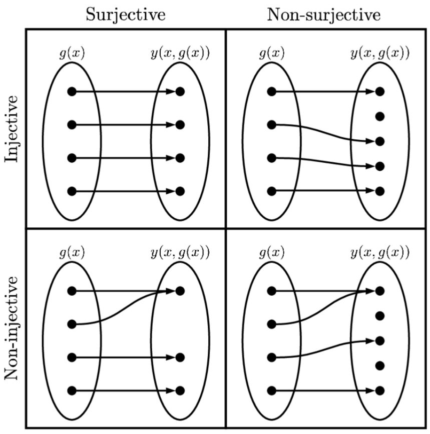
D: Functions

A function $f : X \to Y$ is an assignment of an element of Y to each element of X.

1. f is **injective** if:
$$\forall x_1, x_2 \in X; f(x_1) = f(x_2) \implies x_1 = x_2$$
and this implies that $|X| \leq |Y|$.
2. f is **surjective** if:
$$\forall y \in Y; \exists x \in X : y = f(x)$$
and this implies that $|X| \geq |Y|$.
3. f is **bijective** if it is injective and surjective.



D: Groups

A group G is a set with a composition operator $(\circ)$ such that $\forall x, y, z, \in G$:

1. $x \circ y = xy \in G$
2. $(xy)z = x(yz)$
3. $\exists e \in G : ex = xe = x$
4. $\exists x^{-1} \in G : xx^{-1} = x^{-1}x = e$.

G is **Abelian** if $\forall x, y \in G; xy = yx$.

D1.2.1(i): Fields

A field F is a set defined with addition and multiplication such that:

1. $(+) : F \times F \to F; (\lambda, \mu) \mapsto \lambda + \mu$
2. $(\cdot) : F \times F \to F; (\lambda, \mu) \mapsto \lambda \cdot \mu$
3. $\exists(-\lambda) \in F : \lambda + (-\lambda) = 0_F$
4. $\exists(\lambda^{-1}) \in F : \lambda \cdot (\lambda^{-1}) = 1_F$ except when $\lambda = 0$.
5. $(+)$ and $(\cdot)$ are associative, commutative and distributive.

Remark

$(F, +)$ and $(F \setminus \{0_F\}, \cdot)$ are groups.

Remark

Let n be prime or a prime power. Then $\mathbb{F}_n$ is a finite field with n elements under modulo n. Also, $\mathbb{Q}$ and $\mathbb{R}$ are fields.

D1.2.1(ii): Vector spaces

A vector space V over a field F is an Abelian group $V := (V, +)$ with mapping:
$$F \times V \to V; (\lambda, \boldsymbol{v}) \mapsto \lambda \boldsymbol{v}$$
where for $\forall \lambda, \mu \in F$ and $\forall \boldsymbol{v}, \boldsymbol{w} \in V$:

1. $\lambda(\boldsymbol{v} + \boldsymbol{w}) = (\lambda \boldsymbol{v}) + (\lambda \boldsymbol{w})$
2. $(\lambda + \mu)\boldsymbol{v} = (\lambda \boldsymbol{v}) + (\mu \boldsymbol{v})$
3. $\lambda(\mu \boldsymbol{v}) = (\lambda\mu)\boldsymbol{v}$
4. $1_F \boldsymbol{v} = \boldsymbol{v}$

and is known as a **F-vector space**.

Remark

Let V be a F-vector space and $\boldsymbol{v} \in V$.

1. $0\boldsymbol{v} = 0$
2. $(-1)\boldsymbol{v} = -\boldsymbol{v}$
3. $\lambda \boldsymbol{0} = \boldsymbol{0}$ for $\forall \lambda \in F$.

D: Cartesian products

The **cross product** of sets $X_1, \ldots, X_n$ is:
$$X_1 \times \cdots \times X_n := \{(x_1, \ldots, x_n) : x_i \in X_i\}$$
with bijection $X^n \times X^m \to X^{n+m}$.

The **projection** of a cross product is:
$$\text{pr}_i : X_1 \times \cdots \times X_n \to X_i; \quad (x_1, \ldots, x_n) \mapsto x_i.$$

D1.4.1: Vector subspaces

A vector subspace U of F-vector space V has the following properties:

1. $U \subset V$ and $\boldsymbol{0} \in U$.
2. Let $\boldsymbol{u}, \boldsymbol{v} \in U$ and $\lambda \in F$. Then $\boldsymbol{u} + \boldsymbol{v} \in U$ and $\lambda \boldsymbol{u} \in U$.

and is also a vector space.

