Faculty of Science, Technology, Engineering and Mathematics
M820 Calculus of variations and advanced calculus

M820

TMA 04

Covers Chapters 13 and 14,
and practice in examination-style questions

TMA 04 is a formative assignment that does not count towards your final grade. However, in order to pass the module, you are required to submit at least three TMAs and score at least 30% on at least three of the TMAs submitted.

The substitution rule does not apply to this module.

To be sure of passing this module, you need to achieve a score of at least 40% in the examination and score at least 30% on three out of four TMAs. The final rank score will be completely determined by your overall exam score (OES).

You are strongly encouraged to submit your assignment online using the electronic TMA service. Please read the instructions under the 'Assessment' tab of the module website before starting your assignment.

The assignment cut-off date can be found on the module website.

There are 100 marks available for this assignment.

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3.1

TMA 04

The questions are of varying difficulty and length: the marks allocated to a question provide some indication of its difficulty. Questions or parts of questions marked with * are more challenging.

Question 1 – 24 marks

Consider the functional

$$S[y] = \alpha\, y(1)^2 + \int_0^1 dx\, \beta y'^2, \quad y(0) = 0,$$

with a natural boundary condition at $x = 1$ and subject to the constraint

$$C[y] = \gamma\, y(1)^2 + \int_0^1 dx\, w(x)\, y^2 = 1,$$

where α, β and γ are nonzero constants.

(a) Show that the stationary paths of this system satisfy the Euler–Lagrange equation

$$\beta\, \frac{d^2y}{dx^2} + \lambda\, w(x)\, y = 0, \quad y(0) = 0,\ (\alpha - \gamma\lambda)\, y(1) + \beta\, y'(1) = 0,$$

where λ is a Lagrange multiplier. [*9*]

(b) Let $w(x) = 1$ and $\alpha = \beta = \gamma = 1$. Find the nontrivial stationary paths, stating clearly the eigenfunctions y (normalised so that $C[y] = 1$) and the values of the associated Lagrange multiplier. [*15*]

Question 2 – 10 marks

Let k be a constant positive integer. Show that the nontrivial solutions of the equation

$$\frac{d^2y}{dx^2} + y(1 + x^k) = 0$$

have infinitely many zeros in $(0, \infty)$. Show also that the separation between adjacent zeros tends to zero as $x \to \infty$. [*10*]

Question 3 – 22 marks

(a) Show that the equation and boundary conditions

$$\frac{d}{dx}\left(x^2\,\frac{dy}{dx}\right) + \lambda xy = 0, \quad y(1) = 0,\ y'(2) = 0, \tag{1}$$

form a regular Sturm–Liouville system.

Show further that the system can be written as a constrained variational problem with functional

$$S[y] = \int_1^2 dx\, x^2 y'^2, \quad y(1) = 0,$$

and constraint

$$C[y] = \int_1^2 dx\, xy^2 = 1.$$

[9]

(b) Assume that the eigenvalues λ_k and eigenfunctions y_k, $k = 1, 2, \ldots,$ exist. Working from equation (1), derive the relationship

$$\lambda_k = \int_1^2 dx\, x^2 y_k'^2, \quad k = 1, 2, \ldots.$$

[4]

(c) Using the trial function $z = A\sin(\pi(x-1)/2)$, show that the smallest eigenvalue, λ_1, satisfies the inequality

$$\lambda_1 \le \frac{(7\pi^2 - 18)\pi^2}{6(4 + 3\pi^2)}.$$

Justify your answer briefly. [9]

Question 4 – 22 marks

This is a past examination question.

Consider the functional

$$S[y] = \int_a^v dx\, F(x, y, y'), \quad y(a) = A,$$

where the right-hand end point v is determined so that the path lies on the curve $\tau(x, y) = 0$, i.e. $\tau(v, y(v)) = 0$, and a and A are constants. Consider a perturbed path $y_\epsilon = y + \epsilon h$, where h is an admissible perturbation, i.e. satisfying $h(a) = 0$ and $\tau(v_\epsilon, y_\epsilon(v_\epsilon)) = 0$, where $v_\epsilon = v + \epsilon\xi + O(\epsilon^2)$ is the perturbed value of v.

