The Generalized Delta Rule aka Back-Propagation

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"Everything that is possible demands to exist" – Gottfried Wilhelm Leibniz

Multi-Layered Perceptrons

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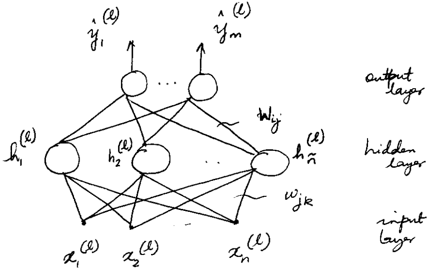

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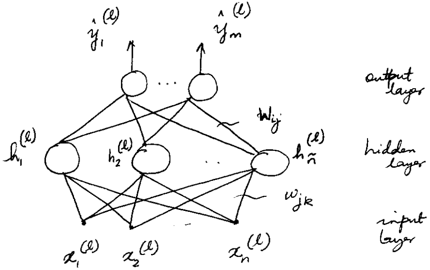

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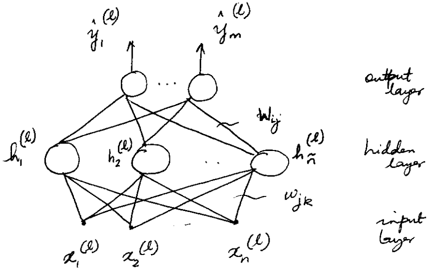

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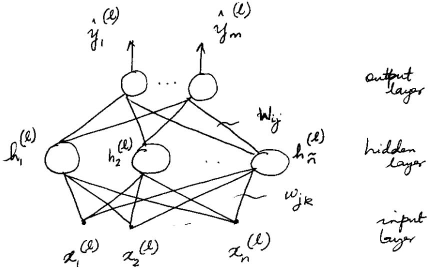

- A bias is present in each neuron (not shown explicitly)
- Output is indexed using i, hidden neurons are indexed using j and the inputs are indexed using k
- W_{ij} is the weight from output i and hidden neuron j. w_{jk} is the weight from hidden neuron j and input k

The Generalized Delta Rule aka Back-Propagation

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- As before, let the training data be denoted by,

$$\left\{ \left(x^{(l)}, y^{(l)} \right) \right\}_{l=1}^{N} \qquad ; x^{(l)} \in \mathcal{R}^n, \quad y^{(l)} \in \mathcal{R}^m$$

The Network Output

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- The output of a hidden neuron is,

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- Similarly, the output can be defined as,

$$\hat{y}_i^{(l)} = f\left(S_i^{(l)}\right) = f\left(\sum_{j=0}^{\tilde{n}} W_{ij} h_j^{(l)}\right)$$

The Cost Function

The Cost Function

- $$\begin{aligned} J^{(l)} &= \sum_{i=1}^{m} \left(y_i^{(l)} - \hat{y}_i^{(l)} \right)^2 \\ &= \sum_{i=1}^{m} {e_i^{(l)}}^2 \end{aligned}$$

The Cost Function

- $$J^{(l)} = \sum_{i=1}^{m} \left(y_i^{(l)} - \hat{y}_i^{(l)} \right)^2 = \sum_{i=1}^{m} e_i^{(l)2}$$

- $$J = \sum_{l=1}^{N} J^{(l)} = \sum_{l=1}^{N} \sum_{i=1}^{m} \left(y_i^{(l)} - \hat{y}_i^{(l)} \right)^2$$

The Cost Function

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$$
\begin{aligned}
J &= \sum_{l=1}^{N}\sum_{i=1}^{m}\left(y_i^{(l)} - \hat{y}_i^{(l)}\right)^2 \\
&= \sum_{l=1}^{N}\sum_{i=1}^{m}\left[y_i^{(l)} - f\left(\sum_{j=0}^{\tilde{n}} W_{ij}\, f\left(\sum_{k=0}^{n} w_{jk} x_k^{(l)}\right)\right)\right]^2
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\end{aligned}
$$

So, we can start with random values of w's and adopt the weights by moving opposite to the gradient computed from the equation above, i.e.

$$\Delta w = -\eta \nabla J$$

The Generalized Delta Rule Derived

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In batch learning, we aggregate the gradients computed for each pattern, i.e.,

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In online learning, we make the approximation that as long as η is small, we can,

