Reciprocal Identities

immediate

August 27, 2020

Let $-\rho^2\Delta u + u = 0$ and

$$\mathbf{T} = \rho^{-1}u^2\mathbf{I} + \rho(|\nabla u|^2\mathbf{I} - 2\nabla u \otimes \nabla u) \tag{1}$$

Using indices and Einstein summation

$$T_{ij} = \rho^{-1}u^2\delta_{ij} + \rho(\nabla_k u \nabla_k u \delta_{ij} - 2\nabla_i u \nabla_j u) \tag{2}$$

There are solid bodies U_p for $p = 1, 2, \dots N_b$ and $\Sigma_p = \partial U_p$ with outward pointing normal $\mathbf{n}$. The force and torque on the pth body are

$$\mathbf{F}_p = \int_{\Sigma_p} \mathbf{T}\mathbf{n}\,\mathrm{d}S, \quad \mathbf{G}_p = \int_{\Sigma_p} \mathbf{x} \times \mathbf{T}\mathbf{n}\,\mathrm{d}S. \tag{3}$$

The stress $\mathbf{T}$ is smooth in $\Omega = \mathbb{R}^3 \setminus \cup_{p=1}^{N_b} U_p$.

We observe that

$$\begin{aligned}(\nabla\cdot\mathbf{T})_i &= \nabla_j T_{ij} \\ &= \rho^{-1}2u\nabla_i u + \rho(\nabla^2_{ik}u\nabla_k u + \nabla_k u \nabla^2_{ik}u - 2\nabla^2_{ij}u\nabla_j u - 2\nabla_i u \nabla^2_{jj}u) \\ &= 0.\end{aligned}$$

so that $\nabla\cdot\mathbf{T} = \mathbf{0}$. Furthermore, we have the identity (Leal, Problems 2-1)

$$\nabla\cdot(\mathbf{T}\times\mathbf{x}) = -\mathbf{x}\times\nabla\cdot\mathbf{T} + \varepsilon : \mathbf{T} \tag{4}$$

where ε is the alternating tensor. Because $\nabla\cdot\mathbf{T} = \mathbf{0}$ and $\mathbf{T}$ is symmetric, we have $\nabla\cdot(\mathbf{T}\times\mathbf{x}) = \mathbf{0}$ as well.

Suppose that $U_p \subset U'_p$ where U'_p is disjoint from all other bodies. Then $\nabla\cdot\mathbf{T} = \nabla\cdot(\mathbf{T}\times\mathbf{x}) = \mathbf{0}$ in $U'_p \setminus U_p$, and we get

$$\mathbf{F}_p = \int_{\Sigma'_p} \mathbf{T}\mathbf{n}\,\mathrm{d}S, \quad \mathbf{G}_p = \int_{\Sigma'_p} \mathbf{x} \times \mathbf{T}\mathbf{n}\,\mathrm{d}S \tag{5}$$

where $\Sigma'_p = \partial U'_p$ and $\mathbf{n}$ having the same outward orientation as on Σ_p. The benefit of this identity is that we can transfer the calculation of the force and torque away from the surface Σ_p, where the double layer gradient integrations are inaccurate, to a nearby surface where they are accurate.

Now consider the situation where

$$u = \sum_{p=1}^{N_b} u_p \tag{6}$$

where $-\rho^2 \Delta u_p + u_p = 0$ and $u_p \in \mathbf{C}^\infty(\mathbb{R}^3 \setminus U_p)$. This situation arises, for example, when u is the double layer potential for separate bodies. Based on this, we expand

$$\begin{aligned}
\mathbf{T} &= \rho^{-1} u^2 \mathbf{I} + \rho(|\nabla u|^2 \mathbf{I} - 2\nabla u \otimes \nabla u) \\
&= \sum_{p,q=1}^{N_b} \rho^{-1} u_p u_q \mathbf{I} + \rho(\nabla u_p \cdot \nabla u_q \mathbf{I} - 2\nabla u_p \otimes \nabla u_q) \\
&= \sum_{p,q=1}^{N_b} \mathbf{T}_{p,q}
\end{aligned}$$

We also get $\nabla \cdot (\mathbf{T}_{pq} + \mathbf{T}_{qp}) = \mathbf{0}$ because

$$\begin{aligned}
&(\nabla \cdot (\mathbf{T}_{pq} + \mathbf{T}_{qp}))_i \\
&= \nabla_j(\rho^{-1} u_p u_q \delta_{ij} + \rho(\nabla_k u_p \nabla_k u_q \delta_{ij} - 2\nabla_i u_p \otimes \nabla_j u_q)) \\
&+ \nabla_j(\rho^{-1} u_q u_p \delta_{ij} + \rho(\nabla_k u_q \nabla_k u_p \delta_{ij} - 2\nabla_i u_q \otimes \nabla_j u_p)) \\
&= \rho^{-1}(2\nabla_i u_p u_q + 2u_p \nabla_i u_q) \\
&+ \rho(2\nabla^2_{ik} u_p \nabla_k u_q + 2\nabla_k u_p \nabla^2_{ik} u_q \\
&- 2\nabla^2_{ij} u_p \nabla_j u_q - 2\nabla_i u_p \nabla^2_{jj} u_q - 2\nabla^2_{ij} u_q \nabla_j u_p - 2\nabla_i u_q \nabla^2_{jj} u_p) \\
&= 0.
\end{aligned}$$

