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1 Toy Version

1.1 Promote elliptic containment to equality

1 Definition (Simplicial Map). A map between simplicial complexes is simplicial if the image of a set of vertices that spans a simplex also spans as simplex.

2 Definition (Morphism). We say a simplicial map between trees is a morphism if edges go to edges.

3 Definition (Collapse Map). Given a G-tree, collapsing components of an invariant forest to points gives a G-map.

1 Proposition (Subgraph Collapse). Replacing a connected subgraph in a graph of groups decomposition with a vertex group with the same group as the collapsed graph corresponds to collapsing an invariant forest to points.

2 Proposition (Folds "factor" as an elementary collapse and a collapse map). If $X \to Y$ is a fold, then there exists a tree Z along with collapse maps to X and Y. Moreover, the map to X is a collapse map corresponding to an elementary collapse.

1.1 Lemma (Hyperbolic elements are preserved). Suppose $X \to Y$ is a collapse map bewteen locally finite G-trees. Suppose Y is not a single point. Then, if an element is hyperbolic for X it is also hyperbolic for Y.

Proof. Suppose for sake of a contradiction that $g \in G$ were hyperbolic for X and elliptic for Y. Let $y \in Y$ be some vertex fixed by the elliptic element g and G_y it's stabilizer. Since Y is not a single point, there is another vertex $y \neq z \in Y$. Because Y is locally finite, G_y and G_z are commensurable. For G-maps, preimages are invariant. By the construction of a collapse map, the preimage of vertices are connected and non-empty. Putting these together we have that the preimages of vertices are invariant trees. This means that the minimal subtrees of G_y and G_z acting on X are contained in the disjoint preimages of y and z respectively. However, since they are commensurable and G_y contains the hyperbolic element g, these minimal trees are non-empty and equal. This is a contradiction. $\square$

1.1 Theorem (Elliptic elements determine elliptic subgroups). Let X and Y be cocompact G-trees with finitely generated vertex groups. Then the following are equivalent:

1. X and Y define the same partition of G into elliptic and hyperbolic elements.

2. X and Y have the same elliptic subgroups.

Proof. By Proposition 2.6, Theorem 4.2, and Corollary 4.3 of [F, deformation and rigidity]. $\square$

1.2 Theorem (Factoring as folds, from Bestvina paper, p455). Let G be a finitely generated group. Suppose that $\alpha : T' \to T$ is a simplicial equivariant map from a G-tree T' to a minimal G-tree T such that no edge in T' is mapped to a point by α. If all edge stabilizers of T are finitely generated and if T'/G is finite, then α can be represented as a finite composition of folds.

1.3 Theorem (Elliptic containment implies equality). If X and Y are locally finite cocompact G-trees with finitely generated vertex and edge stabilizers then $\mathcal{E}(X) \subseteq \mathcal{E}(Y) \Longrightarrow \mathcal{E}(X) = \mathcal{E}(Y)$.

Proof. We first collapse our general G-map to a morphism. Then we use 1.2 to factor the result as a sequence of folds (followed by an immersion if we're not assuming minimality). Then we use 2 and finite generation (of edge and vertex groups?) to get that same elliptic elements give us same elliptic subgroups which gives us same deformation space which means the second collapse was an elementary one (in theory possibly a sequence of expand and collapses) which means we keep local finiteness so we can repeatedly use 1.1. (fix) Add the final bit about the immersion, note that for trees immersions are inclusions. Note that if you're including Z into a tree you can think of the action of Z as the restriction of the action on the target tree. Just need to check that the axis stays... essentially because the entire tree stays $\square$

1.2 Toy version of main result

1.2.1 Definitions

4 Definition (Finite Type). We say a tree action is of *finite type* if the action is non-trivial, the tree is locally finite, and the vertex stabilizers are FP, with finite quotient graph.

1.1 Remark. We will assume a group of dimension 2 and in that case Bieri gives that actions of finite type have vertex and edge stabilizers that are finitely generated free groups.

1.2 Remark. Bieri gives us that if we have an action of finite type then the group is FP.

5 Definition (Transverse). We say that two tree actions X and Y are *transverse* if they are not in the same deformation space and there exist two stabilizers, one for each tree, such that their intersection is FP.

1.3 Remark. The definition of transverse does not depend on the vertices chosen and remains unchanged up to deformation spaces

1.2.2 Result

3 Proposition. Let G a group of cohomological dimension 2. If X and Y are non-trivial G-trees of finite type that are in different deformation spaces then the following are equivalent:

1. X, Y transverse
2. $x \in V(X), y \in V(Y) \Longrightarrow G_x \cap G_y = \{1\}$
3. There exists a cocompact VH-complex K with $\pi_1(K) = G$ whose horizontal and vertical splittings are X and Y.

