

Honours Differential Equations

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Winter 2023

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1 ODE systems

1.1 Integrating factors

Consider linear DE of form

$$y' + P(x)y = Q(x)$$

The integrating factor for this DE is:

$$I(x) = \exp\left(\int P(x)dx\right)$$

and the solution to the linear DE is:

$$y(x) = \frac{1}{I(x)} \int Q(x)I(x)dx + \frac{\alpha}{I(x)}$$

where here α is a constant.

1.2 Change of variables

For higher order differential equations of form

$$y^{(n)} = F(y, y', \dots, y^{(n-1)}, t),$$

consider **change of variables** $x_{i+1} = y^{(i)}$ for $i \in \{0, 1, \dots, n-1\}$.

Taking derivatives with respect to time yields a first order matrix ODE system:

$$x'_j = F_j(t, x_1, \dots, x_n)$$

for $j = 1, \dots, n$. We either immediately write this as a matrix system or linearise near a critical point.

1.3 Existence and uniqueness for IVPs

An initial value problem (**IVP**) is defined as

$$\frac{dx}{dt} = f(x, t)$$

for **initial** condition $x(t_0) = x_0$. A solution $x : I \rightarrow \mathbb{R}$ is a differentiable function that satisfies the IVP. Similarly for a first order system

$$x'_i = F_i(t, x_1, \dots, x_n)$$

to have a **unique** solution, F_i and $\frac{\partial F_i}{\partial x_j}$ must be continuous in a region. Here $i, j \in \{1, \dots, n\}$. This is known as the Picard-Lindelöf theorem.

1.4 Homogeneous systems

1.4.1 Unique eigenvalues

Now consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where \mathbf{A} is a $n \times n$ matrix. Substituting $\mathbf{x} = e^{rt}\boldsymbol{\xi}$ results in an eigenvalue problem:

$$(\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\xi}^{(i)} = \mathbf{0}$$

where $i \in \{1, 2, \dots, n\}$. Our general solution is then:

$$\begin{aligned}\mathbf{x}(t) &= \sum_{i=1}^n c_i e^{r_i t} \boldsymbol{\xi}^{(i)} \\ &= \sum_{i=1}^n c_i \mathbf{x}^{(i)} \\ &= \boldsymbol{\Psi}(t)\mathbf{c}\end{aligned}$$

where $\boldsymbol{\Psi}(t)$ is our fundamental matrix satisfying $\boldsymbol{\Psi}' = \mathbf{A}\boldsymbol{\Psi}$ and that:

$$\boldsymbol{\Psi}(t) = [\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}].$$

Furthermore if initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ are given we then have that:

$$\mathbf{c} = \boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}(t_0)$$

and

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}_0.$$

1.4.2 Matrix exponentials

We can also write our solutions as a matrix exponential, defined as such:

$$\begin{aligned}e^{\mathbf{A}t} &= \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} \\ &= \mathbf{I}_n + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \dots\end{aligned}$$

and since an exponential power series is infinitely differentiable:

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}.$$

Therefore it is then deduced that the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0).$$

and that $e^{\mathbf{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$ where we are finding the coefficients to the general solution $\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}$.

1.4.3 Diagonalisation

Consider again $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where \mathbf{A} is a $n \times n$ matrix that is diagonalisable:

$$\mathbf{A}\boldsymbol{\xi}^{(i)} = r_i\boldsymbol{\xi}^{(i)}$$

$$\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$$

for here \mathbf{D} is our diagonal matrix containing our eigenvalues r_i and

$$\mathbf{T} = [\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}].$$

Then let $\mathbf{x} = \mathbf{T}\mathbf{y}$. After some algebra we have that:

$$\mathbf{y}' = \mathbf{D}\mathbf{y}$$

which have particular solutions $\mathbf{y} = e^{r_i t} \mathbf{e}^{(i)}$ for $i \in \{1, \dots, n\}$.

Since our fundamental matrix with respect to \mathbf{y} is $\mathbf{Q} = e^{\mathbf{D}t}$, the fundamental matrix with respect to \mathbf{x} is:

$$\boldsymbol{\Psi}(t) = \mathbf{T}e^{\mathbf{D}t}$$

and we get an expression for the matrix exponential of \mathbf{A} :

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$$

where $e^{\mathbf{D}t}$ is a diagonal matrix with entries $e^{r_i t}$.

1.4.4 Generalised eigenvectors

If eigenvalues do not generate enough eigenvectors to form n linearly independent solutions for a $n \times n$ matrix \mathbf{A} we try the following ansatz:

$$\mathbf{x} = te^{r_i t}\boldsymbol{\xi} + e^{r_i t}\boldsymbol{\eta}$$

which gives

$$(\mathbf{A} - r_i \mathbf{I}_n)\boldsymbol{\eta}^{(i)} = \boldsymbol{\xi}^{(i)}.$$

Therefore we end up with:

$$\mathbf{x}^{(1)} = e^{r_i t}\boldsymbol{\xi}$$

and

$$\mathbf{x}^{(2)} = te^{r_i t}\boldsymbol{\xi} + e^{r_i t}\boldsymbol{\eta}.$$

1.5 Non-homogeneous systems

Consider non-homogeneous ODE system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}.$$

There are a couple of different approaches we can take to solve such a system.

