CS 302 Computer Security and Privacy

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Lecture 5 Part 1
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Tools Needed for RSA

Algorithms
  Computing with Big Numbers
  Fast Exponentiation Algorithms

Number Theory
  Factoring Assumption
  Number Theory for RSA
  Division of Integers

Tools Needed for RSA

Two kinds of tools are needed to understand and implement RSA.

Algorithms: Need clever algorithms for primality testing, fast exponentiation, and modular inverse computation.

Number theory: Need some theory of Z_n, the integers modulo n, and some special properties of numbers n that are the product of two primes.

Algorithms

Algorithms for arithmetic on big numbers

The arithmetic built into typical computers can handle only 32-bit or 64-bit integers. Hence, all arithmetic on large integers must be performed by software routines.

The straightforward algorithms for addition and multiplication have time complexities $O(N)$ and $O(N^2)$, respectively, where N is the length (in bits) of the integers involved.

Asymptotically faster multiplication algorithms are known, but they involve large constant factor overheads. It's not clear whether they are practical for numbers of the sizes we are talking about.

Outline	Tools Needed for RSA	**Algorithms**	Number Theory
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Bignums

Big number libraries

A lot of cleverness *is* possible in the careful implementation of even the $O(N^2)$ multiplication algorithms, and a good implementation can be many times faster in practice than a poor one. They are also hard to get right because of many special cases that must be handled correctly!

Most people choose to use big number libraries written by others rather than write their own code.

Two such libraries that you can use in this course:

1. GMP (GNU Multiple Precision Arithmetic Library);
2. The big number routines in the openssl crypto library.

GMP

GMP provides a large number of highly-optimized function calls for use with C and C++.

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Bignums

Openssl crypto package

OpenSSL is a cryptography toolkit implementing the Secure Sockets Layer (SSL v2/v3) and Transport Layer Security (TLS v1) network protocols and related cryptography standards required by them.

It is widely used and pretty well debugged. The implementation requires computation on big numbers. OpenSSL implements its own big number routines which are contained in its crypto library.

Type man crypto for general information about the crypto library. Details on the hundreds of individual functions are summarized here.

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Outline	Tools Needed for RSA	**Algorithms**	Number Theory
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Exp

Modular exponentiation

The basic operation of RSA is modular exponentiation of big numbers, i.e., computing $m^e \bmod n$ for big numbers m, e, and n.

The obvious way to compute this would be to compute first $t = m^e$ and then compute $t \bmod n$.

This has two serious drawbacks.

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Exp

Computing m^e the conventional way is too slow

The simple iterative loop to compute m^e requires e multiplications, or about 2^{1024} operations in all. This computation would run longer than the current age of the universe (which is estimated to be 15 billion years).

Assuming one loop iteration could be done in one microsecond (very optimistic seeing as each iteration requires computing a product and remainder of big numbers), only about 30×10^{12} iterations could be performed per year, and only about 450×10^{21} iterations in the lifetime of the universe. But $450 \times 10^{21} \approx 2^{79}$, far less than $e - 1$.

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Exp

The result of computing m^e is too big to write down.

The number m^e is too big to store! This number, when written in binary, is about $1024 * 2^{1024}$ bits long, a number far larger than the number of atoms in the universe (which is estimated to be only around $10^{80} \approx 2^{266}$).

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Exp

Controlling the size of intermediate results

The trick to get around the second problem is to do all arithmetic modulo n, that is, reduce the result modulo n after each arithmetic operation.

The product of two length ℓ numbers is only length 2ℓ before reduction mod n, so in this way, one never has to deal with numbers longer than about 2048 bits.

Question to think about: Why is it correct to do this?

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Exp

Efficient exponentiation

The trick to avoiding the first problem is to use a more efficient exponentiation algorithm based on repeated squaring.

For the special case of $e = 2^k$, one computes $m^e \bmod n$ as follows:

$$
\begin{aligned}
m_0 &= m \\
m_1 &= (m_0 * m_0) \bmod n \\
m_2 &= (m_1 * m_1) \bmod n \\
&\vdots \\
m_k &= (m_{k-1} * m_{k-1}) \bmod n.
\end{aligned}
$$

Clearly, $m_i = m^{2^i} \bmod n$ for all i.

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Exp

Combining the m_i for general e

For values of e that are not powers of 2, $m^e \bmod n$ can be obtained as the product modulo n of certain m_i's.