Proof.
1. $1 \Rightarrow 2$: Fix $x_0 \in V(X)$ Let $y \in V(Y)$. Then $G_{x_0} \cap G_y = (G_{x_0})_y$. By (1) X is transverse to Y hence $G_{x_0} \cap G_y$ is FP. Since the choice of $y \in V(Y)$ was arbitrary, the vertex groups of the G_{x_0} action on Y are FP. Note, Y locally finite implies it's edge groups are finite index subgroups of it's vertex groups. Hence the edge groups are also FP. We claim that the action of G_{x_0} on Y is non-trivial. Given this we apply Bieri to get:

$$\begin{aligned} 2 &= dG \\ &= dG_{x_0} + 1 \\ &= d(G_{x_0})_y + 1 + 1 \\ &= d(G_{x_0} \cap G_y) + 2 \end{aligned}$$

Hence, $d(G_{x_0} \cap G_y) = 0$ so $G_{x_0} \cap G_y$ is trivial.

Claim. The action of G_{x_0} on Y is non-trivial.

Proof. Suppose the action were trivial. That is, there exists some $y \in V(Y)$ such that $(G_{x_0})_y = G_{x_0}$. Hence, G_{x_0} is elliptic for the action of G on Y. By the local finiteness of Y, for all $x \in V(X)$ G_x acts elliptically on Y. Hence, $\mathcal{E}(X) \subset \mathcal{E}(Y)$. Again by local finiteness we can promote this using 1.1 to $\mathcal{E}(X) = \mathcal{E}(Y)$ which by theorem 3.1 gives $X \sim Y$ which contradicts the fact that X and Y were assumed to be in different deformation spaces. $\square$

2. $2 \Rightarrow 1$: Trivial groups are FP.

3. $2 \Rightarrow 3$: Take $X \times Y$ and give it the VH-structure where X and Y correspond to horizontal and vertical edges respectively. Condition (2) says that G acts freely on $X \times Y$. By 3.1 the core is connected. Guirardel gives us a simply-connected cocompact core (warning) needs connected implies simply connected $C \subset X \times Y$. The action is also free on subsets of $X \times Y$. Since the action is cellular and free it's a covering space action. (We avoid situations like irrational rotations on a circle that are free but not covering space actions)

   It remains to show that C/G is VH. Is it enough to say that the action respects the tree factors. (The edge partition on the cover descends to a well-defined edge partition on the quotient and attaching maps constructed in the standard way for the quotient alternate between vertical and horizontal edges)

4. $3 \Rightarrow 2$: Suppose $1 \neq g \in G$ is an element of $G_x \cap G_y$ where $x \in V(X)$ and $y \in V(Y)$. Since K is a VH-complex it has a decomposition as a graph of groups where the vertex spaces are connected subgraphs whose edges are all vertical. Each $x \in V(X)$ is in correspondence with the inclusion of a vertex space (composed entirely of vertical edges) into K. The inclusion of vertex spaces is always injective on fundamental groups. Lastly, after picking basepoints the image of the induced map is the stabilizer of x.

   By Wise, we have that the universal cover of K is contained in $X \times Y$. Because the action respects the product structure, and because vertex spaces (in our case, graphs) are covered by embedded copies of their own universal covers we have that the non-trivial element g when represented by a loop in a vertex space lifts to a path with distinct endpoints in a tree consisting entirely of vertical edges in $X \times Y$. In fact, G_x acts on $\{x\} \times Y$ and freely on $\widetilde{K} \cap (\{x\} \times Y)$ because the action on $\widetilde{K}$ is a covering space action. This means that g acts hyperbolically on Y and we get an axis in Y. This remains an axis in $X \times Y$.

   Thus from looking at the vertical splitting (where vertex spaces were made of vertical edges) we obtained an axis consisting entirely of vertical edges. Similarly, after looking at the horizontal splitting we obtain an axis consisting entirely of horizontal edges.

Finally, since $X \times Y$ is CAT(0) these axes would have to be parallel. This is a contradiction.

□

2 Problem Statement

2.1 Theorem. If G has cohomological dimension 2 then there are at most two deforamtion spaces of finite type pairwise transverse G-trees.

Corollary. If a VH-complex has horizontal and vertical splittings of finite type then up to deformation, these are the only such G-trees.

