

Q 1. Let  $\gamma$  be a real constant with  $\gamma^2 \neq 1$  for the parametric functional

$$S[x, y] = \int_0^1 dt \left[ \sqrt{\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2} - \lambda(\dot{x}y - x\dot{y}) \right], \quad \lambda > 0,$$

with the boundary conditions  $x(0) = y(0) = 0, x(1) = R > 0$  and  $y(1) = 0$ .

(a) In order to show that the stationary paths of this parametric functional are given by the solutions of the equations

$$\frac{dx}{ds} + \gamma \frac{dy}{ds} = 2(c - \lambda y), \text{ and}$$

$$\gamma \frac{dx}{ds} + \frac{dy}{ds} = 2(d + \lambda x),$$

where  $c$  and  $d$  are constants and

$$s(t) = \int_0^t dt \sqrt{\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2} \quad (1.1)$$

consider the following.

From the parametric functional,

$$\Phi = (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{\frac{1}{2}} - \lambda(\dot{x}y - x\dot{y}), \quad \lambda > 0,$$

$$\frac{\partial \Phi}{\partial \dot{x}} = \frac{1}{2} (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (2\dot{x} + 2\gamma\dot{y}) + \lambda y.$$

$$\frac{\partial \Phi}{\partial x} = -\lambda \dot{y}.$$

The Euler-Lagrange equation is given by

$$\frac{d}{dt} \left( \frac{\partial \Phi}{\partial \dot{x}} \right) - \frac{\partial \Phi}{\partial x} = 0.$$

Substituting into the Euler-Lagrange equation for the derivatives obtained above gives,

$$\frac{d}{dt} \left( \frac{1}{2} (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (2\dot{x} + 2\gamma\dot{y}) + \lambda y \right) + \lambda \dot{y} = 0,$$

$$\frac{d}{dt} \left( (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\dot{x} + \gamma\dot{y}) + \lambda y \right) + \lambda \dot{y} = 0,$$

and integrating the above express with respect to  $t$  gives,

$$\left[ (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\dot{x} + \gamma\dot{y}) + \lambda y \right] + \lambda y = 2c,$$

where  $c$  is an arbitrary constant.

$$(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\dot{x} + \gamma\dot{y}) + 2\lambda y = 2c,$$

So,

$$(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\dot{x} + \gamma\dot{y}) = 2(c - \lambda y). \quad (1.2)$$

Now, from (1.1) the definition of  $s$ ,

$$\frac{ds}{dt} = (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{\frac{1}{2}},$$

and

$$\frac{dt}{ds} = \left(1 / \frac{ds}{dt}\right) = (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}}. \quad (1.3)$$

Substituting (1.3) into (1.2) gives,

$$\frac{dt}{ds} (\dot{x} + \gamma\dot{y}) = 2(c - \lambda y),$$

and noting that

$$\frac{dx}{dt} = \dot{x} \quad \text{and} \quad \frac{dy}{dt} = \dot{y},$$

then

$$\begin{aligned} \frac{dt}{ds} \left( \frac{dx}{dt} + \gamma \frac{dy}{dt} \right) &= 2(c - \lambda y) \\ \frac{dt}{ds} \cdot \frac{dx}{dt} + \gamma \frac{dt}{ds} \cdot \frac{dy}{dt} &= 2(c - \lambda y), \end{aligned}$$

thus, by the chain rule,

$$\boxed{\frac{dx}{ds} + \gamma \frac{dy}{ds} = 2(c - \lambda y)}. \quad (1.4)$$

Similarly for  $\dot{y}$

$$\Phi = (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{\frac{1}{2}} - \lambda(x\dot{y} - \dot{x}y), \quad \lambda > 0,$$

$$\frac{\partial \Phi}{\partial \dot{y}} = \frac{1}{2} (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (2\gamma\dot{x} + 2\dot{y}) - \lambda x.$$

$$\frac{\partial \Phi}{\partial y} = \lambda \dot{x}.$$

The Euler-Lagrange equation is given by

$$\frac{d}{dt} \left( \frac{\partial \Phi}{\partial \dot{y}} \right) - \frac{\partial \Phi}{\partial y} = 0.$$

Substituting into the Euler-Lagrange equation for the derivatives obtained above gives,

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (2\gamma\dot{x} + 2\dot{y}) - \lambda x \right] - \lambda \dot{x} = 0,$$

$$\frac{d}{dt} \left[ (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\gamma\dot{x} + \dot{y}) - \lambda x \right] - \lambda\dot{x} = 0,$$

and integrating the above express with respect to  $t$  gives,

$$(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\gamma\dot{x} + \dot{y}) - \lambda x - \lambda x = 2d,$$

where  $d$  is an arbitrary constant. So,

$$(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\gamma\dot{x} + \dot{y}) - 2\lambda x = 2d,$$

and thus,

$$(\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}} (\gamma\dot{x} + \dot{y}) = 2(d + \lambda x).$$

Recall, (1.3) from above,

$$\frac{dt}{ds} = (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{-\frac{1}{2}},$$

and hence,

$$\begin{aligned} \frac{dt}{ds} (\gamma\dot{x} + \dot{y}) &= 2(d + \lambda x), \\ \frac{dt}{ds} \left( \gamma \frac{dx}{dt} + \frac{dy}{dt} \right) &= 2(d + \lambda x), \\ \gamma \frac{dt}{ds} \cdot \frac{dx}{dt} + \frac{dt}{ds} \cdot \frac{dy}{dt} &= 2(d + \lambda x), \end{aligned}$$

Thus, by the chain rule,

$$\boxed{\gamma \frac{dx}{ds} + \frac{dy}{ds} = 2(d + \lambda x).} \quad (1.5)$$

Hence, the stationary paths of the given parametric functional are given by the solutions to equations (1.4) and (1.5).

