Practice Problems Set #1 (*Probability and Statistics*)

1. For a fix natural number $n \in \mathbb{N}$, let $\Omega = \{0,1\}^n$ (the set of all n bit strings). Let p, q be two real numbers such that $0 \leq p, q \leq 1$ and $p + q = 1$.
   - Show that the following function $\mathbb{P} : \Omega \to [0,1]$ is a probability distribution on Ω, where $\mathbb{P}$ is defined as follows: for any $\boldsymbol{w} = (w_1, \ldots, w_n) \in \Omega$, $\mathbb{P}(\boldsymbol{w}) = \prod_{i=1}^{n} p^{w_i} q^{1-w_i}$.
2. Let $\Omega = \{1,2,3,4\} \times \{1,2,3,4\}$. Let $\mathbb{P} : \Omega \to [0,1]$ be the uniform distribution on Ω. Let $B = \{(s,t) \mid s + t = 6\}$. Describe the conditional distribution $\mathbb{P}_B : \Omega \to [0,1]$.
3. For i in $1 \leq i \leq 2$, let $\mathbb{P}_i : \Omega_i \to [0,1]$ be a probability distribution. Define a function $\mathbb{P} : \Omega = \Omega_1 \times \Omega_2 \to [0,1]$ as follows: for every $\boldsymbol{w} = (w_1, w_2) \in \Omega$, $\mathbb{P}(\boldsymbol{w}) = \mathbb{P}_1(w_1) \times \mathbb{P}_2(w_2)$. Show that $\mathbb{P}$ is a probability distribution on Ω.
4. Let $\mathbb{P} : \Omega \to [0,1]$ be a probability distribution and $A, B \subseteq \Omega$. Show that $\mathbb{P}(\bar{A}/B) = 1 - \mathbb{P}(A/B)$.
5. Let $\mathbb{P} : \Omega \to [0,1]$ be a probability distribution and $A, B \subseteq \Omega$. It is give that $\mathbb{P}(A) = \frac{1}{5}$, $\mathbb{P}(A/B) = \frac{1}{3}$, and $\mathbb{P}(B/A) = \frac{1}{7}$. Compute $\mathbb{P}(B)$.
6. Let $\mathbb{P} : \Omega \to [0,1]$ be a probability distribution and $A, B \subseteq \Omega$. Show that $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$, where $B \backslash A = \{w \in B \mid w \notin A\}$.
7. Let $\mathbb{P} : \Omega \to [0,1]$ be a probability distribution and $A, B \subseteq \Omega$. Show that $\mathbb{P}(A \cup B)\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\mathbb{P}(B)$.
8. Let $\mathbb{P} : \Omega \to [0,1]$ be a probability distribution and $A, B, C \subseteq \Omega$ such that $A \cap \bar{C} = B \cap \bar{C}$, where $\bar{C}$ denotes the compliment of C in Ω. Show that $|\mathbb{P}(A) - \mathbb{P}(B)| \leq \mathbb{P}(C)$.
9. Let $\mathbb{P} : \Omega \to [0,1]$ be a probability distribution and $A, B \subseteq \Omega$. Show that if A and B are independent, i.e., $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$, then so are $\bar{A}$ and B.