2.1 Dramatis Personæ

- X a compact connected VH-complex
- $G = \pi_1(X)$
- T_1 the tree from the horizontal splitting of X
- T_2 the tree from the vertical splitting of X
- T_3 an interloping locally finite G-tree with FP vertex stabilizers
- $\mathscr{T} = T_1 \times T_2 \times T_3$
- X^+ a certain complex containing X
- $f : X^+ \to \mathscr{T}$ an equivariant map
- Γ a certain compact connected subgraph of $X^{(1)}$ the 1-skeleton of X
- $J = \mathrm{Im}(f)$
- $K = \mathrm{cell}(J)$
- $C = \mathrm{fill}(K)$ the hero of our story, the simply-connected core on which G acts

2.2 Definitions

6 Definition (Split Maps). Given a graph of spaces we obtain an invariant map as follows ...

7 Definition (Setup). We say that $K = (T, x_0, \sigma = \sigma_1 \cdots \sigma_N, \lambda)$ satisfies property *setup* if:

1. T is a simplicial tree
2. We have that x_0 is a valence one vertex

3. σ is a concatenation of non-degenerate edgepaths
4. $\sigma(t) = x_0 \implies t \in \{0, 1\}$
5. $\lambda(1) \neq \lambda(N)$

8 Definition (Bad). Given $K = (T, x_0, \sigma = \sigma_1 \cdots \sigma_N, \lambda)$ we say that an edge is *bad* if:

1. The edge e separates the basepoint from some endpoint. i.e.

$$e^+ \cap \text{endpoints} \neq \varnothing$$

   where e^+ is the halfspace of e not containing x_0.

2. The edge e always has the same color. i.e. $|\{\lambda(k) \mid e \subseteq \text{Im}\sigma_k\}| = 1$.

9 Definition (bad edge snipping). Let K satisfy property setup with e a bad edge. Let σ_i and σ_j be the first and second subpaths of σ that use e. Take σ' to be the concatenation of σ_k for $k < i$ with the subpath denoted σ'_{ij} that is the concatentation of the largest initial subpath of σ_i not using e and the longest tail of σ_j not using e with σ_k for $k > j$. The unmodified subpaths of σ in σ' receive the same colors as before and we take σ'_{ij} to have the color that σ_i and σ_j shared.

2.2.1 Miscellaneous

2.1 Lemma (Geometric Condition). (Theorem 0.6 in LP97) A minimal simplicial action of a finitely generated group is geometric if and only if all edge groups are finitely generated.

10 Definition (Fiberwise Connected). Let $S \subseteq X \times Y \times Z$. If $S \cap \{\text{pt}_1\} \times \{\text{pt}_2\} \times Z$ and all similar sets as well as permutations are connected then we say S is one dimensional fiberwise connected.

2.2 Lemma (Extension). (Lemma 8.9 in Guirardel) Consider a geometric action of a finitely generated group G on an $\mathbb{R}$-tree T, and let X be a 2-complex endowed with a free properly discontinuous cocompact action of G. Let $\mathscr{F}$ be a G-invariant measured foliation on X. Consider a map $f : X \to T$ which is constant on leaves of $\mathscr{F}$, and isometric in restriction to transverse edges of X. Then there exists a 2-complex X' containing X, endowed with a free properly discontinuous cocompact action of G, a measured foliation $\mathscr{F}'$ extending $\mathscr{F}$, and which induces an isometry between $X'/\mathscr{F}'$ and T. Moreover, the inclusion $X \subseteq X'$ induces an epimorphism of fundamental groups.

2.3 Lemma (Affine Equivariant Map). Suppose that G acts freely on a simplicial complex K and acts on a simplicial tree T. Then there exists an equivariant map $f : K \to T$ where $\mathscr{F}$ the connected components of the fibration from f is a measured foliation and f is an isometry on edges transverse to $\mathscr{F}$.

Proof. Part 1: Construct an equivariant map.

We start by defining f on $K^{(0)}$ the 1-skeleton. By equivariance it is enough to define the map on a single vertex in each vertex orbit. These choices can be arbitrary. Next we check that the resulting map is well-defined. Indeed, if $gv = hv$ then $g^{-1}h = 1$ by freeness and so

$$f(v) = g^{-1}hf(v) = g^{-1}f(hv)$$

but then

$$f(gv) = gf(v) = f(hv).$$

Next we define the map on edges. If vw is an edge, map it to the geodesic $[f(v), g(w)]$.

Lastly, for 2-cells we use the standard fibration from mapping triangles to tripods.

Part 2: Fibration details. □

2.4 Lemma (Guirardel Lemma 5.4, Corollary 5.5). Let T_1, T_2 be two $\mathbb{R}$-trees and let F be a nonempty connected subset of $T_1 \times T_2$ with convex fibers. Then the complement of $\overline{F}$ is a union of quadrants. That is, $\overline{F}$ is also nonempty, connected, and has convex fibers.

Claim. If K satisfies property setup then the first and last edge of σ are equal.

11 Definition (Filling). Let $\{X_k\}_{k\in K}$ be a family of spaces where one can take convex hulls. Given $S \subseteq X := \prod X_k$ define S_k for $k \in K$ via:

$$p \in S_k \iff \exists\, q, r \in S : \forall j \neq k : p_j = q_j = r_j \text{ and } p_k \in \operatorname{cvxhull}_k(\{q_k, r_k\}).$$

12 Definition (Type FP). A group is of type FP if it is (1) type FP_n for all n and (2) finite geometric cohomological dimension.

