Probability distributions

The probablity of an event in a trial is:

$$\mathbb{P}(\text{event}) := \lim_{N\to\infty} \frac{n}{N}$$

given n occurrences in N trials.
For discrete probabilities:

$$\sum_{i=1}^{q} \mathbb{P}(i) = 1$$

$$\mathbb{P}(i \text{ or } j) = \mathbb{P}(i) + \mathbb{P}(j)$$
$$\mathbb{P}(i \text{ and } j) = \mathbb{P}(i)\mathbb{P}(j).$$

Given continuous random variables:

$$\mathbb{P}([x, x + \mathrm{d}x]) = P(x)\mathrm{d}x$$

for P is the probability density function:

$$\int_{-\infty}^{\infty} P(x)\mathrm{d}x = 1.$$

We define the **mean** and **variance** as:

$$\overline{x} = \sum_{i=1}^{q} x_i P_i \text{ or } \int_{-\infty}^{\infty} xP(x)\mathrm{d}x$$

$$\begin{aligned}\overline{\Delta x^2} &= \sum_{i=1}^{q} (x_i - \overline{x})^2 P_i \\ &= \int_{-\infty}^{\infty} (x - \overline{x})^2 P(x)\mathrm{d}x \\ &= \overline{x^2} - (\overline{x})^2.\end{aligned}$$

The **standard deviation** is the square root of the variance $\left(\overline{\Delta x^2}\right)^{1/2}$ and:

$$\overline{f(x)} = \int_{-\infty}^{\infty} f(x)P(x)\mathrm{d}x.$$

Binomial distribution

The probability of observing n events each with probability p in N trials is:

$$P_n = \binom{N}{n} p^n (1-p)^{N-n}$$

where $\binom{N}{n} = \dfrac{N!}{n!(N-n)!}$ with:

$$\overline{n} = Np \text{ and } \overline{\Delta n^2} = Np(1-p)$$

since we have that:

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}$$

$$\begin{aligned}f(\alpha) &= \sum_{n=0}^{N} \binom{N}{n} (p\alpha)^n (1-p)^{N-n} \\ &= (p\alpha + 1 - p)^N.\end{aligned}$$

Note that $\binom{N}{n}$ denotes ways to pick n items from N items. For large N:

$$\ln(N!) \approx N\ln(N) - N$$

known as **Stirling's approximation**.

We also define the **fractional deviation** as the deviation on the scale of the mean:

$$\frac{\left(\overline{\Delta x^2}\right)^{1/2}}{\overline{n}} = \frac{1}{N^{1/2}}.$$

Taylor expansions

Let $s(n)$ be expanded at $n = a$:

$$\begin{aligned}s(n) &= s(a) + s'(a)(n-a) \\ &\quad + \frac{1}{2}s''(a)(n-a)^2 + \mathcal{O}[(n-a)^3].\end{aligned}$$

Poisson distribution

Let $N \gg n$ and let p be the probability of an event in a trial. Assume that as $N \to \infty$, $p \to 0$. Under such conditions the binomial probability of observing n events in N trials is:

$$P_n \approx (\overline{n})^n \frac{\exp(-\overline{n})}{n!}$$

with mean and variance Np.

Gaussian distribution

Let N be very large. Then the binomial distribution becomes Gaussian:

$$P_n \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(n-Np)^2}{2\sigma^2}\right)$$

via Stirling's approximation and Taylor expansions with variance $\sigma^2 = Np(1-p)$ and mean $\mu = Np$.

Microstates and macrostates

A **microstate** is a complete specification of all degrees of freedoms in a system, with respect to a microscopic model.

A **macrostate** is a limited description by the values of observables, like pressure.

Given N molecules with total energy E the **Boltzmann law** defines entropy:

$$S(N, E, \{\alpha\}) := k_B \ln\Big[\Omega(N, E, \{\alpha\})\Big]$$

$$k_B = 1.381 \times 10^{-23} \mathrm{JK}^{-1}$$

where Ω is the number of corresponding microstates to a macrostate N, E with a set of observables α.

i.e. for any macrostate there correspond many microstates. If one molecule has $k \propto V$ microstates in volume V then N molecules have $k^N \propto V^N$ in V.

Two-state model magnets

Consider an array of N magnetic dipoles and total energy E that is subject to a magnetic field $\boldsymbol{H}$.

$$\{\uparrow\downarrow\uparrow\uparrow \dots \downarrow\downarrow\uparrow\uparrow\}$$

Define n to be the number of dipoles with energy $\epsilon_\uparrow = +mH$ (excited state) and the remaining in ground state $\epsilon_\downarrow = -mH$.

Since we can write the total energy E as:

$$mH\big(n - (N-n)\big) = E$$

$$\therefore n = \frac{1}{2}\left(N + \frac{E}{mH}\right)$$

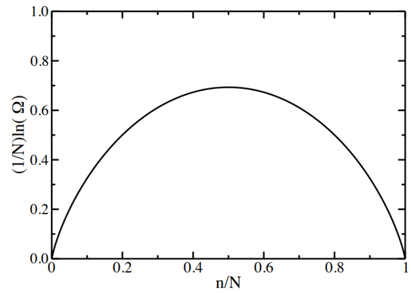
and the **weight** of this macrostate is:

$$\Omega(N, E, n) = \binom{N}{n}.$$

If $N \gg 1$ we use Stirling's approximation and define $x = n/N$:

$$\Omega(N, E, n) \approx \exp\Big[Ns(x)\Big]$$

$$s(x) = -(1-x)\ln(1-x) - x\ln x.$$



For in the $s(x)$ plot above our end points are computed via limits.

Now let the number of excited dipoles be $n = N/2$ and denote n_L as the number of n in the left, from the center of our array.

$$\{\underbrace{\dots\uparrow\downarrow\uparrow\dots}_{n_L}|\dots\downarrow\downarrow\uparrow\dots\}$$

The weight of macrostate n_L now is:

$$\Omega(N, E = 0)$$