FROM STOKES FLOW TO DARCY'S LAW

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1. Introduction

1.1. **Porous media.** Consider a fluid moving in a bounded, open, connected set $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with smooth boundary. If the fluid is viscous and incompressible, we model the flow with the incompressible Navier-Stokes equations:

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) = \Delta u - \nabla p + f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ is the fluid velocity, $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is the fluid pressure, $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ is the external force acting on the fluid, $\rho > 0$ is the constant fluid density, and $\mu > 0$ is the constant fluid viscosity. The incompressibility condition corresponds to the second equation: the divergence-free condition guarantees that the fluid flow is volume-preserving

The term $\rho(\partial_t u + u \cdot \nabla u)$ is the acceleration of the fluid. If f doesn't depend on time, then it's reasonable to assume that the net acceleration on the fluid vanishes everywhere in Ω, i.e. that the fluid is in an equilibrium configuration. This leads to the Stokes equations:

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here we have set $\mu = 1$ for convenience. Doing so is no loss of generality, as we can always employ a scaling argument so that $\mu = 1$ in the scaled coordinate system. Note that we would also arrive at the Stokes system (1.2) by linearizing (1.1), assuming f is time-independent, and looking for a static solution for which $\partial_t u = 0$.

Our goal in these notes is to study the behavior of a fluid flowing in a porous medium such as sand, soil, or porous rocks. The idea is that, while the fluid domain is assumed to be connected, it is permeated by a solid microstructure through which the fluid flows. This structure is assumed to be periodic and very small relative to the size of the domain. The aim, then, is to derive an equation for the effective dynamics of the fluid in the limit as the size of the microstructure vanishes.

1.2. **Defining the microstructure.** Our aim now is to describe how to obtain a model of a porous medium that is amenable to analysis via homogenization methods. We must first describe the microstructure.

The periodic unit cell is defined as $Y = \mathbb{R}^d/\mathbb{Z}^d$. To any subset $A \subseteq Y$ we associate the lifted set $L(A) \subseteq \mathbb{R}^d$, defined by

$$L(A) = \{x \in \mathbb{R}^d \mid [x] \in A\}, \tag{1.3}$$

where $[x] \in Y$ is the equivalence class associated to $x \in \mathbb{R}^d$. We assume that the cell Y is the disjoint union $Y = Y_s \cup Y_f$, where Y_f denotes the "fluid part" of the cell and Y_s denotes the "solid part." To model a porous medium, it is natural to allow $L(Y_s)$ to be a connected set, corresponding to an infinite periodic solid structure in which the fluid flows. For simplicity, in these notes we will consider only the case in which Y_s is strictly contained within the unit cell. This is the case considered by Tartar [4], who was the first to rigorously derive the effective dynamics. Later work by Allaire [1] and Polisevsky [2] generalized Tartar's result to handle the more general case in which the microstructure extends to the boundary of each cell, generating a connected periodic solid network.

We make the following assumptions on Y_s.

(1) $Y_s \subset Y$ is a non-empty closed set of positive measure.
(2) $L(Y_s) \cap \partial[0, 1]^d = \varnothing$. That is, the solid part of Y is strictly contained in the unit cell.
(3) ∂Y_s is smooth.
(4) $\mathbb{R}^d \backslash L(Y_s)$ is connected.

We then define $Y_f = Y \backslash Y_s$, which is non-empty and open by virtue of (1) and (2) above. Item (3) implies that ∂Y_f and $\partial L(Y_f)$ are both smooth. Item (4) guarantees that $L(Y_f) = \mathbb{R}^d \backslash L(Y_s)$

is connected. This means that fluid may flow anywhere throughout the porous medium; it does not get trapped by the infinite solid structure.

We will let $\varepsilon > 0$ denote the scale of the microstructure. It is convenient to decompose $\mathbb{R}^d$ into $\varepsilon-$ cells. For $m \in \mathbb{Z}^d$ we write

$$Y_m^\varepsilon = [\varepsilon m_1, \varepsilon(m_1+1)) \times \cdots \times [\varepsilon m_d, \varepsilon(m_d+1)), \tag{1.4}$$

where we view $Y_m^\varepsilon \subset \mathbb{R}^d$ as being endowed with the subset topology of $\mathbb{R}^d$, not with the periodic topology of Y itself. This then allows us to tile

$$\mathbb{R}^d = \bigcup_{m \in \mathbb{Z}^d} Y_m^\varepsilon. \tag{1.5}$$

We will often write

$$Y_{m;f}^\varepsilon = \varepsilon L(Y_f) \cap Y_m^\varepsilon \subset \mathbb{R}^d \text{ and } Y_{m;s}^\varepsilon = \varepsilon L(Y_s) \cap Y_m^\varepsilon \subset \mathbb{R}^d \tag{1.6}$$

to denote the fluid and solid parts of the $\varepsilon-$cells, respectively.

Recall that we have assumed that Ω is bounded, open, and connected. Define the set of interior $\varepsilon-$cells to be

$$\mathcal{I}_\varepsilon(\Omega) = \{m \in \mathbb{Z}^d \mid Y_m^\varepsilon \subset \Omega\}. \tag{1.7}$$

This set is useful because if we restrict to using it to define the microstructure, we never have to worry about the solid part of Y_m^ε intersecting $\partial\Omega$ and complicating the regularity of the fluid domain's boundary. We define

$$\Omega_\varepsilon = \Omega \backslash \bigcup_{m \in \mathcal{I}_\varepsilon(\Omega)} Y_{m;s}^\varepsilon. \tag{1.8}$$

Notice in particular that Ω_ε is bounded, open, and connected, and $\partial\Omega_\varepsilon$ is smooth.

We will assume henceforth that $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 < 1$ chosen small enough to guarantee that the following hold for all $0 < \varepsilon < \varepsilon_0$:

(1) $\mathcal{I}_\varepsilon(\Omega) \neq \varnothing$, and
(2) $\{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\} \neq \varnothing$.

The first assumption guarantees that $\Omega_\varepsilon \neq \Omega$, i.e. there is microstructure embedded in Ω. The second is a technical assumption made for convenience in proving some lemmas. Since we will ultimately send $\varepsilon \to 0$, neither of these assumptions induces a loss of generality.

1.3. **Darcy's law.** Consider the Stokes problem in Ω_ε for every $\varepsilon \in (0, \varepsilon_0)$:

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

where $f \in L^2(\Omega)$ is some fixed function. Rather than bother with the fine details of each $\varepsilon-$problem, it's natural to seek an effective equation for u_ε and p_ε that is valid in the limit $\varepsilon \to 0$. This is the problem of homogenizing the Stokes system.

A formal two-scale analysis of this problem (see for instance [3]) suggests that

$$u_\varepsilon \approx \varepsilon^2 u \text{ and } p_\varepsilon \approx p \text{ as } \varepsilon \to 0, \tag{1.10}$$

where u and p solve Darcy's law in Ω:

$$\begin{cases} u = K(f - \nabla p) & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

Here $K \in \mathbb{R}^{d \times d}$ is a constant symmetric positive definite tensor called the permeability tensor that can be computed in terms of the microstructure (see Definition 6.1). The system (1.11) gives rise to a single elliptic equation for p, which determines it uniquely (under the assumption of zero-average). This in turn determines u uniquely.

Our goal in these notes is to prove this result by following the proof of Tartar [4], who was the first to derive Darcy's law rigorously as the homogenization of the Stokes problem. We refer to Theorem 6.6 for a precise statement of the result. In order to make the notes more self-contained, we have expanded on the arguments used in [4], and we have included a discussion of how to solve the Stokes problem.

2. Function spaces

2.1. **Definitions.** We need to define various function spaces that will be useful in these notes. Throughout this section we assume that Γ denotes either a bounded, open, connected subset of $\mathbb{R}^d$ with smooth boundary, or else Y_f. In the latter case, the functions on Γ inherit the periodicity of $Y = \mathbb{R}^d \backslash \mathbb{Z}^d$.

We begin with the usual L^2 based Sobolev space

$$H^1(\Gamma) = \{u \in L^2(\Gamma) \mid \nabla u \in L^2(\Gamma)\}. \tag{2.1}$$

Here we make no notational distinction between scalar or vector valued Sobolev spaces: which space is intended will always be clear from the context. We write

$$H_0^1(\Gamma) = \{u \in H^1(\Gamma) \mid u = 0 \text{ on } \partial\Gamma\} \tag{2.2}$$

and

$$H_{0,\sigma}^1(\Gamma) = \{u \in H_0^1(\Gamma) \mid \operatorname{div} u = 0\}. \tag{2.3}$$

The symbol σ is used here because these vector fields are often called "solenoidal." Clearly $H_{0,\sigma}^1(\Gamma)$ is only for vector-valued functions. We endow $H_0^1(\Gamma)$ with the inner-product

$$(u, v)_1 = \int_\Gamma \nabla u : \nabla v, \tag{2.4}$$

where $:$ denotes either the Frobenius inner-product when u is vector valued, or else the usual dot-product when u is scalar. This is an inner-product by virtue of the Poincaré inequality. We write

$$\|u\|_1 = \|u\|_{H^1} = \|u\|_{H^1(\Gamma)} = \sqrt{(u,u)_1} \tag{2.5}$$

for the norm generated by $(\cdot,\cdot)_1$. We adopt the usual convention of writing

$$H^{-1}(\Gamma) = (H_0^1(\Gamma))^*, \tag{2.6}$$

where $*$ denotes the dual space.

We will need the subspace of L^2 orthogonal to constants:

$$\mathring{L}^2(\Gamma) = \{u \in L^2(\Gamma) \mid \int_\Gamma u = 0\}. \tag{2.7}$$

We will denote the inner-product on L^2 by

$$(u, v)_0 = \int_\Gamma u \cdot v, \tag{2.8}$$

where the 0 refers to the fact that L^2 is the Sobolev space with 0 derivatives in L^2. We write

$$\|u\|_0 = \|u\|_{L^2} = \|u\|_{L^2(\Gamma)} = \sqrt{(u,u)_0} \tag{2.9}$$

for the norm generated by $(\cdot,\cdot)_0$. We will also need the solenoidal vectors in L^2:

$$L_\sigma^2(\Gamma) = \{u \in L^2(\Gamma) \mid \operatorname{div} u = 0 \text{ in the sense of distributions}\}. \tag{2.10}$$

When we need to specify the dependence of $(u,v)_1$ or $(u,v)_0$ on the space Γ over which integration is performed, we will sometimes write $(u,v)_{1,\Gamma}$ and $(u,v)_{0,\Gamma}$.

2.2. **Some properties.** The solenoidal spaces behave well with respect to smooth approximation.

Proposition 2.1 (See §1.4 of Temam's book [5])**.** *Let*

$$\mathcal{D}_\sigma(\Gamma) = \{\varphi \in C_c^\infty(\Gamma; \mathbb{R}^d) \mid \operatorname{div} \varphi = 0\}. \tag{2.11}$$

The following hold.

(1) $\mathcal{D}_\sigma(\Gamma)$ *is dense in* $L^2_\sigma(\Gamma)$ *with respect to the* L^2 *norm.*
(2) $\mathcal{D}_\sigma(\Gamma)$ *is dense in* $H^1_{0,\sigma}(\Gamma)$ *with respect to the* H^1 *norm.*

The space $\mathring{L}^2(\Gamma)$ is nice in the sense that it comes with a version of the trace theorem, even though the space provides very little control of the derivatives.

