Mid-term Exam Wed 8th March 1 hr 20 min

Defining sets using recursion:

Set S : $3 \in S$

if $x, y \in S$ then $x+y \in S$.

$$S = \{3, 6, 9, 12, \ldots\}$$

Proof: (i) $S \subseteq \{3, 6, \ldots\}$ to show

(ii) $\{3, 6, 9, \ldots\} \subseteq S$

Structural induction

- where we use the recursive definition of a set to prove some property of the set'

(i) Basis: prove the property for an initial element of S

(ii) Inductive step: if $x, y \in S$ satisfy the property then the recursive construction involving x & y also satisfies the property.

Set has the property

Induction:

$$\boxed{1} + \boxed{k \to (k+1)} \longrightarrow \forall\, k \in \mathbb{Z}^+$$

Structural ind:

$$\boxed{\alpha} + \boxed{\begin{array}{l}\text{if } \alpha, \beta \text{ then} \\ z = \text{recursive step}(\alpha, \beta) \\ \quad \text{holds}\end{array}} \longrightarrow \begin{array}{l}\text{holds} \\ \text{for all} \\ \alpha, \beta, \gamma \ldots\end{array}$$

Σ = set of alphabets

λ = special symbol → reserved for "empty" string

language → Σ^* = set of strings

length of a string $len(\omega)$ = no of alphabets in ω.

$len(\lambda) = 0$.

let $\Sigma = \{a, b, c, d\}$

$\Sigma^* = \{a, ab, ba, aaa, bbacd, \lambda, \ldots\}$

Defn: The set Σ^* of strings over alphabet Σ is defined as follows —

(i) $\lambda \in \Sigma^*$ (empty string)

(ii) if $\omega \in \Sigma^*$ and $x \in \Sigma$ then $\omega \cdot x \in \Sigma^*$ (concatenation)

Defn of l:

$l(\lambda) = 0$

$l(\omega x) = l(\omega) + 1$ where $x \in \Sigma$

Using the recursive defn of Σ^*, prove that

<u>Property (P)</u>: $\ell(x \cdot y) = \ell(x) + \ell(y)$

where $x, y \in \Sigma^*$

& ℓ = len function.

<u>Basis:</u>

$$\left[\begin{array}{l} \underline{\ell(\lambda) = 0} \\ \ell(\omega \cdot x) = \ell(\omega) + 1 \end{array}\right.$$

where $\omega \in \Sigma^*$

$x \in \Sigma$

Basis: $P(\lambda;\lambda) =$ true

Recursive: $P(\omega a)$ is true ✓ ($a \in \Sigma$)

to show: if $P(\omega y)$ is true then $P(\omega y a)$ is true. ($y \in \Sigma^*$)

$$\ell(\omega y a) = \ell(xy) + 1$$

$$\ell(ya) = \ell(y) + 1$$

$$\ell(\omega y) = \ell(\omega) + \ell(y)$$

$$\Downarrow$$

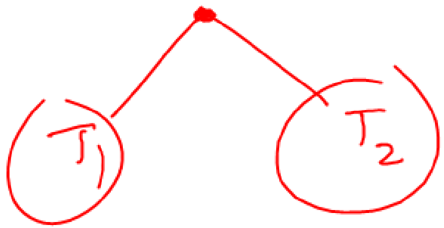
$$\ell(\omega y a) = \ell(\omega) + \ell(y) + 1$$

$$= \ell(\omega) + \ell(ya)$$

Binary tree:

(i) has a root (node) •

(ii) if T_1 & T_2 are binary trees then

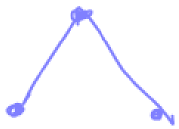


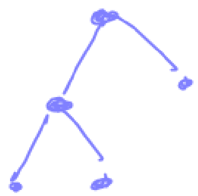
is also a binary tree

$T_1 \cap T_2 = \phi$

$h=0, n=1$

$h=1$
$n=3$

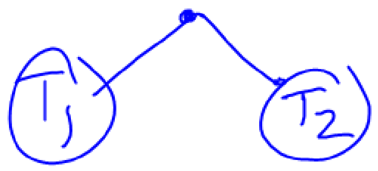




Height of a binary tree :

(i) if T contains only a root then $h(T) = 0$

(ii) T_1 & T_2 are binary trees then

$T =$ 

$$h(T) = 1 + \max(h(T_1), h(T_2)).$$

[for a binary tree T, the no of nodes are denoted by $n(T)$.

