

# Honours Analysis Workshops

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## Workshop 6

1. Consider real function  $f = x^2$ . We know that  $f$  is continuous on  $\mathbb{R}$ , since it is a polynomial. So for  $f : \mathbb{R} \rightarrow \mathbb{R}$  the  $\epsilon - \delta$  definition states:

$$\forall \alpha \in \mathbb{R}; \forall \epsilon > 0; \exists \delta > 0; \forall x \in \mathbb{R}; \\ |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon$$

Now we set  $\alpha > 1$  and  $\epsilon = 1$ . Find **best** possible  $\delta = \delta(\epsilon)$ .

So whilst we can choose  $1 > \delta(\delta + 2\alpha)$ , this is certainly not the best bound.

Consider this approach instead:

$$\begin{aligned} \alpha - \delta < x < \alpha + \delta &\implies \alpha^2 - 1 < x^2 < \alpha^2 + 1 \\ &\implies \sqrt{\alpha^2 - 1} < x < \sqrt{\alpha^2 + 1} \end{aligned}$$

i.e. since we have an implication:

$$\sqrt{\alpha^2 - 1} < \alpha - \delta < x < \alpha + \delta < \sqrt{\alpha^2 + 1}$$

This is really 4 inequalities, and we need to choose the best 2. So:

$$\sqrt{\alpha^2 - 1} < \alpha - \delta \implies \delta < \alpha - \sqrt{\alpha^2 - 1}$$

$$\alpha + \delta < \sqrt{\alpha^2 + 1} \implies \delta < -\alpha + \sqrt{\alpha^2 + 1}$$

We can prove that  $-\alpha + \sqrt{\alpha^2 + 1} > \alpha - \sqrt{\alpha^2 - 1}$  by contradiction. Now for the lower bound:

$$\sqrt{\alpha^2 - 1} < \alpha + \delta \implies -\alpha + \sqrt{\alpha^2 - 1} < \delta$$

$$\alpha - \delta < \sqrt{\alpha^2 + 1} \implies \alpha - \sqrt{\alpha^2 + 1} < \delta$$

By contradiction we have  $\alpha - \sqrt{\alpha^2 + 1} > -\alpha + \sqrt{\alpha^2 - 1}$ . Hence:

$$\alpha - \sqrt{\alpha^2 + 1} < \delta < -\alpha + \sqrt{\alpha^2 - 1}$$

? our lower bound is wrong ?

**Another approach**

We begin from here:

$$\begin{aligned}\alpha - \delta < x < \alpha + \delta &\implies \alpha^2 - 1 < x^2 < \alpha^2 + 1 \\ &\implies \sqrt{\alpha^2 - 1} < x < \sqrt{\alpha^2 + 1}.\end{aligned}$$

It is clear from a graph that the distance from any  $x$  to  $\alpha$  cannot exceed either  $\alpha - \sqrt{\alpha^2 - 1}$  or  $\sqrt{\alpha^2 + 1} - \alpha$  for our function to be continuous.

2. Now define  $f : [0, 1] \rightarrow \mathbb{R}$  with rule  $f(x) = x^2$ .

Show:  $\forall \epsilon > 0; \exists \delta = \epsilon/2; \forall x, \alpha \in [0, 1]; |x - \alpha| < \delta \implies |f(x) - f(\alpha)| < \epsilon$

*Proof.* We really should take the hint when given. Note that  $\forall x, \alpha \in [0, 1]$ :

$$|x + \alpha| < |x| + |\alpha| \leq 2$$

by the triangle inequality. (helpful to think of a triangle)

Since polynomials are continuous we apply the  $\epsilon - \delta$  continuity definition:

$$\forall \epsilon > 0; \exists \delta > 0; \forall x, \alpha \in [0, 1]; |x - \alpha| < \delta \implies |x^2 - \alpha^2| < \epsilon$$

Consider the final line:

$$\begin{aligned} |x^2 - \alpha^2| &= |x + \alpha||x - \alpha| \\ &< 2|x - \alpha| \end{aligned}$$

So if we choose  $\delta = \frac{\epsilon}{2}$  given any  $\epsilon$  then  $|x - \alpha| < \delta$  and

$$|x^2 - \alpha^2| < \epsilon.$$

□

3. Consider function  $f : (0, \infty) \rightarrow \mathbb{R}$  with rule  $f(x) = \frac{1}{x}$ .

Is this function **uniformly continuous**?

