Q 1.

(a) Given,

$$S[y] = \int_a^b \mathrm{d}x \left(y^2 \sinh x - \frac{2y'^2}{\sinh x} \right), \quad y(a) = A, \quad y(b) = B, \quad 0 < a < b,$$

then

$$\text{Let } F = y^2 \sinh x - \frac{2y'^2}{\sinh x}.$$

The following derivatives are required to determine the Euler-Lagrange equation

Making use of the quotient rule here.

$$\frac{\partial F}{\partial y'} = -\frac{4y'}{\sinh x}.$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{4y'}{\sinh x}\right), \\
&= \frac{\sinh x\,(-4y'') - (-4y')\cosh x}{\sinh^2 x}, \\
&= \frac{4}{\sinh x}\left(y'\frac{\cosh x}{\sinh x} - y''\right),
\end{aligned}$$

$$\text{and} \qquad \frac{\partial F}{\partial y} = 2y \sinh x.$$

The Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0,$$

and substituting into this equation the expressions for the derivatives obtained above gives the Euler-Lagrange equation associated with the functional $S[y]$,

$$\frac{4}{\sinh x}\left(y'\frac{\cosh x}{\sinh x} - y''\right) - 2y\sinh x = 0,$$

which can be rearranged into the following,

$$\begin{gathered}
2\sinh(x) \cdot y'' - 2\cosh(x) \cdot y' + \sinh^3(x) \cdot y = 0, \qquad (1.1) \\
y(a) = A,\ y(b) = B,\ 0 < a < b.
\end{gathered}$$

(b) The given functional from part (a) is

$$S[y] = \int_a^b \mathrm{d}x \left(y^2 \sinh x - \frac{2y'^2}{\sinh x} \right), \quad y(a) = A, \quad y(b) = B, \quad 0 < a < b,$$

In the functional the term y', which is a function of the independent variable x, needs to be changed to a function of u which is in turn a function of x. Thus, letting the variable u be a function of x, i.e. $u = u(x)$, then, using the chain rule to change the independent variable from x to u of $y'(u)$ gives the following expression for y'

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

So, if y' in the functional is replaced for the expression above,

$$\begin{aligned}
S[y] &= \int_{x_1=a}^{x_2=b} \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \left(y^2 \sinh x - \frac{2}{\sinh x} \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \right),\\
S[y] &= \int_{x_1=a}^{x_2=b} \frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \left(y^2 \sinh x - \frac{2}{\sinh x} \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 \left(\frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\right) \right),\\
S[y] &= \int_{u_1}^{u_2} \mathrm{d}u \left(y^2 \frac{\mathrm{d}x}{\mathrm{d}u} \sinh x - \frac{2}{\sinh x} \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 \left(\frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\right) \frac{\mathrm{d}x}{\mathrm{d}u} \right),\\
S[y] &= \int_{u_1}^{u_2} \mathrm{d}u \left(y^2 \frac{\mathrm{d}x}{\mathrm{d}u} \sinh x - \frac{2}{\sinh x} \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 \frac{\mathrm{d}u}{\mathrm{d}x} \right),\\
S[y] &= \int_{u(a)}^{u(b)} \mathrm{d}u \left(y^2 \frac{\mathrm{d}x}{\mathrm{d}u} \sinh x - 2 \left[\frac{1}{\sinh x} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\right] \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 \right).
\end{aligned}$$

If

$$\frac{1}{\sinh x} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

then

$$1 = \sinh x \cdot \frac{\mathrm{d}x}{\mathrm{d}u},$$

and so

$$\mathrm{d}u = \sinh x \,\mathrm{d}x.$$

Integrating the above

$$\begin{aligned}
u(x) &= \int \mathrm{d}x \sinh x = \cosh x,\\
u(x) &= \cosh x.
\end{aligned}$$

Thus, the functional becomes

$$\begin{aligned}
S[y] &= \int_{\cosh a}^{\cosh b} \mathrm{d}u \left(y^2 - 2 \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 \right),\\
S[y] &= \int_{\cosh a}^{\cosh b} \mathrm{d}u \left(y^2 - 2\left(y'(u)\right)^2 \right), \qquad (1.2)
\end{aligned}$$

as required, with $u_1 = \cosh a$ and $u_2 = \cosh b$.

(c) Solving for the associated Euler-Lagrange equation for the functional (1.2):

$$\text{Let} \quad \mathcal{F} = y^2 - 2y'^2,$$

then

$$\frac{\partial \mathcal{F}}{\partial y'} = -4y',$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial \mathcal{F}}{\partial y'}\right) = \frac{\mathrm{d}}{\mathrm{d}u}(-4y') = -4y'',$$

$$\text{and} \quad \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}y} = 2y.$$

The Euler-Lagrange equation is

$$-4\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} - 2y = 0,$$

which after dividing through by -4 becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} + \frac{1}{2}y = 0.$$

The auxiliary equation for the above Euler-Lagrange equation is

$$\lambda^2 + \frac{1}{2} = 0, \quad \text{and the roots are} \quad \lambda_{1,2} = \pm i\frac{1}{\sqrt{2}}.$$

Hence, the general solution is

$$y = c_1 \cos\left(\frac{1}{\sqrt{2}} \cdot u\right) + c_2 \sin\left(\frac{1}{\sqrt{2}} \cdot u\right), \tag{1.3}$$

where, c_1 and c_2 are arbitrary constants.

(d) Substituting into (1.3) for $u = \cosh x$ gives,

$$y(x) = c_1 \cos\left(\frac{1}{\sqrt{2}}\cosh x\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}\cosh x\right), \tag{1.4}$$

Now, when $x = 1$, $y = 0$ and when $x = 2$, $y = 2$. Substituting these boundary conditions into the above expression for y (1.4) gives the following pair of simultaneous equations,

$$\begin{aligned} 0 =& c_1 \cos\left(\frac{1}{\sqrt{2}}\cosh 1\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}\cosh 1\right), \quad \text{and} \\ 2 =& c_1 \cos\left(\frac{1}{\sqrt{2}}\cosh 2\right) + c_2 \sin\left(\frac{1}{\sqrt{2}}\cosh 2\right). \end{aligned}$$

Solving for the constants c_1 and c_2 gives,

$$c_1 = \frac{-2\sin\left(\frac{1}{\sqrt{2}}\cosh 1\right)}{\sin\left(\frac{1}{\sqrt{2}}\cosh 2\right)\cos\left(\frac{1}{\sqrt{2}}\cosh 1\right) - \sin\left(\frac{1}{\sqrt{2}}\cosh 1\right)\cos\left(\frac{1}{\sqrt{2}}\cosh 2\right)},$$

$$c_2 = \frac{2\cos\left(\frac{1}{\sqrt{2}}\cosh 1\right)}{\sin\left(\frac{1}{\sqrt{2}}\cosh 2\right)\cos\left(\frac{1}{\sqrt{2}}\cosh 1\right) - \sin\left(\frac{1}{\sqrt{2}}\cosh 1\right)\cos\left(\frac{1}{\sqrt{2}}\cosh 2\right)}.$$