(a) Show that to first order in ϵ,

$$y_\epsilon(v_\epsilon) = y(v) + \epsilon(y'(v)\,\xi + h(v)) + O(\epsilon^2),$$

and hence that at $(x, y) = (v, y(v))$,

$$\xi(\tau_x + y'(v)\,\tau_y) + h(v)\,\tau_y = 0.$$ [5]

(b) Using the formula for the Gâteaux differential

$$\Delta S[y, h] = \xi F|_v + \int_a^v dx\, \left(hF_y + h'F_{y'}\right),$$

show, by integrating by parts or otherwise, that a stationary path must satisfy the transversality condition

$$\tau_x F_{y'} + \tau_y(y'F_{y'} - F) = 0$$

at $(x, y) = (v, y(v))$. [8]

(c) Consider the case when

$$S[y] = \int_0^v dx\, \frac{\sqrt{1 + y'^2}}{y}, \quad y(0) = \delta,$$

where $y(v) > 0$, $\delta > 0$ and the right-hand end point $(v, y(v))$ lies on the line $\alpha y + \beta x + \gamma = 0$, where α, β, γ are constants with $\beta \neq 0$.

Show that the first-integral may be written as

$$y\sqrt{1 + y'^2} = c$$

for some constant $c > 0$, and that the solutions of the first-integral are circles centred at $(-c_\delta, 0)$ with radius c, i.e.

$$y^2 + (x + c_\delta)^2 = c^2,$$

where $c_\delta^2 = c^2 - \delta^2$.

* Using the transversality condition and by differentiating implicitly the equation $y^2 + (x + c_\delta)^2 = c^2$, show that $y' = \alpha/\beta$ and $c_\delta = \gamma/\beta$.

Finally, show that in the limit as $\delta \to 0$, the stationary path becomes

$$\beta^2 y^2 + (\beta x + \gamma)^2 = \gamma^2.$$ [9]

Question 5 – 22 marks

This is a past examination question.

Throughout this question, the 'dot' notation indicates 'differentiation with respect to t', so that, for example, $\dot{x} = dx/dt$.

Let $b > a$, and consider the functional

$$S[x] = \int_a^b dt\, L(t, x, \dot{x}),$$

where the function L is called the Lagrangian, and $x(t)$ is at least twice continuously differentiable.

The *conjugate momentum* p is defined by

$$p = \frac{\partial L}{\partial \dot{x}}. \tag{2}$$

(a) Show that the Euler–Lagrange equation for S may be written as

$$\dot{p} = \frac{\partial L}{\partial x}.$$ [2]

(b) Show that

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x}.$$ [2]

The *Hamiltonian* $H = H(t, x, p)$ is defined by $H(t, x, p) = p\dot{x} - L(t, x, \dot{x})$, where (implicitly) $\dot{x}$ is eliminated using equation (2) to give a function of t, x and p.

(c) By using the result of part (b), or otherwise, show that for a stationary path of S,

$$\frac{\partial L}{\partial t} = -\dot{H}.$$ [5]

Consider a one-parameter family of transformations

$$\overline{t} = \overline{t}(t, x, \dot{x}; \delta),$$
$$\overline{x} = \overline{x}(t, x, \dot{x}; \delta),$$

with S invariant under the family. Let

$$\overline{t} = t + \tau\delta + O(\delta^2),$$
$$\overline{x} = x + \xi\delta + O(\delta^2),$$

where $\tau = \tau(t, x, \dot{x})$ and $\xi = \xi(t, x, \dot{x})$.

The *Rund–Trautman identity* (which you are not required to prove) is

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0. \tag{3}$$

(d) From the Rund–Trautman identity (3), show that

$$(\xi - \dot{x}\tau)\left[\dot{p} - \frac{\partial L}{\partial x}\right] = \frac{d}{dt}\left[p\xi - H\tau\right].$$

Deduce that along a stationary path $x(t)$,

$$p\xi - H\tau = \text{constant}.$$ [8]

Consider a particle of constant mass m moving along the x-axis in a potential $V(x)$. The Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - V(x)$, and the path of the particle from $t = a$ to $t = b$ is a stationary path of S.

(e) Calculate the conjugate momentum p and hence the Hamiltonian $H(t, x, p)$.

The Lagrangian L (and hence S) is invariant under translation of the time coordinate t. From the result of part (d), show that the Hamiltonian H is constant along the path of the particle. [5]