

Q 1.

(a) The functional is

$$S[y] = \alpha y(1)^2 + \int_0^1 dx \beta y'^2, \quad y(0) = 0,$$

with natural boundary condition at $x = 1$ and subject to the constraint

$$C[y] = \gamma y(1)^2 + \int_0^1 dx w(x) y^2 = 1,$$

where α , β and γ are non-zero constants.

To show that the stationary paths of this system satisfy the Euler-Lagrange equation

$$\beta \frac{d^2 y}{dx^2} + \lambda w(x) y = 0, \quad y(0) = 0, \quad (\alpha - \gamma \lambda) y(1) + \beta y'(1) = 0,$$

where λ is a Lagrange multiplier, consider the following.

The auxiliary functional is

$$\begin{aligned} \bar{S}[y] &= \alpha y(1)^2 + \int_0^1 dx \beta y'^2 - \lambda \gamma y(1)^2 - \lambda \int_0^1 dx w(x) y^2, \\ &= \alpha y(1)^2 - \lambda \gamma y(1)^2 + \int_0^1 dx (\beta y'^2 - \lambda w(x) y^2), \\ &= (\alpha - \lambda \gamma) y(1)^2 + \int_0^1 dx (\beta y'^2 - \lambda w(x) y^2). \end{aligned}$$

The Gâteaux differential $\Delta \bar{S}[y, h]$ of the functional $\bar{S}[y]$ is given by the expression

See HB p16.

$$\Delta \bar{S}[y, h] = \left. \frac{d}{d\epsilon} S[y + \epsilon H] \right|_{\epsilon=0}.$$

Rewriting the expression for $\bar{S}[y]$ and replacing each occurrence of y with $y + \epsilon h$ gives,

$$\begin{aligned} \Delta \bar{S}[y + \epsilon h] &= (\alpha - \lambda \gamma) (y(1) + \epsilon h(1))^2 + \\ &\quad \int_0^1 dx (\beta (y + \epsilon h)^2 - \lambda w(x) (y + \epsilon h)^2), \end{aligned}$$

$$\begin{aligned} \Delta \bar{S}[y, h] &= \left. \frac{d}{d\epsilon} \left((\alpha - \lambda \gamma) (y(1) + \epsilon h(1))^2 \right) \right|_{\epsilon=0} + \\ &\quad \left. \frac{d}{d\epsilon} \left(\int_0^1 dx (\beta (y + \epsilon h)^2 - \lambda w(x) (y + \epsilon h)^2) \right) \right|_{\epsilon=0}. \end{aligned}$$

which, after carrying out the differentiation becomes,

$$\begin{aligned}\Delta\bar{S}[y, h] &= \left(2(\alpha - \lambda\gamma)(y(1) + \epsilon h(1))h(1) \right) \Big|_{\epsilon=0} + \\ &\quad \left(\int_0^1 dx \, 2(\beta(y + \epsilon h)'h' - 2\lambda w(x)(y + \epsilon h)h) \right) \Big|_{\epsilon=0}, \\ &= 2(\alpha - \lambda\gamma)y(1)h(1) + 2 \int_0^1 dx \, \beta y' h' - 2 \int_0^1 dx \, \lambda w(x) y h.\end{aligned}$$

Integrating the left-most integral by parts,

$$2\beta \int_0^1 dx \, y' h' = 2\beta \left(\left[y' h \right]_0^1 - \int_0^1 dx \, y'' h \right),$$

and from Table 1 $h(0) = 0$ so,

Boundary condition 1
$y(1) = 0$
$y(1) + \epsilon h(1) = 0$
$0 + \epsilon h(1) = 0$
$h(1) = 0$

Table 1: Determination of $h(1)$.

$$2\beta \int_0^1 dx \, y' h' = 2\beta y'(1)h(1) - 2\beta \int_0^1 dx \, y'' h.$$

Thus,

$$\Delta\bar{S}[y, h] = 2(\alpha - \lambda\gamma)y(1)h(1) + 2\beta y'(1)h(1) - 2 \int_0^1 dx \, (\beta y'' + \lambda w(x)y) h. \quad (1.1)$$

The Euler-Lagrange equation can be found from the above expression for the Gâteaux differential; by definition it is required that the Gâteaux differential be equal to zero. So by setting the expression shown in (1.1) to zero and dividing through by 2 gives the following for a stationary path

$$(\alpha - \lambda\gamma)y(1)h(1) + \beta y'(1)h(1) - \int_0^1 dx \, (\beta y'' + \lambda w(x)y) h = 0.$$

Then, by the Fundamental Lemma of the Calculus of Variations (meaning that because the result has to be true for all admissible paths the term that h multiplies must be equal to zero) and setting $h(1) = 0$, then the Euler-Lagrange equation is given by

$$\beta \frac{d^2 y}{dx^2} + \lambda w(x)y = 0, \quad y(0) = 0, \quad (\alpha - \gamma\lambda)y(1) + \beta y'(1) = 0. \quad (1.2)$$

λ is the Lagrange multiplier.

Equation (1.1) is the Gâteaux differential and (1.2) is the Euler-Lagrange equation (together with the boundary conditions) that the stationary paths must satisfy.

(b) Let $w(x) = 1$ and $\alpha = \beta = \gamma = 1$, so that (1.2) becomes

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(0) = 0, \quad (1 - \lambda)y(1) + y'(1) = 0. \quad (1.3)$$

Then to find the non-trivial stationary paths, the eigenfunctions of y (normalised so that $C[y] = 1$) and the values of the Lagrange multipliers, consider the following.

There are three cases to consider regarding λ , namely $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. So, considering each case in turn as follows.

if $\lambda = 0$ set $\lambda = 0$ in (1.3):

$$\begin{aligned} \frac{d^2y}{dx^2} &= 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0. \\ \frac{dy}{dx} &= A, \quad \text{where } A \text{ is an arbitrary constant.} \\ y &= Ax + B, \quad \text{where } B \text{ is also an arbitrary constant.} \end{aligned}$$

Applying the boundary conditions to y' and y gives $A = 0$ and $B = 0$ and therefore $y(x) = 0$. This is a trivial solution.

If $\lambda < 0$ set $\lambda = -\mu^2$ ($\mu > 0$) in (1.3):

$$\frac{d^2y}{dx^2} - \mu^2 y = 0, \quad y(0) = 0, \quad (1 - \mu^2)y(1) + y'(1) = 0.$$

The auxiliary equation is

$$\begin{aligned} r^2 - \mu^2 &= 0, \quad \text{with } r_1 = -\mu, \quad r_2 = \mu. \\ y(x) &= Ae^{-\mu x} + Be^{\mu x}. \end{aligned}$$

Applying the first boundary conditions to $y(x)$ with $x = 0$ and $y(0) = 0$ gives

$$0 = A + B \quad \text{therefore} \quad A = -B,$$

and so,

$$y(x) = -Be^{-\mu x} + Be^{\mu x}.$$

Differentiating y ,

$$y'(x) = B\mu e^{-\mu x} + B\mu e^{\mu x},$$

Applying the second boundary condition to determine B ,

$$\begin{aligned}(1 - \mu^2)y(1) + y'(1) &= (1 - \mu^2)(-Be^{-\mu} + Be^{\mu}) + B\mu e^{-\mu} + B\mu e^{\mu} = 0, \\ &= B\{(1 - \mu^2)(-e^{-\mu} + e^{\mu}) + \mu e^{-\mu} + \mu e^{\mu}\} = 0.\end{aligned}$$

By assumption $\mu \neq 0$, so the expression immediately above can only be satisfied if $B = 0$ and therefore $y(x) = 0$. This also is a trivial solution.