Proposition 2.2. *Let* $u \in L^2_\sigma(\Gamma)$ *and write* $\nu : \partial\Gamma \to \mathbb{S}^{d-1}$ *for the outward-pointing unit normal on* $\partial\Gamma$*. Then* $u \cdot \nu$ *is well-defined as an element of* $H^{-1/2}(\partial\Gamma) = (H^{1/2}(\Gamma))^*$*, and moreover* $u \cdot \nu = 0$*.*

Proof. Assume initially that u is smooth and satisfies $u, \operatorname{div} u \in L^2$. Let $f \in H^{1/2}(\Gamma)$ and let $\tilde{f} \in H^1(\Gamma)$ denote an extension of f such that $\left\|\tilde{f}\right\|_1 \lesssim \|f\|_{1/2}$. Then

$$\begin{aligned}\left|\langle u \cdot \nu, f\rangle_{-1/2}\right| = \left|\int_{\partial\Gamma} u \cdot \nu f\right| = \left|\int_\Gamma \operatorname{div}(\tilde{f} u)\right| = \left|\int_\Gamma \nabla \tilde{f} \cdot u + \tilde{f} \operatorname{div} u\right| \\ \leq \left\|\tilde{f}\right\|_1 (\|u\|_0 + \|\operatorname{div} u\|_0) \lesssim \|f\|_{1/2} (\|u\|_0 + \|\operatorname{div} u\|_0).\end{aligned} \tag{2.12}$$

This shows that the normal trace map $u \cdot \nu$ is well defined in $H^{-1/2}(\partial\Gamma)$ for every u in the closure of $C^\infty(\bar{\Gamma})$ with respect to the norm

$$\|u\|^2 = \int_\Gamma |u|^2 + |\operatorname{div} u|^2 . \tag{2.13}$$

In particular, Proposition 2.1 implies that $u \cdot \nu \in H^{-1/2}(\partial\Gamma)$. But since $u \in \mathcal{D}_\sigma(\Gamma)$ requires $\langle u \cdot \nu, f\rangle_{-1/2} = 0$ for all $f \in H^{1/2}(\partial\Gamma)$, we find that $u \cdot \nu = 0$.

□

3. Pressure as a Lagrange multiplier

Here we continue to let Γ be a set of the form described at the start of Section 2.

3.1. **A splitting of $H^1_0(\Gamma)$.** Our goal here is to orthogonally decompose $H^1_0(\Gamma)$ in a way that lets us understand the role of the pressure in the Stokes system. We begin by defining a special operator. Let $Q : \mathring{L}^2(\Gamma) \to H^1_0(\Gamma)$ be defined via

$$(p, \operatorname{div} u)_0 = (Qp, u)_1 \text{ for every } u \in H^1_0(\Gamma). \tag{3.1}$$

The operator Q is clearly linear, and it is bounded since

$$\|Qp\|_1^2 = (Qp, Qp)_1 = (p, \operatorname{div} Qp)_0 \leq \|p\|_0 \|\operatorname{div} Qp\|_0 \lesssim \|p\|_0 \|Qp\|_1 \tag{3.2}$$

implies that

$$\|Qp\|_1 \lesssim \|p\|_0 . \tag{3.3}$$

Our aim is to show that the range of Q splits $H^1_0(\Gamma)$ nicely. Before we can prove this we need a technical lemma.

Lemma 3.1. *Let* $p \in \mathring{L}^2(\Gamma)$*. Then there exists* $u \in H^1_0(\Gamma)$ *such that* $\operatorname{div} u = p$*, and*

$$\|u\|_1 \lesssim \|p\|_0 . \tag{3.4}$$

Proof. Let φ solve

$$\begin{cases} \Delta\varphi = p & \text{in } \Gamma \\ \partial_\nu \varphi = 0 & \text{on } \partial\Gamma. \end{cases} \tag{3.5}$$

A unique weak solution exists in $H^1(\Gamma)\cap \mathring{L}^2(\Gamma)$ since $p \in \mathring{L}^2(\Gamma)$ guarantees that the compatibility condition

$$\int_\Gamma p = 0 \tag{3.6}$$

is satisfied. The usual elliptic regularity guarantees that actually $u \in H^2(\Gamma)$, and we have the estimate

$$\|\varphi\|_2 \lesssim \|p\|_0 . \tag{3.7}$$

This H^2 estimate guarantees that $\nabla\varphi \times \nu \in H^{1/2}(\partial\Gamma)$, and so the usual trace theory in $H^2(\Gamma)$ allows us to find $v \in H^2(\Gamma)$ such that

$$\begin{cases} v = 0 & \text{on } \partial\Gamma \\ \partial_\nu v = -\nabla\varphi \times \nu & \text{on } \partial\Gamma. \end{cases} \tag{3.8}$$

Moreover, we may do so in a bounded way:

$$\|v\|_2 \lesssim \|\nabla\varphi \times \nu\|_{1/2} \lesssim \|\varphi\|_2 \lesssim \|p\|_0 . \tag{3.9}$$

Set $u = \nabla\varphi + \operatorname{curl} v$. By construction

$$\operatorname{div} u = \Delta\varphi = p \text{ in } \Gamma, \text{ and } u = 0 \text{ on } \partial\Gamma \tag{3.10}$$

(the latter equality is most easily seen by writing $\operatorname{curl} v$ in an orthogonal frame determined by ν and tangent vectors). We know that $u \in H^1(\Gamma)$ since $\varphi \in H^2(\Gamma)$ and $v \in H^2(\Gamma)$, and we have the estimate

$$\|u\|_1 \le \|\nabla\varphi\|_1 + \|\operatorname{curl} v\|_1 \lesssim \|\varphi\|_2 + \|v\|_2 \lesssim \|p\|_0 . \tag{3.11}$$

□

With this lemma in hand, we can study the range of Q in $H_0^1(\Gamma)$.

Proposition 3.2. *The following hold.*

(1) *We have the inequalities*

$$\|p\|_0 \lesssim \|Qp\|_1 \lesssim \|p\|_0 \tag{3.12}$$

for every $p \in \mathring{L}^2(\Gamma)$.

(2) *Let* $R(Q) = \{Qp \mid p \in \mathring{L}^2(\Gamma)\}$. *Then* $R(Q) \subseteq H_0^1(\Gamma)$ *is closed.*

(3) $Q : \mathring{L}^2(\Gamma) \to R(Q)$ *is an isomorphism.*

Proof. It suffices to prove the first item since the second and third follow easily form it. Let $p \in \mathring{L}^2(\Gamma)$. Set $u \in H_0^1(\Gamma)$ to be the function given by Lemma 3.1. Since $\operatorname{div} u = p$, we may estimate

$$\|p\|_0^2 = (p,p)_0 = (p, \operatorname{div} u)_0 = (Qp, u)_1 \le \|Qp\|_1 \|u\|_1 \lesssim \|Qp\|_1 \|p\|_0 , \tag{3.13}$$

where the last inequality follows from the estimate in Lemma 3.1. Since Q is a bounded operator, we deduce that

$$\|p\|_0 \lesssim \|Qp\|_1 \lesssim \|p\|_0 \text{ for all } p \in \mathring{L}^2(\Gamma). \tag{3.14}$$

□

Now we decompose $H_0^1(\Gamma)$.

Proposition 3.3. *We have the orthogonal decomposition*

$$H_0^1(\Gamma) = R(Q) \oplus H_{0,\sigma}^1(\Gamma). \tag{3.15}$$

Proof. Since $R(Q)$ is a closed subspace of $H_0^1(\Gamma)$, it suffices to prove that $R(Q)^\perp = H_{0,\sigma}^1(\Gamma)$.

Let $u \in H_{0,\sigma}^1(\Gamma)$. Then

$$0 = (p, \operatorname{div} u)_0 = (Qp, u)_1 \text{ for every } p \in \mathring{L}^2(\Gamma), \tag{3.16}$$

and so $u \in R(Q)^\perp$. Hence $H_{0,\sigma}^1(\Gamma) \subseteq R(Q)^\perp$.

On the other hand, if $u \in R(Q)^\perp$, then

$$(p, \operatorname{div} u)_0 = (Qp, u)_1 = 0 \text{ for all } p \in \mathring{L}^2(\Gamma), \tag{3.17}$$

and so $\operatorname{div} u = C$ for some constant $C \in \mathbb{R}$. Then

$$C\,|\Gamma| = \int_\Gamma C = \int_\Gamma \operatorname{div} u = \int_{\partial\Gamma} u \cdot \nu = 0 \Rightarrow C = 0, \tag{3.18}$$

and so we find that $u \in H_{0,\sigma}^1(\Gamma)$. Hence $R(Q)^\perp \subseteq H_{0,\sigma}^1(\Gamma)$.

□

3.2. **Finding the pressure.** With the orthogonal decomposition of Proposition 3.3 in hand, we can completely characterize the functionals that vanish on $H_{0,\sigma}^1(\Gamma)$.

Theorem 3.4. *Suppose that $\Lambda \in H^{-1}(\Gamma)$ satisfies $\Lambda(v) = 0$ for all $v \in H_{0,\sigma}^1(\Gamma)$. Then there exists a unique $p \in \mathring{L}^2(\Gamma)$ such that $\Lambda(v) = (p, \operatorname{div} v)_0$ for all $v \in H_0^1(\Gamma)$. Also,*

$$\|p\|_0 \lesssim \|\Lambda\|_{H^{-1}(\Gamma)}\,. \tag{3.19}$$

Proof. According to the Riesz representation theorem, there exists $u \in H_0^1(\Gamma)$ such that $\Lambda(v) = (u, v)_1$ for all $v \in H_0^1(\Gamma)$. The decomposition given by Proposition 3.3 guarantees that $u \in R(Q)$. Then $u = Qp$ for some unique $p \in \mathring{L}^2(\Gamma)$, and

$$\Lambda(v) = (u, v)_1 = (Qp, v)_1 = (p, \operatorname{div} v)_0 \text{ for all } v \in H_0^1(\Gamma). \tag{3.20}$$

Let $v \in H_0^1(\Gamma)$ be given by Lemma 3.1. Then

$$\|p\|_0^2 = (p, \operatorname{div} v)_0 = \Lambda(v) \le \|\Lambda\|_{H^{-1}(\Gamma)} \|v\|_1 \lesssim \|\Lambda\|_{H^{-1}(\Gamma)} \|p\|_0\,, \tag{3.21}$$

which implies (3.19). □

Remark 3.5. *It is often said that this theorem shows that the pressure in the Stokes problem arises as a Lagrange multiplier. This will be justified in the next section (see Remark 4.4).*

4. Solving the Stokes problem

Our goal now is to produce weak solutions to the Stokes problem (1.2) in domains Γ of the form considered above. Let's assume for the moment that we have a smooth solution pair (u, p) to (1.2) for some smooth f. Taking the dot-product of the first equation with $v \in H_0^1(\Gamma)$ and integrating over Γ, we find that

$$\begin{aligned}\int_\Gamma f \cdot v = \int_\Gamma (-\Delta u + \nabla p) \cdot v = \int_\Gamma \nabla u : \nabla v - p \operatorname{div} v + \int_{\partial\Gamma} -v \cdot \partial_\nu u + pv \cdot \nu \\ = \int_\Gamma \nabla u : \nabla v - p \operatorname{div} v.\end{aligned} \tag{4.1}$$

We may rewrite this in a manner suitable for defining weak solutions:

$$(u, v)_1 - (p, \operatorname{div} v)_0 = \langle f, v \rangle_{-1}\,, \tag{4.2}$$

where $\langle \cdot, \cdot \rangle_{-1}$ denotes the dual pairing between $H_0^1(\Gamma)$ and $H^{-1}(\Gamma)$. The equality (4.2) seems like a good starting point for defining a notion of weak solution. We also want to enforce the incompressibility condition $\operatorname{div} u = 0$, so it's natural to require $u \in H_{0,\sigma}^1(\Gamma)$. This leads us to a definition.

