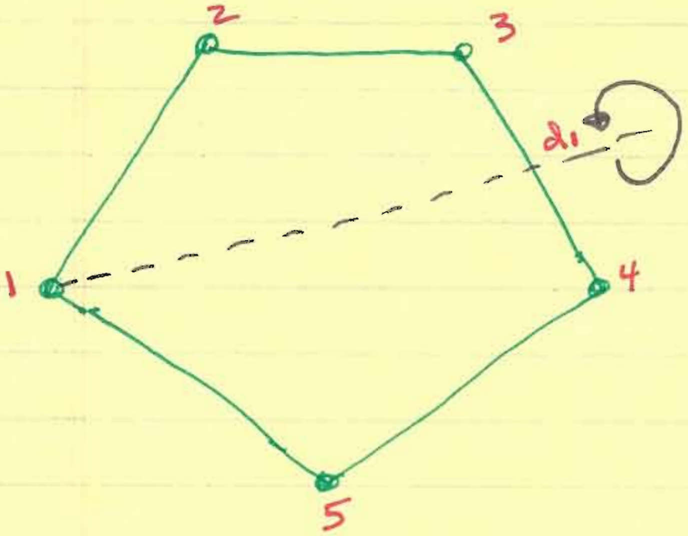
hw2

Homework - cp. 2

p. 45 #1) A group can be constructed by using the rotations and reflections of a pentagon into itself.

a) How many elements are in this group? Is it an abelian group?



Rotations are through $k \cdot \frac{360°}{5}$, $k \in \mathbb{Z}$

$72° \cdot k$ where $k = 0, 1, 2, 3, 4$

geometrically — algebraically

$\pi_0 = 0°$ cw rotation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

$\pi_1 = 72°$ rot. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$

$\pi_2 = 144°$ rot. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$

$\pi_3 = 216°$ rot. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$

$\pi_4 = 288°$ rot. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

Reflections (or flips)

let d_i, $i = 1, \dots 5$ be the flip about vertex i

$d_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$

$d_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$

$d_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

$d_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$

$d_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}$

This group contains 10 elements
5 rotations
5 flips.

D_5: dihedral group of degree 5 and order 10

b) Construct the group

$\circ$ is composition of permutations

$P_1 \circ P_2$ first then second

$\circ$	π_0	π_1	π_2	π_3	π_4	d_1	d_2	d_3	d_4	d_5
π_0	π_0	π_1	π_2	π_3	π_4	d_1	d_2	d_3	d_4	d_5
π_1	π_1	π_2	π_3	π_4	π_0	d_2	d_3	d_4	d_5	d_1
π_2	π_2	π_3	π_4	π_0	π_1	d_3	d_4	d_5	d_1	d_2
π_3	π_3	π_4	π_0	π_1	π_2	d_4	d_5	d_1	d_2	d_3
π_4	π_4	π_0	π_1	π_2	π_3	d_5	d_1	d_2	d_3	d_4
d_1	d_1	d_4				π_0				
d_2	d_2						π_0			
d_3	d_3							π_0		
d_4	d_4								π_0	
d_5	d_5									π_0

Observe that $d_1 \circ \pi_1 = d_4$

whereas $\pi_1 \circ d_1 = d_2$

hence D_5 is not abelian.

c) • Subgroup with 5 elements:

$C_5 = \langle \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}, \circ \rangle$ the cyclic group with 5 elements

C_5 is a subgroup of D_5

• Subgroup with 2 elements : Let $G = \langle \{\pi_0, d_1\}, \circ \rangle$

note - there are four other such subgroups

d) No subgroups with 4 elements. $|H| \,/\, |G|$ Lagrange's Thm.

5.) Show that $\langle \mathbb{Z}, - \rangle$ is not a group

i) closure $\forall x, y \in \mathbb{Z}, \; x - y \in \mathbb{Z}$ ✓

ii) associativity

$$3 - (2 - 1) \stackrel{?}{=} (3 - 2) - 1$$
$$3 - 1 \stackrel{?}{=} 1 - 1$$
$$2 \neq 0 \quad \text{No!}$$

$\therefore \langle \mathbb{Z}, - \rangle$ not a group.

8.) Let G be an arbitrary group (not necessarily finite) $\langle G, \cdot \rangle$, identity $= 1$.

$g \in G$ with $g^{\nu} = 1$, ν minimal

$g^{\nu} = \underbrace{g \cdot g \cdot \cdots g}_{\nu \text{ times}}$ — ν is the order of g.

Prove that $\langle \{ g, g^2, g^3, \dots g^{\nu-1}, g^{\nu} \}, \cdot \rangle$ subgroup of G

Call this subset $\{ g, g^2, \cdots, g^{\nu} \}$ H.

We must verify:

i) Closure : $g^i \cdot g^j = g^{(i+j) \bmod \nu}$ ✓

ii) Associativity :

$$g^i \cdot (g^j \cdot g^k) \stackrel{?}{=} (g^i \cdot g^j) \cdot g^k$$
$$g^i \cdot (g^{(j+k) \bmod \nu}) \stackrel{?}{=} g^{(i+j) \bmod \nu} \cdot g^k$$
$$g^{(i+j+k) \bmod \nu} = g^{(i+j+k) \bmod \nu} \checkmark$$

iii) Identity : $g^{\nu} = 1$, hence g^{ν} is the identity

iv) Inverses : $(g^i)^{-1} = g^{\nu - i}$

as $g^i \cdot g^{\nu-i} = g^{\nu} = g^{\nu-i} \cdot g^i = 1.$ ✓

$\langle H, \cdot \rangle$ is always abelian. $g^i \cdot g^j = g^j \cdot g^i = g^{(i+j) \bmod \nu}$