Prove:

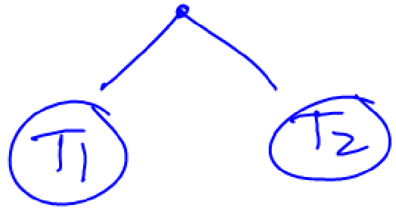
$$n(T) \le 2^{h(T)+1} - 1.$$

Basis:

- $n = 1, \; h = 0$

$$1 \le 2^{0+1} - 1 = 1$$ ✓

Inductive:



$= T$

$$n(T) = 1 + n(T_1) + n(T_2)$$

$$\leq 1 + \left(2^{h(T_1)+1} - 1\right) + \left(2^{h(T_2)+1} - 1\right)$$

$$\leq 1 + \left(2^{\max(h(T_1), h(T_2))+1} - 1\right) + \left(\cdots \right)$$

$$\left(2^4 + 2^6 \le 2^{\max(4,6)} + 2^{\max(4,6)} \right)$$

Counting (Combinatorics)

Rule of addition:

if A can be done in m ways

& B " " n ways

then either A or B can be done in

$(m+n)$ ways

(assuming that both A & B can't be done together).

Rule of multiplication:

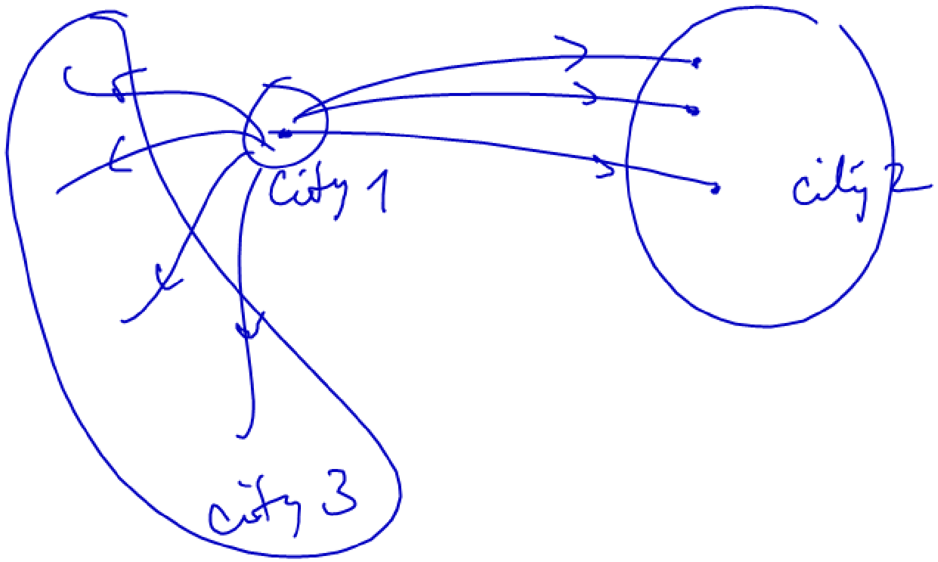
if A can be done in m ways

& followed by any one of these,

B can be done in n ways,

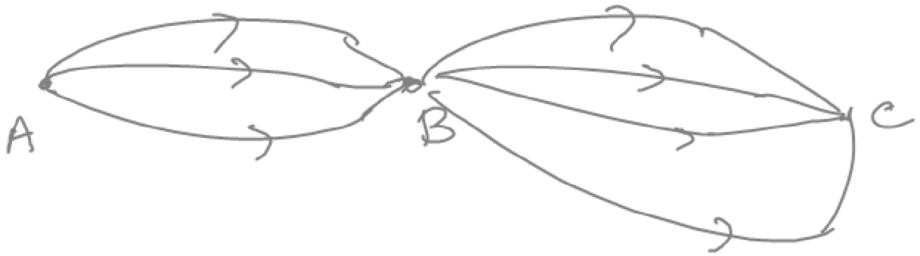
then A followed by B in this order can

be done in $m \times n$ ways.



How many ways can some one leave city 1 ?

$= 3 + 4 = 7$



How many ways from A to C.

$$= 3 \times 4 = 12$$