Q 1. Let γ be a real constant with $\gamma^2 \neq 1$ for the parametric functional

$$S\left[x,y\right] = \int_0^1 \mathrm{d}t\,\left[\sqrt{\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2} - \lambda\left(x\dot{y} - \dot{x}y\right)\right], \quad \lambda > 0,$$

with the boundary conditions $x(0) = y(0) = 0, x(1) = R > 0$ and $y(1) = 0$.

(a) In order to show that the stationary paths of this parametric functional are given by the solutions of the equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} + \gamma\frac{\mathrm{d}y}{\mathrm{d}s} = 2\left(c - \lambda y\right), \text{ and}$$

$$\gamma\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\mathrm{d}y}{\mathrm{d}s} = 2\left(d + \lambda x\right),$$

where c and d are constants and

$$s(t) = \int_0^t \mathrm{d}t\,\sqrt{\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2} \tag{1.1}$$

consider the following.

From the parametric functional,

$$\Phi = \left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{\frac{1}{2}} - \lambda\left(x\dot{y} - \dot{x}y\right), \quad \lambda > 0,$$

$$\frac{\partial\Phi}{\partial\dot{x}} = \frac{1}{2}\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(2\dot{x} + 2\gamma\dot{y}\right) + \lambda y.$$

$$\frac{\partial\Phi}{\partial x} = -\lambda\dot{y}.$$

The Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Phi}{\partial\dot{x}}\right) - \frac{\partial\Phi}{\partial x} = 0.$$

Substituting into the Euler-Lagrange equation for the derivatives obtained above gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\cancelto{1}{2}}\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(\cancelto{1}{2}\dot{x} + \cancelto{1}{2}\gamma\dot{y}\right) + \lambda y\right) + \lambda\dot{y} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(\dot{x} + \gamma\dot{y}\right) + \lambda y\right) + \lambda\dot{y} = 0,$$

and integrating the above express with respect to t gives,

$$\left[\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(\dot{x} + \gamma\dot{y}\right) + \lambda y\right] + \lambda y = 2c,$$

where c is an arbitrary constant.

$$\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}\left(\dot{x} + \gamma\dot{y}\right) + 2\lambda y = 2c,$$

So,

$$\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}(\dot{x} + \gamma\dot{y}) = 2(c - \lambda y). \tag{1.2}$$

Now, from (1.1) the definition of s,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{\frac{1}{2}},$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \left(1 \Big/ \frac{\mathrm{d}s}{\mathrm{d}t}\right) = \left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}. \tag{1.3}$$

Substituting (1.3) into (1.2) gives,

$$\frac{\mathrm{d}t}{\mathrm{d}s}(\dot{x} + \gamma\dot{y}) = 2(c - \lambda y),$$

and noting that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y},$$

then

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}s}\left(\frac{\mathrm{d}x}{\mathrm{d}t} + \gamma\frac{\mathrm{d}y}{\mathrm{d}t}\right) &= 2(c - \lambda y) \\
\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \gamma\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} &= 2(c - \lambda y),
\end{aligned}$$

thus, by the chain rule,

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}s} + \gamma\frac{\mathrm{d}y}{\mathrm{d}s} = 2(c - \lambda y).} \tag{1.4}$$

Similarly for $\dot{y}$

$$\begin{aligned}
\Phi &= \left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{\frac{1}{2}} - \lambda(x\dot{y} - \dot{x}y), \quad \lambda > 0, \\
\frac{\partial\Phi}{\partial\dot{y}} &= \frac{1}{2}\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}(2\gamma\dot{x} + 2\dot{y}) - \lambda x. \\
\frac{\partial\Phi}{\partial y} &= \lambda\dot{x}.
\end{aligned}$$

The Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Phi}{\partial\dot{y}}\right) - \frac{\partial\Phi}{\partial y} = 0.$$

Substituting into the Euler-Lagrange equation for the derivatives obtained above gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2\right)^{-\frac{1}{2}}(2\gamma\dot{x} + 2\dot{y}) - \lambda x\right] - \lambda\dot{x} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}(\gamma\dot{x}+\dot{y})-\lambda x\right]-\lambda\dot{x}=0,$$

and integrating the above express with respect to t gives,

$$\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}(\gamma\dot{x}+\dot{y})-\lambda x-\lambda x=2d,$$

where d is an arbitrary constant. So,

$$\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}(\gamma\dot{x}+\dot{y})-2\lambda x=2d,$$

and thus,

$$\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}}(\gamma\dot{x}+\dot{y})=2\left(d+\lambda x\right).$$

Recall, (1.3) from above,

$$\frac{\mathrm{d}t}{\mathrm{d}s}=\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{-\frac{1}{2}},$$

and hence,

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}s}(\gamma\dot{x}+\dot{y})&=2\left(d+\lambda x\right),\\ \frac{\mathrm{d}t}{\mathrm{d}s}\left(\gamma\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}\right)&=2\left(d+\lambda x\right),\\ \gamma\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}t}{\mathrm{d}s}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}&=2\left(d+\lambda x\right),\end{aligned}$$

Thus, by the chain rule,

$$\boxed{\gamma\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\mathrm{d}y}{\mathrm{d}s}=2\left(d+\lambda x\right).}\qquad(1.5)$$

Hence, the stationary paths of the given parametric functional are given by the solutions to equations (1.4) and (1.5).

