# Q 1.

(a) The functional is

$$S[y] = \alpha y(1)^2 + \int_0^1 \mathrm{d}x\, \beta y'^2, \quad y(0) = 0,$$

with natural boundary condition at $x = 1$ and subject to the constraint

$$C[y] = \gamma y(1)^2 + \int_0^1 \mathrm{d}x\, w(x) y^2 = 1,$$

where $\alpha$, $\beta$ and $\gamma$ are non-zero constants.

To show that the stationary paths of this system satisfy the Euler-Lagrange equation

$$\beta \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda w(x) y = 0, \quad y(0) = 0, \ \ (\alpha - \gamma\lambda)\, y(1) + \beta y'(1) = 0,$$

where $\lambda$ is a Lagrange multiplier, consider the following.

The auxiliary functional is

$$\begin{aligned}\overline{S}[y] &= \alpha y(1)^2 + \int_0^1 \mathrm{d}x\, \beta y'^2 - \lambda\gamma y(1)^2 - \lambda \int_0^1 \mathrm{d}x\, w(x) y^2, \\ &= \alpha y(1)^2 - \lambda\gamma y(1)^2 + \int_0^1 \mathrm{d}x\, \left(\beta y'^2 - \lambda w(x) y^2\right), \\ &= (\alpha - \lambda\gamma)\, y(1)^2 + \int_0^1 \mathrm{d}x\, \left(\beta y'^2 - \lambda w(x) y^2\right).\end{aligned}$$

The Gâteaux differential $\Delta\overline{S}[y,h]$ of the functional $\overline{S}[y]$ is given by the expression See HB p16.

$$\Delta\overline{S}[y,h] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} S[y + \epsilon H]\bigg|_{\epsilon=0}.$$

Rewriting the expression for $\overline{S}[y]$ and replacing each occurrence of $y$ with $y + \epsilon h$ gives,

$$\begin{aligned}\Delta\overline{S}[y + \epsilon h] = (\alpha - \lambda\gamma)\left(y(1) + \epsilon h(1)\right)^2 + \\ \int_0^1 \mathrm{d}x\, \left(\beta (y + \epsilon h)'^2 - \lambda w(x)(y + \epsilon h)^2\right),\end{aligned}$$

$$\begin{aligned}\Delta\overline{S}[y,h] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left( (\alpha - \lambda\gamma)\left(y(1) + \epsilon h(1)\right)^2 \right)\bigg|_{\epsilon=0} + \\ \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left( \int_0^1 \mathrm{d}x\, \left(\beta (y + \epsilon h)'^2 - \lambda w(x)(y + \epsilon h)^2\right) \right)\bigg|_{\epsilon=0}.\end{aligned}$$

which, after carrying out the differentiation becomes,

$$\Delta\overline{S}[y,h] = \left(2\left(\alpha-\lambda\gamma\right)\left(y(1)+\epsilon h(1)\right)h(1)\right)\Big|_{\epsilon=0} + \left(\int_0^1 \mathrm{d}x\, 2\left(\beta(y+\epsilon h)'h' - 2\lambda w(x)(y+\epsilon h)h\right)\right)\Big|_{\epsilon=0},$$

$$= 2\left(\alpha-\lambda\gamma\right)y(1)h(1) + 2\int_0^1 \mathrm{d}x\, \beta y'h' - 2\int_0^1 \mathrm{d}x\, \lambda w(x)yh.$$

Integrating the left-most integral by parts,

$$2\beta\int_0^1 \mathrm{d}x\, y'h' = 2\beta\left(\Big[y'h\Big]_0^1 - \int_0^1 \mathrm{d}x\, y''h\right),$$

and from Table 1 $h(0)=0$ so,

| Boundary condition 1 |
|---|
| $y(1)=0$ |
| $y(1)+\epsilon h(1)=0$ |
| $0+\epsilon h(1)=0$ |
| $\boxed{h(1)=0}$ |

Table 1: Determination of $h(1)$.

$$2\beta\int_0^1 \mathrm{d}x\, y'h' = 2\beta y'(1)h(1) - 2\beta\int_0^1 \mathrm{d}x\, y''h.$$

Thus,

$$\Delta\overline{S}[y,h] = 2\left(\alpha-\lambda\gamma\right)y(1)h(1) + 2\beta y'(1)h(1) - 2\int_0^1 \mathrm{d}x\, \left(\beta y'' + \lambda w(x)y\right)h. \tag{1.1}$$

The Euler-Lagrange equation can be found from the above expression for the Gâteaux differential; by definition it is required that the Gâteaux differential be equal to zero. So by setting the expression shown in (1.1) to zero and dividing through by 2 gives the following for a stationary path

$$\left(\alpha-\lambda\gamma\right)y(1)h(1) + \beta y'(1)h(1) - \int_0^1 \mathrm{d}x\, \left(\beta y'' + \lambda w(x)y\right)h = 0.$$

Then, by the Fundamental Lemma of the Calculus of Variations (meaning that because the result has to be true for all admissible paths the term that $h$ multiplies must be equal to zero) and setting $h(1)=0$, then the Euler-Lagrange equation is given by

$$\beta\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \lambda w(x)y = 0, \quad y(0)=0, \quad \left(\alpha-\gamma\lambda\right)y(1) + \beta y'(1) = 0. \tag{1.2}$$

$\lambda$ is the Lagrange multiplier.

Equation (1.1) is the Gâteaux differential and (1.2) is the Euler-Lagrange equation (together with the boundary conditions) that the stationary paths must satisfy.

(b) Let $w(x) = 1$ and $\alpha = \beta = \gamma = 1$, so that (1.2) becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0, \quad y(0) = 0, \;\; (1-\lambda)y(1) + y'(1) = 0. \tag{1.3}$$

Then to find the non-trivial stationary paths, the eigenfunctions of $y$ (normalised so that $C[y] = 1$) and the values of the Lagrange multipliers, consider the following.

There are three cases to consider regarding $\lambda$, namely $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. So, considering each case in turn as follows.

**if** $\lambda = 0$ set $\lambda = 0$ in (1.3):

$$\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= 0, \quad y(0) = 0, \;\; y(1) + y'(1) = 0. \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= A, \quad \text{where } A \text{ is an arbitrary constant.} \\
y &= Ax + B, \quad \text{where } B \text{ is also an arbitrary constant.}
\end{aligned}$$

Applying the boundary conditions to $y'$ and $y$ gives $A = 0$ and $B = 0$ and therefore $y(x) = 0$. This is a trivial solution.

