Semantics of Not Knowing

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Abstract

Many Bayesian epistemologists now accept that it is not necessary for a rational agent to hold sharp credences. There are various compelling formal theories how such a non-traditional view of credences can accommodate decision making and updating. They are motivated by a common complaint: that sharp credences can fail to represent incomplete evidence and exaggerate the information contained in it. I show that the critical response of those who want to maintain the traditional view with only sharp credences is insufficient. Their criticism, however, opens the door to semantic questions which make indeterminate credences vulnerable on a deeper level. This paper uses both conceptual arguments and hands-on examples to demonstrate that rational agents always have sharp credences. Moreover, it is articulated and substantiated that agents with indeterminate credal states make long term gains betting against agents with sharp credences. This on the surface inconvenient fact can be explained and does not favour indeterminate credences over sharp credences.

1 Introduction

Traditionally, Bayesians have maintained that a rational agent, if she holds a credence, holds a sharp credence. It has recently become popular to drop the requirement for credence functions to be sharp. There are now Bayesians who permit a rational agent to hold indeterminate credal states (from now on instates) based on incomplete or ambiguous evidence. I will refer to Bayesians who continue to adhere to the classical theory of sharp credences for rational agents as 'Laplaceans' (e.g. Adam Elga and Roger White). I will refer to Bayesians who do not believe that a rational agent always holds sharp credences as 'Booleans' (e.g. Peter Walley and James Joyce).

There is some terminological confusion around the adjectives 'imprecise,' 'indeterminate,' and 'mushy' credences. In the following, I will exclusively refer to indeterminate credences or credal states (abbreviated 'instates') and mean by them a a set of credence functions (which some Booleans require to be convex) which it may be rational for an agent to hold within an otherwise orthodox Bayesian framework. After describing the appeal of indeterminacy and showing how contemporary Laplacean objections fail, I will point to more serious failings of indeterminacy in semantic terms and show how a proper semantics of not knowing, which we could also call a semantics of partial belief, solves the problems for sharp credences that Booleans address by introducing instates. There is a sense in which, by linking knowledge of chances to its reflection in credences, Booleans seek to reconcile (a) traditional knowledge epistemology concerned with full belief and (b) formal epistemology concerned with partial belief. There are other more recent reconciliation projects (see Spohn, 2012; and Moss, 2013). If my paper is correct then the Boolean approach will not contribute to this reconciliation because it mixes full belief and partial belief metaphors in ways that are semantically problematic.

I use the word semantics in a narrow sense. It may have been better to use something like the 'fixing of the terms of a theory' instead, but 'semantics' reads better once I have told you what I mean by it. The main complaint of this paper is that instates represent in one credence both degree of belief and properties of the evidence. Instates thus incorporate both doxastic and evidential features of the credal state. Representing multiple features of a state is not per se a bad thing when we fix the terms of a theory, in this case the terms of our theory about beliefs. In colour theory, the term 'red' represents both a phenomenological quality and a neighbourhood on the visible light spectrum. When we say one thing is more red than another thing, we effectively describe a relation based on both phenomenological and physical properties.

My examples and conceptual arguments show that instates, in spite of how plausible they initially sound in representing credal states, fail to give us a coherent terminology in our theory about beliefs. After investigation, it turns out that we want to separate the doxastic and evidential components, just as the scientific mineralogist wants to separate jadeite and nephrite instead of lumping them together under the single term 'jade.' When we first hear of the advantages of instates, two of them sound particularly compelling: (1) that they represent the possibility range for objective chances; and (2) that they fully represent incompleteness or ambiguity of the evidence. Roger White introduced an objection to instates, providing some counter-intuitive examples involving dilation (which I will explain in detail). Booleans can meet this objection, but only at the price of giving up (1) and (2). I will call these Boolean concessions to the problem of dilation (J1) and (J2), after James Joyce, who defends instates but rejects (1) and (2).

The choice is then to defend (1) and (2), which I consider implausible given the problem of dilation; accept (J1) and (J2) while remaining in the Boolean camp; or switch to the Laplacean camp. My examples and conceptual arguments show that once (J1) and (J2) are accepted, there are good reasons to accept that rational agents hold sharp credences to represent credal states. To put this provocatively in a hands-on example, this paper defends a 0.5 sharp credence in heads in all three cases: for a coin of whose bias we are completely ignorant; for a coin whose fairness is supported by a lot of evidence; and even for a coin about whose bias we know that it is either 1/3 or 2/3 for heads. Note that the idea is that the credence represents the doxastic features of the credal state and filters out some evidential features, which need to be independently represented. If instates could successfully integrate the two sets of features, they would be more successful in representing the credal state. As we will see, they cannot.

One potential Boolean claim is that agents who use instates do better than Laplaceans when they bet on the truth of events for which they have varying degrees of evidence. Walley gives an example where a Laplacean does much worse at predicting Soccer World Cup games than Boolean peers who use upper and lower previsions (see Walley, 1991, appendix I). Upper and lower previsions are instates for which it is rational to accept or reject bets if they fall within the margin of indeterminacy.

First I will show on a much more general level how Walley's claims are justified in practice (bettors using upper and lower previsions do on average better than bettors using linear previsions, i.e. sharp credences). Then I will explain why this is the case, how a Laplacean can protect herself against this disadvantage by drawing proper distinctions between credence and evidence, and how indeterminacy emerges as the loser when the contest is about clarity in one's semantics. The problem will be mixed metaphors: the Boolean mixes semantic levels that ought for good reasons to remain separate. Indeterminacy imposes a double task on credences (representing both uncertainty and available evidence) that they cannot coherently fulfill.

I will present several examples where this double task stretches instates to the limits of plausibility. Joyce's idea that credences can represent balance, weight, and specificity of the evidence is inconsistent with the use of indeterminacy (and Joyce himself, in response to the dilation problem, gives the argument why this is the case). The Boolean claim that certain properties of the evidence (its ambiguity, its completeness, conflicts within it) can be recovered from instates is inconsistent with an effective Boolean answer to the dilation problem.

