

Electromagnetism and Relativity

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1 Suffix notation

2 Cartesian tensors

2.1 True tensors

tensor algebra

2.1.1 Rank 2 quotient theorem

The **quotient theorem** is as an alternative definition for tensors. In the context of rank 2 tensors it states that if b_i always transforms as a vector in

$$b_i = T_{ij}a_j$$

and that a_j is also a vector then T_{ij} is a rank 2 tensor.

Proof. We egregiously define entity T_{ij} in frame S and T'_{ij} in frame S' .

The usual transformation laws apply, namely $e'_i = \ell_{ij}e_j$. By definition:

$$\begin{aligned} b'_i &= T'_{ij}a'_j \\ &= T'_{ij}\ell_{jk}a_k \end{aligned}$$

Also directly from transformation laws:

$$\begin{aligned} b'_i &= \ell_{ij}b_j \\ &= \ell_{ij}T_{jk}a_k \end{aligned}$$

$$\therefore (T'_{ij}\ell_{jk} - \ell_{ij}T_{jk})a_k = 0$$

Since a_k are constants of our vector it must then be that:

$$\begin{aligned} T'_{ij}\ell_{jk} &= \ell_{ij}T_{jk} \\ \therefore T'_{ij}\ell_{jk}\ell_{mk} &= \ell_{ij}\ell_{mk}T_{jk} \end{aligned}$$

Where here we aim to eliminate the first two ℓ s. Finally:

$$T'_{im} = \ell_{ij}\ell_{mk}T_{jk}$$

□

2.1.2 General quotient theorem

Let $R_{ij\dots r}$ be a rank m tensor, and $T_{ij\dots s}$ be a set of 3^n numbers where $n > m$.

If $R_{ij\dots r}T_{ij\dots s}$ is a rank $n - m$ tensor then $T_{ij\dots s}$ is a rank n tensor.

symmetric and anti symmetric tensors

2.2 Matrices as tensors

2.3 Pseudotensors

Firstly note that $\det L = +1$ for rotations, and $\det L = -1$ for reflections and inversions. Recall the transformation law $\mathbf{e}'_i = \ell_{ij} \mathbf{e}_j$.

A second rank **pseudotensor** is defined:

$$T'_{ij} = (\det L) \ell_{ip} \ell_{jq} T_{pq}.$$

Furthermore a rank 1 pseudotensor is a **pseudovector** and is defined as:

$$T'_i = (\det L) \ell_{ip} T_p.$$

Finally a **pseudoscalar** is a rank 0 pseudotensor:

$$a' = (\det L) \cdot a,$$

and changes sign under transformation.

2.4 Invariant tensors

2.5 Rotation tensors

2.6 Reflections, inversions and projections

active and passive transformations

maybe merge with rotations?

2.7 Inertia tensors

3 Taylor expansions

3.1 1D expansions

3.2 3D expansions

4 Vector calculus

4.1 Vector operators

4.1.1 Gradient

4.1.2 Divergence

4.1.3 Curl

chain rules, important identities

4.2 Integrals theorems

4.2.1 Line, volume and surface integrals

4.2.2 Divergence theorem

Consider closed surface S with volume V . We have that:

$$\int_V \nabla \cdot \mathbf{E} dV = \oint_S \mathbf{E} \cdot d\mathbf{S}.$$

4.2.3 Stokes's theorem

Consider open surface S with line C as its boundary. We then have that:

$$\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint_C \mathbf{E} \cdot d\mathbf{r}.$$

5 Curvilinear coordinates

5.1 Orthogonal curvilinear coordinates

5.1.1 Scale factors and basis vectors

Consider change of variables:

$$(x_1, x_2, x_3) \leftrightarrow (u_1, u_2, u_3)$$

where u_i are our curvilinear coordinates, and

$$u_i = u_i(x_1, x_2, x_3)$$

$$x_i = x_i(u_1, u_2, u_3).$$

Then we define:

$$\begin{aligned} d\mathbf{r}_i &= \frac{\partial \mathbf{r}}{\partial u_i} du_i \\ &= h_i \mathbf{e}_i du_i \end{aligned}$$

where $h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|$ is our **scale factor** and

$$\mathbf{e}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_i}$$

is our **basis vector** of unit length for a specific set of curvilinear coordinates.

