Q 1.

(a) The functional is

$$S[y] = \alpha y(1)^2 + \int_0^1 \mathrm{d}x\, \beta y'^2, \quad y(0) = 0,$$

with natural boundary condition at $x = 1$ and subject to the constraint

$$C[y] = \gamma y(1)^2 + \int_0^1 \mathrm{d}x\, w(x) y^2 = 1,$$

where α, β and γ are non-zero constants.

To show that the stationary paths of this system satisfy the Euler-Lagrange equation

$$\beta \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda w(x) y = 0, \quad y(0) = 0, \ \ (\alpha - \gamma\lambda)\, y(1) + \beta y'(1) = 0,$$

where λ is a Lagrange multiplier, consider the following.

The auxiliary functional is

$$\begin{aligned}\overline{S}[y] &= \alpha y(1)^2 + \int_0^1 \mathrm{d}x\, \beta y'^2 - \lambda\gamma y(1)^2 - \lambda \int_0^1 \mathrm{d}x\, w(x) y^2, \\ &= \alpha y(1)^2 - \lambda\gamma y(1)^2 + \int_0^1 \mathrm{d}x\, \left(\beta y'^2 - \lambda w(x) y^2\right), \\ &= (\alpha - \lambda\gamma)\, y(1)^2 + \int_0^1 \mathrm{d}x\, \left(\beta y'^2 - \lambda w(x) y^2\right).\end{aligned}$$

The Gâteaux differential $\Delta\overline{S}[y,h]$ of the functional $\overline{S}[y]$ is given by the expression

See HB p16.

$$\Delta\overline{S}[y,h] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} S[y + \epsilon H]\bigg|_{\epsilon=0}.$$

Rewriting the expression for $\overline{S}[y]$ and replacing each occurrence of y with $y + \epsilon h$ gives,

$$\begin{aligned}\Delta\overline{S}[y + \epsilon h] = (\alpha - \lambda\gamma)\left(y(1) + \epsilon h(1)\right)^2 + \\ \int_0^1 \mathrm{d}x\, \left(\beta (y + \epsilon h)'^2 - \lambda w(x)(y + \epsilon h)^2\right),\end{aligned}$$

$$\begin{aligned}\Delta\overline{S}[y,h] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left((\alpha - \lambda\gamma)\left(y(1) + \epsilon h(1)\right)^2 \right)\bigg|_{\epsilon=0} + \\ \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\int_0^1 \mathrm{d}x\, \left(\beta (y + \epsilon h)'^2 - \lambda w(x)(y + \epsilon h)^2\right) \right)\bigg|_{\epsilon=0}.\end{aligned}$$

which, after carrying out the differentiation becomes,

$$\Delta\overline{S}[y,h] = \left(2\left(\alpha-\lambda\gamma\right)\left(y(1)+\epsilon h(1)\right)h(1)\right)\Big|_{\epsilon=0} + \left(\int_0^1 \mathrm{d}x\, 2\left(\beta(y+\epsilon h)'h' - 2\lambda w(x)(y+\epsilon h)h\right)\right)\Big|_{\epsilon=0},$$

$$= 2\left(\alpha-\lambda\gamma\right)y(1)h(1) + 2\int_0^1 \mathrm{d}x\,\beta y'h' - 2\int_0^1 \mathrm{d}x\,\lambda w(x)yh.$$

Integrating the left-most integral by parts,

$$2\beta\int_0^1 \mathrm{d}x\, y'h' = 2\beta\left(\Big[y'h\Big]_0^1 - \int_0^1 \mathrm{d}x\, y''h\right),$$

and from Table 1 $h(0)=0$ so,

Boundary condition 1
$y(1)=0$
$y(1)+\epsilon h(1)=0$
$0+\epsilon h(1)=0$
$\boxed{h(1)=0}$

Table 1: Determination of $h(1)$.

$$2\beta\int_0^1 \mathrm{d}x\, y'h' = 2\beta y'(1)h(1) - 2\beta\int_0^1 \mathrm{d}x\, y''h.$$

Thus,

$$\Delta\overline{S}[y,h] = 2\left(\alpha-\lambda\gamma\right)y(1)h(1) + 2\beta y'(1)h(1) - 2\int_0^1 \mathrm{d}x\,\left(\beta y'' + \lambda w(x)y\right)h. \tag{1.1}$$

The Euler-Lagrange equation can be found from the above expression for the Gâteaux differential; by definition it is required that the Gâteaux differential be equal to zero. So by setting the expression shown in (1.1) to zero and dividing through by 2 gives the following for a stationary path

$$\left(\alpha-\lambda\gamma\right)y(1)h(1) + \beta y'(1)h(1) - \int_0^1 \mathrm{d}x\,\left(\beta y'' + \lambda w(x)y\right)h = 0.$$

Then, by the Fundamental Lemma of the Calculus of Variations (meaning that because the result has to be true for all admissible paths the term that h multiplies must be equal to zero) and setting $h(1)=0$, then the Euler-Lagrange equation is given by

$$\beta\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \lambda w(x)y = 0, \quad y(0)=0, \;\; \left(\alpha-\gamma\lambda\right)y(1) + \beta y'(1) = 0. \tag{1.2}$$

λ is the Lagrange multiplier.

Equation (1.1) is the Gâteaux differential and (1.2) is the Euler-Lagrange equation (together with the boundary conditions) that the stationary paths must satisfy.

(b) Let $w(x) = 1$ and $\alpha = \beta = \gamma = 1$, so that (1.2) becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \quad y(0) = 0, \ (1-\lambda)y(1) + y'(1) = 0. \tag{1.3}$$

Then to find the non-trivial stationary paths, the eigenfunctions of y (normalised so that $C[y] = 1$) and the values of the Lagrange multipliers, consider the following.

There are three cases to consider regarding λ, namely $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. So, considering each case in turn as follows.

if $\lambda = 0$ set $\lambda = 0$ in (1.3):

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0, \quad y(0) = 0, \ y(1) + y'(1) = 0.$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = A, \quad \text{where } A \text{ is an arbitrary constant.}$$
$$y = Ax + B, \quad \text{where } B \text{ is also an arbitrary constant.}$$

Applying the boundary conditions to y' and y gives $A = 0$ and $B = 0$ and therefore $y(x) = 0$. This is a trivial solution.