**If** $\lambda < 0$ set $\lambda = -\mu^2$ $(\mu > 0)$ in (1.3):

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \mu^2 y = 0, \quad y(0) = 0, \;\; (1-\mu^2)y(1) + y'(1) = 0.$$

The auxiliary equation is

$$\begin{aligned}
&r^2 - \mu^2 = 0, \quad \text{with} \quad r_1 = -\mu, \;\; r_2 = \mu. \\
&y(x) = A\mathrm{e}^{-\mu x} + B\mathrm{e}^{\mu x}.
\end{aligned}$$

Applying the first boundary conditions to $y(x)$ with $x = 0$ and $y(o) = 0$ gives

$$0 = A + B \quad \text{therefore} \quad A = -B,$$

and so,

$$y(x) = -B\mathrm{e}^{-\mu x} + B\mathrm{e}^{\mu x}.$$

Differentiating $y$,

$$y'(x) = B\mu\mathrm{e}^{-\mu x} + B\mu\mathrm{e}^{\mu x},$$

Applying the second boundary condition to determine $B$,

$$\begin{aligned}(1-\mu^2)y(1)+y'(1) &= (1-\mu^2)\left(-B\mathrm{e}^{-\mu}+B\mathrm{e}^{\mu}\right)+B\mu\mathrm{e}^{-\mu}+B\mu\mathrm{e}^{\mu}=0,\\ &= B\left\{(1-\mu^2)\left(-\mathrm{e}^{-\mu}+\mathrm{e}^{\mu}\right)+\mu\mathrm{e}^{-\mu}+\mu\mathrm{e}^{\mu}\right\}=0.\end{aligned}$$

By assumption $\mu \neq 0$, so the expression immediately above can only be satisfied if $B=0$ and therefore y(x) = 0. This also is a trivial solution.

**If** $\lambda > 0$ set $\lambda = \mu^2$ ($\mu > 0$) in (1.3):

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mu^2y=0, \quad y(0)=0, \quad (1-\mu^2)y(1)+y'(1)=0.$$

The auxiliary equation is

$$r^2+\mu^2=0, \quad \text{with} \quad r_1=-i\mu, \quad r_2=i\mu.$$

and the general solution is given by

$$y(x)=A\sin\mu x+B\cos\mu x.$$

Applying the first boundary condition $y(0)=0$, gives,

$$y(0)=\cancelto{0}{A\sin 0}+\cancelto{1}{B\cos 0}, \quad \text{and therefore, } B=0.$$

Thus, the updated solution is,

$$y(x)=A\sin\mu x.$$

Differentiating $y(x)$ gives,

$$y'(x)=A\mu\cos\mu x.$$

Applying the second boundary conditions $(1-\mu^2)y(1)+y'(1)=0$:

$$(1-\mu^2)A\sin\mu + A\,\mu\cos\mu = 0,$$

and factoring out $A$ and assuming $A \neq 0$, gives,

$$(1-\mu^2)\sin\mu + \mu\cos\mu = 0, \tag{1.4}$$

$$\frac{(1-\mu^2)}{\mu} = -\frac{\cos\mu}{\sin\mu},$$

$$\frac{(\mu^2-1)}{\mu} = \cot\mu. \tag{1.5}$$

To solve (1.5) for $\mu$ a graph of both sides of (1.5) can be used as shown in Figure 1. From the graphs shown in Figure 1 $\mu_1 \approx 1.208$, $\mu_2 \approx 3.448$, $\mu_3 \approx 6.441$, $\mu_4 \approx 9.530$, $\mu_5 \approx 12.646$ and it is seen that as $n$ increases $\mu_n$ approaches $(n-1)\pi$, $n=1,2,\ldots$. Thus,

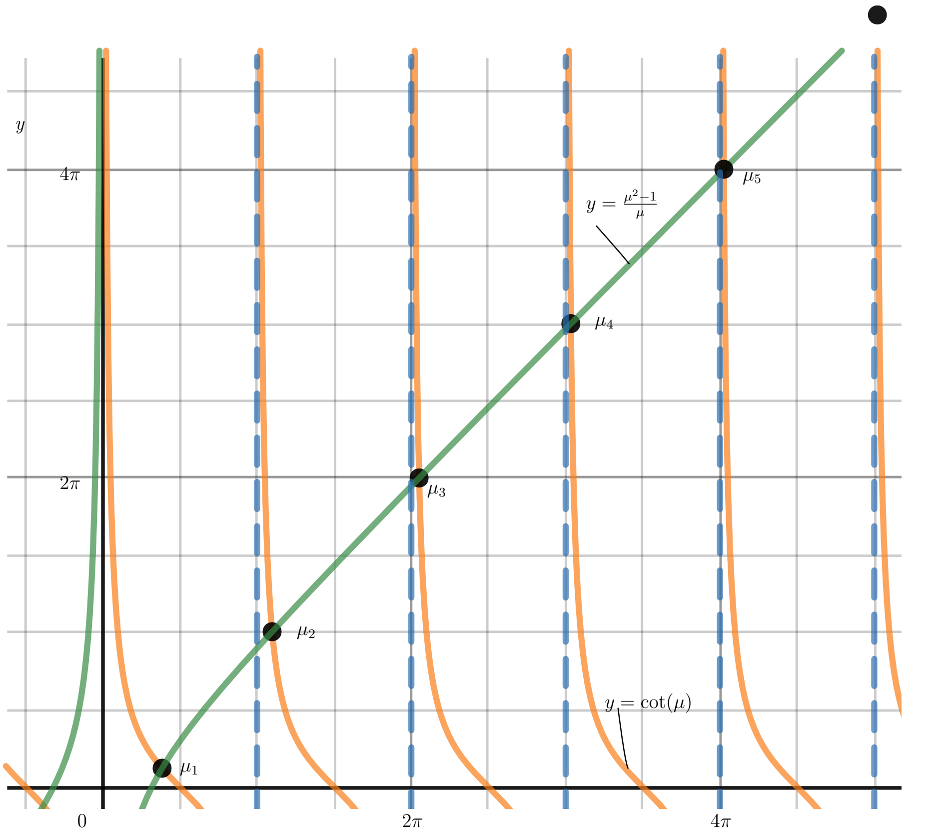



Figure 1: Graphs of $y = \left(\dfrac{\mu^2-1}{\mu}\right)$ and $y = \cot(\mu)$.

the eigenvalues $\lambda_n$ approach $(n-1)^2\pi^2$ as $n$ increases. Hence, the eigenfunctions corresponding to the eigenvalues $\lambda_n$ are,

$$y_n(x) = A_n \sin\left(\sqrt{\lambda_n}x\right), \qquad n = 1, 2, \ldots. \tag{1.6}$$

To normalise the eigenfunctions (1.6) consider the following.

The inner product (with unit weight function $w$) where $(y, y)_w = 1$ will be used to normalise the expression for $y$ given in (1.6),

See HB p31.

$$(y, y)_w = \int_a^b \mathrm{d}x\, y(x)^2 = 1, \tag{1.7}$$

by determining the value of $A_n$ as follows.