Making use of the identity $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$, then the above pair of equations can be simplified to

$$c_1 = \frac{-2\sin\left(\frac{1}{\sqrt{2}}\cosh 1\right)}{\sin\left(\frac{1}{\sqrt{2}}\cosh 2 - \frac{1}{\sqrt{2}}\cosh 1\right)},$$

$$c_2 = \frac{2\cos\left(\frac{1}{\sqrt{2}}\cosh 1\right)}{\sin\left(\frac{1}{\sqrt{2}}\cosh 2 - \frac{1}{\sqrt{2}}\cosh 1\right)}.$$

Thus, substituting into (1.4) for c_1 and c_2 gives,

$$y(x) = \frac{-2\sin\left(\frac{1}{\sqrt{2}}\cosh 1\right)\cos\left(\frac{1}{\sqrt{2}}\cosh x\right) + 2\cos\left(\frac{1}{\sqrt{2}}\cosh 1\right)\sin\left(\frac{1}{\sqrt{2}}\cosh x\right)}{\sin\left(\frac{1}{\sqrt{2}}\cosh 2 - \frac{1}{\sqrt{2}}\cosh 1\right)}.$$

Simplifying the above using the same trig identity as before gives,

$$y(x) = \frac{2\sin\left(\frac{1}{\sqrt{2}}(\cosh x - \cosh 1)\right)}{\sin\left(\frac{1}{\sqrt{2}}\cosh 2 - \frac{1}{\sqrt{2}}\cosh 1\right)}$$

as required.

Q 2.

(a) Given

$$S\left[y_1, y_2\right] = \int \mathrm{d}x \, \left(y_1'^2 + 2y_2'^2 + \left(2y_1 + y_2\right)^2\right),$$

the first coupled Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_1'}\right) - \frac{\partial F}{\partial y_1} = 0,$$

where

$$F = y_1'^2 + 2y_2'^2 + \left(2y_1 + y_2\right)^2.$$

So,

$$\begin{aligned} \frac{\partial F}{\partial y_1'} =& 2y_1', \quad \frac{\mathrm{d}\left(2y_1'\right)}{\mathrm{d}x} = 2y_1'', \quad \text{and} \\ \frac{\partial F}{\partial y_1} =& 2\left(2y_1 + y_2\right) 2 = 4\left(2y_1 + y_2\right). \end{aligned}$$

Thus, the Euler-Lagrange equation for the first coupled equation is

$$2y_1'' - 4\left(2y_1 + y_2\right) = 0. \tag{2.1}$$

The second coupled Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_2'}\right) - \frac{\partial F}{\partial y_2} = 0.$$

So,

$$\begin{aligned} \frac{\partial F}{\partial y_2'} =& 4y_2', \quad \frac{\mathrm{d}\left(4y_2'\right)}{\mathrm{d}x} = 4y_2'', \quad \text{and} \\ \frac{\partial F}{\partial y_2} =& 2\left(2y_1 + y_2\right). \end{aligned}$$

Thus, the Euler-Lagrange equation for the second coupled equation is

$$4y_2'' - 2\left(2y_1 + y_2\right) = 0. \tag{2.2}$$

(b) Making the linear transformation

$$\begin{aligned} z_1 =& y_1 + ay_2, \\ z_2 =& 2y_1 + y_2 \end{aligned}$$

to the new dependent variables z_1, z_2, where a is a constant, the value of a can be found in such a way that the functional can be written as the sum of two functionals, one of which depends only on z_1 and the other only on z_2 in the following way:

$$\begin{aligned} z_1 &= y_1 + a y_2, \\ z_1' &= y_1' + a y_2', \\ y_1' &= z_1' - a y_2', \quad \text{and} \qquad (2.3) \\ y_2' &= \frac{z_1' - y_1'}{a}. \qquad (2.4) \end{aligned}$$

Similarly,

$$\begin{aligned} z_2 &= 2y_1 + y_2, \\ z_2' &= 2y_1' + y_2', \\ y_1' &= \frac{z_2' - y_2'}{2}, \quad \text{and} \qquad (2.5) \\ y_2' &= z_2' - 2y_1'. \qquad (2.6) \end{aligned}$$

Equating (2.3) and (2.5) gives

$$z_1' - a y_2' = \frac{z_2' - y_2'}{2}, \qquad (2.7)$$

and rearranging (2.7) in terms of y_2' gives,

$$y_2' = \frac{2z_1' - z_2'}{2a - 1}. \qquad (2.8)$$

Now equating (2.4) and (2.6)

$$\frac{z_1' - y_1'}{a} = z_2' - 2y_1', \qquad (2.9)$$

and rearranging (2.9) in terms of y_1' gives,

$$y_1' = \frac{z_1' - a z_2'}{1 - 2a}. \qquad (2.10)$$

What is required is $y_1'^2 + 2y_2'^2$ in terms of just z_1 and z_2. Thus, from (2.10)

$$y_1'^2 = \frac{(z_1' - a z_2')^2}{(1 - 2a)^2} = \frac{z_1'^2 - 2a z_1' z_2' + a^2 z_2'^2}{(1 - 2a)^2},$$

and from (2.8) after multiplying the numerator and denominator by -1 so that the denominator becomes $1 - 2a$

$$y_2'^2 = \frac{(z_2' - 2z_1')^2}{(1 - 2a)^2} = \frac{z_2'^2 - 4z_1' z_2' + 4z_1'^2}{(1 - 2a)^2}.$$

Therefore,

$$
\begin{aligned}
y_1'^2 + 2y_2'^2 =& \frac{z_1'^2 - 2az_1'z_2' + a^2 z_2'^2}{(1-2a)^2} + \frac{2\left(z_2'^2 - 4z_1'z_2' + 4z_1'^2\right)}{(1-2a)^2}, \\
=& \frac{9z_1'^2 - 2z_1'z_2'(a+4) + z_2'^2\left(a^2+2\right)}{(1-2a)^2}. \qquad (2.11)
\end{aligned}
$$

To eliminate the $z_1'z_2'$ term in (2.11) set $a+4=0$. Thus,

$$
\begin{aligned}
y_1'^2 + 2y_2'^2 =& \frac{9z_1'^2 + 18z_2'^2}{9^2}, \\
=& \frac{z_1'^2 + 2z_2'^2}{9}. \qquad (2.12)
\end{aligned}
$$

Consequently,

$$a = -4,$$

and is the value of a such that the given functional can be written as the sum of two functionals, one which depends only on z_1 and the other only on z_2. Thus,

$$S[z_1, z_2] = \frac{1}{9}\int \mathrm{d}x\, z_1'^2 + \frac{1}{9}\int \mathrm{d}x\, \left(2z_2'^2 + 9z_2^2\right).$$

(c)

$$S[z_1, z_2] = \int \mathrm{d}x\, \underbrace{\frac{1}{9}z_1'^2}_{=F_1} + \int \mathrm{d}x\, \underbrace{\left(\frac{2}{9}z_2'^2 + z_2^2\right)}_{=F_2}.$$

The Euler-Lagrange equation for F_2 is determined as follows.

$$\frac{\partial F_2}{\partial z_2'} = \frac{4}{9}z_2', \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{4}{9}z_2'\right) = \frac{4}{9}z_2'', \quad \frac{\partial F_2}{\partial z_2} = 2z_2.$$

Therefore, the Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F_2}{\partial z_2'}\right) - \frac{\partial F_2}{\partial z_2} = 0.$$

Substituting into the above Euler-Lagrange equation gives,

$$\frac{4}{9}z_2'' - 2z_2 = 0.$$

This differential equation has solution

$$z_2 = c_1 \mathrm{e}^{\frac{3}{\sqrt{2}}x} + c_2 \mathrm{e}^{-\frac{3}{\sqrt{2}}x}, \qquad (2.13)$$

where c_1 and c_2 are arbitrary constants.