The Laplacean approach of assigning subjective probabilities to partitions of the event space (e.g. objective chances) and then aggregating them by David Lewis' summation formula into a single precise credence function is semantically tidy and shares many of the formal virtues of Boolean theories. If the bad taste about numerical precision in our fuzzy and nebulous world lingers, I will point to philosophical projects in other domains where the concepts we use are sharply bounded, even though our ability to conceive of those sharp boundaries or know them is limited.

2 Motivation for Instates

We want to motivate indeterminacy for the credences of a rational agent, independent of how they are elicited, as forcefully as possible so that the reader will see (a) the appeal of such indeterminacy, (b) the insufficiency of the critical response, and (c) the need for careful articulation of the Laplacean approach that mandates a rational agent to hold sharp credences together with an explanation of how it addresses the concerns which motivate some to resort to indeterminacy.

Our conclusion is that a rational agent is best off in terms of her own goals when she entertains sharp credences with respect to propositions about events that come her way. Whether this is advisable for human or machine intelligence is a different kettle of fish. My topic is the logic of partial beliefs, and I readily admit that such a logic may be computationally intractable or, given finite resources, be an irrational way of keeping track of beliefs.

Let a COIN be a Bernoulli generator that produces successes and failures with probability p for success, labeled H, and $1-p$ for failure, labeled T. Physical coins may serve as examples, if we are willing to set aside that most of them are approximately fair. Imagine three COINs for which we have evidence that COIN$_I$ is fair, COIN$_{II}$ has an unknown bias, and COIN$_{III}$ has as bias either $p = 1/3$ or $p = 2/3$. The Laplacean approach permits a sharp 0.5 credence in H for a rational agent in all three cases. A Boolean approach wants to see the difference in the evidential situation reflected in a rational agent's credal state and at least permit, as credence in H, $\{x|x = 0.5\}$ for $\textsc{coin}_I$, $\{x|0 \leq x \leq 1\}$ for $\textsc{coin}_{II}$, and $\{x|1/3 \leq x \leq 2/3\}$ or $\{1/3, 2/3\}$ for $\textsc{coin}_{III}$.

Here are a few concise statements by Booleans:

> [A] refusal to make a determinate probability judgment does not derive from a lack of clarity about one's credal state. To the contrary, it may derive from a very clear and cool judgment that on the basis of the available evidence, making a numerically determinate judgment would be unwarranted and arbitrary. (Levi, 1985, 395.)

> If there is little evidence concerning [a claim,] then beliefs about [that claim] should be indeterminate, and probability models imprecise, to reflect the lack of information. We regard this as the most important source of imprecision. (Walley, 1991, 212–213.)

> Imprecise probabilities and related concepts [...] provide a powerful language which is able to reflect the partial nature of the knowledge suitably and to express the amount of ambiguity adequately. (Augustin, 2003, 34.)

> As sophisticated Bayesians like Isaac Levi (1980), Richard Jeffrey (1983), Mark Kaplan (1996), have long recognized, the proper response to symmetrically ambiguous or incomplete evidence is not to assign probabilities symmetrically, but to refrain from assigning precise probabilities at all. Indefiniteness in the evidence is reflected not in the values of any single credence function, but in the spread of values across the family of all credence functions that the evidence does not exclude. This is why modern Bayesians represent credal states using sets of credence functions. It is not just that sharp degrees of belief are psychologically unrealistic (though they are). Imprecise credences have a clear epistemological motivation: they are the proper response to unspecific evidence. (Joyce, 2005, 170f.)

Consider the following reasons that motivate Booleans to permit instates for rational agents:

(A) The greatest emphasis motivating indeterminacy rests on lack of evidence or conflicting evidence and the assumption that single probability measures (sharp credences) do not represent such evidence as well as credal states composed by sets of probability measures (instates).

(B) The preference structure of a rational agent may be incomplete so that representation theorems do not yield single probability measures to represent such incomplete structures.

(C) There are more technical and paper-specific reasons, such as Thomas Augustin's attempt to mediate between the minimax pessimism of objectivists and the Bayesian optimism of subjectivists using interval probability (see Augustin, 2003, 35f); Alan Hájek and Michael Smithson's belief that there may be objectively indeterminate chances in the physical world (see Hájek and Smithson, 2012, 33); and Jake Chandler's claim that "the sharp model is at odds with a trio of plausible propositions regarding agnosticism" (Chandler, 2014, 4).

This paper mostly addresses (A), while taking (B) seriously as well and pointing towards solutions for it. I am leaving (C) to more specific responses to the issues presented in the cited articles. I am adding a reason (D) that is poorly documented in the literature: The Boolean rational agent may systematically do better accepting bets than the agent who on principle rejects instates. Walley conducted an experiment in which Boolean participants did significantly better than Laplacean participants, betting on soccer games played in the Soccer World Cup 1982 in Spain (see Walley, 1991, Appendix I). I replicated the experiment using two computer players with rudimentary artificial intelligence and made them specify betting parameters (previsions) for games played in the Soccer World Cup 2014 in Brazil. I used the Poisson distribution (which is an excellent predictor for the outcome of soccer matches) and the FIFA ranking to simulate millions of counterfactual World Cup results and their associated bets, using Walley's evaluation method. The Boolean player had a slight but systematic advantage. In section 5, I will provide an explanation and show how it undermines any support the experiment might give to the Boolean position.

3 Dilation

There are in the literature two kinds of objections to the Boolean position. One is semantic, the other derives unacceptable consequences from instates and then urges to give them up in favour of sharp credences. A paradigm example for the latter kind is Adam Elga's objection, which leads him to the following conclusion: "Perfectly rational agents always have perfectly sharp probabilities" (Elga, 2010, 1).

Elga tries to show that instates are not coherent in the sense that they allow bets which will lead to a sure loss (for example, if an agent has 0.4 as a lower prevision and 0.6 as an upper prevision, then her credal state permits a 45 cent bet on the proposition in question, with $1 being the prize money, and a 56 cent bet against the proposition in question). Joyce addresses Elga's point (see Joyce, 2010, 314) and successfully defends the Boolean position.