If $\lambda < 0$ set $\lambda = -\mu^2$ $(\mu > 0)$ in (1.3):

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \mu^2 y = 0, \quad y(0) = 0, \ (1-\mu^2)y(1) + y'(1) = 0.$$

The auxiliary equation is

$$r^2 - \mu^2 = 0, \quad \text{with} \quad r_1 = -\mu, \ r_2 = \mu.$$
$$y(x) = A\mathrm{e}^{-\mu x} + B\mathrm{e}^{\mu x}.$$

Applying the first boundary conditions to $y(x)$ with $x = 0$ and $y(o) = 0$ gives

$$0 = A + B \quad \text{therefore} \quad A = -B,$$

and so,

$$y(x) = -B\mathrm{e}^{-\mu x} + B\mathrm{e}^{\mu x}.$$

Differentiating y,

$$y'(x) = B\mu\mathrm{e}^{-\mu x} + B\mu\mathrm{e}^{\mu x},$$

Applying the second boundary condition to determine B,

$$\begin{aligned}(1-\mu^2)y(1)+y'(1) &= (1-\mu^2)\left(-B\mathrm{e}^{-\mu}+B\mathrm{e}^{\mu}\right)+B\mu\mathrm{e}^{-\mu}+B\mu\mathrm{e}^{\mu}=0,\\ &= B\left\{(1-\mu^2)\left(-\mathrm{e}^{-\mu}+\mathrm{e}^{\mu}\right)+\mu\mathrm{e}^{-\mu}+\mu\mathrm{e}^{\mu}\right\}=0.\end{aligned}$$

By assumption $\mu \neq 0$, so the expression immediately above can only be satisfied if $B=0$ and therefore y(x) $=0$. This also is a trivial solution.

If $\lambda > 0$ set $\lambda = \mu^2$ $(\mu > 0)$ in (1.3):

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mu^2y=0, \quad y(0)=0, \quad (1-\mu^2)y(1)+y'(1)=0.$$

The auxiliary equation is

$$r^2+\mu^2=0, \quad \text{with} \quad r_1=-i\mu, \quad r_2=i\mu.$$

and the general solution is given by

$$y(x)=A\sin\mu x+B\cos\mu x.$$

Applying the first boundary condition $y(0)=0$, gives,

$$y(0)=\cancel{A\sin 0}^{\,0}+\cancel{B\cos 0}^{\,1}, \quad \text{and therefore, } B=0.$$

Thus, the updated solution is,

$$y(x)=A\sin\mu x.$$

Differentiating $y(x)$ gives,

$$y'(x)=A\mu\cos\mu x.$$

Applying the second boundary conditions $(1-\mu^2)y(1)+y'(1)=0$:

$$(1-\mu^2)A\sin\mu+A\,\mu\cos\mu=0,$$

and factoring out A and assuming $A \neq 0$, gives,

$$(1-\mu^2)\sin\mu+\ \mu\cos\mu=0, \tag{1.4}$$

$$\frac{(1-\mu^2)}{\mu}=-\frac{\cos\mu}{\sin\mu},$$

$$\frac{(\mu^2-1)}{\mu}=\cot\mu. \tag{1.5}$$

To solve (1.5) for μ a graph of both sides of (1.5) can be used as shown in Figure 1. From the graphs shown in Figure 1 $\mu_1 \approx 1.208$, $\mu_2 \approx 3.448$, $\mu_3 \approx 6.441$, $\mu_4 \approx 9.530$, $\mu_5 \approx 12.646$ and it is seen that as n increases μ_n approaches $(n-1)\pi$, $n=1,2,\ldots$. Thus,

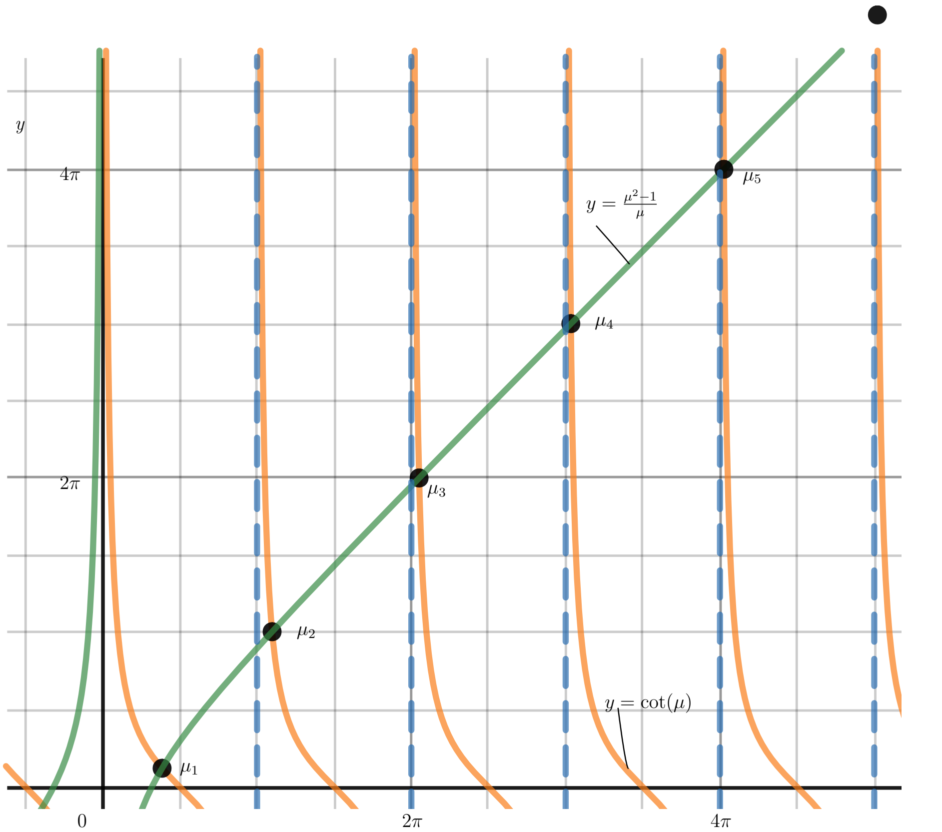


Figure 1: Graphs of $y = \left(\dfrac{\mu^2-1}{\mu}\right)$ and $y = \cot(\mu)$.

the eigenvalues λ_n approach $(n-1)^2\pi^2$ as n increases. Hence, the eigenfunctions corresponding to the eigenvalues λ_n are,

$$y_n(x) = A_n \sin\left(\sqrt{\lambda_n} x\right), \qquad n = 1, 2, \ldots. \tag{1.6}$$

To normalise the eigenfunctions (1.6) consider the following.

The inner product (with unit weight function w) where $(y, y)_w = 1$ will be used to normalise the expression for y given in (1.6),

See HB p31.

$$(y, y)_w = \int_a^b \mathrm{d}x\, y(x)^2 = 1, \tag{1.7}$$

by determining the value of A_n as follows.

