Math 110A HW §1.5 – Solutions

1& 2(m) Exhibit a left and right inverse of f if it exists.

$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ x+2 & \text{if } x \text{ is odd} \end{cases}.$

Note that $f(1) = 3 = f(6)$, so f is not one-to-one. Therefore f does not have a left inverse.

Let $b \in \mathbb{Z}$, then $f(2b) = b$. Therefore f is onto and hence has a right inverse.

Let's find a right inverse. Since in the above argument we only need the first rule, namely $\frac{x}{2}$, to show onto, we probably only need to look there for the inverse. The inverse of the function $\frac{x}{2}$ is $2x$. So let $g(x) = 2x$. Let's check that this is a right inverse of f. Let $x \in \mathbb{Z}$.

$$f \circ g(x) = f(g(x)) = f(2x) = \frac{2x}{2} = x$$

Thus g is a the right inverse of f.

6. Prove that if f is a permutation on A, then f^{-1} is a permutation on A.

Proof. Let $f : A \to A$ be a permutation on A. In other words, f is one-to-one and onto. Therefore f has an inverse mapping, f^{-1}. We now must show that the inverse function is itself one-to-one and onto. Let $a \in A$, then $f^{-1}(f(a)) = a$. Hence f^{-1} is onto. Suppose $f^{-1}(a_1) = f^{-1}(a_2)$. Then $a_1 = f(f^{-1}(a_1) = f(f^{-1}(a_2) = a_2$. Hence f^{-1} is one-to-one. Therefore f^{-1} is a permutation on A. □

8. (a) Prove that the set of onto mappings from A to A is closed under composition of mappings.

Proof. Let f, g be onto mappings from A to A. We need to show that $f \circ g$ is an onto mapping.
Let $b \in A$. Since f is onto there exists $c \in A$ such that $f(c) = b$. Since g is onto there exists $a \in A$ such that $g(a) = c$. Therefore $f \circ g(a) = f(g(a)) = f(c) = b$. Thus $f \circ g$ is onto. □

(b) Prove that the set of one-to-one mappings from A to A is closed under composition of mappings.

Proof. Let f, g be one-to-one mappings from A to A. We need to show that $f \circ g$ is a one-to-one mapping. Let $a_1, a_2 \in A$ such that $f \circ g(a_1) = f \circ g(a_2)$. So $f(g(a_1)) = f(g(a_2))$. Since f is one-to-one, $g(a_1) = g(a_2)$. Since g is one-to-one, $a_1 = a_2$. Thus $f \circ g$ is one-to-one. □

10. Let f and g be mappings from A to A. Prove that if $f \circ g$ is invertible, then f is onto and g is one-to-one.

Proof. Assume that $f \circ g$ is invertible and let h be the inverse of $f \circ g$. So $h \circ (f \circ g) = I$ and $(f \circ g) \circ h = I$, where I is the identity function. In other words, $h \circ (f \circ g)(x) = x$ and $(f \circ g) \circ h(x) = x$ for all $x \in A$.

First we will show f is onto. Let $b \in A$. Then $(f \circ g) \circ h(b) = b$, but mapping composition is associative, so $f \circ (g \circ h)(b) = b$. Thus $f\Big(g(h(b))\Big) = b$. Hence f is onto.

Now we will show f is one-to-one. Let $a_1, a_2 \in A$ such that $g(a_1) = g(a_2)$. Then we have the following.

$$\begin{aligned} g(a_1) &= g(a_2) \\ h(f(g(a_1))) &= h(f(g(a_2))) \\ h \circ (f \circ g)(a_1) &= h \circ (f \circ g)(a_2) \\ a_1 &= a_2 \end{aligned}$$

Thus g is one-to-one. $\square$