Honours Analysis Workshops

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Workshop 6

1. Consider real function $f = x^2$. We know that f is continuous on $\mathbb{R}$, since it is a polynomial. So for $f : \mathbb{R} \to \mathbb{R}$ the $\epsilon - \delta$ definition states:

   $\forall \alpha \in \mathbb{R}; \forall \epsilon > 0; \exists \delta > 0; \forall x \in \mathbb{R};$
   $|x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon$

   Now we set $\alpha > 1$ and $\epsilon = 1$. Find **best** possible $\delta = \delta(\epsilon)$.

   So whilst we can choose $1 > \delta(\delta+2\alpha)$, this is certainly not the best bound.

   Consider this approach instead:

$$\begin{aligned} \alpha - \delta < x < \alpha + \delta &\implies \alpha^2 - 1 < x^2 < \alpha^2 + 1 \\ &\implies \sqrt{\alpha^2 - 1} < x < \sqrt{\alpha^2 + 1} \end{aligned}$$

   i.e. since we have an implication:

$$\sqrt{\alpha^2 - 1} < \alpha - \delta < x < \alpha + \delta < \sqrt{\alpha^2 + 1}$$

   This is really 4 inequalities, and we need to choose the best 2. So:

$$\sqrt{\alpha^2 - 1} < \alpha - \delta \implies \delta < \alpha - \sqrt{\alpha^2 - 1}$$

$$\alpha + \delta < \sqrt{\alpha^2 + 1} \implies \delta < -\alpha + \sqrt{\alpha^2 + 1}$$

   We can prove that $-\alpha + \sqrt{\alpha^2 + 1} > -\alpha + \sqrt{\alpha^2 + 1}$ by contradiction. Now for the lower bound:

$$\sqrt{\alpha^2 - 1} < \alpha + \delta \implies -\alpha + \sqrt{\alpha^2 - 1} < \delta$$

$$\alpha - \delta < \sqrt{\alpha^2 + 1} \implies \alpha - \sqrt{\alpha^2 + 1} < \delta$$

   By contradiction we have $\alpha - \sqrt{\alpha^2 + 1} > -\alpha + \sqrt{\alpha^2 - 1}$. Hence:

$$\alpha - \sqrt{\alpha^2 + 1} < \delta < -\alpha + \sqrt{\alpha^2 + 1}$$

   ? our lower bound is wrong ?

Another approach

We begin from here:

$$\begin{aligned}\alpha - \delta < x < \alpha + \delta &\implies \alpha^2 - 1 < x^2 < \alpha^2 + 1\\ &\implies \sqrt{\alpha^2 - 1} < x < \sqrt{\alpha^2 + 1}.\end{aligned}$$

It is clear from a graph that the distance from any x to α cannot exceed either $\alpha - \sqrt{\alpha^2 - 1}$ or $\sqrt{\alpha^2 + 1} - \alpha$ for our function to be continuous.

2. Now define $f : [0,1] \to \mathbb{R}$ with rule $f(x) = x^2$.

   Show: $\forall \epsilon > 0;\ \exists \delta = \epsilon/2;\ \forall x, \alpha \in [0,1];\ |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon$

   Proof. We really should take the hint when given. Note that $\forall x, \alpha \in [0,1]$:

   $$|x + \alpha| < |x| + |\alpha| \leq 2$$

   by the triangle inequality. (helpful to think of a triangle)

   Since polynomials are continuous we apply the $\epsilon - \delta$ continuity definition:

   $\forall \epsilon > 0;\ \exists \delta > 0;\ \forall x, \alpha \in [0,1];\ |x - \alpha| < \delta \implies |x^2 - \alpha^2| < \epsilon$

   Consider the final line:

   $$\begin{aligned} |x^2 - \alpha^2| &= |x + \alpha||x - \alpha| \\ &< 2|x - \alpha| \end{aligned}$$

   So if we choose $\delta = \frac{\epsilon}{2}$ given any ϵ then $|x - \alpha| < \epsilon$ and

   $$|x^2 - \alpha^2| < \epsilon.$$

   $\square$

3. Consider function $f : (0, \infty) \to \mathbb{R}$ with rule $f(x) = \frac{1}{x}$.

   Is this function **uniformly continuous**?

   So firstly uniform continuity only makes sense if our function is already continuous. Since $x = 0$ is removed, our function is continuous and we may consider uniform continuity.

   Here we claim that f is **not** uniformly continuous.

   Note that the following two notions of uniform continuity is equivalent:

   - $\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in I; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$
   - $\forall s_n, t_n \in I; \lim_{n\to\infty} |s_n - t_n| = 0 \implies \lim_{n\to\infty} |f(s_n) - f(t_n)| = 0$

   given function $f : I \to \mathbb{R}$. This makes our life easy. To disprove uniform continuity we just need to negate the second condition:

$$\exists s_n, t_n \in I; \lim_{n\to\infty} |s_n - t_n| = 0 \quad \text{and} \quad \lim_{n\to\infty} |f(s_n) - f(t_n)| \neq 0.$$

   Choose $s_n = \frac{1}{n}$ and $t_n = \frac{2}{n}$ and we are finished.