After substituting into (1.7) for $y_n(x)$ from (1.6) the following integral is obtained.

$$\int_0^1 \mathrm{d}x\, A_n^2 \sin^2\left(\sqrt{\lambda_n} x\right) = 1,$$

$$A_n^2 \int_0^1 \mathrm{d}x \, \sin^2\left(\sqrt{\lambda_n} x\right) = 1.$$

Using the trig identity $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ gives,

See HB p38 for trig identities.

$$\frac{A_n^2}{2} \int_0^1 \mathrm{d}x \, \left(1 - \cos\left(2\sqrt{\lambda_n} x\right)\right) = 1.$$

$$\frac{A_n^2}{2} \left[x - \frac{1}{2\sqrt{\lambda_n}} \sin\left(2\sqrt{\lambda_n} x\right)\right]_0^1 = 1.$$

$$\frac{A_n^2}{2} \left(1 - \frac{1}{2\sqrt{\lambda_n}} \sin\left(2\sqrt{\lambda_n}\right)\right) = 1.$$

$$\frac{A_n^2}{4\sqrt{\lambda_n}} \left(2\sqrt{\lambda_n} - \sin\left(2\sqrt{\lambda_n}\right)\right) = 1.$$

Using the trig identity $\sin 2\alpha = 2 \sin\alpha \cos\alpha$ then,

$$\frac{A_n^2}{4\sqrt{\lambda_n}} \left(2\sqrt{\lambda_n} - 2\sin\left(\sqrt{\lambda_n}\right) \cos\left(\sqrt{\lambda_n}\right)\right) = 1.$$

From (1.4)

$$-\sin\left(\sqrt{\lambda_n}\right) = \frac{\sqrt{\lambda_n}}{1 - \lambda_n} \cos\left(\sqrt{\lambda_n}\right),$$

and so,

$$\frac{A_n^2}{\cancelto{2}{4\sqrt{\lambda_n}}} \left(\cancelto{1}{2\sqrt{\lambda_n}} + \frac{\cancelto{1}{2\sqrt{\lambda_n}}}{1 - \lambda_n} \cos\left(\sqrt{\lambda_n}\right) \cos\left(\sqrt{\lambda_n}\right)\right) = 1.$$

$$\frac{A_n^2}{2} \left(1 + \frac{1}{1 - \lambda_n} \cos^2\left(\sqrt{\lambda_n}\right)\right) = 1.$$

$$\frac{A_n^2}{2(1 - \lambda_n)} \left((1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)\right) = 1.$$

$$A_n^2 = \frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)}.$$

Therefore,

$$A_n = \sqrt{\frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)}},$$

and the normalised eigenfunctions are given by,

$$y_n(x) = \sqrt{\frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)}} \; \sin\left(\sqrt{\lambda_n} x\right).$$

Q 2. To show that the nontrivial solutions of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y(1 + x^k) = 0, \tag{2.1}$$

where k is a positive integer, has infinitely many zeros in $(0, \infty)$ consider the following.

Use will be made of Sturm's comparison theorem II (Theorem 31.3). See HB p30.

Since $(1 + x^k) \geq 2$ for all $x \geq 1$ let $Q_2 = 2$ and $Q_1(x) = (1 + x^k)$. So that,

$$y_1'' + y_1(1 + x^k) = 0, \quad \text{and} \quad y_2'' + 2y_2 = 0.$$

A solution to $y_2''(x) + 2y_2(x) = 0$ is $y_2(x) = \sin\left(\sqrt{2}x\right)$ and

$$y_2\left(\frac{1}{\sqrt{2}} n\pi\right) = 0 \quad \text{for} \quad n = 1, 2, \ldots.$$

Since Sturm's comparison theorem states that

"... if $y_1(x)$ is a solution of the first equation and $y_2(x)$ is any solution to the second equation, between any two adjacent zeros of y_2 there lies at least on zero of y_1."

Thus, as y_2 has infinitely many zeros in $(0, \infty)$ so too do the nontrivial solutions of the given expression (2.1).

To show that the separation between adjacent zeros tends to zero as $x \to \infty$ a similar argument to that above will be given.

Since $(1 + x^k) \geq \alpha^k$ for all $x \geq \alpha$, $\alpha, x \in (0, \infty)$, let $Q_2 = (1 + x^k)$ and $Q_1(x) = \alpha^k$. So that,

$$y_1'' + y_1(1 + x^k) = 0, \quad \text{and} \quad y_2'' + \alpha^k y_2 = 0.$$

A solution to $y_2''(x) + \alpha^k y_2(x) = 0$ is $y_2(x) = \sin\left(\alpha^{k/2} x\right)$ and

$$y_2\left(\frac{1}{\alpha^{k/2}} n\pi\right) = 0 \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad \alpha \in (0, \infty).$$

Thus, as $x \to \infty$ so does $\alpha \to \infty$ and the separation between zeros $\dfrac{\pi}{\alpha^{k/2}} \to 0$.

Recall that k is a constant positive integer

Q 3.

(a) To show that the following equation and boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda xy = 0, \quad y(1) = 0, y'(2) = 0 \tag{3.1}$$

forms a regular Sturm-Liouville system, consider the following.

A Sturm-Liouville system is a linear, second-order homogeneous differential equation of the form:

see HB p28.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \left(q(x) + \lambda w(x)\right)y = 0, \tag{3.2}$$

which is defined on a finite interval of the real axis $a < x > b$ and satisfies the following three conditions:

1. the functions p(x), q(x) and w(x) are real and continuous for $a < x < b$;
2. $p(x)$ and $w(x)$ are strictly positive for $a < x < b$;
3. $p'(x)$ exists and is continuous for $a \leq x \leq b$,

together with the boundary conditions.

Comparing equations (3.1) and (3.2) it is seen that $q(x) = 0$ with $p(x) = x^2$ and $w(x) = x$. As $p(x)$ and $w(x)$:

1. are real and continuous for $1 < x < 2$;
2. are strictly positive for $1 < x < 2$; and
3. $p'(x)$ exists ($p'(x) = 2x$) and is continuous for $1 \leq x \leq 2$,

and the two boundary conditions are given as $y(1) = 0$ and $y'(2) = 0$, then the system is a regular Sturm-Liouville system.

It can be shown that the system can be written as a constrained variational problem with functional

$$S[y] = \int_1^2 \mathrm{d}x\, x^2 y'^2, \quad y(1) = 0, \tag{3.3}$$

and constraint

$$C[y] = \int_1^2 \mathrm{d}x\, xy^2 = 1, \tag{3.4}$$

as follows.

For the given constrained variational problem

$$\overline{F} = x^2 y'^2 - \lambda x y^2,$$

λ is the Lagrange multiplier.

and

$$\overline{F}_{y'} = \frac{\partial \overline{F}}{\partial y'} = 2x^2 y' \quad \text{and} \quad \overline{F}_y = \frac{\partial \overline{F}}{\partial y} = -2\lambda x y.$$

The Euler-Lagrange equation is then,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \overline{F}}{\partial y'}\right) - \frac{\partial \overline{F}}{\partial y} = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^2 y'\right) - (-2\lambda x y) = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\cancel{2}x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \cancel{2}\lambda x y = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda x y = 0, \quad y(1) = 0, y'(2) = 0.
\end{aligned} \tag{3.5}$$

Equations (3.1) and (3.5) are identical showing that system (3.1) can be written as a constrained variational problem.

