

## Vector products

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

## Suffix notation

1. A suffix that appears twice implies a summation.
2. Any suffix cannot appear more than twice in any term.

We define the **Kronecker delta** as:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the **Levi-Civita** as:

$$\epsilon_{ijk} = \begin{cases} +1 & 123, 312, 231 \\ -1 & 132, 213, 321 \\ 0 & \text{repeat indices.} \end{cases}$$

Consequently:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \\ &= -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \end{aligned}$$

and we have the following identities:

$$\mathbf{a} = \sum_{i=1}^3 a_i \mathbf{e}_i = a_i \mathbf{e}_i$$

$$A\mathbf{x} = a_{ij}x_j \mathbf{e}_i \text{ for } m \times n \text{ matrix } A$$

$$\delta_{ii} = 3$$

$$[\dots]_j \delta_{jk} = [\dots]_k$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k$$

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \mathbf{e}_i$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl} \text{ and } \epsilon_{ijk} \epsilon_{ijk} = 6.$$

## Transformations

Let matrix  $L$  relate basis  $\{\mathbf{e}_i\}$  to basis  $\{\mathbf{e}'_i\}$  with rule:

$$\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j \text{ where } (L)_{ij} = \ell_{ij}.$$

Then  $L^T L = L L^T = I$ , and:

$$\ell_{ik} \ell_{jk} = \ell_{ki} \ell_{kj} = \delta_{ij}$$

$$p'_i = \ell_{ij} p_j \text{ for } \mathbf{p} = p_i \mathbf{e}_i = p'_i \mathbf{e}'_i.$$

## Tensors

A rank 3 tensor is defined as:

$$T'_{ijk} = \ell_{ip} \ell_{jq} \ell_{kr} T_{pqr}$$

which relates frame  $S$  in  $\{\mathbf{e}_i\}$  to frame  $S'$  in  $\{\mathbf{e}'_i\}$  with rule  $\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j$ , etc.

Properties of tensors:

1. The addition of two rank  $n$  tensors is also a rank  $n$  tensor.
2. The multiplication of a rank  $m$  tensor with a rank  $n$  tensor yields a rank  $m + n$  tensor.
3. If  $T_{ijk\dots s}$  is a rank  $m$  tensor then  $T_{\mathbf{ii}k\dots s}$  is a rank  $m - 2$  tensor.
4. If  $T_{ij}$  is a tensor then  $T_{ji}$  is also a tensor. Explicitly:

$$\begin{aligned} T'_{ij} &= \ell_{ip} \ell_{jq} T_{pq} \implies T' = L T L^T \\ T'_{ji} &= \ell_{jp} \ell_{iq} T_{pq}. \end{aligned}$$

## Symmetric tensors

$T_{ij}$  is a symmetric tensor when  $T_{ij} = T_{ji}$  in frame  $S$ . Then  $T'_{ij} = T'_{ji}$  in frame  $S'$ .

Similarly  $T_{ij}$  is an anti-symmetric tensor if  $T_{ij} = -T_{ji}$  and  $T'_{ij} = -T'_{ji}$ .

Finally **any tensor** can be written as a sum of symmetric and anti-symmetric parts:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}).$$

## Quotient theorem

Consider 9 entities  $T_{ij}$  in frame  $S$  and  $T'_{ij}$  in frame  $S'$ . Let  $b_i = T_{ij} a_j$  where  $a_j$  is a vector. If  $b_i$  always transforms as a vector then  $T_{ij}$  is a rank 2 tensor.

Generalising, let  $R_{ijk\dots r}$  be a rank  $m$  tensor and  $T_{ijk\dots s}$  a set of  $3^n$  numbers where  $n > m$ . If  $T_{ijk\dots s} R_{ijk\dots r}$  is a rank  $n - m$  tensor then  $T_{ijk\dots s}$  is a rank  $n$  tensor.

## Matrices

We define a  $m \times n$  matrix  $A$  as  $(A)_{ij} = a_{ij}$  where  $i = 1, \dots, m$  and  $j = 1, \dots, n$ .

- $\text{Tr } A = a_{ii}$
- $(A^T)_{ij} = a_{ji}$
- $(AB)^T = B^T A^T$
- $(I)_{ij} = \delta_{ij}$

The determinant of a  $3 \times 3$  matrix  $A$  is:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \epsilon_{lmn} a_{1l} a_{2m} a_{3n} \\ &= \epsilon_{lmn} a_{l1} a_{m2} a_{n3}. \end{aligned}$$

Furthermore:

$$\begin{aligned} \epsilon_{ijk} \det A &= \epsilon_{lmn} a_{il} a_{jm} a_{kn} \\ \epsilon_{lmn} \det A &= \epsilon_{ijk} a_{il} a_{jm} a_{kn} \\ \det A &= \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} a_{il} a_{jm} a_{kn}. \end{aligned}$$