4. Consider function $f : [a, \infty) \to \mathbb{R}$ for $\boldsymbol{a > 0}$ and $f(x) = \frac{1}{x}$.

   Is this function uniformly continuous?

   Proof. We claim that f is uniformly continuous.

   So we need:

$$\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in [a, \infty) : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

   Substituting f we have that $|x - y| < \delta \implies |\frac{1}{x} - \frac{1}{y}| < \epsilon$.

   Firstly without loss of generality define $x > y \geq a > 0$.

$$\therefore \frac{1}{a} > \frac{1}{y} > \frac{1}{x} \implies \frac{1}{a^2} > \frac{1}{xy}.$$

   Now consider:

$$\begin{aligned} |\frac{1}{x} - \frac{1}{y}| &= \frac{|x - y|}{xy} \\ &< \frac{|x - y|}{a^2} \\ &< \epsilon \end{aligned}$$

   if we choose $\delta = a^2\epsilon$. Therefore:

$$\forall \epsilon > 0; \exists \delta = a^2\epsilon; \forall x, y \in [a, \infty) : |x - y| < \delta \implies |\frac{1}{x} - \frac{1}{y}| < \epsilon,$$

   or that f is uniformly continuous on $[a, \infty)$ where $a > 0$. $\square$

Mean value theorem approach
Not finished!
Firstly $f(x) = \frac{1}{x}$ is differentiable on $[a, \infty)$ as we have the following limit:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= -\frac{1}{x^2} \quad \text{if} \quad x \neq 0. \end{aligned}$$

Define $x > y \geq a > 0$. By the mean value theorem we have:

$$\forall x, y \in [a, \infty); \exists c \in [y, x]; f'(c) = \frac{f(x) - f(y)}{x - y},$$

or that

$$\begin{aligned} |\frac{1}{x} - \frac{1}{y}| &= \frac{1}{x} - \frac{1}{y} \\ &= -\frac{1}{c^2}(x - y). \end{aligned}$$

Now for fixed $[y, x]$, the mean value theorem states that $c > \min\{x, y\}$ and hence $c > a$. (this holds $\forall x, y$) $\therefore \frac{1}{a^2} > \frac{1}{c^2} \implies -\frac{1}{a^2} > -\frac{1}{c^2}$ and:

$$\begin{aligned} |\frac{1}{x} - \frac{1}{y}| &= \frac{1}{x} - \frac{1}{y} \\ &= -\frac{1}{c^2}(x - y) \\ &< -\frac{1}{a^2}(x - y). \end{aligned}$$

Since x and y are non-negative $x - y = |x - y|$ and:

$$|\frac{1}{x} - \frac{1}{y}| < -\frac{1}{a^2}|x - y| = -\frac{\delta}{a^2} = \epsilon.$$

Now pick $\delta = -a^2\epsilon$ and we are finished.

5. 5

6. 6

7. Prove that the following statements are equivalent:
   - $f : I \to \mathbb{R}$ is uniformly continuous.

     i.e. $\forall \epsilon > 0;\ \exists \delta > 0;\ \forall x, y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$
   - $\forall s_n, t_n \in I : \lim_{n\to\infty} |s_n - t_n| = 0 \implies \lim_{n\to\infty} |f(s_n) - f(t_n)| = 0$

   Proof. $\to$ direction

   Direct proof. Assume that f is uniformly continuous:

   $$\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

   Also assume that:

   $$\forall s_n, t_n \in I : \lim_{n\to\infty} |s_n - t_n| = 0.$$

   But this may also be written as:

   $$\forall \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N \implies |s_n - t_n| < \delta,$$

   and since the definition of uniform continuity holds $\forall x, y \in I$ we may set $x = s_n$ and $y = t_n$. Combining our assumptions we get:

   $$\begin{aligned}\forall \epsilon > 0; \exists \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N &\implies |s_n - t_n| < \delta \\ &\implies |f(s_n) - f(t_n)| < \epsilon\end{aligned}$$

   But really what we want is:

   $$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |f(s_n) - f(t_n)| < \epsilon$$

   Or that:

   $$\lim_{n\to\infty} |f(s_n) - f(t_n)| = 0.$$

   □

Proof. $\leftarrow$ direction

Proof by contradiction. Assume that if:

$$\forall s_n, t_n \in I : \lim_{n\to\infty} |s_n - t_n| = 0 \implies \lim_{n\to\infty} |f(s_n) - f(t_n)| = 0,$$

then f is **not** uniformly continuous. i.e. that:

$$\exists \epsilon > 0; \forall \delta > 0; \exists x, y \in I : |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon.$$