13 Definition (Finite Type). An action of *finite type* is one on a locally finite tree where vertex stabilizers are of type FP.

14 Definition (Open Direction). An open direction is a connected component of an $\mathbb{R}$-tree minus a point.

15 Definition (Closed Direction). A closed directon is a connected component of an $\mathbb{R}$-tree minus a point, union that point.

16 Definition (Open Halfspace). An open halfspace is an open direction obtained from deleting the midpoint of an edge.

17 Definition (Closed Halfspace). A closed halfspace is a closed direction obtained from deleting the midpoint of an edge.

18 Definition (Halfspaces of a product). An open (resp. closed) halfspace of a product (at a certain index) is a subset where exactly one projection is an open (resp. closed) halfspace in it's factor and the others are onto.

19 Definition (Generalized quadrants). A generalized open (resp. closed) quadrant with respect to a product of k spaces is an intersection of k open (resp. closed) product halfspaces where each one is at a different index.

20 Definition (cellular-product-convex). We say that $K \subset X$ is cellular-product-convex if it's complement is the open cellular neighborhood of a union of generalized closed quadrants.

3 Outlines

3.1 Misc lemmas

3.1 Theorem (F, "On uniqueness...", Thm 3.2). Let G be a group and let X and Y be cocompact G-trees. Then X and Y are related by an elementary deformation if and only if they have the same elliptic subgroups.

3.1 Lemma (core is connected). Let X and Y be two locally finite G-trees that lie in different deformation spaces. Then the core is connected.

Proof. [warning] still need connected implies simply-connected By Guirardel Proposition 4.14, the core is disconnected if and only if the two trees have a common refinement. This corresponds to collapse maps. On the graph of groups level this means there is an edge common in both quotient graphs. Let K be the stabilizer of this edge. It appears in all three trees. Since the trees are all locally finite, we have that all vertex groups of all trees involved are commensurable. But the property of fixing a point is invariant under commensurability. Therefore, all vertex groups of the first tree are elliptic in the second tree and vice versa. Hence, both actions have the same elliptic subgroups which means they are in the same deformation space which contradics our initial assumptions. □

3.2 Lemma (core is nonempty). Let X and Y be two non-trivial actions of a finitely generated group G with finitely generated vertex groups that lie in different deformation spaces. Then the core of X and Y is non-empty.

Proof. By Guirardel Proposition 3.1, if they were in the same deformation space then they would have homothetic length functions i.e. one length function is a multiple of the other. Since elliptic elemenets fix a point, they have length zero. This means that X and Y have the same elliptic elements. From theorem 4.2 and corollary 4.3 in [F] we get that they have the same elliptic subgroups. This implies they lie in the same deformation space which contradicts our original assumption. □

3.3 Lemma (eqivariant map from elliptic inclusion). If X and Y are G-trees such that $\mathcal{E}(X) \subset \mathcal{E}(Y)$, then there exists an equivariant map $X \to Y$.

Proof. Consider the vertex orbits in X. Pick a vertex from each orbit. Consider how the stabilizers of these vertices in X act on Y. Because of the elliptic subgroup containment, each G_{x_i} we picked out fixes a non-empty set of vertices in Y. Begin to define a map on the 0-skeleton by sending x_i to something in Y fixed by G_{x_i}. There are several choices, but a fixed set of choices plus the invariant condition defines a map on the 0-skeleton.

The containment says such a map is well-defined. Indeed, pick $x \in X^{(0)}$. Suppose $x = gx_0 = hx_0$, then $(h^{-1}g)x_0 = x_0$ and so $h^{-1}g \in G_{x_0}$. Then $f(x_0) = f((h^{-1}g)x_0)$ which gives $hf(x_0) = f(gx_0)$ and so $f(hx_0) = f(gx_0)$ as needed.

Once we have a map on the 0-skeleton we can extend it to the entire tree by drawing unique geodesics in the trees. □

3.4 Lemma (Invariant to bounded distance). Let A and B be invariant subcomplexes of X with G acting cocompactly on both A and B after restricting the action on X. Then A and B are Hausdorff equivalent.