Definition 4.1. *Let $f \in H^{-1}(\Gamma)$. A weak solution to the Stokes problem* (1.2) *is a pair* (u, p) *with*

$$u \in H^1_{0,\sigma}(\Gamma) \text{ and } p \in \mathring{L}^2(\Gamma) \tag{4.3}$$

such that

$$(u, v)_1 - (p, \operatorname{div} v)_0 = \langle f, v \rangle_{-1} \text{ for every } v \in H^1_0(\Gamma). \tag{4.4}$$

Remark 4.2. *The inclusion $p \in \mathring{L}^2(\Gamma)$ is natural since* (1.2) *only requires that p is determined up to a constant. As such, we might as well require that $\int_\Gamma p = 0$ in order to choose that constant.*

The difficulty with the weak formulation is evident when we realize that we have to find u and p simultaneously, and the usual weak solution machinery (namely Lax-Milgram), produces only a single function. A resolution to this problem comes when we think about how we would get energy estimates: we use u as a test function and notice that

$$\|u\|_1^2 = (u, u)_1 = (u, u)_1 - (p, \operatorname{div} u)_0 = \langle f, u \rangle_{-1}. \tag{4.5}$$

The key to this equality is that if we use $v \in H^1_{0,\sigma}(\Gamma)$ as a test function, then the pressure term completely disappears. That is, if (4.2) holds, then

$$(u, v)_1 = \langle f, v \rangle_{-1} \text{ for every } v \in H^1_{0,\sigma}(\Gamma). \tag{4.6}$$

Since $u \in H^1_{0,\sigma}(\Gamma)$ as well, this looks a lot more amenable to an application of Lax-Milgram (or Riesz or the direct method in the calculus of variations). The big question is: can we work directly with (4.6) and then somehow recover the pressure to produce a weak solution? The answer is yes, and the reason is entirely due to Theorem 3.4.

Theorem 4.3. *Let $f \in H^{-1}(\Gamma)$. Then there exists a unique weak solution (u, p) to the Stokes problem* (1.2). *Moreover,*

$$\|u\|_1 + \|p\|_0 \lesssim \|f\|_{H^{-1}(\Gamma)}. \tag{4.7}$$

Proof. Since $H^1_{0,\sigma}(\Gamma) \subset H^1_0(\Gamma)$, we have the natural inclusion

$$H^{-1}(\Gamma) \subset (H^1_{0,\sigma}(\Gamma))^*. \tag{4.8}$$

As such, f defines a bounded linear function on $H^1_{0,\sigma}(\Gamma)$. The Riesz representation theorem, applied to the Hilbert space $H^1_{0,\sigma}(\Gamma)$, guarantees the existence of a unique $u \in H^1_{0,\sigma}(\Gamma)$ such that

$$(u, v)_1 = \langle f, v \rangle_{-1} \text{ for every } v \in H^1_{0,\sigma}(\Gamma). \tag{4.9}$$

Choosing $v = u$, we arrive at the estimate

$$\|u\|_1^2 = (u, u)_1 = \langle f, u \rangle_{-1} \le \|f\|_{H^{-1}(\Gamma)} \|u\|_1, \tag{4.10}$$

and hence

$$\|u\|_1 \le \|f\|_{H^{-1}(\Gamma)}. \tag{4.11}$$

Notice here that we don't have equality due to the fact that $H^{-1}(\Gamma)$ is smaller than the dual of $H^1_{0,\sigma}(\Gamma)$.

Now define $\Lambda \in H^{-1}(\Gamma)$ via

$$\Lambda(v) = (u, v)_1 - \langle f, v \rangle_{-1} \text{ for all } v \in H^1_0(\Gamma). \tag{4.12}$$

We know that Λ is bounded since

$$|\Lambda(v)| \le \|u\|_1 + \|f\|_{H^{-1}(\Gamma)} \lesssim \|f\|_{H^{-1}(\Gamma)} \tag{4.13}$$

whenever $v \in H^1_0(\Gamma)$ satisfies $\|v\|_1 = 1$, and hence

$$\|\Lambda\|_{H^{-1}(\Gamma)} \lesssim \|f\|_{H^{-1}(\Gamma)}. \tag{4.14}$$

According to (4.9), $\Lambda(v) = 0$ for all $v \in H^1_{0,\sigma}(\Gamma)$. Theorem 3.4 then provides us with the existence of a unique $p \in \overset{\circ}{L}{}^2(\Gamma)$ such that

$$(p, \operatorname{div} v) = \Lambda(v) = (u, v)_1 - \langle f, v\rangle_{-1} \text{ for all } v \in H^1_0(\Gamma) \tag{4.15}$$

and

$$\|p\|_0 \lesssim \|\Lambda\|_{H^{-1}(\Gamma)} \lesssim \|f\|_{H^{-1}(\Gamma)} . \tag{4.16}$$

We have now found a pair $(u, p) \in H^1_{0,\sigma}(\Gamma) \times \overset{\circ}{L}{}^2(\Gamma)$ such that

$$(u, v)_1 - (p, \operatorname{div} v)_0 = \langle f, v\rangle_{-1} \text{ for every } v \in H^1_0(\Gamma). \tag{4.17}$$

This constitutes a weak solution to the Stokes system (1.2). Combining (4.11) and (4.16) yields the estimate

$$\|u\|_1 + \|p\|_0 \lesssim \|f\|_{H^{-1}(\Gamma)} . \tag{4.18}$$

It remains to prove uniqueness. If (u_1, p_1) and (u_2, p_2) are both weak solutions, then $u = u_1 - u_2$ and $p = p_1 - p_2$ satisfy

$$(u, v)_1 - (p, \operatorname{div} v)_0 = 0 \text{ for every } v \in H^1_0(\Gamma). \tag{4.19}$$

Choosing $v = u \in\in H^1_{0,\sigma}(\Gamma)$ leads us to the equality

$$\|u\|_1^2 = (u, u)_1 = (u, u)_1 - (p, \operatorname{div} u)_0 = 0, \tag{4.20}$$

and hence $u = 0$. Choosing $v \in H^1_0(\Gamma)$ such that $\operatorname{div} v = p$ as in Lemma 3.1 shows that

$$\|p\|_0^2 = (p, p) = -(u, v)_1 + (p, \operatorname{div} v)_0 = 0, \tag{4.21}$$

and hence $p = 0$.

□

Remark 4.4. *As we mentioned above, it is often said that the pressure in the Stokes problem is determined as a Lagrange multiplier. This would have been more evident if we had employed the direct method to produce $u \in H^1_{0,\sigma}(\Gamma)$ in the first step of the proof. Indeed, we could have found u via a minimization argument:*

$$E(u) = \min_{v\in H^1_{0,\sigma}(\Gamma)} E(v), \text{ where } E(v) = \frac{1}{2}\|v\|_1^2 - \langle f, v\rangle_{-1} . \tag{4.22}$$

We can view this as a constrained minimization on $H^1_0(\Gamma)$, where the constraint is the solenoidal condition $\operatorname{div} u = 0$*. Constrained minimization gives rise to Lagrange multipliers, and so the equation*

$$(u, v)_1 - \langle f, v\rangle_{-1} = (p, \operatorname{div} v)_0 \text{ for every } v \in H^1_0(\Gamma) \tag{4.23}$$

tells us that the pressure term on the right is the Lagrange multiplier corresponding to the divergence-free constraint.

5. Technical preliminaries for Darcy's law

Here we develop the some of the main technical results from Tartar's paper [4].

5.1. The Poincaré inequality in Ω_ε.

The first result specifies the dependence of the constant in Poincaré's inequality on ε.

Lemma 5.1. *Let Ω_ε be given by (1.8). There exists a constant $C \geq 0$, independent of ε, such that*

$$\int_{\Omega_\varepsilon} |u|^2 \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u|^2 \tag{5.1}$$

for every $u \in H^1_0(\Omega_\varepsilon)$.

Proof. Notice first that, by construction,

$$K^\varepsilon := \bigcup_{m\in\mathcal{I}_\varepsilon(\Omega)} Y^\varepsilon_{m;f} \subset \Omega_\varepsilon. \tag{5.2}$$

The usual Poincaré inequality yields a constant $C_0 > 0$ such that

$$\int_{Y^1_{0;f}} |u|^2 \le C_0 \int_{Y^1_{0;f}} |\nabla u|^2 \tag{5.3}$$

for every $u \in H^1(Y^1_{0;f})$ such that $u = 0$ on the solid boundary, $\partial Y^1_{0;s}$. Upon making the change of variables $x \mapsto \varepsilon(x+m)$, we find that

$$\int_{Y^\varepsilon_{m;f}} |u|^2 \le C_0\varepsilon^2 \int_{Y^\varepsilon_{m;f}} |\nabla u|^2 \tag{5.4}$$

for every $u \in H^1(Y^\varepsilon_{m;f})$ such that $u = 0$ on the solid boundary, $\partial Y^\varepsilon_{m;s}$.

On the other hand, if we write $S^\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \sqrt{d}\varepsilon\}$, then we have the inclusion

$$\Omega\backslash\bar{K}^\varepsilon \subset S^\varepsilon. \tag{5.5}$$

By flattening the boundary and employing the fundamental theorem of calculus and an approximation argument, we can prove a variant of the Poincaré inequality in S^ε: there exists a constant $C_1 > 0$, independent of ε, such that

$$\int_{S^\varepsilon} |u|^2 \le C_1\varepsilon^2 \int_{S^\varepsilon} |\nabla u|^2 \tag{5.6}$$

for every $u \in H^1(S^\varepsilon)$ such that $u = 0$ on $\partial\Omega$.

Suppose now that $u \in H^1_0(\Omega_\varepsilon)$. Then $u = 0$ on $\partial Y^\varepsilon_{m;s}$ for every $m \in \mathcal{I}_\varepsilon(\Omega)$, and so we can sum (5.4) over $m \in \mathcal{I}_\varepsilon(\Omega)$ to find that

$$\int_{K^\varepsilon} |u|^2 \le C_0\varepsilon^2 \int_{K^\varepsilon} |\nabla u|^2. \tag{5.7}$$

The restriction of u to $S^\varepsilon \cap \Omega_\varepsilon$ vanishes on $\partial\Omega$ and on $\partial Y^\varepsilon_{m;s}$ for each $m \in \mathcal{I}_\varepsilon(\Omega)$ such that $Y^\varepsilon_{m;s} \cap S^\varepsilon \neq 0$. As such, we may extend u by 0 in each such $Y^\varepsilon_{m;s}$ to view it as element of $H^1(S^\varepsilon)$ such that $u = 0$ on $\partial\Omega$. Applying (5.6) then gives the estimate

$$\int_{S^\varepsilon\cap\Omega_\varepsilon} |u|^2 = \int_{S^\varepsilon} |u|^2 \le C_1\varepsilon^2 \int_{S^\varepsilon} |\nabla u|^2 = C_1\varepsilon^2 \int_{S^\varepsilon\cap\Omega_\varepsilon} |\nabla u|^2. \tag{5.8}$$

We sum (5.7) and (5.8) to arrive at the estimate

$$\int_{\Omega_\varepsilon} |u|^2 = \int_{K^\varepsilon\cup(S^\varepsilon\cap\Omega_\varepsilon)} |u|^2 \le C_0\varepsilon^2 \int_{K^\varepsilon} |\nabla u|^2 + C_1\varepsilon^2 \int_{S^\varepsilon\cap\Omega_\varepsilon} |\nabla u|^2 \le (C_0 + C_1)\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u|^2. \tag{5.9}$$

Here we have used the fact that $\Omega_\varepsilon \subseteq K^\varepsilon \cup (S^\varepsilon \cap \Omega_\varepsilon)$. This is the inequality (5.1). □

5.2. **Extension and restriction operators.** Our goal in this section is to construct a restriction operator $\mathcal{R}_\varepsilon : H^1_0(\Omega) \to H^1_0(\Omega_\varepsilon)$ and an extension operator $\mathcal{E}_\varepsilon : \mathring{L}^2(\Omega_\varepsilon) \to \mathring{L}^2(\Omega)$. The latter is essential in Tartar's derivation of Darcy's law since it allows us to extend the pressure from the Stokes problem in Ω_ε to all of Ω.