So our first assumption gives us:

$$\forall \epsilon > 0; \exists N_1 \in \mathbb{N} : \forall n \geq N_1 \implies |f(s_n) - f(t_n)| < \epsilon$$

and holds true $\forall s_n, t_n \in I$ with condition:

$$\forall \delta > 0; \exists N_2 \in \mathbb{N} : \forall n \geq N_2 \implies |s_n - t_n| < \delta.$$

So taking $N = \max\{N_1, N_2\}$ and combining the previous two statements:

$$\begin{aligned} \forall \epsilon > 0; \forall \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N &\implies |s_n - t_n| < \delta \\ &\implies |f(s_n) - f(t_n)| < \epsilon \end{aligned}$$

The definition for **not** uniformly continuous only makes sense if x and y are sequences, since if they are real numbers then the following implies that they must be equal:

$$\forall \delta > 0; \exists x, y \in I : |x - y| < \delta,$$

and we reach a contradiction from the implication $\exists! \epsilon > 0 : 0 \geq \epsilon$. So for sequences x_n and y_n the previous statement implies:

$$\lim_{n\to\infty} |x_n - y_n| = 0$$

But we also assumed that $\forall s_n, t_n \in I : \lim_{n\to\infty} |s_n - t_n| = 0$.

This justifies setting $x_n = s_n$ and $y_n = t_n$ with condition $\forall n \geq N$.

$$\therefore \exists \epsilon > 0; \forall \delta > 0; \exists s_n, t_n \in I; |s_n - t_n| < \delta \text{ and } |f(s_n) - f(t_n)| \geq \epsilon.$$

But clearly this means that:

$$\lim_{n\to\infty} |f(s_n) - f(t_n)| \neq 0.$$

Then by truth tables f must be uniformly continuous. $\square$

Workshop 7

1. Let $f(x) = [x]$ for $\forall x \in \mathbb{R}$. Find the following integrals:

$$\int_{(0,5)} f$$

and

$$\int_{(-\frac{7}{3},\frac{12}{5}]} f.$$

Note that here we denote $[x]$ as the **floor function**. The floor function **rounds down** its input to the closest integer. So for example we have that $[3.5] = 3$ and $[-2.5] = -3$.

Let's first consider the open interval $(0,5)$. Notice that we can write f as a sum of characteristic functions each with interval of length 1:

$$f(x) = \sum_{j=1}^{5}(j-1)\chi_{[j-1,j)}(x).$$

Recall the characteristic function definition:

$$\chi_{[j-1,j)} = \begin{cases} 1 & x \in [j-1,j) \\ 0 & \text{otherwise.} \end{cases}$$

Integrating this gives us:

$$\begin{aligned}
\int_{(0,5)} f &= \int \sum_{j=1}^{5}(j-1)\chi_{[j-1,j)}(x) \\
&= \sum_{j=1}^{5}(j-1)\int \chi_{[j-1,j)}(x) \\
&= \sum_{j=1}^{5}(j-1) \\
&= 10.
\end{aligned}$$

So now consider the semi-open interval $(-\frac{7}{3}, \frac{12}{5}]$, for $f(x) = [x]$.

We write this function as a sum of characteristic functions:

$$\begin{aligned} f(x) &= -3\chi_{(-\frac{7}{3},-2)} + -2\chi_{[-2,-1)} + -1\chi_{[-1,0)} + 0 + 1\chi_{[1,2)} + 2\chi_{[2,\frac{12}{5}]} \\ &= -3\chi_{(-\frac{7}{3},-2)} + \sum_{j=-2}^{1} j\chi_{[j,j+1)} + 2\chi_{[2,\frac{12}{5}]}. \end{aligned}$$

Then integrating gives:

$$\begin{aligned} \int_{(-\frac{7}{3},\frac{12}{5}]} f &= -1 + -2 + \frac{4}{5} \\ &= -\frac{11}{5}. \end{aligned}$$

2. Let $f(x) = [nx]^2$ for $\forall x \in \mathbb{R}$ and $n \in \mathbb{N}$.

   Show that:
$$\int_{(0,1)} f = \frac{1}{n}\sum_{j=1}^{n-1} j^2.$$

   So first consider:
$$x \in [\frac{j}{n}, \frac{j+1}{n})$$
   for $j \in \{1, \ldots, n-1\}$. Multiplying each of these intervals by n gives:
$$nx \in [j, j+1),$$
   and this also works for negative n values. Taking the floor for each interval:
$$[nx] = j \quad \text{for} \quad \forall nx \in [j, j+1)$$
   and squaring this gives $[nx]^2 = j^2$ for $\forall j \in \{1, \ldots, n-1\}$.