The Euler-Lagrange equation for F_1 is determined as follows.

$$\frac{\partial F_1}{\partial z_1'} = \frac{2}{9}z_1', \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{9}z_1'\right) = \frac{2}{9}z_1'', \quad \frac{\partial F_1}{\partial z_1} = 0.$$

Therefore, the Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F_1}{\partial z_1'}\right) - \frac{\partial F_1}{\partial z_1} = 0.$$

Substituting into the above Euler-Lagrange equation gives,

$$\frac{2}{9}z_1'' = 0.$$

This differential equation has solution

$$z_1 = c_3x + c_4, \tag{2.14}$$

where c_3 and c_4 are arbitrary constants.

From part (b) $a = -4$, thus,

$$z_1 = y_1 - 4y_2 \quad \text{and so} \quad c_3x + c_4 = y_1 - 4y_2.$$

Rearranging the above in terms of y_1 gives,

$$y_1 = c_3x + c_4 + 4y_2.$$

$$z_2 = 2y_1 + y_2 \quad \text{and so} \quad c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} = 2y_1 + y_2.$$

Rearranging the above in terms of y_1 gives

$$y_1 = \frac{1}{2}c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + \frac{1}{2}c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} - \frac{1}{2}y_2.$$

Equating the expressions for y_1

$$c_3x + c_4 + 4y_2 = \frac{1}{2}c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + \frac{1}{2}c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} - \frac{1}{2}y_2,$$

and rearranging the above in terms of y_2

$$\frac{1}{2}c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + \frac{1}{2}c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} - c_3x - c_4 = \frac{1}{2}y_2 + 4y_2 = \frac{9}{2}y_2,$$
$$y_2 = \frac{2}{9}\left[\frac{1}{2}c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + \frac{1}{2}c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} - c_3x - c_4\right].$$

From above,

$$c_3x + c_4 - y_1 = -4y_2 \quad \text{and} \quad c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} - 2y_1 = y_2.$$

Multiplying the expression immediately above for y_2 by -4 gives,

$$-4c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} - 4c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} + 8y_1 = -4y_2.$$

Thus,

$$c_3x + c_4 - y_1 = -4c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} - 4c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} + 8y_1,$$

and rearranging the above in terms of y_1 gives,

$$c_3x + c_4 + 4c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + 4c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x} = 8y_1 + y_1 = 9y_1,$$

$$y_1 = \frac{1}{9}\left[c_3x + c_4 + 4c_1\mathrm{e}^{\frac{3}{\sqrt{2}}x} + 4c_2\mathrm{e}^{-\frac{3}{\sqrt{2}}x}\right].$$

(d) The Euler-Lagrange equation (2.1) was determined to be

$$2y_1'' - 4\left(2y_1 + y_2\right) = 0.$$

Let $\gamma = \dfrac{3}{\sqrt{2}}$ so that $\gamma^2 = \dfrac{9}{2}$, then,

$$\begin{aligned}
y_1 =& \frac{1}{9}\left[c_3x + c_4 + 4c_1\mathrm{e}^{\gamma x} + 4c_2\mathrm{e}^{-\gamma x}\right], \\
y_1' =& \frac{1}{9}\left[c_3 + 4c_1\gamma\mathrm{e}^{\gamma x} - 4c_2\gamma\mathrm{e}^{-\gamma x}\right], \quad \text{and} \\
y_1'' =& \frac{1}{9}\left[4c_1\gamma^2\mathrm{e}^{\gamma x} + 4c_2\gamma^2\mathrm{e}^{-\gamma x}\right].
\end{aligned}$$

$$y_2 = -\frac{2}{9}\left[c_3x + c_4 - \frac{1}{2}c_1\mathrm{e}^{\gamma x} - \frac{1}{2}c_2\mathrm{e}^{-\gamma x}\right].$$

So,

$$\begin{aligned}
2y_1'' - 4\left(2y_1 + y_2\right) =& \cancelto{1}{\frac{2}{9}}\left(4c_1\cancelto{1}{\frac{9}{2}}\mathrm{e}^{\gamma x} + 4c_2\cancelto{1}{\frac{9}{2}}\mathrm{e}^{-\gamma x}\right) \\
&- \frac{8}{9}\left(\cancelto{0}{c_3x} + \cancelto{0}{c_4} + 4c_1\mathrm{e}^{\gamma x} + 4c_2\mathrm{e}^{-\gamma x}\right) \\
&+ \frac{8}{9}\left(\cancelto{0}{c_3x} + \cancelto{0}{c_4} - \frac{1}{2}c_1\mathrm{e}^{\gamma x} - \frac{1}{2}c_2\mathrm{e}^{-\gamma x}\right), \\
=& \cancelto{0}{\left(4c_1\mathrm{e}^{\gamma x} - \frac{32}{9}c_1\mathrm{e}^{\gamma x} - \frac{4}{9}c_1\mathrm{e}^{\gamma x}\right)} \\
&+ \cancelto{0}{\left(4c_2\mathrm{e}^{-\gamma x} - \frac{32}{9}c_2\mathrm{e}^{-\gamma x} - \frac{4}{9}c_2\mathrm{e}^{-\gamma x}\right)}, \\
=& 0.
\end{aligned}$$

Consequently, (y_1, y_2) satisfies the first Euler-Lagrange equation.

Similarly for the second Euler-Lagrange equation equation.

Euler-Lagrange equation (2.2) was determined to be

$$4y_2'' - 2\left(2y_1 + y_2\right) = 0.$$

$$\begin{aligned}
y_2 =& -\frac{2}{9}\left(c_3x + c_4 - \frac{1}{2}c_1\mathrm{e}^{\gamma x} - \frac{1}{2}c_2\mathrm{e}^{-\gamma x}\right),\\
y_2' =& -\frac{2}{9}\left(c_3 - \frac{1}{2}c_1\gamma\mathrm{e}^{\gamma x} + \frac{1}{2}c_2\gamma\mathrm{e}^{-\gamma x}\right),\\
y_2'' =& -\frac{2}{9}\left(-\frac{1}{2}c_1\gamma^2\mathrm{e}^{\gamma x} - \frac{1}{2}c_2\gamma^2\mathrm{e}^{-\gamma x}\right) = \frac{2}{9}\left(\frac{1}{2}c_1\gamma^2\mathrm{e}^{\gamma x} + \frac{1}{2}c_2\gamma^2\mathrm{e}^{-\gamma x}\right), \quad \text{and}\\
y_1 =& \frac{1}{9}\left(c_3x + c_4 + 4c_1\mathrm{e}^{\gamma x} + 4c_2\mathrm{e}^{-\gamma x}\right).
\end{aligned}$$