Proof. Because we are dealing with cocompact actions on cell complexes for the action on A there exists a finite subcomplex $F \subseteq A$ such that the orbit of F covers A. We call F a fundamental domain. Let D_1 be the diameter of F. Pick some $b_0 \in B$ and let D_2 be the distance from F to b_0. Pick an arbitrary $a \in A$. Because the orbit of F is A there exists some $g \in G$ such that $a \in gF$. Then a is within $D_1 + D_2$ of gy. Hence, A is contained in the $D_1 + D_2$ neighborhood of B. Switching A and B in this argument and taking the maximum of the distances shows that A and B are both contained in R neighborhoods of each other for some R; that is they are Hausdorff equivalent. □

3.2 Main Equivariant Map Construction

Let X be a compact VH-complex and set $G = \pi_1 X$. Form the cover $\widetilde{X} \to X$. Note that G acts on $\widetilde{X}$ freely and PDC. Since X is a VH-complex we get two actions of G on trees T_1 and T_2 along with invariant 6 maps f_1 and f_2 from the splitting. Suppose we had a third action of G on a tree T_3 with property *nice*. Given the covering space action and the action on T_3 we use the affine construction 2.3 to get an equivariant map $f_3 : \widetilde{X} \to T_3$. Using Guirardel 8.9 2.2 we extend f_3 to $\widehat{f_3}$ a map with connected fibers. We also need the proof of lemma 2.2 to ensure certain properties hold. Then, in order to extend the f_1 and f_2 maps we use the coning off construction. The product of these extensions gives f.

3.3 Simply connected hyperplanes

To be a valid decomposition we need that the edge spaces are injective on fundamental groups. For this we need the corollary below.

Corollary. The hyperplanes of C are simply connected.

Which will follow from:

3.2 Theorem. The hyperplanes of C are quadrant convex.

One way to prove the theorem is to first show that C is one dimensional fiberwise connected 10 and then apply Guirardel's lemma 2.4 in each hyperplane to conclude that they're quadrant convex as needed.

3.3 Theorem (Connected fibers). The one dimensional fibers of C are connected.

To achieve this, we'll build C from a suitable G-invariant subcomplex $K \subseteq \mathscr{T}$ by taking $C = ((K_x)_y)_z$.

3.4 Combining into main argument

Note first we establish that C is itself simply connected.

3.5 Lemma. The core C is simply connected.

Proof. We observe that C is a graph of spaces, in particular it is a tree of simply connected spaces. Hence by scott and wall it's simply connected □

Finally, having enough dimension will follow from some assumptions about our actions. With this splitting in hand we want to verify that it's made of successive graphs of groups of items of a certain dimension so we can apply Bieri.

21 Definition (Not All Trees). There is an iterated splitting that doesn't end in trees. (Ideally, we will show that this assumption only rules out Z x Z, less ideally a statement about parabolics, less ideally we just assume it)

Corollary. Due to 21 we get that we can apply the dimension argument using Bieri to complete our result.

3.6 Lemma. The action of G on $T_1 \times T_2$ is free.

Proof. ...look in section II.6 of BH... Need enough facts to avoid non-proper spaces, get a semisimple action, and axes. □

3.7 Lemma (Cocompact beginning). We directly show that our S_{ij} are cocompact x factor.

3.8 Lemma (Cocompact factor after neighborhood). K x T with K cocompact taking a neighborhood we get again K' x T with K' cocompact

3.9 Lemma (Filling preserves cocompactness). Since S_{23} is a cocompact G-invariant subcomplex that's contained in $A \times B$ where A and B are subcomplexes of T_1 and $T_2 \times T_3$ respectively and B is itself cocompact and G acts freely on $T_2 \times T_3$ we have that $(S_{23})_x$ is cocompact as well.

Proof. For each vertex orbit of B choose a particular vertex. By cocompactness this list is finite; call them $\{b_1, \ldots, b_n\}$. By freeness in the second factor, the stabilizer of $S' := S_{23} \cap (T_1 \times \{b_1\})$ is trivial. This means that S' injects into S_{23}/G which is compact by assumption. Since we are dealing with cell complexes, S' is also compact. Repeating this argument a finite number of times we have that $S_{23} \cap (T_1 \times \{b_k\})$ is compact for each $1 \leq k \leq n$. Compact items have a well defined diameter. Taking the maximum diamater (in the product metric) and noting we're acting by isometries gives that there is a bound D on the diameters of $S_{23} \cap (T_1 \times \{v\})$ as v ranges over vertices of B.

Let $a \times b$ be a vertex of $(S_{23})_x$. By construction, a lies within the convex hull of the projection of $S_{23} \cap (T_1 \times \{b\})$ (a slice of S_{23}) to the T_1 factor, hence slices of $(S_{23})_x$ have diameter no greater than D in the product metric. This means each slice of $(S_{23})_x$ has a finite number of vertices. It remains to show that there is a unviersal bound on the number of vertices as we range over all slices. Since S_{23} is G-invariant and we're acting by a product, (so the slice at gb is the slice at b acted on by g) there are a finite number of isometry classes of slices since B is cocompact. Hence, there are a finite number of vertex orbits so $(S_{23})_x$ is cocompact as needed. $\square$