We begin with a technical construction from [4].

Lemma 5.2. *Suppose that $\Sigma \subset Y^1_0$ is a smooth surface that encloses $Y^1_{0;s} \subset Y^1_0$ such that $\partial Y^1_{0;s} \cap \Sigma = \varnothing$. Write Y_M for the open set contained between $\partial Y^1_{0;s}$ and Σ. Then for each $u \in H^1(Y^1_0)$ there exist a unique pair (v, q) with $v \in H^1(Y_M)$ and $q \in \mathring{L}^2(Y_M)$ such that*

$$-\Delta v + \nabla q = -\Delta u \text{ in } Y_M \tag{5.10}$$

in the weak sense:

$$(v,w)_1 - (q, \operatorname{div} w)_0 = (u,w)_1 \ \textit{ for all } w \in H^1_0(Y_M). \tag{5.11}$$

Additionally,

$$\operatorname{div} v = \operatorname{div} u + \frac{1}{|Y_M|}\int_{Y^1_{0;s}} \operatorname{div} u \ \textit{ in } Y_M \tag{5.12}$$

and

$$v = u \ \textit{ on } \Sigma \ \textit{ and } v = 0 \ \textit{ on } \partial Y^1_{0;s}. \tag{5.13}$$

Moreover,

$$\|v\|_{H^1(Y_M)} \lesssim \|u\|_{H^1(Y^1_0)}\,. \tag{5.14}$$

Proof. We will exploit linearity to construct v as a sum

$$v = \Phi + \Psi + \Xi. \tag{5.15}$$

We first use the usual trace theory to choose $\Phi \in H^1(Y_M)$ such that

$$\Phi = u \text{ on } \Sigma \text{ and } \Phi = 0 \text{ on } \partial Y^1_{0;s} \tag{5.16}$$

with

$$\|\Phi\|_1 \lesssim \|u\|_{H^{1/2}(\partial Y^1_{0;s})} \lesssim \|u\|_1\,. \tag{5.17}$$

Next we notice that (writing ν for the normal vector pointing out of Y_M)

$$\begin{aligned}
&\int_{Y_M}\left(-\operatorname{div}\Phi + \operatorname{div} u + \frac{1}{|Y_M|}\int_{Y^1_{0;s}} \operatorname{div} u\right)\\
&= \int_{\partial Y^1_{0;s}} (-\Phi\cdot\nu + u\cdot\nu) + \int_{\Sigma}(-\Phi\cdot\nu + u\cdot\nu) + \int_{Y^1_{0;s}} \operatorname{div} u\\
&= \int_{\partial Y^1_{0;s}} u\cdot\nu + \int_\Sigma 0 + \int_{\partial Y^1_{0;s}} -u\cdot\nu = 0.
\end{aligned} \tag{5.18}$$

Here the negative sign appears in the third term of the third inequality because $-\nu$ is the vector pointing out of $Y^1_{0;s}$. This means that

$$-\operatorname{div}\Phi + \operatorname{div} u + \frac{1}{|Y_M|}\int_{Y^1_{0;s}} \operatorname{div} u \in \mathring{L}^2(Y_m). \tag{5.19}$$

This allows us to use Lemma 3.1 in order to find $\Psi \in H^1_0(Y_M)$ satisfying

$$\operatorname{div}\Psi = -\operatorname{div}\Phi + \operatorname{div} u + \frac{1}{|Y_M|}\int_{Y^1_{0;s}} \operatorname{div} u \text{ in } Y_M \tag{5.20}$$

and

$$\|\Psi\|_1 \lesssim \|\Phi\|_1 + \|u\|_1 \lesssim \|u\|_1\,. \tag{5.21}$$

Finally, we use $f = -\Delta(u - \Phi - \Psi) \in H^{-1}(Y_M)$ in Theorem 4.3 to find $\Xi \in H^1_{0,\sigma}(Y_M)$ and $q \in \mathring{L}^2(Y_M)$ such that

$$(\Xi, w)_1 - (q, \operatorname{div} w)_0 = (u - \Phi - \Psi, w)_1 \ \text{ for all } w \in H^1_0(Y_M) \tag{5.22}$$

and

$$\|\Xi\|_1 + \|q\|_0 \lesssim \|-\Delta(u-\Phi-\Psi)\|_{-1} \lesssim \|u\|_1 + \|\Phi\|_1 + \|\Psi\|_1 \lesssim \|u\|_1\,. \tag{5.23}$$

Then $v = \Phi + \Psi + \Xi$ satisfies

$$(v,w)_1 - (q, \operatorname{div} w)_0 = (u,w)_1 \ \text{ for all } w \in H^1_0(Y_M), \tag{5.24}$$

$$\operatorname{div} v = \operatorname{div}(\Phi + \Psi + \Xi) = \operatorname{div} u + \frac{1}{|Y_M|}\int_{Y^1_{0;s}} \operatorname{div} u \text{ in } Y_M, \tag{5.25}$$

and

$$v = \Phi + \Psi + \Xi = u + 0 + 0 = u \text{ on } \Sigma \text{ and } v = 0 \text{ on } \partial Y^1_{0;s} \tag{5.26}$$

in addition to the estimate

$$\|v\|_1 \leq \|\Phi\|_1 + \|\Psi\|_1 + \|\Xi\|_1 \lesssim \|u\|_1 . \tag{5.27}$$

It remains only to prove uniqueness. This can be done as in the uniqueness proof from Theorem 4.3. We leave the details as an exercise.

□

This construction allows us to define a special restriction operator.

Theorem 5.3. *There exists a restriction operator $\mathcal{R}_\varepsilon : H_0^1(\Omega) \to H_0^1(\Omega_\varepsilon)$ satisfying the following properties.*

(1) *We have the estimates*

$$\|\mathcal{R}_\varepsilon w\|_{L^2(\Omega_\varepsilon)} \lesssim \|w\|_{L^2(\Omega)} + \varepsilon \|\nabla w\|_{L^2(\Omega)} \tag{5.28}$$

and

$$\|\nabla \mathcal{R}_\varepsilon w\|_{L^2(\Omega_\varepsilon)} \lesssim \frac{1}{\varepsilon} \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \tag{5.29}$$

for all $w \in H_0^1(\Omega)$.

(2) *If $w \in H_0^1(\Omega_\varepsilon)$ is extended by zero to $\tilde{w} \in H_0^1(\Omega)$, then $\mathcal{R}_\varepsilon \tilde{w} = w$.*

(3) *If $w \in H_{0,\sigma}^1(\Omega)$, then $\mathcal{R}_\varepsilon w \in H_{0,\sigma}^1(\Omega_\varepsilon)$.*

Proof. We again write

$$K^\varepsilon := \bigcup_{m \in \mathcal{I}_\varepsilon(\Omega)} Y_{m;f}^\varepsilon \subset \Omega_\varepsilon. \tag{5.30}$$

We also write

$$Z^\varepsilon := \bigcup_{m \in \mathcal{I}_\varepsilon(\Omega)} Y_m^\varepsilon \subset \Omega_\varepsilon \tag{5.31}$$

for the union of all the $\varepsilon-$cells contained in Ω.

Let Σ and Y_M be as in Lemma 5.2. Given a function $u \in H^1(Y_0^1)$ we define $\mathcal{R}u \in H^1(Y_0^1)$ via

$$\mathcal{R}u(x) = \begin{cases} 0 & \text{if } x \in Y_{0;s}^1 \\ v(x) & \text{if } x \in Y_M \\ u(x) & \text{if } x \in Y_0^1 \backslash (Y_M \cup Y_{0;s}^1), \end{cases} \tag{5.32}$$

where v is the function constructed in Lemma 5.2. The lemma guarantees that

$$\|\mathcal{R}u\|_{H^1(Y_0^1)} \lesssim \|u\|_{H^1(Y_0^1)} . \tag{5.33}$$

Clearly $\mathcal{R}u \in H^1(Y_{0;f}^1)$, $\mathcal{R}u = 0$ on $\partial Y_{0;s}^1$, and

$$\|\mathcal{R}u\|_{H^1(Y_{0;f}^1)} \lesssim \|u\|_{H^1(Y_0^1)} . \tag{5.34}$$

Consider the mapping

$$\phi_m^\varepsilon : Y_0^1 \to Y_m^\varepsilon \text{ given by } \phi_m^\varepsilon(y) = \varepsilon(y + m). \tag{5.35}$$

By employing the change of variables $x = \phi_m^\varepsilon(y)$, we find that that for any $w \in H^1(Y_m^\varepsilon)$,

$$\varepsilon^{-d} \int_{Y_m^\varepsilon} |w|^2 \, dx = \int_{Y_0^1} |w \circ \phi_m^\varepsilon|^2 \, dy \tag{5.36}$$

and

$$\varepsilon^{2-d} \int_{Y_m^\varepsilon} |\nabla w|^2 \, dx = \int_{Y_0^1} |\nabla (w \circ \phi_m^\varepsilon)|^2 \, dy. \tag{5.37}$$

Then (5.34) implies that

$$\|\mathcal{R}(w \circ \phi_m^\varepsilon)\|_{H^1(Y_{0;f}^1)}^2 \lesssim \|w \circ \phi_m^\varepsilon\|_{H^1(Y_0^1)}^2 \lesssim \varepsilon^{-d} \|w\|_{L^2(Y_m^\varepsilon)}^2 + \varepsilon^{2-d} \|\nabla w\|_{L^2(Y_m^\varepsilon)}^2 . \tag{5.38}$$

On the other hand, for $g \in H^1(Y_{0;f}^1)$ the change of variables $y = (\phi_m^\varepsilon)^{-1}(x)$ shows that

$$\int_{Y_{m;f}^\varepsilon} \left|g \circ (\phi_m^\varepsilon)^{-1}\right|^2 dx = \varepsilon^d \int_{Y_{0;f}^1} |g|^2 \, dy \tag{5.39}$$

and

$$\int_{Y^\varepsilon_{m;f}} \left|\nabla(g\circ(\phi^\varepsilon_m)^{-1})\right|^2 dx = \varepsilon^{d-2}\int_{Y^1_{0;f}} |\nabla g|^2\, dy. \tag{5.40}$$

