Q 1.

(a) From the parametric functional,

$$\Phi = \left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{\frac{1}{2}} - \lambda\left(x\dot{y} - \dot{x}y\right), \quad \lambda > 0,$$

$$\frac{\partial\Phi}{\partial\dot{x}} = \frac{1}{2}\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(2\dot{x} + 2\gamma\dot{y}\right) + \lambda y.$$

$$\frac{\partial\Phi}{\partial x} = -\lambda\dot{y}.$$

The Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Phi}{\partial\dot{x}}\right) - \frac{\partial\Phi}{\partial x} = 0.$$

Substituting into the Euler-Lagrange equation for the derivatives obtained above gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(2\dot{x} + 2\gamma\dot{y}\right) + \lambda y\right) + \lambda\dot{y} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(\dot{x} + \gamma\dot{y}\right) + \lambda y\right) + \lambda\dot{y} = 0,$$

and integrating the above express with respect to t gives,

$$\left[\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(\dot{x} + \gamma\dot{y}\right) + \lambda y\right] + \lambda y = 2c,$$

where c is an arbitrary constant. So,

$$\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(\dot{x} + \gamma\dot{y}\right) = 2\left(c - \lambda y\right). \tag{1.1}$$

Now from the definition of s

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{\frac{1}{2}},$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \left(1\Big/\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}. \tag{1.2}$$

Substituting (1.2) into (1.1) gives,

$$\frac{\mathrm{d}t}{\mathrm{d}s}\left(\dot{x} + \gamma\dot{y}\right) = 2\left(c - \lambda y\right),$$

and noting that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y},$$

then

$$\frac{\mathrm{d}t}{\mathrm{d}s}\left(\frac{\mathrm{d}x}{\mathrm{d}t}+\gamma\frac{\mathrm{d}y}{\mathrm{d}t}\right)=2\left(c-\lambda y\right)$$
$$\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\gamma\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=2\left(c-\lambda y\right),$$

thus, by the chain rule,

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}s}+\gamma\frac{\mathrm{d}y}{\mathrm{d}s}=2\left(c-\lambda y\right).} \tag{1.3}$$

Similarly for $\dot{y}$

$$\Phi=\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{\frac{1}{2}}-\lambda\left(x\dot{y}-\dot{x}y\right),\quad \lambda>0,$$
$$\frac{\partial\Phi}{\partial\dot{y}}=\frac{1}{2}\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}\left(2\gamma\dot{x}+2\dot{y}\right)-\lambda x.$$
$$\frac{\partial\Phi}{\partial y}=\lambda\dot{x}.$$

The Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Phi}{\partial\dot{y}}\right)-\frac{\partial\Phi}{\partial y}=0.$$

Substituting into the Euler-Lagrange equation for the derivatives obtained above gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}\left(2\gamma\dot{x}+2\dot{y}\right)-\lambda x\right]-\lambda\dot{x}=0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}\left(\gamma\dot{x}+\dot{y}\right)-\lambda x\right]-\lambda\dot{x}=0,$$

and integrating the above express with respect to t gives,

$$\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}\left(\gamma\dot{x}+\dot{y}\right)-\lambda x-\lambda x=2d,$$

where d is an arbitrary constant. So,

$$\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}\left(\gamma\dot{x}+\dot{y}\right)=2\left(d+\lambda x\right).$$

Recall, (1.2) from above,

$$\frac{\mathrm{d}t}{\mathrm{d}s}=\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}},$$

and hence,

$$\frac{\mathrm{d}t}{\mathrm{d}s}\left(\gamma\dot{x}+\dot{y}\right)=2\left(d+\lambda x\right),$$
$$\frac{\mathrm{d}t}{\mathrm{d}s}\left(\gamma\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}\right)=2\left(d+\lambda x\right),$$
$$\gamma\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=2\left(d+\lambda x\right),$$

Thus, by the chain rule,

$$\boxed{\gamma\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\mathrm{d}y}{\mathrm{d}s}=2\left(d+\lambda x\right).} \tag{1.4}$$

Hence, the stationary paths of the given parametric functional are given by the solutions to equations (1.3) and (1.4).

(b)

(c)

Q 2.

(a)

(b)

(c)

Q 3.

(a)

(b)

Q 4.

Q 5.

Q 6.