(b) It is assumed that the eigenvalues λ_k and the eigenfunctions y_k, $k = 1, 2, \ldots$ exist. By working from (3.1), the following relationship will be derived.

$$\lambda_k = \int_1^2 \mathrm{d}x\, x^2 y_k'^2, \quad k = 1, 2, \ldots.$$

Recall that (3.1) is given as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda x y = 0, \quad y(1) = 0, y'(2) = 0$$

and compare (3.1) with (3.2) repeated below,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x) \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \left(q(x) + \lambda w(x)\right) y = 0.$$

As was determined in part (a) $q(x) = 0$ with $p(x) = x^2$ and $w(x) = x$ and the system is defined on the interval $(1, 2)$.

Now the functional is of the form:

$$S[y] = -\alpha p(a) y(a)^2 + \beta p(b) y(b)^2 + \int_a^b \mathrm{d}x \, \left(p y' - q y^2\right),$$

See HB p29 (SLF).

which, becomes the following after substituting for the appropriate terms found above,

$$S[y] = -\alpha p(1) y(1)^2 + \beta p(2) y(2)^2 + \int_1^2 \mathrm{d}x \, \left(x^2 y' - \cancelto{0}{q}\, y^2\right),$$

$p(x) = x^2$,
$p(1) = 1, p(2) = 4$

$$S[y] = -\alpha y(1)^2 + \beta 4y(2)^2 + \int_1^2 \mathrm{d}x\, x^2 y'.$$

The natural boundary conditions of a Sturm-Liouville system are of the form:

$$\alpha y(1) + y'(1) = 0 \quad \text{and} \quad \beta y(2) + y'(2) = 0.$$

The given boundary conditions are $y(1) = 0$ and $y'(2) = 0$, so this means that $\alpha = 1$ and $y'(1) = 0$ and $\beta y(2) = 0$. Thus,

$$\begin{aligned} S[y] &= \cancelto{0}{-\alpha y(1)^2} + \cancelto{0}{\beta 4y(2)^2} + \int_1^2 \mathrm{d}x\, x^2 y', \\ S[y] &= \int_1^2 \mathrm{d}x\, x^2 y'. \end{aligned}$$

Hence, from the general theory of Sturm-Liouville Systems,

$$\lambda_k = S[y_k] = \int_1^2 \mathrm{d}x\, x^2 y_k',$$

as required.

(c) Given the function $z = A\sin\left(\pi\left(x-1\right)/2\right)$, it will be that the smallest eigenvalue, λ_1, satisfies the inequality

$$\lambda_1 \leq \frac{\left(7\pi^2 - 18\right)\pi^2}{6\left(4 + 3\pi^2\right)},$$

as follows.

Substituting $z = A\sin\left(\pi\left(x-1\right)/2\right)$ into the constraint (3.4) gives

$$\begin{aligned} 1 &= A^2 \int_1^2 \mathrm{d}x\, x \sin^2\left(\frac{\pi\left(x-1\right)}{2}\right) = A^2 \int_1^2 \mathrm{d}x\, x \cos^2\left(\frac{1}{2}\pi x\right), && \text{Using the identity for } \sin\left(\alpha \pm \beta\right). \\ &= A^2 \int_1^2 \mathrm{d}x\, x \frac{1}{2}\left(\cos\left(\pi x\right) + 1\right) = \frac{A^2}{2} \int_1^2 \mathrm{d}x\, x\cos\left(\pi x\right) + x, && \text{Using the identity of } \cos^2\left(\alpha\right) = \tfrac{1}{2}\left(1 + \cos\left(2\alpha\right)\right). \\ &= \frac{A^2}{2} \int_1^2 \mathrm{d}x\, x\cos\left(\pi x\right) + \frac{A^2}{2} \int_1^2 \mathrm{d}x\, x, \\ &= \frac{A^2}{2\pi}\Big[x\sin\left(\pi x\right)\Big]_1^2 - \frac{A^2}{2\pi} \int_1^2 \mathrm{d}x\, \sin\left(\pi x\right) + \frac{A^2}{2} \int_1^2 \mathrm{d}x\, x, && \text{Integrating the first integral by parts.} \end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{2\pi}\left(\cancelto{0}{2\sin(2\pi) - \sin(\pi)}\right) - \frac{A^2}{2\pi}\int_1^2 \mathrm{d}x\,\sin(\pi x) + \frac{A^2}{2}\int_1^2 \mathrm{d}x\,x,\\
&= -\frac{A^2}{2\pi}\int_1^2 \mathrm{d}x\,\sin(\pi x) + \frac{A^2}{2}\int_1^2 \mathrm{d}x\,x,\\
&= -\frac{A^2}{2\pi}\left[-\frac{1}{\pi}\cos(\pi x)\right]_1^2 + \frac{A^2}{2}\left[\frac{x^2}{2}\right]_1^2,\\
&= \frac{A^2}{2\pi}\left[\frac{1}{\pi}\cos(\pi x)\right]_1^2 + \frac{A^2}{2}\left[\frac{x^2}{2}\right]_1^2,\\
&= \frac{A^2}{2\pi^2}\left[\cancelto{1}{\cos(2\pi)} - \cancelto{-1}{\cos(\pi)}\right] + \frac{A^2}{2}\left[\frac{2^2}{2} - \frac{1^2}{2}\right],\\
&= \frac{A^2}{2\pi^2}(2) + \frac{A^2}{2}\left(\frac{3}{2}\right),\\
&= \frac{A^2}{2}\left(\frac{2}{\pi^2} + \frac{3}{2}\right),\\
\therefore 1 &= A^2\left(\frac{1}{\pi^2} + \frac{3}{4}\right). \qquad (3.6)
\end{aligned}
$$

Now,

$$z = A\sin\left(\frac{\pi(x-1)}{2}\right)$$

and

$$\lambda_1 \le S[z] = \int_1^2 \mathrm{d}x\,x^2 z'^2. \qquad (3.7)$$

Differentiating z,

$$
\begin{aligned}
z' &= \frac{\mathrm{d}}{\mathrm{d}x}\left(A\sin\left(\frac{\pi(x-1)}{2}\right)\right),\\
&= A\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\pi(x-1)}{2}\right)\cos\left(\frac{\pi(x-1)}{2}\right),\\
&= A\frac{\pi}{2}\cos\left(\frac{\pi(x-1)}{2}\right),\\
\therefore z' &= A\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right). \qquad (3.8)
\end{aligned}
$$

Making use of the chain rule.

Using the identity for $\cos(\alpha \pm \beta)$. HB p38.

