Contents

1 Problem Statement

1.1 Theorem. If G has algebraic cohomological dimension 2 12 then there are at most two non-trivial finite type 5 pairwise transverse 6 G-trees up to deformation 4.

Proof. Suppose for sake of a contradiction there were three non-trivial G-trees T_1, T_2, and T_3 of finite type 5 that were pairwise transverse 6 and no two are in the same deformation space. By 1.5 without loss of generality, we may assume that these are minimal G-trees. Applying the tranverse construction lemma 3 we obtain X_{12}. Let $\widetilde{X_{12}}$ denote it's universal cover. Applying the affine equivariant map construction 2.9 to the actions on $\widetilde{X_{12}}$ and T_3 gives a map and a fibration that we denote by f_{123} and $\mathscr{F}_{123}$ respectively. Due to 2.7 we have geomeric actions. Additional technical assumptions are handled here 1.3. Then using 2.8 and the proof theoreof 1.1 we obtain $\widetilde{X_{12}}^+$, an extension f_{123}^+, and a fibration $\mathscr{F}_{123}$.

Next let f_{121} and f_{122} denote the projection maps from $\widetilde{X_{12}}$ to T_1 and T_2 respectively. (aka Bass-Serre maps) After coning off as described in 2.6 we get extensions $f_{121}^\wedge$ and $f_{122}^\wedge$ and similarly $\mathscr{F}_{121}^\wedge$ and $\mathscr{F}_{122}^\wedge$. Lastly we form the product map $f := f_{121}^\wedge \times f_{122}^\wedge \times f_{123}^+$.

Consider $J := \mathrm{Im}(f)$. By lemma 4.2 we get that J has property CCP 25, since it is the image of a cocompact set under a continuous G-map it is cocompact and is in fact cocompact by factor 7 since the action is diagonal.

Let S_{12} denote the cellular neighborhood of J, because this set is defined in an invariant way as the smallest such complex with a property it is G invariant and because our trees are locally finite and a cellular neighborhood is contained in a bounded neighborhood we have that S_{12} is cocompact and therefore cocompact by factor 7. Taking a cellular neighborhood respects slices by 1.2 so S_{12} has property CCP 25.

Next we repeatedly apply lemma 3.9 and use the fact that filling preserves CCP 25 by construction to get that $C := S_{12}xyz$ is cocomapct and has CCP 25. By switching 4.4 C has connected 1-dimensional fibers.

By lemma 1.4 we have that C is simply connected and C/G comes with three splittings as a graph of spaces.

We will now iterate the splitting process and then apply Bieri. To begin, by assumption each splitting is non-trivial. Now, consider the vertex groups of a given splitting. By lemma 1.6 those vertex groups act non-trivially on the two trees coming from splittings of the VH-complexes by Wise. These last splittings are themselves splittings over trivial groups - but are they trivial splittings? Here we need to assume that one such iterated sequence of splittings ends in a graph with positive rank.

Then by the Bieri dimenson argument, G has dimenson 3 a contradiction. □

1.1 Lemma (Shortcircuit Guirardel Proof). Stopping after a certain point in Guirardel is beneficial.

1.2 Lemma (Cell Respects Slices). by set theory we are taking cell of a 2D thing and so connected really

1.3 Lemma (Technical assumptions for Guirardel). we use a result of leveitt and paulin for geomeric (see elsewhere) and prove the rest here

1.4 Lemma (Expanded Core is simply connected). Our C inherits a VHD structure from the product of three trees that it sits in. Because the action is diagonal we also get that the quotient C/G is VHD. Our tree actions do not invert edges so we get that the hyperplanes of C/G are two-sided. Recall, a given hyperplane only touches a single parallelism class of edges.

Next, we need the push maps from the hyperplanes to be injective on fundamental groups. The push map from a hyperplane to a vertex space (we necessarily send hyperplanes - itself a connected set - to a component i.e. a vertex space) followed by inclusion is the same (up to homotopy) as globally including the hyperplane into C/G. We will show that the composition is injective so that the induced map from the vertex to the edge is injective as needed. (fix) needs details, see diagram tracking how we consider this set

Corollary. If a VH-complex has horizontal and vertical splittings of finite type then up to deformation, these are the only such G-trees.

1.5 Lemma (Minimality is enough). If the theroem is true for minimal trees then it is true for trees in general

1.6 Lemma (Iterated Splitting). Suppose x were a vertex of T_V, Let K be it's stabilizer. Now K is a subgroup of G and so also acts on T_H. If K had a global fixed point in T_H then by local finiteness of T_V every vertex group of T_V would as well. Then if you are elliptic for T_V you are elliptic for T_H which by a previous lemma in our setting gives that T_H and T_V are in the same deformation space; a contradiction.

2 Toy Version

2.1 Promote elliptic containment to equality

2.1 Lemma (Minimality Argument).

1 Definition (Simplicial Map). A map between simplicial complexes is simplicial if the image of a set of vertices that spans a simplex also spans as simplex.

2 Definition (Morphism). We say a simplicial map between trees is a morphism if edges go to edges.

3 Definition (Collapse Map). Given a G-tree, collapsing components of an invariant forest to points gives a G-map.

1 Proposition (Subgraph Collapse). Replacing a connected subgraph in a graph of groups decomposition with a vertex group with the same group as the collapsed graph corresponds to collapsing an invariant forest to points.

2 Proposition (Folds "factor" as an elementary collapse and a collapse map). If $X \to Y$ is a fold, then there exists a tree Z along with collapse maps to X and Y. Moreover, the map to X is a collapse map corresponding to an elementary collapse.