So firstly uniform continuity only makes sense if our function is already continuous. Since  $x = 0$  is removed, our function is continuous and we may consider uniform continuity.

Here we claim that  $f$  is **not** uniformly continuous.

Note that the following two notions of uniform continuity is equivalent:

- $\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in I; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$
- $\forall s_n, t_n \in I; \lim_{n \rightarrow \infty} |s_n - t_n| = 0 \implies \lim_{n \rightarrow \infty} |f(s_n) - f(t_n)| = 0$

given function  $f : I \rightarrow \mathbb{R}$ . This makes our life easy. To disprove uniform continuity we just need to negate the second condition:

$$\exists s_n, t_n \in I; \lim_{n \rightarrow \infty} |s_n - t_n| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} |f(s_n) - f(t_n)| \neq 0.$$

Choose  $s_n = \frac{1}{n}$  and  $t_n = \frac{2}{n}$  and we are finished.

4. Consider function  $f : [a, \infty) \rightarrow \mathbb{R}$  for  $a > 0$  and  $f(x) = \frac{1}{x}$ .

Is this function uniformly continuous?

*Proof.* We claim that  $f$  is uniformly continuous.

So we need:

$$\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in [a, \infty) : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Substituting  $f$  we have that  $|x - y| < \delta \implies \left| \frac{1}{x} - \frac{1}{y} \right| < \epsilon$ .

Firstly without loss of generality define  $x > y \geq a > 0$ .

$$\therefore \frac{1}{a} > \frac{1}{y} > \frac{1}{x} \implies \frac{1}{a^2} > \frac{1}{xy}.$$

Now consider:

$$\begin{aligned} \left| \frac{1}{x} - \frac{1}{y} \right| &= \frac{|x - y|}{xy} \\ &< \frac{|x - y|}{a^2} \\ &< \epsilon \end{aligned}$$

if we choose  $\delta = a^2 \epsilon$ . Therefore:

$$\forall \epsilon > 0; \exists \delta = a^2 \epsilon; \forall x, y \in [a, \infty) : |x - y| < \delta \implies \left| \frac{1}{x} - \frac{1}{y} \right| < \epsilon,$$

or that  $f$  is uniformly continuous on  $[a, \infty)$  where  $a > 0$ . □

**Mean value theorem approach****Not finished!**

Firstly  $f(x) = \frac{1}{x}$  is differentiable on  $[a, \infty)$  as we have the following limit:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= -\frac{1}{x^2} \quad \text{if } x \neq 0. \end{aligned}$$

Define  $x > y \geq a > 0$ . By the mean value theorem we have:

$$\forall x, y \in [a, \infty); \exists c \in [y, x]; f'(c) = \frac{f(x) - f(y)}{x - y},$$

or that

$$\begin{aligned} \left| \frac{1}{x} - \frac{1}{y} \right| &= \frac{1}{x} - \frac{1}{y} \\ &= -\frac{1}{c^2}(x - y). \end{aligned}$$

Now for fixed  $[y, x]$ , the mean value theorem states that  $c > \min\{x, y\}$  and hence  $c > a$ . (this holds  $\forall x, y) \therefore \frac{1}{a^2} > \frac{1}{c^2} \implies -\frac{1}{a^2} > -\frac{1}{c^2}$  and:

$$\begin{aligned} \left| \frac{1}{x} - \frac{1}{y} \right| &= \frac{1}{x} - \frac{1}{y} \\ &= -\frac{1}{c^2}(x - y) \\ &< -\frac{1}{a^2}(x - y). \end{aligned}$$

Since  $x$  and  $y$  are non-negative  $x - y = |x - y|$  and:

$$\left| \frac{1}{x} - \frac{1}{y} \right| < -\frac{1}{a^2}|x - y| = -\frac{\delta}{a^2} = \epsilon.$$

Now pick  $\delta = -a^2\epsilon$  and we are finished.