P1.4.5

Let $T \subset V$ where V is a F-vector space. Then for all vector subspaces containing T, there exists a smallest vector subspace:
$$\text{span}(T) = \langle T \rangle_F \subset V$$
known as the vector subspace generated by T, or the span of T.

D1.4.7: Generating set

Let $T \subset V$ where V is a F-vector space. Set T is a **generating set** of V if:
$$\text{span}(T) = V$$
and is the linear combination of vectors in T over field F. V is **finitely generated** if its generating set T is finite.

D1.4.9: Power sets

The power set of set X is:
$$\mathcal{P}(X) := \{U : U \subseteq X\}.$$
Let $\mathcal{U} \subseteq \mathcal{P}(X)$. Then:
$$\bigcup_{U \in \mathcal{U}} U := \{x \in X : (\exists U \in \mathcal{U} : x \in U)\}$$
$$\bigcap_{U \in \mathcal{U}} U := \{x \in X : \forall U \in \mathcal{U}; x \in U\}.$$

D1.5.1: Linear independence

Let V be a F-vector space and $L \subseteq V$. Subset L is **linearly independent** if:
$$\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_r \boldsymbol{v}_r = \boldsymbol{0} \implies \alpha_1 = \cdots = \alpha_r = 0$$
for $\boldsymbol{v}_i \in L$ and is pairwise distinct.

Remark

L is linearly dependent if some $\alpha_i \neq 0$.

D1.5.8: Basis

A basis of a vector space V is a **linearly independent** generating set in V.

T1.5.11: Basis evaluation mappings

Let V be a F-vector space.

Then $A = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is a basis of V **iff** the following evaluation mapping:
$$\Phi_A : F^r \to V; \quad (\alpha_1, \ldots, \alpha_r) \mapsto \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_r \boldsymbol{v}_r$$
is a bijection.

Remark

Φ is surjective if A is generating.

T1.5.12

Let V be a vector space and $E \subseteq V$. Then the following statements are equivalent:

1. E is a basis of V.
2. E is minimal among all generating sets, or that $E \setminus \{\boldsymbol{e}\}$ is not a basis for $\forall \boldsymbol{e} \in E$.
3. E is maximal amongst all linearly independent subsets. i.e. $E \cup \{\boldsymbol{v}\}$ is linearly dependent for $\forall \boldsymbol{v} \in V$.

C1.5.13

Every finitely generated vector space has a finite basis.

T1.5.14

Let V be a vector space.

1. Let $L \subseteq V$ be linearly independent and set E be minimal amongst all generating sets of V. Let $L \subseteq E$. Then E is a basis of V.
2. Let $E \subseteq V$ be a generating set and L be maximal amongst all linearly independent subsets of V.

   Let $L \subseteq E$. Then E is a basis of V.

D1.5.15

Let X be a set and F be a field. Then:

$$\mathrm{maps}(X, F) := \{f : (\forall f : X \to F)\}$$

and is a F-vector space under pointwise addition and multiplication via scalars.

Let $F\langle X\rangle \subseteq \mathrm{maps}(X, F)$ be the subset of all mappings that sends all but finitely many elements of X to 0:

$$F\langle X\rangle := \left\{f : \left(\forall f : X \to \{0\}\right)\right\}.$$

It contains all linear combinations of X in F and forms a vector subspace.

T1.5.16

Let V be a F-vector space.

Then $(\boldsymbol{v}_i)_{i\in I}$ is a basis for V **iff**:

$$\forall \boldsymbol{v} \in V; \exists!(a_i)_{i\in I} \subseteq F : \boldsymbol{v} = \sum_{i\in I} a_i \boldsymbol{v}_i.$$

T1.6.1

Let V be a vector space. Let $L \subset V$ be a linearly independent subset and $E \subseteq V$ a generating set. Then $|L| \leq |E|$.

T1.6.2: Steinitz exchange theorem

Let V be a vector space, $L \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set.

Then there exists an **injective** function $\phi : L \to E$ such that:

$\big(E \setminus \phi(L)\big) \cup L$ is a generating set for V.