If $\lambda > 0$ set $\lambda = \mu^2$ ($\mu > 0$) in (1.3):

$$\frac{d^2y}{dx^2} + \mu^2 y = 0, \quad y(0) = 0, \quad (1 - \mu^2)y(1) + y'(1) = 0.$$

The auxiliary equation is

$$r^2 + \mu^2 = 0, \quad \text{with} \quad r_1 = -i\mu, \quad r_2 = i\mu.$$

and the general solution is given by

$$y(x) = A \sin \mu x + B \cos \mu x.$$

Applying the first boundary condition $y(0) = 0$, gives,

$$y(0) = A \sin 0 + B \cos 0, \quad \text{and therefore, } B = 0.$$

Thus, the updated solution is,

$$y(x) = A \sin \mu x.$$

Differentiating $y(x)$ gives,

$$y'(x) = A\mu \cos \mu x.$$

Applying the second boundary conditions $(1 - \mu^2)y(1) + y'(1) = 0$:

$$\begin{aligned}(1 - \mu^2)A \sin \mu + A \mu \cos \mu &= 0, \\ \text{and factoring out } A \text{ and assuming } A \neq 0, \text{ gives,} \\ (1 - \mu^2) \sin \mu + \mu \cos \mu &= 0,\end{aligned}\tag{1.4}$$

$$\begin{aligned}\frac{(1 - \mu^2)}{\mu} &= -\frac{\cos \mu}{\sin \mu}, \\ \frac{(\mu^2 - 1)}{\mu} &= \cot \mu.\end{aligned}\tag{1.5}$$

To solve (1.5) for μ a graph of both sides of (1.5) can be used as shown in Figure 1. From the graphs shown in Figure 1 $\mu_1 \approx 1.208$, $\mu_2 \approx 3.448$, $\mu_3 \approx 6.441$, $\mu_4 \approx 9.530$, $\mu_5 \approx 12.646$ and it is seen that as n increases μ_n approaches $(n - 1)\pi$, $n = 1, 2, \dots$. Thus,

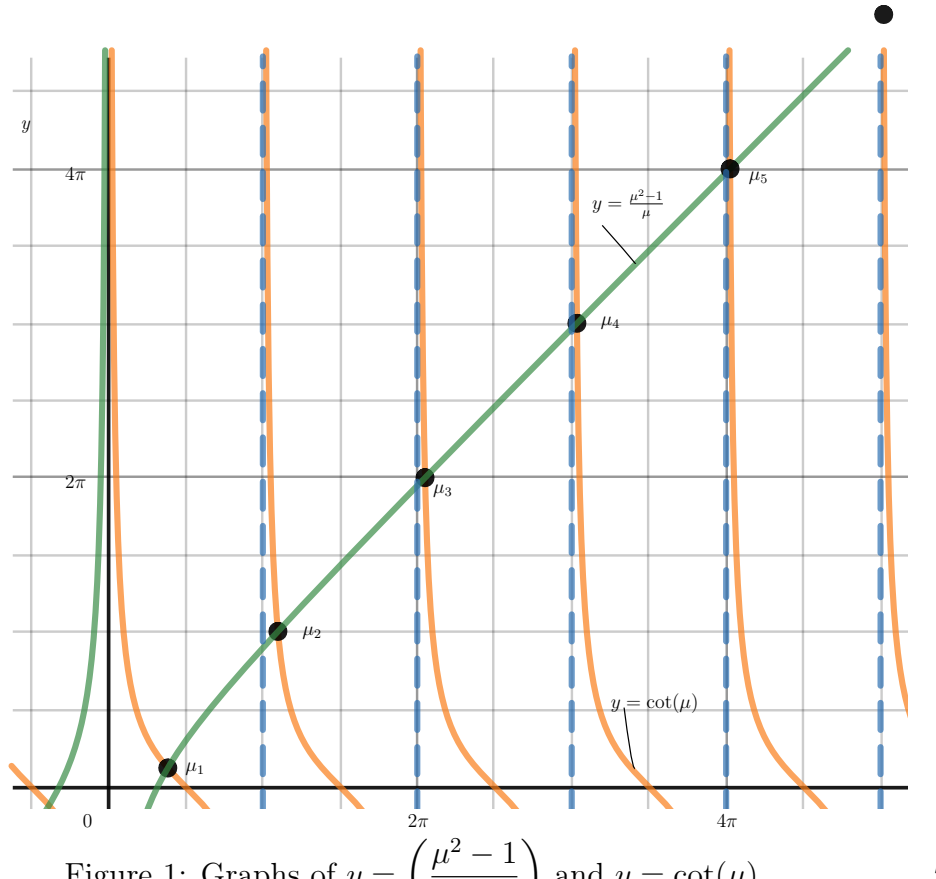


Figure 1: Graphs of $y = \left(\frac{\mu^2 - 1}{\mu}\right)$ and $y = \cot(\mu)$.

the eigenvalues λ_n approach $(n - 1)^2\pi^2$ as n increases. Hence, the eigenfunctions corresponding to the eigenvalues λ_n are,

$$y_n(x) = A_n \sin\left(\sqrt{\lambda_n}x\right), \quad n = 1, 2, \dots \quad (1.6)$$

To normalise the eigenfunctions (1.6) consider the following.

The inner product (with unit weight function w) where $(y, y)_w = 1$ will be used to normalise the expression for y given in (1.6),

See HB p31.

$$(y, y)_w = \int_a^b dx y(x)^2 = 1, \quad (1.7)$$

by determining the value of A_n as follows.