2.2 Lemma (Hyperbolic elements are preserved). Suppose $X \to Y$ is a collapse map bewteen locally finite G-trees. Suppose Y is not a single point. Then, if an element is hyperbolic for X it is also hyperbolic for Y.

Proof. Suppose for sake of a contradiction that $g \in G$ were hyperbolic for X and elliptic for Y. Let $y \in Y$ be some vertex fixed by the elliptic element g and G_y it's stabilizer. Since Y is not a single point, there is another vertex $y \neq z \in Y$. Because Y is locally finite, G_y and G_z are commensurable. For G-maps, preimages are invariant. By the construction of a collapse map, the preimage of vertices are connected and non-empty. Putting these together we have that the preimages of vertices are invariant trees. This means that the minimal subtrees of G_y and G_z acting on X are contained in the disjoint preimages of y and z respectively. However, since they are commensurable and G_y contains the hyperbolic element g, these minimal trees are non-empty and equal. This is a contradiction. $\square$

2.1 Theorem (Elliptic elements determine elliptic subgroups). Let X and Y be cocompact G-trees with finitely generated vertex groups. Then the following are equivalent:

1. X and Y define the same partition of G into elliptic and hyperbolic elements.

2. X and Y have the same elliptic subgroups.

Proof. By Proposition 2.6, Theorem 4.2, and Corollary 4.3 of [F, deformation and rigidity]. □

2.2 Theorem (Factoring as folds, from Bestvina paper, p455). Let G be a finitely generated group. Suppose that $\alpha : T' \to T$ is a simplicial equivariant map from a G-tree T' to a minimal G-tree T such that no edge in T' is mapped to a point by α. If all edge stabilizers of T are finitely generated and if T'/G is finite, then α can be represented as a finite composition of folds.

2.3 Lemma (Minimal trees invariant under commensurability). Suppose G acts on a tree X and H and K are commensurable subgroups. If H contains a hyperbolic element, then so does K and the minimal subtrees for H and for K are equal.

2.4 Lemma (Local finiteness preserves hyperbolically). Suppose $f : X \to Y$ is a G-map of G-trees which is onto, where Y is locally finite and not a single point. Then no element of G is hyperbolic for X and elliptic for Y.

2.1 Remark. Note, if the above Y is minimal then we get surjectivity for free.

2.3 Theorem (Elliptic containment implies equality). If X and Y are locally finite cocompact G-trees with finitely generated vertex and edge stabilizers then $\mathcal{E}(X) \subseteq \mathcal{E}(Y) \Longrightarrow \mathcal{E}(X) = \mathcal{E}(Y)$.

Proof. (fix) this is now fixed due to "an easier way" email using 2.4 and 2.3. □

Proof. (fix) outdated: We first collapse our general G-map to a morphism (edges to to edges and vertices to vertices). Then we use 2.2 to factor the result as a sequence of folds (followed by an immersion if we're not assuming minimality). Then we use 2 and finite generation (of edge and vertex groups?) to get that same elliptic elements give us same elliptic subgroups which gives us same deformation space which means the second collapse was an elementary one (in theory possibly a sequence of expand and collapses) which means we keep local finiteness so we can repeatedly use 2.2. (fix) Add the final bit about the immersion, note that for trees immersions are inclusions. Note that if you're including Z into a tree you can think of the action of Z as the restriction of the action on the target tree. Just need to check that the axis stays... essentially because the entire tree stays □

2.2 Toy version of main result

2.2.1 Definitions

2.5 Lemma (Core non-empty).

4 Definition (Deformation). Definition of deformation moves and the corresponding equivalence classes.

5 Definition (Finite Type). We say a tree action is of *finite type* if the action is non-trivial, the tree is locally finite, and the vertex stabilizers are FP, with finite quotient graph.

2.2 Remark. We will assume a group of dimension 2 and in that case Bieri gives that actions of finite type have vertex and edge stabilizers that are finitely generated free groups.

2.3 Remark. Bieri gives us that if we have an action of finite type then the group is FP.

6 Definition (Transverse). We say that two tree actions X and Y are *transverse* if they are not in the same deformation space and there exist two stabilizers, one for each tree, such that their intersection is FP.

2.4 Remark. The definition of transverse does not depend on the vertices chosen and remains unchanged up to deformation spaces

2.2.2 Result

3 Proposition (Transverse Construction). Let G a group of cohomological dimension 2. If X and Y are non-trivial G-trees of finite type that are in different deformation spaces then the following are equivalent:

1. X, Y transverse
2. $x \in V(X), y \in V(Y) \implies G_x \cap G_y = \{1\}$
3. There exists a cocompact VH-complex K with $\pi_1(K) = G$ whose horizontal and vertical splittings are X and Y.

Proof. 1. $1 \Rightarrow 2$: Fix $x_0 \in V(X)$ Let $y \in V(Y)$. Then $G_{x_0} \cap G_y = (G_{x_0})_y$. By (1) X is transverse to Y hence $G_{x_0} \cap G_y$ is FP. Since the choice of $y \in V(Y)$ was arbitrary, the vertex groups of the G_{x_0} action on Y are FP. Note, Y locally finite implies it's edge groups are finite index subgroups of it's vertex groups. Hence the edge groups are also FP. We claim that the action of G_{x_0} on Y is non-trivial. Given this we apply Bieri to get:

$$\begin{aligned} 2 &= dG \\ &= dG_{x_0} + 1 \\ &= d(G_{x_0})_y + 1 + 1 \\ &= d(G_{x_0} \cap G_y) + 2 \end{aligned}$$

Hence, $d(G_{x_0} \cap G_y) = 0$ so $G_{x_0} \cap G_y$ is trivial.