After substituting into (1.7) for $y_n(x)$ from (1.6) the following integral is obtained.

$$\int_0^1 \mathrm{d}x\, A_n^2 \sin^2\left(\sqrt{\lambda_n}x\right) = 1,$$

$$A_n^2 \int_0^1 \mathrm{d}x\, \sin^2\left(\sqrt{\lambda_n}x\right) = 1.$$

Using the trig identity $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ gives,

See HB p38 for trig identities.

$$\frac{A_n^2}{2} \int_0^1 \mathrm{d}x \left(1 - \cos\left(2\sqrt{\lambda_n}x\right)\right) = 1.$$

$$\frac{A_n^2}{2} \left[x - \frac{1}{2\sqrt{\lambda_n}} \sin\left(2\sqrt{\lambda_n}x\right)\right]_0^1 = 1.$$

$$\frac{A_n^2}{2} \left(1 - \frac{1}{2\sqrt{\lambda_n}} \sin\left(2\sqrt{\lambda_n}\right)\right) = 1.$$

$$\frac{A_n^2}{4\sqrt{\lambda_n}} \left(2\sqrt{\lambda_n} - \sin\left(2\sqrt{\lambda_n}\right)\right) = 1.$$

Using the trig identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$ then,

$$\frac{A_n^2}{4\sqrt{\lambda_n}} \left(2\sqrt{\lambda_n} - 2\sin\left(\sqrt{\lambda_n}\right)\cos\left(\sqrt{\lambda_n}\right)\right) = 1.$$

From (1.4)

$$-\sin\left(\sqrt{\lambda_n}\right) = \frac{\sqrt{\lambda_n}}{1 - \lambda_n} \cos\left(\sqrt{\lambda_n}\right),$$

and so,

$$\frac{A_n^2}{\cancelto{2}{4\sqrt{\lambda_n}}} \left(\cancelto{1}{2\sqrt{\lambda_n}} + \frac{\cancelto{1}{2\sqrt{\lambda_n}}}{1 - \lambda_n} \cos\left(\sqrt{\lambda_n}\right)\cos\left(\sqrt{\lambda_n}\right)\right) = 1.$$

$$\frac{A_n^2}{2} \left(1 + \frac{1}{1 - \lambda_n} \cos^2\left(\sqrt{\lambda_n}\right)\right) = 1.$$

$$\frac{A_n^2}{2(1 - \lambda_n)} \left((1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)\right) = 1.$$

$$A_n^2 = \frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)}.$$

Therefore,

$$A_n = \sqrt{\frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)}},$$

and the normalised eigenfunctions are given by,

$$y_n(x) = \sqrt{\frac{2(1 - \lambda_n)}{(1 - \lambda_n) + \cos^2\left(\sqrt{\lambda_n}\right)}}\ \sin\left(\sqrt{\lambda_n}x\right).$$

Q 2. To show that the nontrivial solutions of the equation

$$\frac{d^2y}{dx^2} + y(1 + x^k) = 0, \tag{2.1}$$

where $k$ is a positive integer, has infinitely many zeros in $(0, \infty)$ consider the following.

Use will be made of Sturm's comparison theorem II (Theorem 31.3). See HB p30.

Since $(1 + x^k) \geq 2$ for all $x \geq 1$ let $Q_2 = 2$ and $Q_1(x) = (1 + x^k)$. So that,

$$y_1'' + y_1(1 + x^k) = 0, \quad \text{and} \quad y_2'' + 2y_2 = 0.$$

A solution to $y_2''(x) + 2y_2(x) = 0$ is $y_2(x) = \sin\left(\sqrt{2}x\right)$ and

$$y_2\left(\frac{1}{\sqrt{2}}n\pi\right) = 0 \quad \text{for} \quad n = 1, 2, \ldots.$$

Since Sturm's comparison theorem states that

"...if $y_1(x)$ is a solution of the first equation and $y_2(x)$ is any solution to the second equation, between any two adjacent zeros of $y_2$ there lies at least on zero of $y_1$."

Thus, as $y_2$ has infinitely many zeros in $(0, \infty)$ so too do the nontrivial solutions of the given expression (2.1).

To show that the separation between adjacent zeros tends to zero as $x \to \infty$ a similar argument to that above will be given.

Since $(1 + x^k) \geq \alpha^k$ for all $x \geq \alpha$, $\alpha, x \in (0, \infty)$, let $Q_2 = (1 + x^k)$ and $Q_1(x) = \alpha^k$. So that,

$$y_1'' + y_1(1 + x^k) = 0, \quad \text{and} \quad y_2'' + \alpha^k y_2 = 0.$$

A solution to $y_2''(x) + \alpha^k y_2(x) = 0$ is $y_2(x) = \sin\left(\alpha^{k/2}x\right)$ and

$$y_2\left(\frac{1}{\alpha^{k/2}}n\pi\right) = 0 \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad \alpha \in (0, \infty).$$

Thus, as $x \to \infty$ so does $\alpha \to \infty$ and the separation between zeros $\dfrac{\pi}{\alpha^{k/2}} \to 0$.

Recall that $k$ is a constant positive integer

Q 3.

(a) To show that the following equation and boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\lambda xy=0,\quad y(1)=0, y'(2)=0 \tag{3.1}$$

forms a regular Sturm-Liouville system, consider the following.

A Sturm-Liouville system is a linear, second-order homogeneous differential equation of the form: see HB p28.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(q(x)+\lambda w(x)\right)y=0, \tag{3.2}$$

which is defined on a finite interval of the real axis $a < x > b$ and satisfies the following three conditions:

1. the functions p(x), q(x) and w(x) are real and continuous for $a < x < b$;
2. $p(x)$ and $w(x)$ are strictly positive for $a < x < b$;
3. $p'(x)$ exists and is continuous for $a \le x \le b$,

together with the boundary conditions.

Comparing equations (3.1) and (3.2) it is seen that $q(x) = 0$ with $p(x) = x^2$ and $w(x) = x$. As $p(x)$ and $w(x)$:

1. are real and continuous for $1 < x < 2$;
2. are strictly positive for $1 < x < 2$; and
3. $p'(x)$ exists ($p'(x) = 2x$) and is continuous for $1 \le x \le 2$,

and the two boundary conditions are given as $y(1) = 0$ and $y'(2) = 0$, then the system is a regular Sturm-Liouville system.

It can be shown that the system can be written as a constrained variational problem with functional

$$S[y]=\int_1^2 \mathrm{d}x\, x^2y'^2,\quad y(1)=0, \tag{3.3}$$

and constraint

$$C[y]=\int_1^2 \mathrm{d}x\, xy^2=1, \tag{3.4}$$

as follows.

For the given constrained variational problem

$$\overline{F} = x^2 y'^2 - \lambda x y^2,$$

$\lambda$ is the Lagrange multiplier.

and

$$\overline{F}_{y'} = \frac{\partial \overline{F}}{\partial y'} = 2x^2 y' \quad \text{and} \quad \overline{F}_y = \frac{\partial \overline{F}}{\partial y} = -2\lambda x y.$$

The Euler-Lagrange equation is then,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \overline{F}}{\partial y'}\right) - \frac{\partial \overline{F}}{\partial y} = 0, \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^2 y'\right) - (-2\lambda x y) = 0, \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\cancel{2}x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \cancel{2}\lambda x y = 0, \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda x y = 0, \quad y(1) = 0, y'(2) = 0. \qquad (3.5)
\end{aligned}$$

Equations (3.1) and (3.5) are identical showing that system (3.1) can be written as a constrained variational problem.

(b) It is assumed that the eigenvalues $\lambda_k$ and the eigenfunctions $y_k$, $k = 1, 2, \ldots$ exist. By working from (3.1), the following relationship will be derived.

$$\lambda_k = \int_1^2 \mathrm{d}x\, x^2 y_k'^2, \quad k = 1, 2, \ldots.$$

Recall that (3.1) is given as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda x y = 0, \quad y(1) = 0, y'(2) = 0$$

and compare (3.1) with (3.2) repeated below,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x) \frac{\mathrm{d}y}{\mathrm{d}x}\right) + \left(q(x) + \lambda w(x)\right) y = 0.$$

As was determined in part (a) $q(x) = 0$ with $p(x) = x^2$ and $w(x) = x$ and the system is defined on the interval $(1, 2)$.