After substituting into (1.7) for $y_n(x)$ from (1.6) the following integral is obtained.

$$\int_0^1 dx A_n^2 \sin^2\left(\sqrt{\lambda_n}x\right) = 1,$$

$$A_n^2 \int_0^1 dx \sin^2 \left(\sqrt{\lambda_n} x \right) = 1.$$

Using the trig identity $\sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha)$ gives,

See HB p38 for trig identities.

$$\frac{A_n^2}{2} \int_0^1 dx \left(1 - \cos \left(2\sqrt{\lambda_n} x \right) \right) = 1.$$

$$\frac{A_n^2}{2} \left[x - \frac{1}{2\sqrt{\lambda_n}} \sin \left(2\sqrt{\lambda_n} x \right) \right]_0^1 = 1.$$

$$\frac{A_n^2}{2} \left(1 - \frac{1}{2\sqrt{\lambda_n}} \sin \left(2\sqrt{\lambda_n} \right) \right) = 1.$$

$$\frac{A_n^2}{4\sqrt{\lambda_n}} \left(2\sqrt{\lambda_n} - \sin \left(2\sqrt{\lambda_n} \right) \right) = 1.$$

Using the trig identity $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ then,

$$\frac{A_n^2}{4\sqrt{\lambda_n}} \left(2\sqrt{\lambda_n} - 2 \sin \left(\sqrt{\lambda_n} \right) \cos \left(\sqrt{\lambda_n} \right) \right) = 1.$$

From (1.4)

$$- \sin \left(\sqrt{\lambda_n} \right) = \frac{\sqrt{\lambda_n}}{1 - \lambda_n} \cos \left(\sqrt{\lambda_n} \right),$$

and so,

$$\frac{A_n^2}{4\sqrt{\lambda_n}} \left(2\sqrt{\lambda_n} + \frac{2\sqrt{\lambda_n}}{1 - \lambda_n} \cos \left(\sqrt{\lambda_n} \right) \cos \left(\sqrt{\lambda_n} \right) \right) = 1.$$

$$\frac{A_n^2}{2} \left(1 + \frac{1}{1 - \lambda_n} \cos^2 \left(\sqrt{\lambda_n} \right) \right) = 1.$$

$$\frac{A_n^2}{2(1 - \lambda_n)} \left((1 - \lambda_n) + \cos^2 \left(\sqrt{\lambda_n} \right) \right) = 1.$$

$$A_n^2 = \frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2 \left(\sqrt{\lambda_n} \right)}.$$

Therefore,

$$A_n = \sqrt{\frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2 \left(\sqrt{\lambda_n} \right)}},$$

and the normalised eigenfunctions are given by,

$$y_n(x) = \sqrt{\frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2 \left(\sqrt{\lambda_n} \right)}} \sin \left(\sqrt{\lambda_n} x \right).$$

Q 2.

Q 3.

- (a) To show that the following equation and boundary conditions:

$$\frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) + \lambda xy = 0, \quad y(1) = 0, y'(2) = 0 \quad (3.1)$$

forms a regular Sturm-Liouville system, consider the following.

A Sturm-Liouville system is a linear, second-order homogeneous differential equation of the form:

see HB p28.

$$\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + (q(x) + \lambda w(x)) y = 0, \quad (3.2)$$

which is defined on a finite interval of the real axis $a < x < b$ and satisfies the following three conditions:

1. the functions $p(x)$, $q(x)$ and $w(x)$ are real and continuous for $a < x < b$;
2. $p(x)$ and $w(x)$ are strictly positive for $a < x < b$;
3. $p'(x)$ exists and is continuous for $a \leq x \leq b$,

together with the boundary conditions.

Comparing equations (3.1) and (3.2) it is seen that $q(x) = 0$ with $p(x) = x^2$ and $w(x) = x$. As $p(x)$ and $w(x)$:

1. are real and continuous for $1 < x < 2$;
2. are strictly positive for $1 < x < 2$; and
3. $p'(x)$ exists ($p'(x) = 2x$) and is continuous for $1 \leq x \leq 2$,

and the two boundary conditions are given as $y(1) = 0$ and $y'(2) = 0$, then the system is a regular Sturm-Liouville system.

It can be shown that the system can be written as a constrained variational problem with functional

$$S[y] = \int_1^2 dx x^2 y'^2, \quad y(1) = 0, \quad (3.3)$$

and constraint

$$C[y] = \int_1^2 dx xy^2 = 1, \quad (3.4)$$

as follows.

For the given constrained variational problem

$$\bar{F} = x^2 y'^2 - \lambda x y^2,$$

λ is the Lagrange multiplier.

and

$$\bar{F}_{y'} = \frac{\partial \bar{F}}{\partial y'} = 2x^2 y' \quad \text{and} \quad \bar{F}_y = \frac{\partial \bar{F}}{\partial y} = -2\lambda x y.$$

The Euler-Lagrange equation is then,

$$\begin{aligned} \frac{d}{dx} \left(\frac{\partial \bar{F}}{\partial y'} \right) - \frac{\partial \bar{F}}{\partial y} &= 0, \\ \frac{d}{dx} (2x^2 y') - (-2\lambda x y) &= 0, \\ \frac{d}{dx} \left(2x^2 \frac{dy}{dx} \right) + 2\lambda x y &= 0, \\ \frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) + \lambda x y &= 0, \quad y(1) = 0, y'(2) = 0. \end{aligned} \quad (3.5)$$

Equations (3.1) and (3.5) are identical showing that system (3.1) can be written as a constrained variational problem.

- (b) It is assumed that the eigenvalues λ_k and the eigenfunctions $y_k, k = 1, 2, \dots$ exist. By working from (3.1), the following relationship will be derived.

$$\lambda_k = \int_1^2 dx \, x^2 y_k'^2, \quad k = 1, 2, \dots$$

Recall that (3.1) is given as:

$$\frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) + \lambda x y = 0, \quad y(1) = 0, y'(2) = 0$$

and compare (3.1) with (3.2) repeated below,

$$\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + (q(x) + \lambda w(x)) y = 0.$$

As was determined in part (a) $q(x) = 0$ with $p(x) = x^2$ and $w(x) = x$ and the system is defined on the interval $(1, 2)$.

Now the functional is of the form:

$$S[y] = -\alpha p(a)y(a)^2 + \beta p(b)y(b)^2 + \int_a^b dx \, (py' - qy^2), \quad \text{See HB p29 (SLF).}$$

which, becomes the following after substituting for the appropriate terms found above,

$$S[y] = -\alpha p(1)y(1)^2 + \beta p(2)y(2)^2 + \int_1^2 dx \, \left(x^2 y' - \cancel{x^0} y^2 \right), \quad \begin{aligned} p(x) &= x^2, \\ p(1) &= 1, p(2) = 4 \end{aligned}$$

$$S[y] = -\alpha y(1)^2 + \beta 4y(2)^2 + \int_1^2 dx x^2 y'.$$

The natural boundary conditions of a Sturm-Liouville system are of the form:

$$\alpha y(1) + y'(1) = 0 \quad \text{and} \quad \beta y(2) + y'(2) = 0.$$

The given boundary conditions are $y(1) = 0$ and $y'(2) = 0$, so this means that $\alpha = 1$ and $y'(1) = 0$ and $\beta y(2) = 0$. Thus,

$$S[y] = \cancel{-\alpha y(1)^2}^0 + \cancel{\beta 4y(2)^2}^0 + \int_1^2 dx x^2 y',$$

$$S[y] = \int_1^2 dx x^2 y'.$$

Hence, from the general theory of Sturm-Liouville Systems,

$$\lambda_k = S[y_k] = \int_1^2 dx x^2 y'_k,$$

as required.

- (c) Given the function $z = A \sin(\pi(x-1)/2)$, it will be that the smallest eigenvalue, λ_1 , satisfies the inequality

$$\lambda_1 \leq \frac{(7\pi^2 - 18)\pi^2}{6(4 + 3\pi^2)},$$

as follows.