Claim. The action of G_{x_0} on Y is non-trivial.

Proof. Suppose the action were trivial. That is, there exists some $y \in V(Y)$ such that $(G_{x_0})_y = G_{x_0}$. Hence, G_{x_0} is elliptic for the action of G on Y. By the local finiteness of Y, for all $x \in V(X)$ G_x acts elliptically on Y. Hence, $\mathcal{E}(X) \subset \mathcal{E}(Y)$. Again by local finiteness we can promote this using 2.3 to $\mathcal{E}(X) = \mathcal{E}(Y)$ which by theorem 3.1 gives $X \sim Y$ which contradicts the fact that X and Y were assumed to be in different deformation spaces. □

2. $2 \Rightarrow 1$: Trivial groups are FP.

3. $2 \Rightarrow 3$: Take $X \times Y$ and give it the VH-structure where X and Y correspond to horizontal and vertical edges respectively. By 2.5 the core is non-empty. Condition (2) says that G acts freely on $X \times Y$. By 3.2 the core is connected. By the remark after Proposition 4.17 in Guirardel the core is simply connected. The action is also free on subsets of $X \times Y$. Since the action is cellular and free it's a covering space action. (We avoid situations like irrational rotations on a circle that are free but not covering space actions)

   It remains to show that C/G is VH. Is it enough to say that the action respects the tree factors. (The edge partition on the cover descends to a well-defined edge partition on the quotient and attaching maps constructed in the standard way for the quotient alternate between vertical and horizontal edges)

4. $3 \Rightarrow 2$: Suppose $1 \neq g \in G$ is an element of $G_x \cap G_y$ where $x \in V(X)$ and $y \in V(Y)$. Since K is a VH-complex it has a decomposition as a graph of groups where the vertex spaces are connected subgraphs whose edges are all vertical. Each $x \in V(X)$ is in correspondence with the inclusion of a vertex space (composed entirely of vertical edges) into K. The inclusion of vertex spaces is always injective on fundamental groups. Lastly, after picking basepoints the image of the induced map is the stabilizer of x.

   By Wise, we have that the universal cover of K is contained in $X \times Y$. Because the action respects the product structure, and because vertex spaces (in our case, graphs) are covered by embedded copies of their own universal covers we have that the non-trivial element g when represented by a loop in a vertex space lifts to a path with distinct endpoints in a tree consisting entirely of vertical edges in $X \times Y$. In fact, G_x acts on $\{x\} \times Y$ and freely on $\widetilde{K} \cap (\{x\} \times Y)$ because the action on $\widetilde{K}$ is a covering space action. This means that g acts hyperbolically on Y and we get an axis in Y. This remains an axis in $X \times Y$.

   Thus from looking at the vertical splitting we obtained an axis consisting entirely of vertical edges. Similarly, after looking at the horizontal splitting we obtain an axis consisting entirely of horizontal edges.

Finally, since $X \times Y$ is CAT(0) these axes would have to be parallel. This is a contradiction.

$\square$

2.3 Dramatis Personæ

- X a compact connected VH-complex
- $G = \pi_1(X)$
- T_1 the tree from the horizontal splitting of X
- T_2 the tree from the vertical splitting of X
- T_3 an interloping locally finite G-tree with FP vertex stabilizers
- $\mathscr{T} = T_1 \times T_2 \times T_3$
- X^+ a certain complex containing X
- $f : X^+ \to \mathscr{T}$ an equivariant map
- Γ a certain compact connected subgraph of $X^{(1)}$ the 1-skeleton of X
- $J = \mathrm{Im}(f)$
- $K = \mathrm{cell}(J)$
- $C = \mathrm{fill}(K)$ the hero of our story, the simply-connected core on which G acts

2.4 Definitions

2.6 Lemma (Cone extension). Call the original complex X, pick out a subcomplex Y with non-overlapping orbits, take products with $[0, 1]$, send the 1 factor copy of Y to somewhere in the $\mathrm{Im}(f)$ and extend using the product fibers and geodesics in the target tree.

7 Definition (Cocompact by factor). Given a diagonal action of G on a product, we say that a G-invariant subset S is cocompact by factor (with respect to some index α in the product) if $S \subseteq S' \times X_\alpha$ where S' is a G-invariant subset of the product restricted to every index except α and S' is cocompact.

4 Proposition (Filling respects slices).

8 Definition (Property Setup). We say that $K = (T, x_0, \sigma = \sigma_1 \cdots \sigma_N, \lambda)$ satisfies property *setup* if:

1. T is a simplicial tree
2. We have that x_0 is a valence one vertex

3. σ is a concatenation of non-degenerate edgepaths
4. $\sigma(t) = x_0 \implies t \in \{0, 1\}$
5. $\lambda(1) \neq \lambda(N)$

9 Definition (Bad). Given $K = (T, x_0, \sigma = \sigma_1 \cdots \sigma_N, \lambda)$ we say that an edge is *bad* if:

1. The edge e separates the basepoint from some endpoint. i.e.

$$e^+ \cap \text{endpoints} \neq \varnothing$$

where e^+ is the halfspace of e not containing x_0.