   So we can now write our function f as the sum of characteristic functions:
$$f(x) = 0 \cdot \chi_{(0,\frac{j}{n})} + \sum_{j=1}^{n-1} j^2 \chi_{[\frac{j}{n},\frac{j+1}{n})}(x)$$
   and integrating this gives:
$$\begin{aligned}\int_{(0,1)} f &= \sum_{j=1}^{n-1} j^2 \int \chi_{[\frac{j}{n},\frac{j+1}{n})}(x)\\ &= \sum_{j=1}^{n-1} j^2 \cdot \frac{1}{n},\end{aligned}$$
   since we have defined n intervals each of length $\frac{1}{n}$. Finally:
$$\begin{aligned}\int_{(0,1)} f &= \frac{1}{n}\sum_{j=1}^{n-1} j^2\\ &= \frac{1}{n} \cdot \frac{n(n-1)(2n-1)}{6}\\ &= \frac{1}{6}(n-1)(2n-1).\end{aligned}$$

3. Let $f(x) = \frac{1}{[x]^2}$ for $\forall x \geq 1$. Show that f is integrable on $[1, \infty)$ and:

$$\int_{[1,\infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

Choose $J_j = [j, j+1)$ for $j \in \mathbb{N}$. Firstly we verify that:

$$\begin{aligned}\sum_{j=1}^{\infty} |c_j| \lambda(J_j) &= \sum_{j=1}^{\infty} \frac{1}{j^2} \\ &= \frac{\pi^2}{6} < \infty\end{aligned}$$

with $c_j = \frac{1}{j^2}$ and our interval of choice being of length 1.

Now $\forall x \in J_i$ where $i \in \mathbb{N}$, we have that:

$$\begin{aligned}\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) &= \sum_{j=1}^{\infty} \frac{1}{j^2} \chi_{J_j}(x) \\ &= \frac{1}{i^2} < \infty\end{aligned}$$

Hence we have proven that f is Lebesgue integrable on $[1, \infty)$, and:

$$\begin{aligned}\int_{[1,\infty)} f &= \sum_{j=1}^{\infty} \frac{1}{j^2} \int \chi_{J_j}(x) \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2} \lambda(J_j) \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2},\end{aligned}$$

and we are finished.

4. So now consider the function:

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}.$$

Show that f is integrable on every **bounded** interval I and:

$$\int_I f = \lambda(I).$$

Proof. Firstly choose:

$$c_j = \begin{cases} 1 & j = 1 \\ -1 & j > 1 \end{cases},$$

and

$$J_j = \begin{cases} I & j = 1 \\ q_{j-1} & j > 1 \end{cases},$$

where we define $I \cap \mathbb{Q} = \{q_1, q_2, \dots\}$ and $j \in \mathbb{N}$.

Then:

$$\begin{aligned} \sum_{j=1}^{\infty} |c_j|\lambda(J_j) &= \lambda(I) + \sum_{j=2}^{\infty} -\lambda(\{q_{j-1}\}) \\ &= \lambda(I) \\ &< \infty \end{aligned}$$

since interval I is bounded and hence of finite length.

Finally $\forall x \in I$:

$$\begin{aligned} f(x) &= \sum_{j=1}^{\infty} |c_j|\chi_{J_j}(x) \\ &= \chi_I(x) + \sum_{j=1}^{\infty} -\chi_{\{q_j\}}(x) \\ &< \infty \end{aligned}$$

So if $x \in \mathbb{Q}$ then $f(x) = 0$, and vice versa.

Therefore our function f is integrable on bounded I with formula:

$$\int_I f = \lambda(I).$$

$\square$

5. Let $f : [a, b] \to \mathbb{R}$ be continuous.

   Let $M = \sup_{x\in[a,b]} |f(x)|$ and $p > 0$. For part (a) show that:

   $\forall \epsilon : (0 < \epsilon < M/2); \exists(\alpha, \beta) \subset [a, b]$:

$$(M - \epsilon)^p(\beta - \alpha) \leq \int_a^b |f(x)|^p dx \leq M^p(b - a).$$

Proof. Direct proof.

Using the approximation property for suprema, $\exists x_0 \in [a, b]$:

$$\sup_{x\in[a,b]} |f(x)| - \epsilon < |f(x_0)|.$$

Then choose a $(\alpha, \beta) \subset [a, b]$ such that $\forall \epsilon > 0; \forall x \in (\alpha, \beta)$:

$$\sup_{x\in[a,b]} |f(x)| - \epsilon < |f(x)|.$$

Since these are strictly positive values taking the power of p preserves signs:

$$(M - \epsilon)^p < |f(x)|^p.$$

Also by the definition of supremum, for $\forall x \in [a, b]$:

$$|f(x)| \leq \sup_{x\in[a,b]} |f(x)|$$

and taking the pth gives:

$$|f(x)|^p \leq M^p.$$

Assuming the integrability of f we use the integral comparison test:

$$\int_a^b |f(x)|^p \leq M^p(b - a).$$

Similarly:

$$(M - \epsilon)^p(\beta - \alpha) < \int_\alpha^\beta |f(x)|^p dx.$$

But because $(\alpha, \beta) \subset [a, b]$:

$$\therefore (M - \epsilon)^p(\beta - \alpha) < \int_\alpha^\beta |f(x)|^p dx \leq \int_a^b |f(x)|^p dx \leq M^p(b - a).$$