Hence (5.38) implies that

$$\left\|\mathcal{R}(w\circ\phi^\varepsilon_m)\circ(\phi^\varepsilon_m)^{-1}\right\|^2_{L^2(Y^\varepsilon_{m;f})} = \varepsilon^d \left\|\mathcal{R}(w\circ\phi^\varepsilon_m)\right\|^2_{L^2(Y^1_{0;f})} \lesssim \|w\|^2_{L^2(Y^\varepsilon_m)} + \varepsilon^2 \|\nabla w\|^2_{L^2(Y^\varepsilon_m)} \tag{5.41}$$

and

$$\begin{aligned}\left\|\nabla(\mathcal{R}(w\circ\phi^\varepsilon_m)\circ(\phi^\varepsilon_m)^{-1})\right\|^2_{L^2(Y^\varepsilon_{m;f})} &= \varepsilon^{d-2}\left\|\nabla(\mathcal{R}(w\circ\phi^\varepsilon_m))\right\|^2_{L^2(Y^1_{0;f})}\\ &\lesssim \varepsilon^{-2}\|w\|^2_{L^2(Y^\varepsilon_m)} + \|\nabla w\|^2_{L^2(Y^\varepsilon_m)}.\end{aligned} \tag{5.42}$$

We now define $\mathcal{R}_\varepsilon w$ for any $w \in H^1_0(\Omega)$ according to

$$\mathcal{R}_\varepsilon w(x) = \begin{cases} w(x) & \text{if } x\in\Omega_\varepsilon\backslash Z^\varepsilon \\ \mathcal{R}(w\circ\phi^\varepsilon_m)\circ(\phi^\varepsilon_m)^{-1}(x) & \text{if } x\in Y^\varepsilon_{m;f} \text{ for } m\in\mathcal{I}_\varepsilon(\Omega).\end{cases} \tag{5.43}$$

Upon summing (5.41) over $m\in\mathcal{I}_\varepsilon(\Omega)$, we find that

$$\|\mathcal{R}_\varepsilon w\|^2_{L^2(K^\varepsilon)} \lesssim \|w\|^2_{L^2(Z^\varepsilon)} + \varepsilon^2\|\nabla w\|^2_{L^2(Z^\varepsilon)}. \tag{5.44}$$

Similarly, we may sum (5.42) to find that

$$\|\nabla\mathcal{R}_\varepsilon w\|^2_{L^2(K^\varepsilon)} \lesssim \varepsilon^{-2}\|w\|^2_{L^2(Z^\varepsilon)} + \|\nabla w\|^2_{L^2(Z^\varepsilon)}. \tag{5.45}$$

Since

$$\|\mathcal{R}_\varepsilon w\|^2_{L^2(\Omega_\varepsilon\backslash Z^\varepsilon)} = \|w\|^2_{L^2(\Omega_\varepsilon\backslash Z^\varepsilon)} \text{ and } \|\nabla\mathcal{R}_\varepsilon w\|^2_{L^2(\Omega_\varepsilon\backslash Z^\varepsilon)} = \|\nabla w\|^2_{L^2(\Omega_\varepsilon\backslash Z^\varepsilon)}, \tag{5.46}$$

$\Omega_\varepsilon = K^\varepsilon \cup (\Omega_\varepsilon\backslash K^\varepsilon)$, and $\varepsilon<1$, we deduce from (5.44) that

$$\|\mathcal{R}_\varepsilon w\|^2_{L^2(\Omega_\varepsilon)} \lesssim \|w\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla w\|^2_{L^2(\Omega)}, \tag{5.47}$$

while we deduce from (5.45) that

$$\|\nabla\mathcal{R}_\varepsilon w\|^2_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{-2}\|w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)}. \tag{5.48}$$

The estimates (5.28) and (5.29) follow easily from (5.47) and (5.48), and the fact that $w=0$ on $\partial\Omega_\varepsilon$ follows from the fact that $w=0$ on $\partial\Omega$ and $\mathcal{R}_\varepsilon w = 0$ on $\partial Y^\varepsilon_{m;s}$ for each $m\in\mathcal{I}_\varepsilon(\Omega)$. This completes the proof of the first item.

To prove the second item we first note that if $u \in H^1(Y^1_{0;f})$ vanishes on $\partial Y^1_{0;s}$, then its extension by zero $\tilde{u}$ belongs to $H^1(Y^1_0)$. Then the pair (v,q) produced by Lemma 5.2 is in fact $(u,0)$, as can easily be checked. Then $\mathcal{R}u = u$. Accordingly, whenever $w\in H^1_0(\Omega_\varepsilon)$ is extended by zero to $\tilde{w}\in H^1_0(\Omega)$, the construction of $\mathcal{R}_\varepsilon\tilde{w}$ guarantees that $\mathcal{R}_\varepsilon\tilde{w} = w$ in each cell $Y^\varepsilon_{m;f}$, and hence $\mathcal{R}_\varepsilon\tilde{w} = w$ in Ω_ε.

The third item follows from a similar argument. If $u\in H^1(Y^1_0)$ satisfies $\operatorname{div} u = 0$ in Y^1_0, then the function v produced by Lemma 5.2 also satisfies $\operatorname{div} v = 0$. This in turn guarantees that $\operatorname{div}\mathcal{R}_\varepsilon w = 0$ in Ω_ε whenever $w\in H^1_0(\Omega)$ satisfies $\operatorname{div} w = 0$ in Ω.

□

The main point of the restriction operator $\mathcal{R}_\varepsilon : H^1_0(\Omega)\to H^1_0(\Omega_\varepsilon)$ is that it induces an extension operator for the pressure.

Proposition 5.4. *There exists an extension operator $\mathcal{E}_\varepsilon : \mathring{L}^2(\Omega_\varepsilon)\to\mathring{L}^2(\Omega)$ such that*

$$(\mathcal{E}_\varepsilon p, \operatorname{div} v)_{0,\Omega} = \int_\Omega \mathcal{E}_\varepsilon p \operatorname{div} v = \int_{\Omega_\varepsilon} p \operatorname{div}\mathcal{R}_\varepsilon v = (p, \operatorname{div}\mathcal{R}_\varepsilon v)_{0,\Omega_\varepsilon} \text{ for all } v\in H^1_0(\Omega). \tag{5.49}$$

Moreover,

$$\|\mathcal{E}_\varepsilon p\|_{L^2(\Omega)} \lesssim \frac{1}{\varepsilon}\|p\|_{L^2(\Omega_\varepsilon)} \text{ for every } p\in\mathring{L}^2(\Omega_\varepsilon), \tag{5.50}$$

and

$$\mathcal{E}_\varepsilon p|_{\Omega_\varepsilon} = p + C_\varepsilon \tag{5.51}$$

for a constant $C_\varepsilon \in \mathbb{R}$ that depends on ε.

Proof. For each $p \in \mathring{L}^2(\Omega_\varepsilon)$ we set $\Lambda_p(v) = (p, \operatorname{div} \mathcal{R}_\varepsilon v)_{0,\Omega_\varepsilon}$ for $v \in H_0^1(\Omega)$. Then the first item of Theorem 5.3 allows us to estimate

$$|\Lambda_p(v)| \leq \|p\|_0 \|\operatorname{div} \mathcal{R}_\varepsilon v\|_0 \lesssim \|p\|_0 \|\nabla \mathcal{R}_\varepsilon v\|_0 \lesssim \|p\|_0 \left(\frac{1}{\varepsilon} \|v\|_0 + \|\nabla v\|_0 \right), \tag{5.52}$$

and hence $\Lambda_p \in H^{-1}(\Omega)$. The third item of Theorem 5.3 guarantees that if $v \in H^1_{0,\sigma}(\Omega)$, then $\mathcal{R}_\varepsilon v \in H^1_{0,\sigma}(\Omega_\varepsilon)$, and hence

$$\Lambda_p(v) = 0 \text{ for all } v \in H^1_{0,\sigma}(\Omega). \tag{5.53}$$

Theorem 3.4 then provides us with a unique $\mathcal{E}_\varepsilon p \in \mathring{L}^2(\Omega)$ such that

$$(p, \operatorname{div} \mathcal{R}_\varepsilon v)_{0,\Omega_\varepsilon} = \Lambda_p(v) = (\mathcal{E}_\varepsilon p, \operatorname{div} v)_{0,\Omega} \text{ for all } v \in H_0^1(\Omega) \tag{5.54}$$

and

$$\|\mathcal{E}_\varepsilon p\|_{L^2(\Omega)} \lesssim \|\Lambda_p\|_{H^{-1}(\Omega)} \lesssim \frac{1}{\varepsilon} \|p\|_{L^2(\Omega_\varepsilon)}. \tag{5.55}$$

It remains only to prove that $\mathcal{E}_\varepsilon p|_{\Omega_\varepsilon}$ and p differ by a constant. For this we use the second item of Theorem 5.3 on an element $w \in H_0^1(\Omega_\varepsilon)$ extended by 0 to an element $\tilde{w} \in H_0^1(\Omega)$. We find that

$$\int_{\Omega_\varepsilon} p \operatorname{div} w = \int_{\Omega_\varepsilon} p \operatorname{div} \mathcal{R}_\varepsilon \tilde{w} = \int_{\Omega} \mathcal{E}_\varepsilon p \operatorname{div} \tilde{w} = \int_{\Omega_\varepsilon} \mathcal{E}_\varepsilon p \operatorname{div} w, \tag{5.56}$$

and hence

$$\int_{\Omega_\varepsilon} (p - \mathcal{E}_\varepsilon p) \operatorname{div} w \text{ for all } w \in H_0^1(\Omega_\varepsilon). \tag{5.57}$$

According to Lemma 3.1, for any $q \in \mathring{L}^2(\Omega_\varepsilon)$ we may find $w \in H_0^1(\Omega_\varepsilon)$ such that $\operatorname{div} w = q$. Using this above shows that

$$\int_{\Omega_\varepsilon} (p - \mathcal{E}_\varepsilon p) q = 0 \text{ for all } q \in \mathring{L}^2(\Omega_\varepsilon), \tag{5.58}$$

and so

$$p - \mathcal{E}_\varepsilon = C_\varepsilon \text{ in } \Omega_\varepsilon \text{ for some constant } C_\varepsilon \in \mathbb{R}. \tag{5.59}$$

□

6. Deriving Darcy's law

6.1. **The cell problem and the permeability tensor.** The cell problem seeks solutions $(v_k, q_k) \in H^1_{0,\sigma}(Y_f) \times \mathring{L}^2(Y_f)$ for $k = 1, 2, 3$ to the Stokes problem

$$\begin{cases} -\Delta v_k + \nabla q_k = e_k & \text{in } Y_f \\ \operatorname{div} v_k = 0 & \text{in } Y_f \\ v_k = 0 & \text{on } \partial Y_s \end{cases} \tag{6.1}$$

where e_k is the unit vector in the k^{th} direction. Unique solutions exist for $k = 1, 2, 3$ by virtue of Theorem 4.3, and the solutions (v_k, q_k) inherit the periodicity of Y.

Definition 6.1. *We define the constant permeability tensor $K \in \mathbb{R}^{d \times d}$ via*

$$K_{ij} = \fint_{Y_f} v_j \cdot e_i = \frac{1}{|Y|} \int_{Y_f} v_j \cdot e_i. \tag{6.2}$$

Our next result shows that the permeability tensor is symmetric and positive definite, which is essential, as it will appear as the coefficient matrix in an elliptic problem later.

