Q 1.

(a)

(b)

Q 2.

Q 3.

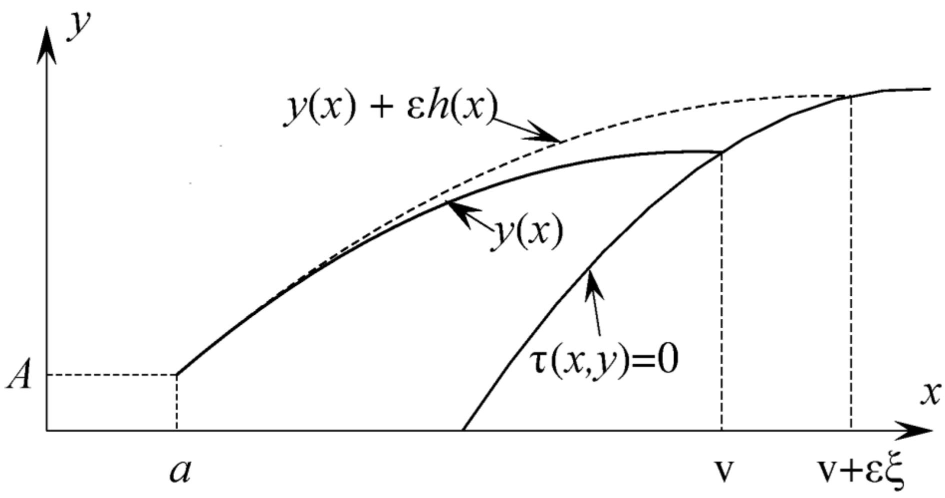
(a)

(b)

(c)

Q 4.

(a)



This Figure 10.6 taken from the module notes p225.

Figure 1: Diagram showing the stationary path (solid line) and a varied path (dashed line) for a problem in which the left-hand end is fixed, but the other end is free to move along the line defined by $\tau(x, y) = 0$.

Given the perturbed path

$$y_\epsilon(x) = y(x) + \epsilon h(x), \tag{4.1}$$

the Taylor series to the first-order of (4.1) at point $x = v$ (see figure 1) is given in (4.2).

The point $x = v$ is known as the point of expansion. HB p8.

$$y_\epsilon(x) = (y(v) + \epsilon h(v)) + (y'(v) + \epsilon h'(v))\,(x - v) + \mathcal{O}\left((x - v)^2\right). \tag{4.2}$$

Now, determining the value of (4.2) at $v_\epsilon = v + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right)$ where v_ϵ is the perturbed value of v:

$$\begin{aligned} y_\epsilon(v_\epsilon) &= y(v) + \epsilon h(v) \\ &\quad + \left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{- v}\right)\left(y'(v) + \epsilon h'(v)\right) \\ &\quad + \mathcal{O}\left(\left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{- v}\right)^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) &= y(v) + \epsilon h(v) \\ &\quad + \left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)\left(y'(v) + \epsilon h'(v)\right) \\ &\quad + \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) = y(v) &+ \epsilon h(v) \\ &+ \epsilon\xi\left(y'(v) + \epsilon h'(v)\right) \\ &+ \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) \\ &+ \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) = y(v) &+ \epsilon\left(h(v) + \xi y'(v)\right) \\ &+ \underbrace{\epsilon^2\xi h'(v) + \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) + \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right)}_{\text{These are all second-order terms in } \epsilon.}. \end{aligned}$$

Thus,

$$y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(h(v) + \xi y'(v)\right) + \mathcal{O}\left(\epsilon^2\right), \tag{4.3}$$

as required.

