v} = 0$
2. $(-1)\boldsymbol{v} = -\boldsymbol{v}$
3. $\lambda\mathbf{0} = \mathbf{0}$ for $\forall \lambda \in F$.

D: Cartesian products

The Cartesian product of sets $X_1, \dots, X_n$ is defined as:
$$X_1 \times \dots \times X_n := \{(x_1, \dots, x_n) : x_i \in X_i\}$$
where $1 \leq i \leq n$.

The projection of a Cartesian product is:
$$\mathrm{pr}_i : X_1 \times \dots \times X_n \to X_i; \quad (x_1, \dots, x_n) \mapsto x_i$$

D1.4.1: Vector subspaces

A vector subspace U of F-vector space V has the follwing properties:

1. $U \subset V$ and $\mathbf{0} \in U$.
2. Let $\boldsymbol{u}, \boldsymbol{v} \in U$ and $\lambda \in F$. Then $\boldsymbol{u} + \boldsymbol{v} \in U$ and $\lambda\boldsymbol{u} \in U$.

and is also a vector space.

P1.4.5

Let $T \subset V$ where V is a F-vector space. Then for all vector subspaces containing T, there exists a smallest vector subspace:
$$\mathrm{span}(T) = \langle T \rangle_F \subset V$$
known as the vector subspace generated by T, or the span of T.

D1.4.7: Generating set

Let $T \subset V$ where V is a F-vector space. T is a generating set of V if:
$$\mathrm{span}(T) = V$$
and is the linear combination of vectors in T over field F.

D1.4.9: Power sets

The power set of set X is:
$$\mathcal{P}(X) := \{U : U \subseteq X\}.$$
Let $\mathcal{U} \subseteq \mathcal{P}(X)$. Then:
$$\bigcup_{U \in \mathcal{U}} U := \{x \in X : (\exists U \in \mathcal{U} : x \in U)\}$$
$$\bigcap_{U \in \mathcal{U}} U := \{x \in X : \forall U \in \mathcal{U}; x \in U\}.$$

D1.5.1: Linear independence

Let V be a F-vector space and $L \subseteq V$. L is linearly independent if:
$$\alpha_1\boldsymbol{v}_1 + \dots + \alpha_r\boldsymbol{v}_r = \mathbf{0} \implies \alpha_1 = \dots = \alpha_r = 0$$
where $\boldsymbol{v}_i \in L$.

D1.5.8: Basis

A basis of a vector space V is a linearly independent generating set in V.

T1.5.11

Let V be a F-vector space.

Then $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ is a basis of V **iff**:
$$\Phi : F^r \to V;$$
$$(\alpha_1, \dots, \alpha_r) \mapsto \alpha_1\boldsymbol{v}_1 + \dots + \alpha_r\boldsymbol{v}_r$$
is a bijection.

T1.5.12

Let V be a vector space and $E \subseteq V$. Then the following statements are equivalent:

1. E is a basis of V.
2. E is minimal among all generating sets, or that $E \backslash \{\boldsymbol{v}\}$ is not a basis for $\forall \boldsymbol{v} \in V$.
3. E is maximal amongst all linearly independent subsets. i.e. $E \cup \{\boldsymbol{v}\}$ is not linearly independent.

C1.5.13

Every finitely generated vector space has a finite basis. (any vector space too!)

T1.5.14

Let V be a vector space.

1. Let $L \subseteq V$ be linearly independent and set E be minimal amongst all generating sets of V. Let $L \subseteq E$. Then E is a basis of V.
2. Let $E \subseteq V$ be a generating set and L be maximal amongst all linearly independent subsets of V.

   Let $L \subseteq E$. Then E is a basis of V.

D1.5.15

Let X be a set and F be a field. Then:
$$\mathrm{maps}(X, F) := \{f : (\forall f : X \to F)\}$$
and is a F-vector space under pointwise addition and multiplication via scalars.

Remark

The subset of all mappings which sends almost all elements of X to 0 is defined: (all but finitely many)

$$F\langle X\rangle \subseteq \mathrm{maps}(X, F)$$

and is a vector subspace.

T1.5.16

T1.6.1

Let V be a vector space. Let $L \subset V$ be a linearly independent subset and $E \subseteq V$ a generating set. Then $|L| \leq |E|$.

T1.6.2

L1.6.3

C1.6.4

D1.6.5: Dimension

The dimension of finite F-vector space V is the cardinality of one its basis.

For infinite vector spaces: $\dim(V) = \infty$.

C1.6.7

Let V be a finitely generated vector space.

1. Every linearly independent $L \subseteq V$ has **at most** $\dim(V)$ elements and if $|L| = \dim(V)$ then L is a basis.
2. Every generating set $E \subseteq V$ has **at least** $\dim(V)$ elements and if $|E| = \dim(V)$ then E is a basis.

C1.6.8

A proper vector subspace of a vector space with finite dimension has itself a strictly smaller dimension.

T1.6.10

Let V be a vector space and $U, W \subseteq V$ be vector subspaces. Then:

$$\begin{aligned}&\dim(U + W) + \dim(U \cap W)\\ &= \dim(U) + \dim(W).\end{aligned}$$