Substituting for z' given by (3.8) into (3.7) gives,

$$
\begin{aligned}
\lambda_1 \leq S[z] &= \int_1^2 \mathrm{d}x\, x^2 \left(A\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)\right)^2, \\
&= \left(\frac{A\pi}{2}\right)^2 \int_1^2 \mathrm{d}x\, x^2 \sin^2\left(\frac{\pi x}{2}\right), \\
&= \frac{A^2\pi^2}{4} \int_1^2 \mathrm{d}x\, x^2 \frac{1}{2}\Bigg(1 - \cos(\pi x)\Bigg), \\
&= \frac{A^2\pi^2}{8} \int_1^2 \mathrm{d}x\, x^2 \left(1 - \cos(\pi x)\right), \\
&= \frac{A^2\pi^2}{8} \int_1^2 \mathrm{d}x\, \left(x^2 - x^2\cos(\pi x)\right), \\
&= \frac{A^2\pi^2}{8} \int_1^2 \mathrm{d}x\, x^2 - \frac{A^2\pi^2}{8} \int_1^2 \mathrm{d}x\, x^2\cos(\pi x), \\
&= \frac{A^2\pi^2}{8} \left[\frac{x^3}{3}\right]_1^2 - \frac{A^2\pi^2}{8} \int_1^2 \mathrm{d}x\, x^2\cos(\pi x), \\
&= \frac{A^2\pi^2 7}{24} - \frac{A^2\pi^2}{8} \int_1^2 \mathrm{d}x\, x^2\cos(\pi x), \\
&= \frac{A^2\pi^2 7}{24} - \frac{A^2\pi^2}{8} \left(\cancelto{0}{\left[\frac{x^2}{\pi}\sin(\pi x)\right]_1^2} - \frac{2}{\pi}\int_1^2 \mathrm{d}x\, x\sin(\pi x) \right), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4} \int_1^2 \mathrm{d}x\, x\sin(\pi x), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4} \left(\cancelto{-\frac{3}{\pi}}{\left[-\frac{x}{\pi}\cos(\pi x)\right]_1^2} - \int_1^2 \mathrm{d}x\, \cos(\pi x) \right), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4} \left(-\frac{3}{\pi} + \frac{1}{\pi} \cancelto{0}{\Big[\sin(\pi x)\Big]_1^2} \right), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4} \left(-\frac{3}{\pi} \right), \\
&= \frac{A^2\pi^2 7}{24} - \frac{A^2 3}{4}, \\
&= A^2 \left(\frac{7\pi^2}{24} - \frac{3}{4} \right), \\
&= \frac{A^2}{4} \left(\frac{7\pi^2 - 18}{6} \right). \qquad (3.9)
\end{aligned}
$$

Using the identity for $\sin^2(\alpha)$. HB p38.

From (3.6)

$$A^2 = \frac{4\pi^2}{4 + 3\pi^2},$$

and substituting for A^2 in (3.9) gives,

$$\lambda_1 \leq \frac{4\pi^2}{4\,(4+3\pi^2)}\left(\frac{7\pi^2-18}{6}\right),$$

$$\lambda_1 \leq \frac{\pi^2}{(4+3\pi^2)}\left(\frac{7\pi^2-18}{6}\right),$$

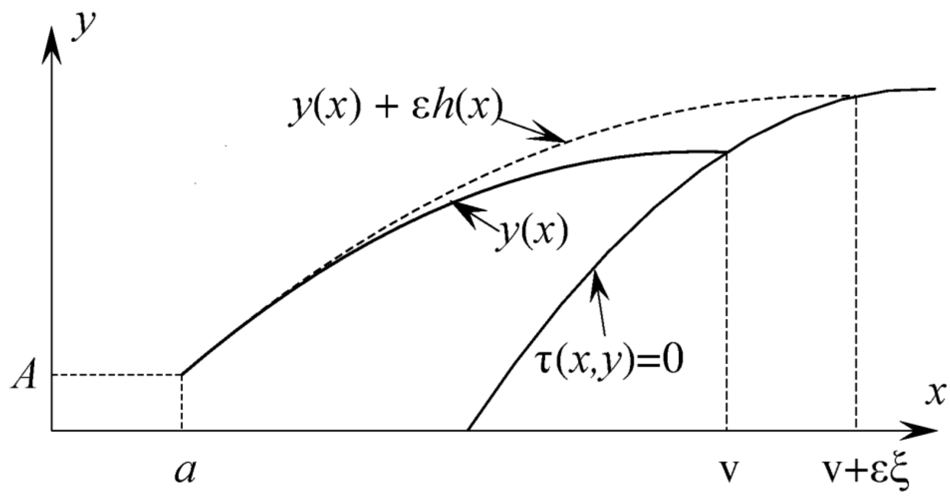
Finally,

$$\lambda_1 \leq \frac{(7\pi^2-18)\,\pi^2}{6\,(4+3\pi^2)},$$

as required.

Q 4.

(a)



This Figure 10.6 taken from the module notes p225.

Figure 2: Diagram showing the stationary path (solid line) and a varied path (dashed line) for a problem in which the left-hand end is fixed, but the other end is free to move along the line defined by $\tau(x, y) = 0$.

Given the perturbed path

$$y_\epsilon(x) = y(x) + \epsilon h(x), \tag{4.1}$$

the Taylor series to the first-order of (4.1) at point $x = v$ (see figure 2) is given in (4.2).

The point $x = v$ is known as the point of expansion. HB p8.

$$y_\epsilon(x) = (y(v) + \epsilon h(v)) + (y'(v) + \epsilon h'(v))(x - v) + \mathcal{O}\left((x - v)^2\right). \tag{4.2}$$

Now, determining the value of (4.2) at $v_\epsilon = v + \epsilon\xi + \mathcal{O}(\epsilon^2)$ where v_ϵ is the perturbed value of v:

$$\begin{aligned} y_\epsilon(v_\epsilon) = {} & y(v) + \epsilon h(v) \\ & + (\cancel{v} + \epsilon\xi + \mathcal{O}(\epsilon^2) \cancel{- v})(y'(v) + \epsilon h'(v)) \\ & + \mathcal{O}\left((\cancel{v} + \epsilon\xi + \mathcal{O}(\epsilon^2) \cancel{- v})^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) = {} & y(v) + \epsilon h(v) \\ & + (\epsilon\xi + \mathcal{O}(\epsilon^2))(y'(v) + \epsilon h'(v)) \\ & + \mathcal{O}\left((\epsilon\xi + \mathcal{O}(\epsilon^2))^2\right), \end{aligned}$$

$$
\begin{aligned}
y_\epsilon(v_\epsilon) = y(v) &+ \epsilon h(v) \\
&+ \epsilon\xi\left(y'(v) + \epsilon h'(v)\right) \\
&+ \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) \\
&+ \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
y_\epsilon(v_\epsilon) = y(v) &+ \epsilon\left(h(v) + \xi y'(v)\right) \\
&+ \underbrace{\epsilon^2\xi h'(v) + \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) + \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right)}_{\text{These are all second-order terms in } \epsilon.}.
\end{aligned}
$$

Thus,

$$
y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(h(v) + \xi y'(v)\right) + \mathcal{O}\left(\epsilon^2\right), \tag{4.3}
$$

as required.

