Q 1.

(a)

(b)

Q 2.

Q 3.

(a)

(b)

(c)

Q 4.

(a)

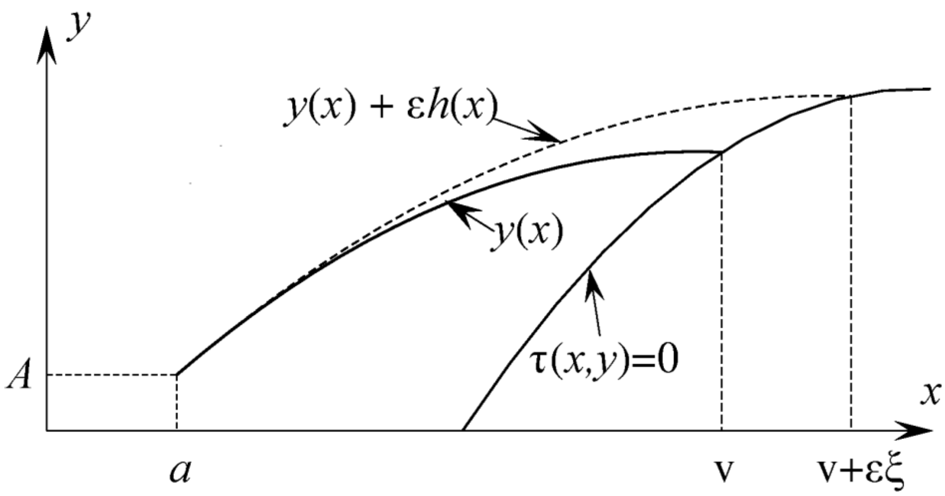

Figure 1: Diagram showing the stationary path (solid line) and a varied path (dashed line) for a problem in which the left-hand end is fixed, but the other end is free to move along the line defined by $\tau(x, y) = 0$.

Given the perturbed path

$$y_\epsilon(x) = y(x) + \epsilon h(x), \tag{4.1}$$

the Taylor series to the first-order of (4.1) at point $x = v$ (see figure 1) is given in (4.2).

The point $x = v$ is known as the point of expansion. HB p8.

$$y_\epsilon(x) = (y(v) + \epsilon h(v)) + (y'(v) + \epsilon h'(v))\,(x - v) + \mathcal{O}\left((x - v)^2\right). \tag{4.2}$$

Now, determining the value of (4.2) at $v_\epsilon = v + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right)$ where v_ϵ is the perturbed value of v:

$$\begin{aligned} y(v_\epsilon) = {} & y(v) + \epsilon h(v) \\ & + \left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{-v}\right)\left(y'(v) + \epsilon h'(v)\right) \\ & + \mathcal{O}\left(\left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{-v}\right)^2\right), \end{aligned}$$

$$\begin{aligned} y(v_\epsilon) = {} & y(v) + \epsilon h(v) \\ & + \left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)\left(y'(v) + \epsilon h'(v)\right) \\ & + \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right), \end{aligned}$$

$$
\begin{aligned}
y(v_\epsilon) = y(v) &+ \epsilon h(v) \\
&+ \epsilon\xi\left(y'(v) + \epsilon h'(v)\right) \\
&+ \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) \\
&+ \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
y(v_\epsilon) = y(v) &+ \epsilon\left(h(v) + \xi y'(v)\right) \\
&+ \underbrace{\epsilon^2\xi h'(v) + \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) + \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right)}_{\text{These are all second-order terms in } \epsilon.}.
\end{aligned}
$$

Thus,

$$
y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(h(v) + \xi y'(v)\right) + \mathcal{O}\left(\epsilon^2\right), \tag{4.3}
$$

as required.

To show that at $(x, y) = (v, y(v))$,

$$
\xi\left(\tau_x + y'(v)\tau_y\right) + h(v)\tau_y = 0
$$

the 2D Taylor expansion to the first-order of $\tau(x, y)$ at point $x = a, y = b$ is required, namely,

$$
\tau(x, y) = \tau(a, b) + \tau_x(a, b)\left[x - a\right] + \tau_y(a, b)\left[y - b\right]. \tag{4.4}
$$

Evaluating (4.4) with $x = v + \epsilon\xi$ and $y = y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(y'(v)\xi + h(v)\right)$ gives,

$$
\begin{aligned}
\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) &= \tau(v + \epsilon\xi, y(v) + \epsilon\left(y'(v)\xi + h(v)\right) \\
&= \cancelto{0}{\tau\left(v, y(v)\right)} + \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\
&= \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\
&= \tau_x\left(v, y(v)\right)\left[\cancel{v} + \epsilon\xi \cancel{- v}\right] + \tau_y\left(v, y(v)\right)\left[\cancel{y(v)} + \epsilon\left(y'(v)\xi + h(v)\right) \cancel{- y(v)}\right], \\
&= \tau_x\left(v, y(v)\right)\epsilon\xi + \tau_y\left(v, y(v)\right)\epsilon\left(y'(v)\xi + h(v)\right), \\
&= \epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right],
\end{aligned}
$$

Recall that $\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) = 0$ and therefore,

$$
\begin{aligned}
\epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right] &= 0, \\
\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right) &= 0, \\
\xi\left[\tau_x\left(v, y(v)\right) + \tau_y\left(v, y(v)\right)y'(v)\right] + \tau_y\left(v, y(v)\right)h(v) &= 0,
\end{aligned}
$$

as required.

(b)

(c)

Q 5.

(a)

(b)

(c)

(d)