Now if the basis vectors satisfy

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

we have an orthogonal set of curvilinear coordinates.

5.1.2 Cylindrical coordinates

We define cylindrical coordinates as

$$(u_1, u_2, u_3) = (\rho, \phi, z)$$

and with the following relation to Cartesian coordinates:

$$\mathbf{r} = \rho \cos \phi \mathbf{e}_x + \rho \sin \phi \mathbf{e}_y + z \mathbf{e}_z.$$

Furthermore:

$$\begin{aligned} h_\rho &= 1 \quad \text{and} \quad \mathbf{e}_\rho = \cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y \\ h_\phi &= \rho \quad \text{and} \quad \mathbf{e}_\phi = -\sin \phi \mathbf{e}_x + \cos \phi \mathbf{e}_y \\ h_z &= 1 \quad \text{and} \quad \mathbf{e}_z = \mathbf{e}_z. \end{aligned}$$

Here ϕ is the anticlockwise rotation of the xy -plane.

5.1.3 Spherical coordinates

We define the spherical coordinates as

$$(u_1, u_2, u_3) = (r, \theta, \phi)$$

$$\mathbf{r} = r \sin \theta \cos \phi \mathbf{e}_x + r \sin \theta \sin \phi \mathbf{e}_y + r \cos \theta \mathbf{e}_z$$

where \mathbf{e}_x , \mathbf{e}_y and \mathbf{e}_z represent the Cartesian unit vectors.

Now $\phi \in [0, 2\pi]$ is the rotation angle in xy -plane, and $\theta \in [0, \pi]$ in z -plane. We also have that:

$$h_r = 1 \quad \text{and} \quad \mathbf{e}_r = \sin \theta \cos \phi \mathbf{e}_x + \sin \theta \sin \phi \mathbf{e}_y + \cos \theta \mathbf{e}_z$$

$$h_\theta = r \quad \text{and} \quad \mathbf{e}_\theta = \cos \theta \cos \phi \mathbf{e}_x + \cos \theta \sin \phi \mathbf{e}_y - \sin \theta \mathbf{e}_z$$

$$h_\phi = r \sin \theta \quad \text{and} \quad \mathbf{e}_\phi = -\sin \phi \mathbf{e}_x + \cos \phi \mathbf{e}_y.$$

We note that spherical coordinates are orthongonal:

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi, \quad \mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_r \quad \text{and} \quad \mathbf{e}_\phi \times \mathbf{e}_r = \mathbf{e}_\theta.$$

5.2 Length, area and volume

5.2.1 Vector and arc length

Firstly the **vector length** due to infinitesimal change in all directions is

$$d\mathbf{r} = \sum_{i=1}^3 h_i du_i \mathbf{e}_i.$$

It is important to note that summation notation does not work here.

Now the **arc length** of $d\mathbf{r}$ is:

$$\begin{aligned} ds &= |d\mathbf{r}| \\ &= \sqrt{d\mathbf{r} \cdot d\mathbf{r}} \end{aligned}$$

and we define the **metric tensor** as

$$\begin{aligned} g_{ij} &= \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \\ &= \frac{\partial \mathbf{r}}{\partial u_i} \cdot \frac{\partial \mathbf{r}}{\partial u_j}. \end{aligned}$$

Since $d\mathbf{r} = dx_k$ we then the following relation:

$$(ds)^2 = g_{ij} du_i du_j.$$

5.2.2 Vector area**5.2.3 Volume**

The volume of the infinitesimal parallelepiped defined by $d\mathbf{r}_1$, $d\mathbf{r}_2$ and $d\mathbf{r}_3$ is:

$$\begin{aligned}dV &= |(d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3| \\&= h_1 h_2 h_3 du_1 du_2 du_3 |(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3| \\&= \sqrt{g} du_1 du_2 du_3\end{aligned}$$

where g is the determinant of the metric tensor.