- **Change of basis**

Let $\mathbf{x} = \mathbf{T}\mathbf{y}$, where \mathbf{T} is our eigenvector matrix from diagonalisation. So $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$, and after some algebra we obtain:

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}$$

which can be solved by integrating factors. Finally revert back to \mathbf{x} .

- **Variation of parameters**

So $\mathbf{x}_H = \Psi\mathbf{c}$ solves the $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where \mathbf{c} is a constant vector.

We then assume that the solution to our non-homogeneous system takes the form:

$$\mathbf{x} = \Psi\mathbf{u}$$

for here $\mathbf{u} = \mathbf{u}(t)$. We then get $\Psi\mathbf{u}' = \mathbf{g}$, which can be solved by eliminating variables and integrating.

- **Method of undetermined coefficients**

Our non-homogeneous ODE system has solutions of form:

$$\mathbf{x} = \mathbf{x}_H + \mathbf{x}_p$$

Solving the homogeneous ODE gives us \mathbf{x}_H .

On the other hand we just need to find a **particular solution** \mathbf{x}_p that satisfies our non-homogeneous ODE. Then our solution is complete.

Whilst the fastest, this method is not guaranteed to work.

1.6 Critical points & linearisation

Consider non-linear ODE system

$$x' = F(x, y),$$

$$y' = G(x, y).$$

We define $\mathbf{x}^0 = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}$ as a **critical point** when $F(\mathbf{x}^0) = G(\mathbf{x}^0) = 0$.

Non-linear systems may then be linearised by Taylor expanding them around a critical point \mathbf{x}^0 , and discarding higher order terms.

i.e. let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_1 = x - x^0$ and $u_2 = y - y^0$.

$$\therefore u'_1 = x'$$

$$\approx F(x^0, y^0) + \left(\frac{\partial F}{\partial x} \right)_{x^0} (x - x^0) + \left(\frac{\partial F}{\partial y} \right)_{y^0} (y - y^0)$$

$$\therefore u'_2 = y'$$

$$\approx G(x^0, y^0) + \left(\frac{\partial G}{\partial x} \right)_{x^0} (x - x^0) + \left(\frac{\partial G}{\partial y} \right)_{y^0} (y - y^0)$$

Then we end up with the following linear system:

$$\mathbf{u}' = \mathbf{A}\mathbf{u}$$

where $\mathbf{A} = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}_{\mathbf{x}=\mathbf{x}^0}$ and is a 2×2 Jacobian matrix.

Our critical points \mathbf{x}^0 may also be classified:

Eigenvalues	Critical Points	Stability
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asympt. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asympt. stable
$r_1, r_2 = \lambda \pm i\mu$ ($\lambda > 0$)	focus	unstable
$r_1, r_2 = \lambda \pm i\mu$ ($\lambda < 0$)	focus	asympt. stable
$r_1 = i\mu, r_2 = -i\mu$	center	stable

Linearisation preserves critical point behaviour **except** when eigenvalues are of $r = \pm i\mu$ form, for which then classification is unknown.

1.7 Stability of critical points

Stable critical points \mathbf{x}^0 : All solutions start and stay near \mathbf{x}^0 .

$$\forall \epsilon > 0; \exists \delta > 0; \forall \mathbf{x}_{\text{solution}} \text{ to } \mathbf{x}' = \mathbf{F}(\mathbf{x}, t): \\ |\mathbf{x}(0) - \mathbf{x}^0| < \delta \implies |\mathbf{x}(t) - \mathbf{x}^0| < \epsilon \text{ for } \forall t \geq 0$$

Attracting critical points \mathbf{x}^0 : All solutions tends to \mathbf{x}^0 .

$$\forall \delta > 0 : |\mathbf{x}(0) - \mathbf{x}^0| < \delta \implies \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^0$$

Asymptotically stable critical points \mathbf{x}^0 : Attracting **and** stable

1.8 Lyapunov's theory and limit cycles

In this section $\dot{\mathbf{x}}$ means its first time derivative. So consider:

$$\dot{x} = F(x, y),$$

$$\dot{y} = G(x, y)$$

defined in \mathbb{R}^2 . Let $\mathbf{x}^0 \in D$ be a critical point.

The function $E : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Lyapunov function where $E(x^0, y^0) = 0$, whenever it exists. Note that the time derivative of E is:

$$\frac{dE}{dt} = \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G$$

- Let $E > 0$ for $\forall \mathbf{x} \neq \mathbf{x}^0$.
If $\frac{dE}{dt} \leq 0$ then \mathbf{x}^0 is stable.
If $\frac{dE}{dt} < 0$ then \mathbf{x}^0 is asymptotically stable.
- If every neighbourhood of \mathbf{x}^0 contains \mathbf{x}^* such that $E(\mathbf{x}^*) > 0$
and if $\frac{dE}{dt} > 0$ then \mathbf{x}^0 is unstable.