(a) The model of tumour growth under radiotherapy from time $t = 0$ to $t = t_1 > 0$ is given as

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -C \log\left(\frac{C}{C_{max}}\right) - \frac{DC}{1+D}, \tag{6.1}$$

where, at time t, $C(t)$ is the size of the tumour, $C_{max} > 0$ is the maximum size of the tumour, a constant, so that $0 < C(t) \leq C_{max}$, and $D(t) \geq 0$ is the rate at which the drug is administered.

Now, making a change of variable

$$x = \log\left(\frac{C}{C_{max}}\right),$$

then (6.1) becomes

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}C}{\mathrm{d}t} \cdot \frac{\mathrm{d}x}{\mathrm{d}C}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}C}{\mathrm{d}t} \cdot \frac{1}{C}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \left[\cancelto{-1}{-C} \log\left(\frac{C}{C_{max}}\right) - \frac{D\cancelto{1}{C}}{1+D}\right] \cdot \frac{1}{\cancelto{1}{C}}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\log\left(\frac{C}{C_{max}}\right) - \frac{D}{1+D}, \\
\text{thus,} \quad \frac{\mathrm{d}x}{\mathrm{d}t} &= -x - \frac{D}{1+D}, \qquad \text{where } -\infty < x \leq 0.
\end{aligned}$$

(b) We are told that:

> It is required to reduce the size of the tumour from C_0 at $t = 0$ to C_1 at $t = t_1$. Let $x_0 = \log(C_0/C_{max})$ and $x_1 = \log(C_1/C_{max})$. For the health of the patient, it is desired to minimise the total amount of drug administered, which is given by the functional
>
> $$S[D] = \int_0^{t_1} \mathrm{d}t\ D(t).$$

So, by expressing D in terms of x and $\dot{x} = \mathrm{d}x/\mathrm{d}t$ it should be possible to show that $S[D]$ may be written as

$$S[x] = -\int_0^{t_1} \mathrm{d}t\ \frac{\dot{x} + x}{1 + \dot{x} + x}.$$

From part (a)

$$\begin{aligned}
\dot{x} &= -x - \frac{D}{1+D}, \\
\dot{x} + x &= -\frac{D}{1+D}, \\
(\dot{x}+x)(1+D) &= -D, \\
(\dot{x}+x) + D(\dot{x}+x) &= -D, \\
D(\dot{x}+x) + D &= -(\dot{x}+x), \\
D(\dot{x}+x+1) &= -\dot{x} - x, \\
D &= \frac{-\dot{x}-x}{\dot{x}+x+1} \ .
\end{aligned}$$

So

$$S[x] = -\int_0^{t_1} \mathrm{d}t \ \frac{\dot{x}+x}{\dot{x}+x+1} \qquad \text{as required.}$$

(c) To show that the Euler-Lagrange equation for $S[x]$ is given by

$$2\ddot{x} + 3\dot{x} + x = -1, \qquad \text{where } \ddot{x} = \mathrm{d}^2 x/\,\mathrm{d}t^2,$$

consider the following.

The Euler-Lagrange equation is given by HB p17.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} = 0, \qquad y(a) = A, \quad y(b) = B.$$

$$\text{Let, } F = \frac{\dot{x}+x}{\dot{x}+x+1},$$

then using the quotient rule to determine $\partial F/\partial \dot{x}$ and $\partial F/\partial x$

$$\frac{\partial}{\partial \dot{x}}\left(\frac{u}{v}\right) = \left[v\frac{\partial u}{\partial \dot{x}} - u\frac{\partial v}{\partial \dot{x}}\right] \Big/ \ v^2 \quad \text{where, } u = \dot{x}+x \text{ and } v = \dot{x}+x+1.$$

$$\begin{aligned}
\frac{\partial F}{\partial \dot{x}} &= \frac{(\dot{x}+x+1)\cdot 1 - (\dot{x}+x)\cdot 1}{(\dot{x}+x+1)^2} = \frac{1}{(\dot{x}+x+1)^2}, \\
\frac{\partial F}{\partial x} &= \frac{(\dot{x}+x+1)\cdot 1 - (\dot{x}+x)\cdot 1}{(\dot{x}+x+1)^2} = \frac{1}{(\dot{x}+x+1)^2}.
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}+x+1\right)^{-2}, \\
&= -2\left(\dot{x}+x+1\right)^{-3}\left(\ddot{x}+\dot{x}\right), \\
&= \frac{-2\left(\ddot{x}+\dot{x}\right)}{\left(\dot{x}+x+1\right)^3}.
\end{aligned}$$

