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THE COMPARISON METHOD FOR STOCHASTIC PROCESSES[1]

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A relationship between the path structure of two real discrete time stochastic processes is deduced from inequalities between their transition functions. The approach is to define processes equivalent to the two on a common space so that pointwise inequalities are possible. An iterated logarithm type law for random walks is given as a particular application of the general method.

1. Introduction. Our aim in this work is to develop a tool for studying real-valued, discrete time stochastic processes. The idea is to investigate a given process by comparing it with another process whose behavior is simpler or better understood. Where feasible, this comparison is made prossible by linking the two processes via a common probability space on which are constructed two new processes equivalent to the two given processes. Inequalities between the transition functions of the two given processes then imply inequalities between the sample paths of the new processes.

Hodges and Rosenblatt (1953) used such a technique in their study of random walks. Lamperti (1970) linked his maximal branching processes to associated sums of independent random variables in order to deduce properties of the former. Jacobs and Schach (1972) applied a similar technique to a queueing theory problem.

Kalmykov (1962) made the first study of the general method of comparing Markov processes. His approach was analytic, involving the transition functions but not the spaces on which the processes were defined. In O'Brien (1972) we gave an elementary analytic proof of his result. Daley (1968) gave a weaker version of Kalmykov's result. Applications of Daley's results to queueing theory problems can be found in Daley and Moran (1968).

The technique below, linking the two given processes via a common probability space, leads to a comparison theorem of a more general type than the distributional inequalities of Kalmykov. Moreover, our results are formulated for general discrete time processes, not merely Markov processes (see Section 3). Kalmykov's (1962), (1969) results are discussed in Section 4 and are generalized to the nonMarkovian situation; the proofs are relatively simple.

The main results do not require the Markov property, but the notation is

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more complicated in the nonMarkovian case. For the sake of tidiness, we therefore restrict our attention to Markov processes in Sections 5–8. In Sections 5 and 6 we discuss conditions for the hypotheses of the comparison theorem to hold. In particular we generalize in Section 6 Daley's (1968) useful concept of stochastic monotonicity. Section 7 contains a few simple examples for Markov processes, while in Section 8 a more substantial application proves an iterated logarithm type law for a class of simple random walks.

Although the theorems are stated for processes on the real line, R, obvious variations are possible for processes on subsets of R; our random walk application is just such a case. Theorem 11 also exemplifies another modification which allows for the periodic nature of simple random walks.

2. Preliminaries. Let N denote the positive integers and let $N_0 = N \cup \{0\}$. A distribution function (df) $F(\cdot)$ is right-continuous and is frequently considered as a function on the extended real line $R^* = R \cup \{-\infty, \infty\}$, with $F(-\infty) = 0 = 1 - F(\infty)$. We write $x^n = (x_0, \cdots, x_n)$, $X^n = (X_0, \cdots, X_n)$, etc., and $x^n \leqq y^n$ for $x_0 \leqq y_0, \cdots, x_n \leqq y_n$.

A sequence of stochastic transition functions $\{p_n\}$ is a sequence of Borel measurable functions $p_n\colon R^n \times R^* \to [0, 1]$, $n \in N$, each of which is a df in its last variable when the others are fixed. It is convenient to call a stochastic process a quadruple $(\{X_n\}, \Omega, \{p_n\}, F)$, where the X_n are random variables on $(\Omega, \mathscr{F}, P)$, F is a df and $\{p_n\}$ is a sequence of stochastic transition functions, such that for all $n \in N_0$ and $a^n \in (R^*)^{n+1}$, the joint df $P(X^n \leqq a^n)$ is given by

$$\int_{-\infty}^{a_0} \cdots \int_{-\infty}^{a_n} p_n(x^{n-1}; dx_n) \cdots p_1(x^0; dx_1)F(dx_0). \tag{2.1}$$

A Markov transition function is a Borel measurable function $p\colon R \times R^* \to [0, 1]$ which is a df in its second variable. A Markov process is a stochastic process for which

$$p_n(x^n) = p(x_{n-1}, x_n) \qquad (\text{all } n \in N)\,. \tag{2.2}$$

Our assumption that Markov transition functions are stationary is one of convenience, and could be dispensed with at the expense of complicating the algebra.

Suppose we are given a df F and a sequence of stochastic transition functions $\{p_n\}$. It is well known that there exist stochastic processes that have initial distribution F and transition functions $\{p_n\}$. The following explicit construction has order preserving properties which will be useful in the proof of Theorem 2.