- Feed pattern I
- Compute $\nabla J^{(I)}$
- Update W's and w's and repeat

Output–Hidden Weights

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$$
\begin{aligned}
J &= \sum_{l=1}^{N} \sum_{i=1}^{m} {e_i^{(l)}}^2 = \sum_{l=1}^{N} \sum_{i=1}^{m} \left(y_i^{(l)} - \hat{y}_i^{(l)} \right)^2 \\
&= \sum_{l=1}^{N} \sum_{i=1}^{m} \left[y_i^{(l)} - f \left(\sum_{j=0}^{\tilde{n}} W_{ij}\, f \left(\sum_{k=0}^{n} w_{jk} x_k^{(l)} \right) \right) \right]^2
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$$

$$
\begin{aligned}
\frac{\partial J^{(l)}}{\partial W_{ij}} &= \frac{\partial J^{(l)}}{\partial e_i^{(l)}} \frac{\partial e_i^{(l)}}{\partial \hat{y}_i^{(l)}} \frac{\partial \hat{y}_i^{(l)}}{\partial S_i^{(l)}} \frac{\partial S_i^{(l)}}{\partial W_{ij}} \\
&= e_i^{(l)} \cdot (-1) \cdot f'(S_i^{(l)}) \cdot h_j^{(l)} \\
&= -\left(y_i^{(l)} - \hat{y}_i^{(l)}\right) \cdot f'(S_i^{(l)}) \cdot h_j^{(l)}
\end{aligned}
$$

Output–Hidden (Online) Weight Updates

$$\begin{aligned}
W_{ij}^{(l)}(\mathrm{new}) &= W_{ij}^{(l)}(\mathrm{old}) + \Delta W_{ij}^{(l)} \\
&= W_{ij}^{(l)}(\mathrm{old}) - \eta \cdot \frac{\partial J^{(l)}}{\partial W_{ij}} \\
&= W_{ij}^{(l)}(\mathrm{old}) + \eta \cdot e_i^{(l)} \cdot f'(S_i^{(l)}) \cdot h_j^{(l)} \\
&= W_{ij}^{(l)}(\mathrm{old}) + \eta \cdot \left(y_i^{(l)} - \hat{y}_i^{(l)}\right) \cdot f'(S_i^{(l)}) \cdot h_j^{(l)} \\
&= W_{ij}^{(l)}(\mathrm{old}) + \eta \cdot \delta_i^{(l)} \cdot h_j^{(l)}
\end{aligned}$$

where, $\delta_i^{(l)} = \left(y_i^{(l)} - \hat{y}_i^{(l)}\right) \cdot f'(S_i^{(l)})$

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\frac{\partial J^{(l)}}{\partial w_{jk}} &= \sum_{i=1}^{m} \frac{\partial J^{(l)}}{\partial e_i^{(l)}} \frac{\partial e_i^{(l)}}{\partial \hat{y}_i^{(l)}} \frac{\partial \hat{y}_i^{(l)}}{\partial S_i^{(l)}} \frac{\partial S_i^{(l)}}{\partial h_j^{(l)}} \frac{\partial h_j^{(l)}}{\partial S_j^{(l)}} \frac{\partial S_j^{(l)}}{\partial w_{jk}} \\
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where, $\delta_j^{(l)} = \left[\sum_{i=1}^{m} \delta_i^{(l)} \cdot W_{ij} \right] \cdot f'(S_j^{(l)})$

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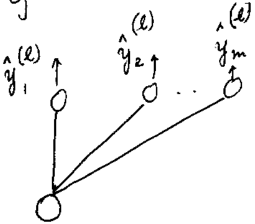
So, $\delta_j^{(l)}$ is computed as



$\delta_i^{(l)}$ is back-propagated through W_{ij} and summed to give $\delta_j^{(l)}$

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  - for $l = 1, 2, \ldots, N$
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$$h_j^{(l)} = f\left(S_j^{(l)}\right) = f\left(\sum_{k=0}^{n} w_{jk} x_k^{(l)}\right) \text{ and } \hat{y}_i^{(l)} = f\left(S_i^{(l)}\right) = f\left(\sum_{j=0}^{\tilde{n}} W_{ij} h_j^{(l)}\right)$$

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and update the weights,

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- There are multiple variations tat help speed up learning e.g. use of momentum, conjugate gradient, Levenburg-Marquardt etc.
- A large number of regularizers are possible e.g. the second derivative of the output relative to the weights to help minimize sudden changes in the output i.e. promote smooth variations

An Application - Autonomous Navigation

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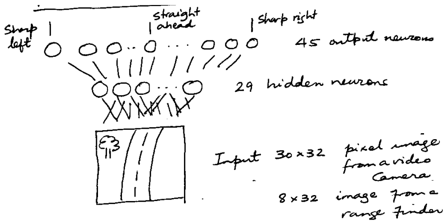

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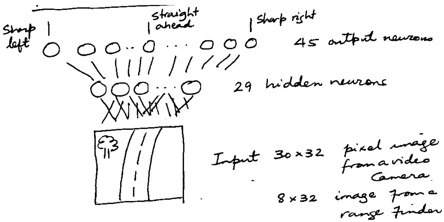

