Recursion:

$$\text{factorial}(n) = 1 \times 2 \times 3 \times \cdots \times n$$

factorial (n):

$$\begin{cases} \text{factorial}(1) = 1 \\ \text{factorial}(n) = n \times \text{factorial}(n-1). \end{cases}$$

Fibonacci:

$f(n)$ = no. of rabbits at the end of n^{th} month

$= f(n-1) + \left(\text{rabbits born at the end of } n^{th} \text{ month}\right)$

$= f(n-1) + f(n-2)$

Inductive $f(n) = f(n-1) + f(n-2)$

Basis: $f(1) = f(2) = 1$

Recursive method to define Fibonacci series.

GCD of two integers:

a, b

$p_1 \times p_2 \times \cdots$ $\qquad$ $q_1 \times q_2 \times \cdots \longrightarrow$ common factors.

$\boxed{x \mid y \equiv \text{"}x \text{ is a divisor of } y\text{"}}$

$2 \mid 6$

① if $g|a$, $g|b$ then $g|a+b$

$g|a-b$

$g|k_1a+k_2b$

where $k_1, k_2 \in \mathbb{Z}$

② $a = qb + r$

where $0 \leq r < b$

(ess dividing a by b)

Euclid's algo:

Aim: $\gcd(a, b)$ $\quad (a \geq b)$

$$\left\{ \begin{array}{l} r_0 = a \\ r_1 = b \end{array} \right\}$$

Steps

$$\left[\begin{array}{ll} r_0 = q_0 r_1 + r_2 & : \ 0 \leq r_2 < r_1 \\ r_1 = q_1 r_2 + r_3 & : \ 0 \leq r_3 < r_2 \\ \vdots & \quad \vdots \\ r_{t-1} = q_{t-1} \cdot r_t + r_{t+1} & \ldots \ 0 \leq r_{t+1} < r_t \\ r_t = q_t \, r_{t+1} + 0 & \end{array} \right.$$

$q_1, q_2, \ldots q_{t-1} \geq 1.$

$$\boxed{\gcd(a, b) = r_{t+1}.}$$

$q_t \geq 2$

(Because $r_{t+1} < r_t$)

$$r_{t+1} \geq 1 = f_1$$

$$r_t \geq 2 \cdot r_{t+1} \geq 2 = f_2$$

$$r_{t-1} = f_2 + f_1 = f_3$$

$$r_{t-2} = f_3 + f_2 = f_4$$

$$\vdots$$

$$b = r_1 = f_t + f_{t-1} = f_{t+1}$$

<u>Lemma</u>:

$$f_{t+1} > \left(\frac{1+\sqrt{5}}{2}\right)^{t-1}$$

$\Rightarrow$ The no of steps in Euclid's algo is $\leq (\log b)$ $\times$ const'

Fib.

$$f(1) = f(2) = 1$$

$$f(n) = f(n-1) + f(n-2)$$

Prove that $f(n) > \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}$ for $n \geq 3$.

Basis: $f(3) = 2 > \left(\frac{1+\sqrt{5}}{2}\right)^{1}$ ✓

Hypothesis: $f(n) > \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \quad \forall\ n \in \{1, 2, \dots k\}$

Ind

$$f(k+1) = f(k) + f(k-1)$$

$$> \left(\frac{1+\sqrt{5}}{2}\right)^{k-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{k-3}$$

$$> \left(\frac{1+\sqrt{5}}{2}\right)^{k-3} \left(\frac{1+\sqrt{5}}{2} + 1\right)$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-3} \left(\frac{3+\sqrt{5}}{2}\right)$$

Check:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+5+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

$$\geq \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}$$

Done.

Structural Induction:

Definition of a set by recursion & then prove some properties of the set by using the recursive definition.

Set S

(i) $3 \in S$

(ii) if $x, y \in S$ then $x+y \in S$

Prove that $S =$ (set of all multiples of 3) $= A$

Aim: (1) $S \subseteq A$ (all elements of S are multiple of 3)

(2) $A \subseteq S$ (every multiple of 3 is in S)

Induction:

Base: $3 \in A$

if $x, y \in S \Rightarrow x+y \in S$

if $3 \mid x$, $3 \mid y \Rightarrow 3 \mid (x+y)$

Base: $3 \in S$

$r \in S$

$(r-3) + 3 \in S$

$\Rightarrow r \in A$