The tensor $\mathbf{T}_{pq} + \mathbf{T}_{qp}$ is symmetric and so $\nabla \cdot ([\mathbf{T}_{pq} + \mathbf{T}_{qp}] \times \mathbf{x}) = \mathbf{0}$ as well.

The divergence free decomposition carries the following ramifications. Consider the force and torque on particle p:

$$
\begin{aligned}
\mathbf{F}_p &= \int_{\Sigma_p} \mathbf{T}\mathbf{n}\ \mathrm{d}S \\
&= \int_{\Sigma_p} \sum_{q,r=1}^{N_b} \mathbf{T}_{qr}\mathbf{n}\ \mathrm{d}S \\
&= \sum_{q=1}^{N_b} \int_{\Sigma_p} \mathbf{T}_{qq}\mathbf{n}\ \mathrm{d}S + \sum_{q<r} \int_{\Sigma_p} (\mathbf{T}_{qr} + \mathbf{T}_{rq})\mathbf{n}\ \mathrm{d}S.
\end{aligned}
$$

We write SDF for smooth and divergence free.

1. $\mathbf{T}_{pp}$ is SDF in U_p^c and vanishes exponentially in the far-field. Therefore

$$\int_{\Sigma_p} \mathbf{T}_{pp}\mathbf{n}\ \mathrm{d}S = -\int_{U_p^c} \nabla \cdot \mathbf{T}_{pp}\ \mathrm{d}x = 0.$$

2. $\mathbf{T}_{qq}$ is SDF in U_p for $q \neq p$. Therefore

$$\int_{\Sigma_p} \mathbf{T}_{qq}\mathbf{n}\ \mathrm{d}S = \int_{U_p} \nabla \cdot \mathbf{T}_{qq}\ \mathrm{d}x = 0.$$

3. Similarly, $\mathbf{T}_{qr} + \mathbf{T}_{rq}$ is SDF in U_p for $q \neq p$, $r \neq p$. Therefore

$$\int_{\Sigma_p} (\mathbf{T}_{qr} + \mathbf{T}_{rq})\mathbf{n}\ \mathrm{d}S = \int_{U_p} \nabla \cdot (\mathbf{T}_{qr} + \mathbf{T}_{rq})\ \mathrm{d}x = 0.$$

The same line of argumentation holds for the torque. That leaves

$$\mathbf{F}_p = \sum_{q\neq p} \int_{\Sigma_p} (\mathbf{T}_{pq} + \mathbf{T}_{qp})\mathbf{n}\ \mathrm{d}S, \quad \mathbf{G}_p = \sum_{q\neq p} \int_{\Sigma_p} \mathbf{x} \times (\mathbf{T}_{pq} + \mathbf{T}_{qp})\mathbf{n}\ \mathrm{d}S. \tag{7}$$

1 Jump relations

Suppose that $u = Dh$ where D is the double layer operator for the screened Laplace equation problem and h a surface density. We have the jump relations

1. $\displaystyle\lim_{x\to x_0^{\pm}\in\Sigma_p} u(x) = \pm\frac{1}{2}h(x_0) + (Dh)(x_0)$

2. $\lim_{x \to x_0^+} \nabla u(x) \cdot \mathbf{n}(x_0) = \lim_{x \to x_0^-} \nabla u(x) \cdot \mathbf{n}(x_0)$

The layer potential $u = Dh$ satisfies the Dirichlet problem so that

$$f(x_0) = \lim_{x \to x_0^+ \in \Sigma_p} u(x) = \frac{1}{2} h(x_0) + (Dh)(x_0). \tag{8}$$

We can therefore calculate tangential derivatives as well.