Now **limit cycles** are defined as periodic solutions such that at least one other **non-closed trajectory** approaches the limit cycle as $t \rightarrow \infty$.

2 Fourier series

2.1 Real Fourier series

Let $f(x)$ and $f'(x)$ be **piecewise continuous** in $[-L, L]$ with **period** $2L$.
i.e. $f(x) = f(x + 2L)$ for $\forall x$. Then the Fourier series for $f(x)$ is

$$f_{FS}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

The **convergence** of our Fourier series depends on the continuity of $f(x)$:

- If $f(x)$ is continuous then $f_{FS}(x) = f(x)$.
- If $f(\alpha)$ is discontinuous then at point α we have

$$f_{FS}(\alpha) = \frac{f(\alpha^+) + f(\alpha^-)}{2}.$$

Note that $f(x)$ is continuous at α if $f(\alpha) = \lim_{x \rightarrow \alpha} f(x)$ and we define:

$$f(\alpha^-) = \lim_{x \rightarrow \alpha^-} f(x)$$

and

$$f(\alpha^+) = \lim_{x \rightarrow \alpha^+} f(x),$$

i.e. limits from left and right respectively. It is important to also note that the derivative of a Fourier series is **not necessarily convergent**.

Now consider $S_n = \sin \frac{n\pi x}{L}$ and $C_n = \cos \frac{n\pi x}{L}$. We then have the following **orthogonality relations**:

$$\langle S_n, S_m \rangle = \langle C_n, C_m \rangle = L\delta_{mn}$$

$$\langle S_n, C_m \rangle = 0$$

where we define the inner product as:

$$\langle u(x), v(x) \rangle = \int_{-L}^L u(x)v(x)dx$$

and use the following trigonometric identities:

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

Now integrating the following expression:

$$\int_{-L}^L \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right) dx = \int_{-L}^L f(x) dx$$

gives:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx.$$

Similarly:

$$a_n = \frac{1}{L} \int_{-L}^L \cos \frac{n\pi x}{L} f(x) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L \sin \frac{n\pi x}{L} f(x) dx.$$

Note that δ_{mn} is the **Kronecker delta** and is defined as:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Furthermore notice that:

- The Fourier series of **even** functions contains only **cosines**.
- The Fourier series of **odd** functions contains only **sines**.

Even functions are defined $f(-x) = f(x)$, and:

$$\int_{-L}^L f_{\text{even}} dx = 2 \int_0^L f_{\text{even}} dx.$$

Similarly **odd** functions are defined $f(-x) = -f(x)$, and:

$$\int_{-L}^L f_{\text{odd}} dx = 0.$$

We can also extend a function defined in $[0, L]$ in several ways:

1. Define even function:

$$g(x) = \begin{cases} f(x) & x \in [0, L] \\ f(-x) & x \in (-L, 0) \end{cases}$$

where its Fourier expansion is a cosine series.

2. Define odd function:

$$h(x) = \begin{cases} f(x) & x \in (0, L) \\ 0 & x = 0, L \\ -f(-x) & x \in (-L, 0) \end{cases}$$

where its Fourier expansion is a sine series.

2.2 Complex Fourier series

Expanding $f(x)$ defined in $[-L, L]$ with period $2L$:

$$f_{FS}(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi}{L}x\right)$$

using Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$. Its coefficients are:

$$c_n = \frac{1}{2L} \int_{-L}^L \exp\left(-\frac{in\pi}{L}x\right) f(x) dx$$

for $\forall n \in \mathbb{Z}$ and:

$$c_n = \begin{cases} (a_n - ib_n)/2 & n > 0 \\ (a_0)/2 & n = 0 \\ (a_n + ib_n)/2 & n < 0. \end{cases}$$

Here we define the **inner product** for complex functions as

$$\langle f, g \rangle = \int_{\alpha}^{\beta} f^*(x)g(x)dx$$

where $f^*(x)$ is the complex conjugate of $f(x)$. Then:

$$\begin{aligned} \left\langle \exp\left(\frac{im\pi}{L}x\right), \exp\left(\frac{in\pi}{L}x\right) \right\rangle &= \int_{-L}^L \exp\left(-\frac{im\pi}{L}x\right) \exp\left(\frac{in\pi}{L}x\right) dx \\ &= 2L\delta_{mn} \end{aligned}$$

and since $f(x) = f_{FS}(x)$ we obtain our formula.

2.3 Parseval's theorem

Parseval's theorem states that given a periodic $f(x)$ with convergent Fourier series we have that

$$\begin{aligned} \langle f, f \rangle &= \int_{-L}^L |f(x)|^2 dx \\ &= 2L \sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= L \left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \right] \end{aligned}$$

and is derived by orthogonality.