5.3 Vector operators in OCCs

5.3.1 Gradient

Let $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$. The gradient of a scalar field in terms of this OCC is:

$$\nabla f(\mathbf{r}) = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3.$$

So let there be the following infinitesimal change:

$$u_1 \rightarrow u_1 + du_1, \quad u_2 \rightarrow u_2 + du_2, \quad u_3 \rightarrow u_3 + du_3$$

and consider the following:

$$\begin{aligned} df(\mathbf{r}) &= \nabla f(\mathbf{r}) \cdot d\mathbf{r} \\ &= \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3 \\ &= \left[\frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3 \right] \cdot [h_1 \mathbf{e}_1 du_1 + h_2 \mathbf{e}_2 du_2 + h_3 \mathbf{e}_3 du_3] \\ &= \left[\frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3 \right] \cdot d\mathbf{r} \end{aligned}$$

and our claim follows from equating terms.

Here h_i are our scale factors and \mathbf{e}_i our orthogonal basis vectors.

6 Electrostatics

6.1 Dirac delta function

The one dimensional **Dirac delta** is defined:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0, \end{cases}$$

and can be thought of as infinitely sharp at $x = 0$ and zero elsewhere.

It satisfies some useful properties:

- $\delta(x - a) = \lim_{\sigma \rightarrow 0} \left[\frac{1}{|\sigma|\sqrt{\pi}} \exp\left(-\frac{(x - a)^2}{\sigma^2}\right) \right]$
i.e. an infinitely sharp Gaussian. (generalised functions)

- **Sift property**

$$\int_{\mathbb{R}} f(x) \delta(x - a) dx = f(a)$$

- Let x_i be the solutions to $g(x_i) = 0$. Then:

$$\int_{\mathbb{R}} f(x) \delta[g(x)] dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

Now we consider the **3D Dirac delta**, which is defined as follows:

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0)$$

given Cartesian coordinates (x_1, x_2, x_3) . It also satisfies the **sift** property:

$$\int_{\mathbb{R}^3} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_0) = f(\mathbf{r}_0).$$

The three dimensional Dirac delta defined in a orthogonal curvilinear coordinate system (u_1, u_2, u_3) is as follows:

$$\delta(\mathbf{r} - \mathbf{a}) = \frac{1}{h_1 h_2 h_3} \delta(u_1 - a_1) \delta(u_2 - a_2) \delta(u_3 - a_3)$$

for h_1, h_2 and h_3 are the scale factors.

6.2 Coulomb's law

Consider the force on charge q at \mathbf{r} due to charge q_1 at \mathbf{r}_1 :

$$\mathbf{F}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{qq_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3},$$

for here $\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$ in vacuum.

Physically, like charges ($qq_1 > 0$) repel while opposite charges ($qq_1 < 0$) attract.

We then define an **electric field** as the force on a small positive test charge:

$$\mathbf{E}(\mathbf{r}) = \lim_{q \rightarrow 0} \left(\frac{1}{q} \mathbf{F}(\mathbf{r}) \right).$$

The force on a charge q at \mathbf{r} from the origin in this electric field is:

$$\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}).$$

A negative point charge is a sink whereas a positive point charge is a source.

Consider a collection of charges q_i at position \mathbf{r}_i . The **principle of superposition** tells us that:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \left(\frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \right).$$

Now consider a continuous charged object with volume V and **charge density** $\rho(\mathbf{r}')$. It generates the following electric field:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV'.$$

Returning to the electric field generated by a point charge q_1 at position \mathbf{r}_1 :

$$\mathbf{E}(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3},$$

this is a **conservative field**, and we may write it as:

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}),$$

where:

$$\phi(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_1|}.$$

Conservative fields have zero curl, and their line integrals are path independent. This namely applies to finding work done.

6.3 Electrostatic Maxwell's equations

6.3.1 Curl equation

For a continuous charge distribution:

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV' \\ &= -\nabla \left(\frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV' \right)\end{aligned}$$

and therefore $\nabla \times \mathbf{E} = \mathbf{0}$ for static electric fields.