5. 5

6. 6

7. Prove that the following statements are equivalent:

- $f : I \rightarrow \mathbb{R}$  is uniformly continuous.  
i.e.  $\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$
- $\forall s_n, t_n \in I : \lim_{n \rightarrow \infty} |s_n - t_n| = 0 \implies \lim_{n \rightarrow \infty} |f(s_n) - f(t_n)| = 0$

*Proof.*  $\rightarrow$  direction

Direct proof. Assume that  $f$  is uniformly continuous:

$$\forall \epsilon > 0; \exists \delta > 0; \forall x, y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Also assume that:

$$\forall s_n, t_n \in I : \lim_{n \rightarrow \infty} |s_n - t_n| = 0.$$

But this may also be written as:

$$\forall \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N \implies |s_n - t_n| < \delta,$$

and since the definition of uniform continuity holds  $\forall x, y \in I$  we may set  $x = s_n$  and  $y = t_n$ . Combining our assumptions we get:

$$\begin{aligned} \forall \epsilon > 0; \exists \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N &\implies |s_n - t_n| < \delta \\ &\implies |f(s_n) - f(t_n)| < \epsilon \end{aligned}$$

But really what we want is:

$$\forall \epsilon > 0; \exists N \in \mathbb{N} : \forall n \geq N \implies |f(s_n) - f(t_n)| < \epsilon$$

Or that:

$$\lim_{n \rightarrow \infty} |f(s_n) - f(t_n)| = 0.$$

□

*Proof.*  $\leftarrow$  direction

Proof by contradiction. Assume that if:

$$\forall s_n, t_n \in I : \lim_{n \rightarrow \infty} |s_n - t_n| = 0 \implies \lim_{n \rightarrow \infty} |f(s_n) - f(t_n)| = 0,$$

then  $f$  is **not** uniformly continuous. i.e. that:

$$\exists \epsilon > 0; \forall \delta > 0; \exists x, y \in I : |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon.$$

So our first assumption gives us:

$$\forall \epsilon > 0; \exists N_1 \in \mathbb{N} : \forall n \geq N_1 \implies |f(s_n) - f(t_n)| < \epsilon$$

and holds true  $\forall s_n, t_n \in I$  with condition:

$$\forall \delta > 0; \exists N_2 \in \mathbb{N} : \forall n \geq N_2 \implies |s_n - t_n| < \delta.$$

So taking  $N = \max\{N_1, N_2\}$  and combining the previous two statements:

$$\begin{aligned} \forall \epsilon > 0; \forall \delta > 0; \forall s_n, t_n \in I; \exists N \in \mathbb{N} : \forall n \geq N &\implies |s_n - t_n| < \delta \\ &\implies |f(s_n) - f(t_n)| < \epsilon \end{aligned}$$

The definition for **not** uniformly continuous only makes sense if  $x$  and  $y$  are sequences, since if they are real numbers then the following implies that they must be equal:

$$\forall \delta > 0; \exists x, y \in I : |x - y| < \delta,$$

and we reach a contradiction from the implication  $\exists! \epsilon > 0 : 0 \geq \epsilon$ . So for sequences  $x_n$  and  $y_n$  the previous statement implies:

$$\lim_{n \rightarrow \infty} |x_n - y_n| = 0$$

But we also assumed that  $\forall s_n, t_n \in I : \lim_{n \rightarrow \infty} |s_n - t_n| = 0$ .

This justifies setting  $x_n = s_n$  and  $y_n = t_n$  with condition  $\forall n \geq N$ .

$$\therefore \exists \epsilon > 0; \forall \delta > 0; \exists s_n, t_n \in I; |s_n - t_n| < \delta \text{ and } |f(s_n) - f(t_n)| \geq \epsilon.$$

But clearly this means that:

$$\lim_{n \rightarrow \infty} |f(s_n) - f(t_n)| \neq 0.$$

Then by truth tables  $f$  must be uniformly continuous.  $\square$

## Workshop 7

1. Let  $f(x) = [x]$  for  $\forall x \in \mathbb{R}$ . Find the following integrals:

$$\int_{(0,5)} f$$

and

$$\int_{(-\frac{7}{3}, \frac{12}{5}]} f.$$

Note that here we denote  $[x]$  as the **floor function**. The floor function **rounds down** its input to the closest integer. So for example we have that  $[3.5] = 3$  and  $[-2.5] = -3$ .