Roger White, in the meantime, articulates his objections on both levels: in terms of semantics and in terms of undesirable consequences of instates. Let us abbreviate the former objection as (Anti-CGT) and the latter objection as (Anti-DIL). CGT stands for Chance Grounding Thesis. It is a semantic claim about instates, and according to White it is semantically unacceptable—some of White's objections are echoed and re-articulated throughout this paper.

> **Chance Grounding Thesis:** Only on the basis of known chances can one legitimately have sharp credences. Otherwise one's spread of credence should cover the range of possible chance hypotheses left open by your evidence. (White, 2010, 174)[TBD]

It is not sufficient for the Laplacean, however, to criticize the Boolean position on the basis of the CGT, so White's (Anti-CGT) argument will not do. This problem is intimately linked to White's (Anti-DIL), to which Laplaceans can respond by giving up on the claims of the CGT. DIL stands for dilation, which provides the grounds for White's objection of the latter kind: not semantic, but drawing out undesirable consequences from the Boolean approach. I will describe dilation in a moment, but for the argument of this paper it is important to understand that Booleans have a line of defence against (Anti-DIL), and (Anti-CGT) to boot. They can, as Joyce does in his response to White, give up on the CGT. More precisely, they can make two concessions, which in honour of Joyce, who makes these concessions, I call (J1) and (J2).

(J1) Credences do not adequately represent evidence (the same instate can reflect different evidential states).

(J2) Instates do not reflect knowledge claims about objective chances (the CGT is not an appropriate semantic characterization of instates).

Just as Joyce successfully defends the Boolean position against Elga's objection, he successfully defends the Boolean position against (Anti-CGT) and (Anti-DIL) by making concessions (J1) and (J2). The rest of this paper shows how (J1) and (J2) are necessary to defend the Boolean position against White's objections and how they leave Booleans in a situation so semantically vulnerable that their position must be rejected anyways, despite their successful defences against objections made on the grounds of undesirable consequences. We will look at a few hands-on examples where Booleans, given (J1) and (J2), under weak assumptions give the wrong answers.

Now we must make good on our promise to explain dilation (using White, 2010, 175ff) and why Joyce needs (J1) and (J2) in order to save the Boolean position from its intuitive appeal.

> **Example 1: Dilation.** You have two Bernoulli Generators, $\textsc{coin}_a$ and $\textsc{coin}_b$. You have good evidence that $\textsc{coin}_a$ is fair and no evidence about the bias of $\textsc{coin}_b$. Furthermore, the two generators are not necessarily independent. Their results could be 100% correlated or anticorrelated, they may be independent, or the correlation could be anywhere between the two extremes. You toss $\textsc{coin}_a$. Without looking at the result, your credence in H_a ($\textsc{coin}_a$ coming up heads) is a sharp 0.5, even if you are open to instates, because you have good evidence for the fairness of $\textsc{coin}_a$. Then you toss $\textsc{coin}_b$. This time you look at the result and the moment you learn it, your credence in H_a dilates from a sharp 0.5 to the vacuous credal state covering the whole interval $[0, 1]$ (provided that this was your credence in H_b, as stipulated). Even though you have just received information (the result of $\textsc{coin}_b$'s toss), your credence in H_a dilates.

Usually, we would expect more information to sharpen our credal states (see Walley's "the more information the more precision" principle and his response to this problem in 1991, 207 and 299). White did not discover the phenomenon of dilation (see the detailed study in Seidenfeld and Wasserman, 1993), but he was able to find examples where the consequences appear grotesque, especially in his chocolates case (see White, 2010, 183).

> **Example 2: White Chocolates.** Four out of five chocolates in the box have cherry fillings, while the rest have caramel. Picking one at random, what should my credence be that it is cherry-filled? Everyone, including the staunchest [Booleans], seems to agree on the answer 4/5. Now of course the chocolate I've chosen has many other features, for example this one is circular with a swirl on top. Noticing such features could hardly make a

> difference to my reasonable credence that it is cherry filled (unless of course I have some information regarding the relation between chocolate shapes and fillings). Often chocolate fillings do correlate with their shapes, but I haven't the faintest clue how they do in this case or any reason to suppose they correlate one way rather than another ... the further result is that while my credence that the chosen chocolate is cherry-filled should be 4/5 prior to viewing it, once I see its shape (whatever shape it happens to be) my credence that it is cherry-filled should dilate to become [indeterminate]. But this is just not the way we think about such matters. (White, 2010, 183.)

White's claim is also that dilation contradicts Bas van Fraassen's reflection principle (see van Fraassen, 1984). If you know that soon you will take a dilated doxastic attitude towards a proposition without loss of information and no surprising information coming in, you can just as well assume the dilated doxastic attitude now, which is clearly counter-intuitive (see White, 2010, 178).

Joyce does an admirable job showing both that (J1) and (J2) address White's dilation problem and that they are necessary in order to avoid White's counter-intuitive results (see Joyce, 2010, 13ff). Joyce rejects the CGT on the grounds that it would make learning impossible (see Joyce, 2010, 7f); and he rejects the notion that identical instates encode identical beliefs by giving a simple example:

> **Example 3: Three-Sided Die.** Suppose $\mathcal{C}$ and $\mathcal{C}^*$ are defined on a partition $\{X, Y, Z\}$ corresponding to the result of a roll of a three sided-die. Let $\mathcal{C}$ contain all credence functions defined on $\{X, Y, Z\}$ such that $\mathbf{c}(Z) \geq 1/2$, and let $\mathcal{C}^*$ be the subset of $\mathcal{C}$ whose members also satisfy $\mathbf{c}(X) = \mathbf{c}(Y)$. (Joyce, 2010, 12.)

$\mathcal{C}$ and $\mathcal{C}^*$ generate the same instates, but they surely differ in the beliefs that they encode, as $\mathcal{C}^*$ regards X and Y as equiprobable, whereas $\mathcal{C}$ does not. To say that dilation is a problem for the Boolean position presupposes that instates encode beliefs, since if they do not it becomes clear why the correlation between the two coin tosses is evidentially relevant to H_a in Example 1. If instates encoded the evidential basis for a belief, however, there could be no weights attached to the various credence functions represented by the instate based on the evidence. All 'committee members,' as Joyce quite helpfully calls them in illustrating how instates work by committee rather than one single credence function, would be equally enfranchised, which would inhibit learning and either introduce regress problems on the non-trivial (neither 0 nor 1) margins of the indeterminate intervals or render all instates vacuous.

