Artificial Neural Networks

Ravi Kothari, Ph.D.
ravi.kothari@ashoka.edu.in

"People worry that computers will get too smart and take over the world, but the real problem is that they're too stupid and they've already taken over the world - Pedro Domingos"

Basic Model of a Neuron

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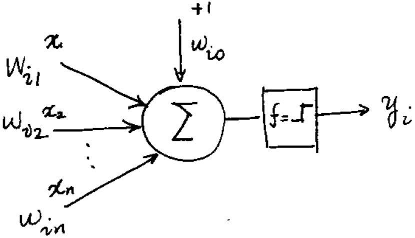

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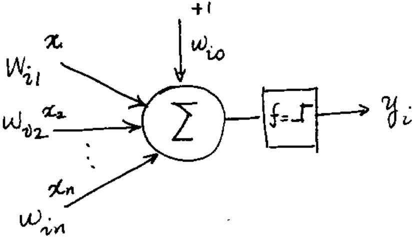

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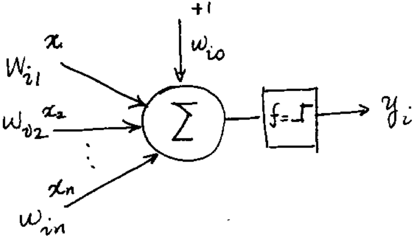

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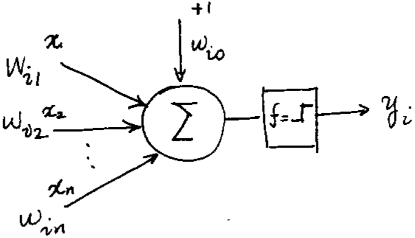

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- Abstraction of a biological neuron. Other abstractions are possible including spiking neurons etc.
- The output of a neuron can be the input to another neuron
- $f(\cdot)$ is the activation function, S_i is the weighted sum, w_{ij} is the weight from the j^{th} input and w_{i0} is the "bias"

Binary Sigmoid Activation Function

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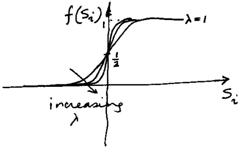

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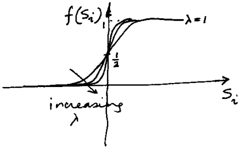

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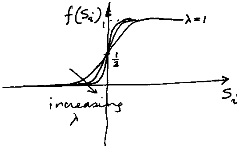

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- With increasing λ, the binary sigmoid becomes more and more like the step function

Bipolar Sigmoid Activation Function

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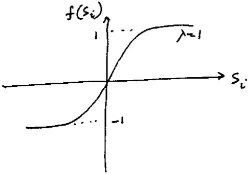

$$y_i = f\left(S_i\right) = \frac{2}{1 + e^{-\lambda S_i}} - 1$$

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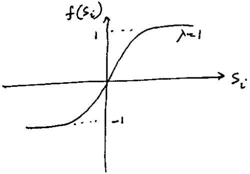

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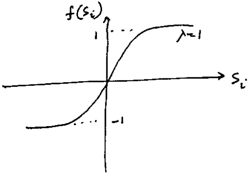

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- Bipolar Sigmoid - Differentiable
- With increasing λ, the bipolar sigmoid becomes more and more steeper

What does a Simple Neuron Do?

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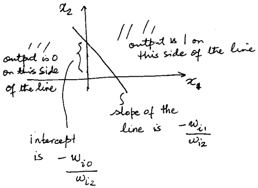

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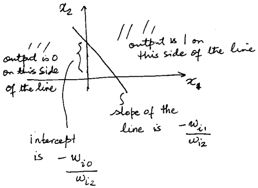

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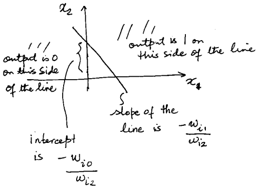

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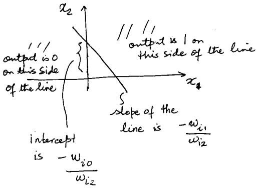