(b) It can be shown that

$$\left( \frac{dx}{ds} \right)^2 + 2\gamma \frac{dx}{ds} \cdot \frac{dy}{ds} + \left( \frac{dy}{ds} \right)^2 = 1,$$

as follows.

From part (a)

$$\frac{ds}{dt} = (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2)^{\frac{1}{2}},$$

and squaring both sides of the above expression gives,

$$\frac{ds}{dt} \cdot \frac{ds}{dt} = (\dot{x}^2 + 2\gamma\dot{x}\dot{y} + \dot{y}^2), \quad (1.6)$$

Again, recall that

$$\frac{dx}{dt} = \dot{x} \quad \text{and} \quad \frac{dy}{dt} = \dot{y},$$

so that (1.6) becomes

$$\frac{ds}{dt} \cdot \frac{ds}{dt} = \left( \frac{dx}{dt} \cdot \frac{dx}{dt} + 2\gamma \frac{dx}{dt} \cdot \frac{dy}{dt} + \frac{dy}{dt} \cdot \frac{dy}{dt} \right), \quad (1.7)$$

multiplying both sides of equation (1.7) by

$$\frac{dt}{ds} \cdot \frac{dt}{ds}$$

gives,

$$\frac{dt}{ds} \cdot \frac{dt}{ds} \cdot \frac{ds}{dt} \cdot \frac{ds}{dt} = \frac{dt}{ds} \cdot \frac{dt}{ds} \cdot \frac{dx}{dt} \cdot \frac{dx}{dt} + 2\gamma \frac{dt}{ds} \cdot \frac{dt}{ds} \cdot \frac{dx}{dt} \cdot \frac{dy}{dt} + \frac{dt}{ds} \cdot \frac{dt}{ds} \cdot \frac{dy}{dt} \cdot \frac{dy}{dt}, \quad (1.8)$$

then by the chain rule (1.8) becomes

$$1 = \left( \frac{dx}{ds} \right)^2 + 2\gamma \frac{dx}{ds} \cdot \frac{dy}{ds} + \left( \frac{dy}{ds} \right)^2, \quad (1.9)$$

as required.

(c) From equation (1.4)

$$\frac{dx}{ds} = 2(c - \lambda y) - \gamma \frac{dy}{ds}$$

and from equation (1.5)

$$\frac{dy}{ds} = 2(d + \lambda x) - \gamma \frac{dx}{ds}$$

Substituting into equation (1.4) for  $dy/ds$  gives,

$$\begin{aligned} \frac{dx}{ds} + \gamma \left( 2(d + \lambda x) - \gamma \frac{dx}{ds} \right) &= 2(c - \lambda y), \\ \frac{dx}{ds} + 2\gamma(d + \lambda x) - \gamma^2 \frac{dx}{ds} &= 2(c - \lambda y), \\ \frac{dx}{ds} (1 - \gamma^2) + 2\gamma(d + \lambda x) &= 2(c - \lambda y), \\ \frac{dx}{ds} (1 - \gamma^2) &= 2(c - \lambda y) - 2\gamma(d + \lambda x), \\ \frac{dx}{ds} (1 - \gamma^2) &= 2c - 2\lambda y - 2\gamma d - 2\gamma \lambda x, \\ \frac{dx}{ds} (1 - \gamma^2) &= -2\lambda(y + \gamma x) + 2(c - \gamma d), \\ x' (1 - \gamma^2) &= -X + D = D - X. \end{aligned}$$

Therefore,

$$\boxed{(1 - \gamma^2)x' = D - X}, \quad (1.10)$$

as required.

Equation (1.5) is

$$\gamma \frac{dx}{ds} + \frac{dy}{ds} = 2(d + \lambda x).$$

Substituting into equation (1.5) for  $dx/ds$  from equation (1.4) gives,

$$\begin{aligned} \gamma \left( 2(c - \lambda y) - \gamma \frac{dy}{ds} \right) + \frac{dy}{ds} &= 2(d + \lambda x), \\ 2\gamma(c - \lambda y) - \gamma^2 \frac{dy}{ds} + \frac{dy}{ds} &= 2(d + \lambda x), \\ \frac{dy}{ds} (1 - \gamma^2) &= 2(d + \lambda x) - 2\gamma(c - \lambda y), \\ \frac{dy}{ds} (1 - \gamma^2) &= 2d + 2\lambda x - 2\gamma c + 2\gamma \lambda y, \\ \frac{dy}{ds} (1 - \gamma^2) &= 2(d - \gamma c) + 2\lambda(x + \gamma y). \end{aligned}$$

$$\therefore \boxed{(1 - \gamma^2)y' = C + Y}. \quad (1.11)$$

as required.

Substituting for  $x'$  and  $y'$  from equations (1.10) and (1.11), respectively, into equation (1.9)

$$\frac{(D - X)^2}{(1 - \gamma^2)^2} + 2\gamma \frac{(D - X)(C + Y)}{(1 - \gamma^2)(1 - \gamma^2)} + \frac{(C + Y)^2}{(1 - \gamma^2)^2} = 1$$

Expanding the above and cross multiplying by  $(1 - \gamma^2)^2$  gives,

$$\begin{aligned} D^2 - 2DX + X^2 + 2\gamma(D - X)(C + Y) + C^2 + 2CY + Y^2 &= (1 - \gamma^2)^2, \\ D^2 - 2DX + X^2 + (2\gamma D - 2\gamma X)(C + Y) + C^2 + 2CY + Y^2 &= (1 - \gamma^2)^2, \\ D^2 - 2DX + X^2 + 2\gamma CD + 2\gamma DY - 2\gamma CX - 2\gamma XY + C^2 + 2CY + Y^2 &= (1 - \gamma^2)^2, \\ X^2 + Y^2 - 2\gamma XY + C^2 + D^2 + 2\gamma CD - 2DX + 2\gamma DY - 2\gamma CX + 2CY &= (1 - \gamma^2)^2, \end{aligned}$$

Collecting terms in just  $X$  and just  $Y$  gives,

$$X^2 + Y^2 - 2\gamma XY + C^2 + D^2 + 2\gamma CD - 2X(D + \gamma C) + 2Y(C + \gamma D) = (1 - \gamma^2)^2. \quad (1.12)$$

Where,  
 $D = 2(c - \gamma d)$ ,  
 $X = 2\lambda(x - y)$  and  
 $x' = \frac{dy}{ds}$ .