Now the functional is of the form:

$$S[y] = -\alpha p(a) y(a)^2 + \beta p(b) y(b)^2 + \int_a^b \mathrm{d}x\, \left(p y' - q y^2\right),$$

See HB p29 (SLF).

which, becomes the following after substituting for the appropriate terms found above,

$$S[y] = -\alpha p(1) y(1)^2 + \beta p(2) y(2)^2 + \int_1^2 \mathrm{d}x\, \left(x^2 y' - \cancelto{0}{q}\, y^2\right),$$

$p(x) = x^2$,
$p(1) = 1, p(2) = 4$

$$S[y] = -\alpha y(1)^2 + \beta 4y(2)^2 + \int_1^2 \mathrm{d}x\, x^2 y'.$$

The natural boundary conditions of a Sturm-Liouville system are of the form:

$$\alpha y(1) + y'(1) = 0 \quad \text{and} \quad \beta y(2) + y'(2) = 0.$$

The given boundary conditions are $y(1) = 0$ and $y'(2) = 0$, so this means that $\alpha = 1$ and $y'(1) = 0$ and $\beta y(2) = 0$. Thus,

$$\begin{aligned} S[y] &= \cancelto{0}{-\alpha y(1)^2} + \cancelto{0}{\beta 4y(2)^2} + \int_1^2 \mathrm{d}x\, x^2 y', \\ S[y] &= \int_1^2 \mathrm{d}x\, x^2 y'. \end{aligned}$$

Hence, from the general theory of Sturm-Liouville Systems,

$$\lambda_k = S[y_k] = \int_1^2 \mathrm{d}x\, x^2 y_k',$$

as required.

(c) Given the function $z = A\sin\left(\pi\left(x-1\right)/2\right)$, it will be that the smallest eigenvalue, $\lambda_1$, satisfies the inequality

$$\lambda_1 \le \frac{\left(7\pi^2 - 18\right)\pi^2}{6\left(4 + 3\pi^2\right)},$$

as follows.

Substituting $z = A\sin\left(\pi\left(x-1\right)/2\right)$ into the constraint (3.4) gives

$$\begin{aligned} 1 &= A^2 \int_1^2 \mathrm{d}x\, x \sin^2\left(\frac{\pi\left(x-1\right)}{2}\right) = A^2 \int_1^2 \mathrm{d}x\, x \cos^2\left(\frac{1}{2}\pi x\right), && \text{Using the identity for } \sin\left(\alpha \pm \beta\right). \\ &= A^2 \int_1^2 \mathrm{d}x\, x \frac{1}{2}\left(\cos\left(\pi x\right) + 1\right) = \frac{A^2}{2}\int_1^2 \mathrm{d}x\, x\cos\left(\pi x\right) + x, && \text{Using the identity of } \cos^2\left(\alpha\right) = \tfrac{1}{2}\left(1 + \cos\left(2\alpha\right)\right). \\ &= \frac{A^2}{2}\int_1^2 \mathrm{d}x\, x\cos\left(\pi x\right) + \frac{A^2}{2}\int_1^2 \mathrm{d}x\, x, \\ &= \frac{A^2}{2\pi}\Big[x\sin\left(\pi x\right)\Big]_1^2 - \frac{A^2}{2\pi}\int_1^2 \mathrm{d}x\, \sin\left(\pi x\right) + \frac{A^2}{2}\int_1^2 \mathrm{d}x\, x, && \text{Integrating the first integral by parts.} \end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{2\pi}\left(\cancelto{0}{2\sin(2\pi)-\sin(\pi)}\right) - \frac{A^2}{2\pi}\int_1^2 \mathrm{d}x\,\sin(\pi x) + \frac{A^2}{2}\int_1^2 \mathrm{d}x\,x, \\
&= -\frac{A^2}{2\pi}\int_1^2 \mathrm{d}x\,\sin(\pi x) + \frac{A^2}{2}\int_1^2 \mathrm{d}x\,x, \\
&= -\frac{A^2}{2\pi}\left[-\frac{1}{\pi}\cos(\pi x)\right]_1^2 + \frac{A^2}{2}\left[\frac{x^2}{2}\right]_1^2, \\
&= \frac{A^2}{2\pi}\left[\frac{1}{\pi}\cos(\pi x)\right]_1^2 + \frac{A^2}{2}\left[\frac{x^2}{2}\right]_1^2, \\
&= \frac{A^2}{2\pi^2}\left[\cancelto{1}{\cos(2\pi)} - \cancelto{-1}{\cos(\pi)}\right] + \frac{A^2}{2}\left[\frac{2^2}{2} - \frac{1^2}{2}\right], \\
&= \frac{A^2}{2\pi^2}(2) + \frac{A^2}{2}\left(\frac{3}{2}\right), \\
&= \frac{A^2}{2}\left(\frac{2}{\pi^2} + \frac{3}{2}\right), \\
\therefore 1 &= A^2\left(\frac{1}{\pi^2} + \frac{3}{4}\right). \qquad (3.6)
\end{aligned}
$$

Now,

$$z = A\sin\left(\frac{\pi(x-1)}{2}\right)$$

and

$$\lambda_1 \le S[z] = \int_1^2 \mathrm{d}x\,x^2 z'^2. \qquad (3.7)$$

Differentiating $z$,

$$
\begin{aligned}
z' &= \frac{\mathrm{d}}{\mathrm{d}x}\left(A\sin\left(\frac{\pi(x-1)}{2}\right)\right), \\
&= A\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\pi(x-1)}{2}\right)\cos\left(\frac{\pi(x-1)}{2}\right), \\
&= A\frac{\pi}{2}\cos\left(\frac{\pi(x-1)}{2}\right), \\
\therefore z' &= A\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right). \qquad (3.8)
\end{aligned}
$$

Making use of the chain rule.

Using the identity for $\cos(\alpha \pm \beta)$. HB p38.