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- On one side of the line, $S_i > 0$ and on the other side $S_i < 0$. So, this simple neuron establishes a decision boundary in the input space
- If the bias (w_{i0}) was not there, the line would always be constrained to pass through the origin. Hence the bias

Learning (Adjusting the Weights)

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- Based on the training data, we would like to come up with an algorithm that would “evolve” the weights such that the network starts producing the desired output

The Delta Rule

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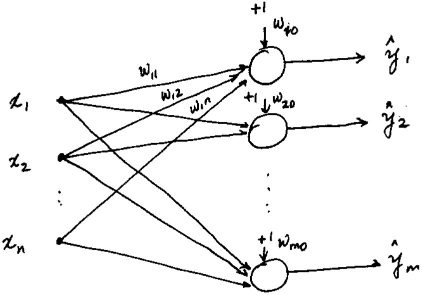

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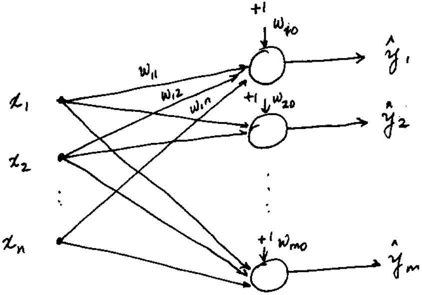

$$\left\{ \left(x^{(l)}, y^{(l)} \right) \right\}_{l=1}^{N} \qquad ; x^{(l)} \in \mathcal{R}^n, \quad y^{(l)} \in \mathcal{R}^m$$

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$$\begin{aligned} J &= \sum_{l=1}^{N} J^{(l)} \\ &= \sum_{l=1}^{N} \sum_{i=1}^{m} \left(y_i^{(l)} - \hat{y}_i^{(l)} \right)^2 \end{aligned} \tag{1}$$

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$$\mathrm{x}^{(l)} = \begin{bmatrix} 1 \\ x_1^{(l)} \\ x_2^{(l)} \\ \vdots \\ x_n^{(l)} \end{bmatrix} \qquad \mathrm{w}_i = \begin{bmatrix} w_{i0} \\ w_{i1} \\ w_{i2} \\ \vdots \\ w_{in} \end{bmatrix}$$

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$$\hat{y}_i^{(l)} = f\left(\sum_{j=0}^{n} w_{ij} x_j^{(l)}\right)$$

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$$
\begin{aligned}
J &= \sum_{l=1}^{N} \sum_{i=1}^{m} \left(y_i^{(l)} - \hat{y}_i^{(l)} \right)^2 \\
&= \sum_{l=1}^{N} \sum_{i=1}^{m} \left(y_i^{(l)} - f \left(\sum_{j=0}^{n} w_{ij} x_j \right) \right)^2
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So, we can start with random values of w's and adopt the weights by moving opposite to the gradient computed from the equation above, i.e.

$$\Delta w = -\eta \nabla J$$

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In online learning, we make the approximation that as long as η is small, we can,

- Feed pattern l
- Compute $\nabla J^{(l)}$
- Update w's and repeat

The Delta Rule Derived

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$$
\begin{aligned}
J &= \sum_{l=1}^{N} \sum_{i=1}^{m} \left(y_i^{(l)} - f\left(\sum_{j=0}^{n} w_{ij} x_j \right) \right)^2 \\
&= \sum_{l=1}^{N} \sum_{i=1}^{m} \left(y_i^{(l)} - \hat{y}_i^{(l)} \right)^2 = \sum_{l=1}^{N} \sum_{i=1}^{m} e_i^{(l)^2}
\end{aligned}
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$$\frac{\partial J^{(l)}}{\partial w_{ij}} = \frac{\partial J^{(l)}}{\partial e_i^{(l)}} \frac{\partial e_i^{(l)}}{\partial \hat{y}_i^{(l)}} \frac{\partial \hat{y}_i^{(l)}}{\partial S_i^{(l)}} \frac{\partial S_i^{(l)}}{\partial w_{ij}}$$
$$= 2e_i^{(l)} \cdot (-1) \cdot f'(S_i^{(l)}) \cdot x_j^{(l)}$$
$$= -2 \left(y_i^{(l)} - \hat{y}_i^{(l)} \right) \cdot f'(S_i^{(l)}) \cdot x_j^{(l)}$$