Let's first consider the open interval  $(0, 5)$ . Notice that we can write  $f$  as a sum of characteristic functions each with interval of length 1:

$$f(x) = \sum_{j=1}^5 (j-1) \chi_{[j-1, j)}(x).$$

Recall the characteristic function definition:

$$\chi_{[j-1, j)} = \begin{cases} 1 & x \in [j-1, j) \\ 0 & \text{otherwise.} \end{cases}$$

Integrating this gives us:

$$\begin{aligned} \int_{(0,5)} f &= \int \sum_{j=1}^5 (j-1) \chi_{[j-1, j)}(x) \\ &= \sum_{j=1}^5 (j-1) \int \chi_{[j-1, j)}(x) \\ &= \sum_{j=1}^5 (j-1) \\ &= 10. \end{aligned}$$

So now consider the semi-open interval  $(-\frac{7}{3}, \frac{12}{5}]$ , for  $f(x) = [x]$ .

We write this function as a sum of characteristic functions:

$$\begin{aligned} f(x) &= -3\chi_{(-\frac{7}{3}, -2)} + -2\chi_{[-2, -1)} + -1\chi_{[-1, 0)} + 0 + 1\chi_{[1, 2)} + 2\chi_{[2, \frac{12}{5}]} \\ &= -3\chi_{(-\frac{7}{3}, -2)} + \sum_{j=-2}^1 j\chi_{[j, j+1)} + 2\chi_{[2, \frac{12}{5}]}. \end{aligned}$$

Then integrating gives:

$$\begin{aligned} \int_{(-\frac{7}{3}, \frac{12}{5}]} f &= -1 + -2 + \frac{4}{5} \\ &= -\frac{11}{5}. \end{aligned}$$

2. Let  $f(x) = [nx]^2$  for  $\forall x \in \mathbb{R}$  and  $n \in \mathbb{N}$ .

Show that:

$$\int_{(0,1)} f = \frac{1}{n} \sum_{j=1}^{n-1} j^2.$$

So first consider:

$$x \in \left[\frac{j}{n}, \frac{j+1}{n}\right)$$

for  $j \in \{1, \dots, n-1\}$ . Multiplying each of these intervals by  $n$  gives:

$$nx \in [j, j+1),$$

and this also works for negative  $n$  values. Taking the floor for each interval:

$$[nx] = j \quad \text{for } \forall nx \in [j, j+1)$$

and squaring this gives  $[nx]^2 = j^2$  for  $\forall j \in \{1, \dots, n-1\}$ .

So we can now write our function  $f$  as the sum of characteristic functions:

$$f(x) = 0 \cdot \chi_{(0, \frac{1}{n})} + \sum_{j=1}^{n-1} j^2 \chi_{[\frac{j}{n}, \frac{j+1}{n})}(x)$$

and integrating this gives:

$$\begin{aligned} \int_{(0,1)} f &= \sum_{j=1}^{n-1} j^2 \int \chi_{[\frac{j}{n}, \frac{j+1}{n})}(x) \\ &= \sum_{j=1}^{n-1} j^2 \cdot \frac{1}{n}, \end{aligned}$$

since we have defined  $n$  intervals each of length  $\frac{1}{n}$ . Finally:

$$\begin{aligned} \int_{(0,1)} f &= \frac{1}{n} \sum_{j=1}^{n-1} j^2 \\ &= \frac{1}{n} \cdot \frac{n(n-1)(2n-1)}{6} \\ &= \frac{1}{6} (n-1)(2n-1). \end{aligned}$$

3. Let  $f(x) = \frac{1}{x^2}$  for  $\forall x \geq 1$ . Show that  $f$  is integrable on  $[1, \infty)$  and:

$$\int_{[1, \infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

Choose  $J_j = [j, j+1)$  for  $j \in \mathbb{N}$ . Firstly we verify that:

$$\begin{aligned} \sum_{j=1}^{\infty} |c_j| \lambda(J_j) &= \sum_{j=1}^{\infty} \frac{1}{j^2} \\ &= \frac{\pi^2}{6} < \infty \end{aligned}$$

with  $c_j = \frac{1}{j^2}$  and our interval of choice being of length 1.

Now  $\forall x \in J_i$  where  $i \in \mathbb{N}$ , we have that:

$$\begin{aligned} \sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) &= \sum_{j=1}^{\infty} \frac{1}{j^2} \chi_{J_j}(x) \\ &= \frac{1}{i^2} < \infty \end{aligned}$$

Hence we have proven that  $f$  is Lebesgue integrable on  $[1, \infty)$ , and:

$$\begin{aligned} \int_{[1, \infty)} f &= \sum_{j=1}^{\infty} \frac{1}{j^2} \int \chi_{J_j}(x) \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2} \lambda(J_j) \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2}, \end{aligned}$$

and we are finished.