Hence electrostatic fields are conservative fields:

$$\int_{C_1} \mathbf{E} \cdot d\mathbf{r} = \int_{C_2} \mathbf{E} \cdot d\mathbf{r}$$

and we have a generalisation of the fundamental theorem of calculus:

$$-\int_a^b \mathbf{E} \cdot d\mathbf{r} = \phi(b) - \phi(a)$$

where $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$. Therefore our potential takes the expression:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV'$$

and has units Volts V or JC^{-1} . We define the **potential difference**:

$$\begin{aligned}V_{A \rightarrow B} &= \phi_B - \phi_A \\ &= -\int_C \mathbf{E} \cdot d\mathbf{r}\end{aligned}$$

and is the energy per unit charge to move small test charge from A to B :

$$V_{A \rightarrow B} = \lim_{q \rightarrow 0} \frac{1}{q} W_{A \rightarrow B} = -\frac{1}{q} \int_C \mathbf{F} \cdot d\mathbf{r}$$

for here work is done against force.

Now consider charge q at \mathbf{r} subject to external electrostatic field.

$$\therefore \mathbf{E}_{ext}(\mathbf{r}) = -\nabla\phi_{ext}(\mathbf{r})$$

$$\therefore W_{ext} = \int_V \rho(\mathbf{r}) \phi_{ext}(\mathbf{r}) dV$$

Note that W_{ext} is the interaction energy.

6.3.2 Divergence equation

Now consider:

$$\begin{aligned}
 \nabla \cdot \mathbf{E} &= \nabla \cdot [-\nabla \phi(\mathbf{r})] \\
 &= -\nabla^2 \phi(\mathbf{r}) \\
 &= -\nabla^2 \left(\frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV' \right) \\
 &= -\frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') dV' \left[\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] \\
 &= -\frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') dV' [-4\pi\delta(\mathbf{r} - \mathbf{r}')] \\
 &= \frac{\rho(\mathbf{r})}{\epsilon_0}
 \end{aligned}$$

due to the sift and symmetric properties of the delta delta function.

Now previously we also used the following result:

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\mathbf{r}).$$

When $\mathbf{r} \neq \mathbf{0}$ we have that:

$$\begin{aligned}
 \nabla^2 \left[\frac{1}{r} \right] &= \frac{\partial}{\partial x_i} \left[-\frac{x_i}{r^3} \right] \\
 &= \left[-\frac{1}{r^3} \delta_{ii} - x_i \frac{3}{2} r^{-5} 2x_i \right] \\
 &= 0.
 \end{aligned}$$

Now if $\mathbf{r} = \mathbf{0}$ consider the following volume integral of an ϵ sized sphere:

$$\begin{aligned}
 \int_{V_\epsilon} \nabla^2 \left[\frac{1}{r} \right] dV &= - \int_{V_\epsilon} \nabla \cdot \left[\frac{\mathbf{r}}{r^3} \right] dV \\
 &= - \int_{S_\epsilon} \frac{\mathbf{r}}{r^3} \cdot d\mathbf{S}
 \end{aligned}$$

where in the final step we used the divergence theorem. On the surface of our sphere, $\mathbf{r} = \epsilon \mathbf{e}_r$ and since:

$$\begin{aligned}
 d\mathbf{S} &= \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv \\
 &= r^2 \sin \theta \mathbf{e}_r d\theta d\phi
 \end{aligned}$$

evaluating our surface integral yields -4π . We then deduce the Dirac delta because at $\mathbf{r} = \mathbf{0}$ the charge is unbounded.

6.4 Electric dipoles

6.4.1 Potential and electric field

Dipoles consist of two equal and **opposite point charges** that are \mathbf{d} apart.

An **ideal dipole** is defined as when the following **dipole limit** is finite and constant:

$$\mathbf{p} = \lim_{\substack{q \rightarrow \infty \\ \mathbf{d} \rightarrow 0}} q\mathbf{d}.$$

A **dipole moment** is simply $\mathbf{p} = q\mathbf{d}$. The **dipole potential** at \mathbf{r}_0 is:

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0 - \mathbf{d}|} - \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) \\ &= \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3},\end{aligned}$$

where we have Taylor expanded the first term about $|\mathbf{r} - \mathbf{r}_0|$. For simplicity we set $\mathbf{r}_0 = \mathbf{0}$. Then the **electric field** generated by our dipole at the origin is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3\mathbf{p} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{1}{r^3} \mathbf{p} \right),$$

since $\mathbf{E} = -\nabla\phi(\mathbf{r})$. Note that these formulae are in Cartesian coordinates.