L1.6.3: Exchange lemma

Let V be a vector space. Let $M \subset V$ be a finite linearly independent subset and $E \subseteq V$ be a generating set where $M \subseteq E$.

If $\exists \boldsymbol{w} \in V \setminus M$ such that set $M \cup \{\boldsymbol{w}\}$ is linearly independent then:

$\exists \boldsymbol{e} \in E \setminus M : \big(E \setminus \boldsymbol{e}\big) \cup \{\boldsymbol{w}\}$ is generating.

C1.6.4

Let V be a finitely generated vector space.

1. V has finite basis.
2. V cannot have infinite basis.
3. Any two basis of V have the same number of elements.

D1.6.5: Dimension

The dimension of finite F-vector space V is the cardinality of one of its basis.

For infinite vector spaces: $\dim(V) = \infty$. We also define $\dim(\{\mathbf{0}\}) := 0$.

C1.6.7

Let V be a finitely generated vector space.

1. Every linearly independent $L \subseteq V$ has **at most** $\dim(V)$ elements and if $|L| = \dim(V)$ then L is a basis.
2. Every generating set $E \subseteq V$ has **at least** $\dim(V)$ elements and if $|E| = \dim(V)$ then E is a basis.

C1.6.8

A proper vector subspace of a vector space with finite dimension has itself a strictly smaller dimension.

T1.6.10: Dimension theorem

Let V be a vector space and $U, W \subseteq V$ be vector subspaces. Then:

$$\begin{aligned}&\dim(U + W) + \dim(U \cap W)\\ &= \dim(U) + \dim(W)\end{aligned}$$

where $U + W := \langle U \cup W\rangle \subseteq V$.

D1.7.1: Linear mappings

Let V and W be F-vector spaces.
A mapping $f : V \to W$ is F**-linear** or a **homomorphism** of vector spaces if for $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $\forall \lambda \in F$:

1. $f(\boldsymbol{v}_1 + \boldsymbol{v}_2) = f(\boldsymbol{v}_1) + f(\boldsymbol{v}_2)$
2. $f(\lambda \boldsymbol{v}_1) = \lambda f(\boldsymbol{v}_1)$.

Furthermore bijective linear mappings are an **isomorphism** of vector spaces.

A homomorphism from a vector space to itself is an **endomorphism**.

An isomorphism of a vector space to itself is an **automorphism**.

D1.7.5: Fixed points

In a linear mapping a fixed point is sent to itself. Given mapping $f : X \to X$ the **set of fixed points** is:

$$X^f = \{x \in X : f(x) = x\}.$$

D1.7.6: Complementary subspaces

Vector subspaces V_1, V_2 of vector space V are **complementary** if the direct sum of vector subspaces is bijective:

$$\oplus : V_1 \times V_2 \to V; (\boldsymbol{v}_1, \boldsymbol{v}_2) \mapsto \boldsymbol{v}_1 + \boldsymbol{v}_2.$$

i.e. $V_1 \oplus V_2 = V$.

T1.7.7

Let $n \in \mathbb{N}$ and V a F-vector space.
V is isomorphic to F^n **iff** $\dim(V) = n$.

L1.7.8

Let V, W be F-vector spaces and let B be a basis of V. Then the following mapping:

$$\mathrm{hom}_F(V, W) \to \mathrm{maps}(B, W); f \mapsto f_B$$

is a bijection. The set of all linear maps or homomorphisms from V to W is:

$$\mathrm{hom}_F(V, W) \subseteq \mathrm{maps}(B, W).$$

P1.7.9

Let $f : V \to W$ be a linear mapping, where V, W are vector spaces.

1. If f is injective, there exists map $g : W \to V$ such that $g \circ f = \mathrm{id}_V$. i.e. it has a **left inverse**.
2. If f is surjective, there exists map $g : W \to V$ such that $f \circ g = \mathrm{id}_W$. i.e. it has a **right inverse**.

D1.8.1: Image and kernel

Let $f : V \to W$ be a linear mapping.
The **image** of this linear mapping f is:

$$\begin{aligned}\mathrm{im}(f) &:= f(V)\\ &= \{\boldsymbol{w} \in W : \forall \boldsymbol{v} \in V; \boldsymbol{w} = f(\boldsymbol{v})\}\end{aligned}$$

and is a vector subspace of W.