4. So now consider the function:

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}.$$

Show that  $f$  is integrable on every **bounded** interval  $I$  and:

$$\int_I f = \lambda(I).$$

*Proof.* Firstly choose:

$$c_j = \begin{cases} 1 & j = 1 \\ -1 & j > 1 \end{cases},$$

and

$$J_j = \begin{cases} I & j = 1 \\ q_{j-1} & j > 1 \end{cases},$$

where we define  $I \cap \mathbb{Q} = \{q_1, q_2, \dots\}$  and  $j \in \mathbb{N}$ .

Then:

$$\begin{aligned} \sum_{j=1}^{\infty} |c_j| \lambda(J_j) &= \lambda(I) + \sum_{j=2}^{\infty} -\lambda(\{q_{j-1}\}) \\ &= \lambda(I) \\ &< \infty \end{aligned}$$

since interval  $I$  is bounded and hence of finite length.

Finally  $\forall x \in I$ :

$$\begin{aligned} f(x) &= \sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) \\ &= \chi_I(x) + \sum_{j=1}^{\infty} -\chi_{\{q_j\}}(x) \\ &< \infty \end{aligned}$$

So if  $x \in \mathbb{Q}$  then  $f(x) = 0$ , and vice versa.

Therefore our function  $f$  is integrable on bounded  $I$  with formula:

$$\int_I f = \lambda(I).$$

□

5. Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous.

Let  $M = \sup_{x \in [a, b]} |f(x)|$  and  $p > 0$ . For part (a) show that:

$\forall \epsilon : (0 < \epsilon < M/2); \exists (\alpha, \beta) \subset [a, b] :$

$$(M - \epsilon)^p(\beta - \alpha) \leq \int_a^b |f(x)|^p dx \leq M^p(b - a).$$

*Proof.* Direct proof.

Using the approximation property for suprema,  $\exists x_0 \in [a, b]$ :

$$\sup_{x \in [a, b]} |f(x)| - \epsilon < |f(x_0)|.$$

Then choose a  $(\alpha, \beta) \subset [a, b]$  such that  $\forall \epsilon > 0; \forall x \in (\alpha, \beta) :$

$$\sup_{x \in [a, b]} |f(x)| - \epsilon < |f(x)|.$$

Since these are strictly positive values taking the power of  $p$  preserves signs:

$$(M - \epsilon)^p < |f(x)|^p.$$

Also by the definition of supremum, for  $\forall x \in [a, b] :$

$$|f(x)| \leq \sup_{x \in [a, b]} |f(x)|$$

and taking the  $p$ th gives:

$$|f(x)|^p \leq M^p.$$

Assuming the integrability of  $f$  we use the integral comparison test:

$$\int_a^b |f(x)|^p dx \leq M^p(b - a).$$

Similarly:

$$(M - \epsilon)^p(\beta - \alpha) < \int_\alpha^\beta |f(x)|^p dx.$$

But because  $(\alpha, \beta) \subset [a, b]$ :

$$\therefore (M - \epsilon)^p(\beta - \alpha) < \int_\alpha^\beta |f(x)|^p dx \leq \int_a^b |f(x)|^p dx \leq M^p(b - a).$$

□

For part (b) we want:

$$\lim_{p \rightarrow \infty} \left( \int_a^b |f(x)|^p dx \right)^{1/p} = M$$

*Proof.* From part (a) we have that  $\forall \epsilon : 0 < \epsilon < M/2$ :

$$(M - \epsilon)^p (\beta - \alpha) \leq \int_a^b |f(x)|^p dx \leq M^p (b - a)$$

where  $a < \alpha < \beta < b$ . Taking the  $p$ th root gives:

$$(M - \epsilon)(\beta - \alpha)^{1/p} \leq \left( \int_a^b |f(x)|^p dx \right)^{1/p} \leq M(b - a)^{1/p}.$$