2. The edge e always has the same color. i.e. $|\{\lambda(k) \mid e \subseteq \text{Im}\sigma_k\}| = 1$.

10 Definition (bad edge snipping). Let K satisfy property setup 8 with e a bad edge. Let σ_i and σ_j be the first and second subpaths of σ that use e. Take σ' to be the concatenation of σ_k for $k < i$ with the subpath denoted σ'_{ij} that is the concatentation of the largest initial subpath of σ_i not using e and the longest tail of σ_j not using e with σ_k for $k > j$. The unmodified subpaths of σ in σ' receive the same colors as before and we take σ'_{ij} to have the color that σ_i and σ_j shared.

2.4.1 Miscellaneous

11 Definition (VHD-complex). We say a cube complex is VHD if it's edges can be partitioned into three sets each with a different color such that the the link of each vertex is a tripartite simplicial graph.

12 Definition (Cohomological Dimension). We are using the minimal resolution size over the integers

2.5 Remark. The only case when geo is not alg is possibly when geo=3 and alg=2

2.7 Lemma (Geometric Condition). (Theorem 0.6 in LP97) A minimal simplicial action of a finitely generated group is geometric if and only if all edge groups are finitely generated.

13 Definition (Fiberwise Connected). Let $S \subseteq X \times Y \times Z$. If $S \cap \{\text{pt}_1\} \times \{\text{pt}_2\} \times Z$ and all similar sets as well as permutations are connected then we say S is one dimensional fiberwise connected.

2.8 Lemma (Guirardel Extension). (Lemma 8.9 in Guirardel) Consider a geometric action of a finitely generated group G on an $\mathbb{R}$-tree T, and let X be a 2-complex endowed with a free properly discontinuous cocompact action of G. Let $\mathscr{F}$ be a G-invariant measured foliation on X. Consider a map $f : X \to T$ which is constant on leaves of $\mathscr{F}$, and isometric in restriction to transverse edges of X. Then there exists a 2-complex X' containing X, endowed with a free properly discontinuous cocompact action of G, a measured foliation $\mathscr{F}'$ extending $\mathscr{F}$, and which induces an isometry between $X'/\mathscr{F}'$ and T. Moreover, the inclusion $X \subseteq X'$ induces an epimorphism of fundamental groups.

2.9 Lemma (Affine Equivariant Map). Suppose that G acts freely on a simplicial complex K and acts on a simplicial tree T. Then there exists an equivariant map $f : K \to T$ where $\mathscr{F}$ the connected components of the fibration from f is a measured foliation and f is an isometry on edges transverse to $\mathscr{F}$.

Proof. Part 1: Construct an equivariant map.

We start by defining f on $K^{(0)}$ the 1-skeleton. By equivariance it is enough to define the map on a single vertex in each vertex orbit. These choices can be arbitrary. Next we check that the resulting map is well-defined. Indeed, if $gv = hv$ then $g^{-1}h = 1$ by freeness and so

$$f(v) = g^{-1}hf(v) = g^{-1}f(hv)$$

but then

$$f(gv) = gf(v) = f(hv).$$

Next we define the map on edges. If vw is an edge, map it to the geodesic $[f(v), g(w)]$.

Lastly, for 2-cells we use the standard fibration from mapping triangles to tripods.

Part 2: Fibration details. □

2.10 Lemma (Guirardel Lemma 5.4, Corollary 5.5). Let T_1, T_2 be two $\mathbb{R}$-trees and let F be a nonempty connected subset of $T_1 \times T_2$ with convex fibers. Then the complement of $\overline{F}$ is a union of quadrants. That is, $\overline{F}$ is also nonempty, connected, and has convex fibers.

Claim. If K satisfies property setup 8 then the first and last edge of σ are equal.

14 Definition (Filling). Let $\{X_k\}_{k\in K}$ be a family of spaces where one can take convex hulls. Given $S \subseteq X := \prod X_k$ define S_k for $k \in K$ via:

$$p \in S_k \iff \exists\, q, r \in S : \forall j \neq k : p_j = q_j = r_j \text{ and } p_k \in \text{cvxhull}_k(\{q_k, r_k\}).$$

15 Definition (Type FP). A group is of type FP if it is (1) type FP_n for all n and (2) finite geometric cohomological dimension.

16 Definition (Finite Type). An action of *finite type* is one on a locally finite tree where vertex stabilizers are of type FP.

17 Definition (Open Direction). An open direction is a connected component of an $\mathbb{R}$-tree minus a point.

18 Definition (Closed Direction). A closed directon is a connected component of an $\mathbb{R}$-tree minus a point, union that point.

19 Definition (Open Halfspace). An open halfspace is an open direction obtained from deleting the midpoint of an edge.

20 Definition (Closed Halfspace). A closed halfspace is a closed direction obtained from deleting the midpoint of an edge.

21 Definition (Halfspaces of a product). An open (resp. closed) halfspace of a product (at a certain index) is a subset where exactly one projection is an open (resp. closed) halfspace in it's factor and the others are onto.

22 Definition (Generalized quadrants). A generalized open (resp. closed) quadrant with respect to a product of k spaces is an intersection of k open (resp. closed) product halfspaces where each one is at a different index.

23 Definition (cellular-product-convex). We say that $K \subset X$ is cellular-product-convex if it's complement is the open cellular neighborhood of a union of generalized closed quadrants.

3 Outlines

3.1 Misc lemmas

3.1 Lemma (Parallel edges). Suppose X is a VHD-complex. Then parallel edges have the same color and edges that share a vertex have different colors.

