Q 1.

(a) In order to show that

$$H_{ham} = \begin{pmatrix} 0 & 1 & 2 & 4 & 6 & 4 & 3 & 5 \\ 3 & 2 & 2 & 6 & 1 & 2 & 2 & 0 \end{pmatrix}$$

is a parity check matrix for a Hamming code Ham(2,7) consider the following.

For Ham(2,7) we have $r = 2$ and $q = 7$ so any non-zero vector $\boldsymbol{v}$ in $V(2,7)$ has exactly $7 - 1 = 6$ non-zero scalar multiples, forming the set $\{\lambda\boldsymbol{v}|\lambda \in GF(7), \lambda \neq 0\}$. The $(7^2 - 1)/(7 - 1) = 8$ such sets or classes are given below.

$$\left\{\begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}0\\6\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\5\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\4\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}3\\6\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}6\\5\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}4\\4\end{pmatrix}, \begin{pmatrix}6\\6\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}, \begin{pmatrix}5\\5\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}4\\6\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}5\\4\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}6\\2\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}6\\1\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}, \begin{pmatrix}1\\6\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}5\\6\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}6\\3\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}, \begin{pmatrix}2\\6\end{pmatrix}, \begin{pmatrix}5\\1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}4\\5\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}5\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}6\\0\end{pmatrix}, \begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\right\}$$

Each class contains exactly one column vector from the parity-check matrix H and as such each column vectors of H is linearly independent of any other. Thus, the given parity-check matrix, H, is that for a Hamming code Ham(2,7).

(b) Using just row operations only the parity check matrix of Ham(2,7) can be transformed into the generator matrix, G, for the simplex code, Sim(2,7), in standard form as shown below.

$$G_{sim} = \begin{pmatrix} 0 & 1 & 2 & 4 & 6 & 4 & 3 & 5 \\ 3 & 2 & 2 & 6 & 1 & 2 & 2 & 0 \end{pmatrix} \xrightarrow[r_2 \to 5r_2]{r_1 \to 3r_1} \begin{pmatrix} 0 & 3 & 6 & 5 & 4 & 5 & 2 & 1 \\ 1 & 3 & 3 & 2 & 5 & 3 & 3 & 0 \end{pmatrix},$$

$$\xrightarrow{r_2 \to r_2 - r_1} \begin{pmatrix} 0 & 3 & 6 & 5 & 4 & 5 & 2 & 1 \\ 1 & 0 & 4 & 4 & 1 & 5 & 1 & 6 \end{pmatrix},$$

$$\xrightarrow{r_1 \to 5r_1} \begin{pmatrix} 0 & 1 & 2 & 4 & 6 & 4 & 3 & 5 \\ 1 & 0 & 4 & 4 & 1 & 5 & 1 & 6 \end{pmatrix},$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 0 & 4 & 4 & 1 & 5 & 1 & 6 \\ 0 & 1 & 2 & 4 & 6 & 4 & 3 & 5 \end{pmatrix}.$$

Now using Theorem 7.6 of **H** page 70: if $G_{sim} = [I_2|A]$ then $H_{sim} = [-A^T|I_6]$ and given G_{sim} in standard form above we have,

$$A = \begin{pmatrix} 4 & 4 & 1 & 5 & 1 & 6 \\ 2 & 4 & 6 & 4 & 3 & 5 \end{pmatrix},$$

$$-A = \begin{pmatrix} 3 & 3 & 6 & 2 & 6 & 1 \\ 5 & 3 & 1 & 3 & 4 & 2 \end{pmatrix},$$

$$-A^T = \begin{pmatrix} 3 & 5 \\ 3 & 3 \\ 6 & 1 \\ 2 & 3 \\ 6 & 4 \\ 1 & 2 \end{pmatrix},$$

$$I_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\therefore H_{sim} = \left[-A^T|I_6\right] = \begin{pmatrix} 3 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) Sim(2,7) has $|V(8,7)| \,/\, |\mathrm{Sim}(2,7)| = 7^8/7^2 = 7^6$ cosets.

Now in our case, $d(C) = 2t + 1 = 7$, so we are guaranteed that $<= t, <= 3$ errors can be corrected in any codeword. Thus, in the top part of the Slepian standard array the coset-leaders will be those that have a weight of three or less. Thus, in the top part of the Slepian standard array we will have one vector of weight zero, namely 00000000. For the case where the weight is one we can choose for a given coordinate position in a vector of length eight any one of the values from $\{1, 2, \ldots, 6\}$. As there are eight coordinate positions in the vector we have

$$8 \times \binom{8}{1} = 8 \times \frac{8!}{(8-1)! \times 1!} = 6 \cdot 8 = 48,$$

coset-leaders of weight one. Continuing with this logic we can build an expression for the number of coset-leaders of weight two and weight three.

