Naohito,

Your 1993 paper, Geometrical Structures of Some Non-Distance Models for Asymmetric MDS, is very helpful. I was wondering about transitivity. If d is an asymmetric distance measure and

$$\Delta(p,q) = d(q,p) - d(p,q) \tag{1}$$

then one would usually expect that $\Delta(P,Q) > 0$ and $\Delta(Q,R) > 0$ implies $\Delta(P,R) > 0$. That is if the asymmetry is well-behaved.

I am particularly interested in the Kullback-Leibler divergence:

$$D_{\mathrm{KL}}(P,Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \tag{2}$$

where P, Q are finite probability distributions with

$$\sum_{i=1}^{n} p_i = 1. \tag{3}$$

It turns out that the KL divergence is not well-behaved. For example, if

$$\begin{aligned} P &= (1/3, 1/3, 1/3) && (4) \\ Q &= (1/2, 1/4, 1/4) && (5) \\ R &= (2/5, 2/5, 1/5) && (6) \end{aligned}$$

$$\begin{aligned} D(P, Q) &> 0 && (7) \\ D(Q, R) &> 0 && (8) \\ D(P, R) &< 0 && (9) \end{aligned}$$

You made the following suggestion:

> Very interesting! I suppose that your finding may be explained by examining the definiteness of the Hermitian matrix $H = (S+S')/2+$

$i * (S - S')/2$ constructed from the proximity matrix S which you specified above. Here, S' denotes the transposed matrix of S, and i denotes the pure imaginary number. This inspection comes from the theory of the Hermitian Form Model (HFM) proposed by Chino and Shiraiwa (1993), Behaviormetrika. Our theory states that members (or objects) are embedded in a finite-dimensional Hilbert space if and only if the matrix H is positive-semi definite.

I would love to show that this is the case and your hunch is right. Transitivity is violated if for S' (your skew-symmetric matrix)

$$\begin{bmatrix} 0 & k_{12} & k_{13} \\ k_{21} & 0 & k_{23} \\ k_{31} & k_{32} & 0 \end{bmatrix}. \tag{10}$$

k_{12} and k_{23} agree in their sign against k_{13} (for example: $k_{12} > 0, k_{23} > 0, k_{13} < 0$).

Take, for example the following proximity data matrix

$$\begin{bmatrix} 0 & 2 & 3 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}. \tag{11}$$

Let H be defined as in your paper on page 36. Then

$$\det(H - \lambda I) = -\lambda^3 + 14\lambda - 1 \tag{12}$$

There are then three real-valued eigenvalues $\lambda_1, \lambda_2, \lambda_3$. What you call Λ and U_1 on page 37 looks as follows:

$$\Lambda = \begin{bmatrix} -3.78 & 0 & 0 \\ 0 & 0.0715 & 0 \\ 0 & 0 & 3.71 \end{bmatrix}, \tag{13}$$

$$U_1 = \begin{bmatrix} 0.0185 + 0.639i & -0.375 + 0.199i & 0.514 + 0.386i \\ 0.279 - 0.494i & -0.169 - 0.573i & 0.503 + 0.260i \\ -0.519 & 0.681 & 0.516 \end{bmatrix}. \tag{14}$$

All of this makes good sense, but then I get confused on page 38. In your 1978 paper, the idea was to map the objects whose distance are measured into some kind of space in order to model them (first, you did this for two-dimensional spaces, then for three, and then you generalized it to higher-dimensional spaces in later papers). The size of the parallelogram between them was indicative of asymmetries (cross product), whereas the inner product was indicative of their mutual proximity or distance. The orientation indicated which way the asymmetry went.

I take the 1993 paper to propose a non-spatial model where the objects are mapped onto an f.d.c. Hilbert space. Cross product and inner product are still indicative of the skew-symmetry and symmetry. The inner product, which chracterizes the Hilbert space, is

$$\phi(\zeta, \tau) = \zeta \Lambda \tau^*. \tag{15}$$

What I don't see explained is the following question:

Where are the original objects in the Hilbert space? In my proximity data matrix (11), for example, the distance measured from object 1 to object 2 is 2 and the distance measured from object 2 to object 1 is 3. Where are these objects in the f.d.c. Hilbert space characterized by (15)? Do we have to go through the long and painful iteration process of your 1978 paper? I thought for a moment that your x_i in your formula (13) on page 38 may have something to do with this, but their dimension is double of what we have in the Hilbert space. In any case, why did you need X, Ω_s, Ω_{sk} on page 38? Once we have the objects mapped onto the Hilbert space, we have the inner product defined by (15), but how do we define the cross-product? It is usually only defined for three dimensional space. All of this is not clear to me.