3 PDEs

3.1 Separation of variables

The only methodology considered is separation of variables. So for PDE:

$$\hat{D}[u(x_1, \dots, x_n)] = 0$$

where \hat{D} is our differential operator, we look for solutions of form:

$$u(x_1, \dots, x_n) = X_1(x_1) \cdots X_n(x_n)$$

subject to **initial** and **boundary** conditions.

3.2 Heat equation

The heat equation is an equation of the following form:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

where α^2 is the thermal diffusivity constant.

3.2.1 Standard boundary conditions

We firstly define:

- **Initial condition:** $u(x, 0) = f(x)$ for $0 \leq x \leq L$
- **Boundary condition:** $u(0, t) = u(L, t) = 0$ for $\forall t > 0$

Let solutions be of form:

$$\begin{aligned} u(x, t) &= X(x) \cdot T(t) \\ \therefore X(x) \cdot \dot{T}(t) &= \alpha^2 X''(x) \cdot T(t) \end{aligned}$$

Only a constant function may satisfy the first equality:

$$\frac{1}{\alpha^2} \frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda.$$

Writing this as two ODEs:

$$\begin{aligned} \dot{T} + \alpha^2 \lambda T &= 0 \\ X'' + \lambda X &= 0. \end{aligned}$$

The first one we can directly integrate, yielding:

$$T(t) = a_1 \exp(-\alpha^2 \lambda t).$$

The second ODE is a spring system, hence it has solution of form:

$$X(x) = b_1 \cos \lambda^{1/2} x + b_2 \sin \lambda^{1/2} x.$$

However this time before proceeding we need to consider boundary conditions:

$$X(0) = X(L) = 0.$$

We find $X(0) = b_1 = 0$ and $X(L) = b_2 \sin \lambda^{1/2} L = 0$.

The second equation implies that λ must of the following form:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for } \forall n \in \mathbb{N}$$

and so

$$X'' + \lambda X = 0 \implies X_n = b_2 \sin \lambda_n^{1/2} x.$$

Since λ is discretised:

$$\therefore T_n = a_1 \exp(-\alpha^2 \lambda_n t).$$

Our general solution must then be:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Using initial condition $u(x, 0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \lambda_n^{1/2} x$$

and we recognise this as an odd Fourier series with period $2L$.

$$\begin{aligned} \therefore \int_{-L}^L \sin(\lambda_n^{1/2} x) f(x) dx &= \sum_{n=1}^{\infty} c_n \int_{-L}^L \left(\sin(\lambda_n^{1/2} x)\right)^2 dx \\ \therefore 2 \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx &= c_n L \end{aligned}$$

The final step we split the integration range and use $x = -x^*$.

$$\therefore c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) f(x) dx$$

This is fine because we can extend $u(x, t)$ via reflection for negative x .

3.2.2 Fixed boundary temperatures

We reconsider the heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

but now with the following **non-homogeneous boundary conditions**:

- $u(0, t) = T_1$
- $u(L, t) = T_2$
- $u(x, 0) = f(x)$

Physically our rod has fixed boundary temperatures, namely T_1 and T_2 .

We approach this problem with a change of variables:

$$v(x) = \lim_{t \rightarrow \infty} u(x, t).$$

Using our boundary conditions v must be linear:

$$\therefore v(x) = \frac{T_2 - T_1}{L}x + T_1$$

since $v'' = 0$, $v(0) = T_1$ and $v(L) = T_2$. We then deduce that:

$$u(x, t) = v(x) + \omega(x, t)$$

for $\omega(x, t)$ satisfies the same heat equation with initial conditions:

- $\omega(0, t) = \omega(L, t) = 0$
- $\omega(x, 0) = f(x) - v(x)$

Recognising this as our initial example:

$$\omega(x, t) = \sum_{n=1}^{\infty} c_n \exp(-\alpha^2 \lambda_n t) \sin \lambda_n^{1/2} x$$

where again $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and because $\omega(x, t)$ is a Fourier series with period $2L$:

$$c_n = \frac{2}{L} \int_0^L \sin(\lambda_n^{1/2} x) \{f(x) - v(x)\} dx.$$

3.2.3 Insulated rod ends

For the final example we consider:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

and define the following conditions:

- $\frac{\partial}{\partial x} u(0, t) = \frac{\partial}{\partial x} u(L, t) = 0$
- $u(x, 0) = f(x)$

We begin again with a separation of variables:

3.3 Wave equation

3.4 Laplace's equation

Laplace's equation takes the form $\nabla^2 u = 0$. In two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and we only consider boundary conditions. (Dirichlet conditions)

3.4.1 Rectangular boundary conditions

We open with the following example:

- **Boundary for y:** $u(x, 0) = u(x, b) = 0$
- **Boundary for x:** $u(0, y) = 0$ and $u(a, y) = f(y)$

where $x \in [0, a]$ and $y \in [0, b]$. Begin by separation of variables:

$$\begin{aligned} u(x, y) &= X(x) \cdot Y(y) \\ \therefore \frac{X''}{X} &= -\frac{Y''}{Y} = \lambda. \end{aligned}$$

Recognising the previous statement as two ODEs:

$$X'' - \lambda X = 0 \text{ for } X(0) = 0$$

$$Y'' + \lambda Y = 0 \text{ for } Y(0) = Y(b) = 0$$

The second ODE we have already solved in the heat equation. It has solution:

$$Y_n = a_1 \sin(\lambda_n^{1/2} y) \text{ for } \lambda_n = \frac{n^2 \pi^2}{b^2}.$$

The first ODE has solutions of form:

$$X_n = a_2 \cosh(\lambda_n^{1/2} x) + a_3 \sinh(\lambda_n^{1/2} x)$$

where these are the hyperbolic functions:

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}).$$

Using our boundary condition $X(0) = 0$ gives:

$$X_n = a_3 \sinh(\lambda_n^{1/2} x).$$

Now putting all of this together we get:

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} x) \sin(\lambda_n^{1/2} y)$$

To find coefficients c_n we use $u(a, y) = f(y)$.

$$\therefore f(y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} a) \sin(\lambda_n^{1/2} y)$$

Since we have a Fourier series with period $2b$:

$$\begin{aligned} \int_{-b}^b \sin(\lambda_n^{1/2} y) f(y) dy &= \sum_{n=1}^{\infty} c_n \sinh(\lambda_n^{1/2} a) \int_{-b}^b \sin(\lambda_n^{1/2} y) dy \\ &= c_n \sinh(\lambda_n^{1/2} a) \cdot b \end{aligned}$$

We can split the first integral to give us:

$$c_n = \frac{2}{b \sinh(\lambda_n^{1/2} a)} \int_0^b \sin(\lambda_n^{1/2} y) f(y) dy$$

where $\lambda_n = \left(\frac{n\pi}{b}\right)^2$ and our solution is complete.

3.4.2 Circular boundary conditions

Now we solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

but with a circular boundary. In polar coordinates (r, θ) :

- $u(a, \theta) = f(\theta)$
- $u(r, \theta)$ is bounded

where a is the radius of our circle and $\theta \in [0, 2\pi]$. Since $u = u(x, y)$:

$$u'_\theta = u'_x x'_\theta + u'_y y'_\theta$$

$$u''_{\theta\theta} = (u''_{xx} x'_\theta + u''_{xy} y'_\theta) x'_\theta + u'_x x''_{\theta\theta} + (u''_{yy} y'_\theta + u''_{xy} x'_\theta) y'_\theta + u'_y y''_{\theta\theta}$$

$$u'_r = u'_x x'_r + u'_y y'_r$$

$$u''_{rr} = (u''_{xx} x'_r + u''_{xy} y'_r) x'_r + u'_x x''_{rr} + (u''_{xy} x'_r + u''_{yy} y'_r) y'_r + u'_y y''_{rr}$$

and here we have used the chain rule.

Applying these derivatives we obtain the following equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0.$$

Using separation of variables:

$$u(r, \theta) = R(r)\Theta(\theta)$$

$$\therefore r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\ddot{\Theta}}{\Theta} = \lambda$$

where λ is our separation constant.

$$\therefore \ddot{\Theta} + \lambda\Theta = 0$$

$$\therefore r^2 R'' + rR' = \lambda R$$

4 Sturm-Liouville theory

4.1 Regular S-L problems

Sturm-Liouville theory is a general theory for 2nd order ODEs.

Consider the following eigenvalue ODE:

$$-\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

where $r(x)$ is our weight function. We define the following boundary conditions:

1. $a_1 y(0) + a_2 y'(0) = 0$
2. $b_1 y(1) + b_2 y'(1) = 0$.

This is a **regular Sturm-Liouville** problem, where $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous functions and $p(x)$, $r(x)$ are strictly positive functions for $\forall x \in [0, 1]$.

Eigenvalues λ_n yield **eigenfunctions** $\phi_n(x)$ which are nontrivial solutions to our S-L problem. Important consequences include:

- Eigenvalues λ_n of a S-L problem are **real**.

Furthermore each eigenvalue corresponds to one eigenfunction.

- Eigenfunctions $\phi_n(x)$ are orthogonal:

$$\langle \phi_m, \phi_n \rangle = \int_0^1 r(x) \phi_m(x) \phi_n(x) dx = \delta_{mn}$$

in Hilbert space $L^2([0, 1], r(x)dx)$.

