Review of the Manuscript "Integral equation methods for the Yukawa-Beltrami equation on the sphere"

This manuscript presents an integral equation method for solving the Yukawa-Beltrami equation on the sphere. All the material in the manuscript are straightforward and fairly standard. That is, the Green's formula on a Riemannian manifold is well-known to any researcher working on PDEs and differential geometry. The jump relations of layer potentials in Riemannian manifolds are also well-known (as a local property, it is obvious that the jump relations are the same for the Yukawa-Beltrami and Laplace-Beltrami operators). Just to name a reference, Mitrea and Taylor [1] discussed in detail the boundary layer methods for Lipschitz domains in Riemannian manifolds. Indeed, the paper actually discussed a much harder problem in greater generality. The authors should first try to find all relevant work that has alrealdy been done by analysts (for example, do a simple search over the work by Mitrea and Taylor and check the references therein), instead of trying to produce unsatisfactory proofs for much simpler and specific cases. The numerical part of the manuscript uses the Alpert quadrature for logarithmically singular kernels, which is also quite standard and well-known.

Several specific comments are as follows.

1. In the introduction part, the authors mentioned very briefly that the motivation of their work is solving the heat equation on the sphere. I would like to see at least three **application** papers which really need to solve the heat equation on the sphere. Otherwise, the motivation is not justified and it does not seem to be appropriate to publish the manuscript since one could then write so many papers with each particular PDE on a specific manifold without any real applications.

2. All the theoretical results in the manucript are straightforward and standard and analysts have proven them in a more general form. It is strongly recommended that the authors study previous work done by analysts and cite their work. The authors should also shorten the manuscript by deleting the "proofs" of Proposition 1, Lemmas 2 and 3 in the manucript.

3. It seems that the kernel of the double layer potential has high-order logarithmically singular terms even though it is continuous at the diagonal. Instead of presenting numerical example 1, the authors should

do a little analysis to show that this is indeed true. Numerical example 2 is also not needed since the operator becomes increasingly coercive as k increases.

4. The authors need to clarify the real distinction on the integral equation method for $k > \frac{1}{2}$ and $k \leq \frac{1}{2}$. The PDE itself does not seem to have any essential differences between these two cases. The comment that $k^2 = \mathcal{O}(1/\Delta t)$ does not really justify the neglect of the discussion of the case $k \leq \frac{1}{2}$.

5. Lemma 1 is a little unsatisfactory as the authors do not explain why $\lim_{\theta \to 0^+} F(\theta)$ should be $\frac{1}{2\pi}$. Standard approach of finding the constant C_k would utilize the fact that the Green's function satifies the PDE with the Dirac delta function as the right hand side and carry out an integration procedure. The authors could add one sentence or two for this.

6. It seems weird to use $\tilde{V}$ and $\tilde{W}$ instead of simple S and D to denote the single and double layer potentials, respectively. Is there any particular reason to do so? Also, the justification of single and double layer potentials satisfying the PDE is totally unnecessary since it uses just simple exchange of order of differentiation and integration. The authors should remove it.

In summary, the authors should justify the application of the manuscript and shorten the manuscript (definitely less than ten pages) in order to get it published. I would recommend the publication of the manuscript only if the changes are made to accommodate the above comments.

References

[1] M. Mitrea, M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds.* J. Funct. Anal., **163** (1999), pp. 181-251.