1. Find solutions to **BVPs**.

   For part (i) we have that:

   $$y'' + y = x \quad \text{for} \quad y(0) = y(\pi) = 0.$$

   We first find the homogeneous solution:

   $$y'' + y = 0,$$

   and this gives: $y_H = \alpha \cos x + \beta \sin x$. The particular solution for the differential equation is $y_p = x$, and therefore our general solution is:

   $$y = x + \alpha \cos x + \beta \sin x.$$

   However if we substitute our **boundary conditions** we find that:

   $$y(0) = \alpha = 0$$

   yet

   $$y(\pi) = \pi - \alpha = 0.$$

   This is clearly a contradiction and there are **no solutions** to this problem.

   For part (ii) we are asked:

   $$y'' + 4y = \cos x \quad \text{for} \quad y'(0) = y'(\pi) = 0.$$

   The homogeneous equation is:

   $$y'' + 4y = 0$$

   and has eigenvalues $\lambda = \pm 2i$, which corresponds to solution:

   $$y_H = \alpha \cos 2x + \beta \sin 2x.$$

   We try for a particular solution of form $y_p = \gamma \cos x + \eta \sin x$ and after substituting our general solution takes the form:

   $$y = \alpha \cos 2x + \beta \sin 2x + \frac{1}{3} \cos x.$$

   Taking derivatives and substituting boundary conditions we find $\beta = 0$, and so for this **BVP** there are infinitely many solutions of form:

   $$y = \alpha \cos 2x + \frac{1}{3} \cos x,$$

   where $\alpha \in \mathbb{R}$.

2. Consider the following function:

$$g(x) = \begin{cases} 1+x & x \in [-1,0) \\ 1-x & x \in [0,1). \end{cases}$$

Find its Fourier series with period $2L$, where $L = 1$.

Firstly our function g is an <u>even</u> function and so its Fourier series is purely in cosine form:

$$g_{FS}(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos n\pi x$$

since $L = 1$. Its coefficients are:

$$\begin{aligned} c_0 &= \frac{1}{L}\int_{-L}^{L} g(x)\mathrm{d}x \\ &= \int_{-1}^{1} g(x)\mathrm{d}x \\ &= \int_{-1}^{0} (1+x)\mathrm{d}x + \int_{0}^{1} (1-x)\mathrm{d}x \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} c_n &= \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} g(x)\mathrm{d}x \\ &= \int_{-1}^{1} \cos(n\pi x) g(x)\mathrm{d}x \\ &= \frac{2}{(n\pi)^2}\Big(1-(-1)^n\Big) \end{aligned}$$

for $n \in \{1, 2, \dots\}$ and we note that c_n is nonzero in only <u>odd</u> terms.

$$\therefore g_{FS}(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi x}{(2m+1)^2}$$

3. For part (i) find the Fourier series of the following:

$$f(x) = \begin{cases} -1 & x \in [-L, 0) \\ 1 & x \in (0, L] \end{cases}$$

with period $2L$ and $L = \pi$.

Firstly our function is <u>odd</u> as hence:

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=}^{\infty} a_n \sin \frac{n\pi x}{L}$$

where its coefficients are:

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x)\mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{0} (-1)\mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} \mathrm{d}x \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} \sin \frac{n\pi x}{L} f(x)\mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x)\mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{0} -\sin(nx)\mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} \sin(nx)\mathrm{d}x \\ &= \frac{2}{n\pi}\big[1 - (-1)^n\big]. \end{aligned}$$

Then our Fourier series takes the following form:

$$f_{FS}(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{2m+1}.$$