Note that our eigenfunctions are orthonormal and so we define:

$$\phi_n(x) = k_n y_n(x)$$

where k_n is our scale factor. Since $\langle \phi_n, \phi_n \rangle = 1$:

$$\therefore \int_0^1 r(x) k_n^2 y_n^2(x) dx = 1$$

and so we have that:

$$\begin{aligned} k_n &= \frac{1}{\sqrt{\langle y_n, y_n \rangle}} \\ &= \left(\int_0^1 r(x) y_n^2(x) dx \right)^{-1/2}. \end{aligned}$$

4.1.1 General 2nd order ODEs

Consider the following general 2nd order eigenvalue ODE:

$$-P(x)\frac{d^2y}{dx^2} - \omega(x)\frac{dy}{dx} + q(x)y = \lambda r(x)y.$$

Multiply this by the following integrating factor:

$$F(x) = \exp\left[\int_0^x \frac{\omega(s) - p'(s)}{p(s)} ds\right]$$

yields an ODE of S-L form:

$$-\frac{d}{dx}\left[F(x)P(x)\frac{dy}{dx}\right] + F(x)q(x)y = \lambda F(x)r(x)y.$$

4.1.2 Lagrange's identity

Our previous definition is motivated by the **Lagrange's identity**:

$$\begin{aligned}\langle \mathcal{L}[u], v \rangle - \langle u, \mathcal{L}[v] \rangle &= -\left[p(u'v^* - u(v')^*)\right]_0^1 \\ &= -\left[p(x)\left(\frac{du}{dx} \cdot v^* - u \cdot \frac{dv^*}{dx}\right)\right]_0^1\end{aligned}$$

where $u = u(x)$, $v = v(x)$ are complex functions and

$$\mathcal{L}[u] = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u.$$

Here the inner product is defined as

$$\langle u, v \rangle = \int_0^1 uv^* dx$$

and we have integrated by parts using the following identities:

$$[pu'v^*]' = (pu')'v^* + pu'(v^*)'$$

$$[pu(v^*)']' = (p(v^*)')'u + pu'(v^*)'.$$

For a S-L problem its properties follow from:

$$\langle \mathcal{L}[u], v \rangle = \langle u, \mathcal{L}[v] \rangle$$

where functions u and v satisfy its boundary conditions.

4.1.3 Series expansion

Now the set of orthonormal eigenfunctions $\{\phi_n(x)\}$ from a S-L problem with boundary conditions may be used to expand function $f(x)$:

$$f_\phi(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

for $\forall x \in [0, 1]$. Integrating this on both sides:

$$\begin{aligned} \int_0^1 r(x) \phi_m(x) f(x) dx &= \int_0^1 r(x) \phi_m(x) \sum_{n=1}^{\infty} c_n \phi_n(x) dx \\ &= \sum_{n=1}^{\infty} c_n \int_0^1 r(x) \phi_m(x) \phi_n(x) dx \\ &= \sum_{n=1}^{\infty} c_n \delta_{mn} \\ &= c_m \end{aligned}$$

and so we get a general formula for our coefficients:

$$c_n = \int_0^1 r(x) \phi_n(x) f(x) dx.$$

If $f(x)$ and $f'(x)$ are piecewise continuous on $x \in [0, 1]$ then:

$$\forall x \in (0, 1); \sum_{n=1}^{\infty} c_n \phi_n(x) = \frac{f(x^-) + f(x^+)}{2}$$

where these are the left and right handed limits.

4.1.4 General Parseval's identity for S-L problems

We have that:

$$\int_0^1 r(x) [f(x)]^2 dx = \sum_{n=1}^{\infty} c_n^2$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

and

$$c_n = \int_0^1 r(x) \phi_n(x) f(x) dx.$$

4.2 Non-homogeneous S-L problems

Consider:

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

where $f(x)$ is our non-homogeneous term, and that we define

$$\mathcal{L}[y] = -\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + q(x)y.$$

Assume that this ODE satisfies regular S-L boundary conditions.

Firstly we solve the corresponding homogeneous S-L problem:

$$\mathcal{L}[y] = \lambda r(x)y$$

for non-trivial λ_n and $\phi_n(x)$.

Now let the general solution to our non-homogeneous S-L problem be:

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

and substituting this yields:

$$\begin{aligned} \sum_{n=1}^{\infty} b_n \mathcal{L}[\phi_n(x)] &= r(x) \sum_{n=1}^{\infty} b_n \lambda_n \phi_n(x) \\ &= \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x). \end{aligned}$$

Rearranging then gives:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} b_n (\lambda_n - \mu) \phi_n(x).$$

Now we can also expand $\frac{f(x)}{r(x)}$ in terms of our eigenfunctions:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

with

$$\begin{aligned} c_n &= \int_0^1 r(x) \phi_n(x) \frac{f(x)}{r(x)} dx \\ &= \int_0^1 \phi_n(x) f(x) dx \end{aligned}$$

and so equating yields the following relation

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

4.3 Non-homogeneous PDEs

Consider the following non-homogeneous PDE:

$$r(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[p(x) \frac{\partial u}{\partial x} \right] - q(x)u(x, t) + F(x, t)$$

with boundary and initial conditions:

- $\frac{\partial}{\partial x} u(0, t) - h_1 u(0, t) = 0$
- $\frac{\partial}{\partial x} u(1, t) - h_2 u(1, t) = 0$
- $u(x, 0) = f(x).$

Firstly we solve the homogenous case via separation of variables. Let:

$$u(x, t) = X(x)T(t)$$

and after some algebra:

$$\frac{1}{rX} \left[p'X' + pX'' - qX \right] = \frac{\dot{T}}{T} = -\lambda.$$

This yields two ODEs:

$$\begin{aligned} \dot{T} + \lambda T &= 0 \\ -[pX']' + qX &= \lambda rX \end{aligned}$$

with boundary conditions:

$$\begin{aligned} X'(0) - h_1 X(0) &= 0 \\ X'(1) - h_2 X(1) &= 0 \end{aligned}$$

Clearly our second ODE is of S-L form. We are now going to assume that this regular S-L problem has non-trivial λ_n and orthonormal eigenfunctions $\phi_n(x)$.

Let the general solution to our PDE be:

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \phi_n(x).$$

Substituting this into our PDE yields:

$$r(x) \sum_{n=1}^{\infty} \dot{b}_n(t) \phi_n(x) = \sum_{n=1}^{\infty} b_n(t) \left[\left(p(x) \phi_n'(x) \right)' - q(x) \phi_n(x) \right] + F(x, t).$$

Now since we have a S-L problem:

$$\left(p(x)\phi'_n(x)\right)' - q(x)\phi_n(x) = -\lambda_n\phi_n(x)r(x)$$

and after dividing through our PDE by $r(x)$ we get:

$$\sum_{n=1}^{\infty} \dot{b}_n(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t) \left[-\lambda_n\phi_n(x)\right] + \frac{F(x,t)}{r(x)}.$$

But we can also expand our non-homogeneous term as a sum:

$$\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x)$$

and its coefficients are:

$$\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)dx$$

in $L^2([0,1], r(x))$. Therefore after some algebra:

$$\sum_{n=1}^{\infty} \left[\dot{b}_n(t) + \lambda_n b_n(t) - \gamma_n(t)\right]\phi_n(x) = 0$$

and yields the following ODE:

$$\dot{b}_n(t) + \lambda_n b_n(t) - \gamma_n(t) = 0.$$

Solution is via integrating factors:

$$b_n(t) = e^{-\lambda_n t} \int_0^t \gamma_n(s)e^{\lambda_n s} ds + e^{-\lambda_n t} b_n(0).$$

Finally using $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) = f(x)$$

and we get

$$b_n(0) = \int_0^1 r(x)f(x)\phi_n(x)dx.$$

4.4 Singular S-L problems

Consider the following ODE:

$$-\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y = \lambda r(x)y$$

but now $p(x)$, $q(x)$ and $r(x)$ are discontinuous at $x = 0$ and/or $x = 1$.
This is a **singular** S-L problem with boundary condition:

$$b_1 y(1) + b_2 y'(1) = 0$$

or

$$a_1 y(0) + a_2 y'(0) = 0.$$

Now singular S-L problems at $x = 0$ may be self-adjoint or that they yield:

- $\lambda_n \in \mathbb{R}$ (Real eigenvalues)
- $\langle \phi_m, \phi_n \rangle = \delta_{mn}$ (Orthogonal eigenfunctions)

if they satisfy Lagrange's identity. Consider singular S-L problem at $x = 0$:

$$\begin{aligned} \int_{\epsilon}^1 (\mathcal{L}[u]v - u\mathcal{L}[v]) dx &= \left[-p(x) \left(u'(x)v(x) - u(x)v'(x) \right) \right]_{\epsilon}^1 \\ &= p(\epsilon) \left(u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon) \right) \end{aligned}$$

and tends to zero if and only if:

$$\lim_{\epsilon \rightarrow \infty} \left[p(\epsilon) \left(u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon) \right) \right] = 0,$$

where we define:

$$\mathcal{L}[u] = -\frac{d}{dx} \left[p(x) \frac{du}{dx} \right] + q(x)u.$$

Therefore this is the condition for our singular S-L problem to have real eigenvalues and orthogonal eigenfunctions.

Similarly for singular S-L problem at $x = 1$ it is self-adjoint if:

$$\lim_{\epsilon \rightarrow \infty} \left[p(1-\epsilon) \left(u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon) \right) \right] = 0.$$

Finally whilst eigenvalues may be real they are not necessarily discrete!

5 Laplace transforms

So let $f(t)$ be defined for $t \in [0, \infty)$. Its Laplace transform is:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt.$$

Let functions of exponential order be defined as:

$$\forall t \in [0, \infty); \exists A > 0 \text{ and } B \in \mathbb{R} : |f(t)| \leq Ae^{Bt}$$

where f is piecewise continuous and we denote such functions as $f \in E$.