(b) It can be shown that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2+2\gamma\frac{\mathrm{d}x}{\mathrm{d}s}\cdot\frac{\mathrm{d}y}{\mathrm{d}s}+\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2=1,$$

as follows.

From part (a)

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right)^{\frac{1}{2}},$$

and squaring both sides of the above expression gives,

$$\frac{\mathrm{d}s}{\mathrm{d}t}\cdot\frac{\mathrm{d}s}{\mathrm{d}t}=\left(\dot{x}^2+2\gamma\dot{x}\dot{y}+\dot{y}^2\right),\qquad(1.6)$$

Again, recall that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y},$$

so that (1.6) becomes

$$\frac{\mathrm{d}s}{\mathrm{d}t} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 2\gamma \frac{\mathrm{d}x}{\mathrm{d}t} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \right), \tag{1.7}$$

multiplying both sides of equation (1.7) by

$$\frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}t}{\mathrm{d}s}$$

gives,

$$\frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 2\gamma \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}, \tag{1.8}$$

then by the chain rule (1.8) becomes

$$1 = \left(\frac{\mathrm{d}x}{\mathrm{d}s} \right)^2 + 2\gamma \frac{\mathrm{d}x}{\mathrm{d}s} \cdot \frac{\mathrm{d}y}{\mathrm{d}s} + \left(\frac{\mathrm{d}y}{\mathrm{d}s} \right)^2, \tag{1.9}$$

as required.

(c) From equation (1.4)

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 2\left(c - \lambda y\right) - \gamma \frac{\mathrm{d}y}{\mathrm{d}s}$$

and from equation (1.5)

$$\frac{\mathrm{d}y}{\mathrm{d}s} = 2\left(d + \lambda y\right) - \gamma \frac{\mathrm{d}x}{\mathrm{d}s}$$

Substituting into equation (1.4) for $\mathrm{d}y/\,\mathrm{d}s$ gives,

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}s} + \gamma \left(2\left(d + \lambda x\right) - \gamma \frac{\mathrm{d}x}{\mathrm{d}s} \right) = 2\left(c - \lambda y\right), \\
&\frac{\mathrm{d}x}{\mathrm{d}s} + 2\gamma\left(d + \lambda x\right) - \gamma^2 \frac{\mathrm{d}x}{\mathrm{d}s} = 2\left(c - \lambda y\right), \\
&\frac{\mathrm{d}x}{\mathrm{d}s}\left(1 - \gamma^2\right) + 2\gamma\left(d + \lambda x\right) = 2\left(c - \lambda y\right), \\
&\frac{\mathrm{d}x}{\mathrm{d}s}\left(1 - \gamma^2\right) = 2\left(c - \lambda y\right) - 2\gamma\left(d + \lambda x\right), \\
&\frac{\mathrm{d}x}{\mathrm{d}s}\left(1 - \gamma^2\right) = 2c - 2\lambda y - 2\gamma d - 2\gamma\lambda x, \\
&\frac{\mathrm{d}x}{\mathrm{d}s}\left(1 - \gamma^2\right) = -2\lambda\left(y + \gamma x\right) + 2\left(c - \gamma d\right), \\
&x'\left(1 - \gamma^2\right) = -X + D = D - X.
\end{aligned}$$

Therefore,

$$\boxed{(1-\gamma^2)x' = D - X,} \tag{1.10}$$

Where, $D = 2\,(c-\gamma d)$, $X = 2\lambda(x-y)$ and $x' = \frac{\mathrm{d}y}{\mathrm{d}s}$.

as required.

Equation (1.5) is

$$\gamma\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\mathrm{d}y}{\mathrm{d}s} = 2\,(d+\lambda x)\,.$$