The **kernel** of this linear mapping f is:

$$\ker(f) := f^{-1}(\mathbf{0}) = \{\boldsymbol{v} \in V : f(\boldsymbol{v}) = \mathbf{0}\}$$

and is a vector subspace of V.

L1.8.2

A linear mapping $f : V \to W$ is injective **iff** $\ker(f) = \{\mathbf{0}\}$.

T1.8.4: Rank-nullity theorem

Let $f : V \to W$ be a linear mapping and V, W are vector spaces. Then:

$$\dim(V) = \dim\Big(\ker(f)\Big) + \dim\Big(\mathrm{im}(f)\Big).$$

T2.1.1: Matrix mappings

Let F be a field and $m, n \in \mathbb{N}$.

Then there exists a bijection:

$$M : \hom_F(F^m, F^n) \to \mathrm{mat}(n \times m; F);$$
$$f \mapsto [f]$$

and attaches each linear mapping f with its **representing matrix** $M(f) := [f]$.

Remark

The set of $n \times m$ matrices in F is defined:

$$\mathrm{mat}(n \times m; F).$$

i.e. matrices with n rows and m columns.

D2.1.6: Matrix products

The product $A \circ B = AB$ for A is $n \times m$, B is $m \times \ell$ and AB is $n \times \ell$ is defined as:

$$(AB)_{ik} = \sum_{j=1}^{m} A_{ij} B_{jk}$$

with the following mapping:

$$\mathrm{mat}(n \times m; F) \times \mathrm{mat}(m \times \ell; F)$$
$$\to \mathrm{mat}(n \times \ell; F); (A, B) \mapsto AB.$$

T2.1.8

Let $g : F^\ell \to F^m$ and $f : F^m \to F^n$ be linear mappings. Then $[f \circ g] = [f] \circ [g]$.

P2.1.9

Let A, A' be $n \times m$, B, B' be $m \times \ell$ and C, C' be $\ell \times k$. Denote $I = I_m$ as the $m \times m$ identity matrix. Then:

1. $(A + A')B = AB + A'B$
2. $A(B + B') = AB + AB'$
3. $IB = B$
4. $AI = A$
5. $(AB)C = A(BC)$.

D2.2.1: Invertible matrices

A matrix A is **invertible** if:

$$\exists B, C : BA = I \text{ and } AC = I.$$

D2.2.2: Elementary matrices

Elementary matrices are square matrices that differs from the identity matrix by at most one entry.

T2.2.3

Every square matrix with entries in a field can be written as a product of elementary matrices.

D2.2.4: Smith normal form

Matrices with **only** non-zero entries along the diagonal are in Smith normal form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

T2.2.5

Let A be an $n \times m$ matrix. Then:

PAQ is of Smith normal form

where P and Q are invertible.

Remark

$\mathrm{rank}(A) = \mathrm{rank}(PAQ)$.

D2.2.7: Column and row rank

Let matrix $A \in \mathrm{mat}(n \times m; F)$.

The column rank of A is the dimension of the subspace of F^n generated by the columns of A.

Similarly the row rank of A is the dimension of the subspace of F^m generated by the rows of A.

T2.2.8

Column and row ranks are equal.

D2.2.9: Full rank matrices

Let A be $n \times m$ with entries in F.
A is **full rank** if $\mathrm{rank}(A) = \min(m, n)$.

Let $A = [a]$ with mapping $a : F^m \to F^n$. Then $\dim\big(\mathrm{im}(a)\big) := \mathrm{rank}(A)$.

T2.3.1: Representing matrices

Let V and W be F-vector spaces with bases $A = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ s.t. $\langle A \rangle = V$ and $B = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ s.t. $\langle B \rangle = W$.

Then for every linear map $f : V \to W$ there exists a **representing matrix**:

$$({}_B[f]_A)_{ij} = a_{ij}$$
$$f(\boldsymbol{v}_j) = a_{1j}\boldsymbol{w}_1 + \cdots + a_{nj}\boldsymbol{w}_n \in W$$

which produces the following bijection:

$$M_B^A : \hom_F(V, W) \to \mathrm{mat}(n \times m; F);$$
$$f \mapsto {}_B[f]_A$$

and $M_B^A(f) = {}_B[f]_A$ is the representing matrix of linear mapping f with respect to bases A and B.