This is false, for a single cube see picture in phone taken on 2020-08-22 The only property we need is that every cube is colored VHD, so we can reference splittings dual to an edge and that the sides are VH complexes

3.1 Theorem (F, "On uniqueness...", Thm 3.2). Let G be a group and let X and Y be cocompact G-trees. Then X and Y are related by an elementary deformation if and only if they have the same elliptic subgroups.

3.2 Lemma (core is connected). Let X and Y be two locally finite G-trees that lie in different deformation spaces. Then the core is connected.

Proof. By Guirardel Proposition 4.14, the core is disconnected if and only if the two trees have a common refinement. This corresponds to collapse maps. This means there is an edge in the refinement that has an edge above it in both trees. Let K be the stabilizer of this edge. It appears in all three trees. Since the trees are all locally finite, we have that all vertex groups of all trees involved are commensurable. But the property of fixing a point is invariant under commensurability. Therefore, all vertex groups of the first tree are elliptic in the second tree and vice versa. Hence, both actions have the same elliptic subgroups which means they are in the same deformation space which contradics our initial assumptions. $\square$

3.3 Lemma (core is nonempty). Let X and Y be two non-trivial actions of a finitely generated group G with finitely generated vertex groups that lie in different deformation spaces. Then the core of X and Y is non-empty.

Proof. By Guirardel Proposition 3.1, if they were in the same deformation space then they would have homothetic length functions i.e. one length function is a multiple of the other. Since elliptic elemenets fix a point, they have length zero. This means that X and Y have the same elliptic elements. From theorem 4.2 and corollary 4.3 in [F] we get that they have the same elliptic subgroups. This implies they lie in the same deformation space which contradicts our original assumption. □

3.4 Lemma (eqivariant map from elliptic inclusion). If X and Y are G-trees such that $\mathcal{E}(X) \subset \mathcal{E}(Y)$, then there exists an equivariant map $X \to Y$.

Proof. Consider the vertex orbits in X. Pick a vertex from each orbit. Consider how the stabilizers of these vertices in X act on Y. Because of the elliptic subgroup containment, each G_{x_i} we picked out fixes a non-empty set of vertices in Y. Begin to define a map on the 0-skeleton by sending x_i to something in Y fixed by G_{x_i}. There are several choices, but a fixed set of choices plus the invariant condition defines a map on the 0-skeleton.

The containment says such a map is well-defined. Indeed, pick $x \in X^{(0)}$. Suppose $x = gx_0 = hx_0$, then $(h^{-1}g)x_0 = x_0$ and so $h^{-1}g \in G_{x_0}$. Then $f(x_0) = f((h^{-1}g)x_0)$ which gives $hf(x_0) = f(gx_0)$ and so $f(hx_0) = f(gx_0)$ as needed.

Once we have a map on the 0-skeleton we can extend it to the entire tree by drawing unique geodesics in the trees. □

3.5 Lemma (Invariant to bounded distance). Let A and B be invariant subcomplexes of X with G acting cocompactly on both A and B after restricting the action on X. Then A and B are Hausdorff equivalent.

Proof. Because we are dealing with cocompact actions on cell complexes for the action on A there exists a finite subcomplex $F \subseteq A$ such that the orbit of F covers A. We call F a fundamental domain. Let D_1 be the diameter of F. Pick some $b_0 \in B$ and let D_2 be the distance from F to b_0. Pick an arbitrary $a \in A$. Because the orbit of F is A there exists some $g \in G$ such that $a \in gF$. Then a is within $D_1 + D_2$ of gy. Hence, A is contained in the $D_1 + D_2$ neighborhood of B. Switching A and B in this argument and taking the maximum of the distances shows that A and B are both contained in R neighborhoods of each other for some R; that is they are Hausdorff equivalent. □

3.2 Main Equivariant Map Construction

Let X be a compact VH-complex and set $G = \pi_1 X$. Form the cover $\widetilde{X} \to X$. Note that G acts on $\widetilde{X}$ freely and PDC. Since X is a VH-complex we get two actions of G on trees T_1 and T_2 along with invariant maps f_1 and f_2 from the splitting. Suppose we had a third action of G on a tree T_3 with property *nice*. Given the covering space action and the action on T_3 we use the affine construction 2.9 to get an equivariant map $f_3 : \widetilde{X} \to T_3$. Using Guirardel 8.9 2.8 we extend f_3 to $\widehat{f}_3$ a map with connected fibers. We also need the proof of lemma 2.8 to ensure certain properties hold. Then, in order to extend the f_1 and f_2 maps we use the coning off construction. The product of these extensions gives f.

3.3 Combining into main argument

Finally, having enough dimension will follow from some assumptions about our actions. With this splitting in hand we want to verify that it's made of successive graphs of groups of items of a certain dimension so we can apply Bieri.

24 Definition (Not All Trees). There is an iterated splitting that doesn't end in trees. (Ideally, we will show that this assumption only rules out Z x Z, less ideally a statement about parabolics, less ideally we just assume it)

Corollary. Due to 24 we get that we can apply the dimension argument using Bieri to complete our result.

3.6 Lemma. The action of G on $T_1 \times T_2$ is free.

Proof. ...look in section II.6 of BH... Need enough facts to avoid non-proper spaces, get a semisimple action, and axes. $\square$

3.7 Lemma (Cocompact beginning). We directly show that our S_{ij} are cocompact x factor.

3.8 Lemma (Cocompact factor after neighborhood). K x T with K cocompact taking a neighborhood we get again K' x T with K' cocompact

3.9 Lemma (Filling preserves cocompactness). Since S_{23} is a cocompact G-invariant subcomplex that's contained in $A \times B$ where A and B are subcomplexes of T_1 and $T_2 \times T_3$ respectively and B is itself cocompact and G acts freely on $T_2 \times T_3$ we have that $(S_{23})_x$ is cocompact as well.