To show that at $(x, y) = (v, y(v))$,

$$\xi\left(\tau_x + y'(v)\tau_y\right) + h(v)\tau_y = 0$$

the 2D Taylor expansion to the first-order of $\tau(x, y)$ at point $x = a, y = b$ is required, namely,

$$\tau(x, y) = \tau(a, b) + \tau_x(a, b)\left[x - a\right] + \tau_y(a, b)\left[y - b\right]. \tag{4.4}$$

Evaluating (4.4) with $x = v_\epsilon = v + \epsilon\xi$ and $y = y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(y'(v)\xi + h(v)\right)$ gives,

$$\begin{aligned} \tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) &= \tau(v + \epsilon\xi, y(v) + \epsilon\left(y'(v)\xi + h(v)\right) \\ &= \cancelto{0}{\tau\left(v, y(v)\right)} + \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\ &= \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\ &= \tau_x\left(v, y(v)\right)\left[\cancel{v} + \epsilon\xi \cancel{- v}\right] + \tau_y\left(v, y(v)\right)\left[\cancel{y(v)} + \epsilon\left(y'(v)\xi + h(v)\right) \cancel{- y(v)}\right], \\ &= \tau_x\left(v, y(v)\right)\epsilon\xi + \tau_y\left(v, y(v)\right)\epsilon\left(y'(v)\xi + h(v)\right), \\ &= \epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right], \end{aligned}$$

Recall that $\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) = 0$ and therefore,

$$\begin{aligned} \epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right] &= 0, \\ \tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right) &= 0, \end{aligned}$$

$$\xi\left[\tau_x\left(v, y(v)\right) + \tau_y\left(v, y(v)\right)y'(v)\right] + \tau_y\left(v, y(v)\right)h(v) = 0, \tag{4.5}$$

as required.

(b) The Gâteaux differential is given as

$$\Delta S\left(y, h\right) = \xi F\big|_v + \int_a^v \mathrm{d}x\ \left(hF_y + h'F_{y'}\right), \tag{4.6}$$

and it will be shown, by integrating by parts the integral of (4.6), that a stationary path must satisfy the transversality condition given in (4.7):

$$\tau_x F_{y'} + \tau_y \left(y' F_{y'} - F\right) = 0 \quad \text{at} \quad (x, y) = (v, y(v)) \tag{4.7}$$

as follows.

Equation (4.6) can be rewritten as in (4.8),

$$\Delta S\,(y, h) = \xi F\big|_v + \int_a^v \mathrm{d}x\, hF_y + \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.8}$$

and integrating by parts the right-most integral in (4.8):

$$I = \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.9}$$

$$\begin{aligned}
&\text{Let } u = F_{y'} \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) \\
&\text{Let } \frac{\mathrm{d}v}{\mathrm{d}x} = h'(x) \quad \text{then} \quad v = \int \mathrm{d}x\, h'(x) = h(x).
\end{aligned}$$

For clarity, here v is not the same as that for the upper limit of integration in (4.9).

For integration by parts:

$$\begin{aligned}
I &= \int_a^v \mathrm{d}x\, u\frac{\mathrm{d}v}{\mathrm{d}x} = [uv]_a^v - \int_a^v \mathrm{d}x\, v\frac{\mathrm{d}u}{\mathrm{d}x}, \\
&= \left[F_{y'} h(x)\right]_a^v - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= \left(F_{y'} h(x)\big|_{x=v} - F_{y'} \cancelto{0}{h(a)}\right) - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x).
\end{aligned}$$

The Gâteaux differential (4.8) becomes,

$$\begin{aligned}
\Delta S\,(y, h) &= \xi F\big|_v + \int_a^v \mathrm{d}x\, h(x) F_y + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, h(x) F_y - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, \left(h(x) F_y - \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x)\right), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x) - h(x) F_y\right), \\
&= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) - F_y\right) h(x)
\end{aligned}$$

$$= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right) h(x). \tag{4.10}$$

On a stationary path $\Delta S\,(y,h) = 0$ for all allowed h and the Euler-Lagrange equation equation is satisfied and so the integrand of (4.10) is zero, thus the Gâteaux differential shown in (4.10) reduces to

$$\Delta S\,(y,h) = \xi F|_{x=v} + F_{y'}h(x)|_{x=v} = 0. \tag{4.11}$$

Rewriting (4.5) more succinctly as

$$\xi\,(\tau_x + \tau_y y'(x))|_{x=v} + \tau_y h(x)|_{x=v} = 0, \tag{4.12}$$

$\tau_x = \tau_x\,(v, y(v))$, and
$\tau_y = \tau_y\,(v, y(v))$

and rearranging (4.12) in terms of $h(x)$ evaluated at $x = v$,

$$h(x)\,|_{x=v} = -\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\,|_{x=v}\,. \tag{4.13}$$