4 Core has quadrant convex hyperplanes

4.1 Lemma (Connected Fibers). Let p in X our 2-complex, let p' be the unique projection. Let Γ be a compact subgraph in our 2-complex X. Take T to be either T_1 or T_2. For now suppose we have defined a map $f : \Gamma \to T$. We will define a map $F : \Gamma \times I \to T$. Choose an arbitrary $t_0 \in f(\Gamma)$ and define:

$$F(x, s) = \begin{cases} f(x) & s = 0 \\ t_0 & s = 1 \\ \gamma_{f(x),t_0}(s|\gamma|)a & \text{else} \end{cases}$$

Let $k_0 \in \Gamma \cap F^{-1}(F(x, s))$ be the nearest point to p'. Put γ_{p',k_0}. Then define $g : \mathrm{Im}(\gamma_{p',k_0}) \to I$ by $g(t) = \frac{a(t)}{a(t)+b}$ where $a(t) = d(f(t), F(x, s))$ and $b = d(F(x, s), t_0$. Then we compute $F(t, g(t)) = F(x, s)$ so $\mathrm{Graph}(g) \subseteq F^{-1}(x, s)$. Now, g is continuous so $\mathrm{Graph}(g)$ is connected. Hence, (x, s) is connected to $(k_0, 0)$ in $F^{-1}(F(x, s))$ as needed.

4.2 Lemma (Fibers homeomorphic to Coordinate planes). Put $f = \hat{f}_1 \times \hat{f}_2 \times \hat{f}_3 : X^+ \to T_1 \times T_2 \times T_3$ and $J = \mathrm{Im}(f)$. We claim that $J \cap T_1 \times T_2 \times \{z\} = \mathrm{Im}_f(\hat{f}_3^{-1}(z))$. Let $p = (p_1, p_2, p_3) \in T_1 \times T_2 \times T_3$ then we have the following.

$$\begin{aligned} p \in \text{ LHS } &\iff p \in \mathrm{Im}(f) \wedge p_3 = z \\ &\iff \exists x \in X^+ (f(x) = p \wedge \hat{f}_3(x) = z) \\ &\iff \exists x \in X^+ (f(x) = p \wedge x \in \hat{f}_3^{-1}(z)) \\ &\iff p \in \mathrm{Im}_f(\hat{f}_3^{-1}(z)) \end{aligned}$$

4.1 Wrap up lemmas

The following is a list of statements, the goal is to prove enough of them to arrive at item number 1 for a suitably chosen core C.

1. $C \subseteq \mathscr{T}$ has simply connected hyperplanes
2. C has hyperplanes that are (1) connected (2) quadrant convex
3. The hyperplanes and one dimensional fibers of C are connected
4. If $S \subseteq \mathscr{T}$ connected in all coordinate planes, then so are S_x, S_y, and S_z.
5. If $R \subseteq T_1 \times T_2$ has convex fibers then $(R_x)_y = (R_y)_x$.
6. If $S \subseteq \mathscr{T}$ is connected in all coordinate planes then $((K_x)_y)_z$ is one dimensional fiberwise convex.
7. If $S \subseteq \mathscr{T}$ and is connected in all coordinate planes then $(S_x)_y = (S_y)_x$.
8. Guirardel's Lemma

4.1.1 Using Wrap-up lemmas to prove main statement

The implications are as follows:

$$(4),(7)\boxed{\Rightarrow}(6) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \tag{1}$$
$$(8) \Rightarrow (4) \tag{2}$$
$$(5) \Rightarrow (7) \tag{3}$$

4.3 Lemma (Reduction to vertical subpath). Let S be a subset of $T_1 \times T_2$ and let $p, q, r \in S$ satisfy:

1. $r \notin S$
2. $p, q \in S$
3. $p_2 = q_2 = r_2$
4. $r_1 \in \operatorname{cvxhull}_{T_1}(\{p_1, q_1\})$

Let $\sigma : [0,1] \to S$ be a path from p to q, then there exists a path σ' such that taking $p, q := \sigma'(0), \sigma'(1)$ satisfies properties (2)-(4) and $\pi_2(\operatorname{Im}(\sigma' \mid_{(0,1)}))$ is contained in an open direction of T_2 at r_2.

Proof. Consider the inverse image of $T_1 \times \{r_2\}$ under σ. After subdividing $[0,1]$ and noting that we're taking edge paths we get that this set is a finite union of closed intervals in $[0,1]$. We also have that it contains 0 and 1. Because r_1 is separating in T_1 we have that each closed interval lies within an open direction in T_1 at r_1. Also, because r_1 lies in the convex hull of p_1 and q_1 we get that the number of interals is at least 2. Consider pairs of endpoints of the closed intervals making up the closure of the complement of the inverse image. If each pair is in the same direction in T_1 at r_1 then because closed intervals map to horizontal directions we would have that 0 and 1 were in the same direction which contradicts our assumptions; hence there exists a subpath σ' where only the endpoints map into $T_1 \times \{r_2\}$. Consider $\sigma' |_{(0,1)}$, it's image lies in a product of a finite subtree of T_1 and a disjoint union of directions of T_2 at r_2. Because σ' is continuous it must lie in a single connected component and so projects to a single verticle direction of T_2 at r_2.

