Naohito,

Thank you very much for responding to my inquiries so far. I have been working through your 1993 paper, Geometrical Structures of Some Non-Distance Models for Asymmetric MDS. Here is something I am wondering about. You define the seminorm (page 38)

$$\|\zeta\| = \sqrt{\varphi(\zeta, \zeta)} \tag{1}$$

and then say, "in particular, if φ is positive for any $\zeta \neq 0$, $\|\zeta\|$ defines a *norm*." Let me defend the claim that $\sqrt{\varphi(\zeta, \zeta)}$ **never** defines a norm because the diagonal matrix Λ is always indefinite (i.e. it always contains at least one negative and one positive real number). Let's call this claim ALWAYS-INDEF.

Recall that φ was defined as follows:

$$\varphi(\zeta, \tau) = \zeta \Lambda \tau^* \tag{2}$$

and

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \tag{3}$$

where the λ_j are the eigenvalues (repeated according to multiplicity) of H as defined on page 36. H is a Hermitian matrix with $\operatorname{tr}(H) = 0$, and according to a well-known theorem in linear algebra (see link in email)

$$\sum_{j=1}^{n} \lambda_j = \operatorname{tr}(H) = 0. \tag{4}$$

In other words, the trace of Λ is 0. ALWAYS-INDEF follows immediately unless $\Lambda = 0$, which is a trivial case.

This appears to be sad news for both you and me. You were trying to model asymmetries. The Hermitian Form Model looks initially very promising and elegant, but now it turns out that the seminorm defined on the Hilbert Space is always indefinite. I was hoping that I could classify asymmetry as follows: (null) no asymmetry, for example Euclidean distance; (well-behaved) asymmetry obeying the triangle inequality and transitivity, for example Tobler's wind model; (ill-behaved) asymmetry violating the triangle inequality and transitivity, for example the Kullback-Leibler divergence. Your distinction between only seminorm-inducing asymmetries and norm-inducing asymmetries may have delivered such a classification, as you expressed in your email:

> Very interesting! I suppose that your finding may be explained by examining the definiteness of the Hermitian matrix $H = (S + S')/2 + i * (S - S')/2$ constructed from the proximity matrix S which you specified above. Here, S' denotes the transposed matrix of S, and i denotes the pure imaginary number. This inspection comes from the theory of the Hermitian Form Model (HFM) proposed by Chino and Shiraiwa (1993), Behaviormetrika. Our theory states that members (or objects) are embedded in a finite-dimensional Hilbert space if and only if the matrix H is positive-semi definite.

Unofortunately, as it turns out, all cats are grey at night and there are no norm-inducing asymmetries. Even if H is symmetrical and $S_{sk} = 0$ the seminorm on the Hilbert Space is still indefinite.