THEOREM 1. *Let $\{\xi_n, n \in N_0\}$ be a sequence of independent random variables on $(\Omega, \mathscr{F}, P)$, each distributed uniformly on $(0, 1)$. For each $\omega \in \Omega$, define inductively*:

$$\begin{aligned} X_0(\omega) &= \inf\{y \in R\colon F(y) \geqq \xi_0(\omega)\} \\ X_n(\omega) &= \inf\{y \in R\colon p_n(X^{n-1}(\omega); y) \geqq \xi_n(\omega)\}\,. \end{aligned} \tag{2.3}$$

Then $\{X_n\}$ has initial distribution F and transition functions $\{p_n\}$.

The proof of this result is direct and is omitted here (cf. e.g. O'Brien, 1971).

3. The comparison theorems. The statement of our comparison theorem for general stochastic processes involves two sequences of functions which have certain characteristics of the inverse of a distribution function. These functions serve to unify the various specific examples (cf. Sections 4, 7, and 8). Their role may be visualized better in the simpler Markov case (cf. Theorem 3). First let $S_n: R^{n+1} \times (R^*)^n \to R^*$, $n \in N$, denote a sequence of functions, each non-decreasing in its last n arguments. Now, define $T_n: R^{n+1} \times R^* \to R^*$ by $T_1 = S_1$ and

$$T_n(x^n, y) = S_n(x^n, T_{n-1}(x^{n-1}, y), \cdots, T_1(x^1, y), y) . \tag{3.1}$$

THEOREM 2. *Let F and G be* df's *and let $\{p_n\}$ and $\{q_n\}$ be sequences of stochastic transition functions which, for some sequence $\{S_n\}$ as above, satisfy*

$$p_n(x^{n-1}; x_n) \leqq q_n(y^{n-1}; S_n(x^n, y^{n-1})) \tag{3.2}$$

for all $(x^n, y^{n-1}) \in R^{2n+1}$. Then there exist a probability space $(\Omega, \mathscr{F}, P)$ and stochastic processes $(\{X_n\}, \Omega, \{p_n\}, F)$ and $(\{Y_n\}, \Omega, \{q_n\}, G)$ such that with T_n as in (3.1),

$$P(Y_n \leqq T_n(X^n, Y_0), n \in N) = 1 . \tag{3.3}$$

PROOF. Let $(\Omega, \mathscr{F}, P)$ and $\{\xi_n\}$ be as in Theorem 1 and define both $\{X_n\}$ and $\{Y_n\}$ as in that theorem. By the construction and (3.2)

$$q_n(Y^{n-1}; Y_n - \varepsilon) < \xi_n \leqq p_n(X^{n-1}; X_n) \leqq q_n(Y^{n-1}; S_n(X^n, Y^{n-1}))$$

for any $\varepsilon > 0$, whence $Y_n \leqq S_n(X^n, Y^{n-1})$. In particular, $Y_1 \leqq S_1(X^1, Y^0) = T_1(X^1, Y_0)$. An inductive argument using (3.1) and the non-decreasing property of S_n completes the proof of (3.3).

To discuss the case of Markov processes, suppose there exists a function $T: R^2 \times R^* \to R^*$ such that each S_n in Theorem 2 satisfies $S_n(x^n, y^{n-1}) = T(x_{n-1}, x_n, y_{n-1})$. Then $T_1 = T$ and for $n > 1$,

$$T_n(x^n, y_0) = T(x_{n-1}, x_n, T_{n-1}(x^{n-1}, y_0)) . \tag{3.4}$$

Any function $T: R^2 \times R^* \to R^*$ that is non-decreasing in its third variable will be called a *comparison function*. Since for Markov processes (3.2) reduces via (2.2) to $p(x_{n-1}; x_n) \leqq q(y_{n-1}; S_n(x^n, y^{n-1}))$, every S_n may be assumed to have this special form. We then obtain directly from Theorem 2

THEOREM 3. *Let T be a comparison function and T_n as in* (3.4). *Let F and G be* df's *and let p and q be transition functions satisfying*

$$p(x, z) \leqq q(y, T(x, z, y)) , \qquad \textit{all } \ x, y, z \in R . \tag{3.5}$$

Then there are Markov processes $(\{X_n\}, \Omega, p, F)$ and $(\{Y_n\}, \Omega, q, G)$ on a probability space $(\Omega, \mathscr{F}, P)$ such that

$$P(Y_n \leqq T_n(X^n, Y_0), n \in N) = 1 . \tag{3.6}$$

4. Kalmykov's theorems. Our next theorem generalizes to a nonMarkovian context a result of Kalmykov (1962).

THEOREM 4. *Let* $(\{X_n\}, \Omega', \{p_n\}, F)$ *and* $(\{Y_n\}, \Omega'', \{q_n\}, G)$ *be stochastic processes satisfying* $F(z) \leqq G(z)$ *and* $p_n(x^{n-1}; z) \leqq q_n(y^{n-1}; z)$ *whenever* $y^{n-1} \leqq x^{n-1}$. *Then*

$$P(X_n \leqq z) \leqq P(Y_n \leqq z)\,, \qquad \textit{all } z \in R,\ n \in N_0\,. \tag{4.1}$$

