Q 1.

(a)

(b)

Q 2.

Q 3.

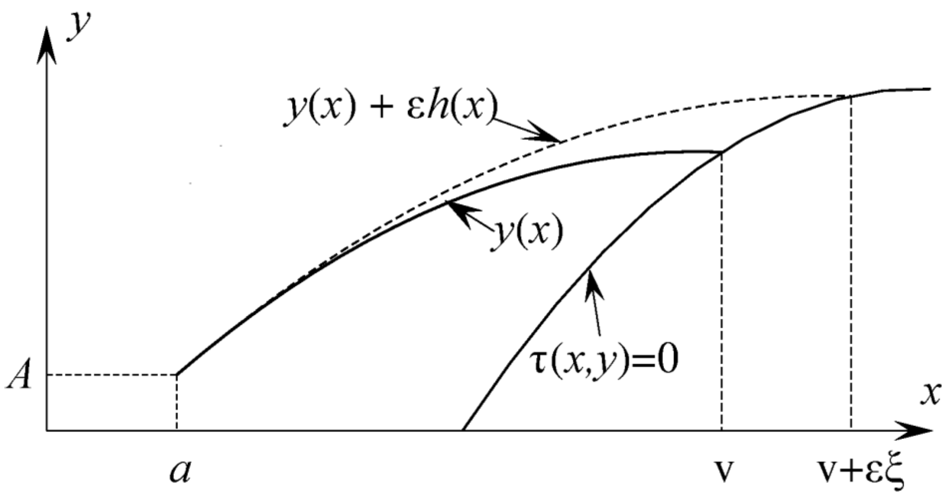
(a)

(b)

(c)

Q 4.

(a)



This Figure 10.6 taken from the module notes p225.

Figure 1: Diagram showing the stationary path (solid line) and a varied path (dashed line) for a problem in which the left-hand end is fixed, but the other end is free to move along the line defined by $\tau(x, y) = 0$.

Given the perturbed path

$$y_\epsilon(x) = y(x) + \epsilon h(x), \tag{4.1}$$

the Taylor series to the first-order of (4.1) at point $x = v$ (see figure 1) is given in (4.2).

The point $x = v$ is known as the point of expansion. HB p8.

$$y_\epsilon(x) = (y(v) + \epsilon h(v)) + (y'(v) + \epsilon h'(v))(x - v) + \mathcal{O}\left((x - v)^2\right). \tag{4.2}$$

Now, determining the value of (4.2) at $v_\epsilon = v + \epsilon\xi + \mathcal{O}(\epsilon^2)$ where v_ϵ is the perturbed value of v:

$$\begin{aligned} y(v_\epsilon) = y(v) &+ \epsilon h(v) \\ &+ \left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{-v}\right)(y'(v) + \epsilon h'(v)) \\ &+ \mathcal{O}\left(\left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{-v}\right)^2\right), \end{aligned}$$

$$\begin{aligned} y(v_\epsilon) = y(v) &+ \epsilon h(v) \\ &+ \left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)(y'(v) + \epsilon h'(v)) \\ &+ \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right), \end{aligned}$$

$$
\begin{aligned}
y(v_\epsilon) = y(v) &+ \epsilon h(v) \\
&+ \epsilon\xi\left(y'(v) + \epsilon h'(v)\right) \\
&+ \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) \\
&+ \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
y(v_\epsilon) = y(v) &+ \epsilon\left(h(v) + \xi y'(v)\right) \\
&+ \underbrace{\epsilon^2\xi h'(v) + \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) + \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right)}_{\text{These are all second-order terms in } \epsilon.}.
\end{aligned}
$$

Thus,

$$
y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(h(v) + \xi y'(v)\right) + \mathcal{O}\left(\epsilon^2\right), \tag{4.3}
$$

as required.

To show that at $(x, y) = (v, y(v))$,

$$
\xi\left(\tau_x + y'(v)\tau_y\right) + h(v)\tau_y = 0
$$

the 2D Taylor expansion to the first-order of $\tau(x, y)$ at point $x = a, y = b$ is required, namely,

$$
\tau(x, y) = \tau(a, b) + \tau_x(a, b)\left[x - a\right] + \tau_y(a, b)\left[y - b\right]. \tag{4.4}
$$

Evaluating (4.4) with $x = v + \epsilon\xi$ and $y = y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(y'(v)\xi + h(v)\right)$ gives,

$$
\begin{aligned}
&\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) = \tau(v + \epsilon\xi, y(v) + \epsilon\left(y'(v)\xi + h(v)\right) \\
&= \cancelto{0}{\tau\left(v, y(v)\right)} + \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\
&= \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\
&= \tau_x\left(v, y(v)\right)\left[\cancel{v} + \epsilon\xi \cancel{- v}\right] + \tau_y\left(v, y(v)\right)\left[\cancel{y(v)} + \epsilon\left(y'(v)\xi + h(v)\right) \cancel{- y(v)}\right], \\
&= \tau_x\left(v, y(v)\right)\epsilon\xi + \tau_y\left(v, y(v)\right)\epsilon\left(y'(v)\xi + h(v)\right), \\
&= \epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right],
\end{aligned}
$$

Recall that $\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) = 0$ and therefore,

$$
\begin{aligned}
\epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right] &= 0, \\
\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right) &= 0,
\end{aligned}
$$

$$
\xi\left[\tau_x\left(v, y(v)\right) + \tau_y\left(v, y(v)\right)y'(v)\right] + \tau_y\left(v, y(v)\right)h(v) = 0, \tag{4.5}
$$

as required.