$\square$

For part (b) we want:

$$\lim_{p\to\infty}\left(\int_a^b |f(x)|^p dx\right)^{1/p} = M$$

Proof. From part (a) we have that $\forall\epsilon : 0 < \epsilon < M/2$:

$$(M-\epsilon)^p(\beta-\alpha) \leq \int_a^b |f(x)|^p dx \leq M^p(b-a)$$

where $a < \alpha < \beta < b$. Taking the pth root gives:

$$(M-\epsilon)(\beta-\alpha)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \leq M(b-a)^{1/p}.$$

Now by definition $\beta-\alpha > 0$ and $b-a > 0$. So taking $p \to \infty$:

$$M-\epsilon \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \leq M < M+\epsilon$$

for all $0 < \epsilon < M/2$, by monotone convergence theorem. Then:

$$|\left(\int_a^b |f(x)|^p dx\right)^{1/p} - M| < \epsilon,$$

or that

$$\lim_{p\to\infty}\left(\int_a^b |f(x)|^p dx\right)^{1/p} = M.$$

□

6. Let $f(x) = n$ for $\forall x \in ((n+1)^{-2}, n^{-2}]$ and $n \in \mathbb{N}$. Show that:

$$\int_{(0,1]} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

We write our function as the following sum:

$$f(x) = \sum_{j=1}^{\infty} \chi_{(0,\frac{1}{j^2}]}(x).$$

This expression is clearly finite. We then check that:

$$\begin{aligned}\sum_{j=1}^{\infty} |c_j|\lambda(J_j) &= \sum_{j=1}^{\infty} \lambda((0,\frac{1}{j^2}]) \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2} \\ &= \frac{\pi^2}{6} < \infty.\end{aligned}$$

Finally:

$$\therefore \int_{(0,1)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

But why not:

$$f(x) = \sum_{j=1}^{\infty} j \cdot \chi_{(\frac{1}{(j+1)^2},\frac{1}{j^2}]}(x)?$$

Workshop 8

1. 1
2. 2
3. 3
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6. 6

7. Define $L(x) = \int_1^x \frac{\mathrm{d}t}{t}$ for $\forall x > 0$. Show:

- $L(xy) = L(x) + L(y)$
- $L'(x) = \frac{1}{x}$
- $L_{inv}(x) = E(x)$, where we define $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

For the first part we want to show:

$$\int_1^{yx} \frac{\mathrm{d}t}{t} = \int_1^x \frac{\mathrm{d}t}{t} + \int_1^y \frac{\mathrm{d}t}{t}.$$

Beginning from the left hand side let $t = x\alpha$.

$$\therefore \int_{t=1}^{t=xy} \implies \int_{\alpha=\frac{1}{x}}^{\alpha=y}$$

$$\therefore \mathrm{d}t = x\mathrm{d}\alpha$$
$$\therefore \frac{1}{t} = \frac{1}{x\alpha}$$

Now splitting this integral via T4.9 gives:

$$\begin{aligned}\int_{t=1}^{t=yx} \frac{\mathrm{d}t}{t} &= \int_{\alpha=\frac{1}{x}}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} \\ &= \int_{\alpha=1}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} + \int_{\alpha=\frac{1}{x}}^{\alpha=1} \frac{\mathrm{d}\alpha}{\alpha} \\ &= \int_{\alpha=1}^{\alpha=y} \frac{\mathrm{d}\alpha}{\alpha} + \int_{\beta=1}^{\beta=x} \frac{\mathrm{d}\beta}{\beta}\end{aligned}$$

where we set $\alpha = \frac{1}{x}\beta$ in the second integral.

$$\therefore L(xy) = L(x) + L(y)$$

Using the fundamental theorem of calculus:

$$L(x) = \int_1^x \frac{\mathrm{d}t}{t} \implies \frac{\mathrm{d}}{\mathrm{d}x} L(x) = \frac{1}{x}$$

since $\forall t > 0$, $\frac{1}{t}$ is continuous.

For the final part let's first define our functions:

$$E : \mathbb{R} \to \mathbb{R}$$

$$L : \mathbb{R}^+ \to \mathbb{R}$$

where $\mathbb{R}^+ = \mathbb{R}\backslash\{0, \dots\}$ represents the positive reals. Then define:

$$E(x) = z$$

for $x, z \in \mathbb{R}$ and:

$$L(y) = x$$

for $y \in \mathbb{R}^+$.

For these two functions to be inverses of each other we must show that:

$$E(L(y)) = y$$

and

$$L(E(x)) = x.$$

Consider

$$\frac{\mathrm{d}}{\mathrm{d}y}E(L(y)) = E(L(y))\frac{1}{y}.$$

Rearranging this and taking integrals:

$$\int_1^{E(L(y))} \frac{1}{E(L(y))}\mathrm{d}E(L(y)) = \int_1^y \frac{1}{y}\mathrm{d}y.$$

This gives:

$$\Big[L(E(L(y)))\Big]_{E(L(y))=1}^{E(L(y))=E(L(y))} = [L(y)]_1^y$$

or that:

$$L(E(L(y))) = L(y).$$

$$\therefore E(L(y)) = y$$

This is fine since $y \in \mathbb{R}^+ \subset \mathbb{R}$. Similarly consider the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}L(E(x)) = \frac{1}{E(x)}E(x) = 1.$$

Here $L(E(x))$ is defined as $\forall x \in \mathbb{R}; E(x) > 0$.