Substituting into equation (1.5) for $\mathrm{d}x/\,\mathrm{d}s$ from equation (1.4) gives,

$$\begin{aligned}
&\gamma\left(2\,(c-\lambda y) - \gamma\frac{\mathrm{d}y}{\mathrm{d}s}\right) + \frac{\mathrm{d}y}{\mathrm{d}s} = 2\,(d+\lambda x)\,,\\
&2\gamma\,(c-\lambda y) - \gamma^2\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\mathrm{d}y}{\mathrm{d}s} = 2\,(d+\lambda x)\,,\\
&\frac{\mathrm{d}y}{\mathrm{d}s}\left(1-\gamma^2\right) = 2\,(d+\lambda x) - 2\gamma\,(c-\lambda y)\,,\\
&\frac{\mathrm{d}y}{\mathrm{d}s}\left(1-\gamma^2\right) = 2d + 2\lambda x - 2\gamma c + 2\gamma\lambda y,\\
&\frac{\mathrm{d}y}{\mathrm{d}s}\left(1-\gamma^2\right) = 2\,(d-\gamma c) + 2\lambda\,(x+\gamma y)\,.
\end{aligned}$$

$$\therefore \qquad \boxed{\left(1-\gamma^2\right)y' = C + Y.} \tag{1.11}$$

Where, $C = 2\,(d-\gamma c)$, $Y = 2\lambda\,(x+\gamma y)$ and $y' = \frac{\mathrm{d}y}{\mathrm{d}s}$

as required.

Substituting for x' and y' from equations (1.10) and (1.11), respectively, into equation (1.9)

$$\frac{(D-X)^2}{(1-\gamma^2)^2} + 2\gamma\frac{(D-X)}{(1-\gamma^2)}\frac{(C+Y)}{(1-\gamma^2)} + \frac{(C+Y)^2}{(1-\gamma^2)^2} = 1$$

Expanding the above and cross multiplying by $(1-\gamma^2)^2$ gives,

$$\begin{aligned}
&D^2 - 2DX + X^2 + 2\gamma\,(D-X)\,(C+Y) + C^2 + 2CY + Y^2 = \left(1-\gamma^2\right)^2,\\
&D^2 - 2DX + X^2 + (2\gamma D - 2\gamma X)\,(C+Y) + C^2 + 2CY + Y^2 = \left(1-\gamma^2\right)^2,\\
&D^2 - 2DX + X^2 + 2\gamma CD + 2\gamma DY - 2\gamma CX - 2\gamma XY + C^2 + 2CY + Y^2 = \left(1-\gamma^2\right)^2,\\
&X^2 + Y^2 - 2\gamma XY + C^2 + D^2 + 2\gamma CD - 2DX + 2\gamma DY - 2\gamma CX + 2CY = \left(1-\gamma^2\right)^2,
\end{aligned}$$

Collecting terms in just X and just Y gives,

$$X^2 + Y^2 - 2\gamma XY + C^2 + D^2 + 2\gamma CD - 2X\,(D+\gamma C) + 2Y\,(C+\gamma D) = \left(1-\gamma^2\right)^2. \tag{1.12}$$

Now, from above (see margin notes)

$$
\begin{aligned}
(D+\gamma C) =& 2(c-\gamma d)+\gamma(2(d-\gamma c)),\\
=& 2c-\cancel{2\gamma d}+\cancel{2\gamma d}-2\gamma^2 c,\\
=& 2c\left(1-\gamma^2\right).
\end{aligned}
$$

$$
\begin{aligned}
(C+\gamma D) =& 2(d-\gamma c)+\gamma(2(c-\gamma d)),\\
=& 2d-\cancel{2\gamma c}+\cancel{2\gamma c}-2\gamma^2 d,\\
=& 2d\left(1-\gamma^2\right).
\end{aligned}
$$

So,

$$
D+\gamma C = 2c\left(1-\gamma^2\right) \quad \text{and}
$$
$$
C+\gamma D = 2d\left(1-\gamma^2\right).
$$

Substituting these expressions into equation (1.12) gives,

$$
X^2+Y^2-2\gamma XY+C^2+D^2+2\gamma CD-4c\left(1-\gamma^2\right)X+4d\left(1-\gamma^2\right)Y=\left(1-\gamma^2\right)^2,
$$

as required.

The boundary conditions are

$$
x(0)=y(0)=0, \quad x(1)=R>0, \quad y(1)=0.
$$

Now, applying these boundary conditions to equation(1.12) and from the margin notes above we have:

$$
X(t)=2\lambda\left(\gamma x(t)+y(t)\right).
$$
$$
Y(t)=2\lambda\left(x(t)+\gamma y(t)\right).
$$

At $t=0$,

$$
X(0)=2\lambda\left(\gamma x(0)+y(0)\right)=0.
$$
$$
Y(0)=2\lambda\left(x(0)+\gamma y(0)\right)=0.
$$

Thus, equation (1.12) at $t=0$ reduces to

$$
C^2+D^2+2\gamma CD=\left(1-\gamma^2\right)^2.
$$

At $t=1$,

$$
X(1)=2\lambda\left(\gamma x(1)+y(1)\right)=2\lambda\gamma R.
$$
$$
Y(1)=2\lambda\left(x(1)+\gamma y(1)\right)=2\lambda R.
$$