Therefore, the Boolean position cannot be upheld without concessions (J1) and (J2). Joyce, for example, clearly rejects the CGT and underlines how 'committee members' can be discriminately enfranchised by a "directionality of the spread" (see Joyce, 2010, 318), which clarifies that instates, as little as sharp credences, adequately represent the evidence supporting the partial belief. Joyce gives a nice formal description of this, but I want to show by example how dilation is unproblematic (and therefore agree with Joyce that instates cannot encode beliefs).

> **Example 4: Dilating Urns.** You draw one ball from an urn with 200 balls (100 red, 100 black) and receive the information that the urn actually had two chambers, one with 99 red balls and 1 black ball, the other with 1 red ball and 99 black balls.

Dilation from a sharp credence of $\{0.5\}$ to an instate of $\{0.01, 0.99\}$ or $[0.01, 0.99]$ (depending on whether convexity is required) is unproblematic, although the example already prefigures that there is something odd about the Boolean semantic approach. The example licences a 99:1 bet for one of the colours (if the instate is interpreted as upper and lower previsions), but this is a problem that arises out of Boolean semantics without dilation, which we will address again in Example 8.

Booleans have the resources to extract themselves from the problem of dilation, but only at the cost of making Joyce's semantic concessions (J1) and (J2) which we will use against them. These semantic concessions are inescapable, not only on account of dilation. If one were to be committed to the principle of regularity, that all states of the world considered possible have positive probability (for a defence see Savage et al., 1963); and to the solution of Henry Kyburg's lottery paradox, that what is rationally accepted should have probability 1 (for a defence of this principle see Douven and Williamson, 2006); and the CGT, that one's spread of credence should cover the range of possible chance hypotheses left open by the evidence (implied by much of Boolean literature); then one's instate would always be vacuous. Booleans must deny at least one of the premises to avoid the conclusion (Joyce denies the CGT).

A sharp credence constrains partial beliefs in objective chances by Lewis' summation formula (which we will provide in the next section). No objective chance is excluded by it (principle of regularity) and any updating will merely change the partial beliefs, but no full beliefs. Instates, on the other hand, by giving ranges of acceptable objective chances suggest that there is a full belief that the objective chance does not lie outside what is indicated by the instate. A Boolean can avoid this situation by accepting Joyce's concession (J2).

Here is a brief example to illustrate the difference between Laplacean semantics of partial beliefs based on the principle of regularity and Boolean semantics which introduce an obscure grey zone between partial beliefs and full belief, update and revision, traditional epistemology and formal epistemology.

> **Example 5: Bavarian King.** Matthias Perth, an Austrian civil servant, observes the Bavarian king at the Congress of Vienna in 1815 and writes in his diary that the king "appears to be a man between 45 and 47 years old" (see `http://www.das-perth-projekt.at`).

If Perth then learns that the king was 49 years old, he must revise, not just update, his earlier judgment. The appropriate formal instrument is belief revision, not probability update, requiring a substantial reconciliation project between formal and traditional epistemology operating in the background. I do not see this project articulated in the Boolean literature (for an example of such a project see Spohn, 2012, especially chapter 10). Sarah Moss also undertakes it and assumes the Boolean approach (see Moss, 2013), but I fail to see how the Boolean approach is essential to her reconciliation or how her reconciliation gives independent arguments for the Boolean approach. If Perth had wanted to express a sharp credence, he would have said, "my best guess is that the king is 46 years old," and the information that the king was 49 would have triggered the appropriate update, without any revision of full beliefs.

In summary, I return to White's objections (Anti-CGT) and (Anti-DIL) which, as they stand, fail because the CGT is not a necessary ingredient of the Boolean position; and dilation for indeterminate credences is in principle not any more surprising than a piece of information that increases the Shannon entropy of a sharp credence (see Example 4). It is true for both sharp and indeterminate credences that information can make us less certain about things, and it is true for both sharp and indeterminate credences that they do not encode the evidence.

Once Booleans have brought their house in order to accommodate White's objection using concessions (J1) and (J2), they open the door to semantic problems. We finally get to inquire what kind of coherence there is in defending indeterminacy when it neither fulfills the promise of adequately representing evidence nor the promise of reconciling traditional full belief 'knowledge' epistemology and Bayesian partial belief epistemology as outlined in the CGT, but only adds another hierarchical layer of uncertainty to a numerical quantity (a sharp credence) whose job it already is to represent uncertainty, thus unnecessarily introducing regress problems. We will turn to these semantic considerations now, show how they display the virtues of sharp credences in responding to the forceful motivations for instates while making those instates look semantically otiose. Then we will show in the last section how sharp credences have an elegant solution for being outperformed by instates in betting scenarios, and there we will rest our case.

4 Semantics of Partial Belief

Instates are suggestive of a measurement that represents numerically the mass of an object and then also make claims about its density. With sharp credences, the semantic roles of evidence, information, and uncertainty are appropriately differentiated. Rational decision making, inference, and betting behaviour are based on sharp credences together with the evidence that is at its foundation. Information represents evidence, and sharp credences represent uncertainty. Measurement is in any case a misleading analogy for credences. Measurements come with imprecision estimates to account for inaccuracy. Credences, however, are not measurements, especially not of objective chances. They represent uncertainty. They are more like logical truth values than they are like measurements.

It is a slippery affair to determine what evidence is, which I will leave to others. My claim is that a rational agent is someone who can distill information from evidence which places numerically precise constraints on relatively prior probability distributions, which then can be updated to form posterior probability distributions and the credences associated with them. Note that relatively prior probability distributions are not ignorance priors or non-informative priors, which I would call absolutely prior probability distributions. I have no answers where absolutely prior probability distributions come from, how they are justified, or in what sense they are objective. I am not concerned whether all rational agents, if they have the same evidence, should arrive at the same credal states; or even if they should all update a given relatively prior credal state to the same posterior credal state, if they have the same evidence. Sometimes there may be different ways to translate or interpret evidence into information.