Properties of determinants:

1. Adding rows to each other does not change the determinant.
2. Interchanging two rows changes determinant signs.
3.  $\det A = \det A^T$
4.  $\det(AB) = \det A \cdot \det B$

These also apply to columns. Finally:

$$\epsilon_{ijk} \epsilon_{lmn} \det A = \begin{vmatrix} a_{il} & a_{im} & a_{in} \\ a_{jl} & a_{jm} & a_{jn} \\ a_{kl} & a_{km} & a_{kn} \end{vmatrix}$$

and setting  $A = I$  yields:

$$\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

## Linear equations

Let  $\mathbf{y} = A\mathbf{x}$ . Then  $x_i = A_{ij}^{-1} y_j$  with:

$$\begin{aligned} A_{ij}^{-1} &= \frac{1}{2 \det A} \epsilon_{imn} \epsilon_{jpk} a_{pm} a_{qn} \\ &= \frac{1}{\det A} C_{ij}^T \end{aligned}$$

where  $C$  is the cofactor matrix of  $A$ .

## Pseudotensors

A rank 2 pseudotensor is defined as:

$$T'_{ij} = (\det L) \ell_{ip} \ell_{jq} T_{pq}$$

where  $(L)_{ij} = \ell_{ij}$  and  $\det L = \pm 1$ .

Pseudovectors are rank 1 pseudotensors.

## Invariant tensors

Tensor  $T$  is invariant or isotropic if:

$$T_{ijk\dots} = \ell_{i\alpha} \ell_{j\beta} \ell_{k\gamma} \dots T_{\alpha\beta\gamma\dots}$$

for every orthogonal matrix  $L$ .

- If  $a_{ij}$  is a rank 2 invariant tensor then  $a_{ij} = \lambda \delta_{ij}$ .
- The most general rank 3 invariant pseudotensor is  $a_{ijk} = \lambda \epsilon_{ijk}$ . There are no rank 3 invariant true tensors.
- Invariant true tensors can only be even ranked.
- Invariant pseudotensors can only be odd ranked.

### Rotation tensors

The clockwise rotation of position vector  $\mathbf{x}$  to  $\mathbf{y}$  about unit vector  $\hat{\mathbf{n}}$  is given by:

$$y_i = R_{ij}(\theta, \hat{\mathbf{n}})x_j$$

$$R_{ij}(\theta, \hat{\mathbf{n}}) = \delta_{ij} \cos \theta + (1 - \cos \theta)n_i n_j - \epsilon_{ijk} n_k \sin \theta$$

and is the rotation tensor.

### Reflections and inversions

The reflection of vector  $\mathbf{x}$  to  $\mathbf{y}$  in plane with unit vector  $\hat{\mathbf{n}}$  is:

$$y_i = \sigma_{ij}x_j$$

$$\sigma_{ij} = \delta_{ij} - 2n_i n_j.$$

The inversion of vector  $\mathbf{x}$  to  $\mathbf{y}$  is given by  $\mathbf{y} = -\mathbf{x}$  and is defined as:

$$y_i = P_{ij}x_j$$

$$P_{ij} = \delta_{ij}.$$

### Projections

We define  $P$  to be a parallel projection operator to vector  $\mathbf{u}$  if:

$$P\mathbf{u} = \mathbf{u} \text{ and } P\mathbf{v} = \mathbf{0}$$

where  $\mathbf{u} \cdot \mathbf{v} = 0$ . Then:

$$P_{ij} = \frac{u_i u_j}{u^2}.$$

Similarly we define  $Q$  to be an orthogonal projection to vector  $\mathbf{u}$  if:

$$Q\mathbf{u} = \mathbf{0} \text{ and } Q\mathbf{v} = \mathbf{v}.$$

Here  $Q = I - P$ .

### Inertia tensors

Let  $\mathbf{L}$  denote the angular momentum of a rigid body about the origin of mass  $m$ , volume  $V$  and density  $\rho$  at position  $\mathbf{r}$  with velocity  $\mathbf{v}$ . Then:

$$L_i = I_{ij}\omega_j$$

$$I_{ij} = I_{ij}(O) = \int_V \rho(r^2 \delta_{ij} - x_i x_j) dV$$

where  $I_{ij}(O)$  is the inertia tensor about the origin. The kinetic energy of such a body is:

$$T = \frac{1}{2} I_{ij} \omega_i \omega_j = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega}.$$

### Parallel axis theorem

Consider the same rigid body now with centre of mass  $G$  and let  $\overrightarrow{OG} = \mathbf{R}$ . Then:

$$I_{ij}(O) = I_{ij}(G) + M(R^2 \delta_{ij} - X_i X_j)$$

$$M = \int_V \rho'(\mathbf{r}') dV'.$$

### Diagonalisation