The definition of u extends into U_p. There, like in the exterior, u is a solution of the screened Laplace equation and the stress is divergence free. Therefore

$$\begin{aligned} \mathbf{0} = \int_{U_p} \nabla \cdot \mathbf{T} \, \mathrm{d}x &= \int_{\Sigma_p} \mathbf{T}_- \mathbf{n} \, \mathrm{d}S \\ &= \int_{\Sigma_p} [\mathbf{T}_- - \mathbf{T}_+] \mathbf{n} \, \mathrm{d}S + \mathbf{F}_p \end{aligned}$$

The $\pm$ subscripts indicates limits from the outside, inside resp., of Σ_p. We can now evaluate the stress jump.

$$\begin{aligned} [\mathbf{T}_+ - \mathbf{T}_-] \mathbf{n} &= \rho^{-1} (u_+^2 - u_-^2) \mathbf{n} \\ &+ \rho (|\nabla u_+|^2 - |\nabla u_-|^2) \mathbf{n} \\ &- 2\rho (\nabla u_+ \nabla u_+ \cdot \mathbf{n} - \nabla u_- \nabla u_- \cdot \mathbf{n}) \\ &= I \rho^{-1} \mathbf{n} + II \rho \mathbf{n} - 2\rho III \end{aligned}$$

We have

1. Using the jump condition,

$$\begin{aligned} I &= u_+^2 - u_-^2 \\ &= (\tfrac{1}{2} h(x_0) + (Dh)(x_0))^2 - (-\tfrac{1}{2} h(x_0) + (Dh)(x_0))^2 \\ &= 2h(x_0)(Dh)(x_0) \end{aligned}$$

   Now $f(x_0) = \frac{1}{2} h(x_0) + (Dh)(x_0)$ implies that $(Dh)(x_0) = f(x_0) - \frac{1}{2} h(x_0)$. This tells us that

$$I = 2h(x_0) f(x_0) - h^2(x_0). \tag{9}$$

2. For II, the normal derivative is continuous and so it remains to evaluate the jump from tangential derivative. Assume that $\{\tau_1, \tau_2, \mathbf{n}\}$ are an

orthonormal frame. Then

$$
\begin{aligned}
II &= |\nabla u_+|^2 - |\nabla u_-|^2 \\
&= (\nabla u_+ \cdot \tau_1)^2 + (\nabla u_+ \cdot \tau_2)^2 + (\nabla u_+ \cdot \mathbf{n})^2 \\
&- (\nabla u_- \cdot \tau_1)^2 - (\nabla u_- \cdot \tau_2)^2 - (\nabla u_- \cdot \mathbf{n})^2 \\
&= (\nabla u_+ \cdot \tau_1)^2 - (\nabla u_- \cdot \tau_1)^2 + (\nabla u_+ \cdot \tau_2)^2 - \nabla u_- \cdot \tau_2)^2.
\end{aligned}
$$

Now for any tangential vector τ,

$$
\begin{gathered}
u_+ = \tfrac{1}{2}h + (Dh), \quad u_- = -\tfrac{1}{2}h + (Dh) \\
\nabla u_+ \cdot \tau = \tfrac{1}{2}\nabla h \cdot \tau + \nabla(Dh) \cdot \tau, \\
\nabla u_- \cdot \tau = -\tfrac{1}{2}\nabla h \cdot \tau + \nabla(Dh) \cdot \tau
\end{gathered}
$$

This gives

$$
\begin{aligned}
&(\nabla u_+ \cdot \tau)^2 - (\nabla u_- \cdot \tau)^2 \\
&= (\tfrac{1}{2}\nabla h \cdot \tau + \nabla(Dh) \cdot \tau)^2 - (-\tfrac{1}{2}\nabla h \cdot \tau + \nabla(Dh) \cdot \tau)^2 \\
&= 2\nabla h \cdot \tau \nabla(Dh) \cdot \tau
\end{aligned}
$$

As above, $\nabla f \cdot \tau = \nabla u \cdot \tau = \frac{1}{2}\nabla h \cdot \tau + \nabla(Dh) \cdot \tau$ allows us to write the previous displayed expression as

$$
2\nabla h \cdot \tau \nabla f \cdot \tau - (\nabla h \cdot \tau)^2. \tag{10}
$$

This finally brings us to

$$
\begin{aligned}
II &= \nabla h \otimes (2\nabla f - \nabla h) : (\tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2) \\
&= 2\nabla h \cdot \tau_1 \nabla f \cdot \tau_1 - (\nabla h \cdot \tau_1)^2 + 2\nabla h \cdot \tau_2 \nabla f \cdot \tau_2 - (\nabla h \cdot \tau_2)^2.
\end{aligned}
$$

3. Lastly, we analyze the term III. The normal derivative is continuous and can therefore be factored out right away.

$$
\begin{aligned}
III &= \nabla u_+ \nabla u_+ \cdot \mathbf{n} - \nabla u_- \nabla u_- \cdot \mathbf{n} \\
&= (\nabla u_+ - \nabla u_-) \nabla u_+ \cdot \mathbf{n} \\
&= (\nabla u_+ \cdot \tau_1 \tau_1 + \nabla u_+ \cdot \tau_2 \tau_2 + \nabla u_+ \cdot \mathbf{n}\mathbf{n} \\
&- \nabla u_- \cdot \tau_1 \tau_1 - \nabla u_- \cdot \tau_2 \tau_2 - \nabla u_- \cdot \mathbf{n}\mathbf{n}) \nabla u_+ \cdot \mathbf{n} \\
&= (\nabla h \cdot \tau_1 \tau_1 + \nabla h \cdot \tau_2 \tau_2) \nabla u_+ \cdot \mathbf{n}
\end{aligned}
$$