If A and B are standard bases then $[f]$.

T2.3.2

Let U, V, W be F-vector spaces with finite dimension and bases A, B, C respectively.

If $f : U \to V$ and $g : V \to W$ are linear mappings then ${}_C[g \circ f]_A = {}_C[g]_B \circ {}_B[g]_A$.

D2.3.3: Vector representations

Let V be a finite dimensional vector space with basis $A = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$. Then:

$$\Phi_A^{-1} : V \to F^r; \boldsymbol{v} \mapsto {}_A[\boldsymbol{v}]$$

is a bijection and the column vector ${}_A[\boldsymbol{v}]$ is known as the representation of vector $\boldsymbol{v}$ with respect to basis A.

T2.3.4

Let V, W be finite dimensional F-vector spaces with bases A and B respectively.

Let $f : V \to W$ be a linear mapping. Then ${}_B[f(\boldsymbol{v})] = {}_B[f]_A \circ {}_A[\boldsymbol{v}]$ for $\forall \boldsymbol{v} \in V$.

D2.4.1

Let V be a F-vector space and let sets $A = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $B = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ be bases of V. The representation matrix of the identity mapping:

$$\mathrm{id}_V : V \to V; \boldsymbol{v} \mapsto \boldsymbol{v}$$

is a **change of basis matrix** ${}_B[\mathrm{id}_V]_A$ with entries a_{ij} given by definition:

$$\boldsymbol{v}_j = \sum_{i=1}^{n} a_{ij} \boldsymbol{w}_i.$$

T2.4.3: Change of basis

Let V and W be finite dimensional vector spaces with linear mapping $f : V \to W$. Let A, A' be ordered bases of V and B, B' be ordered bases of W. Then:

$${}_{B'}[f]_{A'} = {}_{B'}[\mathrm{id}_W]_B \circ {}_B[f]_A \circ {}_A[\mathrm{id}_V]_{A'}.$$

C2.4.4

Let V be a finite dimensional vector space and let $f : V \to V$ be an endomorphism. Let A, A' be bases of V. Then:

$${}_{A'}[f]_{A'} = {}_A[\mathrm{id}_V]_{A'}^{-1} \circ {}_A[f]_A \circ {}_A[\mathrm{id}_V]_{A'}.$$

T2.4.5

Let V and W be finite dimensional vector spaces and let $f : V \to W$ be linear.

Then there exists a basis A of V and a basis B of W such that the representing matrix ${}_B[f]_A$ has nonzero entries only on the diagonal.

D2.4.6: Trace

The trace of a $n \times n$ matrix A is the sum of its diagonal entries:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

D3.1.1: Rings

A **ring** R is a set equipped with addition and multiplication that satisfy:

1. $(R, +)$ is an **Abelian group** with additive identity $0_R \in R$.
2. $(R, \cdot)$ is a **monoid**, meaning that:
$$(\cdot) : R \times R \to R; (a, b) \mapsto a \cdot b$$
is associative with identity element $1 = 1_R \in R$ such that:
$$\forall a, b, c \in R; (a \cdot b) \cdot c = a \cdot (b \cdot c),$$
$$a \cdot 1 = 1 \cdot a = a$$
yet $a \cdot b \neq b \cdot a$ in general.
3. Multiplication in R with respect to addition is distributive:
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$
$$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$$
for $\forall a, b, c \in R$.

For **nonzero** rings $0_R \neq 1_R$.

Division rings are rings where nonzero elements have multiplicative inverses.

P3.1.7

A natural number is divisible by 3 if the sum of its digits is divisble by 3.

D3.1.8: Fields

A **field** F is a nonzero commutative ring with **multiplicative** inverses to every nonzero element:

$$\forall a \in F; \exists a^{-1} \in F : aa^{-1} = a^{-1}a = 1.$$

i.e. a commutative division ring.

P3.1.11

$\mathbb{Z}/m\mathbb{Z}$ is a field **iff** m is prime.

L3.2.1

Let R be a ring and $a, b \in R$. Then:

1. $0a = a0 = 0$
2. $(-a)b = -(ab) = a(-b)$
3. $(-a)(-b) = ab$.

D3.2.3

Let $m \in \mathbb{Z}$. Then mth multiple ma of $a \in (R, +)$ is $ma = \underbrace{a + \cdots + a}_{m \text{ times}}$ if $m > 0$.