To show that at $(x, y) = (v, y(v))$,

$$
\xi\left(\tau_x + y'(v)\tau_y\right) + h(v)\tau_y = 0
$$

the 2D Taylor expansion to the first-order of $\tau(x, y)$ at point $x = a, y = b$ is required, namely,

$$
\tau(x, y) = \tau(a, b) + \tau_x(a, b)\left[x - a\right] + \tau_y(a, b)\left[y - b\right]. \tag{4.4}
$$

Evaluating (4.4) with $x = v_\epsilon = v + \epsilon\xi$ and $y = y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(y'(v)\xi + h(v)\right)$ gives,

$$
\begin{aligned}
&\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) = \tau(v + \epsilon\xi, y(v) + \epsilon\left(y'(v)\xi + h(v)\right) \\
&= \cancelto{0}{\tau\left(v, y(v)\right)} + \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\
&= \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\
&= \tau_x\left(v, y(v)\right)\left[\cancel{v} + \epsilon\xi \cancel{- v}\right] + \tau_y\left(v, y(v)\right)\left[\cancel{y(v)} + \epsilon\left(y'(v)\xi + h(v)\right) \cancel{- y(v)}\right], \\
&= \tau_x\left(v, y(v)\right)\epsilon\xi + \tau_y\left(v, y(v)\right)\epsilon\left(y'(v)\xi + h(v)\right), \\
&= \epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right],
\end{aligned}
$$

Recall that $\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) = 0$ and therefore,

$$
\begin{aligned}
\epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right] &= 0, \\
\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right) &= 0,
\end{aligned}
$$

$$
\xi\left[\tau_x\left(v, y(v)\right) + \tau_y\left(v, y(v)\right)y'(v)\right] + \tau_y\left(v, y(v)\right)h(v) = 0, \tag{4.5}
$$

as required.

(b) The Gâteaux differential is given as

$$
\Delta S\left(y, h\right) = \xi F\big|_v + \int_a^v \mathrm{d}x\ \left(hF_y + h'F_{y'}\right), \tag{4.6}
$$

and it will be shown, by integrating by parts the integral of (4.6), that a stationary path must satisfy the transversality condition given in (4.7):

$$\tau_x F_{y'} + \tau_y \left(y' F_{y'} - F\right) = 0 \quad \text{at} \quad (x, y) = (v, y(v)) \tag{4.7}$$

as follows.

Equation (4.6) can be rewritten as in (4.8),

$$\Delta S\,(y, h) = \xi F\big|_v + \int_a^v \mathrm{d}x\, h F_y + \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.8}$$

and integrating by parts the right-most integral in (4.8):

$$I = \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.9}$$

$$\begin{aligned}
&\text{Let } u = F_{y'} \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) \\
&\text{Let } \frac{\mathrm{d}v}{\mathrm{d}x} = h'(x) \quad \text{then} \quad v = \int \mathrm{d}x\, h'(x) = h(x).
\end{aligned}$$

For clarity, here v is not the same as that for the upper limit of integration in (4.9).

For integration by parts:

$$\begin{aligned}
I &= \int_a^v \mathrm{d}x\, u \frac{\mathrm{d}v}{\mathrm{d}x} = [uv]_a^v - \int_a^v \mathrm{d}x\, v \frac{\mathrm{d}u}{\mathrm{d}x}, \\
&= \left[F_{y'} h(x)\right]_a^v - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= \left(F_{y'} h(x)\big|_{x=v} - F_{y'} \cancelto{0}{h(a)}\right) - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x).
\end{aligned}$$

The Gâteaux differential (4.8) becomes,

$$\begin{aligned}
\Delta S\,(y, h) &= \xi F\big|_v + \int_a^v \mathrm{d}x\, h(x) F_y + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, h(x) F_y - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, \left(h(x) F_y - \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x)\right), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x) - h(x) F_y\right), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) - F_y\right) h(x)
\end{aligned}$$

$$= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right) h(x). \tag{4.10}$$

On a stationary path $\Delta S\,(y,h) = 0$ for all allowed h and the Euler-Lagrange equation equation is satisfied and so the integrand of (4.10) is zero, thus the Gâteaux differential shown in (4.10) reduces to

$$\Delta S\,(y,h) = \xi F|_{x=v} + F_{y'}h(x)|_{x=v} = 0. \tag{4.11}$$

Rewriting (4.5) more succinctly as

$$\xi\,(\tau_x + \tau_y y'(x))|_{x=v} + \tau_y h(x)|_{x=v} = 0, \tag{4.12}$$

$\tau_x = \tau_x\,(v, y(v))$, and
$\tau_y = \tau_y\,(v, y(v))$

and rearranging (4.12) in terms of $h(x)$ evaluated at $x = v$,

$$h(x)\,|_{x=v} = -\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\,|_{x=v}\,. \tag{4.13}$$

Substituting for $h(v)$ from (4.13) into (4.11) gives,

$$\begin{aligned}
\xi F|_{x=v} - F_{y'}\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\Big|_{x=v} &= 0,\\
\left(\xi F - F_{y'}\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\right)\Bigg|_{x=v} &= 0,\\
\xi\left(F - F_{y'}\frac{(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Bigg|_{x=v} &= 0,\\
-\left(-F + F_{y'}\frac{(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Bigg|_{x=v} &= 0,\\
-\left(\frac{-F\tau_y + F_{y'}\,(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Bigg|_{x=v} &= 0,\\
-\,(-F\tau_y + F_{y'}\,(\tau_x + y'(x)\tau_y))|_{x=v} &= 0,\\
-\,(-F\tau_y + F_{y'}\tau_x + F_{y'}y'(x)\tau_y)|_{x=v} &= 0,\\
-\,(\tau_y\,(-F + F_{y'}y'(x)) + F_{y'}\tau_x)|_{x=v} &= 0,\\
-\,(\tau_y\,(F_{y'}y'(x) - F) + F_{y'}\tau_x)|_{x=v} &= 0.
\end{aligned}$$

Finally,

$$\tau_y\,(F_{y'}y'(x) - F)\,|_{x=v} + F_{y'}\tau_x|_{x=v} = 0. \tag{4.14}$$

Equation (4.14) shows that the transversality condition has been satisfied, as required.

(c) Now, considering the case when

$$S[y] = \int_0^v \mathrm{d}x\,\frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta,$$

where $y(v) > 0$, $\delta > 0$ and the right-hand end point $(v, y(v))$ lies on the line $\alpha y + \beta x + \gamma = 0$, where α, β, γ are constants with $\beta \neq 0$, it can be shown that the first-integral may be written as

$$y\sqrt{1+y'^2} = c, \tag{4.15}$$

for some constant $c > 0$, as follows.