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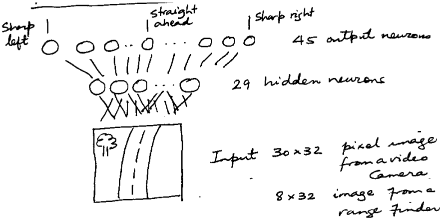

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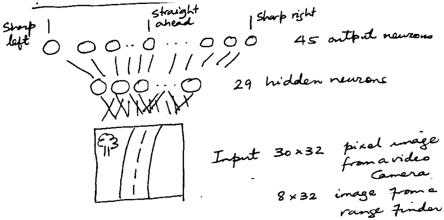

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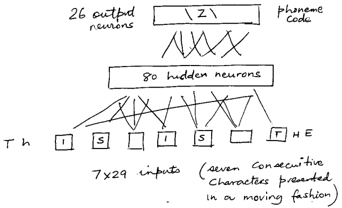

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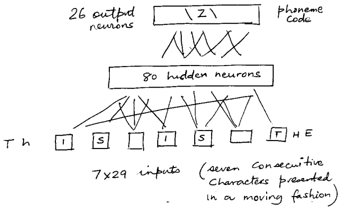

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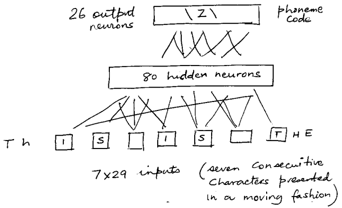

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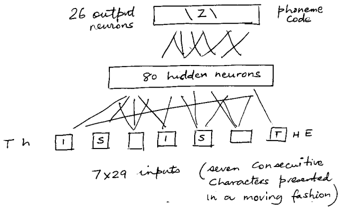

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An Application - Image Compression

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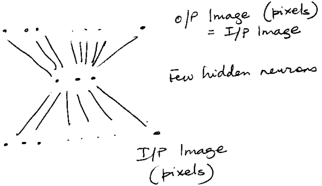

- The input and the desired output are the same i.e. the output is the uncompressed image. The output of the hidden neurons is the compressed image (if the number of neurons in the hidden layer are less than the number of neurons in the input layer)

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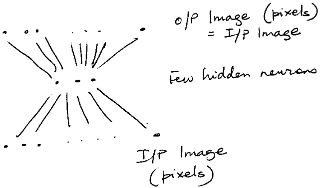

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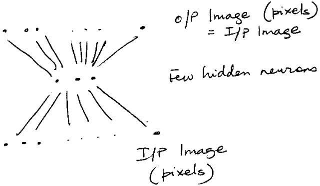

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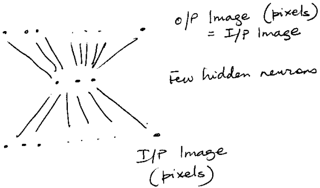

- The input and the desired output are the same i.e. the output is the uncompressed image. The output of the hidden neurons is the compressed image (if the number of neurons in the hidden layer are less than the number of neurons in the input layer)
- Lossy compression since one may not recover the exact output as the input (training error will not likely go to 0 specially for large amounts of compression)

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  where, $0 \leq \alpha \leq 1$ is the momentum coefficient

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- Effect when moving through a plateau,

$$\begin{aligned}
\Delta w(0) &= \frac{\partial J}{\partial w} = -\eta a \\
\Delta w(1) &= \frac{\partial J}{\partial w} + \alpha\left(\Delta w(0)\right) = -\eta a - \eta \alpha a = -\eta a(1+\alpha) \\
\Delta w(2) &= \frac{\partial J}{\partial w} + \alpha\left(-\eta a - \eta \alpha a\right) = -\eta a - \eta \alpha a = -\eta a(1+\alpha+\alpha^2) \\
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- So, effective learning rate is increased. What happens through a rough

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- $$J(w) = J(w_0) + (w - w_0)\nabla J(w_0) + (w - w_0)^T H(w - w_0)$$

  where, $H_{ij} = \frac{\partial^2 J}{\partial w_i \partial w_j}$

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  where, $H_{ij} = \frac{\partial^2 J}{\partial w_i \partial w_j}$
- The Hessian is expensive to compute. So, there are quasi-Newton methods like Conjugate Gradient, Levenberg-Marquardt etc. that can be used

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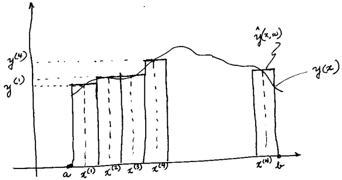

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