$0a := 0$ and if $m < 0$, $(-m)a = -(ma)$.

L3.2.4

Let R be a ring where $a, b \in R$ and $m, n \in \mathbb{Z}$. Then:

1. $m(a + b) = ma + mb$
2. $(m + n)a = ma + na$
3. $m(na) = (mn)a$
4. $m(ab) = (ma)b = a(mb)$
5. $(ma)(nb) = (mn)(ab)$.

D3.2.6: Units

Let R be a ring. An element $r \in R$ is a **unit** if it has a multiplicative inverse:

$$\exists r^{-1} \in R : rr^{-1} = r^{-1}r = 1_R.$$

P3.2.9: Group of units

$R^\times$ is the **set of units** in ring R and forms a group under multiplication.

D3.2.11: Divisor of zero

Let $r \neq 0_R \in R$ where R is a ring. Then element r is a **divisor of zero** if:

$$\exists s \in R : rs = 0_R \text{ or } sr = 0_R.$$

D3.2.12: Integral domains

Intgeral domains are commutative rings with **no** divisors of zeros.

P3.2.15: Cancellation law

Let $a, b, c \in R$ for R is an integral domain. If $ab = ac$ and $a \neq 0$ then $b = c$.

P3.2.16

Let $m \in \mathbb{N}$. Then $\mathbb{Z}/m\mathbb{Z}$ is an integral domain **iff** m is prime.

T3.2.17

Every finite integral domain is a field.

Remark

If $|R| < \infty$ then $f : R \to R$ is surjective.

D3.3.2: Polynomial rings

$R[X]$ is a ring of polynomials over R with zero and identity: $0, 1 \in R$. If $P \in R[X]$:

$$P = a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m$$

with $\deg(P) = m \geq 0$ and $a_i \in R$.

L3.3.3

Let R be a ring and let $P, Q \neq 0 \in R[X]$.

1. $\deg(PQ) = \deg(P) + \deg(Q)$
2. If R is an integral domain then so is polynomial ring $R[X]$.

T3.3.4

Let R be an integral domain and let $P, Q \in R[X]$ where $\deg(Q) \leq \deg(P)$ and that polynomial Q is a **monic**.

Then $\exists! A, B \in R[X] : P = AQ + B$ and either $\deg(B) < \deg(Q)$ or $B = 0$.

Remark

A polynomial Q is monic if:

$$Q = q_0 + \cdots + q_m X^m$$

where $q_m = 1$.

D3.3.6

Let R be a commutative ring and let $P \in R[X]$ be a polynomial. Then:

$$R[X] \to \mathrm{maps}(R, R)$$

where we **evaluate** $P(\lambda)$ for $\lambda \in R$:

$$P(X) \mapsto \{P_\lambda : R \to R; \lambda \mapsto P(\lambda)\}.$$

If $P(\lambda) = 0$ then λ is a **root** of P.

P3.3.9

Let R be a commutative ring, let $\lambda \in R$ and $P(X) \in R[X]$. Then $P(\lambda) = 0$ **iff**:

$$P(X) = (X - \lambda)Q(X)$$

where $Q(X) \in R[X]$.

T3.3.10

Polynomial $P \neq 0 \in R[X] \setminus \{0\}$ has at most $\deg(P)$ roots in integral domain R.

D3.3.11: Algebraically closed field

A field F is algebraically closed if every $P \in F[X] \setminus F$ has a root in field F.

T3.3.13: FTA

Field $\mathbb{C}$ is algebraically closed.

T3.3.14

Let field F be algebraically closed. Then every $P \in F[X] \setminus \{0\}$ decomposes into:

$$P = c(X - \lambda_1) \dots (X - \lambda_n)$$

where $c \in F^\times$ and $\lambda_1, \dots, \lambda_n \in F$.

D3.4.1: Ring homomorphisms

Let R and S be rings. $f : R \to S$ is a ring homomorphism if for all $x, y \in R$:

$$f(x + y) = f(x) + f(y)$$

$$f(xy) = f(x)f(y).$$

L3.4.5

Let R and S be rings. Let $f : R \to S$ be a **ring homomorphism**. Then for all $x, y \in R$ and $m \in \mathbb{Z}$:

1. $f(0_R) = 0_S$
2. $f(-x) = -f(x)$
3. $f(x - y) = f(x) - f(y)$
4. $f(mx) = mf(x)$

since $(R, +)$ is a group.

