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Let $S$ be a non-empty set of +ve integers

non-negative $\geq 0$

$> 0$

then $\exists$ a least integer $\in S$.

Well ordering Principle (W.O.P.)

Least in $S$: $(S \neq \phi)$

$x$ is said to be the least element in $S$
if $\forall\, m \in S$, $x \leq m$.

## Division algorithm

$a$ = any arbitrary integer

$d$ = a +ve integer

then $\exists\ q, r$ s.t.

$$0 \le r < d$$

and

$$a = qd + r.$$

$(q, r)$ = unique

$\frac{17}{4} \Rightarrow 17 = 4 \times 4 + 1$

w.o.p. $\longrightarrow$ division property

$a, d$ are given: in div. algo, we aim to find $q, r$.

$$S = \{ a - kd \ , \text{ where } k = \text{integer} \}$$

$$S' = \{ x \in S \ \text{ s.t } \ x \geq 0 \}$$

Claim

(i) $S' =$ non-empty. $\Rightarrow$ w.o.p. applies on $S'$

($S'$ has integers only)

w.o.p. $\Rightarrow$

$a - kd = r$ is the least integer in $S'$.

$r \geq 0$ (since $r \in S'$)

<u>Upper bound on $r$:</u>

Suppose $r \geq d$

<u>Consider $(r-d)$.</u>

clearly $(r-d) \geq 0$. $\Rightarrow$ $(a-kd)-d \geq 0$

$a-(k+1)d \geq 0$.

By the defn. of $S'$, we have

$a-(k+1)d \in S'$

$a-(k+1)d < (a-kd) = r$

Let $S \subset \mathbb{N}$ be a set satisfying the following two properties :

(i) $1 \in S$

(ii) if $k \in S$ then $(k+1) \in S$

then $S = \mathbb{N}$

$\boxed{\text{Principle of Mathematical Induction}}$

## W.O.P. $\Longleftrightarrow$ P.M.I.

(1) W.O.P. $\longrightarrow$ P.M.I.

Technique : Consider $(\mathbb{N} - S)$ and show that it is empty.

Let $(\mathbb{N} - S)$ be non-empty.

$\Rightarrow$ By W.O.P. $\exists\ x \in (\mathbb{N} - S)$ which is least. $\to (x \neq 1)$ -

$x$ is least $\Rightarrow (x-1) \in S$

By the defn of $S$ : if $(x-1) \in S$ then $x \in S$

$\to$ Contradiction.

Applications.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

$P(n)$ : . _ _ _ _

$P(1)$ : Basis :

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

$$1 = \frac{1 \times 2}{2}$$ ✓

(2) Hypothesis: Let $P(k)$ be true

$$\Rightarrow \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

(3) Induction: $P(k+1)$ ?

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}.$$

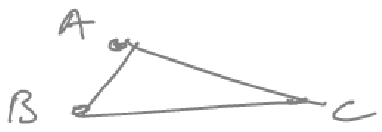
$\Rightarrow$ P.M.I. implies $P(n)$ is true for all $n \in \mathbb{N}$.

## Odd Pie fight:

an odd no. of people stand in a yard at mutually distinct distances.

At the same time, each person throws a pie at his/her nearest neighbour. Prove that there is at least one survivor.

Base:




$A \to B$

$B \to A$

$C \to \ldots$

C is survivor

**Hypothesis** $P(k)$: if $(2k+1)$ people play this game, there is a survivor.

**Induction Step**
$P(k+1)$: if $(2k+3)$ people.

let $\alpha$, $\beta$ be the closest neighbors among these $(2k+3)$ people


**Case 1** Someone from these throw a pie at $\alpha$ or $\beta$

**Case 2** none of these $(2k+1)$ people throw a pie at $\alpha$ or $\beta$