Thus, when applied to equation (1.12),

$$
\begin{aligned}
4\lambda^2\gamma^2R^2+4\lambda^2R^2-2\gamma(2\lambda\gamma R)(2\lambda R)-4c\left(1-\gamma^2\right)(2\lambda\gamma R)\\
+4d\left(1-\gamma^2\right)(2\lambda R)+C^2+D^2+2\gamma CD=\left(1-\gamma^2\right)^2.
\end{aligned}
$$

$$
\begin{aligned}
4\lambda^2\gamma^2R^2+4\lambda^2R^2-8\lambda^2\gamma^2R^2-8c\left(1-\gamma^2\right)\lambda\gamma R\\
+8d\left(1-\gamma^2\right)\lambda R+C^2+D^2+2\gamma CD=\left(1-\gamma^2\right)^2.
\end{aligned}
$$

Note from above that $C^2 + D^2 + 2\gamma CD = (1-\gamma^2)^2$, thus

$$4\lambda^2\gamma^2 R^2 + 4\lambda^2 R^2 - 8\lambda^2\gamma^2 R^2 - 8c\left(1-\gamma^2\right)\lambda\gamma R + 8d\left(1-\gamma^2\right)\lambda R = 0,$$

which, after dividing through by $4\lambda R$, gives

$$\lambda\gamma^2 R + \lambda R - 2\lambda\gamma^2 R - 2c\left(1-\gamma^2\right)\gamma + 2d\left(1-\gamma^2\right) = 0, \qquad \lambda, R \neq 0 \text{ and } \gamma^2 \neq 1$$

$$\lambda R - \lambda\gamma^2 R - 2c\left(1-\gamma^2\right)\gamma + 2d\left(1-\gamma^2\right) = 0,$$

$$\lambda R\left(1-\gamma^2\right) - 2c\left(1-\gamma^2\right)\gamma + 2d\left(1-\gamma^2\right) = 0,$$

and dividing through by $(1-\gamma^2)$, gives

$$\lambda R - 2c\gamma + 2d = 0.$$

Simplifying,

$$\lambda R - 2\left(c\gamma + d\right) = 0,$$

and noting that the quantity $2\left(c\gamma + d\right)$ is equal to C, then

$$\lambda R + C = 0.$$

Finally,

$$\boxed{C = -\lambda R, \quad R > 0,} \tag{1.13}$$

as required.

Recall that $C^2 + D^2 + 2\gamma CD = (1-\gamma^2)^2$ and $C = -\lambda R, \quad R > 0$.

Thus,

$$\lambda^2 R^2 + D^2 + 2\gamma\left(-\lambda R\right) D = \left(1-\gamma^2\right)^2,$$

$$D^2 - 2\lambda\gamma R D + \lambda^2 R^2 - \left(1-\gamma^2\right)^2 = 0.$$

Solving for D in the above quadratic,

$$D = \frac{2\lambda\gamma R \pm \sqrt{4\lambda^2\gamma^2 R^2 - 4\left(\lambda^2 R^2 - (1-\gamma^2)^2\right)}}{2},$$

$$D = \lambda\gamma R \pm \sqrt{\lambda^2\gamma^2 R^2 - \lambda^2 R^2 + (1-\gamma^2)^2},$$

$$D = \lambda\gamma R \pm \sqrt{\lambda^2 R^2(\gamma^2 - 1) + (1-\gamma^2)^2},$$

$$D = \lambda\gamma R \pm \sqrt{-\lambda^2 R^2(1-\gamma^2) + (1-\gamma^2)^2}$$

It is seen that D will always have two real roots if $\gamma^2 > 1$, as in the expression below

$$D = \lambda\gamma R \pm \sqrt{-\lambda^2 R^2(1-\gamma^2) + (1-\gamma^2)^2}$$

$(1-\gamma^2)^2 > 0$ and $(1-\gamma^2) < 0$ and consequently $-\lambda^2 R^2(1-\gamma^2) > 0$ making the entire quantity inside the radical positive.

Also, if $\gamma^2 < 1$ the $(1-\gamma^2) > 0$ with $-\lambda^2 R^2 (1-\gamma^2) < 0$ and the quantity $(1-\gamma^2)^2 - \lambda^2 R^2 (1-\gamma^2) > 0$ provided that $\lambda^2 R^2 (1-\gamma^2) < (1-\gamma^2)^2$, that is if $\lambda R < \sqrt{(1-\gamma^2)}$, there will be two real roots of D.

Conversely, if $(1-\gamma^2)^2 - \lambda^2 R^2 (1-\gamma^2) < 0$ then $\lambda^2 R^2 (1-\gamma^2) > (1-\gamma^2)^2$, that is $\lambda R > \sqrt{(1-\gamma^2)}$, there will not be any real solutions of D.