It is important not to confuse the claim that it is reasonable to hold both X and Y with the claim that it is reasonable to hold either X (without Y) or Y (without X). It is the reasonableness of holding X and Y concurrently that is controversial, not the reasonableness of holding Y (without holding X) when it is reasonable to hold X. We will later talk about anti-luminosity, the fact that a rational agent may not be able to distinguish psychologically between a 54.9 cent bet on an event and a 45.1 bet on its negation, when her sharp credence is 0.55. She must reject one of them not to incur sure loss, so proponents of indeterminacy suggest that she choose one of them freely without being constrained by her credal state or reject both of them. I claim that a sharp credence will make a recommendation between the two so that only one of the bets is rational given her particular credence, but that does not mean that another sharp credence which would give a different recommendation may not also be rational for her to have.

I am sympathetic to the viewpoint that once a rational agent has a relatively prior credal state and has formalized her evidence in terms of information, then the probability distributions forming her posterior credal state should be unique. Joyce, with his 'committee member' approach, shows how this kind of updating can be done for instates (see Joyce, 2010, 288; also Bradley and Steele, 2013, 6).

On the one hand (the Laplacean approach), you can have partial beliefs about how a parameter is distributed and then use Lewis' summation formula (see Lewis, 1981, 266f) to integrate over them and condense them to a sharp credence. Walley comments on this "reduction" in his section on Bayesian second order probabilities (see Walley, 1991, 258f), but he mistakenly represents the Laplacean approach as a second order approach, as if the probability distributions that are summarized by Lewis' formula are of the same kind as the resulting credences. They are not. They refer to partitions of the event space, sometimes corresponding to objective chances, and represent the subjective probabilities that are associated with them. The credence, by contrast, is a quantity representing partial belief and assisting in the making of decisions and inferences. It is the Boolean approach which has elements of a second order approach and thus makes itself vulnerable to regress problems by adding another dimension of uncertainty to a parameter (the credence) which already represents uncertainty.

On the other hand (the Boolean approach), you can try to represent your uncertainty about the distribution of the parameter by an instate. I want to show that the Laplacean approach can be aligned with the forceful motivations we listed in the previous section to introduce instates, as long as we do not require that a sharp credence represent the evidence as well as the agent's state of uncertainty. We have learned that this requirement can be reduced ad absurdum even for instates, see (J1) and (J2).

One of Joyce's complaints is that a sharp credence of 0.5 for a COIN contains too much information if there is little or no evidence that the COIN is fair. This complaint, of course, is only effective if the indeterminacy of the credence is anticorrelated to the amount of information in the evidence. In (J1), however, Joyce admits that instates cannot represent the evidence without violating the reflection principle due to White's dilation problem. He is quite clear that the same instate can represent different evidential scenarios (see, for example, Joyce, 2010, 302). (J1) may not be sufficient to defend against the information argument, but Walley's and Joyce's claim that instates are less informative than sharp credences (see Walley, 1991, 34; and Joyce, 2010, 311 for examples, but this attitude is passim) has no foundation in information theory. To compare instates and sharp credences informationally, we would need a non-additive set function obeying Shannon's axioms for information. This is a non-trivial task. I have not succeeded solving it (nor do I need to carry the Booleans' water), but I am not convinced that it will result in an information measure which assigns, for instance, more information to a sharp credence such as $\{0.5\}$ than to an instate such as $\{x|1/3 \leq x \leq 2/3\}$.

Returning to (J1) and the problem of inadequate representation, Augustin recognizes this long before Joyce, with specific reference to instates: "The imprecise posterior does no longer contain all the relevant information to produce optimal decisions. Inference and decision do not coincide any more" (Augustin, 2003, 41) (see also an example for inadequate representation of evidence by instates in Bradley and Steele, 2013, 16). At best, instates fare no better than sharp credences, at worst they unhelpfully mimic saying something about the evidence that is much better said elsewhere.

Not only can we align sharp credences with the motivations to introduce instates, we can also show that instates perform worse semantically because they mix evidential and doxastic metaphors in deleterious ways. Sharp credences have one task: to represent epistemic uncertainty and serve as a tool for updating, inference, and decision making. They cannot fulfill this task without continued reference to the evidence which operates in the background. To use an analogy, credences are not sufficient statistics with respect to updating, inference, and decision making. What is remarkable about Joyce's response to White's dilation problem is that Joyce recognizes that instates are not sufficient statistics either. But this means that they fail at the double task which has been imposed on them: to represent both epistemic uncertainty and the evidence.

In the following, I will provide a few examples where it becomes clear that instates have difficulty representing uncertainty because they are tangled in a double task which they cannot fulfill.

> **Example 6: Aggregating Expert Opinion.** You have no information whether it will rain tomorrow (R) or not except the predictions of two weather forecasters. One of them forecasts 0.3 on channel GPY, the other 0.6 on channel QCT. You consider the QCT forecaster to be significantly more reliable, based on past experience.

An instate corresponding to this situation may be $[0.3, 0.6]$ (see Walley, 1991, 214), but it will have a difficult time representing the difference in reliability of the experts. We could try $[0.2, 0.8]$ (since the greater reliability of QCT suggests that the chance of rain tomorrow is higher rather than lower) or $[0.1, 0.7]$ (since the greater reliability of QCT suggests that its estimate is more precise), but it remains obscure what the criteria might be.

A sharp credence of $P(R) = 0.53$, for example, does the right thing. Such a credence says nothing about any beliefs that the objective chance is restricted to a subset of the unit interval, but it accurately reflects the degree of uncertainty that the rational agent has over the various possibilities. Beliefs about objective chances make little sense in many situations where we have credences, since it is doubtful even in the case of rain tomorrow that there is an urn of nature from which balls are drawn. What is really at play is a complex interaction between epistemic states (for example, experts evaluating meteorological data) and the evidence which influences them.

