Faculty of Science, Technology, Engineering and Mathematics
M820 Calculus of variations and advanced calculus

M820

TMA 03

Covers Chapters 9–12

TMA 03 is a formative assignment that does not count towards your final grade. However, in order to pass the module, you are required to submit at least three TMAs and score at least 30% on at least three of the TMAs submitted.

The substitution rule does not apply to this module.

To be sure of passing this module, you need to achieve a score of at least 40% in the examination and score at least 30% on three out of four TMAs. The final rank score will be completely determined by your overall exam score (OES).

You are strongly encouraged to submit your assignment online using the electronic TMA service. Please read the instructions under the 'Assessment' tab of the module website before starting your assignment.

The assignment cut-off date can be found on the module website.

There are 100 marks available for this assignment.

3.1

WEB 06827 3

The questions are of varying difficulty and length: the marks allocated to a question provide some indication of its difficulty. Questions or parts of questions marked with * are more challenging.

Question 1 – 24 marks

Let γ be a real constant with $\gamma^2 \neq 1$ for the parametric functional

$$S[x, y] = \int_0^1 dt \left[\sqrt{\dot{x}^2 + 2\gamma\,\dot{x}\dot{y} + \dot{y}^2} - \lambda(x\dot{y} - \dot{x}y)\right], \quad \lambda > 0,$$

with the boundary conditions $x(0) = y(0) = 0$, $x(1) = R > 0$ and $y(1) = 0$.

(a) Show that the stationary paths of this parametric functional are given by the solutions of the equations

$$\frac{dx}{ds} + \gamma\,\frac{dy}{ds} = 2(c - \lambda y) \quad \text{and} \quad \gamma\,\frac{dx}{ds} + \frac{dy}{ds} = 2(d + \lambda x),$$

where c and d are constants and

$$s(t) = \int_0^t dt\,\sqrt{\dot{x}^2 + 2\gamma\,\dot{x}\dot{y} + \dot{y}^2}.$$ [*12*]

(b) Show that

$$\left(\frac{dx}{ds}\right)^2 + 2\gamma\,\frac{dx}{ds}\,\frac{dy}{ds} + \left(\frac{dy}{ds}\right)^2 = 1.$$ [*2*]

(c*) Show that the equations for x and y can be written in the form

$$(1 - \gamma^2)x' = D - X \quad \text{where } D = 2(c - \gamma d),\ X = 2\lambda(\gamma x + y),$$
$$(1 - \gamma^2)y' = C + Y \quad \text{where } C = 2(d - \gamma c),\ Y = 2\lambda(x + \gamma y),$$

where

$$x' = \frac{dx}{ds}, \quad y' = \frac{dy}{ds}.$$

Hence show that X and Y satisfy

$$X^2 + Y^2 - 2\gamma XY - 4c(1 - \gamma^2)X + 4d(1 - \gamma^2)Y$$
$$+ C^2 + D^2 + 2\gamma CD = (1 - \gamma^2)^2.$$

Then use the boundary conditions to show that

$$C^2 + 2\gamma CD + D^2 = (1 - \gamma^2)^2 \quad \text{and} \quad C = -\lambda R.$$

Deduce that if $\gamma^2 > 1$, then there are always two real solutions of these equations, and if $\gamma^2 < 1$, then there are two real solutions if $\lambda R < \sqrt{1 - \gamma^2}$ and none if $\lambda R > \sqrt{1 - \gamma^2}$. [*10*]

Question 2 – 22 marks

(a) Show that the stationary path of the functional

$$S[y] = \int_0^v dx\,(y'^2 + y^2), \quad y(0) = 1,\ y(v) = v,\ v > 0,$$

is given by

$$y = \cosh x + B \sinh x, \quad 0 \le x \le v,$$

where B and v are given by the solutions of the equations

$$v = \cosh v + B \sinh v \quad \text{and} \quad B^2 - 1 = 2 \sinh v + 2B \cosh v.$$ [*10*]

(b) Use these equations to show that v is given by the real solution(s) of $f(v) = 0$, where

$$f(v) = v^2 - 2v(1 + \sinh v) \cosh v + 1 + 2 \sinh v.$$ [*4*]

(c*) Deduce that there is only one stationary path.

Note that a graphical argument alone is insufficient for full marks here; you should give rigorous analysis to back up your explanation. [*8*]

Question 3 – 8 marks

Heron's formula for the area A of a triangle with sides of length $a, b, c > 0$ is

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{a+b+c}{2}.$$

Use the method of Lagrange multipliers to do the following.

(a) Show that for a given fixed perimeter $P = 2s$ of the triangle, the area is maximised when the triangle is equilateral. [*4*]

(b) Find, in terms of the fixed perimeter $P = 2s$, the maximum area of a right-angled triangle with perimeter P. [*4*]

Question 4 – 15 marks

Use the method of Lagrange multipliers to find the function $y(x)$ that makes the functional

$$S[y] = \int_1^2 dx\, x^2 y^2$$

stationary subject to the two constraints

$$\int_1^2 dx\, xy = 1 \quad \text{and} \quad \int_1^2 dx\, x^2 y = 2.$$

Calculate the stationary value of $S[y]$. *[15]*

Question 5 – 14 marks

Let $a > 0$, $b > 0$, $c > 0$ be constants. Show that the stationary path of the functional

$$S[y] = \int_0^a dx\, (y'^2 + 2by), \quad y(a) = 0,$$

with a natural boundary condition at $x = 0$ and subject to the constraint

$$C[y] = \int_0^a dx\, y = c,$$

is given by

$$y(x) = \frac{3c}{2a^3}(a^2 - x^2).$$ *[14]*

Question 6 – 17 marks

A model of tumour growth under chemotherapy from time $t = 0$ to $t = t_1 > 0$ is

$$\frac{dC}{dt} = -C \log\left(\frac{C}{C_{max}}\right) - \frac{DC}{1+D} \qquad (*)$$

where, at time t, $C(t)$ is the size of the tumour, $C_{max} > 0$ is the maximum size of the tumour, a constant, so that $0 < C(t) \le C_{max}$, and $D(t) \ge 0$ is the rate at which the drug is administered.

(a) Show that by making the change of variable, $x = \log(C/C_{max})$, equation $(*)$ becomes

$$\frac{dx}{dt} = -x - \frac{D}{1+D},$$

where $-\infty < x \le 0$. [2]

It is required to reduce the size of the tumour from C_0 at $t = 0$ to C_1 at $t = t_1$. Let $x_0 = \log(C_0/C_{max})$ and $x_1 = \log(C_1/C_{max})$. For the health of the patient, it is desired to minimise the total amount of drug administered, which is given by the functional

$$S[D] = \int_0^{t_1} dt\, D(t).$$

(b) By expressing D in terms of x and $\dot{x} = dx/dt$, show that $S[D]$ may be written

$$S[x] = -\int_0^{t_1} dt\, \frac{\dot{x} + x}{1 + \dot{x} + x}.$$ [4]

(c) Show that the Euler-Lagrange equation for $S[x]$ is given by

$$2\ddot{x} + 3\dot{x} + x = -1 \qquad (**)$$

where $\ddot{x} = d^2x/dt^2$. [4]

(d) Solve $(**)$ with the boundary conditions $x(0) = x_0$ and $x(t_1) = x_1$, to show that the stationary path of $S[x]$ is given by

$$x(t) = \alpha e^{-t} + (x_0 + 1 - \alpha)e^{-t/2} - 1,$$

where

$$\alpha = \frac{(x_0 + 1)e^{-t_1/2} - (x_1 + 1)}{e^{-t_1/2} - e^{-t_1}}.$$ [7]