Proposition 6.2. *The permeability tensor $K \in \mathbb{R}^{d \times d}$ is symmetric and positive definite.*

Proof. For $i, j = 1, \dots, d$ we may use v_j as a test function in the weak formulation of the equation for (v_i, q_i) to see that

$$K_{ij} = \int_{Y_f} v_j \cdot e_i = (v_i, v_j)_1 - (p_i, \operatorname{div} v_j)_0 = (v_i, v_j)_1 = (v_j, v_i)_1$$
$$= (v_j, v_i)_1 - (p_j, \operatorname{div} v_i)_0 = \int_{Y_f} v_i \cdot e_j = K_{ji}. \quad (6.3)$$

Hence K is symmetric.

Let $\xi \in \mathbb{R}^d$. Then, writing ξ^i for the components of ξ, we find that

$$K\xi \cdot \xi = \sum_{i=1}^{d} \sum_{j=1}^{d} K_{ij} \xi^i \xi^j = \sum_{i=1}^{d} \sum_{j=1}^{d} (v_j, v_i)_1 \xi^i \xi^j = \left\| \sum_{i=1}^{d} \xi^i v_i \right\|_1^2. \quad (6.4)$$

This shows that K is positive semi-definite, but we can do better.

Set

$$v := \sum_{i=1}^{d} \xi^i v_i \text{ and } q = \sum_{i=1}^{d} \xi^i q_i. \quad (6.5)$$

Taking linear combinations of the weak formulations of the problems for (v_k, q_k) leads us to the identity

$$(v, w)_1 - (q, \operatorname{div} w)_0 = (\xi, w)_0 \text{ for all } w \in H_0^1(Y_f). \quad (6.6)$$

If $v = 0$, then in particular

$$-(q, \operatorname{div} w)_0 = (\xi, w)_0 \text{ for all } w \in H_0^1(Y_f). \quad (6.7)$$

Choosing $w = \nabla\varphi$ for $\varphi \in C_c^\infty(Y_f)$ proves that

$$(q, -\Delta\varphi)_0 = (\xi, \nabla\varphi)_0 = 0 \text{ for all } \varphi \in C_c^\infty(Y_f), \quad (6.8)$$

which means that q is weakly harmonic. Weyl's lemma then implies that $q \in C^\infty(Y_f)$, and then (6.7) tells us that

$$\nabla q = \xi \text{ in } Y_f. \quad (6.9)$$

Since Y_f is connected, we deduce that

$$q(x) = \xi \cdot x + C \text{ for all } x \in Y_f, \text{ where } C \in \mathbb{R} \text{ is some constant}. \quad (6.10)$$

We now know that q is smooth and respects the periodicity of Y; this is only possible if $\xi = 0$.

Hence $v = 0$ implies that $\xi = 0$. Returning to (6.4), we find that

$$K\xi \cdot \xi > 0 \text{ for all } \xi \neq 0, \quad (6.11)$$

which means that K is positive definite. $\square$

6.2. **Darcy's law.** Assume that $f \in L^2(\Omega)$ is given and is independent of ε. For each $\varepsilon \in (0, \varepsilon_0)$, Theorem 4.3 provides us with a unique pair $(u_\varepsilon, p_\varepsilon) \in H^1_{0,\sigma}(\Omega_\varepsilon) \times \mathring{L}^2(\Omega_\varepsilon)$ that is a weak solution of the Ω_ε–Stokes problem

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.12)$$

We extend these solutions as follows.

Definition 6.3. *Let $(u_\varepsilon, p_\varepsilon) \in H^1_{0,\sigma}(\Omega_\varepsilon) \times \mathring{L}^2(\Omega_\varepsilon)$ be as above. We define their extensions $(\hat{u}_\varepsilon, \hat{p}_\varepsilon) \in H^1_{0,\sigma}(\Omega) \times \mathring{L}^2(\Omega)$ according to*

$$\hat{u}_\varepsilon = \tilde{u}_\varepsilon \text{ and } \hat{p}_\varepsilon = \mathcal{E}_\varepsilon p_\varepsilon, \quad (6.13)$$

where $\tilde{u}_\varepsilon$ denotes the extension by zero of u_ε from Ω_ε to Ω, and $\mathcal{E}_\varepsilon : \mathring{L}^2(\Omega_\varepsilon) \to \mathring{L}^2(\Omega)$ is the extension operator constructed in Proposition 5.4.

Next we derive a priori estimates for $(\hat{u}_\varepsilon, \hat{p}_\varepsilon)$.

Proposition 6.4. *Let $(\hat{u}_\varepsilon, \hat{p}_\varepsilon) \in H^1_{0,\sigma}(\Omega) \times \mathring{L}^2(\Omega)$ be as in Definition 6.3. Then*

$$\frac{1}{\varepsilon^2} \|\hat{u}_\varepsilon\|_{L^2(\Omega)} + \frac{1}{\varepsilon} \|\nabla \hat{u}_\varepsilon\|_{L^2(\Omega)} + \|\hat{p}_\varepsilon\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} . \tag{6.14}$$

Proof. Since $(u_\varepsilon, p_\varepsilon)$ are weak solutions of (6.12) and $f \in L^2(\Omega)$, we know that

$$(u_\varepsilon, v)_{1,\Omega_\varepsilon} - (p_\varepsilon, \operatorname{div} v)_{0,\Omega_\varepsilon} = (f, v)_{0,\Omega_\varepsilon} \text{ for every } v \in H^1_0(\Omega_\varepsilon). \tag{6.15}$$

We may use $v = u_\varepsilon$ to arrive at the estimate

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} f \cdot u_\varepsilon \le \|f\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} . \tag{6.16}$$

Lemma 5.1 them implies that

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon \|f\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \tag{6.17}$$

and hence

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon \|f\|_{L^2(\Omega)} . \tag{6.18}$$

Chaining together this estimate and Lemma 5.1 then yields the estimate

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^2 \|f\|_{L^2(\Omega)} . \tag{6.19}$$

Since $\hat{u}_\varepsilon$ is the extension of u_ε by zero, we deduce from (6.18) and (6.19) that

$$\frac{1}{\varepsilon^2} \|\hat{u}_\varepsilon\|_{L^2(\Omega)} + \frac{1}{\varepsilon} \|\nabla \hat{u}_\varepsilon\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} . \tag{6.20}$$

Since $\hat{p}_\varepsilon = \mathcal{E}_\varepsilon p_\varepsilon$, Proposition 5.4 implies that

$$(\hat{p}_\varepsilon, \operatorname{div} v)_{0,\Omega} = (\mathcal{E}_\varepsilon p_\varepsilon, \operatorname{div} v)_{0,\Omega} = (p_\varepsilon, \operatorname{div} \mathcal{R}_\varepsilon v)_{0,\Omega_\varepsilon} \text{ for all } v \in H^1_0(\Omega), \tag{6.21}$$

where $\mathcal{R}_\varepsilon$ is the restriction operator constructed in Theorem 5.3. Combining (6.15) and (6.21) yields the equality

$$(\hat{p}_\varepsilon, \operatorname{div} v)_{0,\Omega} = (u_\varepsilon, \mathcal{R}_\varepsilon v)_{1,\Omega_\varepsilon} - (f, \mathcal{R}_\varepsilon v)_{0,\Omega_\varepsilon} \text{ for every } v \in H^1_0(\Omega). \tag{6.22}$$

According to Lemma 3.1 there exists $v_\varepsilon \in H^1_0(\Omega)$ such that $\operatorname{div} v_\varepsilon = \hat{p}_\varepsilon$ in Ω and $\|v_\varepsilon\|_{H^1(\Omega)} \le C \|\hat{p}_\varepsilon\|_{L^2(\Omega)}$ for a constant $C > 0$ independent of ε. Using $v = v_\varepsilon$ in (6.22) and recalling the estimates (5.28) and (5.29) of Theorem 5.3 as well as (6.18), we find that

$$\begin{aligned}
\|\hat{p}_\varepsilon\|^2_{L^2(\Omega)} = (\hat{p}_\varepsilon, \operatorname{div} v_\varepsilon)_{0,\Omega} &= (u_\varepsilon, \mathcal{R}_\varepsilon v_\varepsilon)_{1,\Omega_\varepsilon} - (f, \mathcal{R}_\varepsilon v_\varepsilon)_{0,\Omega_\varepsilon} \\
&\le \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla \mathcal{R}_\varepsilon v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega)} \|\mathcal{R}_\varepsilon v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
\lesssim \varepsilon \|f\|_{L^2(\Omega)} \left(\frac{1}{\varepsilon} \|v_\varepsilon\|_{L^2(\Omega)} + \|\nabla v_\varepsilon\|_{L^2(\Omega)} \right) &+ \|f\|_{L^2(\Omega)} \left(\|v_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla v_\varepsilon\|_{L^2(\Omega)} \right) \\
&\lesssim \|f\|_{L^2(\Omega)} \|v_\varepsilon\|_{H^1(\Omega)} \lesssim \|f\|_{L^2(\Omega)} \|\hat{p}_\varepsilon\|_{L^2(\Omega)} .
\end{aligned} \tag{6.23}$$

Then

$$\|\hat{p}_\varepsilon\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} . \tag{6.24}$$

The estimate (6.14) now follows by summing (6.20) and (6.24).

□

An immediate consequence of Proposition 6.4 is that there exist $(u, p) \in L^2(\Omega) \times \mathring{L}^2(\Omega)$ such that, up to the extraction of a subsequence,

$$\frac{\hat{u}_\varepsilon}{\varepsilon}^2 \rightharpoonup u \text{ weakly in } L^2(\Omega) \text{ and } \hat{p}_\varepsilon \rightharpoonup p \text{ weakly in } \mathring{L}^2(\Omega) \text{ as } \varepsilon \to 0. \tag{6.25}$$

Our next result derives some extra information.

Proposition 6.5. *Let $(\hat{u}_\varepsilon, \hat{p}_\varepsilon) \in H^1_{0,\sigma}(\Omega) \times \mathring{L}^2(\Omega)$ be as in Definition 6.3. Assume that (6.25) holds for $(u, p) \in L^2(\Omega) \times \mathring{L}^2(\Omega)$. Then the following hold.*

(1) $u \in L^2_\sigma(\Omega)$. *In particular* $\operatorname{div} u = 0$ *in the sense of distributions and* $u \cdot \nu = 0$ *on* $\partial\Omega$, *where the equality is understood in the sense of* $H^{-1/2}(\partial\Omega)$ *via Proposition 2.2.*

(2) *Actually,* $\hat{p}_\varepsilon \to p$ *strongly in* $\mathring{L}^2(\Omega)$.

Proof. We have the inclusion $H^1_{0,\sigma}(\Omega) \subset L^2_\sigma(\Omega) \subset L^2(\Omega)$. It's easy to see that $L^2_\sigma(\Omega)$ is a weakly closed subspace of $L^2(\Omega)$: if $\{w_k\}_{k=0}^\infty \subset L^2_\sigma(\Omega)$ and $w_\varepsilon \rightharpoonup w$ weakly in $L^2(\Omega)$, then for $\varphi \in C_c^\infty(\Omega)$,

$$0 = \int_\Omega w_k \cdot \nabla\varphi \to \int_\Omega w \cdot \nabla\varphi \Rightarrow \int_\Omega w \cdot \nabla\varphi = 0, \tag{6.26}$$

and so $\operatorname{div} w = 0$ in the sense of distributions. Since $\hat{u}_\varepsilon/\varepsilon^2 \in H^1_{0,\sigma}(\Omega)$ for each ε, we deduce that $u \in L^2_\sigma(\Omega)$.