Where,  
 $C = 2(d - \gamma c)$ ,  
 $Y = 2\lambda(x + \gamma y)$  and  
 $y' = \frac{dy}{ds}$ .

Now, from above (see margin notes)

$$\begin{aligned}(D + \gamma C) &= 2(c - \gamma d) + \gamma(2(d - \gamma c)), \\ &= 2c - 2\gamma d + 2\gamma d - 2\gamma^2 c, \\ &= 2c(1 - \gamma^2).\end{aligned}$$

$$\begin{aligned}(C + \gamma D) &= 2(d - \gamma c) + \gamma(2(c - \gamma d)), \\ &= 2d - 2\gamma c + 2\gamma c - 2\gamma^2 d, \\ &= 2d(1 - \gamma^2).\end{aligned}$$

So,

$$\begin{aligned}D + \gamma C &= 2c(1 - \gamma^2) \quad \text{and} \\ C + \gamma D &= 2d(1 - \gamma^2).\end{aligned}$$

Substituting these expressions into equation (1.12) gives,

$$X^2 + Y^2 - 2\gamma XY + C^2 + D^2 + 2\gamma CD - 4c(1 - \gamma^2)X + 4d(1 - \gamma^2)Y = (1 - \gamma^2)^2,$$

as required.

The boundary conditions are

$$x(0) = y(0) = 0, \quad x(1) = R > 0, \quad y(1) = 0.$$

Now, applying these boundary conditions to equation (1.12) and from the margin notes above we have:

$$X(t) = 2\lambda(\gamma x(t) + y(t)).$$

$$Y(t) = 2\lambda(x(t) + \gamma y(t)).$$

At  $t = 0$ ,

$$X(0) = 2\lambda(\gamma x(0) + y(0)) = 0.$$

$$Y(0) = 2\lambda(x(0) + \gamma y(0)) = 0.$$

Thus, equation (1.12) at  $t = 0$  reduces to

$$C^2 + D^2 + 2\gamma CD = (1 - \gamma^2)^2.$$

At  $t = 1$ ,

$$X(1) = 2\lambda(\gamma x(1) + y(1)) = 2\lambda\gamma R.$$

$$Y(1) = 2\lambda(x(1) + \gamma y(1)) = 2\lambda R.$$

Thus, when applied to equation (1.12),

$$\begin{aligned}4\lambda^2\gamma^2R^2 + 4\lambda^2R^2 - 2\gamma(2\lambda\gamma R)(2\lambda R) - 4c(1 - \gamma^2)(2\lambda\gamma R) \\ + 4d(1 - \gamma^2)(2\lambda R) + C^2 + D^2 + 2\gamma CD = (1 - \gamma^2)^2.\end{aligned}$$

$$\begin{aligned}4\lambda^2\gamma^2R^2 + 4\lambda^2R^2 - 8\lambda^2\gamma^2R^2 - 8c(1 - \gamma^2)\lambda\gamma R \\ + 8d(1 - \gamma^2)\lambda R + C^2 + D^2 + 2\gamma CD = (1 - \gamma^2)^2.\end{aligned}$$

Note from above that  $C^2 + D^2 + 2\gamma CD = (1 - \gamma^2)^2$ , thus

$$4\lambda^2\gamma^2R^2 + 4\lambda^2R^2 - 8\lambda^2\gamma^2R^2 - 8c(1 - \gamma^2)\lambda\gamma R + 8d(1 - \gamma^2)\lambda R = 0,$$

which, after dividing through by  $4\lambda R$ , gives

$$\lambda\gamma^2R + \lambda R - 2\lambda\gamma^2R - 2c(1 - \gamma^2)\gamma + 2d(1 - \gamma^2) = 0, \quad \lambda, R \neq 0 \text{ and } \gamma^2 \neq 1$$

$$\lambda R - \lambda\gamma^2R - 2c(1 - \gamma^2)\gamma + 2d(1 - \gamma^2) = 0,$$

$$\lambda R(1 - \gamma^2) - 2c(1 - \gamma^2)\gamma + 2d(1 - \gamma^2) = 0,$$

and dividing through by  $(1 - \gamma^2)$ , gives

$$\lambda R - 2c\gamma + 2d = 0.$$

Simplifying,

$$\lambda R - 2(c\gamma + d) = 0,$$

and noting that the quantity  $2(c\gamma + d)$  is equal to  $C$ , then

$$\lambda R + C = 0.$$

Finally,

$$\boxed{C = -\lambda R, \quad R > 0,} \quad (1.13)$$

as required.

Recall that  $C^2 + D^2 + 2\gamma CD = (1 - \gamma^2)^2$  and  $C = -\lambda R$ ,  $R > 0$ .