Substituting $z = A \sin(\pi(x-1)/2)$ into the constraint (3.4) gives

$$\begin{aligned} 1 &= A^2 \int_1^2 dx x \sin^2\left(\frac{\pi(x-1)}{2}\right) = A^2 \int_1^2 dx x \cos^2\left(\frac{1}{2}\pi x\right), \\ &= A^2 \int_1^2 dx x \frac{1}{2}(\cos(\pi x) + 1) = \frac{A^2}{2} \int_1^2 dx x \cos(\pi x) + x, \\ &= \frac{A^2}{2} \int_1^2 dx x \cos(\pi x) + \frac{A^2}{2} \int_1^2 dx x, \\ &= \frac{A^2}{2\pi} \left[x \sin(\pi x) \right]_1^2 - \frac{A^2}{2\pi} \int_1^2 dx \sin(\pi x) + \frac{A^2}{2} \int_1^2 dx x, \end{aligned}$$

Using the identity for $\sin(\alpha \pm \beta)$.

Using the identity of $\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$.

Integrating the first integral by parts.

$$\begin{aligned}
&= \frac{A^2}{2\pi} \left(\overset{0}{\cancel{2 \sin(2\pi)} - \sin(\pi)} \right) - \frac{A^2}{2\pi} \int_1^2 dx \sin(\pi x) + \frac{A^2}{2} \int_1^2 dx x, \\
&= -\frac{A^2}{2\pi} \int_1^2 dx \sin(\pi x) + \frac{A^2}{2} \int_1^2 dx x, \\
&= -\frac{A^2}{2\pi} \left[-\frac{1}{\pi} \cos(\pi x) \right]_1^2 + \frac{A^2}{2} \left[\frac{x^2}{2} \right]_1^2, \\
&= \frac{A^2}{2\pi} \left[\frac{1}{\pi} \cos(\pi x) \right]_1^2 + \frac{A^2}{2} \left[\frac{x^2}{2} \right]_1^2, \\
&= \frac{A^2}{2\pi^2} \left[\overset{1}{\cancel{\cos(2\pi)} - \cos(\pi)} \overset{-1}{} \right] + \frac{A^2}{2} \left[\frac{2^2}{2} - \frac{1^2}{2} \right], \\
&= \frac{A^2}{2\pi^2} (2) + \frac{A^2}{2} \left(\frac{3}{2} \right), \\
&= \frac{A^2}{2} \left(\frac{2}{\pi^2} + \frac{3}{2} \right), \\
\therefore 1 &= A^2 \left(\frac{1}{\pi^2} + \frac{3}{4} \right). \tag{3.6}
\end{aligned}$$

Now,

$$z = A \sin \left(\frac{\pi(x-1)}{2} \right)$$

and

$$\lambda_1 \leq S[z] = \int_1^2 dx x^2 z'^2. \tag{3.7}$$

Differentiating z ,

$$\begin{aligned}
z' &= \frac{d}{dx} \left(A \sin \left(\frac{\pi(x-1)}{2} \right) \right), && \text{Making use of the chain rule.} \\
&= A \frac{d}{dx} \left(\frac{\pi(x-1)}{2} \right) \cos \left(\frac{\pi(x-1)}{2} \right), \\
&= A \frac{\pi}{2} \cos \left(\frac{\pi(x-1)}{2} \right), \\
\therefore z' &= A \frac{\pi}{2} \sin \left(\frac{\pi x}{2} \right). && \text{Using the identity for } \cos(\alpha \pm \beta). \text{ HB p38.}
\end{aligned} \tag{3.8}$$

Substituting for z' given by (3.8) into (3.7) gives,

$$\begin{aligned}
 \lambda_1 \leq S[z] &= \int_1^2 dx x^2 \left(A \frac{\pi}{2} \sin \left(\frac{\pi x}{2} \right) \right)^2, \\
 &= \left(\frac{A\pi}{2} \right)^2 \int_1^2 dx x^2 \sin^2 \left(\frac{\pi x}{2} \right), \\
 &= \frac{A^2 \pi^2}{4} \int_1^2 dx x^2 \frac{1}{2} \left(1 - \cos(\pi x) \right), \\
 &= \frac{A^2 \pi^2}{8} \int_1^2 dx x^2 (1 - \cos(\pi x)), \\
 &= \frac{A^2 \pi^2}{8} \int_1^2 dx (x^2 - x^2 \cos(\pi x)), \\
 &= \frac{A^2 \pi^2}{8} \int_1^2 dx x^2 - \frac{A^2 \pi^2}{8} \int_1^2 dx x^2 \cos(\pi x), \\
 &= \frac{A^2 \pi^2}{8} \left[\frac{x^3}{3} \right]_1^2 - \frac{A^2 \pi^2}{8} \int_1^2 dx x^2 \cos(\pi x), \\
 &= \frac{A^2 \pi^2 7}{24} - \frac{A^2 \pi^2}{8} \int_1^2 dx x^2 \cos(\pi x), \\
 &= \frac{A^2 \pi^2 7}{24} - \frac{A^2 \pi^2}{8} \left(\left[\frac{x^2}{\pi} \sin(\pi x) \right]_1^2 - \frac{2}{\pi} \int_1^2 dx x \sin(\pi x) \right), \\
 &= \frac{A^2 \pi^2 7}{24} + \frac{A^2 \pi}{4} \int_1^2 dx x \sin(\pi x), \\
 &= \frac{A^2 \pi^2 7}{24} + \frac{A^2 \pi}{4} \left(\left[-\frac{x}{\pi} \cos(\pi x) \right]_1^2 - \int_1^2 dx \cos(\pi x) \right), \\
 &= \frac{A^2 \pi^2 7}{24} + \frac{A^2 \pi}{4} \left(-\frac{3}{\pi} + \frac{1}{\pi} \left[\sin(\pi x) \right]_1^2 \right), \\
 &= \frac{A^2 \pi^2 7}{24} + \frac{A^2 \pi}{4} \left(-\frac{3}{\pi} \right), \\
 &= \frac{A^2 \pi^2 7}{24} - \frac{A^2 3}{4}, \\
 &= A^2 \left(\frac{7\pi^2}{24} - \frac{3}{4} \right), \\
 &= \frac{A^2}{4} \left(\frac{7\pi^2 - 18}{6} \right). \tag{3.9}
 \end{aligned}$$

Using the identity for $\sin^2(\alpha)$. HB p38.