So,

$$\begin{aligned}
4y_2'' - 2\left(2y_1 + y_2\right) =& \frac{\cancelto{2}{8}}{\cancel{9}}\left(\frac{\cancelto{1}{1}}{\cancel{2}}c_1\frac{\cancelto{1}{9}}{\cancel{2}}\mathrm{e}^{\gamma x} + \frac{\cancelto{1}{1}}{\cancel{2}}c_2\frac{\cancelto{1}{9}}{\cancel{2}}\mathrm{e}^{-\gamma x}\right)\\
& - \frac{4}{9}\left(\cancelto{0}{c_3x} + \cancelto{0}{c_4} + 4c_1\mathrm{e}^{\gamma x} + 4c_2\mathrm{e}^{-\gamma x}\right)\\
& + \frac{4}{9}\left(\cancelto{0}{c_3x} + \cancelto{0}{c_4} - \frac{1}{2}c_1\mathrm{e}^{\gamma x} - \frac{1}{2}c_2\mathrm{e}^{-\gamma x}\right),\\
=& \cancelto{0}{2c_1\mathrm{e}^{\gamma x} - \frac{16}{9}c_1\mathrm{e}^{\gamma x} - \frac{2}{9}c_1\mathrm{e}^{\gamma x}}\\
& \cancelto{0}{+ 2c_2\mathrm{e}^{-\gamma x} - \frac{16}{9}c_2\mathrm{e}^{-\gamma x} - \frac{2}{9}c_2\mathrm{e}^{-\gamma x}},\\
=& 0.
\end{aligned}$$

Consequently, (y_1, y_2) satisfies the second Euler-Lagrange equation, too.

Q 3.

(a) Let $0 < a < b$ and the given functional is

$$S[y] = \int_a^b \mathrm{d}x\, x^5 \left(y'^2 - \frac{2}{3} y^3 \right).$$

So the Euler-Lagrange equation can be determined as follows.

$$\text{Let,} \quad F = x^5 \left(y'^2 - \frac{2}{3} y^3 \right), \quad \text{then,}$$

$$\frac{\partial F}{\partial y'} = 2y'x^5, \quad \frac{\partial F}{\partial y} = -\frac{2}{3} \cdot 3y^2x^5 = -2y^2x^5.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2y'x^5\right) = 2y'5x^4 + x^5 2y'' = 10y'x^4 + 2x^5y''.$$

The Euler-Lagrange equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0,$$

therefore, we have

$$10y'x^4 + 2x^5y'' + 2y^2x^5 = 0,$$

which can be simplified to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{5}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0,$$

as required.

(b) In order to show that $S[y]$ is invariant under the scale transformation $\bar{x} = \alpha x$, $\bar{y} = \beta y$, where α and β are constants satisfying $\alpha^2\beta = 1$, consider the following.

$$\bar{x} = \alpha x, \quad \bar{y} = \beta y, \quad \alpha^2\beta = 1.$$

$$S\left[y\right] = \int_a^b \mathrm{d}x\, x^5 \left(y'^2 - \frac{2}{3} y^3 \right).$$

$$S\left[\bar{y}\right] = \int_{\bar{a}}^{\bar{b}} \mathrm{d}\bar{x}\, \bar{x}^5 \left(\left(\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}\right)^2 - \frac{2}{3} \bar{y}^3 \right),$$

$$S\left[\bar{y}\right] = \int_{\bar{a}}^{\bar{b}} \mathrm{d}\bar{x} \left(\bar{x}^5 \left(\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}\right)^2 - \frac{2}{3} \bar{x}^5 \bar{y}^3 \right).$$

Now, changing the variables, $\bar{x} = \alpha x$, $\bar{y} = \beta y$ and noting that

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \frac{\mathrm{d}\bar{y}}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}\bar{x}},$$

$$
\begin{aligned}
S[\bar{y}] &= \int_a^b \alpha\,\mathrm{d}x \left(\alpha^5 x^5 \left(\frac{\beta\,\mathrm{d}y}{\alpha\,\mathrm{d}x}\right)^2 - \frac{2}{3}\alpha^5 x^5 \beta^3 y^3\right), \\
S[\bar{y}] &= \int_a^b \alpha\,\mathrm{d}x \left(\alpha^5 \frac{\beta^2}{\alpha^2} x^5 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{2}{3}\alpha^5 \beta^3 x^5 y^3\right), \\
S[\bar{y}] &= \int_a^b \mathrm{d}x \left(\alpha^4 \beta^2 x^5 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{2}{3}\alpha^6 \beta^3 x^5 y^3\right), \\
S[\bar{y}] &= \alpha^4\beta^2 \int_a^b \mathrm{d}x \left(x^5 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{2}{3}\alpha^2 \beta x^5 y^3\right), \\
S[\bar{y}] &= \alpha^4\beta^2 \int_a^b \mathrm{d}x\, x^5 \left(y'^2 - \frac{2}{3}\alpha^2 \beta y^3\right).
\end{aligned}
$$

As $\alpha^2\beta = 1$, then the last expression for $S[\bar{y}]$ can be simplified to

$$
S[\bar{y}] = \int_a^b \mathrm{d}x\, x^5 \left(y'^2 - \frac{2}{3}y^3\right).
$$

Hence, $S[\bar{y}] = S[y]$ and thus it has been shown that $S[y]$ is invariant under the scale transformation outlined above.

(c) In order to deduce that a first-integral of $S[y]$ is

$$
4x^5 y y' + x^6 \left(y'^2 + \frac{2}{3}y^3\right) = c
$$

where c is a constant use will be made of Noether's theorem. See definition 7.1, p.165.

$$
\alpha^2\beta = 1, \quad \alpha = (1+\delta), \quad \text{then} \quad (1+\delta)^2\beta = 1.
$$

$$
\Phi = \alpha x = (1+\delta)x = x + \delta x.
$$

$$
\Psi = \beta y = \frac{1}{(1+\delta)^2} y = y - 2y\delta \quad \text{to first order.}
$$

HB p.2 Binomial expansion.

$$
\phi = \left.\frac{\partial \Phi}{\partial \delta}\right|_{\delta=0} = x, \qquad \psi = \left.\frac{\partial \Psi}{\partial \delta}\right|_{\delta=0} = -2y.
$$

$$
F = x^5\left(y'^2 - \frac{2}{3}y^3\right), \qquad \frac{\partial F}{\partial y'} = 2x^5 y'.
$$

Now making use of Noether's theorem HB p.21 c.f. equation (FI).

$$
\frac{\partial F}{\partial y'}\psi + \left(F - y'\frac{\partial F}{\partial y'}\right)\phi = \mathit{constant}.
$$

A first-integral of $S[y]$.

$$\begin{aligned}\frac{\partial F}{\partial y'}\psi + \left(F - y'\frac{\partial F}{\partial y'}\right)\phi =& 2x^5 y'\left(-2y\right) + \left(x^5\left(y'^2 - \frac{2}{3}y^3\right) - y' 2x^5 y'\right)x,\\ =& -4x^5 yy' + x^6\left(y'^2 - \frac{2}{3}y^3 - 2y'^2\right) = -c,\\ =& -4x^5 yy' + x^6\left(-y'^2 - \frac{2}{3}y^3\right) = -c,\\ =& 4x^5 yy' + x^6\left(y'^2 + \frac{2}{3}y^3\right) = c,\end{aligned}$$

as required.

(d)

$$y_1 = Ax^{-2} \quad \text{and} \quad y_2 = \frac{B}{\left(x^2 + \frac{B}{24}\right)^2}.$$

The first-integral is

$$4x^5 yy' + x^6\left(y'^2 + \frac{2}{3}y^3\right) = c.$$

$$y_1' = -2Ax^{-3}, \quad y_1'^2 = \left(-2Ax^{-3}\right)^2 = 4A^2x^{-6}, \quad y_1^3 = A^3x^{-6}.$$

A and B are constants.