Thus,

$$\lambda^2R^2 + D^2 + 2\gamma(-\lambda R)D = (1 - \gamma^2)^2,$$

$$D^2 - 2\lambda\gamma RD + \lambda^2R^2 - (1 - \gamma^2)^2 = 0.$$

Solving for  $D$  in the above quadratic,

$$D = \frac{2\lambda\gamma R \pm \sqrt{4\lambda^2\gamma^2R^2 - 4(\lambda^2R^2 - (1 - \gamma^2)^2)}}{2},$$

$$D = \lambda\gamma R \pm \sqrt{\lambda^2\gamma^2R^2 - \lambda^2R^2 + (1 - \gamma^2)^2},$$

$$D = \lambda\gamma R \pm \sqrt{\lambda^2R^2(\gamma^2 - 1) + (1 - \gamma^2)^2},$$

$$D = \lambda\gamma R \pm \sqrt{-\lambda^2R^2(1 - \gamma^2) + (1 - \gamma^2)^2}$$

It is seen that  $D$  will always have two real roots if  $\gamma^2 > 1$ , as in the expression below

$$D = \lambda\gamma R \pm \sqrt{-\lambda^2R^2(1 - \gamma^2) + (1 - \gamma^2)^2}$$

$(1 - \gamma^2)^2 > 0$  and  $(1 - \gamma^2) < 0$  and consequently  $-\lambda^2R^2(1 - \gamma^2) > 0$  making the entire quantity inside the radical positive.

Also, if  $\gamma^2 < 1$  then  $(1 - \gamma^2) > 0$  with  $-\lambda^2 R^2 (1 - \gamma^2) < 0$  and the quantity  $(1 - \gamma^2)^2 - \lambda^2 R^2 (1 - \gamma^2) > 0$  provided that  $\lambda^2 R^2 (1 - \gamma^2) < (1 - \gamma^2)^2$ , that is if  $\lambda R < \sqrt{1 - \gamma^2}$ , there will be two real roots of  $D$ .

Conversely, if  $(1 - \gamma^2)^2 - \lambda^2 R^2 (1 - \gamma^2) < 0$  then  $\lambda^2 R^2 (1 - \gamma^2) > (1 - \gamma^2)^2$ , that is  $\lambda R > \sqrt{1 - \gamma^2}$ , there will not be any real solutions of  $D$ .



Q 2. In order to show that the stationary path of the functional

$$S[y] = \int_0^v dx (y'^2 + y^2), y(0) = 1, y(v) = v, v > 0,$$

is given by

$$y = \cosh x + B \sinh x, \quad 0 \leq x \leq v,$$

where  $B$  and  $v$  are given by the solutions of the equations

$$v = \cosh v + B \sinh v \quad \text{and} \quad B^2 - 1 = 2 \sinh v + 2B \cosh v,$$

consider the following:

(a) The integrand is

$$F = y'^2 + y^2,$$

and the Euler-Lagrange equation is given by

$$\frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0,$$

where

$$\frac{\partial F}{\partial y'} = 2y', \quad \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = \frac{d}{dx} (2y') = 2y'' \quad \text{and} \quad \frac{\partial F}{\partial y} = 2y.$$

Thus, the Euler-Lagrange equation can be written as

$$2y'' - 2y = 0 \quad \text{or} \quad \frac{d^2 y}{dx^2} - y = 0.$$

The auxiliary equation is  $\lambda^2 - 1 = 0$  and has general solution

$$y = A \cosh(x) + B \sinh(x).$$

When  $x = 0$ ,  $y(0) = 1$ , so

$$1 = A \cosh(0) + B \sinh(0), \quad \text{therefore } A = 1.$$

When  $x = v$ ,  $y(v) = v$ , so

$$v = \cosh(v) + B \sinh(v), \quad \text{therefore } B = \frac{v - \cosh(v)}{\sinh(v)}.$$

Now, using the **Transversality Condition** in which the end of the curve HB p.25 can be expressed in the form  $y = g(x)$  and also noting that  $y(v) = v$ , then:

$$F + (g'(x) - y'(x)) F_{y'} = 0 \text{ at } x = v,$$

Now,  $g(x) = x$  so  $g'(x) = 1$  and from above  $F = y'^2 + y^2$ , hence

$$(y'^2 + y^2) + (1 - y') 2y' = 0,$$

$$\begin{aligned} y'^2 + y^2 + 2y' - 2y'^2 &= 0, \\ -y'^2 + 2y' + y^2 &= 0 \text{ at } x = v. \end{aligned} \quad (2.1)$$

From above  $y(x) = \cosh x + B \sinh x$  and therefore  $y'(x) = \sinh x + B \cosh x$

Substituting these expressions into (2.1) at  $x = v$  gives,

$$-(\sinh v + B \cosh v)^2 + 2(\sinh v + B \cosh v) + (\cosh v + B \sinh v)^2 = 0$$

which simplifies to

$$-\sinh^2 v - B^2 \cosh^2 v + 2 \sinh v + 2B \cosh v + \cosh^2 v + B^2 \sinh^2 v = 0$$

Note:  $\cosh^2 v - \sinh^2 v = 1$ , so

$$1 - B^2 + 2 \sinh v + 2B \cosh v = 0,$$

therefore,  $B^2 - 1 = 2 \sinh v + 2B \cosh v$

- (b) Using the equations found in part (a) it can be shown that  $v$  is given by the real solution(s) of  $f(v) = 0$ , where

$$f(v) = v^2 - 2v(1 + \sinh v) \cosh v + 1 + 2 \sinh v.$$

as follows:

Recall from part (a) that,

$$v = \cosh v + B \sinh v, \quad (2.2)$$

$$B^2 - 1 = 2 \sinh v + 2B \cosh v, \quad (2.3)$$

From (2.2)

$$B = \frac{v - \cosh v}{\sinh v}. \quad (2.4)$$

Substituting (2.4) into (2.3) gives,

$$\left( \frac{v - \cosh v}{\sinh v} \right)^2 - 1 = 2 \sinh v + 2 \left( \frac{v - \cosh v}{\sinh v} \right) \cosh v,$$

$$\begin{aligned} \frac{v^2 - 2v \cosh v + (\cosh^2 v - 2 \sinh^2 v)}{\sinh^2 v} &= \\ 2 \sinh v + 2 \left( \frac{v - \cosh v}{\sinh v} \right) \cosh v, \end{aligned}$$

$$\frac{v^2 - 2v \cosh v + 1}{\sinh^2 v} = \frac{2 \sinh^2 v + 2v \cosh v - 2 \cosh^2 v}{\sinh v},$$

$$\frac{v^2 - 2v \cosh v + 1}{\sinh v} = -2 (\cosh^2 v - \sinh^2 v) + 2v \cosh v,$$

$$\frac{v^2 - 2v \cosh v + 1}{\sinh v} = -2 + 2v \cosh v,$$

$$v^2 - 2v \cosh v + 1 = 2 \sinh v (v \cosh v - 1),$$

$$v^2 = 2v \cosh v - 1 + 2 \sinh v (v \cosh v - 1),$$

$$v^2 = 2v \sinh v \cosh v - 2 \sinh v + 2v \cosh v - 1,$$

$$v^2 = 2v \cosh v (\sinh v + 1) - 2 \sinh v - 1,$$

$$\boxed{\therefore f(v) = v^2 - 2v(1 + \sinh v) \cosh v + 1 + 2 \sinh v = 0.} \quad (2.5)$$

- (c) To deduce that there is only one stationary path, which is equivalent to proving that  $f(v) = 0$  has exactly one solution for  $v > 0$ , consider the following.

First, it is necessary to show that  $f(v) = 0$  has at least one solution in the interval  $0 < v < \infty$ . Now  $f$  is continuous on  $[0, 1]$  with  $f(0) > 0$  and  $f(1) < 0$ . Thus, by the **Intermediate Value Theorem**, it follows that  $f(v) = 0$  for some value of  $v$  in the interval  $(0, 1)$  and hence in  $(0, \infty)$ . Consequently,  $f(v) = 0$  has at least one solution in the interval  $0 < v < \infty$ .

$$f(0) = 1.0$$

$$f(1) = -2.363$$

rounded to 3 figures

Next, it is necessary to show that  $f(v) = 0$  has at most one solution in the interval  $0 < v < \infty$ . Assume that it has more than one solution in this interval and choose two solutions within this interval; say at  $v = a$  and  $v = b$  and suppose  $a < b$ . Then, since  $f$  is continuous on the interval  $[a, b]$  and so differentiable on this interval, with  $f(a) = 0$  and  $f(b) = 0$  and it follows from **Rolle's Theorem** that  $f'(d) = 0$  for some  $d$  in  $(a, b)$ . However,  $f'(v) < 0$  for all  $v > 0$  (see below). Hence this shows that  $f'(v)$  cannot be zero in the interval  $(a, b)$  and there exists a contradiction, thus showing that  $f(x) = 0$  has at most one solution in the interval  $0 < v < \infty$ . This coupled with the fact that  $f(v) = 0$  has at least one solution in the interval  $0 < v < \infty$  leads to the conclusion that this equation has exactly one solution in the given interval.

$$f(v) = v^2 - 2v(1 + \sinh v) \cosh v + 1 + 2 \sinh v,$$

$$f'(v) = \frac{d}{dv} (v^2) - 2 \frac{d}{dv} (v(1 + \sinh v) \cosh v) + \frac{d}{dv} (1) + 2 \frac{d}{dv} (\sinh v),$$

$$= 2v - 2 \frac{d}{dv} (v(1 + \sinh v) \cosh v) + 2 \cosh v,$$

$$= 2v - 2 \frac{d}{dv} (v \cosh v + v \sinh v \cosh v) + 2 \cosh v,$$

$$= 2v - 2 \frac{d}{dv} (v \cosh v) - 2 \frac{d}{dv} (v \sinh v \cosh v) + 2 \cosh v,$$

$$\begin{aligned} &= 2v - 2(v \sinh v + \cosh v) - 2 \frac{d}{dv} (v \sinh v \cosh v) + 2 \cosh v, \\ &= 2v - 2v \sinh v - 2 \cosh v - 2 \frac{d}{dv} (v \sinh v \cosh v) + 2 \cosh v, \end{aligned}$$

Simplifying

$$\begin{aligned} f'(v) &= 2v(1 - \sinh v) - 2 \frac{d}{dv} (v \sinh v \cosh v), \\ &= 2v(1 - \sinh v) - 2(\sinh v \cosh v + v \cosh v \cosh v + v \sinh v \sinh v), \\ &= 2v(1 - \sinh v) - 2 \sinh v \cosh v - 2v(\cosh^2 v + \sinh^2 v), \\ &= 2(v(1 - \sinh v) - \sinh v \cosh v - v(\cosh^2 v + \sinh^2 v)), \\ &= 2(v - v \sinh v - \sinh v \cosh v - v \cosh^2 v - v \sinh^2 v) \end{aligned}$$

Inspection of the last expressions shows that  $f'(v)$  is negative for all  $v > 0$ .

Q 3.

- (a) Heron's formula for the area  $A$  of a triangle with sides of length  $a, b, c > 0$  is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{a+b+c}{2}.$$

The method of Lagrange multipliers will be used to show that for a triangle with a fixed perimeter  $P = a + b + c = 2s$ , then its area is maximised when the triangle is an equilateral, as follows.