Substituting for $h(v)$ from (4.13) into (4.11) gives,

$$\begin{aligned}
\xi F|_{x=v} - F_{y'}\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\Big|_{x=v} &= 0,\\
\left(\xi F - F_{y'}\frac{\xi}{\tau_y}\,(\tau_x + y'(x)\tau_y)\right)\Big|_{x=v} &= 0,\\
\xi\left(F - F_{y'}\frac{(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(-F + F_{y'}\frac{(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(\frac{-F\tau_y + F_{y'}\,(\tau_x + y'(x)\tau_y)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(-F\tau_y + F_{y'}\,(\tau_x + y'(x)\tau_y)\right)|_{x=v} &= 0,\\
-\left(-F\tau_y + F_{y'}\tau_x + F_{y'}y'(x)\tau_y\right)|_{x=v} &= 0,\\
-\left(\tau_y\,(-F + F_{y'}y'(x)) + F_{y'}\tau_x\right)|_{x=v} &= 0,\\
-\left(\tau_y\,(F_{y'}y'(x) - F) + F_{y'}\tau_x\right)|_{x=v} &= 0.
\end{aligned}$$

Finally,

$$\tau_y\,(F_{y'}y'(x) - F)\,|_{x=v} + F_{y'}\tau_x|_{x=v} = 0. \tag{4.14}$$

Equation (4.14) shows that the transversality condition has been satisfied, as required.

(c) Now, considering the case when

$$S[y] = \int_0^v \mathrm{d}x\,\frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta,$$

where $y(v) > 0$, $\delta > 0$ and the right-hand end point $(v, y(v))$ lies on the line $\alpha y + \beta x + \gamma = 0$, where α, β, γ are constants with $\beta \neq 0$, it can be shown that the first-integral may be written as

$$y\sqrt{1+y'^2} = c, \tag{4.15}$$

for some constant $c > 0$, as follows.

The first-integral is given by

$$y'G_{y'} - G = \text{constant}, \qquad \text{HB p17.}$$

with,

$$
\begin{aligned}
G &= \frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta. \\
G_{y'} &= \frac{1}{y}\left(\frac{1}{\cancel{2}_1}\left(1+y'^2\right)^{-\frac{1}{2}} \overset{1}{\cancel{2}}y'\right), \\
&= \frac{\left(1+y'^2\right)^{-\frac{1}{2}} y'}{y}, \\
&= \frac{y'}{y\sqrt{1+y'^2}}.
\end{aligned}
$$

The first-integral becomes,

$$
\begin{aligned}
y'\left(\frac{y'}{y\sqrt{1+y'^2}}\right) - \frac{\sqrt{1+y'^2}}{y} &= c, \quad \text{where } c \text{ is a constant,} \\
\left(\frac{y'^2}{y\sqrt{1+y'^2}}\right) - \frac{\sqrt{1+y'^2}}{y} &= c, \\
\frac{y'^2}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}\sqrt{1+y'^2}}{y\sqrt{1+y'^2}} &= c, \\
\frac{y'^2}{y\sqrt{1+y'^2}} - \frac{1+y'^2}{y\sqrt{1+y'^2}} &= c, \\
\frac{y'^2 - 1 - y'^2}{y\sqrt{1+y'^2}} &= c \\
-\frac{1}{y\sqrt{1+y'^2}} &= c, \\
-\frac{1}{c} = y\sqrt{1+y'^2}&.
\end{aligned}
$$

Redefining the constant c, then the first-integral may be written as,

$$y\sqrt{1+y'^2} = c, \quad \text{for some constant } c > 0, \text{ as required.} \tag{4.16}$$

Now, rearranging (4.16) in terms of y', as follows.

$$
\begin{aligned}
y'^2 = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= \frac{c^2}{y^2} - 1, \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{\frac{c^2}{y^2} - 1}.
\end{aligned}
$$