Substituting these expressions into the first-integral gives,

$$\begin{aligned}4x^5 Ax^{-2}\left(-2Ax^{-3}\right) + x^6\left(4A^2x^{-6} + \frac{2}{3}A^3x^{-6}\right) &= c,\\ -8A^2 + 4A^2 + \frac{2}{3}A^3 &= c,\\ \frac{-24A^2 + 12A^2 + 2A^3}{3} &= c,\\ \underbrace{\frac{-12A^2 + 2A^3}{3}}_{\text{and this is a constant so equation is satisfied.}} &= c.\end{aligned}$$

Recall, c is a constant.

$$y_2 = \frac{B}{\left(x^2 + \frac{B}{24}\right)^2}, \quad y_2 = B\left(x^2 + \frac{B}{24}\right)^{-2}, \quad y_2' = -\frac{4Bx}{\left(x^2 + \frac{B}{24}\right)^3}.$$

Substituting these expressions into the first-integral gives,

$$
\begin{aligned}
c =& 4x^5 \frac{B}{\left(x^2+\frac{B}{24}\right)^2}\frac{(-4Bx)}{\left(x^2+\frac{B}{24}\right)^3} + x^6\left(\frac{16B^2x^2}{\left(x^2+\frac{B}{24}\right)^6} + \frac{2}{3}\frac{B^3}{\left(x^2+\frac{B}{24}\right)^6}\right),\\
c =& -\frac{16x^6B^2}{\left(x^2+\frac{B}{24}\right)^5} + x^6\left(\frac{48B^2x^2+2B^3}{3\left(x^2+\frac{B}{24}\right)^6}\right),\\
c =& -\frac{16x^6B^2}{\left(x^2+\frac{B}{24}\right)^5} + \frac{2x^6B^2}{\left(x^2+\frac{B}{24}\right)^5}\left(\frac{24x^2+B}{3\left(x^2+\frac{B}{24}\right)}\right),\\
c =& -\frac{16x^6B^2}{\left(x^2+\frac{B}{24}\right)^5} + \frac{2x^6B^2}{\left(x^2+\frac{B}{24}\right)^5}\left(\frac{\cancelto{1}{(24x^2+B)}}{\cancelto{\frac{1}{8}}{\frac{3}{24}}\cancelto{1}{(24x^2+B)}}\right),\\
c =& -\frac{16x^6B^2}{\left(x^2+\frac{B}{24}\right)^5} + \frac{2x^6B^2}{\left(x^2+\frac{B}{24}\right)^5}(8),\\
c =& -\frac{16x^6B^2}{\left(x^2+\frac{B}{24}\right)^5} + \frac{16x^6B^2}{\left(x^2+\frac{B}{24}\right)^5},\\
c =& 0.
\end{aligned}
$$

Thus, y_1 and y_2 both give solutions to the first-integral for any constant values A or B.

The Euler-Lagrange equation is given by,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0,$$

and

$$F = x^5\left(y'^2 - \frac{2}{3}y^3\right).$$

$$\frac{\partial F}{\partial y'} = 2x^5y', \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(2x^5y'\right) = 2x^5y'' + y'10x^4.$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^5y'^2 - \frac{2}{3}x^5y^3\right) = -2x^5y^2.$$

The Euler-Lagrange equation is

$$2x^5y'' + 10x^4y' + 2x^5y^2 = 0,$$

which, after dividing through out by $2x^5$ gives,

$$y'' + \frac{5}{x}y' + y^2 = 0, \quad \text{as determined previously in part a above.}$$

Now,

$$y = Ax^{-2}, \quad y' = -2Ax^{-3}, \quad y'' = 6Ax^{-4},$$

and so,

$$
\begin{aligned}
y'' + \frac{5}{x}y' + y^2 =& 6Ax^{-4} + \frac{5}{x}\left(-2Ax^{-3}\right) + A^2x^{-4}, \\
=& 6Ax^{-4} - 10Ax^{-4} + A^2x^{-4}. \\
=& -4Ax^{-4} + A^2x^{-4}, \\
=& \left(A-4\right)Ax^{-4} = 0 \quad \text{for all } A.
\end{aligned}
$$

Therefore, $A = 0$ or $A = 4$ satisfies the Euler-Lagrange equation, so the value of the constant A is not arbitrary.

$$
\begin{aligned}
y =& \frac{B}{\left(x^2 + \frac{B}{24}\right)^2}. \\
y' =& \frac{-2B2x}{\left(x^2 + \frac{B}{24}\right)^3} = \frac{-4Bx}{\left(x^2 + \frac{B}{24}\right)^3}. \\
y'' =& \frac{24Bx^2}{\left(x^2 + \frac{B}{24}\right)^4} - \frac{4B}{\left(x^2 + \frac{B}{24}\right)^3}.
\end{aligned}
$$

The Euler-Lagrange equation is

$$
y'' + \frac{5}{x}y' + y^2 = 0.
$$

Substituting into the Euler-Lagrange equation for the expressions immediately above gives,

$$
\begin{aligned}
&y'' + \frac{5}{2}y' + y^2 = \frac{24Bx^2}{\left(x^2 + \frac{B}{24}\right)^4} - \frac{4B}{\left(x^2 + \frac{B}{24}\right)^3} - \frac{5}{\cancelto{1}{x}}\frac{4B\cancelto{1}{x}}{\left(x^2 + \frac{B}{24}\right)^3} + \frac{B^2}{\left(x^2 + \frac{B}{24}\right)^4}, \\
&= \left(x^2 + \frac{B}{24}\right)^{-4}\left(24Bx^2 - 4B\left(x^2 + \frac{B}{24}\right) - 20B\left(x^2 + \frac{B}{24}\right) + B^2\right), \\
&= \left(x^2 + \frac{B}{24}\right)^{-4}\left(\cancel{24Bx^2} - \cancel{4Bx^2} - \frac{B^2}{6} - \cancel{20Bx^2} - \frac{\cancelto{5}{20}}{\cancelto{6}{24}}B^2 + B^2\right) \\
&= \left(x^2 + \frac{B}{24}\right)^{-4}\left(-\frac{B^2}{6} - \frac{5}{6}B^2 + \frac{6}{6}B^2\right), \\
&= \frac{1}{6}\left(x^2 + \frac{B}{24}\right)^{-4}\left(-6B^2 + 6B^2\right), \\
&\text{so,} \quad y'' + \frac{5}{2}y' + y^2 = 0.
\end{aligned}
$$

Thus, the Euler-Lagrange equation equation is satisfied regardless of the value of the constant B. Consequently, B is an arbitrary constant.

Q 4.

(a) The functional is

$$S[y] = \int_0^1 \mathrm{d}x\ (y')^n\,\mathrm{e}^y, \quad y(0) = 1, \quad y(1) = A > 1,$$

and the integrand $F = (y')^n\,\mathrm{e}^y$ is independent of the independent variable, x, so that (first-integral),

$$y'\frac{\mathrm{d}F}{\mathrm{d}y'} - F = constant, \quad F = (y')^n\,\mathrm{e}^y.$$

$$\frac{\partial F}{\partial y'} = n\mathrm{e}^y\,(y')^{n-1}, \quad y'\frac{\partial F}{\partial y'} = n\mathrm{e}^y\,(y')^n.$$

$$\therefore \quad n\mathrm{e}^y\,(y')^n - (y')^n\,\mathrm{e}^y = constant.$$

$$(n-1)\,\mathrm{e}^y\,(y')^n = constant.$$

$$(n-1)\,(y')^n = constant \cdot \mathrm{e}^{-y}.$$

Let $constant = (n-1)\cdot C_0^n$, where C_0 is an arbitrary constant. Then,

$$(n-1)\,(y')^n = (n-1)\,C_0^n \cdot \mathrm{e}^{-y},$$

and so,

$$y' = C_0 \cdot \mathrm{e}^{-\left(\dfrac{y}{n}\right)}.$$

This differential equation can be solved as follows.