For this problem  $f(a, b, c) = \sqrt{s(s-a)(s-b)(s-c)}$ , and  $g(a, b, c) = a + b + c - 2s$ , so that the auxiliary function is

$$\begin{aligned} \bar{F}(a, b, c, \lambda) &= f(a, b, c) - \lambda g(a, b, c), \\ &= \sqrt{s(s-a)(s-b)(s-c)} - \lambda(a+b+c-2s), \end{aligned} \quad s = \frac{a+b+c}{2}$$

with each of  $a, b$  and  $c > 0$ . The following three equations relating  $a, b, c$  and  $\lambda$  are required:

$$\frac{\partial \bar{F}}{\partial a} = 0, \quad \frac{\partial \bar{F}}{\partial b} = 0 \quad \text{and} \quad \frac{\partial \bar{F}}{\partial c} = 0.$$

So,

$$\begin{aligned} \frac{\partial \bar{F}}{\partial a} &= \frac{-s(s-b)(s-c)}{2\sqrt{s(s-a)(s-b)(s-c)}} - \lambda = 0, \\ \frac{\partial \bar{F}}{\partial b} &= \frac{-s(s-a)(s-c)}{2\sqrt{s(s-a)(s-b)(s-c)}} - \lambda = 0, \\ \frac{\partial \bar{F}}{\partial c} &= \frac{-s(s-a)(s-b)}{2\sqrt{s(s-a)(s-b)(s-c)}} - \lambda = 0. \end{aligned}$$

So,

$$\begin{aligned} s(s-b)(s-c) &= -2\lambda A, \\ s(s-a)(s-c) &= -2\lambda A, \\ s(s-a)(s-b) &= -2\lambda A \quad \text{and} \\ a+b+c-2s &= 0. \end{aligned}$$

Thus,

$$s(s-b)(s-c) = s(s-a)(s-c) = s(s-a)(s-b) \quad \text{and} \quad s \neq 0, a, b, c.$$

Consequently,  $a = b = c$ .

- (b) To find, in terms of the fixed perimeter  $P = 2s$ , the maximum area of a right-angled triangle with perimeter  $P$  consider the following.

This problem has two constraints. Therefore, there are two Lagrange multipliers,  $\lambda$  and  $\mu$ , one for each constraint. The auxiliary equation is

$$\bar{F}(a, b, c, \lambda, \mu) = f(a, b, c) - \lambda g_1(a, b, c) - \mu g_2(a, b, c),$$

and finding the stationary points of this is achieved by finding the solutions of

$$\frac{\partial \bar{F}}{\partial a} = 0, \quad \frac{\partial \bar{F}}{\partial b} = 0, \quad \frac{\partial \bar{F}}{\partial c} = 0,$$

The first constraint relates to the triangle being right angled. Assuming side  $c$  is the hypotenuse, then Pythagoras' Theorem

$$a^2 + b^2 = c^2$$

and the constraint becomes  $g_1(a, b, c) = a^2 + b^2 - c^2$ .

The second constraint relates to the triangle having a fixed perimeter as in part (a), namely  $g_2(a, b, c) = a + b + c - 2s$ .

Thus, the auxiliary equation given above becomes

$$\bar{F} = \sqrt{s(s-a)(s-b)(s-c)} - \lambda(a^2 + b^2 - c^2) - \mu(a + b + c - 2s).$$

$$\frac{d\bar{F}}{da} = \frac{-s(s-b)(s-c)}{2\sqrt{s(s-a)(s-b)(s-c)}} - 2\lambda a - \mu = 0,$$

$$\frac{d\bar{F}}{db} = \frac{-s(s-a)(s-c)}{2\sqrt{s(s-a)(s-b)(s-c)}} - 2\lambda b - \mu = 0,$$

$$\frac{d\bar{F}}{dc} = \frac{-s(s-a)(s-b)}{2\sqrt{s(s-a)(s-b)(s-c)}} - 2\lambda c - \mu = 0.$$

Recall that  $A = \sqrt{s(s-a)(s-b)(s-c)}$ , so

$$\frac{-s(s-b)(s-c)}{2A} - 2\lambda a - \mu = 0, \tag{3.1}$$

$$\frac{-s(s-a)(s-c)}{2A} - 2\lambda b - \mu = 0, \tag{3.2}$$

$$\frac{-s(s-a)(s-b)}{2A} + 2\lambda c - \mu = 0. \tag{3.3}$$

Subtracting (3.2) from (3.1) gives,

$$\left( \frac{-s(s-b)(s-c)}{2A} - 2\lambda a - \mu \right) - \left( \frac{-s(s-a)(s-c)}{2A} - 2\lambda b - \mu \right) = 0$$

$$\frac{-s(s-b)(s-c)}{2A} - 2\lambda a - \mu + \frac{s(s-a)(s-c)}{2A} + 2\lambda b + \mu = 0$$

$$\frac{-s(s-b)(s-c)}{2A} - 2\lambda a + \frac{s(s-a)(s-c)}{2A} + 2\lambda b = 0$$

$$\frac{-s(s-b)(s-c)}{2A} + \frac{s(s-a)(s-c)}{2A} + 2\lambda(b-a) = 0$$

$$\frac{-s(s-b)(s-c) + s(s-a)(s-c)}{2A} + 2\lambda(b-a) = 0$$

$$s(s-c) \frac{-(s-b) + (s-a)}{2A} + 2\lambda(b-a) = 0$$

$$s(s-c) \frac{-s + b + s - a}{2A} + 2\lambda(b-a) = 0$$

$$\frac{s(s-c)(b-a)}{2A} + 2\lambda(b-a) = 0$$

$$(b-a) \left( \frac{s(s-c)}{2A} + 2\lambda \right) = 0$$

$$(b-a)(s(s-c) + 4A\lambda) = 0$$

and so  $b-a=0$  or  $s(s-c) + 4A\lambda = 0$  with  $s \neq 0, c$ .

