Assignment Exam #3 (*Probability and Statistics*)

Definition 1 (Convex Function) *Let $[c, d]$ be an interval of the real line and let $f : [c, d] \to \mathbb{R}$ be a function. The function f is called convex on $[c, d]$ if for all $a_1, a_2 \in [c, d]$ and $t \in [0, 1]$, we have*

$$f(ta_1 + (1-t)a_2) \leq tf(a_1) + (1-t)f(a_2).$$

Fact: For convex $f : [c, d] \to \mathbb{R}$, $a_1, \ldots, a_n \in [c, d]$ and $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$, we have

$$f(\sum_{i=1}^{n} t_i a_i) \leq \sum_{i=1}^{n} t_i f(a_i).$$

1. Let $f : [\alpha, \beta] \to \mathbb{R}$ be a convex function, where $\alpha, \beta \in \mathbb{R}$. Let $S = \{a_1, a_2, \ldots, a_k\}$, where $\alpha \leq a_1 < a_2 < \cdots < a_k \leq \beta$. Let X be a discrete random variable taking values in S. Then show that $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$. [2]

2. Let X be a discrete random variable taking postive real values. Show that $\mathbb{E}[X] \geq e^{\mathbb{E}[\log X]}$. [2]

3. Use problem (2) to derive arithmetic-geometric inequality, i.e., for all positive numbers $a_1, \ldots, a_k$, we have [5]

$$\frac{\sum_{i=1}^{k} a_i}{k} \geq (\prod_{i=1}^{k} a_i)^{\frac{1}{k}}$$

   Hint: Consider random variable X taking values in the set $\{a_1, \ldots, a_k\}$ with appropriate distribution.