D3.4.7: Ideals

Let $I \subset R$ where R is a ring. Then I is an **ideal** of ring R if:

1. $I \neq \emptyset$ and $0_R \in I$
2. I is closed under subtraction.
3. $\forall i \in I; \forall r \in R; ri, ir \in I$

and we denote $I \trianglelefteq R$.

D3.4.11: Ideal of R generated by T

Let R be a commutative ring and $T \subset R$. Then the ideal of R generated by T is:

$${}_R\langle T \rangle = \left\{ \sum_i r_i t_i : t_i \in T; \forall r_i \in R \right\}$$

where $i \in \{1, \dots, m\}$ and $m \leq |T|$.

P3.4.14

${}_R\langle T \rangle$ is the smallest ideal containing T.

D3.4.15: Principle ideal

An ideal of a commutative ring R is the **principle ideal** if:

$$I = {}_R\langle t \rangle \text{ where } t \in R.$$

P3.4.18

Let $f : R \to S$ be a ring homomorphism. Then $\ker(f)$ is an <u>ideal</u> of ring R where:

$$\ker(f) = \{r \in R : f(r) = 0_S\}$$

and is a subgroup of $(R, +)$.

L3.4.21 and L3.4.22

The set intersection and addition of ideals also form ideals.

D3.4.23: Subrings

A subset $R' \subseteq R$ is a subring of ring R if R' also satisfies D3.1.1.

P3.4.26: Subring test

$R' \subseteq R$ is a subring of R **iff** $\forall a, b \in R'$:

1. R' has multiplicative identity.
2. $a - b \in R'$
3. $ab, ba \in R'$

i.e. that R' is closed under subtraction and multiplication.

P3.4.28

Let $f : R \to S$ be a ring homomorphism.

1. If R' is a subring of R then $f(R')$ and $\mathrm{im}(f)$ are subrings of S.
2. Let $f(1_R) = 1_S$. Then:

$$x \in R^\times \implies f(x) \in S^\times.$$

D3.5.1: Relations

A **relation** R on set X is a subset of $X \times X$. We denote $(x, y) = xRy \in X \times X$.

R is an **equivalence relation** on set X if $\forall x, y, z \in X$ the following is true:

1. Reflexive: xRx
2. Symmetric: $xRy \iff yRx$
3. Transitive: $(xRy \wedge yRz) \implies xRz$.

D3.5.3: Equivalence classes

Let $\sim$ be an equivalence relation on X. Then the **equivalence class** of $x \in X$ is:

$$E(x) = \{z \in X : z \sim x\} \subseteq X$$

where an element of an equivalence class is a **representative** of the class.

D3.5.5

Given an equivalence relation $\sim$ on set X, the **set of equivalence classes** is:

$$(X \backslash \sim) := \{E(x) : x \in X\} \subseteq \mathcal{P}(X).$$

We also define a <u>surjective</u> map:

$$\mathrm{can} : X \to (X/ \sim); x \mapsto E(x)$$

known as the **canonical mapping**.

Remark

A mapping $f : X \to Z$ is **well-defined** if there is an equivalence relation $\sim$ on X such that $x \sim y \implies f(x) = f(y)$.

Then there exists a unique mapping $\overline{f}$:

$$\overline{f} : (X/ \sim) \to Z; E(x) \mapsto f(x)$$

where $f = \overline{f} \circ \mathrm{can}$.

D3.6.1: Cosets

Let I be an ideal of ring R. Then:

$$x + I = \{x + i : i \in I\} \subseteq R$$

is the coset of x with respect to I in R.

Remark

1. $x + I$ is both a left and right coset of x since $(R, +)$ is Abelian.
2. Ideals of rings are subgroups.

D3.6.3: Factor rings

Let I be an ideal of ring R and define an equivalence relation on R where:

$$x \sim y \iff x - y \in I.$$

Then the **factor ring** of R by I is the set of cosets of I in R and denoted as R/I:

$$R/I = (R/ \sim)$$

for each element is an equivalence class:

$$\begin{aligned} E(x) &= \{z \in R : z - x \in I\} \\ &= \{x + i \in R : i \in I\} \\ &= x + I. \end{aligned}$$

T3.6.4

Let I be an ideal of ring R. Then R/I is a ring where $\forall x, y \in R$:

$$(x + I) \dot{+} (y + I) = (x + y) + I$$

$$(x + I) \cdot (y + I) = xy + I$$

where $x + I, y + I \in R/I$.