Subtracting (3.3) from (3.1) gives,

$$\frac{-s(s-b)(s-c)}{2A} + \frac{s(s-a)(s-b)}{2A} - 2\lambda(a+c) = 0,$$

$$\frac{-s(s-b)(s-c) + s(s-a)(s-b)}{2A} - 2\lambda(a+c) = 0,$$

$$s(s-b) \frac{-(s-c) + (s-a)}{2A} - 2\lambda(a+c) = 0,$$

$$s(s-b) \frac{-s + c + s - a}{2A} - 2\lambda(a+c) = 0,$$

$$\frac{s(s-b)(c-a)}{2A} - 2\lambda(a+c) = 0,$$

$$s(s-b)(c-a) - 4A\lambda(a+c) = 0,$$

recall that  $-4A\lambda = s(s-c)$ , so

$$s(s-b)(c-a) + s(s-c)(a+c) = 0,$$

$$sc - sa - bc + ab + sa + sc - ac - c^2 = 0,$$

$$2sc - bc + ab - ac - c^2 = 0,$$

$$ab + c(2s - a - b - c) = 0.$$

Now it is noted that  $2s - a - b - c = 0$ , but  $ab = 0$  is not possible so  $a = b$ , and by Pythagoras' Theorem  $c = \sqrt{a^2 + a^2} = a\sqrt{2}$  and the perimeter  $P = a + b + c = a + a + a\sqrt{2} = a(2 + \sqrt{2})$ .

$$s = \frac{a + b + c}{2}$$

The area of a right-angled triangle is given by  $A = \frac{1}{2}ab = \frac{1}{2}a^2$ . From the foregoing  $a = \frac{P}{(2 + \sqrt{2})}$ . So,

$$A = \frac{1}{2} \left( \frac{P}{2 + \sqrt{2}} \right)^2 = \frac{P^2}{12 + 8\sqrt{2}} = \frac{3 - 2\sqrt{2}}{4} P^2.$$

Q 4.



Q 5.

Q 6.

- (a) The model of tumour growth under radiotherapy from time  $t = 0$  to  $t = t_1 > 0$  is given as

$$\frac{dC}{dt} = -C \log \left( \frac{C}{C_{max}} \right) - \frac{DC}{1+D}, \quad (6.1)$$

where, at time  $t$ ,  $C(t)$  is the size of the tumour,  $C_{max} > 0$  is the maximum size of the tumour, a constant, so that  $0 < C(t) \leq C_{max}$ , and  $D(t) \geq 0$  is the rate at which the drug is administered.

Now, making a change of variable

$$x = \log \left( \frac{C}{C_{max}} \right),$$

then (6.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= \frac{dC}{dt} \cdot \frac{dx}{dC}, \\ \frac{dx}{dt} &= \frac{dC}{dt} \cdot \frac{1}{C}, \\ \frac{dx}{dt} &= \left[ -C \log \left( \frac{C}{C_{max}} \right) - \frac{DC}{1+D} \right] \cdot \frac{1}{C}, \\ \frac{dx}{dt} &= -\log \left( \frac{C}{C_{max}} \right) - \frac{D}{1+D}, \\ \text{thus, } \frac{dx}{dt} &= -x - \frac{D}{1+D}, \quad \text{where } -\infty < x \leq 0. \end{aligned}$$

- (b) We are told that:

It is required to reduce the size of the tumour from  $C_0$  at  $t = 0$  to  $C_1$  at  $t = t_1$ . Let  $x_0 = \log(C_0/C_{max})$  and  $x_1 = \log(C_1/C_{max})$ . For the health of the patient, it is desired to minimise the total amount of drug administered, which is given by the functional

$$S[D] = \int_0^{t_1} dt D(t).$$

So, by expressing  $D$  in terms of  $x$  and  $\dot{x} = dx/dt$  it should be possible to show that  $S[D]$  may be written as

$$S[x] = - \int_0^{t_1} dt \frac{\dot{x} + x}{1 + \dot{x} + x}.$$

From part (a)

$$\begin{aligned}
 \dot{x} &= -x - \frac{D}{1+D}, \\
 \dot{x} + x &= -\frac{D}{1+D}, \\
 (\dot{x} + x)(1+D) &= -D, \\
 (\dot{x} + x) + D(\dot{x} + x) &= -D, \\
 D(\dot{x} + x) + D &= -(\dot{x} + x), \\
 D(\dot{x} + x + 1) &= -\dot{x} - x, \\
 D &= \frac{-\dot{x} - x}{\dot{x} + x + 1}.
 \end{aligned}$$

So

$$S[x] = - \int_0^{t_1} dt \frac{\dot{x} + x}{\dot{x} + x + 1} \quad \text{as required.}$$

(c) To show that the Euler-Lagrange equation for  $S[x]$  is given by

$$2\ddot{x} + 3\dot{x} + x = -1, \quad \text{where } \ddot{x} = d^2x/dt^2,$$

consider the following.

