1 Example 2.1 derivation

We start with our state representation of the cartpole. We will refer to the cart position as p, the pole angle as θ, and the input as u. The nonlinear dynamics are then given as:

$$x = \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \dot{p} \\ \dot{\theta} \\ a_p\left(p, \theta, \dot{p}, \dot{\theta}, u\right) \\ a_\theta\left(p, \theta, \dot{p}, \dot{\theta}, u\right) \end{bmatrix} \tag{1}$$

where $a_p\left(p, \theta, \dot{p}, \dot{\theta}, u\right)$ and $a_\theta\left(p, \theta, \dot{p}, \dot{\theta}, u\right)$ are the nonlinear functions representing the accelerations of the cart and pole respectively.

The first derivation step is to linearize the dynamics about the upright equilibrium point (x_0) to get:

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ & \frac{\partial a_p}{\partial x} & & \\ & \frac{\partial a_\theta}{\partial x} & & \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{\partial a_p}{\partial u} \\ \frac{\partial a_\theta}{\partial u} \end{bmatrix} u \tag{2}$$

where $\hat{x} = (x - x_0)$ is the linearized state.

The next step is to update the dynamics to only depend on our measurement of the cart acceleration, and remove dependence on u. This is needed because we don't have info about u and only have an IMU measurement of $\ddot{p}$. We assume our measurement of $\ddot{p}$ is given by:

$$\ddot{p} = y + s \tag{3}$$

with $\ddot{p}$ as the true acceleration of the cart, and $s \sim \mathcal{N}\left(0, \sigma^2\right)$ as the IMU measurement noise.

We can now include this measurement into our linearized dynamics by introducing y as a secondary control input:

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ & \frac{\partial a_\theta}{\partial x} & & \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ \frac{\partial a_\theta}{\partial u} & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s \tag{4}$$

Next we need to eliminate u from the above equation. To do this we take advantage of the relationship between u and $\ddot{p}$ (note this assumes the linearization we performed earlier):

$$\ddot{p} = y + s = \frac{\partial a_p}{\partial x}\hat{x} + \frac{\partial a_p}{\partial u}u \tag{5}$$

$$u = \frac{\partial a_p}{\partial u}^{-1}\left(y + s - \frac{\partial a_p}{\partial x}\hat{x}\right) \tag{6}$$

Note that for this to be well defined, $\frac{\partial a_p}{\partial u}$ must not be 0. This is fine, because its value is very close to the mass of the cart m which is nonzero.

We can now substitute our expression for u back into (4) to get:

$$\begin{aligned}
\dot{\hat{x}} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ & \frac{\partial a_\theta}{\partial x} & & \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ \frac{\partial a_\theta}{\partial u} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial a_p}{\partial u}^{-1}\left(y + s - \frac{\partial a_p}{\partial x}\hat{x}\right) \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s & (7) \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \frac{\partial a_\theta}{\partial x} & - & \frac{\partial a_\theta}{\partial u}\frac{\partial a_p}{\partial u}^{-1}\frac{\partial a_p}{\partial x} & \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{\partial a_\theta}{\partial u}\frac{\partial a_p}{\partial u}^{-1} \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{\partial a_\theta}{\partial u}\frac{\partial a_p}{\partial u}^{-1} \end{bmatrix} s & (8) \\
&= A\hat{x} + By + Cs & (9)
\end{aligned}$$

Finally we discretize these continuous time dynamics to obtain our state update equations:

$$\begin{aligned}
\hat{x}_{k+1} &= (I + Adt)\,\hat{x}_k + (Bdt)\,y + (Cdt)\,s & (10) \\
&= \bar{A}\hat{x} + \bar{B}y + \eta & (11)
\end{aligned}$$

where $\eta = (Cdt)\, s \sim \mathcal{N}(0, \Sigma)$, and $\Sigma = CC^T dt^2 \sigma^2$ (note that this is a good example of a non-diagonal covariance matrix).

This is equivalent to the formulation we use to build our Kalman observer in Example 1, so the derivation of the observer updates follow from here.