□

4.4 Lemma (Statement 4). Let $S \subseteq \mathscr{T}$ be a subcomplex that is connected in all coordinate planes. Then S_x, S_y, and S_z are as well.

Proof. Without loss of generality, consider S_x, note that S_x will be connected in all xy and xz planes because S was. Consider the yz-planes in S_x, if there were no new points added then the planes are connected and we are done. Suppose that $p \in (S_x \smallsetminus S) \cap \pi_1^{-1}(p_1)$, we will show that there is a path in $S_x \cap \pi_2^{-1}(p_2)$ between p and some point in S.

Since p isn't in S there exist distinct points r and s in S that agree in all coordinates except the first where we have that $p_1 \in \mathrm{cvxhull}_{T_1}(\{r_1, s_1\})$. Now, because S is connected in all coordinate planes there is a path σ from r to s that lies in $S \cap \pi_2^{-1}(p_2)$. In fact, we can take σ to be a path that begins and ends at p and r but otherwise has T_3 coordinates lying in exactly one direction of T_3 at p_3. We have factored out this situation into claim 4.3.

Take σ as in the claim 4.3. It remains to show that there is a path connecting p to another point in S. Considering closed quadrants, there exists a sequence $t_1, \ldots, t_n$ such that $v_i := \sigma(t_i)$ are vertices in $p_1 \times p_2 \times \overline{\delta}$ (here δ is the distinguished T_3 direction) with the property that $\sigma(t_1) = p$, $\sigma(t_n) = q$, and $\sigma_i := \sigma |_{[t_i, t_{i+1}]}$ lie in quadrant i for $0 < i < n$. (Here, quadrant i is determined by specifying a direction in T_1 at p_1 since we've already chosen δ above.) Projecting, we obtain a sequence of subpaths in $\overline{\delta}$ a direction in T_3 that are colored based on their quadrant. Note that $\overline{\delta}$ has a valence one vertex at p_3 because our tree is simplicial and subpaths don't end at p_3 unless it's the first or last because the claim has that the interior of the path maps into δ. This situation satisfies the conditions for our coloring lemma 4.7.

Multicolored paths in this vertical direction δ give paths contained in S_x, applying the coloring lemma 4.7 gives the required multicolored path. This completes the proof. □

4.5 Lemma (Statement 5). If $R \subseteq T_1 \times T_2$ is a connected subcomplex then $(R_x)_y = (R_y)_x$.

Proof. We will first show that $(R_x)_y$ has connected fibers. Note, it already has connected y-fibers, so it remains to show that it has connected x-fibers. Suppose that $(R_x)_y \cap (T_1 \times \{y_0\})$ were a disconnected x-fiber. Then because $(R_x)_y$ is a subcomplex we have that the fiber is separated by some edge. Let x_0 be the midpoint of this edge. Let ℓ and r denote the left and right closed halfspaces of $T_1 \times \{y_0\}$ at the midpoint x_0. We will show that either $\ell \cap (R_x)_y = \varnothing$ or $r \cap (R_x)_y = \varnothing$, applying this to all such x_0 will show $(R_x)_y$ has connected x-fibers.

Suppose this were false, and that $\ell \cap (R_x)_y \neq \varnothing$ and $r \cap (R_x)_y \neq \varnothing$. We consider three cases: (1) ℓ and r intersect R_x nontrivially (2) ℓ and r don't intersect R_x, and (3) exactly one of ℓ or r intersects R_x nontrivially.

1. Case 1: If both ℓ and r intersect R_x then both contain points of R because we filled in the x-fiber, but then $x_0 \times y_0 \in R_x$ a contradiction.

2. Case 2: Pick a point x^+ in $r \cap (R_x)_y \setminus R_x$. Let q^+ and q^- be points above and below x^+ in R_x. These points are either in R already, or because we filled in the x-fiber there exist points above and below the line $\ell \cup r$ but this is a contradiction since R is connected.

3. Case 3: Without loss of generality, suppose $\ell \cap R_x = \varnothing$ and $r \cap R_x \neq \varnothing$. This implies there is some $s \in R \cap r$, we will get a contradiction by separating this point from another point in R.