Proof. For each vertex orbit of B choose a particular vertex. By cocompactness this list is finite; call them $\{b_1, \ldots, b_n\}$. By freeness in the second factor, the stabilizer of $S' := S_{23} \cap (T_1 \times \{b_1\})$ is trivial. This means that S' injects into S_{23}/G which is compact by assumption. Since we are dealing with cell complexes, S' is also compact. Repeating this argument a finite number of times we have that $S_{23} \cap (T_1 \times \{b_k\})$ is compact for each $1 \leq k \leq n$. Compact items have a well defined diameter. Taking the maximum diamater (in the product metric) and noting we're acting by isometries gives that there is a bound D on the diameters of $S_{23} \cap (T_1 \times \{v\})$ as v ranges over vertices of B.

Let $a \times b$ be a vertex of $(S_{23})_x$. By construction, a lies within the convex hull of the projection of $S_{23} \cap (T_1 \times \{b\})$ (a slice of S_{23}) to the T_1 factor, hence slices of $(S_{23})_x$ have diameter no greater than D in the product metric. This means each slice of $(S_{23})_x$ has a finite number of vertices. It remains to show that there is a unviersal bound on the number of vertices as we range over all slices. Since S_{23} is G-invariant and we're acting by a product, (so the slice at gb is the slice at b acted on by g) there are a finite number of isometry classes of slices since B is cocompact. Hence, there are a finite number of vertex orbits so $(S_{23})_x$ is cocompact as needed. $\square$

4 Core has quadrant convex hyperplanes

4.1 LEMMA (Coning Connected Fibers). Let p in X our 2-complex, let p' be the unique projection. Let Γ be a compact subgraph in our 2-complex X. Take T to be either T_1 or T_2. For now suppose we have defined a map $f : \Gamma \to T$. We will define a map $F : \Gamma \times I \to T$. Choose an arbitrary $t_0 \in f(\Gamma)$ and define:

$$F(x,s) = \begin{cases} f(x) & s = 0 \\ t_0 & s = 1 \\ \gamma_{f(x),t_0}(s|\gamma|)a & \text{else} \end{cases}$$

Let $k_0 \in \Gamma \cap F^{-1}(F(x,s))$ be the nearest point to p'. Put γ_{p',k_0}. Then define $g : \mathrm{Im}(\gamma_{p',k_0}) \to I$ by $g(t) = \frac{a(t)}{a(t)+b}$ where $a(t) = d(f(t), F(x,s))$ and $b = d(F(x,s), t_0$. Then we compute $F(t, g(t)) = F(x,s)$ so $\mathrm{Graph}(g) \subseteq F^{-1}(x,s)$. Now, g is continuous so $\mathrm{Graph}(g)$ is connected. Hence, (x,s) is connected to $(k_0, 0)$ in $F^{-1}(F(x,s))$ as needed.

4.2 LEMMA (Fibers homeomorphic to Coordinate planes). Put $f = f_1 \times f_2 \times f_3 : X \to T_1 \times T_2 \times T_3$ and $J = \mathrm{Im}(f)$. We claim that $J \cap T_1 \times T_2 \times \{z\} = \mathrm{Im}_f({f_3}^{-1}(z))$. Let $p = (p_1, p_2, p_3) \in T_1 \times T_2 \times T_3$ then we have the following.

$$\begin{aligned} p \in \text{ LHS } &\iff p \in \mathrm{Im}(f) \wedge p_3 = z \\ &\iff \exists x \in X(f(x) = p \wedge f_3(x) = z) \\ &\iff \exists x \in X(f(x) = p \wedge x \in f_3^{-1}(z)) \\ &\iff p \in \mathrm{Im}_f(f_3^{-1}(z)) \end{aligned}$$

4.1 Wrap up lemmas

The following is a list of statements, the goal is to prove enough of them to arrive at item number 1 for a suitably chosen core C.

1. $C \subseteq \mathscr{T}$ has simply connected hyperplanes
2. C has hyperplanes that are (1) connected (2) quadrant convex
3. The hyperplanes and one dimensional fibers of C are connected
4. If $S \subseteq \mathscr{T}$ connected in all coordinate planes, then so are S_x, S_y, and S_z.
5. If $R \subseteq T_1 \times T_2$ has convex fibers then $(R_x)_y = (R_y)_x$.
6. If $S \subseteq \mathscr{T}$ is connected in all coordinate planes then $((K_x)_y)_z$ is one dimensional fiberwise convex.
7. If $S \subseteq \mathscr{T}$ and is connected in all coordinate planes then $(S_x)_y = (S_y)_x$.
8. Guirardel's Lemma

4.1.1 Using Wrap-up lemmas to prove main statement

The implications are as follows:

$$(4),(7)\boxed{\Rightarrow}(6) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \tag{1}$$
$$(8) \Rightarrow (4) \tag{2}$$
$$(5) \Rightarrow (7) \tag{3}$$

4.3 Lemma (Reduction to vertical subpath). Suppose $S \subseteq T_1 \times T_2 \times T_3$ is a subcomplex with CCP. Let $p, q, r \in S$ satisfy

1. $r \notin S$
2. $p, q \in S$
3. $p_2 = q_2 = r_2$ and $p_3 = q_3 = r_3$
4. $r \in \mathrm{cvx}_{T_1}(\{p, q\})$

then there is a path $\sigma : [0,1] \to S$ between p and q such that $\sigma(t)$ is contained in $S \cap (T_1 \times \{r_2\} \times \delta)$ where δ is an open direction in T_3 at r_3 provided $t \neq 0, 1$.