Q 2. In order to show that the stationary path of the functional

$$S[y] = \int_0^v \mathrm{d}x \, \left(y'^2 + y^2\right), y(0) = 1, y(v) = v, v > 0,$$

is given by

$$y = \cosh x + B \sinh x, \quad 0 \le x \le v,$$

where B and v are given by the solutions of the equations

$$v = \cosh v + B \sinh v \quad \text{and} \quad B^2 - 1 = 2\sinh v + 2B\cosh v,$$

consider the following:

(a) The integrand is

$$F = y'^2 + y^2,$$

and the Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0,$$

where

$$\frac{\partial F}{\partial y'} = 2y', \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(2y'\right) = 2y'' \quad \text{and} \quad \frac{\partial F}{\partial y} = 2y.$$

Thus, the Euler-Lagrange equation can be written as

$$2y'' - 2y = 0 \quad \text{or} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y = 0.$$

The auxiliary equation is $\lambda^2 - 1 = 0$ and has general solution

$$y = A\cosh(x) + B\sinh(x).$$

When $x = 0$, $\quad y(0) = 1$, so

$$1 = A\cosh(0) + B\sinh(0), \text{ therefore } A = 1.$$

When $x = v$, $\quad y(v) = v$, so

$$v = \cosh(v) + B\sinh(v), \text{ therefore } B = \frac{v - \cosh(v)}{\sinh(v)}.$$

Now, using the **Transversality Condition** in which the end of the curve can be expressed in the form $y = g(x)$ and also noting that $y(v) = v$, then: HB p.25

$$F + \left(g'(x) - y'(x)\right) F_{y'} = 0 \text{ at } x = v,$$

Now, $g(x) = x$ so $g'(x) = 1$ and from above $F = y'^2 + y^2$, hence

$$\left(y'^2 + y^2\right) + \left(1 - y'\right) 2y' = 0,$$

$$\begin{aligned} y'^2 + y^2 + 2y' - 2y'^2 &= 0, \\ -y'^2 + 2y' + y^2 &= 0 \text{ at } x = v. \end{aligned} \tag{2.1}$$

From above $y(x) = \cosh x + B \sinh x$ and therefore $y'(x) = \sinh x + B \cosh x$

Substituting these expressions into (2.1) at $x = v$ gives,

$$-(\sinh v + B\cosh v)^2 + 2(\sinh v + B\cosh v) + (\cosh v + B\sinh v)^2 = 0$$

which simplifies to

$$-\sinh^2 v - B^2\cosh^2 v + 2\sinh v + 2B\cosh v + \cosh^2 v + B^2\sinh^2 v = 0$$

Note: $\cosh^2 v - \sinh^2 v = 1$, so

$$1 - B^2 + 2\sinh v + 2B\cosh v = 0,$$

$$\boxed{\text{therefore, } B^2 - 1 = 2\sinh v + 2B\cosh v}$$

(b) Using the equations found in part (a) it can be shown that v is given by the real solution(s) of $f(v) = 0$, where

$$f(v) = v^2 - 2v(1 + \sinh v)\cosh v + 1 + 2\sinh v.$$

as follows:

Recall from part (a) that,

$$v = \cosh v + B\sinh v, \tag{2.2}$$

$$B^2 - 1 = 2\sinh v + 2B\cosh v, \tag{2.3}$$

From (2.2)

$$B = \frac{v - \cosh v}{\sinh v}. \tag{2.4}$$

Substituting (2.4) into (2.3) gives,

$$\left(\frac{v - \cosh v}{\sinh v}\right)^2 - 1 = 2\sinh v + 2\left(\frac{v - \cosh v}{\sinh v}\right)\cosh v,$$

$$\frac{v^2 - 2v\cosh v + \left(\cosh^2 v - 2\sinh^2 v\right)}{\sinh^2 v} = 2\sinh v + 2\left(\frac{v - \cosh v}{\sinh v}\right)\cosh v,$$

$$\frac{v^2 - 2v\cosh v + 1}{\sinh^2 v} = \frac{2\sinh^2 v + 2v\cosh v - 2\cosh^2 v}{\sinh v},$$

$$\frac{v^2 - 2v\cosh v + 1}{\sinh v} = -2\left(\cosh^2 v - \sinh^2 v\right) + 2v\cosh v,$$

$$\frac{v^2 - 2v\cosh v + 1}{\sinh v} = -2 + 2v\cosh v,$$

$$v^2 - 2v\cosh v + 1 = 2\sinh v\left(v\cosh v - 1\right),$$
$$v^2 = 2v\cosh v - 1 + 2\sinh v\left(v\cosh v - 1\right),$$
$$v^2 = 2v\sinh v\cosh v - 2\sinh v + 2v\cosh v - 1,$$
$$v^2 = 2v\cosh v\left(\sinh v + 1\right) - 2\sinh v - 1,$$

$$\boxed{\therefore f(v) = v^2 - 2v\left(1 + \sinh v\right)\cosh v + 1 + 2\sinh v = 0.} \tag{2.5}$$

(c) To deduce that there is only one stationary path, which is equivalent to proving that $f(v) = 0$ has exactly one solution for $v > 0$, consider the following.