Now by definition  $\beta - \alpha > 0$  and  $b - a > 0$ . So taking  $p \rightarrow \infty$ :

$$M - \epsilon \leq \left( \int_a^b |f(x)|^p dx \right)^{1/p} \leq M < M + \epsilon$$

for all  $0 < \epsilon < M/2$ , by monotone convergence theorem. Then:

$$\left| \left( \int_a^b |f(x)|^p dx \right)^{1/p} - M \right| < \epsilon,$$

or that

$$\lim_{p \rightarrow \infty} \left( \int_a^b |f(x)|^p dx \right)^{1/p} = M.$$

□

6. Let  $f(x) = n$  for  $\forall x \in ((n+1)^{-2}, n^{-2}]$  and  $n \in \mathbb{N}$ . Show that:

$$\int_{(0,1]} f = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

We write our function as the following sum:

$$f(x) = \sum_{j=1}^{\infty} \chi_{(0, \frac{1}{j^2}]}(x).$$

This expression is clearly finite. We then check that:

$$\begin{aligned} \sum_{j=1}^{\infty} |c_j| \lambda(J_j) &= \sum_{j=1}^{\infty} \lambda((0, \frac{1}{j^2}]) \\ &= \sum_{j=1}^{\infty} \frac{1}{j^2} \\ &= \frac{\pi^2}{6} < \infty. \end{aligned}$$

Finally:

$$\therefore \int_{(0,1)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

But **why not**:

$$f(x) = \sum_{j=1}^{\infty} j \cdot \chi_{(\frac{1}{(j+1)^2}, \frac{1}{j^2}]}(x)?$$

## Workshop 8

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6

7. Define  $L(x) = \int_1^x \frac{dt}{t}$  for  $\forall x > 0$ . Show:

- $L(xy) = L(x) + L(y)$
- $L'(x) = \frac{1}{x}$
- $L_{inv}(x) = E(x)$ , where we define  $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ .

For the first part we want to show:

$$\int_1^{yx} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_1^y \frac{dt}{t}.$$

Beginning from the left hand side let  $t = x\alpha$ .

$$\therefore \int_{t=1}^{t=yx} \Rightarrow \int_{\alpha=\frac{1}{x}}^{\alpha=y}$$

$$\therefore dt = x d\alpha$$

$$\therefore \frac{1}{t} = \frac{1}{x\alpha}$$

Now splitting this integral via T4.9 gives:

$$\begin{aligned} \int_{t=1}^{t=yx} \frac{dt}{t} &= \int_{\alpha=\frac{1}{x}}^{\alpha=y} \frac{d\alpha}{\alpha} \\ &= \int_{\alpha=1}^{\alpha=y} \frac{d\alpha}{\alpha} + \int_{\alpha=\frac{1}{x}}^{\alpha=1} \frac{d\alpha}{\alpha} \\ &= \int_{\alpha=1}^{\alpha=y} \frac{d\alpha}{\alpha} + \int_{\beta=1}^{\beta=x} \frac{d\beta}{\beta} \end{aligned}$$

where we set  $\alpha = \frac{1}{x}\beta$  in the second integral.

$$\therefore L(xy) = L(x) + L(y)$$

Using the fundamental theorem of calculus:

$$L(x) = \int_1^x \frac{dt}{t} \Rightarrow \frac{d}{dx} L(x) = \frac{1}{x}$$

since  $\forall t > 0$ ,  $\frac{1}{t}$  is continuous.

For the final part let's first define our functions:

$$E : \mathbb{R} \rightarrow \mathbb{R}$$

$$L : \mathbb{R}^+ \rightarrow \mathbb{R}$$

where  $\mathbb{R}^+ = \mathbb{R} \setminus \{0, \dots\}$  represents the positive reals. Then define:

$$E(x) = z$$

for  $x, z \in \mathbb{R}$  and:

$$L(y) = x$$

for  $y \in \mathbb{R}^+$ .

For these two functions to be inverses of each other we must show that:

$$E(L(y)) = y$$

and

$$L(E(x)) = x.$$

Consider

$$\frac{d}{dy} E(L(y)) = E(L(y)) \frac{1}{y}.$$

Rearranging this and taking integrals:

$$\int_1^{E(L(y))} \frac{1}{E(L(y))} dE(L(y)) = \int_1^y \frac{1}{y} dy.$$

This gives:

$$\left[ L(E(L(y))) \right]_{E(L(y))=1}^{E(L(y))=E(L(y))} = [L(y)]_1^y$$

or that:

$$L(E(L(y))) = L(y).$$

$$\therefore E(L(y)) = y$$

This is fine since  $y \in \mathbb{R}^+ \subset \mathbb{R}$ . Similarly consider the following:

$$\frac{d}{dx} L(E(x)) = \frac{1}{E(x)} E(x) = 1.$$

Here  $L(E(x))$  is defined as  $\forall x \in \mathbb{R}; E(x) > 0$ .