For sufficiently large s , the Laplace transform $f \in E$ converges.

5.1 Properties

5.1.1 Inversion formula

Now let $F(s) = \mathcal{L}[f(t)]$. We have the following inversion formula:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds. \end{aligned}$$

5.1.2 Reduction of order

Applying the Laplace transform to derivatives:

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

for $\forall f, f' \in E$ and generalising this via induction gives:

$$\begin{aligned} \mathcal{L}[f^{(n)}(t)] &= s^n \mathcal{L}[f(t)] - s^{(n-1)} f^{(0)}(0) - s^{(n-2)} f^{(1)}(0) \\ &\quad - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0). \end{aligned}$$

5.1.3 Shifts, scaling and derivatives

Let $F(s) = \mathcal{L}[f(t)](s)$. We then have that:

1. **s-shift:**

$$\mathcal{L}[e^{-ct} f(t)](s) = F(s + c)$$

where $s + c > \gamma$.

2. **t-shift:**

Let $c \geq 0$ and $f(t) = 0$ if $t < 0$. Then:

$$\mathcal{L}[f(t - c)](s) = e^{-sc} F(s).$$

Furthermore utilising the unit step function:

$$\mathcal{L}[g(t - c)u_c(t)](s) = e^{-sc} G(s)$$

where $G(s) = \mathcal{L}[g(t)](s)$ and $g(t)$ any normal function.

3. s-derivative:

$$\mathcal{L}[tf(t)](s) = -\frac{d}{ds}F(s).$$

We can also extend this to the n th derivative:

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s).$$

4. scaling:

$$\mathcal{L}[f(ct)] = \frac{1}{c}F\left(\frac{s}{c}\right)$$

and

$$\frac{1}{c}\mathcal{L}\left[f\left(\frac{t}{c}\right)\right] = F(cs)$$

where $c > 0$.

5.2 Applications**5.2.1 Higher order ODEs**

So consider the following n th order ODE:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = f(t)$$

for $f \in E$. After taking the Laplace transform of both sides we get:

$$Z(s)\mathcal{L}[y(t)] = \mathcal{L}[f(t)] + Z_0(s)$$

where $Z(s)$ is a degree n polynomial and $Z_0(s)$ a degree $n - 1$ polynomial dependent on our initial conditions.

Now if our source term is of the following form:

$$f(t) = t^n e^{at} (A \cos bt + B \sin bt)$$

then $\mathcal{L}[f(t)]$ is rational and therefore:

$$\mathcal{L}[y(t)] = \frac{\mathcal{L}[f(t)]}{Z(s)} + \frac{Z_0(s)}{Z(s)}$$

where we can solve this via standard transforms.

5.2.2 Discontinuous source terms

Since the Laplace transform is convergent for all piecewise continuous functions, we can use it to solve ODEs with discontinuous source terms:

$$Ay''(t) + By'(t) + Cy(t) = g(t)$$

where $g(t)$ is piecewise continuous and of the following form:

$$g(t) = f(t)[u_a(t) - u_b(t)] = \begin{cases} f(t) & t \in [a, b) \\ 0 & t \notin [a, b) \end{cases}$$

for $b > a$. This is the **unit step function**:

$$u_c(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq c \end{cases}$$

and is also known as the Heaviside function. Its Laplace transform is:

$$\mathcal{L}[u_c(t)](s) = \frac{e^{-sc}}{s}.$$

Furthermore we can define a shift of $f(t)$ by $c > 0$ to the right by:

$$f(t - c)u_c(t)$$

$$\mathcal{L}[f(t - c)u_c(t)] = e^{-sc}F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

5.2.3 Impulse functions

The **Dirac delta** is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

and physically a "strike" on system in infinitely short time. We have that:

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = f(t_0)$$

$$\frac{d}{dt}[u_{t_0}(t)] = \delta(t - t_0)$$

and

$$\mathcal{L}[\delta(t - t_0)] = e^{-st_0}.$$

If $t_0 = 0$ then $\mathcal{L}[\delta(t)] = \lim_{t_0 \rightarrow 0} (e^{-st_0}) = 1$.

Taking the Laplace transform of the following ODE

$$y''(t) + y(t) = \delta(t)$$

with initial conditions $y(0) = y'(0) = 0$:

$$\mathcal{L}[y(t)] = \frac{1}{s^2 + 1}$$

5.2.4 Convolutions

5.3 Standard transforms

- $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ where $n \in \mathbb{N}$ and $s > 0$.
- $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ where $s > a$.
- $\mathcal{L}[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2},$
 $\mathcal{L}[e^{at} \cos bt] = \frac{s-a}{(s-a)^2 + b^2}$ where $s > a$.
- $\mathcal{L}[\sinh at] = \frac{a}{s^2 - a^2},$
 $\mathcal{L}[\cosh at] = \frac{s}{s^2 - a^2}$ where $s > |a|$.