A sharp credence is often associated with probability distributions over chances, while an instate puts chances in sets where they all have an equal voice. This may also be at the bottom of Susanna Rinard's objection (see White, 2010, 184) that Joyce's committee members are all equally enfranchised and so it is not clear how extremists among them could not always be replaced by even greater extremists even after updating on evidence which should serve to consolidate indeterminacy. Joyce has a satisfactory response to this objection (see Joyce, 2010, 291), but I do not see how the response addresses the problem of aggregating expert opinion without the kind of summation that Laplaceans find unobjectionable, even though information is lost and can only be recouped by going back to the evidence. More generally, the two levels for sharp credences, representation of uncertainty and distributions over partitions, tidily differentiate between the doxastic and the evidential dimension; instates, on the other hand, just add another level of uncertainty on top of the uncertainty that is already expressed in the partial belief and thus do not make the appropriate semantic distinctions.

As we will see in the next example, it is an advantage of sharp credences that they do not exclude objective chances, even extreme ones, because they are fully committed to partial belief and do not suggest, as indeterminate credences do, that there is full belief knowledge that the objective chance is a member of a proper subset of the possibilities.

> **Example 7: Precise Credences.** Your sharp credence for rain tomorrow, based on the expert opinion of channel GPY and channel QCT (you have no other information) is 0.53. Is it reasonable, considering how little evidence you have, to reject the belief that the chance of rain tomorrow is 0.52 or 0.54; or to prefer a 52.9 cent bet on rain to a 47.1 cent bet on no rain?

The first question is confused, but in instructive ways (a display of this confusion is found in Hájek and Smithson, 2012, 38f, and their doctor and time of the day analogy). A sharp credence rejects no hypothesis about objective chances (unlike an instate, unless (J2) is firmly in place). It often has a subjective probability distribution operating in the background, over which it integrates to yield the sharp credence (it would do likewise in Hájek and Smithson's example for the prognosis of the doctor or the time of the day, without any problems). This subjective probability distribution may look like this:

$P(\pi(R) = 0.00)$	$=$	0.0001
$P(\pi(R) = 0.01)$	$=$	0.0003
$P(\pi(R) = 0.02)$	$=$	0.0007
...		...
$P(\pi(R) = 0.30)$	$=$	0.0015
$P(\pi(R) = 0.31)$	$=$	0.0016
...		...
$P(\pi(R) = 0.52)$	$=$	0.031
$P(\pi(R) = 0.53)$	$=$	0.032
$P(\pi(R) = 0.54)$	$=$	0.030
...		...

It is condensed by Lewis' summation formula to a sharp credence, without being reduced to it:

$$C(R) = \int_0^1 \zeta P(\pi(R) = \zeta)\, d\zeta \tag{1}$$

Lewis' 1981 paper "A Subjectivist's Guide to Objective Chance" addresses the question what the relationship between π, P, and C is. The point is that we have properly separated the semantic dimensions and that the Laplacean approach is not a second order probability approach. The partial belief epistemology deals with sharp credences and how they represent uncertainty and serve as a tool in inference, updating, and decision making; while Lewis' Humean speculations and his interpretation of the principal principle cover the relationship between subjective probabilities and objective chance.

Instates, by contrast, mix these semantic dimensions so that in the end we get a muddle where a superficial reading of indeterminacy suddenly follows a converse principal principle of sorts, namely that objective chances are constrained by the factivity of a rational agent's credence when this credence is knowledge (Lewis actually talks about such a converse, but in completely different and epistemologically more intelligible terms, see Lewis, 1981, 289). Sharp credences are more, not less, permissive with respect to objective chances operating externally (compared to the internal belief state of the agent, which the credence reflects). By the principle of regularity and in keeping with statistical practice, all objective chances as possible states of the world are given positive subjective probabilities, even though they may be very small. Instates, on the other hand, mix partial belief epistemology with full belief epistemology and presumably exclude objective chances which lie outside the credal state from consideration because they are fully known not to hold (see Levi, 1981, 540, "inference derives credal probability from knowledge of the chances of possible outcomes").

The second question is also instructive: why would we prefer a 52.9 cent bet on rain to a 47.1 cent bet on no rain, given that we do not possess the power of descrimination between these two bets? The answer to this question ties in with the issue of incomplete preference structure referred to above as motiviation (B) for instates.

> It hardly seems a requirement of rationality that belief be precise (and preferences complete); surely imprecise belief (and corresponding incomplete preferences) are at least rationally permissible. (Bradley and Steele, 2013, 2.)

In personal communication, Yang Liu at Columbia University posed this problem to me more forcefully: the development of representation theorems beginning with Frank Ramsey (followed by increasingly more compelling representation theorems in Savage, 1954; and Jeffrey, 1965; and numerous other variants in contemporary literature) puts the horse before the cart and bases probability and utility functions of an agent on her preferences, not the other way around. Once completeness as an axiom for the preferences of an agent is jettisoned, indeterminacy follows automatically. Indeterminacy may thus be a natural consequence of the proper way to think about credences in terms of the preferences that they represent.

In response, preferences may very well logically and psychologically precede an agent's probability and utility functions, but that does not mean that we cannot inform the axioms we use for a rational agent's preferences by undesirable consequences downstream. Completeness may sound like an unreasonable imposition at the outset, but if incompleteness has unwelcome semantic consequences for credences, it is not illegitimate to revisit the issue. Timothy Williamson goes through this exercise with vague concepts, showing that all upstream logical solutions to the problem fail and that it has to be solved downstream with an epistemic solution (see Williamson, 1996). Vague concepts, like sharp credences, are sharply bounded, but not in a way that is luminous to the agent (for anti-luminosity see chapter 4 in Williamson, 2000). Anti-luminosity answers the original question: the rational agent prefers the 52.9 cent bet on rain to a 47.1 cent bet on no rain based on her sharp credence without being in a position to have this preference necessarily or have it based on physical or psychological ability (for the analogous claim about knowledge see Williamson, 2000, 95).

In a way, advocates of indeterminacy have solved this problem for us. There is strong agreement among most of them that the issue of determinacy for credences is not an issue of elicitation (sometimes the term 'indeterminacy' is used instead of 'imprecision' to underline this difference; see Levi, 1985, 395). The appeal of preferences is that we can elicit them more easily than assessments of probability and utility functions. The indeterminacy issue has been raised to the probability level (or moved downstream) by indeterminacy advocates themselves who feel justifiably uncomfortable with an interpretation of their theory in behaviourist terms. So it shall be solved there, and this paper makes an appeal to reject indeterminacy on this level. The solution then has to be carried upstream (or lowered to the logically more basic level of preferences), where we recognize that completeness for preferences is after all a desirable axiom for rationality. Isaac Levi agrees with me on this point: when he talks about indeterminacy, it proceeds from the level of probability judgment to preferences, not the other way around (see Levi, 1981, 533).