Substituting for $z'$ given by (3.8) into (3.7) gives,

$$
\begin{aligned}
\lambda_1 \leq S[z] &= \int_1^2 \mathrm{d}x\, x^2 \left(A\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)\right)^2, \\
&= \left(\frac{A\pi}{2}\right)^2 \int_1^2 \mathrm{d}x\, x^2 \sin^2\left(\frac{\pi x}{2}\right), \\
&= \frac{A^2\pi^2}{4}\int_1^2 \mathrm{d}x\, x^2 \frac{1}{2}\Bigg(1-\cos\left(\pi x\right)\Bigg), \\
&= \frac{A^2\pi^2}{8}\int_1^2 \mathrm{d}x\, x^2\left(1-\cos\left(\pi x\right)\right), \\
&= \frac{A^2\pi^2}{8}\int_1^2 \mathrm{d}x\, \left(x^2 - x^2\cos\left(\pi x\right)\right), \\
&= \frac{A^2\pi^2}{8}\int_1^2 \mathrm{d}x\, x^2 - \frac{A^2\pi^2}{8}\int_1^2 \mathrm{d}x\, x^2\cos\left(\pi x\right), \\
&= \frac{A^2\pi^2}{8}\left[\frac{x^3}{3}\right]_1^2 - \frac{A^2\pi^2}{8}\int_1^2 \mathrm{d}x\, x^2\cos\left(\pi x\right), \\
&= \frac{A^2\pi^2 7}{24} - \frac{A^2\pi^2}{8}\int_1^2 \mathrm{d}x\, x^2\cos\left(\pi x\right), \\
&= \frac{A^2\pi^2 7}{24} - \frac{A^2\pi^2}{8}\left(\cancelto{0}{\left[\frac{x^2}{\pi}\sin\left(\pi x\right)\right]_1^2} - \frac{2}{\pi}\int_1^2 \mathrm{d}x\, x\sin\left(\pi x\right)\right), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4}\int_1^2 \mathrm{d}x\, x\sin\left(\pi x\right), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4}\left(\cancelto{-\frac{3}{\pi}}{\left[-\frac{x}{\pi}\cos\left(\pi x\right)\right]_1^2} - \int_1^2 \mathrm{d}x\, \cos\left(\pi x\right)\right), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4}\left(-\frac{3}{\pi} + \frac{1}{\pi}\cancelto{0}{\left[\sin\left(\pi x\right)\right]_1^2}\right), \\
&= \frac{A^2\pi^2 7}{24} + \frac{A^2\pi}{4}\left(-\frac{3}{\pi}\right), \\
&= \frac{A^2\pi^2 7}{24} - \frac{A^2 3}{4}, \\
&= A^2\left(\frac{7\pi^2}{24} - \frac{3}{4}\right), \\
&= \frac{A^2}{4}\left(\frac{7\pi^2-18}{6}\right). \qquad (3.9)
\end{aligned}
$$

Using the identity for $\sin^2(\alpha)$. HB p38.

From (3.6)

$$A^2 = \frac{4\pi^2}{4+3\pi^2},$$

and substituting for $A^2$ in (3.9) gives,

$$\lambda_1 \leq \frac{4\pi^2}{4\,(4+3\pi^2)}\left(\frac{7\pi^2-18}{6}\right),$$

$$\lambda_1 \leq \frac{\pi^2}{(4+3\pi^2)}\left(\frac{7\pi^2-18}{6}\right),$$

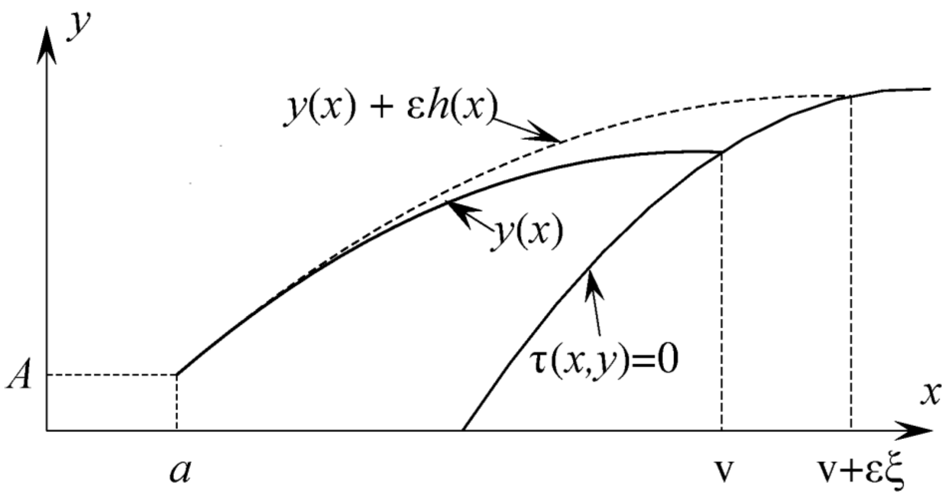
Finally,

$$\lambda_1 \leq \frac{(7\pi^2-18)\,\pi^2}{6\,(4+3\pi^2)},$$

as required.

Q 4.

(a)




This Figure 10.6 taken from the module notes p225.

Figure 2: Diagram showing the stationary path (solid line) and a varied path (dashed line) for a problem in which the left-hand end is fixed, but the other end is free to move along the line defined by $\tau(x, y) = 0$.

Given the perturbed path

$$y_\epsilon(x) = y(x) + \epsilon h(x), \tag{4.1}$$

the Taylor series to the first-order of (4.1) at point $x = v$ (see figure 2) is given in (4.2).

The point $x = v$ is known as the point of expansion. HB p8.

$$y_\epsilon(x) = (y(v) + \epsilon h(v)) + (y'(v) + \epsilon h'(v))(x - v) + \mathcal{O}\left((x - v)^2\right). \tag{4.2}$$

Now, determining the value of (4.2) at $v_\epsilon = v + \epsilon\xi + \mathcal{O}(\epsilon^2)$ where $v_\epsilon$ is the perturbed value of $v$:

$$\begin{aligned} y_\epsilon(v_\epsilon) = y(v) &+ \epsilon h(v) \\ &+ \left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{-v}\right)(y'(v) + \epsilon h'(v)) \\ &+ \mathcal{O}\left(\left(\cancel{v} + \epsilon\xi + \mathcal{O}\left(\epsilon^2\right) \cancel{-v}\right)^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) = y(v) &+ \epsilon h(v) \\ &+ \left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)(y'(v) + \epsilon h'(v)) \\ &+ \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) = y(v) &+ \epsilon h(v) \\ &+ \epsilon\xi\left(y'(v) + \epsilon h'(v)\right) \\ &+ \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) \\ &+ \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right), \end{aligned}$$

$$\begin{aligned} y_\epsilon(v_\epsilon) = y(v) &+ \epsilon\left(h(v) + \xi y'(v)\right) \\ &+ \underbrace{\epsilon^2\xi h'(v) + \mathcal{O}\left(\epsilon^2\right)\left(y'(v) + \epsilon h'(v)\right) + \mathcal{O}\left(\left(\epsilon\xi + \mathcal{O}\left(\epsilon^2\right)\right)^2\right)}_{\text{These are all second-order terms in } \epsilon.}. \end{aligned}$$

Thus,

$$y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(h(v) + \xi y'(v)\right) + \mathcal{O}\left(\epsilon^2\right), \tag{4.3}$$

as required.