First, it is necessary to show that $f(v) = 0$ has at least one solution in the interval $0 < v < \infty$. Now f is continuous on $[0, 1]$ with $f(0) > 0$ and $f(1) < 0$. Thus, by the **Intermediate Value Theorem**, it follows that $f(v) = 0$ for some value of v in the interval $(0, 1)$ and hence in $(0, \infty)$. Consequently, $f(v) = 0$ has at least one solution in the interval $0 < v < \infty$.

$f(0) = 1.0$
$f(1) = -2.363$
rounded to 3 figures

Next, it is necessary to show that $f(v) = 0$ has at most one solution in the interval $0 < v < \infty$. Assume that it has more that one solution in this interval and choose two solutions within this interval; say at $v = a$ and $v = b$ and suppose $a < b$. Then, since f is continuous on the interval $[a, b]$ and so differentiable on this interval, with $f(a) = 0$ and $f(b) = 0$ and it follows from **Rolle's Theorem** that $f'(d) = 0$ for some d in (a, b). However, $f'(v) < 0$ for all $v > 0$ (see below). Hence this shows that $f'(v)$ cannot be zero in the interval (a, b) and there exists a contradiction, thus showing that $f(x) = 0$ has at most one solution in the interval $0 < v < \infty$. This coupled with the fact that $f(v) = 0$ has at least one solution in the interval $0 < v < \infty$ leads to the conclusion that this equation has exactly one solution in the given interval.

$$\begin{aligned}
f(v) =& v^2 - 2v\left(1 + \sinh v\right)\cosh v + 1 + 2\sinh v,\\
f'(v) =& \frac{\mathrm{d}}{\mathrm{d}v}\left(v^2\right) - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v\left(1 + \sinh v\right)\cosh v\right) + \frac{\mathrm{d}}{\mathrm{d}v}\left(1\right) + 2\frac{\mathrm{d}}{\mathrm{d}v}\left(\sinh v\right),\\
=& 2v - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v\left(1 + \sinh v\right)\cosh v\right) + 2\cosh v,\\
=& 2v - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v\cosh v + v\sinh v\cosh v\right) + 2\cosh v,\\
=& 2v - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v\cosh v\right) - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v\sinh v\cosh v\right) + 2\cosh v,
\end{aligned}$$

$$
\begin{aligned}
&=2v - 2\left(v \sinh v + \cosh v\right) - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v \sinh v \cosh v\right) + 2\cosh v, \\
&=2v - 2v \sinh v - 2\cosh v - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v \sinh v \cosh v\right) + 2\cosh v,
\end{aligned}
$$

Simplifying

$$
\begin{aligned}
f'(v) =&2v\left(1 - \sinh v\right) - 2\frac{\mathrm{d}}{\mathrm{d}v}\left(v \sinh v \cosh v\right), \\
=&2v\left(1 - \sinh v\right) - 2\left(\sinh v \cosh v + v \cosh v \cosh v + v \sinh v \sinh v\right), \\
=&2v\left(1 - \sinh v\right) - 2\sinh v \cosh v - 2v\left(\cosh^2 v + \sinh^2 v\right), \\
=&2\left(v\left(1 - \sinh v\right) - \sinh v \cosh v - v\left(\cosh^2 v + \sinh^2 v\right)\right), \\
=&2\left(v - v \sinh v - \sinh v \cosh v - v \cosh^2 v - v \sinh^2 v\right)
\end{aligned}
$$

Inspection of the last expressions shows that $f'(v)$ is negative for all $v > 0$.

Q 3.

(a)

(b)

Q 4.

Q 5. Given that a, b, c are constant and that $a > 0$, $b > 0$, $c > 0$, it will be shown that the stationary path of the functional

$$S[y] = \int_0^a \mathrm{d}x\, \left(y'^2 + 2by\right), \quad y(a) = 0,$$

with a natural boundary condition at $x = 0$ and subject to the constraint

$$C[y] = \int_0^a \mathrm{d}x\, y = c,$$

is given by

$$y(x) = \frac{3c}{2a^3}\left(a^2 - x^2\right),$$

as follows. The Euler-Lagrange equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0.$$

Now,

$$\overline{F} = y'^2 + 2by - \lambda y, \quad \text{where } \lambda \text{ is a Lagrange multiplier.}$$

$$\frac{\partial \overline{F}}{\partial y} = 2b - \lambda \quad \text{and} \quad \frac{\partial \overline{F}}{\partial y'} = 2y'.$$

The Euler-Lagrange equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(2y') - 2b + \lambda = 0,$$
$$2y'' - 2b + \lambda = 0.$$

Hence, $y'' = b - \dfrac{\lambda}{2}$, $\quad y' = \left(b - \dfrac{\lambda}{2}\right)x + c_1$ and $y = \left(\dfrac{b}{2} - \dfrac{\lambda}{4}\right)x^2 + c_1 x + c_2$, where c_1 and c_2 are arbitrary constants. Applying the boundary condition $y(a) = 0$

$$y(a) = \left(\frac{b}{2} - \frac{\lambda}{4}\right)a^2 + c_1 a + c_2 = 0.$$

Q 6.