Express e in binary as $e = (b_s b_{s-1} \ldots b_2 b_1 b_0)_2$. Then $e = \sum_i b_i 2^i$, so

$$m^e = m^{\sum_i b_i 2^i} = \prod_i m^{b_i 2^i} = \prod_i (m^{2^i})^{b_i} = \prod_{i\,:\; b_i=1} m_i.$$

Since each $b_i \in \{0, 1\}$, we include exactly those m_i in the final product for which $b_i = 1$. Hence,

$$m^e \bmod n = \prod_{i\,:\; b_I=1} m_i \bmod n.$$

Outline	Tools Needed for RSA	Algorithms	Number Theory

Exp

Towards greater efficiency

It is not necessary to perform this computation in two phases.

Rather, the two phases can be combined together, resulting in slicker and simpler algorithms that do not require the explicit storage of the m_i's.

We give both a recursive and an iterative version. They're both based on the identities[1]

$$m^e = \begin{cases} (m^2)^{\lfloor e/2 \rfloor} & \text{if } e \text{ is even;} \\ (m^2)^{\lfloor e/2 \rfloor} \times m & \text{if } e \text{ is odd;} \end{cases}$$

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[1]$\lfloor e/2 \rfloor$ is the greatest integer less than or equal to $e/2$.

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Exp

A recursive exponentiation algorithm

Here is a recursive version written in C notation, but it should be understood that the C programs only work for numbers smaller than 2^{16}. To handle larger numbers requires the use of big number functions.

```c
/* computes m^e mod n recursively */
int modexp( int m, int e, int n) {
  int r;
  if ( e == 0 ) return 1;          /* m^0 = 1 */
  r = modexp(m*m % n, e/2, n);     /* r = (m^2)^(e/2) mod n */
  if ( (e&1) == 1 ) r = r*m % n;   /* handle case of odd e */
  return r;
}
```

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Exp

An iterative exponentiation algorithm

This same idea can be expressed iteratively to achieve even greater efficiency.

```
/* computes m^e mod n iteratively */
int modexp( int m, int e, int n) {
  int r = 1;
  while ( e > 0 ) {
    if ( (e&1) == 1 ) r = r*m % n;
    e /= 2;
    m = m*m % n;
  }
  return r;
}
```

Correctness

The loop invariant is

$$e > 0 \land (m_0^{e_0} \bmod n = rm^e \bmod n) \qquad (1)$$

where m_0 and e_0 are the initial values of m and e, respectively.

Proof of correctness:

- It is easily checked that (1) holds at the start of each iteration.
- If the loop exits, then $e = 0$, so $r \bmod n$ is the desired result.
- Termination is ensured since e gets reduced during each iteration.

A minor optimization

Note that the last iteration of the loop computes a new value of m that is never used. A slight efficiency improvement results from restructuring the code to eliminate this unnecessary computation. Following is one way of doing so.

```
/* computes m^e mod n iteratively */
int modexp( int m, int e, int n) {
  int r = ( (e&1) == 1 ) ? m % n : 1;
  e /= 2;
  while ( e > 0 ) {
    m = m*m % n;
    if ( (e&1) == 1 ) r = r*m % n;
    e /= 2;
  }
  return r;
}
```

Number Theory

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Factoring

Factoring assumption

The *factoring problem* is to find a prime divisor of a composite number n.

The *factoring assumption* is that there is no probabilistic polynomial-time algorithm for solving the factoring problem, even for the special case of an integer n that is the product of just two distinct primes

The security of RSA is based on the factoring assumption. No feasible factoring algorithm is known, but there is no proof that such an algorithm does not exist.

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Factoring

How big is big enough?

The security of RSA depends on n, p, q being sufficiently large.

What is sufficiently large? That's hard to say, but n is typically chosen to be at least 1024 bits long, or for better security, 2048 bits long.

The primes p and q whose product is n are generally chosen be roughly the same length, so each will be about half as long as n.

Number theory overview

In this and following sections, we review some number theory that is needed for understanding RSA.

I will provide only a high-level overview. Further details are contained in course handouts and the textbooks.

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Number Theory for RSA

Bare-bones definitions

The following definitions apply to the RSA parameters.

- p, q are distinct large primes of roughly the same length.
- $n = pq$.
- $\mathbf{Z}_n = \{0, 1, \ldots, n-1\}$.
- $\phi(n) = (p-1)(q-1)$. [ϕ is Euler's totient function.]
- $\mathbf{Z}_n^*$ are the numbers in $\mathbf{Z}_n$ that are relatively prime to n (that is, not divisible by either p or q, which is most of them).

