1 VH-complexes

Let X be our complex and X_V be the set of vertical edges. We say gates are connected components of X_V and corridors are cc of $X - X_V$. The complex X is clean if the attaching maps of the corridors are injective.

1.1 Theorem (Wise). If X a complete, VH, clean, connected complex then there is a finite sheeted cover that's a graph product.

1.2 Theorem. If X is NPC, VH, compact, nice(???) then X is complete XOR contains a locally geodesic VH pair of lollipops

2 Definitions

2.1 Definition (Graph - Abstract). An *Abstract Graph* Γ is a set (V, E, ∂, i) where V and E are non-empty sets and $\partial : E \to V$ and $i : E \to E$ are functions satisfying $i^2(e) = e$ and $i(e) \neq e$ for all $e \in E$. (i.e. the function i is a fixed-point free involution)

We set $o(e) := \partial(e)$ and $t(e) := (\partial \circ i)(e)$ for origin and terminal vertices and put $\overline{e} := i(e)$. We call the orbits of i the *undirected edges* of G. (i.e. the set $\{e, \overline{e}\}$ is an undirected edge)

2.2 Definition. A tree action (G, T) is *minimal* if it contains no proper invariant subtree. A graph of groups $\mathcal{G}$ is minimal if there is no proper subgroup carrying the entire group.

2.3 Definition (Collapse Move). If we have a group G acting on a tree T with an edge e such that $G_e = G_v$ where $v = o(e)$ and $o(e)$ and $t(e)$ are in different G-orbits then we can form T_e a new tree with $V(T_e) = V(T) \smallsetminus Gt(e)$ and $E(T_e) = E(T) \smallsetminus Ge$. Then for all edges f with $o(f) = t(ge)$ for some $g \in G$ we define $o(f) = gv$ in T_e.

(Alternatively, if q was the map that paired the inital and terminal vertices of ge and left the others alone then we could define the new tree by taking $E(T_e) := E(T) \smallsetminus Ge$ and $V(T_e) := q(V)$ with attaching map $q \circ \partial$.

If $\mathcal{G}$ is a graph of groups decomposition of G with $\varphi_e : G_e \to G_{\partial e}$ an isomorphism then define $\mathcal{G}_e$ by removing e, $\overline{e}$ and ∂e and for every edge f with $\partial(f) = \partial(e)$ replace φ_f with $\varphi_{\overline{e}} \circ \varphi_e^{-1} \circ \varphi_f$ and set $\partial f = \partial \overline{e}$. This corresponds to taking the edge e from the tree description above and folding $[e]$ in the graph.

2.4 Definition (Fold). Given a tree with $\partial e = \partial f$ identify e with f as well as $\overline{e}$ and $\overline{f}$ and $\overline{\partial} e$ with $\overline{\partial} f$ and do so equivariantly. (In $\mathcal{G}$ this corresponds to moves of type A or B with subtype I, II, or III.)

Claim. A reduced not locally finite tree remains not locally finite after folding.

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3 Claims