Integrating our expression as an indefinite integral:

$$L(E(x)) = x + k$$

and we find that  $k = 0$  by setting  $x = 0$ .

$$\therefore L(E(x)) = x$$

8. Let  $g : [a, b] \rightarrow \mathbb{R}$  be continuous, and that  $g \geq 0$  for  $\forall x \in [a, b]$ . Then let:

$$\int_a^b g(x)dx = 0.$$

Show that  $\forall x \in [a, b]$  we have  $g(x) = 0$ .

Firstly because  $g \geq 0$  splitting the integral using T4.9:

$$\int_a^b g(x)dx = \int_a^c g(x)dx + \int_c^b g(x)dx = 0$$

implies that  $\forall c \in [a, b]$ :

$$\int_a^c g(x)dx = 0$$

as areas of positive functions are always positive.

Since  $g(x)$  is continuous we can use the fundamental theorem of calculus.

Let:

$$G(x) = \int_a^x g(t)dt = 0$$

for  $\forall x \in [a, b]$  as shown above. We then have that:

$$g(x) = \frac{d}{dx}G(x) = 0$$

for  $\forall x \in [a, b]$ .



## Workshop 9

1. Show that  $\chi_E$  is not Riemann-integrable, where  $E = \mathbb{Q} \cap [0, 1]$ .

This is known as the Dirichlet function. Firstly let:

$$\mathbb{Q} \cap [0, 1] = \{q_0, q_1, \dots\}$$

and is the set of rationals between zero and one. Clearly we have that  $q_0 = 0$  and  $q_j \rightarrow 1$ . Then let  $I_j = (q_{j-1}, q_j)$  where  $j \in \mathbb{N}$  which implies:

$$\sup_{x, y \in I_j} |f(x) - f(y)| = 1$$

if  $f(x) = \chi_E$ . We know that  $f(x)$  is Riemann-integrable if and only if:

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

yet we have that:

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| \lambda(I_j) = \sum_{j=1}^n \lambda(I_j) = 1$$

and hence our function is not Riemann-integrable.

2. For part (i) show that:

If  $f$  is Riemann-integrable then  $|f|$  is also Riemann-integrable.

Let  $f$  be Riemann-integrable. Then by L4.1:

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

where  $I_j = (x_{j-1}, x_j)$  and  $a = x_0 < \dots < x_n = b$ .

Now using the **reverse triangle inequality**:

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|$$

for  $\forall x, y \in I_j$  we then have that:

$$\sup_{x, y \in I_j} ||f(x)| - |f(y)|| \leq \sup_{x, y \in I_j} |f(x) - f(y)|$$

and therefore:

$$\sum_{j=1}^n \sup_{x, y \in I_j} ||f(x)| - |f(y)|| \lambda(I_j) < \epsilon$$

or that  $|f|$  is also Riemann-integrable.

For part (ii) disprove that:

Let  $|f|$  be Riemann-integrable. Then  $f$  is also Riemann-integrable.

So consider the following function:

$$\chi_E(x) = \begin{cases} 1 & x \text{ is rational} \\ -1 & \text{otherwise} \end{cases}$$

where  $E = \mathbb{Q} \cap [0, 1]$ . Taking the modulus of this function gives:

$$|\chi_E(x)| = 1$$

for  $\forall x \in [0, 1]$  and clearly:

$$\sup_{x, y \in I_j} ||\chi_E(x)| - |\chi_E(y)|| = 0$$

where  $I_j = (x_{j-1}, x_j)$ ,  $j = 1, 2, \dots$  and  $0 = x_0 < \dots < x_n = 1$ . Then by L4.1,  $|\chi_E|$  is Riemann-integrable. However this is not true without the modulus:

$$\sup_{x, y \in I_j} |\chi_E(x) - \chi_E(y)| = 2$$

and hence again via L4.1 this function is not Riemann-integrable.

3. Let  $-\infty \leq a < b < \infty$  and let  $f$  be integrable on  $(u, b)$  for  $\forall u \in (a, b)$ . Then  $f$  is integrable on interval  $(a, b)$  **if and only if**:

$$\exists m < \infty : \forall u \in (a, b); \int_u^b |f| < m.$$

4. Show that the following statements are equivalent:

- $\exists M < \infty$  such that  $\forall v \in (a, b)$ :

$$\int_a^v |f| \leq M$$

- Consider partition of  $(a, b)$ :

$$a < v_1 < v_2 < \dots < b$$

and define  $I_1 = (a, v_1]$ ,  $I_j = (v_{j-1}, v_j]$  where  $j = 2, 3, \dots$

We then have that  $\exists M < \infty$  such that:

$$\sum_{j=1}^n \int_{I_j} |f| \leq M$$

where  $n \in \mathbb{N}$ .

