Double layer formulation for solid particles in Stokes flow

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Consider the Stokes flow with a single solid particle suspended in the fluid. Let p denote the particle, Γ the boundary of the particle, F_p and T_p the net force and torque exerted on the particle, and c_p the center of mass of the particle. For a force and torque free particle we have

$$
\begin{aligned}
-\mu\Delta u + \nabla p &= 0 && \text{for } x \in \mathbb{R}^2 \backslash p, && (1)\\
\nabla \cdot u &= 0 && \text{for} x \in \mathbb{R}^2 \backslash p, && (2)\\
\lim_{x\to\infty} u &= u^\infty, && && (3)\\
F_p &= 0, && && (4)\\
T_p &= 0. && && (5)
\end{aligned}
$$

Following the formulation of Power [1] one can write $u(x) = u^\infty + \mathcal{D}[\eta](x)$ for $x \in \mathbb{R}^2 \backslash P$. Taking the limit to the surface of the particle we have

$$
\begin{aligned}
U_p + (x - c_p)\omega_p &= u^\infty - \frac{1}{2}\eta + \mathcal{D}[\eta](x) && \text{for } x \in \Gamma, && \text{(6a)}\\
\int_\Gamma \eta \,\mathrm{d}s(y) &= 0 && \text{(force free constraint)}, && \text{(6b)}\\
\int_\Gamma (y - c_p)^\perp \cdot \eta \,\mathrm{d}s(y) &= 0 && \text{(torque free constraint)}, && \text{(6c)}
\end{aligned}
$$

where U_p is the particle velocity, Ω_p is the particle angular velocity, $\mathcal{D}$ is the double layer operator and η is a density. Eq. Set (6) is a complete system for η, U_p and ω_p.

The difference of Eq. Set (6) with that of fixed boundaries is the absence of Stokeslet and Rotlet terms because of zero force and torque. The discrete mobility matrix is

$$
\left[\begin{array}{cc|cc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{$-D + \frac{1}{2}I$}} & 1 & 0 & y_1 - c_1 \\
 & & 0 & 1 & y_2 - c_2 \\
\hline
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\hline
y_2 - c_2 & -(y_1 - c_1) & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} \eta \\ \hline U_p \\ \hline \omega_p \end{array}\right]
=
\left[\begin{array}{c} u^\infty \\ \hline 0 \\ \hline 0 \end{array}\right], \tag{7}
$$

which can be inverted and stored. One only need to track the center of mass and an orientation vector o_p (rotated by ω_p). It is easy to extend this to a suspension of multiple particles.

References

[1] H Power. The completed double layer boundary integral equation method for two-dimensional Stokes flow. *IMA Journal of Applied Mathematics*, 1993. URL http://imamat.oxfordjournals.org/content/51/2/123.short.