From (3.6)

$$A^2 = \frac{4\pi^2}{4 + 3\pi^2},$$

and substituting for A^2 in (3.9) gives,

$$\lambda_1 \leq \frac{4\pi^2}{4(4+3\pi^2)} \left(\frac{7\pi^2-18}{6} \right),$$

$$\lambda_1 \leq \frac{\pi^2}{(4+3\pi^2)} \left(\frac{7\pi^2-18}{6} \right),$$

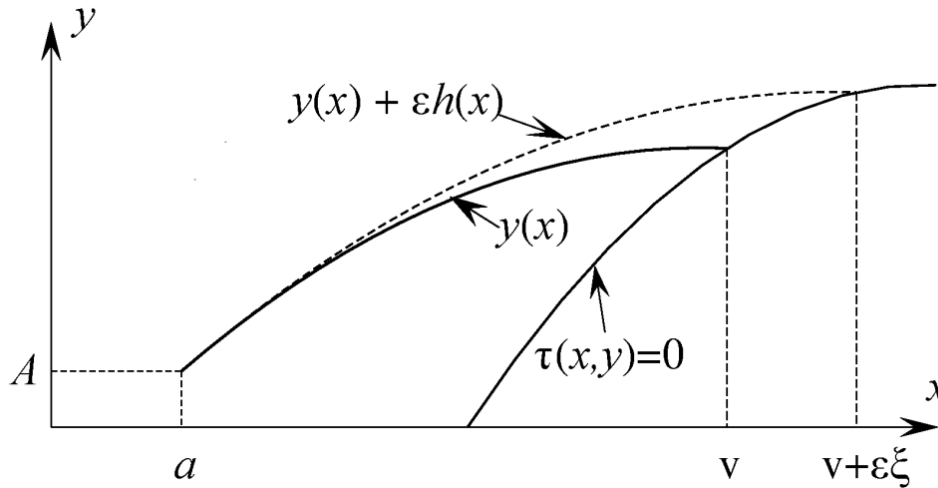
Finally,

$$\lambda_1 \leq \frac{(7\pi^2-18)\pi^2}{6(4+3\pi^2)},$$

as required.

Q 4.

(a)



This Figure 10.6
taken from the
module notes p225.

Figure 2: Diagram showing the stationary path (solid line) and a varied path (dashed line) for a problem in which the left-hand end is fixed, but the other end is free to move along the line defined by $\tau(x, y) = 0$.

Given the perturbed path

$$y_\epsilon(x) = y(x) + \epsilon h(x), \quad (4.1)$$

the Taylor series to the first-order of (4.1) at point $x = v$ (see figure 2) is given in (4.2).

The point $x = v$ is known as the point of expansion. HB p8.

$$y_\epsilon(x) = (y(v) + \epsilon h(v)) + (y'(v) + \epsilon h'(v))(x - v) + \mathcal{O}((x - v)^2). \quad (4.2)$$

Now, determining the value of (4.2) at $v_\epsilon = v + \epsilon \xi + \mathcal{O}(\epsilon^2)$ where v_ϵ is the perturbed value of v :

$$\begin{aligned} y_\epsilon(v_\epsilon) &= y(v) + \epsilon h(v) \\ &\quad + (\cancel{v} + \epsilon \xi + \mathcal{O}(\epsilon^2) \cancel{\nearrow v}) (y'(v) + \epsilon h'(v)) \\ &\quad + \mathcal{O}\left((\cancel{v} + \epsilon \xi + \mathcal{O}(\epsilon^2) \cancel{\nearrow v})^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) &= y(v) + \epsilon h(v) \\ &\quad + (\epsilon \xi + \mathcal{O}(\epsilon^2)) (y'(v) + \epsilon h'(v)) \\ &\quad + \mathcal{O}\left((\epsilon \xi + \mathcal{O}(\epsilon^2))^2\right), \end{aligned}$$

$$\begin{aligned}
y_\epsilon(v_\epsilon) &= y(v) + \epsilon h(v) \\
&+ \epsilon \xi (y'(v) + \epsilon h'(v)) \\
&+ \mathcal{O}(\epsilon^2) (y'(v) + \epsilon h'(v)) \\
&+ \mathcal{O}\left((\epsilon \xi + \mathcal{O}(\epsilon^2))^2\right),
\end{aligned}$$

$$\begin{aligned}
y_\epsilon(v_\epsilon) &= y(v) + \epsilon (h(v) + \xi y'(v)) \\
&+ \underbrace{\epsilon^2 \xi h'(v) + \mathcal{O}(\epsilon^2) (y'(v) + \epsilon h'(v)) + \mathcal{O}\left((\epsilon \xi + \mathcal{O}(\epsilon^2))^2\right)}_{\text{These are all second-order terms in } \epsilon}.
\end{aligned}$$

Thus,

$$y_\epsilon(v_\epsilon) = y(v) + \epsilon (h(v) + \xi y'(v)) + \mathcal{O}(\epsilon^2), \quad (4.3)$$

as required.

To show that at $(x, y) = (v, y(v))$,

$$\xi (\tau_x + y'(v)\tau_y) + h(v)\tau_y = 0$$

the 2D Taylor expansion to the first-order of $\tau(x, y)$ at point $x = a, y = b$ is required, namely,

$$\tau(x, y) = \tau(a, b) + \tau_x(a, b) [x - a] + \tau_y(a, b) [y - b]. \quad (4.4)$$

Evaluating (4.4) with $x = v_\epsilon = v + \epsilon \xi$ and $y = y_\epsilon(v_\epsilon) = y(v) + \epsilon (y'(v)\xi + h(v))$ gives,

$$\begin{aligned}
\tau(v_\epsilon, y_\epsilon(v_\epsilon)) &= \tau(v + \epsilon \xi, y(v) + \epsilon (y'(v)\xi + h(v))) \\
&= \tau(v, y(v)) + \tau_x(v, y(v)) [v_\epsilon - v] + \tau_y(v, y(v)) [y_\epsilon(v_\epsilon) - y(v)], \\
&= \tau_x(v, y(v)) [v_\epsilon - v] + \tau_y(v, y(v)) [y_\epsilon(v_\epsilon) - y(v)], \\
&= \tau_x(v, y(v)) [\cancel{v} + \epsilon \xi - \cancel{v}] + \tau_y(v, y(v)) [\cancel{y(v)} + \epsilon (y'(v)\xi + h(v)) - \cancel{y(v)}], \\
&= \tau_x(v, y(v)) \epsilon \xi + \tau_y(v, y(v)) \epsilon (y'(v)\xi + h(v)), \\
&= \epsilon [\tau_x(v, y(v)) \xi + \tau_y(v, y(v)) (y'(v)\xi + h(v))],
\end{aligned}$$

Recall that $\tau(v_\epsilon, y_\epsilon(v_\epsilon)) = 0$ and therefore,

$$\begin{aligned}
\epsilon [\tau_x(v, y(v)) \xi + \tau_y(v, y(v)) (y'(v)\xi + h(v))] &= 0, \\
\tau_x(v, y(v)) \xi + \tau_y(v, y(v)) (y'(v)\xi + h(v)) &= 0,
\end{aligned}$$

$$\xi [\tau_x(v, y(v)) + \tau_y(v, y(v)) y'(v)] + \tau_y(v, y(v)) h(v) = 0, \quad (4.5)$$

as required.