Proof. Proprty CCP implies that $S \cap (T_1 \times \{r_2\} \times T_3)$ is connected. Let σ be a path in that set from p to q. Consider the pre-image of $T_1 \times r_2 \times r_3$ by σ, call it K. Note that the complement of K is a countable disjoint union of open intervals in $[0,1]$ – we will choose one later. Each open interval is connected so considering projection and the fact that r_3 is separating in T_3 we have that under σ each open interval is mapped so the third coordinate lies in a single direction of T_3 at r_3. After identifying, K maps into T_1. Color the points of K by which direction at r_1 in T_1 they map into. Here we use the fact that σ is a path that is disjoint from r. In fact, because S is a subcomplex it is closed and so there is an open neighborhood of r that is disjoint from S and therefore also σ. Intersecting this neighborhood with $T_1 \times r_2 \times r_3$ gives an open neighborhood in T_1 that is disjoint from σ.

The upside is that each monocolored subset of K is closed by looking at the image of σ in the slice and taking intersections with a closed halfspace pointing away from r_1. Take the smallest pairwise distance between the finite number of colored closed sets. Consider two points in K that achieve that distance. There cannot be any points of K between them because we chose the smallest distance. This picks out an interval with endpoints that map to different directions as needed. $\square$

25 Definition (Connected in coordinate planes). Let S be a subset of a product indexed by $1 \leq k \leq N$. Then S is connected in all coordinate planes if $S \cap \pi_k^{-1}(p)$ is connected for all $p \in X_k$ for all k.

4.4 Lemma (Statement 4). Let $S \subseteq \mathscr{T}$ be a subcomplex that is connected in all coordinate planes 25. Then S_x, S_y, and S_z are as well.

Proof. Without loss of generality, consider S_x, note that S_x will be connected in all xy and xz planes because S was. Consider the yz-planes in S_x, if there were no new points added then the planes are connected and we are done. Suppose that $p \in (S_x \smallsetminus S) \cap \pi_1^{-1}(p_1)$, we need to connect p to a point in S. We will show that there is a path in $S_x \cap \pi_2^{-1}(p_2)$ between p and some point in $p' \in S$.

Since p is in $S_x \smallsetminus S$ there exist distinct points r and s in S that agree in all coordinates except the first where we have that $p_1 \in \mathrm{cvxhull}_{T_1}(\{r_1, s_1\})$. Now, because S is connected in all coordinate planes there is a path σ from r to s that lies in $S \cap \pi_2^{-1}(p_2)$. In fact, we can take σ to be a path that begins at r and ends at s with T_3 coordinates lying in exactly one closed direction of T_3 at p_3. We have factored out this situation into claim 4.3.

Take σ as in the claim 4.3. It remains to show that there is a path connecting p to another point in S. Considering closed quadrants, there exists a sequence $t_1, \ldots, t_n$ such that $v_i := \sigma(t_i)$ are vertices in $p_1 \times p_2 \times \overline{\delta}$ (here δ is the distinguished T_3 direction) with the property that $\sigma(t_1) = p$, $\sigma(t_n) = q$, and $\sigma_i := \sigma\,|_{[t_i, t_{i+1}]}$ lie in quadrant i for $0 < i < n$. (Here, quadrant i is determined by specifying a direction in T_1 at p_1 since we've already chosen δ above.) Projecting, we obtain a sequence of subpaths in $\overline{\delta}$ a direction in T_3 that are colored based on their quadrant. Note that $\overline{\delta}$ has a valence one vertex at p_3 because our tree is simplicial and subpaths don't end at p_3 unless it's the first or last because the claim has that the interior of the path maps into δ. This situation satisfies the conditions for our coloring lemma 4.7.

Multicolored paths in this vertical direction δ give paths contained in S_x, applying the coloring lemma 4.7 gives the required multicolored path. This completes the proof. □

4.5 Lemma (Statement 5). If $R \subseteq T_1 \times T_2$ is a connected subcomplex then $(R_x)_y = (R_y)_x$.

Proof. We will first show that $(R_x)_y$ has connected fibers. Note, it already has connected y-fibers, so it remains to show that it has connected x-fibers. Suppose that $(R_x)_y \cap (T_1 \times \{y_0\})$ were a disconnected x-fiber. Then because $(R_x)_y$ is a subcomplex we have that the fiber is separated by some edge. Let x_0 be the midpoint of this edge. Let ℓ and r denote the left and right closed halfspaces of $T_1 \times \{y_0\}$ at the midpoint x_0. We will show that either $\ell \cap (R_x)_y = \varnothing$ or $r \cap (R_x)_y = \varnothing$, applying this to all such x_0 will show $(R_x)_y$ has connected x-fibers.

Suppose this were false, and that $\ell \cap (R_x)_y \neq \varnothing$ and $r \cap (R_x)_y \neq \varnothing$. We consider three cases: (1) ℓ and r intersect R_x nontrivially (2) ℓ and r don't intersect R_x, and (3) exactly one of ℓ or r intersects R_x nontrivially.

1. Case 1: If both ℓ and r intersect R_x then both contain points of R because we filled in the x-fiber, but then $x_0 \times y_0 \in R_x$ a contradiction.