The Euler-Lagrange equation is given by

HB p17.

$$\frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) - \frac{\partial F}{\partial x} = 0, \quad y(a) = A, \quad y(b) = B.$$

$$\text{Let, } F = \frac{\dot{x} + x}{\dot{x} + x + 1},$$

then using the quotient rule to determine  $\partial F/\partial \dot{x}$  and  $\partial F/\partial x$

$$\frac{\partial}{\partial \dot{x}} \left( \frac{u}{v} \right) = \left[ v \frac{\partial u}{\partial \dot{x}} - u \frac{\partial v}{\partial \dot{x}} \right] / v^2 \quad \text{where, } u = \dot{x} + x \text{ and } v = \dot{x} + x + 1.$$

$$\begin{aligned}
 \frac{\partial F}{\partial \dot{x}} &= \frac{(\dot{x} + x + 1) \cdot 1 - (\dot{x} + x) \cdot 1}{(\dot{x} + x + 1)^2} = \frac{1}{(\dot{x} + x + 1)^2}, \\
 \frac{\partial F}{\partial x} &= \frac{(\dot{x} + x + 1) \cdot 1 - (\dot{x} + x) \cdot 1}{(\dot{x} + x + 1)^2} = \frac{1}{(\dot{x} + x + 1)^2}.
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) &= \frac{d}{dt} (\dot{x} + x + 1)^{-2}, \\
 &= -2(\dot{x} + x + 1)^{-3} (\ddot{x} + \dot{x}), \\
 &= \frac{-2(\ddot{x} + \dot{x})}{(\dot{x} + x + 1)^3}.
 \end{aligned}$$

Therefore, the Euler-Lagrange equation is

$$\begin{aligned}
 -\frac{2(\ddot{x} + \dot{x})}{(\dot{x} + x + 1)^3} - \frac{1}{(\dot{x} + x + 1)^2} &= 0, \\
 \frac{2(\ddot{x} + \dot{x})}{(\dot{x} + x + 1)^3} &= -\frac{1}{(\dot{x} + x + 1)^2}, \\
 \frac{2(\ddot{x} + \dot{x})}{(\dot{x} + x + 1)} &= -1, \\
 2(\ddot{x} + \dot{x}) &= -(\dot{x} + x + 1), \\
 2(\ddot{x} + \dot{x}) + (\dot{x} + x) &= -1, \\
 2\ddot{x} + 3\dot{x} + x &= -1, \quad \text{as required.}
 \end{aligned} \tag{6.2} \quad \ddot{x} = d^2x/dt^2.$$

- (d) Now solving the second-order linear ordinary differential equation (6.2) with the boundary conditions  $x(0) = x_0$  and  $x(t_1) = x_1$  as follows.

The auxiliary equation is

$$2\lambda^2 + 3\lambda + 1 = 0.$$

This has roots  $\lambda_1 = -1$  and  $\lambda_2 = -\frac{1}{2}$ . These roots are real and different so the complementary function is given by

$$y_c = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t}, \quad \text{where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

Using the method of undetermined coefficients to determine the particular solution to (6.2) as follows.

The particular solution to

$$2\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + x(t) = -1,$$

is of the form

$$x_p(t) = a_1,$$

where  $a_1$  is an unknown constant to be determined as follows:

$$\frac{dx(t)}{dt} = \frac{d}{dt}(a_1) = 0,$$

and therefore

$$\frac{d^2x(t)}{dt^2} = 0.$$

Substituting the particular solution  $x_p(t)$  in to (6.2) gives

$$2 \times 0 + 3 \times 0 + a_1 = -1.$$

Therefore,  $a_1 = -1$  and the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t} - 1. \tag{6.3}$$

Now using the boundary conditions to solve for the unknown constants  $c_1$  and  $c_2$  as follows.

At  $t = 0$ ,  $x(0) = x_0$  and substituting this condition into (6.3) gives,

$$c_1 + c_2 - 1 = x_0. \quad (6.4)$$

At  $t = t_1$ ,  $x(t_1) = x_1$  and substituting this condition into (6.3) gives,

$$c_1 e^{-t_1} + c_2 e^{-\frac{1}{2}t_1} - 1 = x_1 \quad (6.5)$$

From (6.4)

$$c_1 = (x_0 + 1) - c_2 \quad (6.6)$$

Substituting into (6.5) for  $c_1$  from (6.6) and rearranging in terms of  $c_2$

$$\begin{aligned} (x_0 + 1) e^{-t_1} - c_2 e^{-t_1} + c_2 e^{-\frac{1}{2}t_1} &= x_1 + 1, \\ (x_0 + 1) e^{-t_1} + c_2 \left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right) &= (x_1 + 1), \\ c_2 \left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right) &= (x_1 + 1) - (x_0 + 1) e^{-t_1}, \\ c_2 &= \frac{(x_1 + 1) - (x_0 + 1) e^{-t_1}}{\left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right)}. \end{aligned} \quad (6.7)$$

Substituting for  $c_2$  in (6.6) from (6.7)

$$\begin{aligned} c_1 &= (x_0 + 1) - \left[ \frac{(x_1 + 1) - (x_0 + 1) e^{-t_1}}{\left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right)} \right], \\ c_1 &= \frac{(x_0 + 1) \left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right) - (x_1 + 1) + (x_0 + 1) e^{-t_1}}{\left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right)} \\ c_1 &= \frac{(x_0 + 1) e^{-\frac{1}{2}t_1} - (x_0 + 1) e^{-t_1} - (x_1 + 1) + (x_0 + 1) e^{-t_1}}{\left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right)} \\ c_1 &= \frac{(x_0 + 1) e^{-\frac{1}{2}t_1} - (x_1 + 1)}{\left( e^{-\frac{1}{2}t_1} - e^{-t_1} \right)} = \alpha. \end{aligned} \quad (6.8)$$

Therefore, from (6.4)

$$c_2 = (x_0 + 1) - \alpha. \quad (6.9)$$

Substituting into (6.3) for  $c_1$  and  $c_2$  from (6.8) and (6.9) respectively gives,

$$x(t) = \alpha e^{-t} + (x_0 + 1 - \alpha) e^{-\frac{1}{2}t} - 1 \quad (\alpha \text{ given in (6.8)}),$$

as required.

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