The first-integral is given by

$$y'G_{y'} - G = \text{constant}, \qquad \text{HB p17.}$$

with,

$$G = \frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta.$$

$$\begin{aligned}
G_{y'} &= \frac{1}{y}\left(\frac{1}{\cancel{2}_1}(1+y'^2)^{-\frac{1}{2}}\,\cancel{2}^{1}y'\right),\\
&= \frac{(1+y'^2)^{-\frac{1}{2}}\,y'}{y},\\
&= \frac{y'}{y\sqrt{1+y'^2}}.
\end{aligned}$$

The first-integral becomes,

$$\begin{aligned}
y'\left(\frac{y'}{y\sqrt{1+y'^2}}\right) - \frac{\sqrt{1+y'^2}}{y} &= c, \quad \text{where } c \text{ is a constant,}\\
\left(\frac{y'^2}{y\sqrt{1+y'^2}}\right) - \frac{\sqrt{1+y'^2}}{y} &= c,\\
\frac{y'^2}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}\sqrt{1+y'^2}}{y\sqrt{1+y'^2}} &= c,\\
\frac{y'^2}{y\sqrt{1+y'^2}} - \frac{1+y'^2}{y\sqrt{1+y'^2}} &= c,\\
\frac{y'^2-1-y'^2}{y\sqrt{1+y'^2}} &= c\\
-\frac{1}{y\sqrt{1+y'^2}} &= c,\\
-\frac{1}{c} &= y\sqrt{1+y'^2}.
\end{aligned}$$

Redefining the constant c, then the first-integral may be written as,

$$y\sqrt{1+y'^2} = c, \quad \text{for some constant } c > 0, \text{ as required.} \tag{4.16}$$

Now, rearranging (4.16) in terms of y', as follows.

$$\begin{aligned}
y'^2 = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= \frac{c^2}{y^2} - 1,\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{\frac{c^2}{y^2} - 1}.
\end{aligned}$$

Then,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 1 \Big/ \frac{\mathrm{d}y}{\mathrm{d}x},$$

so,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{1}{\sqrt{\frac{c^2}{y^2} - 1}} = \frac{1}{\sqrt{\frac{c^2-y^2}{y^2}}} = \frac{y}{\sqrt{c^2 - y^2}}. \\
\int \mathrm{d}y \, \frac{\mathrm{d}x}{\mathrm{d}y} &= \int \mathrm{d}y \, \frac{y}{\sqrt{c^2 - y^2}}, \\
x &= \int \mathrm{d}x \, \frac{y}{\sqrt{c^2 - y^2}}. \qquad (4.17)
\end{aligned}$$

Solving the integral of (4.17),

$$x = \int \mathrm{d}y \, \frac{y}{\sqrt{c^2 - y^2}}.$$

Let $u = c^2 - y^2$,

$$\frac{\mathrm{d}u}{\mathrm{d}y} = -2y, \quad \text{so} \quad \frac{\mathrm{d}y}{\mathrm{d}u} = 1 \Big/ \frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{1}{2y}.$$

$$\begin{aligned}
x &= \int \mathrm{d}u \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right) \frac{y}{\sqrt{u}}, \\
&= \int \mathrm{d}u \left(-\frac{1}{2\cancel{y}}\right) \frac{\cancel{y}}{\sqrt{u}}, \\
&= -\frac{1}{2} \int \mathrm{d}u \, \frac{1}{\sqrt{u}}, \\
&= -\frac{1}{2} \int \mathrm{d}u \, u^{-\frac{1}{2}}, \\
&= -\frac{1}{2} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c_\delta = -\sqrt{u} - c_\delta,
\end{aligned}$$

where c_δ is the constant of integration.

Thus,

$$\begin{aligned}
x &= -\sqrt{c^2 - y^2} - c_\delta, \\
(x + c_\delta)^2 &= c^2 - y^2, \\
y^2 + (x + c_\delta)^2 &= c^2, \quad \text{as required.} \qquad (4.18)
\end{aligned}$$

The solution of the first-integral (4.18) are circles centred at $(-c_\delta, 0)$.

Applying the boundary condition, $y(0) = \delta$, to (4.18) gives,

$$\delta^2 + c_\delta^2 = c^2 \quad \text{and so,} \quad c_\delta^2 = c^2 - \delta^2.$$

Differentiating $y^2 + (x + c_\delta)^2 = c^2$ implicitly:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x + c_\delta\right)^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left(c^2\right),\\
&2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(x + c_\delta\right) = 0,\\
&y\frac{\mathrm{d}y}{\mathrm{d}x} + x + c_\delta = 0,\\
&yy' + x + c_\delta = 0. \qquad (4.19)
\end{aligned}$$

Comparing (4.19) with $\alpha y + \beta x + \gamma = 0$ but first dividing through this expression by β:

$$\frac{\alpha}{\beta}y + x + \frac{\gamma}{\beta} = 0,$$

then it is seen that $y' = \alpha/\beta$ and $c_\delta = \gamma/\beta$.

Finally, it can be shown that in the limit as $\delta \to 0$, the stationary path becomes

$$\beta^2 y^2 + \left(\beta x + \gamma\right)^2 = \gamma^2,$$

as follows.

In the limit as $\delta \to 0$, $c_\delta = c$ and equation (4.18) can be written as,

$$y^2 + \left(x + c_\delta\right)^2 = c_\delta^2. \qquad (4.20)$$

Substituting into (4.20) for $c_\delta = \gamma/\beta$ gives,

$$y^2 + \left(x + \frac{\gamma}{\beta}\right)^2 = \frac{\gamma^2}{\beta^2},$$

cross multiplying by β^2 gives,

$$\begin{aligned}
&\beta^2 y^2 + \beta^2\left(x + \frac{\gamma}{\beta}\right)^2 = \gamma^2 \quad \text{and}\\
&\beta^2 y^2 + \left(\beta x + \gamma\right)^2 = \gamma^2 \quad \text{as required.}
\end{aligned}$$

Q 5. Given

$$S[x] = \int_a^b \mathrm{d}t\, L\left(t, x, \dot{x}\right), \quad \text{with} \quad b > a,$$

where L is called the Lagrangian, and $x(t)$ is at least twice differentiable.

The *conjugate momentum* p is define by

$$p = \frac{\partial L}{\partial \dot{x}}. \tag{5.1}$$

(a) It can be shown that the Euler-Lagrange equation for S is defined by

$$\dot{p} = \frac{\partial L}{\partial x},$$

as follows.

The Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,$$

which becomes after substituting in the *conjugate momentum,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p\right) - \frac{\partial L}{\partial x} = 0, \quad \text{and}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\partial L}{\partial x} = 0,$$

$$\therefore \quad \dot{p} = \frac{\partial L}{\partial x}, \quad \text{as required.}$$

(b) From the handbook, the total derivative can be expressed as: see HB p3.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\frac{\mathrm{d}x_k}{\mathrm{d}t}.$$

Using this result then,

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(L(t, x, \dot{x})\right), \\
&= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}, \\
&= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x}, \\
&= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x}, \quad \text{as required.}
\end{aligned}$$

Recall that p is the *conjugate momentum* define above.

(c) The *Hamiltonian* $H = H(t, x, p)$ is defined by $H(t, x, p) = p\dot{x} - L(t, x, \dot{x})$, where (implicitly) $\dot{x}$ is eliminated using (5.1) to give a function of t, x and p.

Using the result obtained in part (b) it will be shown that for a stationary path of S that

$$\frac{\partial L}{\partial t} = -\dot{H} \tag{5.2}$$

as follows.