T3.6.7

Let I be an ideal of ring R. Then:

1. $\text{can} : R \to R/I$ is a surjective ring homomorphism with kernel I.
2. Let $f : R \to S$ where $f(I) = \{0_S\}$ and that f is a ring homomorphism.

   Then there is a unique $\overline{f} : R/I \to S$ such that $f = \overline{f} \circ \text{can}$ and that $\overline{f}$ is also a ring homomorphism.

$$\begin{tikzcd} R \arrow[r, "\text{can}"] \arrow[dr, "f"'] & R/I \arrow[d, dashed, "\overline{f}"] \\ & S \end{tikzcd}$$

T3.6.9: FIT for rings

Every ring homomorphism $f : R \to S$ induces a ring isomorphism:

$$\overline{f} : R/\ker(f) \to \operatorname{im}(f)$$

where $\overline{f}$ is a **bijection**. This is the first isomorphism theorem for rings.

D3.7.1: Left modules

A left module M over a ring R is a **pair** consisting of an Abelian group $(M, \dot{+})$ and the following mapping:

$$R \times M \to M; (r, a) \mapsto ra$$

such that $\forall r, s \in R$ and $\forall a, b \in M$:

$$r(a \dot{+} b) = (ra) \dot{+} (rb)$$

$$(r + s)a = (ra) \dot{+} (sa)$$

$$r(sa) = (rs)a$$

$$1_R a = a$$

also known as an R**-module**.

L3.7.8

Let R be a ring and M be a R-module. Then $\forall r \in R$ and $\forall a \in M$:

1. $0_R a = 0_M$
2. $r 0_M = 0_M$
3. $(-r)a = r(-a) = -(ra)$.

D3.7.11: Module homomorphisms

Let R be a ring, M and N be R-modules. Then $f : M \to N$ is an R-homomorphism if $\forall r \in R$ and $\forall a, b \in M$:

$$f(a + b) = f(a) + f(b)$$

$$f(ra) = rf(a).$$

f is an R**-isomorphism** if it is bijective and we denote that $M \cong N$.

D3.7.15: Submodules

$M' \subseteq M$ is a submodule if it also satisfies D3.7.11 but restricted to itself.

P3.7.20: Submodule test

Let M be a R-module. M' is a submodule of M **iff** $\forall a, b \in M$ and $\forall r \in R$:

1. $0_M \in M'$
2. $a - b \in M'$
3. $ra \in M'$.

L3.7.21

Let $f : M \to N$ be an R-homomorphism. Then $\ker(f)$ and $\operatorname{im}(f)$ are submodules.

D3.7.23: Submodule generated by T

Let $T \subseteq M$ for M is a R-module. Then:

$${}_R\langle T \rangle = \left\{ \sum_i r_i t_i : t_i \in T; \forall r_i \in R \right\}$$

where $i \in \{1, \ldots, m\}$ and $m \leq |T|$.

Module N is **cyclic** if $N = {}_R\langle t \rangle$.

L3.7.29 and L3.7.30

Intersecting and adding collections of submodules also form submodules.

D3.7.31: Factor modules

Let R be a ring, M be a R-module and N a submodule of M. Let $a \in M$. Then the **coset of** a **with respect to** N **in** M is:

$$a + N = \{a + b : b \in N\}$$

Every coset is an equivalent class of the following equivalence relation:

$$\forall a, b \in M; a \sim b \iff a - b \in N$$

and we define the **factor module** of M by the submodule N as:

$$M/N = (M/\sim)$$

with the following operators:

$$(a + N) \dot{+} (b + N) = (a + b) + N$$

$$r(a + N) = ra + N$$

and additive identity $0_{M/N} = 0_M + N$.

T3.7.33: FIT for modules

Let M and N be R-modules. Then every R-homomorphism $f : M \to N$ induces a R-isomorphism:

$$\overline{f} : M/\ker(f) \to \operatorname{im}(f)$$

where $\overline{f}$ is a bijection.