What is $f'(S_i)$

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Recall that,

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If the activation function is linear, $\hat{y}_i^{(l)} = S_i^{(l)}$

$$f'\left((S_i^{(l)}\right) = \frac{\hat{y}_i^{(l)}}{S_i^{(l)}} = 1 \tag{3}$$

If the activation function is a binary sigmoid, $\hat{y}_i^{(l)} = 1/(1 + e^{-S_i^{(l)}})$

$$f'\left(S_i^{(l)}\right) = \hat{y}_i^{(l)}\left(1 - \hat{y}_i^{(l)}\right) \tag{4}$$

If the activation function is a bipolar sigmoid, $\hat{y}_i^{(l)} = \left(2/(1 + e^{-S_i^{(l)}})\right) - 1$

$$f'\left(S_i^{(l)}\right) = \frac{1}{2}\left(1 - \hat{y}^{(l)2}_i\right) \tag{5}$$

The Overall Algorithm (Online Learning)

Initialize all the weights to small random values
while J is large
    for $l = 1, 2, \ldots, N$
        Do a forward pass i.e. compute $\hat{y}_1^{(l)}, \hat{y}_2^{(l)}, \ldots, \hat{y}_m^{(l)}$
        for $i = 1, 2, \ldots, n$
            for $j = 1, 2, \ldots, m$

$$\frac{\partial J^{(l)}}{\partial w_{ij}} = -\left(y_i^{(l)} - \hat{y}_i^{(l)}\right) \cdot f'\left(S_i^{(l)}\right) \cdot x_j^{(l)}$$

$$w_{ij}(\text{new}) = w_{ij}(\text{old}) - \eta \frac{\partial J^{(l)}}{\partial w_{ij}}$$

            end for
        end for
    end for
end while

The Overall Algorithm (Batch Learning)

Initialize all the weights to small random values

while J is large

$\Delta w_{ij} = 0; \quad i = 1, 2, \ldots n; \; j = 1, 2, \ldots, m$

for $l = 1, 2, \ldots, N$

Do a forward pass i.e. compute $\hat{y}_1^{(l)}, \hat{y}_2^{(l)}, \ldots, \hat{y}_m^{(l)}$

for $i = 1, 2, \ldots, n$

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$$\frac{\partial J^{(l)}}{\partial w_{ij}} = -\left(y_i^{(l)} - \hat{y}_i^{(l)}\right) \cdot f'\left(S_i^{(l)}\right) \cdot x_j^{(l)}$$

$$\Delta w_{ij}(\text{new}) = \Delta w_{ij}(\text{old}) - \eta \frac{\partial J^{(l)}}{\partial w_{ij}}$$

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$w_{ij}(\text{new}) = w_{ij}(\text{old}) | \Delta w_{ij}$

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- The loss function is quadratic
- The activation function needs to be differentiable
- For classification, we can interpret an output of greater than 0 as Class 1 and an output of less than 0 as Class 2 if we are using a bipolar sigmoid
- Note that a neuron establishes a linear decision boundary. Thus, such networks can only perform linearly separable pattern classification

An Illustrative Example – Online Learning

Assume we are given the following training data,

	Input	Desired Output	
$x^{(1)} \rightarrow$	1	1	$\leftarrow y^{(1)}$
$x^{(2)} \rightarrow$	-0.5	-1	$\leftarrow y^{(2)}$
$x^{(3)} \rightarrow$	3	1	$\leftarrow y^{(3)}$
$x^{(4)} \rightarrow$	-1	-1	$\leftarrow y^{(4)}$

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So, we have $n = 1$ input and $m = 1$ output. Let us initialize the parameters as: $w_{10} = -0.5$, $w_{11} = 0.5$, take bipolar activations (outputs are $+1$ and -1), and $\eta = 0.1$

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    - ★ $w_{10}(new) = -0.5 + 0.1\,(1-0)) \cdot \frac{1}{2}\,(1-0) \cdot 1 = -0.45$

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