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(p\dot{x} - L),\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(p\dot{x}) - \frac{\mathrm{d}L}{\mathrm{d}t},\\ &= p\ddot{x} + \dot{p}\dot{x} - \frac{\mathrm{d}L}{\mathrm{d}t},\end{aligned}$$

and substituting into the above expression for $\frac{\mathrm{d}L}{\mathrm{d}t}$ from part (b) gives,

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= p\ddot{x} + \dot{p}\dot{x} - \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x}\right),\\ &= \cancel{p\ddot{x}} + \cancel{\dot{p}\dot{x}} - \left(\frac{\partial L}{\partial t} + \cancel{\dot{p}\dot{x}} + \cancel{p\ddot{x}}\right),\end{aligned}$$

$$\therefore \frac{\mathrm{d}H}{\mathrm{d}t} = \dot{H} = -\frac{\partial L}{\partial t}, \quad \text{as required.}$$

(d) The *Rund-Trautman identity* is given as

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0, \tag{5.3}$$

and from this identity it will be shown that

$$(\xi - \dot{x}\tau)\left[\dot{p} - \frac{\partial L}{\partial x}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right], \tag{5.4}$$

as follows.

The Rund-Trautman Identity:

$$\frac{\partial L}{\partial x}\xi + \boxed{p\dot{\xi}} + \frac{\partial L}{\partial t}\tau - \boxed{H\dot{\tau}} = 0, \tag{5.5}$$

$$-\frac{\partial L}{\partial x}\xi - \frac{\partial L}{\partial t}\tau = \boxed{p\dot{\xi}} - \boxed{H\dot{\tau}} \tag{5.6}$$

The RHS:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = \boxed{p\dot{\xi}} + \dot{p}\xi - \boxed{H\dot{\tau}} - \dot{H}\tau \tag{5.7}$$

$$= \boxed{p\dot{\xi}} - \boxed{H\dot{\tau}} - \dot{H}\tau + \dot{p}\xi \tag{5.8}$$

(5.6) into (5.8):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = -\frac{\partial L}{\partial x}\xi - \frac{\partial L}{\partial t}\tau - \dot{H}\tau + \dot{p}\xi \tag{5.9}$$

collect terms in ξ

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = \xi\left(\dot{p} - \frac{\partial L}{\partial x}\right) - \frac{\partial L}{\partial t}\tau - \dot{H}\tau \tag{5.10}$$

From the question, the *Hamiltonian* is defined as

$$H = p\dot{x} - L \quad \text{and its derivative is given by} \tag{5.11}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \dot{H} = \frac{\mathrm{d}}{\mathrm{d}t}\left(p\dot{x}\right) - \frac{\mathrm{d}L}{\mathrm{d}t} \tag{5.12}$$

$$= p\ddot{x} + \dot{p}\dot{x} - \frac{\mathrm{d}L}{\mathrm{d}t} \tag{5.13}$$

As shown in part (b)

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x} \tag{5.14}$$

(5.14) into (5.13)

$$\dot{H} = p\ddot{x} + \dot{p}\dot{x} - \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x}\right) \tag{5.15}$$

$$= p\ddot{x} + \dot{p}\dot{x} - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial x}\dot{x} - p\ddot{x} \tag{5.16}$$

$$\dot{H} + \frac{\partial L}{\partial t} = p\ddot{x} + \dot{p}\dot{x} - \frac{\partial L}{\partial x}\dot{x} - p\ddot{x} \tag{5.17}$$

Multiply (5.17) by $-\tau$ and cancel like terms

$$-\dot{H}\tau - \frac{\partial L}{\partial t}\tau = \cancelto{1}{-p\ddot{x}\tau} - \dot{p}\dot{x}\tau + \frac{\partial L}{\partial x}\dot{x}\tau \cancelto{1}{+p\ddot{x}\tau} \tag{5.18}$$

$$= -\dot{p}\dot{x}\tau + \frac{\partial L}{\partial x}\dot{x}\tau \tag{5.19}$$

$$= -\left(\dot{p} - \frac{\partial L}{\partial x}\right)\dot{x}\tau \tag{5.20}$$

Substitute (5.20) into (5.10)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = \xi\left(\dot{p} - \frac{\partial L}{\partial x}\right) - \left(\dot{p} - \frac{\partial L}{\partial x}\right)\dot{x}\tau \tag{5.21}$$

Finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = \left(\xi - \dot{x}\tau\right)\left(\dot{p} - \frac{\partial L}{\partial x}\right) \tag{5.22}$$

as required.

For a stationary path

$$\dot{p} = \frac{\partial L}{\partial \dot{t}}$$

and so (5.22) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = (\xi - \dot{x}\tau)\left(\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial x}\right) = 0 \tag{5.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = 0. \tag{5.24}$$

As the differentiation of a constant is zero, then the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - \tau H\right] = 0$$

must mean that

$$p\xi - \tau H = \text{constant}.$$

(e) Now, considering a particle of constant mass m moving along the x-axis in a potential $V(x)$. The Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - V(x)$, and the path of the particle from $t = a$ to $t = b$ is a stationary path of S.

The conjugate momentum p is calculated as follows.

$$p = \frac{\partial L}{\partial \dot{x}} \quad \text{and} \quad L = \frac{1}{2}m\dot{x}^2 - V(x).$$

$$\therefore p = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = 2 \cdot \frac{1}{2}m\dot{x} = m\dot{x}.$$

The Hamiltonian is calculated as follows.

$$H\left(t, x, p\right) = p\dot{x} - L\left(t, x, \dot{x}\right).$$

Substituting into this expression for conjugate momentum p and the Lagrangian L, gives,

$$\begin{aligned} H(x, \dot{x}) &= \left(m\dot{x}\right)\dot{x} - \left(\frac{1}{2}m\dot{x}^2 - V(x)\right), \\ &= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x), \\ &= \frac{1}{2}m\dot{x}^2 + V(x). \end{aligned}$$

From above $p = m\dot{x}$ and therefore $\dot{x} = p/m$, hence,

$$\boxed{H(p, x) = \frac{p^2}{2m} + V(x).} \tag{5.25}$$

As the Lagrangian L (and hence S) is invariant under the *translation of the time coordinate* t, the one-parameter family of transformations (there is only one independent variable) given in the question, namely:

This is the first-order expansion of the 1-parameter family w.r.t. δ at $\delta = 0$.

$$\bar{t} = t + \tau\delta + O\,(\delta^2),$$
$$\overline{x} = x + \xi\delta + O\,(\delta^2),$$

become (dropping the order δ^2 terms):

$$\bar{t} = t + \tau\delta,$$
$$\overline{x} = x.$$

Here, $\xi = 0$ and $\tau \neq 0$ with $\tau =$ constant due to the Lagrangian only being invariant under translation of the time coordinate and not invariant under the translation of position with time. The result of part (d) was

$$p\xi - H\tau = \text{constant},$$

and thus, as $\xi = 0$ becomes

$$-H\tau = \textit{constant}.$$

Therefore, H is constant along the path travelled by the particle in the potential field. This means that the total energy of the system is conserved over the distance travelled by the particle in the potential field.