Then,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 1 \Big/ \frac{\mathrm{d}y}{\mathrm{d}x},$$

so,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{1}{\sqrt{\frac{c^2}{y^2} - 1}} = \frac{1}{\sqrt{\frac{c^2-y^2}{y^2}}} = \frac{y}{\sqrt{c^2 - y^2}}. \\
\int \mathrm{d}y \, \frac{\mathrm{d}x}{\mathrm{d}y} &= \int \mathrm{d}y \, \frac{y}{\sqrt{c^2 - y^2}}, \\
x &= \int \mathrm{d}x \, \frac{y}{\sqrt{c^2 - y^2}}. && (4.17)
\end{aligned}$$

Solving the integral of (4.17),

$$x = \int \mathrm{d}y \, \frac{y}{\sqrt{c^2 - y^2}}.$$

Let $u = c^2 - y^2$,

$$\frac{\mathrm{d}u}{\mathrm{d}y} = -2y, \quad \text{so} \quad \frac{\mathrm{d}y}{\mathrm{d}u} = 1 \Big/ \frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{1}{2y}.$$

$$\begin{aligned}
x &= \int \mathrm{d}u \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right) \frac{y}{\sqrt{u}}, \\
&= \int \mathrm{d}u \left(-\frac{1}{2\cancel{y}}\right) \frac{\cancel{y}}{\sqrt{u}}, \\
&= -\frac{1}{2} \int \mathrm{d}u \, \frac{1}{\sqrt{u}}, \\
&= -\frac{1}{2} \int \mathrm{d}u \, u^{-\frac{1}{2}}, \\
&= -\frac{1}{2} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c_\delta = -\sqrt{u} - c_\delta,
\end{aligned}$$

where c_δ is the constant of integration.

Thus,

$$\begin{aligned}
x &= -\sqrt{c^2 - y^2} - c_\delta, \\
(x + c_\delta)^2 &= c^2 - y^2, \\
y^2 + (x + c_\delta)^2 &= c^2, \quad \text{as required.} && (4.18)
\end{aligned}$$

The solution of the first-integral (4.18) are circles centred at $(-c_\delta, 0)$.

Applying the boundary condition, $y(0) = \delta$, to (4.18) gives,

$$\delta^2 + c_\delta^2 = c^2 \quad \text{and so,} \quad c_\delta^2 = c^2 - \delta^2.$$

Differentiating $y^2 + (x + c_\delta)^2 = c^2$ implicitly:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x + c_\delta\right)^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left(c^2\right),\\
&2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(x + c_\delta\right) = 0,\\
&y\frac{\mathrm{d}y}{\mathrm{d}x} + x + c_\delta = 0,\\
&yy' + x + c_\delta = 0.
\end{aligned} \tag{4.19}$$

Comparing (4.19) with $\alpha y + \beta x + \gamma = 0$ but first dividing through this expression by β:

$$\frac{\alpha}{\beta}y + x + \frac{\gamma}{\beta} = 0,$$

then it is seen that $y' = \alpha/\beta$ and $c_\delta = \gamma/\beta$.

Finally, it can be shown that in the limit as $\delta \to 0$, the stationary path becomes

$$\beta^2 y^2 + \left(\beta x + \gamma\right)^2 = \gamma^2,$$

as follows.

In the limit as $\delta \to 0$, $c_\delta = c$ and equation (4.18) can be written as,

$$y^2 + \left(x + c_\delta\right)^2 = c_\delta^2. \tag{4.20}$$

Substituting into (4.20) for $c_\delta = \gamma/\beta$ gives,

$$y^2 + \left(x + \frac{\gamma}{\beta}\right)^2 = \frac{\gamma^2}{\beta^2},$$

cross multiplying by β^2 gives,

$$\begin{aligned}
&\beta^2 y^2 + \beta^2\left(x + \frac{\gamma}{\beta}\right)^2 = \gamma^2 \quad \text{and}\\
&\beta^2 y^2 + \left(\beta x + \gamma\right)^2 = \gamma^2 \quad \text{as required.}
\end{aligned}$$

Q 5.

(a)

(b)

(c)

(d)