1 Where we left off after Example 1.2

The system dynamics of the double integrator are given by:

$$x_k = Ax_{k-1} + Bu_{k-1} + w_{k-1} \tag{1}$$
$$z_k = Hx_k + v_k \tag{2}$$

with $A = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ dt/m \end{bmatrix}$, and $H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Here x is the position and velocity, u is the input torque, $w \sim \mathcal{N}(0, Q)$ is the process noise (Q was assumed zero for our original examples), z is the measured position, $v \sim \mathcal{N}(0, R)$ is the position measurement noise (assumed normal), dt is the time step and m is the mass.

For example 1.2, we built an observer that used the position measurement and the known input to estimate our current state. The update equations for that observer were:

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_{k-1} \tag{3}$$
$$P_k^- = AP_{k-1}A^T + Q \tag{4}$$
$$K_k = P_k^- H^T \left(HP_k^- H^T + R\right)^{-1} \tag{5}$$
$$\hat{x}_k = \hat{x}_k^- + K_k \left(z_k - H\hat{x}_k^-\right) \tag{6}$$
$$P_k = (I - K_k H) P_k^- \tag{7}$$

2 Example 1.3 derivation

For example 1.3, the task is to build an observer that is blind to the input torque, but instead uses a noisy measurement of the acceleration (e.g. from an IMU). We start by representing the noisy acceleration measurement y_k as:

$$y_k = a_k + s_k$$

with a_k as the true acceleration of the double integrator, and $s_k \sim \mathcal{N}\left(0, \sigma^2\right)$ as the IMU measurement noise.

Now since acceleration is neither an input or a state of our original dynamics, it's not immediately obvious how to integrate this into our above setup. The trick here is to reformulate the dynamics in a way that suits our observer design. Instead of the setup in (1), (2), we will write the equations of motion as:

$$\begin{aligned} x_k &= Ax_{k-1} + Ca_{k-1} \\ &= Ax_{k-1} + C\left(y_{k-1} - s_{k-1}\right) \\ &= Ax_{k-1} + Cy_{k-1} - Cs_{k-1} \\ &= Ax_{k-1} + Cy_{k-1} + \eta_{k-1} \end{aligned} \tag{8}$$
$$z_k = Hx_k + v_k \tag{9}$$

with $C = \begin{bmatrix} 0 \\ dt \end{bmatrix}$, $\eta = -Cs \sim \mathcal{N}(0, \Sigma)$, and $\Sigma = C\sigma^2 C^T = \begin{bmatrix} 0 & 0 \\ 0 & dt^2\sigma^2 \end{bmatrix}$.

Now this looks similar to the form we had in (1), (2), so we can derive our update equations in a similar fashion:

$$\hat{x}_k^- = A\hat{x}_{k-1} + Cy_{k-1} \tag{10}$$
$$P_k^- = AP_{k-1}A^T + \Sigma \tag{11}$$
$$K_k = P_k^- H^T \left(HP_k^- H^T + R\right)^{-1} \tag{12}$$
$$\hat{x}_k = \hat{x}_k^- + K_k \left(z_k - H\hat{x}_k^-\right) \tag{13}$$
$$P_k = (I - K_k H) P_k^- \tag{14}$$