Fact: $|\mathbf{Z}_n^*| = \phi(n)$.

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Number Theory for RSA

Example: Small RSA

- $p = 11$, $q = 13$.
- $n = p \times q = 143$.
- $\mathbf{Z}_{143} = \{0, 1, 2, \ldots, 141, 142\}$.
- $\phi(143) = (p-1)(q-1) = 10 \times 12 = 120$.
- $\mathbf{Z}_{143}^* = \mathbf{Z}_{143} - M_{11} - M_{13}$, where
  $M_{11} = \{0, 11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132\}$,
  $M_{13} = \{0, 13, 26, 39, 52, 65, 78, 91, 104, 117, 130\}$.
- $\begin{aligned} |\mathbf{Z}_{143}^*| &= |\mathbf{Z}_{143}| - |M_{11}| - |M_{13}| + |M_{11} \cap M_{13}| \\ &= 143 - 13 - 11 + 1 \\ &= 120 = \phi(143). \end{aligned}$

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Number Theory for RSA

Summary of what is needed

Here's a summary of the number theory needed to understand RSA and its associated algorithms.

- Greatest common divisor, $\mathbf{Z}_n$, $\bmod n$, $\phi(n)$, $\mathbf{Z}_n^*$, and how to add, subtract, multiply, and find inverses mod n.
- Euler's theorem: $a^{\phi(n)} \equiv 1 \pmod{n}$ for $a \in \mathbf{Z}_n^*$.
- How to generate large prime numbers: density of primes and testing primality.

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Number Theory for RSA

Generating the RSA modulus

To generate an RSA modulus n, we find primes p and q of the desired lengths and compute $n = pq$.

To find a large prime, repeatedly choose numbers of the desired length and *test each for primality*. We must show that the *density of primes* is large enough for this procedure to be feasible.

Finding the RSA key pair

- ▶ The RSA key pair (e, d) is chosen to satisfy the *modular equation* $ed \equiv 1 \pmod{\phi(n)}$.
- ▶ To find (e, d):
    1. Choose random $e \in \mathbf{Z}^*_{\phi(n)}$. Do this by repeatedly choosing e at random from $\mathbf{Z}_{\phi(n)}$ and using gcd to test membership in $\mathbf{Z}^*_{\phi(n)}$.
    2. *Solve* the modular equation $ed \equiv 1 \pmod{\phi(n)}$ for d.

Using *Euler's theorem*, we can show

$$m^{ed} \equiv m \pmod{n}$$

for all $m \in \mathbf{Z}^*_n$. This implies $D_d(E_e(m)) = m$.

To show that decryption works even in the rare case that $m \in \mathbf{Z}_n - \mathbf{Z}^*_n$ requires some more number theory that we will omit.

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Division of Integers

Quotient and remainder

Theorem (Euclidean division)

Let a, b be integers and assume $b > 0$. There are unique integers q (the quotient) and r (the remainder) such that $a = bq + r$ and $0 \leq r < b$.

Write the quotient as $a \div b$ and the remainder as $a \bmod b$. Then

$$a = b \times (a \div b) + (a \bmod b).$$

Equivalently,

$$a \bmod b = a - b \times (a \div b).$$

$$a \div b = \lfloor a/b \rfloor.^2$$

---

^{2}Here, $/$ is ordinary real division and $\lfloor x \rfloor$, the *floor* of x, is the greatest integer $\leq x$. In C, $/$ is used for both $\div$ and $/$ depending on its operand types.

The mod operator for negative numbers

When either a or b is negative, there is no consensus on the definition of $a \bmod b$.

By our definition, $a \bmod b$ is always in the range $[0 \ldots b - 1]$, even when a is negative.

Example,

$$(-5) \bmod 3 = (-5) - 3 \times ((-5) \div 3) = -5 - 3 \times (-2) = 1.$$

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Division of Integers

The mod operator % in C

In the C programming language, the mod operator % is defined differently, so $(a \% b) \neq (a \bmod b)$ when a is negative and b is positive.

The C standard defines $a \% b$ to be the number r satisfying the equation $(a/b) * b + r = a$, so $r = a - (a/b) * b$.

C also defines a/b to be the result of rounding the real number a/b towards zero, so $-5/3 = -1$. Hence,

$$-5 \% 3 = -5 - (-5/3) * 3 = -5 + 3 = -2.$$