To show that at $(x, y) = (v, y(v))$,

$$\xi\left(\tau_x + y'(v)\tau_y\right) + h(v)\tau_y = 0$$

the 2D Taylor expansion to the first-order of $\tau(x, y)$ at point $x = a, y = b$ is required, namely,

$$\tau(x, y) = \tau(a, b) + \tau_x(a, b)\left[x - a\right] + \tau_y(a, b)\left[y - b\right]. \tag{4.4}$$

Evaluating (4.4) with $x = v_\epsilon = v + \epsilon\xi$ and $y = y_\epsilon(v_\epsilon) = y(v) + \epsilon\left(y'(v)\xi + h(v)\right)$ gives,

$$\begin{aligned} \tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) &= \tau(v + \epsilon\xi, y(v) + \epsilon\left(y'(v)\xi + h(v)\right) \\ &= \cancelto{0}{\tau\left(v, y(v)\right)} + \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\ &= \tau_x\left(v, y(v)\right)\left[v_\epsilon - v\right] + \tau_y\left(v, y(v)\right)\left[y_\epsilon(v_\epsilon) - y(v)\right], \\ &= \tau_x\left(v, y(v)\right)\left[\cancel{v} + \epsilon\xi \cancel{- v}\right] + \tau_y\left(v, y(v)\right)\left[\cancel{y(v)} + \epsilon\left(y'(v)\xi + h(v)\right) \cancel{- y(v)}\right], \\ &= \tau_x\left(v, y(v)\right)\epsilon\xi + \tau_y\left(v, y(v)\right)\epsilon\left(y'(v)\xi + h(v)\right), \\ &= \epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right], \end{aligned}$$

Recall that $\tau\left(v_\epsilon, y_\epsilon(v_\epsilon)\right) = 0$ and therefore,

$$\begin{aligned} \epsilon\left[\tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right)\right] &= 0, \\ \tau_x\left(v, y(v)\right)\xi + \tau_y\left(v, y(v)\right)\left(y'(v)\xi + h(v)\right) &= 0, \end{aligned}$$

$$\xi\left[\tau_x\left(v, y(v)\right) + \tau_y\left(v, y(v)\right)y'(v)\right] + \tau_y\left(v, y(v)\right)h(v) = 0, \tag{4.5}$$

as required.

(b) The Gâteaux differential is given as

$$\Delta S\left(y, h\right) = \xi F\big|_v + \int_a^v \mathrm{d}x\ \left(hF_y + h'F_{y'}\right), \tag{4.6}$$

and it will be shown, by integrating by parts the integral of (4.6), that a stationary path must satisfy the transversality condition given in (4.7):

$$\tau_x F_{y'} + \tau_y \left(y' F_{y'} - F\right) = 0 \quad \text{at} \quad (x, y) = (v, y(v)) \tag{4.7}$$

as follows.

Equation (4.6) can be rewritten as in (4.8),

$$\Delta S\,(y, h) = \xi F\big|_v + \int_a^v \mathrm{d}x\, h F_y + \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.8}$$

and integrating by parts the right-most integral in (4.8):

$$I = \int_a^v \mathrm{d}x\, h' F_{y'}, \tag{4.9}$$

$$\begin{aligned} &\text{Let } u = F_{y'} \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) \\ &\text{Let } \frac{\mathrm{d}v}{\mathrm{d}x} = h'(x) \quad \text{then} \quad v = \int \mathrm{d}x\, h'(x) = h(x). \end{aligned}$$

For clarity, here $v$ is not the same as that for the upper limit of integration in (4.9).

For integration by parts:

$$\begin{aligned} I &= \int_a^v \mathrm{d}x\, u \frac{\mathrm{d}v}{\mathrm{d}x} = [uv]_a^v - \int_a^v \mathrm{d}x\, v \frac{\mathrm{d}u}{\mathrm{d}x}, \\ &= \left[F_{y'} h(x)\right]_a^v - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= \left( F_{y'} h(x)\big|_{x=v} - F_{y'} \cancelto{0}{h(a)} \right) - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x). \end{aligned}$$

The Gâteaux differential (4.8) becomes,

$$\begin{aligned} \Delta S\,(y, h) &= \xi F\big|_v + \int_a^v \mathrm{d}x\, h(x) F_y + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, h(x) F_y - \int_a^v \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} + \int_a^v \mathrm{d}x\, \left( h(x) F_y - \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x) \right), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left( \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) h(x) - h(x) F_y \right), \\ &= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left( \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'}\right) - F_y \right) h(x) \end{aligned}$$

$$= \xi F\big|_{x=v} + F_{y'} h(x)\big|_{x=v} - \int_a^v \mathrm{d}x\, \left( \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} \right) h(x). \tag{4.10}$$

On a stationary path $\Delta S\left(y,h\right) = 0$ for all allowed $h$ and the Euler-Lagrange equation equation is satisfied and so the integrand of (4.10) is zero, thus the Gâteaux differential shown in (4.10) reduces to

$$\Delta S\left(y,h\right) = \xi F|_{x=v} + F_{y'}h(x)|_{x=v} = 0. \tag{4.11}$$

Rewriting (4.5) more succinctly as

$$\xi\left(\tau_x + \tau_y y'(x)\right)|_{x=v} + \tau_y h(x)|_{x=v} = 0, \tag{4.12}$$

$\tau_x = \tau_x\left(v, y(v)\right)$, and
$\tau_y = \tau_y\left(v, y(v)\right)$

and rearranging (4.12) in terms of $h(x)$ evaluated at $x = v$,

$$h(x)\mid_{x=v} = -\frac{\xi}{\tau_y}\left(\tau_x + y'(x)\tau_y\right)\mid_{x=v}. \tag{4.13}$$

Substituting for $h(v)$ from (4.13) into (4.11) gives,

$$\begin{aligned}
\xi F|_{x=v} - F_{y'}\frac{\xi}{\tau_y}\left(\tau_x + y'(x)\tau_y\right)\Big|_{x=v} &= 0,\\
\left(\xi F - F_{y'}\frac{\xi}{\tau_y}\left(\tau_x + y'(x)\tau_y\right)\right)\Big|_{x=v} &= 0,\\
\xi\left(F - F_{y'}\frac{\left(\tau_x + y'(x)\tau_y\right)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(-F + F_{y'}\frac{\left(\tau_x + y'(x)\tau_y\right)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(\frac{-F\tau_y + F_{y'}\left(\tau_x + y'(x)\tau_y\right)}{\tau_y}\right)\Big|_{x=v} &= 0,\\
-\left(-F\tau_y + F_{y'}\left(\tau_x + y'(x)\tau_y\right)\right)|_{x=v} &= 0,\\
-\left(-F\tau_y + F_{y'}\tau_x + F_{y'}y'(x)\tau_y\right)|_{x=v} &= 0,\\
-\left(\tau_y\left(-F + F_{y'}y'(x)\right) + F_{y'}\tau_x\right)|_{x=v} &= 0,\\
-\left(\tau_y\left(F_{y'}y'(x) - F\right) + F_{y'}\tau_x\right)|_{x=v} &= 0.
\end{aligned}$$

Finally,

$$\tau_y\left(F_{y'}y'(x) - F\right)\mid_{x=v} + F_{y'}\tau_x|_{x=v} = 0. \tag{4.14}$$

Equation (4.14) shows that the transversality condition has been satisfied, as required.

(c) Now, considering the case when

$$S[y] = \int_0^v \mathrm{d}x\,\frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta,$$

where $y(v) > 0$, $\delta > 0$ and the right-hand end point $(v, y(v))$ lies on the line $\alpha y + \beta x + \gamma = 0$, where $\alpha, \beta, \gamma$ are constants with $\beta \neq 0$, it can be shown that the first-integral may be written as

$$y\sqrt{1+y'^2} = c, \tag{4.15}$$

for some constant $c > 0$, as follows.