Integrating our expression as an indefinite integral:

$$L(E(x)) = x + k$$

and we find that $k = 0$ by setting $x = 0$.

$$\therefore L(E(x)) = x$$

8. Let $g : [a, b] \to \mathbb{R}$ be continuous, and that $g \geq 0$ for $\forall x \in [a, b]$. Then let:

$$\int_a^b g(x)\mathrm{d}x = 0.$$

Show that $\forall x \in [a, b]$ we have $g(x) = 0$.

Firstly because $g \geq 0$ splitting the integral using T4.9:

$$\int_a^b g(x)\mathrm{d}x = \int_a^c g(x)\mathrm{d}x + \int_c^b g(x)\mathrm{d}x = 0$$

implies that $\forall c \in [a, b]$:

$$\int_a^c g(x)\mathrm{d}x = 0$$

as areas of positive functions are always positive.

Since $g(x)$ is continuous we can use the fundamental theorem of calculus.

Let:

$$G(x) = \int_a^x g(t)\mathrm{d}t = 0$$

for $\forall x \in [a, b]$ as shown above. We then have that:

$$g(x) = \frac{\mathrm{d}}{\mathrm{d}x}G(x) = 0$$

for $\forall x \in [a, b]$.

Workshop 9

1. Show that χ_E is not Riemann-integrable, where $E = \mathbb{Q} \cap [0, 1]$.

   This is known as the Dirichlet function. Firstly let:

   $$\mathbb{Q} \cap [0, 1] = \{q_0, q_1, \dots\}$$

   and is the set of rationals between zero and one. Clearly we have that $q_0 = 0$ and $q_j \to 1$. Then let $I_j = (q_{j-1}, q_j)$ where $j \in \mathbb{N}$ which implies:

   $$\sup_{x,y \in I_j} |f(x) - f(y)| = 1$$

   if $f(x) = \chi_E$. We know that $f(x)$ in Riemann-integrable if and only if:

   $$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

   yet we have that:

   $$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) = \sum_{j=1}^{n} \lambda(I_j) = 1$$

   and hence our function is not Riemann-integrable.

2. For part (i) show that:
   If f is Riemann-integrable then $|f|$ is also Riemann-integrable.

   Let f be Riemann-integrable. Then by L4.1:

   $$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

   where $I_j = (x_{j-1}, x_j)$ and $a = x_0 < \cdots < x_n = b$.

   Now using the **reverse triangle inequality**:

   $$\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|$$

   for $\forall x, y \in I_j$ we then have that:

   $$\sup_{x,y \in I_j} \big||f(x)| - |f(y)|\big| \leq \sup_{x,y \in I_j} |f(x) - f(y)|$$

   and therefore:

   $$\sum_{j=1}^{n} \sup_{x,y \in I_j} \big||f(x)| - |f(y)|\big| \lambda(I_j) < \epsilon$$

   or that $|f|$ is also Riemann-integrable.

For part (ii) disprove that:
Let $|f|$ be Riemann-integrable. Then f is also Riemann-integrable.

So consider the following function:

$$\chi_E(x) = \begin{cases} 1 & x \text{ is rational} \\ -1 & \text{otherwise} \end{cases}$$

where $E = \mathbb{Q} \cap [0,1]$. Taking the modulus of this function gives:

$$|\chi_E(x)| = 1$$

for $\forall x \in [0,1]$ and clearly:

$$\sup_{x,y \in I_j} ||\chi_E(x)| - |\chi_E(y)|| = 0$$

where $I_j = (x_{j-1}, x_j)$, $j = 1, 2, \ldots$ and $0 = x_0 < \cdots < x_n = 1$. Then by L4.1, $|\chi_E|$ is Riemann-integrable. However this is not true without the modulus:

$$\sup_{x,y \in I_j} |\chi_E(x) - \chi_E(y)| = 2$$

and hence again via L4.1 this function is not Riemann-integrable.

3. Let $-\infty \leq a < b < \infty$ and let f be integrable on (u, b) for $\forall u \in (a, b)$. Then f is integrable on interval (a, b) **if and only if**:

$$\exists m < \infty : \forall u \in (a,b); \int_u^b |f| < m.$$

4. Show that the following statements are equivalent:
   - $\exists M < \infty$ such that $\forall v \in (a, b)$:
   $$\int_a^v |f| \leq M$$
   - Consider partition of (a, b):
   $$a < v_1 < v_2 < \cdots < b$$
   and define $I_1 = (a, v_1]$, $I_j = (v_{j-1}, v_j]$ where $j = 2, 3, \ldots$.
   We then have that $\exists M < \infty$ such that:
   $$\sum_{j=1}^{n} \int_{I_j} |f| \leq M$$
   where $n \in \mathbb{N}$.