> **Example 8: Monkey-Filled Urns.** E.T. Jaynes describes an experiment with monkeys filling an urn randomly with balls from another urn, for which sampling provides no information and so makes updating vacuous (see Jaynes and Bretthorst, 2003, 160). Here is a variant of this experiment for which a sharp credence provides a more compelling result than the associated instate: Let urn A contain 4 balls, two red and two black. A monkey randomly fills urn B from urn A with two balls. We draw from urn B.

The sharp credence of drawing a red ball is 0.5, following Lewis' summation formula for the different combinations of balls in urn B. This solution is more intuitive in terms of further inference, decision making, and betting behaviour than a credal state of $\{0, 1/2, 1\}$ or $[0, 1]$ (depending on the convexity requirement), since this instate would licence an exorbitant bet in favour of one colour, for example one that costs \$9,999 and pays \$10,000 if red is drawn and nothing if black is drawn.

To make this example more vivid consider a Hand Urn, where you draw by hand from an urn with 100 balls, 50 red balls and 50 black balls. When your hand retreats from the urn, does it not contain either a red ball or a black ball and so serve itself as an urn, from which in a sense you draw a ball? Your hand contains one ball, either red or black, and the indeterminate credal state that it is one or the other should be $[0, 1]$. This contradicts our intuition that our credence should be a sharp 0.5. As is the case with the White Chocolate example, instates appear to be highly contingent on a problem's mode of representation, more so than intuition allows.

> **Example 9: Three Prisoners.** Prisoner X_1 knows that two out of three prisoners (X_1, X_2, X_3) will be executed and one of them pardoned. He asks the warden of the prison to tell him the name of another prisoner who will be executed, hoping to gain knowledge about his own fate. When the warden tells him that X_3 will be executed, X_1 erroneously updates his probability of pardon from 1/3 to 1/2, since either X_1 or X_2 will be spared.

Walley maintains that for the Monty Hall problem and the Three Prisoners problem, the probabilities of a rational agent should dilate rather than settle on the commonly accepted solutions. For the Three Prisoners problem, there is a compelling case for standard conditioning and the result that the credence for prisoner X_1 to have been pardoned ought to be unchanged after the update (see Lukits, 2014, 1421f). Walley's dilated solution would give prisoner X_1 hope on the doubtful possibility (and unfounded assumption) that the warden might prefer to provide X_3's (rather than X_2's) name in case prisoner X_1 was pardoned.

This example brings an interesting issue to the forefront. Sharp credences often reflect independence of variables where such independence is unwarranted. Booleans (more specifically, detractors of the principle of indifference or the principle of maximum entropy, principles which are used to generate sharp credences for rational agents) tend to point this out gleefully. They prefer to dilate over the possible dependence relationships (independence included). White's dilation problem is an instance of this. The fallacy in the argument for instates, illustrated by the Three Prisoners problem, is that the probabilistic independence of sharp credences does not imply independence of variables (only the converse is correct), but only that it is unknown whether there is dependence, and if yes, whether it is correlation or inverse correlation.

In the Three Prisoners problem, there is no evidence about the degree or the direction of the dependence, and so prisoner X_1 should take no comfort in the information that she receives. The prisoner's probabilities will reflect probabilistic independence, but make no claims about causal independence. Walley has unkind things to say about sharp credences and their ability to respond to evidence (for example that their "inferences rarely conform to evidence", see Walley, 1991, 396), but in this case it appears to me that they outperform the Boolean approach.

> **Example 10: Wagner's Linguist.** A linguist hears the utterance of a native and concludes that the native cannot be part of certain population groups, depending on what the utterance means. The linguist is uncertain between some options about the meaning of the utterance. (For full details see Wagner, 1992, 252; and Spohn, 2012, 197.)

The mathematician Carl Wagner proposed a natural generalization of Jeffrey Conditioning for his Linguist example (see Wagner, 1992). Since the principle of maximum entropy is already a generalization of Jeffrey Conditioning, the question naturally arises whether the two generalizations agree. Wagner makes the case that they do not agree and deduces that the principle of maximum entropy is sometimes an inappropriate updating mechanism, in line with many earlier criticisms of the principle of maximum entropy (see van Fraassen, 1981; Shimony, 1985; Skyrms, 1987; and, later on, Grove and Halpern, 1997). What is interesting about this case is that Wagner uses instates for his deduction, so that even if you agree with his natural generalization of Jeffrey Conditioning (which I find plausible), the inconsistency with the principle of maximum entropy can only be inferred assuming instates. Wagner is unaware of this, and I am showing in another paper (in process) how on the assumption of sharp credences Wagner's generalization of Jeffrey conditioning accords with the principle of maximum entropy.

This will not convince Booleans, since they are already unlikely to believe in the general applicability of the principle of maximum entropy (just as Wagner's argument is unlikely to convince a proponent of the principle of maximum entropy, since they have a tendency to reject instates). The battle lines are clearly drawn. Wagner's argument, instead of undermining the principle of maximum entropy, just shows that instates are as wedded to rejecting the claims of the principle of maximum entropy as the principle of maximum entropy is wedded to sharp credences (these marriages are only unilaterally monogamous, however, as it is perfectly coherent to reject both the principle of maximum entropy and the Boolean position; or to reject both the Laplacean position and instates).

Endorsement of instates, however, implies that there are situations of probability update in which the posterior probability distribution is more informative than it might be in terms of information theory. Indeterminate credences violate the relatively natural intuition that we should not gain information from evidence when a less informative updated probability will do the job of responding to the evidence. This is not a strong argument in favour of sharp credences. The principle of maximum of entropy has received a thorough bashing in the last forty years. I consider it to be much easier to convince someone to reject instates on independent (semantic) grounds than to convince them to give the principle of maximum entropy a second chance. But the section on semantics comes to an end here, and we want to proceed to the intriguing issue of who does better in betting situations: instates or sharp credences.