To prove that $\hat{p}_\varepsilon \to p$ strongly it suffices to prove that

$$\lim_{\varepsilon\to 0} \|\hat{p}_\varepsilon\|^2_{L^2(\Omega)} = \|p\|^2_{L^2(\Omega)}. \tag{6.27}$$

We may use Lemma 3.1 to find $v_\varepsilon \in H^1_0(\Omega)$ such that $\operatorname{div} v_\varepsilon = \hat{p}_\varepsilon$ in Ω and

$$\|v_\varepsilon\|_{H^1(\Omega)} \le C \|\hat{p}_\varepsilon\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \tag{6.28}$$

where for the last inequality we have used Proposition 6.4. From weak compactness and Rellich's theorem we know that, up to the extraction of another subsequence,

$$v_\varepsilon \rightharpoonup v \text{ weakly in } H^1_0(\Omega) \text{ and } v_\varepsilon \to v \text{ strongly in } L^2(\Omega). \tag{6.29}$$

Since $\operatorname{div} v_\varepsilon = \hat{p}_\varepsilon$ and $\hat{p}_\varepsilon \rightharpoonup p$ weakly in $L^2(\Omega)$, we find that $\operatorname{div} v = p$. Then

$$\begin{aligned}\|\hat{p}_\varepsilon\|^2_{L^2(\Omega)} - \|p\|^2_{L^2(\Omega)} &= (\hat{p}_\varepsilon, \operatorname{div} v_\varepsilon)_{0,\Omega} - (p, \operatorname{div} v)_{0,\Omega} \\ &= (\hat{p}_\varepsilon - p, \operatorname{div} v)_{0,\Omega} + (\hat{p}_\varepsilon, \operatorname{div}(v_\varepsilon - v))_{0,\Omega}.\end{aligned} \tag{6.30}$$

From (6.22), (5.28), (5.29), and Proposition 6.4 we may estimate

$$\begin{aligned}(\hat{p}_\varepsilon, \operatorname{div}(v_\varepsilon - v))_{0,\Omega} &= (u_\varepsilon, \mathcal{R}_\varepsilon(v_\varepsilon - v))_{1,\Omega_\varepsilon} - (f, \mathcal{R}_\varepsilon(v_\varepsilon - v))_{0,\Omega_\varepsilon} \\ &\le \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla \mathcal{R}_\varepsilon(v_\varepsilon - v)\|_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega)} \|\mathcal{R}_\varepsilon(v_\varepsilon - v)\|_{L^2(\Omega_\varepsilon)} \\ &\lesssim \|f\|_{L^2(\Omega)} \varepsilon \left(\frac{1}{\varepsilon}\|v_\varepsilon - v\|_{L^2(\Omega)} + \|\nabla(v_\varepsilon - v)\|_{L^2(\Omega)}\right) \\ &\quad + \|f\|_{L^2(\Omega)} \left(\|v_\varepsilon - v\|_{L^2(\Omega)} + \varepsilon \|\nabla(v_\varepsilon - v)\|_{L^2(\Omega)}\right) \\ &\lesssim \|f\|_{L^2(\Omega)} \left(\|v_\varepsilon - v\|_{L^2(\Omega)} + \varepsilon \|\nabla(v_\varepsilon - v)\|_{L^2(\Omega)}\right) \\ &\lesssim \|f\|_{L^2(\Omega)} \left(\|v_\varepsilon - v\|_{L^2(\Omega)} + \varepsilon \|v_\varepsilon\|_{H^1(\Omega)} + \varepsilon \|v\|_{H^1(\Omega)}\right) \\ &\lesssim \|f\|_{L^2(\Omega)} \left(\|v_\varepsilon - v\|_{L^2(\Omega)} + \varepsilon \|\hat{p}_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|v\|_{H^1(\Omega)}\right) \\ &\lesssim \|f\|_{L^2(\Omega)} \left(\|v_\varepsilon - v\|_{L^2(\Omega)} + \varepsilon \|f\|_{L^2(\Omega)} + \varepsilon \|v\|_{H^1(\Omega)}\right).\end{aligned} \tag{6.31}$$

Since $v_\varepsilon \to v$ strongly in $L^2(\Omega)$, we see that along this second subsequence

$$\lim_{\varepsilon\to 0} (\hat{p}_\varepsilon, \operatorname{div}(v_\varepsilon - v))_{0,\Omega} = 0. \tag{6.32}$$

On the other hand $(\hat{p}_\varepsilon - p, \operatorname{div} v)_{0,\Omega} \to 0$ by weak convergence. Hence, along this second subsequence

$$\lim_{\varepsilon\to 0} \left(\|\hat{p}_\varepsilon\|^2_{L^2(\Omega)} - \|p\|^2_{L^2(\Omega)}\right) = 0. \tag{6.33}$$

Since this works for any possible choice of second subsequence, we deduce that the convergence holds along the original subsequence, and so $\hat{p}_\varepsilon \to p$ strongly in $L^2(\Omega)$. ☐

We have now developed all of the tools needed to prove the main result, which derives Darcy's law as the homogenization limit of the Ω_ε−Stokes problem.

Theorem 6.6. *Let $(\hat{u}_\varepsilon, \hat{p}_\varepsilon) \in H^1_{0,\sigma}(\Omega) \times \mathring{L}^2(\Omega)$ be as in Definition 6.3. Assume that (6.25) holds for $(u, p) \in L^2(\Omega) \times \mathring{L}^2(\Omega)$. Then the following hold.*

(1) *$p \in H^1(\Omega) \cap \mathring{L}^2(\Omega)$ is the unique zero-average solution to the elliptic problem*

$$\begin{cases} -\operatorname{div}(K\nabla p) = -\operatorname{div}(Kf) & \text{in } \Omega \\ K(f - \nabla p) \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{6.34}$$

in the weak sense:

$$\int_\Omega K\nabla p \cdot \nabla w = \int_\Omega Kf \cdot \nabla w \text{ for all } w \in H^1(\Omega). \tag{6.35}$$

Here $K \in \mathbb{R}^{d\times d}$ is the permeability tensor given by Definition 6.1.

(2) *$u = K(f - \nabla p)$ in Ω.*

Since (u, p) are uniquely determined, we actually have the convergence results

$$\frac{\hat{u}_\varepsilon}{\varepsilon^2} \rightharpoonup u \text{ weakly in } L^2(\Omega) \text{ and } \hat{p}_\varepsilon \to p \text{ strongly in } \mathring{L}^2(\Omega) \tag{6.36}$$

along any sequence of ε values in $(0, \varepsilon_0)$.

Proof. We divide the proof into several steps.

Step 1 – Tilings of the cell problem solutions

Let $(v_k, q_k) \in H^1_0(Y_f) \cap \mathring{L}^2(Y_f)$ be as in (6.1), and write $\tilde{v}_k \in H^1(Y)$ for the extension of v_k by zero. Define the functions $v^\varepsilon_k \in H^1_{loc}(\mathbb{R}^d)$ and $q^\varepsilon_k \in L^2_{loc}(L(Y_f))$ according to

$$v^\varepsilon_k(x) = \tilde{v}_k([x/\varepsilon]) \text{ and } q^\varepsilon_k(x) = q_k([x/\varepsilon]), \tag{6.37}$$

where $[y] \in Y$ is the equivalence class of $y \in \mathbb{R}^d$ under the quotient $\mathbb{R}^d/\mathbb{Z}^d$. Notice that $\operatorname{div} v^\varepsilon_k = 0$ in $\mathbb{R}^d$ and that $v^\varepsilon_k = 0$ on $\partial L(Y_f)$.

Step 2 – Estimates

Let us write $J_\varepsilon(\Omega) = \{m \in \mathbb{Z}^d \mid Y^\varepsilon_m \cap \Omega \neq \varnothing\}$. Since Ω is bounded, it is contained in some large cube, and hence there exists a constant $C_\Omega > 0$ such that

$$\# J_\varepsilon(\Omega) \leq \frac{C_\Omega}{\varepsilon^d}. \tag{6.38}$$

A simple rescaling shows that

$$\varepsilon^{-d} \left\| q^\varepsilon_k \right\|^2_{L^2(Y^\varepsilon_{m;f})} = \left\| q_k \right\|^2_{L^2(Y_f)}, \qquad \varepsilon^{-d} \left\| v^\varepsilon_k \right\|^2_{L^2(Y^\varepsilon_m)} = \left\| v_k \right\|^2_{L^2(Y)}, \\ \text{and } \varepsilon^{2-d} \left\| \nabla v^\varepsilon_k \right\|^2_{L^2(Y^\varepsilon_m)} = \left\| \nabla v_k \right\|^2_{L^2(Y)}. \tag{6.39}$$

We may then use this to estimate

$$\left\| v^\varepsilon_k \right\|^2_{L^2(\Omega)} \leq \sum_{m\in J_\varepsilon(\Omega)} \left\| v^\varepsilon_k \right\|^2_{L^2(Y^\varepsilon_m)} = \sum_{m\in J_\varepsilon(\Omega)} \varepsilon^d \left\| v_k \right\|^2_{L^2(Y)} = \varepsilon^d \left\| v_k \right\|^2_{L^2(Y)} \# J_\varepsilon(\Omega) \leq C_\Omega \left\| v_k \right\|^2_{L^2(Y)} \tag{6.40}$$

and

$$\left\| \nabla v^\varepsilon_k \right\|^2_{L^2(\Omega)} \leq \sum_{m\in J_\varepsilon(\Omega)} \left\| \nabla v^\varepsilon_k \right\|^2_{L^2(Y^\varepsilon_m)} = \sum_{m\in J_\varepsilon(\Omega)} \varepsilon^{d-2} \left\| \nabla v_k \right\|^2_{L^2(Y)} \\ = \varepsilon^{d-2} \left\| \nabla v_k \right\|^2_{L^2(Y)} \# J_\varepsilon(\Omega) \leq \frac{C_\Omega}{\varepsilon^2} \left\| \nabla v_k \right\|^2_{L^2(Y)}. \tag{6.41}$$

Similarly, for $K^\varepsilon = \bigcup_{m\in I_\varepsilon(\Omega)} Y^\varepsilon_{m;f}$, we may estimate

$$\left\| q^\varepsilon_k \right\|^2_{L^2(K^\varepsilon)} = \sum_{m\in I_\varepsilon(\Omega)} \left\| q^\varepsilon_k \right\|^2_{L^2(Y^\varepsilon_{m;f})} = \sum_{m\in I_\varepsilon(\Omega)} \varepsilon^d \left\| q_k \right\|^2_{L^2(Y_f)} \\ = \varepsilon^d \left\| q_k \right\|^2_{L^2(Y_f)} \# I_\varepsilon(\Omega) \leq \varepsilon^d \left\| q_k \right\|^2_{L^2(Y_f)} \# J_\varepsilon(\Omega) \leq C_\Omega \left\| q_k \right\|^2_{L^2(Y_f)}. \tag{6.42}$$

Since v_k and q_k do not depend on ε, we deduce that

$$\|v_k^\varepsilon\|_{L^2(\Omega)} + \varepsilon\,\|\nabla v_k^\varepsilon\|_{L^2(\Omega)} + \|q_k^\varepsilon\|_{L^2(K^\varepsilon)} \lesssim 1. \tag{6.43}$$