   We begin by assuming the first condition. Let:
   $$\exists M < \infty : \forall v \in (a, b); \int_a^v |f| \leq M.$$
   Since this holds for all elements in (a, b) we order our elements as v_m where $m = 1, 2, \ldots$ and notice the following equality:
   $$\int_a^{v_m} |f| = \sum_{j=1}^{m} \int_{I_j} |f| \leq M.$$
   For the opposite direction assume that $\forall I_j$:
   $$\sum_{j=1}^{n} \int_{I_j} |f| \leq M < \infty$$
   and using T4.8(d) in lecture notes implies:
   $$\int_{I=(a,b)} |f| = \sum_{j=1}^{\infty} \int_{I_j} |f|.$$
   Then T4.8(a) implies that the integral of a subinterval is also defined and bounded.

5. Now show the converse of question 4. Let f be integrable on (a, b). Then:

$$\exists M < \infty : \forall v \in (a, b); \int_a^v |f| \leq M.$$

If f is integrable then its modulus $|f|$ must be also integrable. Now its integral value must be defined and bounded. Pick M to bound our integral:

$$\int_{(a,b)} |f| < M < \infty$$

and using T4.8(c) gives us that every subinterval is also integrable and bounded.

Questions 4 and 5 constitutes the proof to the following result.

Theorem 0.1.
Let $-\infty \leq a < b \leq \infty$ and let f be integrable on (a, v) for $\forall v \in (a, b)$.
Then f is integrable on (a, b) ***if and only if****:*

$$\exists M < \infty : \forall v \in (a, b); \int_a^v |f| \leq M$$

Similarly we have that:

Theorem 0.2.
Let $-\infty \leq a < b < \infty$ and let f be integrable on (u, b) for $\forall u \in (a, b)$.
Then f is integrable on interval (a, b) ***if and only if****:*

$$\exists M < \infty : \forall u \in (a, b); \int_u^b |f| < M.$$

6. Show the following:
   Let $-\infty \leq a < b < \infty$ and let f be integrable on (u, b) for $\forall u \in (a, b)$.
   Then f is integrable on interval (a, b) **if and only if**:

$$\exists M < \infty : \forall u \in (a, b); \int_u^b |f| < M.$$

Proof. $\leftarrow$ direction.
Firstly we show that the following are equivalent:

- $\exists M < \infty$ such that $\forall u \in (a, b)$:

$$\int_u^b |f| \leq M$$

- Consider partition of (a, b):

$$a < \cdots < u_2 < u_1 < b$$

  and define $I_1 = [u_1, b)$, $I_i = [u_i, u_{i-1})$ where $i = 2, 3, \ldots$.

  We then have that $\exists M < \infty$ such that:

$$\sum_{j=1}^{n} \int_{I_j} |f| \leq M$$

  where $n \in \mathbb{N}$.

We begin by assuming the first condition. Let:

$$\exists M < \infty : \forall u \in (a, b); \int_u^b |f| \leq M.$$

Since this holds for all elements in (a, b) we order our elements as u_m where $m = 1, 2, \ldots$ and the following equality holds via T4.9:

$$\int_{u_m}^b |f| = \sum_{j=1}^{m} \int_{I_j} |f| \leq M.$$

For the opposite direction assume that $\forall n \in \mathbb{N}$:

$$\sum_{j=1}^{n} \int_{I_j} |f| \leq M < \infty$$

and using T4.8(d):

$$\sum_{j=1}^{\infty} \int_{I_j} |f| = \int_{I=(a,b)} |f| < \infty.$$

Then T4.8(a) implies that the integral of a subinterval is also defined and bounded. This also implies that f is integrable on (a, b). $\square$

Proof. $\rightarrow$ direction.
Let f be Riemann-integrable on (a, b). Then f is also Lebesgue-integrable on (a, b) and so is $|f|$. By definition our integral value must be bounded:

$$\int_a^b |f| \leq M$$

and by T4.8(c) every subinterval of $I = (a, b)$ must also be integrable. This includes subintervals of form (u, b) where $u \in (a, b)$ and therefore:

$$\int_{(u,b)} |f| \leq M.$$

□

7. 7
8. 8
9. 9

10. Show that:

$$f(x) = (-1)^{[x]}\frac{1}{[x]}$$

is not integrable on $[1, \infty)$.

Firstly consider the negation of T4.2(c):
If $|f|$ is not integrable on I, then f is not integrable on I.

We can check for the integrability of $|f|$ via T4.3(b).