   Let u and d be directions of $x_0 \times T_2$ at $x_0 \times y_0$. If both intersected R_x then $x_0 \times y_0 \in R$ a contradiction so without loss of generality, suppose $d \cap R_x \neq \varnothing$. Now let x^- be a point in ℓ, by assumption $x^- \in (R_x)_y \setminus R_x$. This gives a point q^- below ℓ in R_x which is already in R or there exists a point $t^- \in R$ to the left of q^-; but this is a contradiction since $\ell \cup d$ separates these points from $s \in R$.

Hence, either $\ell \cap (R_x)_y = \varnothing$ or $r \cap (R_x)_y = \varnothing$ as needed, so $(R_x)_y$ has connected x-fibers. Similarly, $(R_y)_x$ also has connected fibers. Hence by 2.4 their complements are unions of quadrants and so they contain $QH(R)$. It remains to show that they are contained within $QH(R)$.

Claim. $(R_x)_y \subseteq QH(R)$

Proof. (sketch) The idea is to show that for every point $p \in (R_x)_y$ that for every quadrant Q containing p (i.e. that would be attempting to remove it) we can find a point $r \in R \cap Q$. Picking an open quadrant containing p amounts to picking a point $q = (q_1, q_2)$ with halfspaces at each coordinate that contain the corresponding p_i. The hard case is where $p \in (R_x)_y \setminus R_x$ - so you pick a point in a vertical direction at p pointing towards p, because this point is only there due to filling there's another vertical direction that you're grabbing that contains some R_x. Then you pick a horizintal place and point towards p_1, this must contain at least one of the directions with R in it. □

Proof. We will show for every point $p \in (R_x)_y$ and every open quadrant Q with $p \in Q$ that there exists some $r \in R \cap Q$. Let $q = (q_1, q_2)$ be the point where Q is based and label the halfspaces so that $Q = q_1^+ \times q_2^+$.

Case 1 $p \in R_x \setminus R$: Let $p \in R_x \setminus R$: Let $\ell = (\ell_1, p_2)$ and $u = (u_1, p_2)$ be points in R that cause the vertical filling. Now, q_1^+ contains all but one direction at p_1 and so must contain either ℓ_1 or u_1. Since q_2^+ must contain p_2 we get that $Q = q_1^+ \times q_2^+$ contains a point of R.

Case 2 $p \in (R_x)_y \smallsetminus R_x$: Let $u = (p_1, u_2)$ and $d = (p_1, d_2)$ be points that cause the vertical filling. Now, q_2^+ contains all but one direction at p_2 and so must contain either u_2 or d_2. Without loss of generality, suppose it contains u_2. If $u \in R$ then we are done. Suppose $u \in R_x \smallsetminus R$. Then we can find $u' = (p', u_2)$ and $u'' = (p'', u_2)$ with p' and p'' in different directions at p_1. Now, q_1^+ contains all but one direction at p_1 and so must contain either p' or p''. Suppose without loss of generality that it contains p', then Q contains $p' \times u_2 \in R$. □

□

4.6 Lemma (Boxed Implication). Repeatedly applying statement 7 to both S and S_α where α is one of x, y, or z and noticing that $\langle (12), (23) \rangle = S_3$ we get that $((K_x)_y)_z$ is equal to any of the permutations of the indices. In particular, $((K_x)_y)_z = ((K_y)_z)_x = ((K_z)_x)_y$ which shows that $((K_x)_y)_z$ has connected one dimensional fibers.

4.2 Planar Path Argument

4.7 Lemma (Coloring Lemma). Let K satisfy property setup. Let A be the set of endpoints minus the basepoint x_0. Then there exists some $a \in A$ such that for all edges e in the geodesic $[x_0, a]$ there exist some i, j such that σ_i and σ_j use e and $\lambda(i) \neq \lambda(j)$.

Proof. Suppose the lemma were false. Then there exist counterexamples K satisfying property setup such that for all $a \in A$ there exists an edge e on $[x_0, a]$ such that $\lambda(i) = \lambda(j)$ whenever σ_i and σ_j use e. These K have bad edges and $|\sigma| > 0$. Now take a K such that $|\sigma|$ is minimized. Let K' be the result of snipping a bad edge. By lemma 4.8, K' has property setup so $|\sigma'| > 0$. If K' contained a multicolored path between x_0 and some a' in A' then because the path σ in K an be obtained by inserting subpaths into σ', we have that $A' \subseteq A$ and any edge that was multicolored stays multicolored. Therefore K' has no multicolored path, because that would force K to have one. Then the fact $|\sigma'| < |\sigma|$ gives a contradiction because K was chosen to be minimal.

□

4.8 Lemma (Snip invariant). If K satisfies property setup then K' obtained from snipping a bad edge also satisfies property setup.

Proof. Let e be a bad edge of K, since σ_1 and σ_N have different colors the edge e in at most one of them. Because x_0 has valence one, σ_1 and σ_N share the edge containing x_0; therefore e cannot be the common edge. The rest follows. □