(b) The Gâteaux differential is given as

$$\Delta S(y, h) = \xi F|_v + \int_a^v dx (h F_y + h' F_{y'}), \quad (4.6)$$

and it will be shown, by integrating by parts the integral of (4.6), that a stationary path must satisfy the transversality condition given in (4.7):

$$\tau_x F_{y'} + \tau_y (y' F_{y'} - F) = 0 \quad \text{at} \quad (x, y) = (v, y(v)) \quad (4.7)$$

as follows.

Equation (4.6) can be rewritten as in (4.8),

$$\Delta S(y, h) = \xi F|_v + \int_a^v dx h F_y + \int_a^v dx h' F_{y'}, \quad (4.8)$$

and integrating by parts the right-most integral in (4.8):

$$I = \int_a^v dx h' F_{y'}, \quad (4.9)$$

$$\text{Let } u = F_{y'} \quad \text{then} \quad \frac{du}{dx} = \frac{d}{dx} (F_{y'})$$

$$\text{Let } \frac{dv}{dx} = h'(x) \quad \text{then} \quad v = \int dx h'(x) = h(x).$$

For clarity, here v is not the same as that for the upper limit of integration in (4.9).

For integration by parts:

$$\begin{aligned} I &= \int_a^v dx u \frac{dv}{dx} = [uv]_a^v - \int_a^v dx v \frac{du}{dx}, \\ &= [F_{y'} h(x)]_a^v - \int_a^v dx \frac{d}{dx} (F_{y'}) h(x), \\ &= \left(F_{y'} h(x)|_{x=v} - F_{y'} h(a) \right) - \int_a^v dx \frac{d}{dx} (F_{y'}) h(x), \\ &= F_{y'} h(x)|_{x=v} - \int_a^v dx \frac{d}{dx} (F_{y'}) h(x). \end{aligned}$$

The Gâteaux differential (4.8) becomes,

$$\begin{aligned} \Delta S(y, h) &= \xi F|_v + \int_a^v dx h(x) F_y + F_{y'} h(x)|_{x=v} - \int_a^v dx \frac{d}{dx} (F_{y'}) h(x), \\ &= \xi F|_{x=v} + F_{y'} h(x)|_{x=v} + \int_a^v dx h(x) F_y - \int_a^v dx \frac{d}{dx} (F_{y'}) h(x), \\ &= \xi F|_{x=v} + F_{y'} h(x)|_{x=v} + \int_a^v dx \left(h(x) F_y - \frac{d}{dx} (F_{y'}) h(x) \right), \\ &= \xi F|_{x=v} + F_{y'} h(x)|_{x=v} - \int_a^v dx \left(\frac{d}{dx} (F_{y'}) h(x) - h(x) F_y \right), \\ &= \xi F|_{x=v} + F_{y'} h(x)|_{x=v} - \int_a^v dx \left(\frac{d}{dx} (F_{y'}) - F_y \right) h(x) \\ &= \xi F|_{x=v} + F_{y'} h(x)|_{x=v} - \int_a^v dx \left(\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} \right) h(x). \quad (4.10) \end{aligned}$$

On a stationary path $\Delta S(y, h) = 0$ for all allowed h and the Euler-Lagrange equation is satisfied and so the integrand of (4.10) is zero, thus the Gâteaux differential shown in (4.10) reduces to

$$\Delta S(y, h) = \xi F|_{x=v} + F_{y'} h(x)|_{x=v} = 0. \quad (4.11)$$

Rewriting (4.5) more succinctly as

$$\xi (\tau_x + \tau_y y'(x))|_{x=v} + \tau_y h(x)|_{x=v} = 0, \quad (4.12) \quad \begin{array}{l} \tau_x = \tau_x(v, y(v)), \text{ and} \\ \tau_y = \tau_y(v, y(v)) \end{array}$$

and rearranging (4.12) in terms of $h(x)$ evaluated at $x = v$,

$$h(x)|_{x=v} = -\frac{\xi}{\tau_y} (\tau_x + y'(x)\tau_y)|_{x=v}. \quad (4.13)$$

Substituting for $h(v)$ from (4.13) into (4.11) gives,

$$\begin{aligned} \xi F|_{x=v} - F_{y'} \frac{\xi}{\tau_y} (\tau_x + y'(x)\tau_y)|_{x=v} &= 0, \\ \left(\xi F - F_{y'} \frac{\xi}{\tau_y} (\tau_x + y'(x)\tau_y) \right)|_{x=v} &= 0, \\ \xi \left(F - F_{y'} \frac{(\tau_x + y'(x)\tau_y)}{\tau_y} \right)|_{x=v} &= 0, \\ - \left(-F + F_{y'} \frac{(\tau_x + y'(x)\tau_y)}{\tau_y} \right)|_{x=v} &= 0, \\ - \left(\frac{-F\tau_y + F_{y'} (\tau_x + y'(x)\tau_y)}{\tau_y} \right)|_{x=v} &= 0, \\ - (-F\tau_y + F_{y'} (\tau_x + y'(x)\tau_y))|_{x=v} &= 0, \\ - (-F\tau_y + F_{y'}\tau_x + F_{y'}y'(x)\tau_y)|_{x=v} &= 0, \\ - (\tau_y (-F + F_{y'}y'(x)) + F_{y'}\tau_x)|_{x=v} &= 0, \\ - (\tau_y (F_{y'}y'(x) - F) + F_{y'}\tau_x)|_{x=v} &= 0. \end{aligned}$$

Finally,

$$\tau_y (F_{y'}y'(x) - F)|_{x=v} + F_{y'}\tau_x|_{x=v} = 0. \quad (4.14)$$

Equation (4.14) shows that the transversality condition has been satisfied, as required.

(c) Now, considering the case when

$$S[y] = \int_0^v dx \frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta,$$

where $y(v) > 0$, $\delta > 0$ and the right-hand end point $(v, y(v))$ lies on the line $\alpha y + \beta x + \gamma = 0$, where α, β, γ are constants with $\beta \neq 0$, it can be shown that the first-integral may be written as

$$y\sqrt{1+y'^2} = c, \quad (4.15)$$

for some constant $c > 0$, as follows.