The first-integral is given by

$$y' G_{y'} - G = \text{constant}, \qquad \text{HB p17.}$$

with,

$$
\begin{aligned}
G &= \frac{\sqrt{1+y'^2}}{y}, \quad y(0) = \delta. \\
G_{y'} &= \frac{1}{y}\left(\frac{1}{\cancel{2}_1}\left(1+y'^2\right)^{-\frac{1}{2}} \overset{1}{\cancel{2}} y'\right), \\
&= \frac{\left(1+y'^2\right)^{-\frac{1}{2}} y'}{y}, \\
&= \frac{y'}{y\sqrt{1+y'^2}}.
\end{aligned}
$$

The first-integral becomes,

$$
\begin{aligned}
y'\left(\frac{y'}{y\sqrt{1+y'^2}}\right) - \frac{\sqrt{1+y'^2}}{y} &= c, \quad \text{where } c \text{ is a constant,} \\
\left(\frac{y'^2}{y\sqrt{1+y'^2}}\right) - \frac{\sqrt{1+y'^2}}{y} &= c, \\
\frac{y'^2}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}\sqrt{1+y'^2}}{y\sqrt{1+y'^2}} &= c, \\
\frac{y'^2}{y\sqrt{1+y'^2}} - \frac{1+y'^2}{y\sqrt{1+y'^2}} &= c, \\
\frac{y'^2 - 1 - y'^2}{y\sqrt{1+y'^2}} &= c \\
-\frac{1}{y\sqrt{1+y'^2}} &= c, \\
-\frac{1}{c} &= y\sqrt{1+y'^2}.
\end{aligned}
$$

Redefining the constant $c$, then the first-integral may be written as,

$$y\sqrt{1+y'^2} = c, \quad \text{for some constant } c > 0, \text{ as required.} \tag{4.16}$$

Now, rearranging (4.16) in terms of $y'$, as follows.

$$
\begin{aligned}
y'^2 = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= \frac{c^2}{y^2} - 1, \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{\frac{c^2}{y^2} - 1}.
\end{aligned}
$$

Then,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 1 \Big/ \frac{\mathrm{d}y}{\mathrm{d}x},$$

so,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{1}{\sqrt{\frac{c^2}{y^2} - 1}} = \frac{1}{\sqrt{\frac{c^2-y^2}{y^2}}} = \frac{y}{\sqrt{c^2 - y^2}}. \\
\int \mathrm{d}y \, \frac{\mathrm{d}x}{\mathrm{d}y} &= \int \mathrm{d}y \, \frac{y}{\sqrt{c^2 - y^2}}, \\
x &= \int \mathrm{d}x \, \frac{y}{\sqrt{c^2 - y^2}}. \qquad (4.17)
\end{aligned}$$

Solving the integral of (4.17),

$$x = \int \mathrm{d}y \, \frac{y}{\sqrt{c^2 - y^2}}.$$

Let $u = c^2 - y^2$,

$$\frac{\mathrm{d}u}{\mathrm{d}y} = -2y, \quad \text{so} \quad \frac{\mathrm{d}y}{\mathrm{d}u} = 1 \Big/ \frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{1}{2y}.$$

$$\begin{aligned}
x &= \int \mathrm{d}u \left( \frac{\mathrm{d}y}{\mathrm{d}u} \right) \frac{y}{\sqrt{u}}, \\
&= \int \mathrm{d}u \left( -\frac{1}{2\cancel{y}} \right) \frac{\cancel{y}}{\sqrt{u}}, \\
&= -\frac{1}{2} \int \mathrm{d}u \, \frac{1}{\sqrt{u}}, \\
&= -\frac{1}{2} \int \mathrm{d}u \, u^{-\frac{1}{2}}, \\
&= -\frac{1}{2} \left( \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) + c_\delta = -\sqrt{u} - c_\delta,
\end{aligned}$$

where $c_\delta$ is the constant of integration.

Thus,

$$\begin{aligned}
x &= -\sqrt{c^2 - y^2} - c_\delta, \\
(x + c_\delta)^2 &= c^2 - y^2, \\
y^2 + (x + c_\delta)^2 &= c^2, \quad \text{as required.} \qquad (4.18)
\end{aligned}$$

The solution of the first-integral (4.18) are circles centred at $(-c_\delta, 0)$.

Applying the boundary condition, $y(0) = \delta$, to (4.18) gives,

$$\delta^2 + c_\delta^2 = c^2 \quad \text{and so,} \quad c_\delta^2 = c^2 - \delta^2.$$

Differentiating $y^2 + (x + c_\delta)^2 = c^2$ implicitly:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x + c_\delta\right)^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left(c^2\right),\\
&2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(x + c_\delta\right) = 0,\\
&y\frac{\mathrm{d}y}{\mathrm{d}x} + x + c_\delta = 0,\\
&yy' + x + c_\delta = 0. \qquad (4.19)
\end{aligned}$$

Comparing (4.19) with $\alpha y + \beta x + \gamma = 0$ but first dividing through this expression by $\beta$:

$$\frac{\alpha}{\beta}y + x + \frac{\gamma}{\beta} = 0,$$

then it is seen that $y' = \alpha/\beta$ and $c_\delta = \gamma/\beta$.

Finally, it can be shown that in the limit as $\delta \to 0$, the stationary path becomes

$$\beta^2 y^2 + \left(\beta x + \gamma\right)^2 = \gamma^2,$$

as follows.

In the limit as $\delta \to 0$, $c_\delta = c$ and equation (4.18) can be written as,

$$y^2 + \left(x + c_\delta\right)^2 = c_\delta^2. \qquad (4.20)$$

Substituting into (4.20) for $c_\delta = \gamma/\beta$ gives,

$$\begin{aligned}
&y^2 + \left(x + \frac{\gamma}{\beta}\right)^2 = \frac{\gamma^2}{\beta^2},\\
&\qquad\text{cross multiplying by } \beta^2 \text{ gives,}\\
&\beta^2 y^2 + \beta^2\left(x + \frac{\gamma}{\beta}\right)^2 = \gamma^2 \quad \text{and}\\
&\beta^2 y^2 + \left(\beta x + \gamma\right)^2 = \gamma^2 \quad \text{as required.}
\end{aligned}$$

Q 5. Given

$$S[x] = \int_a^b \mathrm{d}t\, L\,(t, x, \dot{x})\,, \quad \text{with} \quad b > a,$$

where $L$ is called the Lagrangian, and $x(t)$ is at least twice differentiable.

The *conjugate momentum* $p$ is define by

$$p = \frac{\partial L}{\partial \dot{x}}. \tag{5.1}$$

(a) It can be shown that the Euler-Lagrange equation for $S$ is defined by

$$\dot{p} = \frac{\partial L}{\partial x},$$

as follows.

The Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,$$

which becomes after substituting in the *conjugate momentum,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(p) - \frac{\partial L}{\partial x} = 0, \quad \text{and}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\partial L}{\partial x} = 0,$$

$$\therefore \quad \dot{p} = \frac{\partial L}{\partial x}, \quad \text{as required.}$$

(b) From the handbook, the total derivative can be expressed as: see HB p3.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\frac{\mathrm{d}x_k}{\mathrm{d}t}.$$

Using this result then,

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\,(L(t, x, \dot{x}))\,,\\
&= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t},\\
&= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x},\\
&= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x}, \quad \text{as required.}
\end{aligned}$$

Recall that $p$ is the *conjugate momentum* define above.