(b) The Gâteaux differentialis given as

$$
\Delta S\left(y, h\right) = \xi F\big|_v + \int_a^v \mathrm{d}x\ \left(hF_y + h'F_{y'}\right), \tag{4.6}
$$

and it will be shown, by integrating by parts (4.6), that a stationary path must satisfy the transversality condition given in (4.7):

$$\tau_x F_{y'} + \tau_y \left(y' F_{y'} - F\right) = 0 \quad \text{at} \quad (x, y) = (v, y(v)) \tag{4.7}$$

as follows.

Equation (4.6) can be rewritten as in (4.8),

$$\Delta S\left(y, h\right) = \xi F\big|_v + \int_a^v \mathrm{d}x\, h F_y + \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.8}$$

and integrating by parts the right-most integral in (4.8):

$$I = \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.9}$$

$$\begin{aligned} &\text{Let } u = F_{y'} \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) \\ &\text{Let } \frac{\mathrm{d}v}{\mathrm{d}x} = h'(x) \quad \text{then} \quad v = \int \mathrm{d}x\, h'(x) = h(x). \end{aligned}$$

For clarity, here v is not the same as that for the upper limit of integration in (4.9).

For integration by parts:

$$\begin{aligned} I &= \int_a^v \mathrm{d}x\, u \frac{\mathrm{d}v}{\mathrm{d}x} = \left[uv\right]_a^v - \int_a^v \mathrm{d}x\, v \frac{\mathrm{d}u}{\mathrm{d}x}, \\ &= \left[F_{y'} h(x)\right]_a^v - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= \left(F_{y'} h(x)\big|_{x=v} - F_{y'} \cancelto{0}{h(a)} \right) - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x). \end{aligned}$$

The Gâteaux differential (4.8) becomes,

$$\begin{aligned} \Delta S\left(y, h\right) &= \xi F\big|_v + \int_a^v \mathrm{d}x\, h(x) F_y + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, h(x) F_y - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, \left(h(x) F_y - \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x) \right), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x) - h(x) F_y \right), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) - F_y \right) h(x) \end{aligned}$$

$$= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} \right) h(x). \tag{4.10}$$

On a stationary path $\Delta S\,(y,h) = 0$ for all allowed h and the Euler-Lagrange equation equation is satisfied and so the integrand of (4.10) is zero, thus the Gâteaux differential shown in (4.10) reduces to

$$\Delta S\,(y,h) = \xi F|_{x=v} + F_{y'}h(x)|_{x=v} = 0. \tag{4.11}$$

Rewriting (4.5) more succinctly as

$$\xi\,[\tau_x + \tau_y y'(x)] + \tau_y h(x)|_{x=v} = 0, \tag{4.12}$$

and rearranging (4.12) in terms of $h(x)$ evaluated at $x = v$,

$$h(x)\,|_{x=v} = -\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\,|_{x=v}\,. \tag{4.13}$$

Substituting for $h(v)$ from (4.13) into (4.11) gives,

$$\begin{aligned}
\xi F|_{x=v} - F_{y'}\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\Big|_{x=v} &= 0,\\
\left(\xi F - F_{y'}\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\right)\Big|_{x=v} &= 0,\\
\xi\left(F - F_{y'}\frac{(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(-F + F_{y'}\frac{(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(\frac{-F\tau_y + F_{y'}\,(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(-F\tau_y + F_{y'}\,(\tau_x + y'(x)\tau_y)\right)|_{x=v} &= 0,\\
-\left(-F\tau_y + F_{y'}\tau_x + F_{y'}y'(x)\tau_y\right)|_{x=v} &= 0,\\
-\left(\tau_y\,(-F + F_{y'}y'(x)) + F_{y'}\tau_x\right)|_{x=v} &= 0,\\
-\left(\tau_y\,(F_{y'}y'(x) - F) + F_{y'}\tau_x\right)|_{x=v} &= 0.
\end{aligned}$$

Finally,

$$\tau_y\,(F_{y'}y'(x) - F)\,|_{x=v} + F_{y'}\tau_x|_{x=v} = 0. \tag{4.14}$$

Equation (4.14) shows that the transversality condition has been satisfied, as required.

(c)

Q 5.

(a)

(b)

(c)

(d)