Thus, the number of coset-leaders in the top part of the Slepian standard array is given by the following expression:

$$\sum_{k=0}^{t}(q-1)^k\binom{n}{k} = \binom{n}{0} + (q-1)\binom{n}{1} + (q-1)^2\binom{n}{2} +, \ldots, +(q-1)^t\binom{n}{t}.$$

In our case $q = 7$, $n = 8$ and $t = 3$ as $d(C) = 2t + 1 = 7$ and so we have

$$\begin{aligned}\sum_{k=0}^{3}(7-1)^k\binom{8}{k} &= \binom{8}{0} + (7-1)\binom{8}{1} + (7-1)^2\binom{8}{2} + (7-1)^3\binom{8}{3}, \\ &= 1 + 48 + 1008 + 12096 = 13153 \quad \text{coset-leaders.}\end{aligned}$$

Thus, there are 13153 coset leaders in the top part of the Slepian standard array for Sim(2,7).

Shown in Table 1 are the syndromes for the vectors $10000000, 01000000, \ldots .00000001$. To correct the received vector, assuming at most one error, 45632036 which has syndrome 123256 observe that $531342 \times 3 = 123256$. Hence, the received vector 45632036 is in the same coset as that with coset-leader $01000000 \times 3 = 03000000$. So, we decode the received vector as $45632036 - 03000000 = 42632036$.

vector	syndrome
1 0 0 0 0 0 0 0	3 3 6 2 6 1
0 1 0 0 0 0 0 0	5 3 1 3 4 2
0 0 1 0 0 0 0 0	1 0 0 0 0 0
0 0 0 1 0 0 0 0	0 1 0 0 0 0
0 0 0 0 1 0 0 0	0 0 1 0 0 0
0 0 0 0 0 1 0 0	0 0 0 1 0 0
0 0 0 0 0 0 1 0	0 0 0 0 1 0
0 0 0 0 0 0 0 1	0 0 0 0 0 1

Table 1: Syndromes of vectors $10000000, 01000000, \ldots, 00000001$.

(d) The extended binary Hamming code is the code obtained by from $\hat{\mathrm{Ham}}(4, 2)$ by adding an overall parity-check. Thus, parity-check matrix for an extended

Hamming code $\hat{\text{Ham}}(4, 2)$ is

$$\hat{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The incomplete decoding algorithm for this extended binary Hamming code is as follows. Suppose the received vector is $\boldsymbol{y}$. Calculate the syndrome $S(\boldsymbol{y}) = \boldsymbol{y}\hat{H}^T$ such that $S(\boldsymbol{y}) = (s_1, s_2, s_3, s_4, s_5)$. Then

1. If $s_5 = 0$ and $(s_1, s_2, s_3, s_4) = \mathbf{0}$, assume that no errors have occurred,
2. If $s_5 = 0$ and $(s_1, s_2, s_3, s_4) \neq \mathbf{0}$, assume that at least two errors have occurred and request retransmission,
3. If $s_5 = 1$ and $(s_1, s_2, s_3, s_4) = \mathbf{0}$, assume that a single error in the last place of y has occurred,
4. If $s_5 = 1$ and $(s_1, s_2, s_3, s_4) \neq \mathbf{0}$, assume that a single error in the j^{th} place, where j is the number whose binary representation is (s_1, s_2, s_3, s_4).

(e) Applying the algorithm of part (d) to the following received vectors

1. $\boldsymbol{y} = [0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ so $S(\boldsymbol{y}) = \boldsymbol{y}\hat{H}^T = [0\ 1\ 0\ 1\ 1]$. So $s_5 = 1$ with $(s_1, s_2, s_3, s_4) = (0, 1, 0, 1) \neq \mathbf{0}$. Therefore, assume an error in the $j = 5$ place of $\boldsymbol{y}$.
2. $\boldsymbol{y} = [0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1]$ so $S(\boldsymbol{y}) = \boldsymbol{y}\hat{H}^T = [0\ 0\ 0\ 0\ 1]$. So $s_5 = 1$ with $(s_1, s_2, s_3, s_4) = (0, 0, 0, 0) = \mathbf{0}$. Therefore, assume an error in the last place of $\boldsymbol{y}$.
3. $\boldsymbol{y} = [1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1]$ so $S(\boldsymbol{y}) = \boldsymbol{y}\hat{H}^T = [1\ 1\ 0\ 0\ 0]$. So $s_5 = 0$ with $(s_1, s_2, s_3, s_4) = (1, 1, 0, 0) \neq \mathbf{0}$. Therefore, assume that at least two errors have occurred and request retransmission,

Q 2.

(a)

(b)

(c)

(d)

(e)

Q 3.

(a)

(b)

(c)

Q 4.

(a)

(b)

(c)