This gives the simplification

$$
\begin{aligned}
[\mathbf{T}_+ - \mathbf{T}_-] &= (2hf - h^2)\rho^{-1}\mathbf{n} \\
&\quad + [2\nabla h \cdot \tau_1 \nabla f \cdot \tau_1 - (\nabla h \cdot \tau_1)^2 + 2\nabla h \cdot \tau_2 \nabla f \cdot \tau_2 - (\nabla h \cdot \tau_2)^2]\rho\mathbf{n} \\
&\quad - 2\rho[\nabla h \cdot \tau_1 \tau_1 + \nabla h \cdot \tau_2 \tau_2]\nabla u_+ \cdot \mathbf{n}
\end{aligned}
$$

All terms of this expression are known, with the exception of $\nabla u_+ \cdot \mathbf{n}$, which is the normal derivative of the double layer potential.

2 Jumps and reciprocity

We focus now on the stress

$$\mathbf{T}_{pq} + \mathbf{T}_{qp} \tag{11}$$

This stress has a jump across Σ_p and Σ_q, but is continuous on all other surfaces. On Σ_p, u_q and ∇u_q are continuous and the jumps come from u_p and ∇u_p as follows. We recall that

$$
\begin{aligned}
[u_p] &= h_p \\
[\nabla u_p] &= \nabla h_p \cdot (\tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2)
\end{aligned}
$$

As such

$$
\begin{aligned}
J_{pq} &= \{(\mathbf{T}_{pq} + \mathbf{T}_{qp})_+ - (\mathbf{T}_{pq} + \mathbf{T}_{qp})_-\}\mathbf{n} \\
&= \rho^{-1} u_{p+} u_q \mathbf{n} + \rho(\nabla u_{p+} \cdot \nabla u_q \mathbf{n} - 2\nabla u_{p+} \nabla u_q \cdot \mathbf{n}) \\
&\quad + \rho^{-1} u_q u_{p+} \mathbf{n} + \rho(\nabla u_q \cdot \nabla u_{p+} \mathbf{n} - 2\nabla u_q \nabla u_{p+} \cdot \mathbf{n}) \\
&\quad - \rho^{-1} u_{p-} u_q \mathbf{n} - \rho(\nabla u_{p-} \cdot \nabla u_q \mathbf{n} - 2\nabla u_{p-} \nabla u_q \cdot \mathbf{n}) \\
&\quad - \rho^{-1} u_q u_{p-} \mathbf{n} - \rho(\nabla u_q \cdot \nabla u_{p-} \mathbf{n} - 2\nabla u_q \nabla u_{p-} \cdot \mathbf{n}) \\
&= 2\rho^{-1}[u_p] u_q \mathbf{n} + 2\rho[\nabla u_p] \cdot \nabla u_q \mathbf{n} - 2\rho([\nabla u_p] \nabla u_q \cdot \mathbf{n}).
\end{aligned}
$$

The tensorial term cancels the way it does because $\nabla u_p \cdot \mathbf{n}$ is continuous across Σ_p. And so, in one dimension,

$$J_{pq} = 2\rho^{-1} h_p u_q \mathbf{n} + 2\rho(\nabla h_p \cdot \tau \nabla u_q \cdot \tau)\mathbf{n} - 2\rho(\nabla h_p \cdot \tau \nabla u_q \cdot \mathbf{n})\tau$$

To make its use explicit, we calculate

$$
\begin{aligned}
\mathbf{0} &= \sum_{q\neq p} \int_{U_p} \nabla \cdot (\mathbf{T}_{pq} + \mathbf{T}_{qp})\ \mathrm{d}x \\
&= \sum_{q\neq p} \int_{\Sigma_p} (\mathbf{T}_{pq} + \mathbf{T}_{qp})_{-}\mathbf{n}\ \mathrm{d}s \\
&= \sum_{q\neq p} \int_{\Sigma_p} (\mathbf{T}_{pq} + \mathbf{T}_{qp})_{+}\mathbf{n} - J_{pq}\ \mathrm{d}s \\
&= \mathbf{F}_p - \sum_{q\neq p} \int_{\Sigma_p} J_{pq}\ \mathrm{d}s
\end{aligned}
$$

In other words

$$
\mathbf{F}_p = \sum_{q\neq p} \int_{\Sigma_p} J_{pq}\ \mathrm{d}s, \quad \mathbf{G}_p = \sum_{q\neq p} \int_{\Sigma_p} \mathbf{x} \times J_{pq}\ \mathrm{d}s \tag{12}
$$