First rewrite the equation as

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = C_0 \cdot \mathrm{e}^{-\left(\dfrac{y(x)}{n}\right)}.$$

Divide throughout by $\exp\left(-y(x)/n\right)$ to obtain

$$\mathrm{e}^{\left(\dfrac{y(x)}{n}\right)} \cdot \frac{\mathrm{d}y(x)}{\mathrm{d}x} = C_0.$$

Integrate the above with respect to x

$$\int \mathrm{d}x\ e^{\left(\dfrac{y(x)}{n}\right)} \cdot \frac{\mathrm{d}y(x)}{\mathrm{d}x} = \int \mathrm{d}x\,C_0.$$

Thus,

$$n \cdot e^{\left(\dfrac{y(x)}{n}\right)} = C_0 x + C_1, \quad \text{where } C_1 \text{ is an arbitrary constant.}$$

Solving the above for $y(x)$

$$\ln\left(n \cdot e^{\left(\frac{y(x)}{n}\right)}\right) = \ln\left(C_0 x + C_1\right),$$

$$\ln(n) + \ln\left(\mathrm{e}^{\left(\frac{y(x)}{n}\right)}\right) = \ln\left(C_0 x + C_1\right),$$

$$\frac{y(x)}{n} + \ln(n) = \ln\left(C_0 x + C_1\right),$$

$$y(x) = n \cdot \ln\left(C_0 x + C_1\right) - n \cdot \ln(n).$$

Therefore,

$$y(x) = n \cdot \ln\left(\frac{C_0 x + C_1}{n}\right).$$

Now, the boundary conditions $y(0) = 1$ and $y(1) = A > 1$ will allow the constants C_0 and C_1 to be found.

When $x = 0$, $y = 1$,

$$1 = n \cdot \ln\left(\frac{C_1}{n}\right),$$

$$\frac{1}{n} = \ln\left(\frac{C_1}{n}\right),$$

$$\mathrm{e}^{\left(\frac{1}{n}\right)} = \frac{C_1}{n},$$

$$C_1 = n \cdot \mathrm{e}^{\left(\frac{1}{n}\right)}, \quad \text{so}$$

$$y(x) = n \cdot \ln\left(\frac{C_0 x + n \cdot \mathrm{e}^{\left(\frac{1}{n}\right)}}{n}\right).$$

When $x = 1$, $y = A$,

$$A = n \cdot \ln\left(\frac{C_0 + n \cdot \mathrm{e}^{\left(\frac{1}{n}\right)}}{n}\right),$$

$$\frac{A}{n} = \ln\left(\frac{C_0 + n \cdot \mathrm{e}^{\left(\frac{1}{n}\right)}}{n}\right),$$

$$\mathrm{e}^{\left(\frac{A}{n}\right)} = \frac{C_0 + n \cdot \mathrm{e}^{\left(\frac{1}{n}\right)}}{n},$$

$$n \cdot \mathrm{e}^{\left(\frac{A}{n}\right)} = C_0 + n \cdot \mathrm{e}^{\left(\frac{1}{n}\right)}, \quad \text{so}$$

$$C_0 = n\left(e^{\left(\frac{A}{n}\right)} - e^{\left(\frac{1}{n}\right)}\right).$$

Let $C_0/n = c$, a constant, then

$$y(x) = n \cdot \ln\left(cx + \mathrm{e}^{\left(\frac{1}{n}\right)}\right),$$

and

$$c = \left(e^{\left(\frac{A}{n}\right)} - e^{\left(\frac{1}{n}\right)}\right).$$

as required.

(b) The Jacobi equation will be used to determine the nature of this stationary path as follows. HB p.22

Let $n > 1$ be a positive integer.

$$F(y, y') = (y')^n \mathrm{e}^y.$$

$$P(x) = \frac{\partial^2 F}{\partial y'^2} = \frac{\partial^2 \left((y')^n \mathrm{e}^y\right)}{\partial y'^2},$$

$$\frac{\partial F}{\partial y'} = n\,(y')^{n-1} \mathrm{e}^y.$$

$$P(x) = \frac{\partial^2 F}{\partial y'^2} = n\,(n-1)\,(y')^{n-2}\,\mathrm{e}^{y},$$

where $y = n\ln\left(cx + e^{\frac{1}{n}}\right)$. Therefore,

$$y' = \frac{nc}{\left(cx + e^{\frac{1}{n}}\right)}, \qquad \text{where} \quad c = \mathrm{e}^{\frac{A}{n}} - \mathrm{e}^{\frac{1}{n}}.$$

So,

$$\begin{aligned}
P(x) =& n(n-1)\left(\frac{nc}{cx+\mathrm{e}^{\frac{1}{n}}}\right)^{n-2} \mathrm{e}^{n\ln\left(cx+e^{\frac{1}{n}}\right)},\\
=& n(n-1)\left(\frac{nc}{cx+\mathrm{e}^{\frac{1}{n}}}\right)^{n-2} \left(cx+e^{\frac{1}{n}}\right)^{n},\\
=& n(n-1)\left(\frac{nc}{cx+\mathrm{e}^{\frac{1}{n}}}\right)^{n} \frac{\left(cx+e^{\frac{1}{n}}\right)^{2}\left(cx+e^{\frac{1}{n}}\right)^{n}}{(nc)^2},\\
=& n(n-1)\frac{(nc)^n\left(cx+e^{\frac{1}{n}}\right)^{2+n}}{(nc)^2\left(cx+e^{\frac{1}{n}}\right)^{n}},\\
=& n(n-1)(nc)^{n-2}\left(cx+e^{\frac{1}{n}}\right)^{2} > 0.
\end{aligned}$$

Note: $c = \mathrm{e}^{\frac{A}{n}} - \mathrm{e}^{\frac{1}{n}} > 0$ because $A > 1$, $n > 1$, thus, $n(n-1) > 0$ and $nc > 0$. The term $\left(cx + e^{\frac{1}{n}}\right)^2 > 0$ regardless of the terms within the brackets as it is squared.