Now let our dipole with moment $\mathbf{p} = p\mathbf{e}_z$ be at:

$$\mathbf{r} = r \sin \theta \cos \phi \mathbf{e}_x + r \sin \theta \sin \phi \mathbf{e}_y + r \cos \theta \mathbf{e}_z.$$

Then in spherical coordinates (r, θ, χ) we have that:

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \\ &= \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2}\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\left(\frac{\partial\phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial\phi}{\partial\theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial\phi}{\partial\chi} \mathbf{e}_\chi \right) \\ &= \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} \left[2 \cos \theta \mathbf{e}_r + \sin \theta \mathbf{e}_\theta \right]\end{aligned}$$

where χ represents the anticlockwise rotation in the xy -plane.

6.4.2 Force, torque and energy

Consider a dipole at \mathbf{r} with moment $\mathbf{p} = q\mathbf{d}$.

The force on this dipole due to an external electric field \mathbf{E}_{ext} is:

$$\begin{aligned}\mathbf{F}(\mathbf{r}) &= \mathbf{F}_{-q} + \mathbf{F}_{+q} \\ &= -q\mathbf{E}_{ext}(\mathbf{r}) + q\mathbf{E}_{ext}(\mathbf{r} + \mathbf{d})\end{aligned}$$

where we have $-q$ at \mathbf{r} and $+q$ at $\mathbf{r} + \mathbf{d}$. Now in the dipole limit:

$$\mathbf{F}(\mathbf{r}) = (\mathbf{p} \cdot \nabla) \mathbf{E}_{ext}(\mathbf{r})$$

since $\mathbf{d} \rightarrow 0$ and we use the three dimensional Taylor expansion.

The torque on our dipole from external electric field is:

$$\mathbf{G}(\mathbf{r}) = \mathbf{p} \times \mathbf{E}_{ext}(\mathbf{r}).$$

The interaction energy is the following:

$$\begin{aligned}W &= -q\phi_{ext}(\mathbf{r}) + q\phi_{ext}(\mathbf{r} + \mathbf{d}) \\ &= -\mathbf{p} \cdot \mathbf{E}_{ext}(\mathbf{r})\end{aligned}$$

where we Taylor expand the second expression.

6.4.3 Multidipole expansion

potential

work done

6.5 Gauss's law

Gauss's law is the integral form of Maxwell's first equation:

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is the total charge enclosed by volume V . This result follows from the application of the divergence theorem and is useful in problems with symmetry.

6.5.1 Boundaries

Consider surface S with charge density σ separating electric fields \mathbf{E}_1 and \mathbf{E}_2 . Firstly consider the normal component:

6.5.2 Conductors

special case for electrostatics

6.6 Poisson's equation

In electrostatics we have:

$$\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

where ρ is our charge density. This is the **Poisson's equation** and is a consequence of the fact that $\nabla \times \mathbf{E} = \mathbf{0}$ and $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$.

6.6.1 Existence and uniqueness of solutions

The existence of solutions is given by the fact that:

$$\mathbf{E} = -\nabla \phi.$$

Poisson's equation has **unique** solution ϕ if we have volume V bounded by surface S and one of the following boundary conditions:

- 1.

6.6.2 Method of images

6.7 Electrostatic energy

6.8 Capacitors

6.8.1 Parallel plates

6.8.2 Concentric spheres

7 Magnetostatics

charge distribution \implies electric field

current \implies magnetic field

7.1 Currents

Elementary current

Bulk current density

Surface current density

Line current

units!

Infinitesimal current element (dependent on material)

units: $Cs^{-1}m = Am$

Note that $J = Am^{-2}$.

Current flowing through surface and line.