(a) The model of tumour growth under radiotherapy from time $t = 0$ to $t = t_1 > 0$ is given as

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -C \log\left(\frac{C}{C_{max}}\right) - \frac{DC}{1+D}, \tag{6.1}$$

where, at time t, $C(t)$ is the size of the tumour, $C_{max} > 0$ is the maximum size of the tumour, a constant, so that $0 < C(t) \leq C_{max}$, and $D(t) \geq 0$ is the rate at which the drug is administered.

Now, making a change of variable

$$x = \log\left(\frac{C}{C_{max}}\right),$$

then (6.1) becomes

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}C}{\mathrm{d}t} \cdot \frac{\mathrm{d}x}{\mathrm{d}C}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}C}{\mathrm{d}t} \cdot \frac{1}{C}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \left[\cancelto{-1}{-C} \log\left(\frac{C}{C_{max}}\right) - \frac{D\cancelto{1}{C}}{1+D}\right] \cdot \frac{1}{\cancelto{1}{C}}, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\log\left(\frac{C}{C_{max}}\right) - \frac{D}{1+D}, \\
\text{thus,} \quad \frac{\mathrm{d}x}{\mathrm{d}t} &= -x - \frac{D}{1+D}, \qquad \text{where } -\infty < x \leq 0.
\end{aligned}$$

(b) We are told that:

> It is required to reduce the size of the tumour from C_0 at $t = 0$ to C_1 at $t = t_1$. Let $x_0 = \log(C_0/C_{max})$ and $x_1 = \log(C_1/C_{max})$. For the health of the patient, it is desired to minimise the total amount of drug administered, which is given by the functional
>
> $$S[D] = \int_0^{t_1} \mathrm{d}t\ D(t).$$

So, by expressing D in terms of x and $\dot{x} = \mathrm{d}x/\mathrm{d}t$ it should be possible to show that $S[D]$ may be written as

$$S[x] = -\int_0^{t_1} \mathrm{d}t\ \frac{\dot{x} + x}{1 + \dot{x} + x}.$$

From part (a)

$$\begin{aligned}
\dot{x} &= -x - \frac{D}{1+D}, \\
\dot{x} + x &= -\frac{D}{1+D}, \\
(\dot{x}+x)(1+D) &= -D, \\
(\dot{x}+x) + D(\dot{x}+x) &= -D, \\
D(\dot{x}+x) + D &= -(\dot{x}+x), \\
D(\dot{x}+x+1) &= -\dot{x} - x, \\
D &= \frac{-\dot{x}-x}{\dot{x}+x+1} \;.
\end{aligned}$$

So

$$S[x] = -\int_0^{t_1} \mathrm{d}t\, \frac{\dot{x}+x}{\dot{x}+x+1} \qquad \text{as required.}$$

(c) To show that the Euler-Lagrange equation for $S[x]$ is given by

$$2\ddot{x} + 3\dot{x} + x = -1, \qquad \text{where } \ddot{x} = \mathrm{d}^2x/\,\mathrm{d}t^2,$$

consider the following.

The Euler-Lagrange equation is given by HB p17.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} = 0, \qquad y(a) = A, \quad y(b) = B.$$

$$\text{Let, } F = \frac{\dot{x}+x}{\dot{x}+x+1},$$

then using the quotient rule to determine $\partial F/\partial \dot{x}$ and $\partial F/\partial x$

$$\frac{\partial}{\partial \dot{x}}\left(\frac{u}{v}\right) = \left[v\frac{\partial u}{\partial \dot{x}} - u\frac{\partial v}{\partial \dot{x}}\right]\Big/\, v^2 \quad \text{where, } u = \dot{x}+x \text{ and } v = \dot{x}+x+1.$$

$$\begin{aligned}
\frac{\partial F}{\partial \dot{x}} &= \frac{(\dot{x}+x+1)\cdot 1 - (\dot{x}+x)\cdot 1}{(\dot{x}+x+1)^2} = \frac{1}{(\dot{x}+x+1)^2}, \\
\frac{\partial F}{\partial x} &= \frac{(\dot{x}+x+1)\cdot 1 - (\dot{x}+x)\cdot 1}{(\dot{x}+x+1)^2} = \frac{1}{(\dot{x}+x+1)^2}.
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}+x+1\right)^{-2}, \\
&= -2\left(\dot{x}+x+1\right)^{-3}\left(\ddot{x}+\dot{x}\right), \\
&= \frac{-2\left(\ddot{x}+\dot{x}\right)}{\left(\dot{x}+x+1\right)^3}.
\end{aligned}$$