5 Evidence Differentials and Cushioning Credences

I have given away the answer already in the introduction: instates do better. It is surprising that, except for a rudimentary allusion to this in Walley's book, no Boolean has caught on to this yet. After I found out that agents with instates do better betting on soccer games, I let Betsy and Linda play a more basic betting game. An n-sided die is rolled (by the computer). The die is fair, unbeknownst to the players. Their bets are randomly and uniformly drawn from the simplex for which the probabilities attributed to the n results add up to 1. Betsy also surrounds her credences with an imprecision uniformly drawn from the interval $(0, y)$. I used Walley's pay off scheme (see Walley, 1991, 632) to settle the bets.

Here is an example: let $n = 2$, so the die is a fair COIN. Betsy's and Linda's bets are randomly and uniformly drawn from the line segment from $(0, 1)$ to $(1, 0)$ (these are two-dimensional Cartesian coordinates), the two-dimensional simplex (for higher n, the simplex is a pentatope generalized for n dimensions with side length $2^{1/2}$). The previsions (limits at which bets are accepted) may be $(0.21, 0.79)$ for Linda and $(0.35 \pm 0.11, 0.65 \pm 0.11)$ for Betsy, where the indeterminacy ± 0.11 is also randomly and uniformly drawn from the imprecision interval $(0, y) \subseteq (0, 1)$. The first bet is on H, and Linda is willing to pay 22.5 cents for it, while Betsy is willing to pay 77.5 cents against it. The second bet is on T (if $n > 2$, there will not be the same symmetry as in the COIN case between the two bets), for which Betsy is willing to pay 77.5 cents, and against which Linda is willing to pay 22.5 cents. Each bet pays $1 if successful. Often, Linda's credal state will overlap with Betsy's sharp credence so that there will not be a bet.

The computer simulation clearly shows that Linda does better than Betsy in the long run. A defence of sharp credences for rational agents needs to have an explanation for this. We will call it partial belief cushioning, which is based on an evidence differential between the bettors.

In many decision-making contexts, we do not have the luxury of calling off the bet. We have to decide one way or another. This is a problem for instates, as Booleans have to find a way to decide without receiving instructions from the credal state. Booleans have addressed this point extensively (see for example Joyce, 2010, 311ff; for an opponent's view of this see Elga, 2010, 6ff). The problem for sharp credences arises when bets are noncompulsory, for then the data above suggest that agents holding instates systematically do better. Often, decision making happens as betting vis-à-vis uninformed nature or opponents which are at least as uninformed as the rational agent. Sometimes, however, bets are offered by better informed or potentially better informed bookies. In this case, even an agent with sharp credences must cushion her credences and is better off by rejecting bets that look attractive in terms of her partial beliefs.

If an agent does not cushion her partial beliefs (whether they are sharp or indeterminate), she will incur a loss in the long run. Since cushioning is permitted in Walley's experimental setup (the bets are noncompulsory), Laplacean agents should also have access to it and then no longer do worse than Boolean agents. One may ask what sharp credences do if they just end up being cushioned anyway and do not provide sufficient information to decide on rational bets. The answer is that sharp credences are sufficient where betting (or decision making more generally) is compulsory; the cushioning only supplies the information from the evidence inasmuch as betting is noncompulsory and so again properly distinguishes semantic categories. This task is much harder for Booleans, although I do not claim that it is insurmountable: instates can provide a coherent approach to compulsory betting. What they cannot do, once cushioning is introduced, is outperform sharp credences in noncompulsory betting situations.

Here are a few examples: even if I have little evidence on which to base my opinion, someone may force me to either buy Coca Cola shares or short them, and so I have to have a share price p in mind that I consider fair. I will buy Coca Cola shares for less than p, and short them for more than p, if forced to do one or the other. This does not mean that it is now reasonable for me to go (not forced by anyone) and buy Coca Cola shares for p. It may not even be reasonable to go (not forced by anyone) and buy Coca Cola share for $p - \delta$ with $\delta > 0$.

It may in fact be quite unreasonable, since there are many players who have much better evidence than I do and will exploit my ignorance. I suspect that most lay investors in the stock market make this mistake: even though they buy and sell stock at prices that seem reasonable to them, professional investors are much better and faster at exploiting arbitrage opportunities and more subtle regularities. If indices rise, lay investors will make a little less than their professional counterparts; and when they fall, lay investors lose a lot more. In sum, unless there is sustained growth and everybody wins, lay investors lose in the long term.

A case in point is the U.S. Commodity Futures Trading Commission's crack-down on the online prediction market Intrade. Intrade offered fair bets for or against events of public significance, such as election results or other events which had clear yes-or-no outcomes. Even though the bets were all fair and Intrade only received a small commission on all bets, and even though Intrade's predictions were remarkably accurate, the potential for professional arbitrageurs was too great and the CFTC shut Intrade down (see `https://www.intrade.com`).

Cushioning does not stand in the way of holding a sharp credence, even if the evidence is dim. The evidence determines for a rational agent the partial beliefs over possible states of the world operating in the background. The better the evidence, the more pointed the distributions of these partial beliefs will be and the more willing the rational agent will be to enter a bet, if betting is noncompulsory. The mathematical decision rule will be based on the underlying distribution of the partial beliefs, not only on the sharp credence. As we have stated before, the sharp credence is not a sufficient statistic for decision making, inference, or betting behaviour; and neither is an instate.

The rational agent with a sharp credence has resources at her disposal to use just as much differentiation with respect to accepting and rejecting bets as the agent with instates. Often (if she is able to and especially if the bets are offered to her by a better-informed agent), she will reject both of two complementary bets, even when they are fair. On the one hand, any advantage that the agent with an instate has over her can be counteracted based on her distribution over partial beliefs that she has with respect to all possibilities. On the other hand, the agent with instates suffers from a semantic mixing of metaphors between evidential and doxastic dimensions that puts her at a real disadvantage in terms of understanding the sources and consequences of her knowledge and her uncertainties.

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