The first-integral is given by

$$y'G_{y'} - G = \text{constant}, \quad \text{HB p17.}$$

with,

$$\begin{aligned} G &= \frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta. \\ G_{y'} &= \frac{1}{y} \left(\frac{1}{2} (1+y'^2)^{-\frac{1}{2}} \cdot 2y' \right), \\ &= \frac{(1+y'^2)^{-\frac{1}{2}} y'}{y}, \\ &= \frac{y'}{y\sqrt{1+y'^2}}. \end{aligned}$$

The first-integral becomes,

$$\begin{aligned} y' \left(\frac{y'}{y\sqrt{1+y'^2}} \right) - \frac{\sqrt{1+y'^2}}{y} &= c, \quad \text{where } c \text{ is a constant,} \\ \left(\frac{y'^2}{y\sqrt{1+y'^2}} \right) - \frac{\sqrt{1+y'^2}}{y} &= c, \\ \frac{y'^2}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}\sqrt{1+y'^2}}{y\sqrt{1+y'^2}} &= c, \\ \frac{y'^2}{y\sqrt{1+y'^2}} - \frac{1+y'^2}{y\sqrt{1+y'^2}} &= c, \\ \frac{y'^2 - 1 - y'^2}{y\sqrt{1+y'^2}} &= c \\ -\frac{1}{y\sqrt{1+y'^2}} &= c, \\ -\frac{1}{c} &= y\sqrt{1+y'^2}. \end{aligned}$$

Redefining the constant c , then the first-integral may be written as,

$$y\sqrt{1+y'^2} = c, \quad \text{for some constant } c > 0, \text{ as required.} \quad (4.16)$$

Now, rearranging (4.16) in terms of y' , as follows.

$$\begin{aligned} y'^2 &= \left(\frac{dy}{dx} \right)^2 = \frac{c^2}{y^2} - 1, \\ \frac{dy}{dx} &= \sqrt{\frac{c^2}{y^2} - 1}. \end{aligned}$$

Then,

$$\frac{dx}{dy} = 1 \bigg/ \frac{dy}{dx},$$

so,

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{\sqrt{\frac{c^2}{y^2} - 1}} = \frac{1}{\sqrt{\frac{c^2 - y^2}{y^2}}} = \frac{y}{\sqrt{c^2 - y^2}}. \\ \int dy \frac{dx}{dy} &= \int dy \frac{y}{\sqrt{c^2 - y^2}}, \\ x &= \int dx \frac{y}{\sqrt{c^2 - y^2}}. \end{aligned} \quad (4.17)$$

Solving the integral of (4.17),

$$x = \int dy \frac{y}{\sqrt{c^2 - y^2}}.$$

Let $u = c^2 - y^2$,

$$\frac{du}{dy} = -2y, \quad \text{so} \quad \frac{dy}{du} = 1 \bigg/ \frac{du}{dy} = -\frac{1}{2y}.$$

$$\begin{aligned} x &= \int du \left(\frac{dy}{du} \right) \frac{y}{\sqrt{u}}, \\ &= \int du \left(-\frac{1}{2y} \right) \frac{y}{\sqrt{u}}, \\ &= -\frac{1}{2} \int du \frac{1}{\sqrt{u}}, \\ &= -\frac{1}{2} \int du u^{-\frac{1}{2}}, \\ &= -\frac{1}{2} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) + c_\delta = -\sqrt{u} - c_\delta, \end{aligned}$$

where c_δ is the constant of integration.

Thus,

$$\begin{aligned} x &= -\sqrt{c^2 - y^2} - c_\delta, \\ (x + c_\delta)^2 &= c^2 - y^2, \\ y^2 + (x + c_\delta)^2 &= c^2, \quad \text{as required.} \end{aligned} \quad (4.18)$$

Applying the boundary condition, $y(0) = \delta$, to (4.18) gives,

$$\delta^2 + c_\delta^2 = c^2 \quad \text{and so,} \quad c_\delta^2 = c^2 - \delta^2.$$

The solution of the first-integral (4.18) are circles centred at $(-c_\delta, 0)$.

Differentiating $y^2 + (x + c_\delta)^2 = c^2$ implicitly:

$$\begin{aligned}\frac{d}{dx}(y^2) + \frac{d}{dx}(x + c_\delta)^2 &= \frac{d}{dx}(c^2), \\ 2y \frac{dy}{dx} + 2(x + c_\delta) &= 0, \\ y \frac{dy}{dx} + x + c_\delta &= 0, \\ yy' + x + c_\delta &= 0.\end{aligned}\tag{4.19}$$

Comparing (4.19) with $\alpha y + \beta x + \gamma = 0$ but first dividing through this expression by β :

$$\frac{\alpha}{\beta}y + x + \frac{\gamma}{\beta} = 0,$$

then it is seen that $y' = \alpha/\beta$ and $c_\delta = \gamma/\beta$.

Finally, it can be shown that in the limit as $\delta \rightarrow 0$, the stationary path becomes

$$\beta^2 y^2 + (\beta x + \gamma)^2 = \gamma^2,$$

as follows.

In the limit as $\delta \rightarrow 0$, $c_\delta = c$ and equation (4.18) can be written as,

$$y^2 + (x + c_\delta)^2 = c_\delta^2.\tag{4.20}$$

Substituting into (4.20) for $c_\delta = \gamma/\beta$ gives,

$$\begin{aligned}y^2 + \left(x + \frac{\gamma}{\beta}\right)^2 &= \frac{\gamma^2}{\beta^2}, \\ \text{cross multiplying by } \beta^2 \text{ gives,} \\ \beta^2 y^2 + \beta^2 \left(x + \frac{\gamma}{\beta}\right)^2 &= \gamma^2 \quad \text{and} \\ \beta^2 y^2 + (\beta x + \gamma)^2 &= \gamma^2 \quad \text{as required.}\end{aligned}$$

Q 5. Given

$$S[x] = \int_a^b dt L(t, x, \dot{x}), \quad \text{with } b > a,$$

where L is called the Lagrangian, and $x(t)$ is at least twice differentiable.

The *conjugate momentum* p is defined by

$$p = \frac{\partial L}{\partial \dot{x}}. \quad (5.1)$$

(a) It can be shown that the Euler-Lagrange equation for S is defined by

$$\dot{p} = \frac{\partial L}{\partial x},$$

as follows.

The Euler-Lagrange equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0,$$

which becomes after substituting in the *conjugate momentum*,

$$\frac{d}{dt} (p) - \frac{\partial L}{\partial x} = 0, \quad \text{and}$$

$$\frac{dp}{dt} - \frac{\partial L}{\partial x} = 0,$$

$$\therefore \dot{p} = \frac{\partial L}{\partial x}, \quad \text{as required.}$$

(b) From the handbook, the total derivative can be expressed as:

see HB p3.

$$\frac{df}{dt} = \sum_{k=1}^n \frac{\partial f}{\partial x_k} \frac{dx_k}{dt}.$$

Using this result then,

$$\begin{aligned} \frac{dL}{dt} &= \frac{d}{dt} (L(t, x, \dot{x})), \\ &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} \frac{dx}{dt} + \frac{\partial L}{\partial \dot{x}} \frac{d\dot{x}}{dt}, \\ &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} \dot{x} + \frac{\partial L}{\partial \dot{x}} \ddot{x}, \\ &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} \dot{x} + p \ddot{x}, \quad \text{as required.} \end{aligned}$$

Recall that p is the *conjugate momentum* defined above.

- (c) The *Hamiltonian* $H = H(t, x, p)$ is defined by $H(t, x, p) = p\dot{x} - L(t, x, \dot{x})$, where (implicitly) \dot{x} is eliminated using (5.1) to give a function of t, x and p .