Therefore, the Euler-Lagrange equation is

$$\begin{aligned}
-\frac{2(\ddot{x}+\dot{x})}{(\dot{x}+x+1)^3}-\frac{1}{(\dot{x}+x+1)^2} &= 0,\\
\frac{2(\ddot{x}+\dot{x})}{(\dot{x}+x+1)^3} &= -\frac{1}{(\dot{x}+x+1)^2},\\
\frac{2(\ddot{x}+\dot{x})}{(\dot{x}+x+1)} &= -1,\\
2(\ddot{x}+\dot{x}) &= -(\dot{x}+x+1),\\
2(\ddot{x}+\dot{x})+(\dot{x}+x) &= -1,\\
2\ddot{x}+3\dot{x}+x &= -1, \quad \text{as required.} \qquad (6.2)
\end{aligned}$$

$\ddot{x} = \mathrm{d}^2x/\,\mathrm{d}t^2$.

(d) Now solving the second-order linear ordinary differential equation (6.2) with the boundary conditions $x(0) = x_0$ and $x(t_1) = x_1$ as follows.

The auxiliary equation is

$$2\lambda^2 + 3\lambda + 1 = 0.$$

This has roots $\lambda_1 = -1$ and $\lambda_2 = -\frac{1}{2}$. These roots are real and different so the complementary function is given by

$$y_c = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-\frac{1}{2}t}, \quad \text{where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

Using the method of undetermined coefficients to determine the particular solution to (6.2) as follows.

The particular solution to

$$2\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x(t)}{\mathrm{d}t} + x(t) = -1,$$

is of the form

$$x_p(t) = a_1,$$

where a_1 is an unknown constant to be determined as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(a_1) = 0,$$

and therefore

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = 0.$$

Substituting the particular solution $x_p(t)$ in to (6.2) gives

$$2 \times 0 + 3 \times 0 + a_1 = -1.$$

Therefore, $a_1 = -1$ and the general solution is

$$x(t) = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-\frac{1}{2}t} - 1. \qquad (6.3)$$

Now using the boundary conditions to solve for the unknown constants c_1 and c_2 as follows.

At $t = 0$, $x(0) = x_0$ and substituting this condition into (6.3) gives,

$$c_1 + c_2 - 1 = x_0. \tag{6.4}$$

At $t = t_1$, $x(t_1) = x_1$ and substituting this condition into (6.3) gives,

$$c_1 \mathrm{e}^{-t_1} + c_2 \mathrm{e}^{-\frac{1}{2}t_1} - 1 = x_1 \tag{6.5}$$

From (6.4)

$$c_1 = (x_0 + 1) - c_2 \tag{6.6}$$

Substituting into (6.5) for c_1 from (6.6) and rearranging in terms of c_2

$$\begin{aligned}
&(x_0+1)\,\mathrm{e}^{-t_1} - c_2\mathrm{e}^{-t_1} + c_2\mathrm{e}^{-\frac{1}{2}t_1} = x_1 + 1,\\
&(x_0+1)\,\mathrm{e}^{-t_1} + c_2\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right) = (x_1+1)\,,\\
&c_2\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right) = (x_1+1) - (x_0+1)\,\mathrm{e}^{-t_1},\\
&c_2 = \frac{(x_1+1) - (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}. \qquad (6.7)
\end{aligned}$$

Substituting for c_2 in (6.6) from (6.7)

$$\begin{aligned}
c_1 &= (x_0+1) - \left[\frac{(x_1+1) - (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}\right],\\
c_1 &= \frac{(x_0+1)\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right) - (x_1+1) + (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}\\
c_1 &= \frac{(x_0+1)\,\mathrm{e}^{-\frac{1}{2}t_1} - (x_0+1)\,\mathrm{e}^{-t_1} - (x_1+1) + (x_0+1)\,\mathrm{e}^{-t_1}}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)}\\
c_1 &= \frac{(x_0+1)\,\mathrm{e}^{-\frac{1}{2}t_1} - (x_1+1)}{\left(\mathrm{e}^{-\frac{1}{2}t_1} - \mathrm{e}^{-t_1}\right)} = \alpha. \qquad (6.8)
\end{aligned}$$

Therefore, from (6.4)

$$c_2 = (x_0 + 1) - \alpha. \tag{6.9}$$

Substituting into (6.3) for c_1 and c_2 from (6.8) and (6.9) respectively gives,

$$x(t) = \alpha\mathrm{e}^{-t} + (x_0 + 1 - \alpha)\,\mathrm{e}^{-\frac{1}{2}t} - 1 \quad (\alpha \text{ given in (6.8)}),$$

as required.