Our function can be written as a sum of characteristic functions:

$$\begin{aligned} |f(x)| &= \frac{1}{[x]} \\ &= \sum_{n=1}^{\infty} f_n(x) \end{aligned}$$

where

$$f_n(x) = \frac{1}{n}\chi_{[n,n+1)}(x) \geq 0$$

and $n \in \mathbb{N}$. Then let $I = [1, \infty)$ and consider the following:

$$\begin{aligned} \sum_{n=1}^{\infty} \int_I f_n &= \sum_{n=1}^{\infty} \frac{1}{n} \int_I \chi_{[n,n+1)}(x) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &\geq M \end{aligned}$$

for $\forall M \in \mathbb{R}$. By T4.3(b), $|f|$ is not integrable on $[1, \infty)$.

Then by T4.2(c), f is not integrable on $[1, \infty)$.

11. 11

Workshop 10

1. 1
2. 2
3. 3

4. Let $f : [a, b] \to \mathbb{C}$ be an integrable function. Show that:

$$\int_a^b |f(x)|\mathrm{d}x \leq C\Big[\int_a^b |f(x)|^2 \mathrm{d}x\Big]^{1/2}$$

where $C \in (0, \infty)$ and $f(x) \in L^2$.

From the previous problem we have that:

$$\begin{aligned}\int_a^b |f(x)g(x)|\mathrm{d}x &\leq \frac{\lambda}{2}||f(x)||_2^2 + \frac{1}{2\lambda}||g(x)||_2^2 \\ &= ||f(x)||_2 \, ||g(x)||_2\end{aligned}$$

where $\lambda = \Big[||g(x)||_2^2 \, ||f(x)||_2^{-2}\Big]^{1/2}$ and if we set $g(x) = 1$:

$$\int_a^b |f(x)|\mathrm{d}x \leq [b-a]^{1/2}\Big[\int_a^b |f(x)|^2 \mathrm{d}x\Big]^{1/2}.$$

Finally since interval length is nonnegative $\exists C = [b-a]^{1/2} > 0$:

$$\int_a^b |f(x)|\mathrm{d}x \leq C\Big[\int_a^b |f(x)|^2 \mathrm{d}x\Big]^{1/2}.$$

This is also a consequence of Hölder's inequality:

$$||fg||_1 \leq ||f||_p \, ||g||_p$$

where $p \geq 1$.

The converse of this statement is false as the opposite of it would be:

$$\exists C \in (0, \infty) : ||f||_2 > C||f||_1$$

where $C \in (0, \infty)$ and this is true by our previous proof.

Hence this is a contradiction and we have that:

$$\nexists C \in (0, \infty) : ||f||_2 \leq C||f||_1.$$

5. 5

6. For part (a) consider:

$$f_n(x) = n\chi_{[0,\frac{1}{n}]}.$$

Then in the limit we have $f(x) = \delta(x)\chi_{\{0\}}$ and:

$$\begin{aligned}
|f_n(x) - f(x)|^2 &= |n\chi_{[0,\frac{1}{n}]} - \delta(x)\chi_{\{0\}}|^2 \\
&= (n\chi_{[0,\frac{1}{n}]} - \delta(x)\chi_{\{0\}})^2 \\
&= n^2\chi^2_{[0,\frac{1}{n}]} - 2\delta(x)\chi_{[0,\frac{1}{n}]}\chi_{\{0\}} + \delta(x)^2\chi^2_{\{0\}} \\
&= n^2\chi_{[0,\frac{1}{n}]} - 2\delta(x)\chi_{\{0\}} + \delta(x)^2\chi_{\{0\}} \\
&= n^2\chi_{[0,\frac{1}{n}]} - \delta(x)\chi_{\{0\}}.
\end{aligned}$$

where $\delta(x)$ is the Dirac delta and we used the sift property.

Now let's consider the L^2-norm defined in $[0,1]$:

$$\begin{aligned}
||f_n(x) - f(x)||_2 &= \left[\int_0^1 |f_n(x) - f(x)|^2 \mathrm{d}x\right]^{1/2} \\
&= \left[\int_0^1 n^2\chi_{[0,\frac{1}{n}]} - \delta(x)\chi_{\{0\}}\mathrm{d}x\right]^{1/2} \\
&= \left[n^2 \cdot \frac{1}{n}\right]^{1/2} \\
&= n^{1/2}
\end{aligned}$$

and therefore:

$$\lim_{n\to\infty} ||f_n(x) - f(x)||_2 \neq 0.$$

Then our function $f_n(x) = n\chi_{[0,\frac{1}{n}]}$ is not L^2 convergent.

For part (b) show that if $|f_n(x)| \leq 1$ for $\forall x \in [0,1]$ then $f_n \to f$ in L^2.

If $|f_n(x)| \leq 1$ then $|f(x)| \leq 1$. $\therefore |f_n(x) - f(x)| \leq 2$

$$\therefore 0 < |f_n(x) - f(x)|^2 \leq 2|f_n(x) - f(x)|$$

Since $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ by pointwise convergence

$$|f_n(x) - f(x)|^2 \to 0$$

where we used the squeeze theorem here.

$$\therefore \left[\int_0^1 |f_n(x) - f(x)|^2 \mathrm{d}x\right]^{1/2} \to 0$$

if $n \to \infty$ and so $f_n(x)$ is L^2 convergent.