Therefore, the Euler-Lagrange equation is

$$\begin{aligned}
-\frac{2(\ddot{x}+\dot{x})}{(\dot{x}+x+1)^3}-\frac{1}{(\dot{x}+x+1)^2} &= 0,\\
\frac{2(\ddot{x}+\dot{x})}{(\dot{x}+x+1)^3} &= -\frac{1}{(\dot{x}+x+1)^2},\\
\frac{2(\ddot{x}+\dot{x})}{(\dot{x}+x+1)} &= -1,\\
2(\ddot{x}+\dot{x}) &= -(\dot{x}+x+1),\\
2(\ddot{x}+\dot{x})+(\dot{x}+x) &= -1,\\
2\ddot{x}+3\dot{x}+x &= -1, \quad \text{as required.} \qquad (6.2)
\end{aligned}$$

$\ddot{x} = \mathrm{d}^2x/\,\mathrm{d}t^2$.

(d) Now solving the second-order linear ordinary differential equation (6.2) with the boundary conditions $x(0) = x_0$ and $x(t_1) = x_1$ as follows.

The auxiliary equation is

$$2\lambda^2 + 3\lambda + 1 = 0.$$

This has roots $\lambda_1 = -1$ and $\lambda_2 = -\frac{1}{2}$. These roots are real and different so the complementary function is given by

$$y_c = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-\frac{1}{2}t}, \quad \text{where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

Using the method of undetermined coefficients to determine the particular solution to (6.2) as follows.

The particular solution to

$$2\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x(t)}{\mathrm{d}t} + x(t) = -1,$$

is of the form

$$x_p(t) = a_1,$$

where a_1 is an unknown constant to be determined as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(a_1) = 0,$$

and therefore

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = 0.$$

Substituting the particular solution $x_p(t)$ in to (6.2) gives

$$2\times 0 + 3\times 0 + a_1 = -1.$$

Therefore, $a_1 = -1$ and the general solution is

$$x(t) = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-\frac{1}{2}t} - 1. \qquad (6.3)$$

Now using the boundary conditions to solve for the unknown constants c_1 and c_2 as follows.

At $t = 0$, $x(0) = x_0$ and substituting this condition into (6.3) gives,

$$c_1 + c_2 - 1 = x_0. \tag{6.4}$$

At $t = t_1$, $x(t_1) = x_1$ and substituting this condition into (6.3) gives,

$$c_1\mathrm{e}^{-t_1} + c_2\mathrm{e}^{-\frac{1}{2}t_1} - 1 = x_1 \tag{6.5}$$

From (6.4)

$$c_1 = (x_0 + 1) - c_2 \tag{6.6}$$

Substituting into (6.5) for c_1 from (6.6) and rearranging in terms of c_2

$$\begin{aligned}
&(x_0+1)\,\mathrm{e}^{-t_1} - c_2\mathrm{e}^{-t_1} + c_2\mathrm{e}^{-\frac{1}{2}t_1} = x_1 + 1,\\
&(x_0+1)\,\mathrm{e}^{-t_1} + c_2\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right) = (x_1+1)\,,\\
&c_2\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right) = (x_1+1) - (x_0+1)\,\mathrm{e}^{-t_1},\\
&c_2 = \frac{(x_1+1) - (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}. \qquad (6.7)
\end{aligned}$$

Substituting for c_2 in (6.6) from (6.7)

$$\begin{aligned}
c_1 &= (x_0+1) - \left[\frac{(x_1+1) - (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}\right],\\
c_1 &= \frac{(x_0+1)\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right) - (x_1+1) + (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}\\
c_1 &= \frac{(x_0+1)\,\mathrm{e}^{-\frac{1}{2}t_1} - (x_0+1)\,\mathrm{e}^{-t_1} - (x_1+1) + (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}\\
c_1 &= \frac{(x_0+1)\,\mathrm{e}^{-\frac{1}{2}t_1} - (x_1+1)}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)} = \alpha. \qquad (6.8)
\end{aligned}$$

Therefore, from (6.4)

$$c_2 = (x_0 + 1) - \alpha. \tag{6.9}$$

Substituting into (6.3) for c_1 and c_2 from (6.8) and (6.9) respectively gives,

$$x(t) = \alpha\mathrm{e}^{-t} + (x_0 + 1 - \alpha)\,\mathrm{e}^{-\frac{1}{2}t} - 1 \quad (\alpha \text{ given in (6.8)}),$$

as required.