Using the result obtained in part (b) it will be shown that for a stationary path of S that

$$\frac{\partial L}{\partial t} = -\dot{H} \quad (5.2)$$

as follows.

$$\begin{aligned} \frac{dH}{dt} &= \frac{d}{dt}(p\dot{x} - L), \\ &= \frac{d}{dt}(p\dot{x}) - \frac{dL}{dt}, \\ &= p\ddot{x} + \dot{p}\dot{x} - \frac{dL}{dt}, \end{aligned}$$

and substituting into the above expression for $\frac{dL}{dt}$ from part (b) gives,

$$\begin{aligned} &= p\ddot{x} + \dot{p}\dot{x} - \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x} \right), \\ &= \cancel{p\ddot{x}} + \dot{p}\dot{x} - \left(\frac{\partial L}{\partial t} + \cancel{\dot{p}\dot{x}} + \cancel{p\ddot{x}} \right), \end{aligned}$$

$$\therefore \frac{dH}{dt} = \dot{H} = -\frac{\partial L}{\partial t}, \quad \text{as required.}$$

- (d) The *Rund-Trautman identity* is given as

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0, \quad (5.3)$$

and from this identity it will be shown that

$$(\xi - \dot{x}\tau) \left[\dot{p} - \frac{\partial L}{\partial x} \right] = \frac{d}{dt} [p\xi - H\tau], \quad (5.4)$$

as follows.

First it will be shown that the left-hand side of (5.3) is equal to zero by expanding out the bracketed terms and substituting the derivative terms.

$$\begin{aligned} (\xi - \dot{x}\tau) \left[\dot{p} - \frac{\partial L}{\partial x} \right] &= \xi\dot{p} - \xi\frac{\partial L}{\partial x} - \dot{x}\tau\dot{p} + \dot{x}\tau\frac{dL}{dx}, \\ &= \cancel{\xi\dot{p}} - \cancel{\xi\dot{p}} - \cancel{\dot{x}\tau\dot{p}} + \cancel{\dot{x}\tau\dot{p}}, \\ &= 0. \end{aligned}$$

Secondly, it will be shown that (5.3) is equal to the right-hand side of (5.4) which is equal to zero. Substituting into (5.3) for

$$\frac{\partial L}{\partial t} = -\dot{H}, \quad \text{and} \quad \dot{p} = \frac{\partial L}{\partial x} \quad \text{gives the following,}$$

$$\begin{aligned}
\dot{p}\xi + p\dot{\xi} - \dot{H}\tau - H\dot{\tau} &= 0, \\
\underbrace{\dot{p}\xi + p\dot{\xi}} - \underbrace{(\dot{H}\tau + H\dot{\tau})} &= 0, \\
\frac{d}{dt}(p\xi) - \frac{d}{dt}(\tau H) &= 0, \\
\therefore \frac{d}{dt}[p\xi - \tau H] &= 0.
\end{aligned}$$

The product rule has been used here.

Thus,

$$(\xi - \dot{x}\tau) \left[\dot{p} - \frac{\partial L}{\partial x} \right] = \frac{d}{dt}[p\xi - \tau H], \quad \text{as required.}$$

The differentiation of a constant is zero and also noting that zero itself is a constant, then the expression

$$\frac{d}{dt}[p\xi - \tau H] = 0$$

must mean that

$$p\xi - \tau H = \text{constant.}$$

- (e) Now, considering a particle of constant mass m moving along the x -axis in a potential $V(x)$. The Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - V(x)$, and the path of the particle from $t = a$ to $t = b$ is a stationary path of S .

The conjugate momentum p is calculated as follows.

$$\begin{aligned}
p &= \frac{\partial L}{\partial \dot{x}} \quad \text{and} \quad L = \frac{1}{2}m\dot{x}^2 - V(x). \\
\therefore p &= \frac{\partial}{\partial \dot{x}} \left(\frac{1}{2}m\dot{x}^2 - V(x) \right) = 2 \cdot \frac{1}{2}m\dot{x} = m\dot{x}.
\end{aligned}$$

The Hamiltonian is calculated as follows.

$$H(t, x, p) = p\dot{x} - L(t, x, \dot{x}).$$

Substituting into this expression for conjugate momentum p and the Lagrangian L , gives,

$$\begin{aligned}
H &= (m\dot{x})\dot{x} - \left(\frac{1}{2}m\dot{x}^2 - V(x) \right), \\
&= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x), \\
&= \frac{1}{2}m\dot{x}^2 + V(x).
\end{aligned}$$

(f)

$$\frac{d}{dt}(p(t)) = \dot{p}$$

$$\begin{aligned} \frac{d}{dt}(\xi(t, x, \dot{x})) &= \frac{\partial}{\partial t}\xi(t, x, \dot{x}) \\ &\quad + \frac{\partial}{\partial x}\xi(t, x, \dot{x})\frac{dx}{dt} \\ &\quad + \frac{\partial}{\partial \dot{x}}\xi(t, x, \dot{x})\frac{d\dot{x}}{dt}. \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(\xi(t, x, \dot{x})) &= \frac{\partial}{\partial t}\xi(t, x, \dot{x}) \\ &\quad + \frac{\partial}{\partial x}\xi(t, x, \dot{x})\dot{x} \\ &\quad + \frac{\partial}{\partial \dot{x}}\xi(t, x, \dot{x})\ddot{x}. \end{aligned}$$

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\dot{x} + \frac{\partial \xi}{\partial \dot{x}}\ddot{x}.$$

This must be similar for $\tau(t, x, \dot{x})$, too:

$$\frac{d\tau}{dt} = \frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x}\dot{x} + \frac{\partial \tau}{\partial \dot{x}}\ddot{x}.$$

$$\begin{aligned} \frac{d}{dt}(H(t, x(t), p(t))) &= \frac{\partial}{\partial t}H(t, x(t), p(t)) \\ &\quad + \frac{\partial}{\partial x}H(t, x(t), p(t))\frac{dx(t)}{dt} \\ &\quad + \frac{\partial}{\partial \dot{x}}H(t, x(t), p(t))\frac{dp(t)}{dt}. \end{aligned}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \dot{x}}\dot{p}.$$

$$\frac{d}{dt}(p\xi) = p\frac{d\xi}{dt} + \xi\frac{dp}{dt} = p\dot{\xi} + \xi\dot{p}$$

$$\frac{d}{dt}(H\tau) = H\frac{d\tau}{dt} + \tau\frac{dH}{dt} = H\dot{\tau} + \tau\dot{H}$$

$$\begin{aligned} \frac{d}{dt}[p\xi - H\tau] &= \frac{d}{dt}(p\xi) - \frac{d}{dt}(H\tau), \\ &= p\dot{\xi} + \xi\dot{p} - (H\dot{\tau} + \tau\dot{H}), \\ &= p\dot{\xi} + \xi\dot{p} - H\dot{\tau} - \tau\dot{H}, \end{aligned}$$

The *Rund-Trautman identity* is:

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0$$

which can be rearranging to:

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0$$

$$p\dot{\xi} + \frac{\partial L}{\partial x}\xi - H\dot{\tau} + \frac{\partial L}{\partial t}\tau = 0$$

Compare to:

$$\frac{d}{dt} [p\xi - H\tau] = p\dot{\xi} + \xi\dot{p} - H\dot{\tau} - \tau\dot{H}$$