We begin by assuming the first condition. Let:

$$\exists M < \infty : \forall v \in (a, b); \int_a^v |f| \leq M.$$

Since this holds for all elements in  $(a, b)$  we order our elements as  $v_m$  where  $m = 1, 2, \dots$  and notice the following equality:

$$\int_a^{v_m} |f| = \sum_{j=1}^m \int_{I_j} |f| \leq M.$$

For the opposite direction assume that  $\forall I_j$ :

$$\sum_{j=1}^n \int_{I_j} |f| \leq M < \infty$$

and using T4.8(d) in lecture notes implies:

$$\int_{I=(a,b)} |f| = \sum_{j=1}^{\infty} \int_{I_j} |f|.$$

Then T4.8(a) implies that the integral of a subinterval is also defined and bounded.

5. Now show the converse of question 4. Let  $f$  be integrable on  $(a, b)$ . Then:

$$\exists M < \infty : \forall v \in (a, b); \int_a^v |f| \leq M.$$

If  $f$  is integrable then its modulus  $|f|$  must be also integrable. Now its integral value must be defined and bounded. Pick  $M$  to bound our integral:

$$\int_{(a,b)} |f| < M < \infty$$

and using T4.8(c) gives us that every subinterval is also integrable and bounded.

Questions 4 and 5 constitutes the proof to the following result.

**Theorem 0.1.**

Let  $-\infty \leq a < b \leq \infty$  and let  $f$  be integrable on  $(a, v)$  for  $\forall v \in (a, b)$ . Then  $f$  is integrable on  $(a, b)$  **if and only if**:

$$\exists M < \infty : \forall v \in (a, b); \int_a^v |f| \leq M$$

Similarly we have that:

**Theorem 0.2.**

Let  $-\infty \leq a < b < \infty$  and let  $f$  be integrable on  $(u, b)$  for  $\forall u \in (a, b)$ . Then  $f$  is integrable on interval  $(a, b)$  **if and only if**:

$$\exists m < \infty : \forall u \in (a, b); \int_u^b |f| < m.$$

6. Show the following:

Let  $-\infty \leq a < b < \infty$  and let  $f$  be integrable on  $(u, b)$  for  $\forall u \in (a, b)$ .

Then  $f$  is integrable on interval  $(a, b)$  **if and only if**:

$$\exists M < \infty : \forall u \in (a, b); \int_u^b |f| < M.$$

*Proof.*  $\leftarrow$  direction.

Firstly we show that the following are equivalent:

- $\exists M < \infty$  such that  $\forall u \in (a, b)$ :

$$\int_u^b |f| \leq M$$

- Consider partition of  $(a, b)$ :

$$a < \dots < u_2 < u_1 < b$$

and define  $I_1 = [u_1, b)$ ,  $I_i = [u_i, u_{i-1})$  where  $i = 2, 3, \dots$

We then have that  $\exists M < \infty$  such that:

$$\sum_{j=1}^n \int_{I_j} |f| \leq M$$

where  $n \in \mathbb{N}$ .

We begin by assuming the first condition. Let:

$$\exists M < \infty : \forall u \in (a, b); \int_u^b |f| \leq M.$$

Since this holds for all elements in  $(a, b)$  we order our elements as  $u_m$  where  $m = 1, 2, \dots$  and the following equality holds via T4.9:

$$\int_{u_m}^b |f| = \sum_{j=1}^m \int_{I_j} |f| \leq M.$$

For the opposite direction assume that  $\forall n \in \mathbb{N}$ :

$$\sum_{j=1}^n \int_{I_j} |f| \leq M < \infty$$

and using T4.8(d):

$$\sum_{j=1}^{\infty} \int_{I_j} |f| = \int_{I=(a,b)} |f| < \infty.$$

Then T4.8(a) implies that the integral of a subinterval is also defined and bounded. This also implies that  $f$  is integrable on  $(a, b)$ .  $\square$

*Proof.*  $\rightarrow$  direction.  
time is up.

□

7.