2. Case 2: Pick a point x^+ in $r \cap (R_x)_y \smallsetminus R_x$. Let q^+ and q^- be points above and below x^+ in R_x. These points are either in R already, or because we

filled in the x-fiber there exist points above and below the line $\ell \cup r$ but this is a contradiction since R is connected.

3. Case 3: Without loss of generality, suppose $\ell \cap R_x = \varnothing$ and $r \cap R_x \neq \varnothing$. This implies there is some $s \in R \cap r$, we will get a contradiction by separating this point from another point in R.

   Let u and d be directions of $x_0 \times T_2$ at $x_0 \times y_0$. If both intersected R_x then $x_0 \times y_0 \in R$ a contradiction so without loss of generality, suppose $d \cap R_x \neq \varnothing$. Now let x^- be a point in ℓ, by assumption $x^- \in (R_x)_y \smallsetminus R_x$. This gives a point q^- below ℓ in R_x which is already in R or there exists a point $t^- \in R$ to the left of q^-; but this is a contradiction since $\ell \cup d$ separates these points from $s \in R$.

Hence, either $\ell \cap (R_x)_y = \varnothing$ or $r \cap (R_x)_y = \varnothing$ as needed, so $(R_x)_y$ has connected x-fibers. Similarly, $(R_y)_x$ also has connected fibers. Hence by 2.10 their complements are unions of quadrants and so they contain $QH(R)$. It remains to show that they are contained within $QH(R)$.

Claim. $(R_x)_y \subseteq QH(R)$

Proof. (sketch) The idea is to show that for every point $p \in (R_x)_y$ that for every quadrant Q containing p (i.e. that would be attempting to remove it) we can find a point $r \in R \cap Q$. Picking an open quadrant containing p amounts to picking a point $q = (q_1, q_2)$ with halfspaces at each coordinate that contain the corresponding p_i. The hard case is where $p \in (R_x)_y \smallsetminus R_x$ - so you pick a point in a vertical direction at p pointing towards p, because this point is only there due to filling there's another vertical direction that you're grabbing that contains some R_x. Then you pick a horizintal place and point towards p_1, this must contain at least one of the directions with R in it. □

Proof. We will show for every point $p \in (R_x)_y$ and every open quadrant Q with $p \in Q$ that there exists some $r \in R \cap Q$. Let $q = (q_1, q_2)$ be the point where Q is based and label the halfspaces so that $Q = q_1^+ \times q_2^+$.

Case 1 $p \in R_x \smallsetminus R$: Let $p \in R_x \smallsetminus R$: Let $\ell = (\ell_1, p_2)$ and $u = (u_1, p_2)$ be points in R that cause the vertical filling. Now, q_1^+ contains all but one direction at p_1 and so must contain either ℓ_1 or u_1. Since q_2^+ must contain p_2 we get that $Q = q_1^+ \times q_2^+$ contains a point of R.

Case 2 $p \in (R_x)_y \smallsetminus R_x$: Let $u = (p_1, u_2)$ and $d = (p_1, d_2)$ be points that cause the vertical filling. Now, q_2^+ contains all but one direction at p_2 and so must contain either u_2 or d_2. Without loss of generality, suppose it contains u_2. If $u \in R$ then we are done. Suppose $u \in R_x \smallsetminus R$. Then we can find $u' = (p', u_2)$ and $u'' = (p'', u_2)$ with p' and p'' in different directions at p_1. Now, q_1^+ contains all but one direction at p_1 and so must contain either p' or p''. Suppose without loss of generality that it contains p', then Q contains $p' \times u_2 \in R$. □

□

4.6 Lemma (Boxed Implication). Repeatedly applying statement 7 to both S and S_α where α is one of x, y, or z and noticing that $\langle(12),(23)\rangle = S_3$ we get that $((K_x)_y)_z$ is equal to any of the permutations of the indices. In particular, $((K_x)_y)_z = ((K_y)_z)_x = ((K_z)_x)_y$ which shows that $((K_x)_y)_z$ has connected one dimensional fibers.

4.2 Planar Path Argument

4.7 Lemma (Coloring Lemma). Let K satisfy property setup 8. Let A be the set of endpoints minus the basepoint x_0. Then there exists some $a \in A$ such that for all edges e in the geodesic $[x_0, a]$ there exist some i, j such that σ_i and σ_j use e and $\lambda(i) \neq \lambda(j)$.

Proof. Suppose the lemma were false. Then there exist counterexamples K satisfying property setup 8 such that for all $a \in A$ there exists an edge e on $[x_0, a]$ such that $\lambda(i) = \lambda(j)$ whenever σ_i and σ_j use e. These K have bad edges and $|\sigma| > 0$. Now take a K such that $|\sigma|$ is minimized. Let K' be the result of snipping a bad edge. By lemma 4.8, K' has property setup so $|\sigma'| > 0$. If K' contained a multicolored path between x_0 and some a' in A' then because the path σ in K an be obtained by inserting subpaths into σ', we have that $A' \subseteq A$ and any edge that was multicolored stays multicolored. Therefore K' has no multicolored path, because that would force K to have one. Then the fact $|\sigma'| < |\sigma|$ gives a contradiction because K was chosen to be minimal.

□

4.8 Lemma (Snip invariant). If K satisfies property setup 8 then K' obtained from snipping a bad edge also satisfies property setup.

Proof. Let e be a bad edge of K, since σ_1 and σ_N have different colors the edge e in at most one of them. Because x_0 has valence one, σ_1 and σ_N share the edge containing x_0; therefore e cannot be the common edge. The rest follows. □