Step 3 – A weak limit

We claim that $v_k^\varepsilon \rightharpoonup \bar{v}_k$ weakly in $L^2(\Omega)$, where $\bar{v}_k \in \mathbb{R}^d$ is the constant vector

$$\bar{v}_k = \int_Y \tilde{v}_k = \int_{Y_f} v_k. \tag{6.44}$$

Indeed, for $\varphi \in C_c^\infty(\Omega)$

$$\begin{aligned}\int_\Omega (v_k^\varepsilon(x) - \bar{v}_k)\cdot\varphi(x)dx &= \sum_{m\in J_\varepsilon(\Omega)} \int_Y (v_k(y) - \bar{v}_k)\cdot\varphi(\varepsilon(m+y))\varepsilon^d dy \\ &= \int_Y (v_k(y) - \bar{v}_k)\cdot \sum_{m\in J_\varepsilon(\Omega)} \varphi(\varepsilon(m+y))\varepsilon^d dy = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a(n)\cdot\overline{b_\varepsilon(n)}\end{aligned} \tag{6.45}$$

where the Fourier coefficients on Y are given for $n \in \mathbb{Z}^d$ by

$$a(n) = \int_Y (v_k(y) - \bar{v}_k)\exp(-2\pi i n\cdot y)dy \tag{6.46}$$

and

$$b_\varepsilon(n) = \int_Y \sum_{m\in J_\varepsilon(\Omega)} \varphi(\varepsilon(m+y))\varepsilon^d \exp(-2\pi i n\cdot y)dy. \tag{6.47}$$

Notice that $a(0) = 0$ by the choice of $\bar{v}_k$. We then compute

$$\begin{aligned} b_\varepsilon(n) &= \int_Y \sum_{m\in J_\varepsilon(\Omega)} \varphi(\varepsilon(m+y))\varepsilon^d \exp(-2\pi i n\cdot y)dy \\ &= \sum_{m\in J_\varepsilon(\Omega)} \int_{Y_m^\varepsilon} \varphi(x)\exp(-2\pi i n\cdot(x/\varepsilon - m))dx \\ &= \sum_{m\in J_\varepsilon(\Omega)} \int_{Y_m^\varepsilon} \varphi(x)\exp(-2\pi i n\cdot x/\varepsilon)dx \\ &= \int_{\mathbb{R}^d} \varphi(x)\exp(-2\pi i n\cdot x/\varepsilon)dx \\ &= \mathcal{F}\varphi(n/\varepsilon)\end{aligned} \tag{6.48}$$

where $\mathcal{F}$ denotes the Fourier transform, and the second-to-last equality follows from the fact that

$$\operatorname{supp}(\varphi) \subseteq \Omega \subseteq \bigcup_{m\in J_\varepsilon(\Omega)} Y_m^\varepsilon. \tag{6.49}$$

Since $\varphi \in C_c^\infty(\Omega)$, it is also in the Schwartz class, which means that $\mathcal{F}\varphi$ is Schwartz class as well. Then there exists $C_\varphi > 0$ such that

$$|\mathcal{F}\varphi(\xi)| \le \frac{C_\varphi}{(1+|\xi|^d)} \text{ for all } \xi \in \mathbb{R}^d, \tag{6.50}$$

which in particular implies that

$$|b_\varepsilon(n)| = |\mathcal{F}\varphi(n/\varepsilon)| \le \frac{C_\varphi \varepsilon^d}{(\varepsilon^d + |n|^d)}. \tag{6.51}$$

Now, since

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |a(n)\cdot b_\varepsilon(n)| \le \left(\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |a(n)|^2\right)^{1/2} \left(\sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{C_\varphi^2 \varepsilon^{2d}}{(\varepsilon^d+|n|^d)^2}\right)^{1/2}$$
$$\le C_\varphi \varepsilon^d \|v_k-\bar{v}_k\|_{L^2(Y)} \left(\sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|n|^{2d}}\right)^{1/2} \lesssim \varepsilon^d C_\varphi \|v_k-\bar{v}_k\|_{L^2(Y)} \tag{6.52}$$

we deduce from (6.43) and the fact that v_k is independent of ε that

$$\left|\int_\Omega (v_k^\varepsilon(x)-\bar{v}_k)\cdot\varphi(x)dx\right| \le \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |a(n)\cdot b_\varepsilon(n)| \lesssim \varepsilon^d C_\varphi \|v_k-\bar{v}_k\|_{L^2(Y)} \lesssim \varepsilon^d C_\varphi. \tag{6.53}$$

Hence

$$\int_\Omega (v_k^\varepsilon(x)-\bar{v}_k)\cdot\varphi(x)dx \to 0. \tag{6.54}$$

The density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$ then guarantees that $v_k^\varepsilon \rightharpoonup \bar{v}_k$ weakly in $L^2(\Omega)$, as claimed.

Step 4 – Passing to the limit

Let $\varphi\in C_c^\infty(\Omega)$. When ε is sufficiently small we must have the inclusion $\operatorname{supp}(\varphi)\subseteq K^\varepsilon$. This allows us to deduce from (6.1) that

$$(v_k^\varepsilon,\varphi\hat{u}_\varepsilon)_{1,\Omega} = (v_k^\varepsilon,\varphi u_\varepsilon)_{1,\Omega_\varepsilon} = \frac{1}{\varepsilon}(q_k^\varepsilon,\operatorname{div}(\varphi u_\varepsilon))_{0,\Omega_\varepsilon} + \frac{1}{\varepsilon^2}(f,\varphi u_\varepsilon)_{0,\Omega_\varepsilon}$$
$$= \varepsilon\left(q_k^\varepsilon,\nabla\varphi\cdot\frac{\hat{u}_\varepsilon}{\varepsilon^2}\right)_{0,\Omega_\varepsilon} + \left(e_k,\varphi\frac{\hat{u}_\varepsilon}{\varepsilon^2}\right)_{0,\Omega}. \tag{6.55}$$

The estimate (6.43) and Proposition 6.4 then imply that

$$\lim_{\varepsilon\to 0}(v_k^\varepsilon,\varphi\hat{u}_\varepsilon)_{1,\Omega} = (e_k,\varphi u)_{0,\Omega}. \tag{6.56}$$

On the other hand, we may use $v=\varphi v_k^\varepsilon$ as a test function in the weak formulation (6.12) to see that (using the fact that $\hat{p}_\varepsilon = p_\varepsilon + C_\varepsilon$ on Ω_ε)

$$(\hat{u}_\varepsilon,\varphi v_k^\varepsilon)_{1,\Omega} = (u_\varepsilon,\varphi v_k^\varepsilon)_{1,\Omega_\varepsilon} = (p_\varepsilon,\operatorname{div}(\varphi v_k^\varepsilon))_{0,\Omega_\varepsilon} + (f,\varphi v_k^\varepsilon)_{0,\Omega_\varepsilon}$$
$$= (\hat{p}_\varepsilon,\operatorname{div}(\varphi v_k^\varepsilon))_{0,\Omega_\varepsilon} + (f,\varphi v_k^\varepsilon)_{0,\Omega_\varepsilon} = (\hat{p}_\varepsilon,\nabla\varphi\cdot v_k^\varepsilon)_{0,\Omega_\varepsilon} + (f,\varphi v_k^\varepsilon)_{0,\Omega_\varepsilon}$$
$$= (\hat{p}_\varepsilon,\nabla\varphi\cdot v_k^\varepsilon)_{0,\Omega} + (f,\varphi v_k^\varepsilon)_{0,\Omega}. \tag{6.57}$$

Since $\hat{p}_\varepsilon\to p$ strongly in $L^2(\Omega)$ and $v_k^\varepsilon\rightharpoonup\bar{v}_k$ weakly in $L^2(\Omega)$, we deduce that

$$\lim_{\varepsilon\to 0}(\hat{u}_\varepsilon,\varphi v_k^\varepsilon)_{1,\Omega} = (p,\nabla\varphi\cdot\bar{v}_k)_{0,\Omega} + (f,\varphi\bar{v}_k)_{0,\Omega}. \tag{6.58}$$

We rewrite

$$(\hat{u}_\varepsilon,\varphi v_k^\varepsilon)_{1,\Omega} - (v_k^\varepsilon,\varphi\hat{u}_\varepsilon)_{1,\Omega} = \int_\Omega \nabla\hat{u}_\varepsilon : \nabla(\varphi v_k^\varepsilon) - \nabla v_k^\varepsilon : \nabla(\varphi\hat{u}_\varepsilon)$$
$$= \int_\Omega \nabla\hat{u}_\varepsilon : (v_k^\varepsilon\otimes\nabla\varphi + \varphi\nabla v_k^\varepsilon) - \nabla v_k^\varepsilon : (\hat{u}_\varepsilon\otimes\nabla\varphi + \varphi\nabla\hat{u}_\varepsilon)$$
$$= \int_\Omega \nabla\hat{u}_\varepsilon : v_k^\varepsilon\otimes\nabla\varphi - \nabla v_k^\varepsilon : \hat{u}_\varepsilon\otimes\nabla\varphi$$
$$= \varepsilon\int_\Omega \nabla\frac{\hat{u}_\varepsilon}{\varepsilon} : v_k^\varepsilon\otimes\nabla\varphi - \varepsilon^2\int_\Omega \nabla v_k^\varepsilon : \frac{\hat{u}_\varepsilon}{\varepsilon^2}\otimes\nabla\varphi. \tag{6.59}$$

Then (6.43) and Proposition 6.4 imply that

$$\lim_{\varepsilon\to 0}\Big((\hat{u}_\varepsilon,\varphi v_k^\varepsilon)_{1,\Omega} - (v_k^\varepsilon,\varphi\hat{u}_\varepsilon)_{1,\Omega}\Big) = 0. \tag{6.60}$$

Combining (6.56), (6.58), and (6.60) then yields the equality

$$(e_k, \varphi u)_{0,\Omega} = (p, \nabla\varphi \cdot \bar{v}_k)_{0,\Omega} + (f, \varphi\bar{v}_k)_{0,\Omega} \text{ for every } \varphi \in C_c^\infty(\Omega). \tag{6.61}$$

This implies that

$$\int_\Omega \varphi u = K \int_\Omega p\nabla\varphi + f\varphi \text{ for every } \varphi \in C_c^\infty(\Omega), \tag{6.62}$$

where K is the permeability tensor.

Step 5 – Equations for (u, p)

Proposition 6.2 guarantees that K is symmetric and positive definite, so we deduce from (6.62) that

$$\int_\Omega \varphi(K^{-1}u - f) = \int_\Omega p\nabla\varphi \text{ for every } \varphi \in C_c^\infty(\Omega), \tag{6.63}$$

which means that p is weakly differentiable on Ω, and

$$-\nabla p = K^{-1}u - f \in L^2(\Omega). \tag{6.64}$$

Since we already knew that $p \in \mathring{L}^2(\Omega)$, we have that $p \in H^1(\Omega)$. Then for any $v \in H^1(\Omega)$ we have

$$\int_\Omega K\nabla p \cdot \nabla v = \int_\Omega Kf \cdot \nabla v - \int_\Omega u \cdot \nabla v, \tag{6.65}$$

but since $u \in L^2_\sigma(\Omega)$ and $C^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$, we have that

$$-\int_\Omega u \cdot \nabla v = 0. \tag{6.66}$$

Then $p \in H^1(\Omega) \cap \mathring{L}^2(\Omega)$ satisfies

$$\int_\Omega K\nabla p \cdot \nabla v = \int_\Omega Kf \cdot \nabla v \text{ for all } v \in H^1(\Omega), \tag{6.67}$$

which means that p is the unique solution to (6.34). Then $u = K(f - \nabla p)$ is also uniquely determined, and we deduce the convergence results along any sequence $\varepsilon \to 0$.

□

References

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