Thus, if the stationary path $y(x)$ is an extremum then it will be a local minimum.

$$Q(x) = \frac{\partial^2 F}{\partial y^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial^2 F}{\partial y \partial y'}\right).$$

$$F = (y')^n\,\mathrm{e}^y, \qquad \frac{\partial F}{\partial y} = (y')^n\,\mathrm{e}^y, \qquad \frac{\partial^2 F}{\partial y^2} = (y')^n\,\mathrm{e}^y.$$

$$\frac{\partial F}{\partial y'} = n\,(y')^{n-1}\,\mathrm{e}^y, \qquad \frac{\partial^2 F}{\partial y \partial y'} = n\,(y')^{n-1}\,\mathrm{e}^y.$$

So,

$$\begin{aligned}
Q(x) &= (y')^n \mathrm{e}^y - \frac{\mathrm{d}}{\mathrm{d}x}\left(n\,(y')^{n-1}\,\mathrm{e}^y\right),\\
&= (y')^n \mathrm{e}^y - \left(n\,(y')^{n-1}\,y'\mathrm{e}^y + \mathrm{e}^y n(n-1)\,(y')^{n-2}\,y''\right),\\
&= (y')^n \mathrm{e}^y - \left(n\,(y')^{n}\,\mathrm{e}^y + \mathrm{e}^y n(n-1)\,(y')^{n-2}\,y''\right),\\
&=\mathrm{e}^y\left((y')^n - n\,(y')^n - n(n-1)\,(y')^{n-2}\,y''\right),\\
&=\mathrm{e}^y\left((1-n)\,(y')^n - n(n-1)\,(y')^{n-2}\,y''\right),\\
&=\mathrm{e}^y\left((y')^n\left[(1-n) - n(n-1)\,(y')^{-2}\,y''\right]\right),\\
&=\mathrm{e}^y\left((y')^n\left[(1-n) - \frac{n(n-1)}{(y')^2}y''\right]\right).
\end{aligned}$$

Now,

$$y' = \frac{nc}{cx+\mathrm{e}^{\frac{1}{n}}}, \qquad y'' = -\frac{nc^2}{\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^2}.$$

Substituting into the last equation for $Q(x)$ for y' and y'' gives,

$$\begin{aligned}
Q(x) =&\mathrm{e}^y\left(\frac{(nc)^n}{\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^n}\left[(1-n) - \frac{n(n-1)}{\frac{nc^2}{\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^2}}\left(-\frac{nc^2}{\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^2}\right)\right]\right),\\
=&\mathrm{e}^y\left(\frac{(nc)^n}{\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^n}\left[(1-n) - \frac{n(n-1)(-nc)^2}{(nc)^2}\right]\right),\\
=&\mathrm{e}^y\left(\frac{(nc)^n}{\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^n}\left[(1-n)+(n-1)\right]\right),\\
=&0.
\end{aligned}$$

The Jacobi equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) - Q(x)u = 0, \quad u(a)=0, \quad u'(a)=1. \qquad \text{HB p.22}$$

Thus,

$$P(x)\frac{\mathrm{d}u}{\mathrm{d}x} = k, \quad \text{where } k \text{ is a constant.}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = k\frac{1}{P(x)} = k\cdot\frac{\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^{-2}}{n(n-1)(nc)^{n-2}}, \quad n>1.$$

$$\text{Redefining } k, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = k\left(cx+\mathrm{e}^{\frac{1}{n}}\right)^{-2}.$$

$$u = k \int \mathrm{d}x \, \frac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)^2}.$$

$$\text{Let } s = cx + \mathrm{e}^{\frac{1}{n}}, \quad \text{then } \frac{\mathrm{d}s}{\mathrm{d}x} = c,$$

$$\text{and } \quad \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}x}} = \frac{1}{c}.$$

$$\text{Thus, } u = k \int \mathrm{d}s \, \frac{1}{s^2} \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{k}{c} \int \mathrm{d}s \, s^{-2}.$$

$$u = A - \frac{k}{c} s^{-1}, \qquad \text{where } A \text{ is a constant.}$$

Substituting back for s gives, $u = A - \dfrac{k}{c} \dfrac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)}$.

Applying the initial condition $u(0) = 0$ enables the constant A to be found,

$$A = \frac{k}{c \cdot \mathrm{e}^{\frac{1}{n}}}.$$

Hence,

$$u(x) = \frac{k}{c \cdot \mathrm{e}^{\frac{1}{n}}} - \frac{k}{c} \frac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)}.$$

Differentiating the above expression with respect to x gives,

$$u'(x) = (-1) \cdot \left(-\frac{k}{c}\right) \cdot \frac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)^2} \cdot c = k \cdot \frac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)^2}.$$

Applying the initial condition $u'(0) = 1$ gives,

$$1 = k \cdot \frac{1}{\left(\mathrm{e}^{\frac{1}{n}}\right)^2}, \quad \text{thus, } k = \mathrm{e}^{\frac{2}{n}}.$$

Hence,

$$\begin{aligned}
u(x) =& \frac{k}{c\mathrm{e}^{\frac{1}{n}}} - \frac{k}{c} \frac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)}, \quad \text{where, } k = \mathrm{e}^{\frac{2}{n}}, \\
u(x) =& \frac{\mathrm{e}^{\frac{2}{n}}}{c\mathrm{e}^{\frac{1}{n}}} - \frac{\mathrm{e}^{\frac{2}{n}}}{c} \frac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)}, \\
u(x) =& \frac{\mathrm{e}^{\frac{1}{n}}}{c} - \frac{\mathrm{e}^{\frac{2}{n}}}{c} \frac{1}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)},
\end{aligned}$$

$$
\begin{aligned}
u(x) =& \frac{\mathrm{e}^{\frac{1}{n}}}{c}\left[1 - \frac{\mathrm{e}^{\frac{1}{n}}}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)}\right], \\
u(x) =& \frac{\mathrm{e}^{\frac{1}{n}}}{c}\left[\frac{cx + \mathrm{e}^{\frac{1}{n}} - \mathrm{e}^{\frac{1}{n}}}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)}\right], \\
u(x) =& \frac{\mathrm{e}^{\frac{1}{n}}}{c}\left[\frac{cx}{\left(cx + \mathrm{e}^{\frac{1}{n}}\right)}\right], \\
u(x) =& \frac{\mathrm{e}^{\frac{1}{n}}x}{cx + \mathrm{e}^{\frac{1}{n}}}.
\end{aligned}
$$

Equating $u(x)$ to zero gives,

$$
\begin{aligned}
0 =& \frac{e^{\frac{1}{n}}x}{cx + \mathrm{e}^{\frac{1}{n}}}, \\
0 =& \mathrm{e}^{\frac{1}{n}}x,
\end{aligned}
$$

which can only be true when $x = 0$. Therefore, the closed interval $[0, 1]$ does not contain points conjugate to the point $x = 0$.

From Theorem 8.4:

A sufficient condition, Notes, p.185.

- The function $y(x)$ satisfies the Euler-Lagrange equation;
- Along the curve $y(x), P(x) = F_{y'y'} > 0$ for $0 \leq x \leq 1$; and
- The closed interval $[0, 1]$ contains no points conjugate to the point $x = 0$.

Hence, the functional has a weak minimum along $y(x)$.

Q 5.