(c) The *Hamiltonian* $H = H(t, x, p)$ is defined by $H(t, x, p) = p\dot{x} - L(t, x, \dot{x})$, where (implicitly) $\dot{x}$ is eliminated using (5.1) to give a function of $t, x$ and $p$.

Using the result obtained in part (b) it will be shown that for a stationary path of $S$ that

$$\frac{\partial L}{\partial t} = -\dot{H} \tag{5.2}$$

as follows.

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(p\dot{x} - L\right), \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(p\dot{x}\right) - \frac{\mathrm{d}L}{\mathrm{d}t}, \\ &= p\ddot{x} + p\dot{x} - \frac{\mathrm{d}L}{\mathrm{d}t},\end{aligned}$$

and substituting into the above expression for $\frac{\mathrm{d}L}{\mathrm{d}t}$ from part (b) gives,

$$\begin{aligned}&= p\ddot{x} + p\dot{x} - \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + p\ddot{x}\right), \\ &= \cancel{p\ddot{x}} + \cancel{p\dot{x}} - \left(\frac{\partial L}{\partial t} + \cancel{\dot{p}\dot{x}} + \cancel{p\ddot{x}}\right),\end{aligned}$$

$$\therefore \frac{\mathrm{d}H}{\mathrm{d}t} = \dot{H} = -\frac{\partial L}{\partial t}, \quad \text{as required.}$$

(d) The *Rund-Trautman identity* is given as

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0, \tag{5.3}$$

and from this identity it will be shown that

$$\left(\xi - \dot{x}\tau\right)\left[\dot{p} - \frac{\partial L}{\partial x}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right], \tag{5.4}$$

as follows.

First it will be shown that the left-hand side of (5.3) is equal to zero by expanding out the bracketed terms and substituting the derivative terms.

$$\begin{aligned}\left(\xi - \dot{x}\tau\right)\left[\dot{p} - \frac{\partial L}{\partial x}\right] &= \xi\dot{p} - \xi\frac{\partial L}{\partial x} - \dot{x}\tau\dot{p} + \dot{x}\tau\frac{\mathrm{d}L}{\mathrm{d}x}, \\ &= \cancel{\xi\dot{p} - \xi\dot{p} - \dot{x}\tau\dot{p} + \dot{x}\tau\dot{p}}, \\ &= 0.\end{aligned}$$

Secondly, it will be shown that (5.3) is equal to the right-hand side of (5.4) which is equal to zero. Substituting into (5.3) for

$$\frac{\partial L}{\partial t} = -\dot{H}, \quad \text{and} \quad \dot{p} = \frac{\partial L}{\partial x} \quad \text{gives the following,}$$

$$\begin{aligned}
&\dot{p}\xi + p\dot{\xi} - \dot{H}\tau - H\dot{\tau} = 0,\\
&\underbrace{\dot{p}\xi + p\dot{\xi}} - \underbrace{\left(\dot{H}\tau + H\dot{\tau}\right)} = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(p\xi\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\tau H\right) = 0,\\
&\therefore \frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - \tau H\right] = 0.
\end{aligned}$$

The product rule has been used here.

Thus,

$$(\xi - \dot{x}\tau)\left[\dot{p} - \frac{\partial L}{\partial x}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - \tau H\right], \quad \text{as required.}$$

The differentiation of a constant is zero and also noting that zero itself is a constant, then the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - \tau H\right] = 0$$

must mean that

$$p\xi - \tau H = \text{constant}.$$

(e) Now, considering a particle of constant mass $m$ moving along the $x$-axis in a potential $V(x)$. The Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - V(x)$, and the path of the particle from $t = a$ to $t = b$ is a stationary path of $S$.

The conjugate momentum $p$ is calculated as follows.

$$\begin{aligned}
p &= \frac{\partial L}{\partial \dot{x}} \quad \text{and} \quad L = \frac{1}{2}m\dot{x}^2 - V(x).\\
\therefore p &= \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = 2\cdot\frac{1}{2}m\dot{x} = m\dot{x}.
\end{aligned}$$

The Hamiltonian is calculated as follows.

$$H\left(t, x, p\right) = p\dot{x} - L\left(t, x, \dot{x}\right).$$

Substituting into this expression for conjugate momentum $p$ and the Lagrangian $L$, gives,

$$\begin{aligned}
H &= \left(m\dot{x}\right)\dot{x} - \left(\frac{1}{2}m\dot{x}^2 - V(x)\right),\\
&= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x),\\
&= \frac{1}{2}m\dot{x}^2 + V(x).
\end{aligned}$$

(f)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p(t)\right) = \dot{p}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\xi(t,x,\dot{x})\right) &= \frac{\partial}{\partial t}\xi(t,x,\dot{x}) \\ &\quad + \frac{\partial}{\partial x}\xi(t,x,\dot{x})\frac{\mathrm{d}x}{\mathrm{d}t} \\ &\quad + \frac{\partial}{\partial \dot{x}}\xi(t,x,\dot{x})\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}.\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\xi(t,x,\dot{x})\right) &= \frac{\partial}{\partial t}\xi(t,x,\dot{x}) \\ &\quad + \frac{\partial}{\partial x}\xi(t,x,\dot{x})\dot{x} \\ &\quad + \frac{\partial}{\partial \dot{x}}\xi(t,x,\dot{x})\ddot{x}.\end{aligned}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\dot{x} + \frac{\partial \xi}{\partial \dot{x}}\ddot{x}.$$

This must be similar for $\tau(t,x,\dot{x})$, too:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x}\dot{x} + \frac{\partial \tau}{\partial \dot{x}}\ddot{x}.$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(H(t,x(t),p(t))\right) &= \frac{\partial}{\partial t}H(t,x(t),p(t)) \\ &\quad + \frac{\partial}{\partial x}H(t,x(t),p(t))\frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ &\quad + \frac{\partial}{\partial \dot{x}}H(t,x(t),p(t))\frac{\mathrm{d}p(t)}{\mathrm{d}t}.\end{aligned}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \dot{x}}\dot{p}.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p\xi\right) = p\frac{\mathrm{d}\xi}{\mathrm{d}t} + \xi\frac{\mathrm{d}p}{\mathrm{d}t} = p\dot{\xi} + \xi\dot{p}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H\tau\right) = H\frac{\mathrm{d}\tau}{\mathrm{d}t} + \tau\frac{\mathrm{d}H}{\mathrm{d}t} = H\dot{\tau} + \tau\dot{H}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left(p\xi\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(H\tau\right), \\ &= p\dot{\xi} + \xi\dot{p} - \left(H\dot{\tau} + \tau\dot{H}\right), \\ &= p\dot{\xi} + \xi\dot{p} - H\dot{\tau} - \tau\dot{H},\end{aligned}$$

The *Rund-Trautman identity* is:

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0$$

which can be rearranging to:

$$\frac{\partial L}{\partial x}\xi + p\dot{\xi} + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0$$

$$p\dot{\xi} + \frac{\partial L}{\partial x}\xi - H\dot{\tau} + \frac{\partial L}{\partial t}\tau = 0$$

Compare to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p\xi - H\tau\right] = p\dot{\xi} + \xi\dot{p} - H\dot{\tau} - \tau\dot{H}$$