(a)
$$F = f(x)\sqrt{1+y'^2} = f(x)\ \left(1+y'^2\right)^{\frac{1}{2}}.$$

The Euler-Lagrange equation is,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0, \quad \text{so,} \quad \frac{\partial F}{\partial y'} = \beta, \quad \text{where } \beta \text{ is a constant.}$$

$$\frac{\partial F}{\partial y'} = \frac{1}{2}\,f(x)\left(1+y'^2\right)^{-\frac{1}{2}}\ 2y' = \beta.$$

$$\begin{aligned}
f(x)\,\frac{y'}{(1+y'^2)^{\frac{1}{2}}} &= \beta, \\
f(x)^2\,\frac{y'^2}{(1+y'^2)} &= \beta^2, \\
f(x)^2\,y'^2 = \beta^2\left(1+y'^2\right) &= \beta^2 + \beta y'^2, \\
f(x)^2\,y'^2 - \beta^2 - \beta^2 y'^2 &= 0, \\
y'^2\ \left(f(x)^2 - \beta^2\right) &= \beta^2, \\
y'^2 = \frac{\beta^2}{(f(x)^2 - \beta^2)}&. \\
\text{Thus,}\quad y' = \frac{\beta}{\sqrt{f(x)^2 - \beta^2}}&.
\end{aligned}$$

Integrating the last expression gives,

$$y(x) = C + \beta\int_a^x \mathrm{d}w\,\frac{1}{\sqrt{f(w)^2 - \beta^2}}, \quad \text{where } C \text{ is a constant.}$$

Now, $y(a) = A$, and applying this to the above expression for $y(x)$ gives,

$$A = C + \beta\,\cancelto{0}{\int_a^a \mathrm{d}w\,\frac{1}{\sqrt{f(w)^2 - \beta^2}}}, \qquad \text{therefore, } C = A.$$

$$\text{Hence,} \qquad y(x) = A + \beta\int_a^x \mathrm{d}w\,\frac{1}{\sqrt{f(w)^2 - \beta^2}}, \qquad \text{as required.}$$

Also, $y(b) = B$ so,

$$B = A + \beta\int_a^b \mathrm{d}w\,\frac{1}{\sqrt{f(w)^2 - \beta^2}}.$$

Hence, the constant β satisfies,

$$B - A = \beta\int_a^b \mathrm{d}w\,\frac{1}{\sqrt{f(w)^2 - \beta^2}}.$$

(b) Jacobi's equation is given by HB p.22.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P\,\frac{\mathrm{d}u}{\mathrm{d}x}\right) - Qu = 0, \quad a \le x \le b,$$

where

$$P(x) = \frac{\partial^2 F}{\partial y'^2} = F_{y'y'},$$

and

$$Q(x) = \frac{\partial^2 F}{\partial y^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial^2 F}{\partial y \partial y'}\right) = F_{yy} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{yy'}\right).$$

From part (a)

$$F = f(x)\sqrt{1+y'^2} = f(x)\ \left(1+y'^2\right)^{\frac{1}{2}}.$$

$$F_y = 0, \qquad F_{yy} = 0, \quad F_{yy'} = 0 \quad \text{and hence,} \quad Q(x) = 0.$$

The express for $P(x)$ is determined as follows,

$$\begin{aligned}
F =& f(x)\ \left(1+y'^2\right)^{\frac{1}{2}}.\\
F_{y'} =& \frac{1}{2} f(x)\left(1+y'^2\right)^{-\frac{1}{2}}\ 2y',\\
F_{y'} =& f(x)\left(1+y'^2\right)^{-\frac{1}{2}}\ y'.
\end{aligned}$$

Using the product rule to take the partial derivative of the express for $F_{y'}$ with respect to y', and to be clear about its derivation,

$$\text{Let} \quad u = f(x)\left(1+y'^2\right)^{-\frac{1}{2}} \quad \text{and} \quad v = y'.$$

$$\frac{\partial F_{y'}}{\partial y'} = F_{y'y'} = u\,\frac{\mathrm{d}v}{\mathrm{d}y'} + u\,\frac{\mathrm{d}u}{\mathrm{d}y'}.$$

Then, using the chain rule to obtain the partial derivative of u with respect to y',

$$\frac{\mathrm{d}u}{\mathrm{d}y'} = -\frac{1}{2}\,f(x)\left(1+y'^2\right)^{-\frac{3}{2}} \cdot 2y' = -f(x)\left(1+y'^2\right)^{-\frac{3}{2}} \cdot y',$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}y'} = 1.$$

Thus,

$$
\begin{aligned}
F_{y'y'} =& f(x)\left(1+y'^2\right)^{-\frac{1}{2}} + y'\left[-f(x)\left(1+y'^2\right)^{-\frac{3}{2}}\cdot y'\right],\\
=& f(x)\left(1+y'^2\right)^{-\frac{1}{2}} - f(x)\left(1+y'^2\right)^{-\frac{3}{2}}\cdot y'^2,\\
=& \frac{f(x)}{(1+y'^2)^{\frac{1}{2}}}\left[1-\frac{y'^2}{(1+y'^2)}\right],\\
=& \frac{f(x)}{(1+y'^2)^{\frac{1}{2}}}\left[\frac{1}{(1+y'^2)}\right],\\
=& \frac{f(x)}{(1+y'^2)^{\frac{3}{2}}}.
\end{aligned}
$$

Hence, we have

$$P(x) = \frac{f(x)}{(1+y'^2)^{\frac{3}{2}}} > 0, \quad \text{as} \quad f(x) > 0.$$

To determine if conjugate points to $x = a$ exist or not consider the following.

$$
\begin{aligned}
Q(x) = 0 \quad &\text{so } \frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\,u'\right) = 0.\\
u' = \frac{k}{P(x)} \quad &\text{and } u(a) = 0, u'(a) = 1.\\
\text{Therefore, } k = P(a), \quad &\text{so } u' = \frac{P(a)}{P(x)} > 0.
\end{aligned}
$$

Consequently, u is strictly increasing, so $u(x) > u(a) = 0$ for all $x > a$. Thus, $u \neq 0$ on the interval $(a, b]$, which means there are no conjugate points.

This shows that the stationary path found in part (a) gives a weak local minimum (as $P(x) > 0$ for all $a \leq x \leq b$) of the functional $S[y]$.

(c) If $y(x)$ is the stationary path ($y'(x) \neq 0$ and $f(x) \neq 0$) the value of the functional along another admissible path $x + h$ is

$$S[y+h] = \int_a^b \mathrm{d}x\, f(x)\sqrt{1+(y'+h')^2}.$$

Using the result

$$\sqrt{1+(z+u)^2} - \sqrt{1+z^2} \geq \frac{zu}{\sqrt{1+z^2}},$$

for the case where $z = y'$ and $u = h'$, it is seen that

$$\sqrt{1+(y'+h')^2} - \sqrt{1+y'^2} \geq \frac{y'h'}{\sqrt{1+y'^2}}.$$

$$S[y+h] - S[y] = \int_a^b \mathrm{d}x \left\{ f(x)\sqrt{1+(y'+h')^2} - f(x)\sqrt{1+y'^2} \right\},$$

$$\begin{aligned} S[y+h] - S[y] &= \int_a^b \mathrm{d}x\, f(x) \left\{ \sqrt{1+(y'+h')^2} - \sqrt{1+y'^2} \right\}, \\ &\geq \int_a^b \mathrm{d}x\, f(x) \frac{y'}{\sqrt{1+y'^2}} h', \quad f(x) > 0. \end{aligned}$$

Inequality sign passes through integral here as $f(x) > 0$.

The Euler-Lagrange equation for the functional gave

$$f(x)\frac{y'}{\sqrt{1+y'^2}} = \beta,$$

where β is a constant. Hence,

$$S[y+h] - S[y] \geq \beta \int_a^b \mathrm{d}x\, h' = 0$$

as $h(a) = 0$ and $h(b) = 0$ since $y(a) = A$ and $y(b) = B$.

As $y'(x) \neq 0$ and $f(x) \neq 0$ the stationary path found in part (a) gives a global minimum of the functional $S[y]$, since $S[y] \leq S[y+h]$ for **all** h.