PROOF. Define the functions S_n of Theorem 2 by $S_n(x^n, y^{n-1}) = x_n$ if $y^{n-1} \leqq x^{n-1}$, $= \infty$ otherwise. Then by that theorem, there are processes $\{U_n\}$ and $\{V_n\}$, defined on a common probability space and equivalent to $\{X_n\}$ and $\{Y_n\}$ respectively, such that $V_n \leqq U_n$ a.s., whence (4.1).

The Markov version of this theorem, obtainable directly from Theorem 3 by putting $T(x, z, y) = z$ if $y \leqq x$, $= \infty$ otherwise, is exactly Kalmykov's result which we quote as

THEOREM 5. *Let* $(\{X_n\}, \Omega, p, F)$ *and* $(\{Y_n\}, \Omega', q, G)$ *be Markov processes which satisfy* $F(z) \leqq G(z)$ *and* $p(x, z) \leqq q(y, z)$ *whenever* $y \leqq x$. *Then* (4.1) *holds.*

The probabilistic method of proof for Theorems 4 and 5 has the advantage of giving some immediate results on limit probabilities, hitting probabilities, and first passage times. Specifically, we get

COROLLARY 4.1. *Assume the hypotheses of Theorems* 4 *or* 5 *hold. For any* $a \in R$ *and* $k \in N$:

(a) $P(X_n \to \infty \textit{ as } n \to \infty) \geqq P(Y_n \to \infty \textit{ as } n \to \infty)$;
(b) $P(X_n \in [a, \infty) \textit{ for some } n \geqq k) \geqq P(Y_n \in [a, \infty) \textit{ for some } n \geqq k)$;
(c) $P(\min\{n: X_n \geqq a\} \leqq k) \geqq P(\min\{n: Y_n \geqq a\} \leqq k)$.

The derivation and extension to the nonMarkovian case of another result of Kalmykov (1969) is equally simple. Let $\{a_n\}$ and $\{b_n\}$ be sequences for which $b_n \leqq a_n$ for all $n \in N_0$ and assume $b_0 \leqq X_0 \leqq a_0$, $b_0 \leqq Y_0 \leqq a_0$. Define the first passage time random variables $\tau_X = \inf\{n: X_n \geqq a_n\}$ ($= \infty$ if $X_n < a_n$ for all n), $\sigma_X = \inf\{n: X_n \leqq b_n\}$ and similarly τ_Y and σ_Y. Then under the hypotheses of Theorems 4 or 5, we have for all $k \in N$ that

$$P(\tau_X \leqq \min(\sigma_X, k)) \geqq P(\tau_Y \leqq \min(\sigma_Y, k))\,.$$

5. Properties of comparison functions. It is clear from Theorem 3 that comparison functions play a key role in the potential applicability of comparison techniques to Markov processes: we study them more fully in this section and the next.

For any given comparison function T, define a binary relation T^* on the space of (Markov) transition functions as follows:

DEFINITION. pT^*q if and only if $p(x, z) \leqq q(y, T(x, z, y))$, all $x, y, z \in R$.

Let p and q be any transition functions. Trivially, if T is identically $+\infty$, then pT^*q, so

$$S(x, z, y) = \inf\{T(x, z, y): pT^*q\} \tag{5.1}$$

defines a comparison function, and pS^*q by the right-continuity in z of $q(y, z)$. Inspection of Theorem 3 shows that replacing T by S yields a tighter bound on Y_n in (3.3).

Now suppose that p, q and T are such that pT^*q. Then $V(x, z, y) = \inf_{u \geqq z} T(x, u, y)$ defines a comparison function, and pV^*q. Thus, when pT^*q, we may assume that $T(x, z, y)$ is non-decreasing in z; certainly, the S of (5.1) has this property.

Let T be a comparison function such that the functions T_n as defined by (3.4) satisfy $T_n(x^n, y_0) = T(x_0, x_n, y_0)$. For example, the functions T_n of Section 4 have this property; further examples are in Section 7. The conclusion of Theorem 3 is simpler in this case (or indeed, whenever T_n depends only on x_0, x_n and y_0 for each n). The following theorem, proved in Lamperti and O'Brien (1972), gives conditions for such a situation to hold.

THEOREM 6. *Let T be a comparison function of the form*

$$T(x, z, y) = f^{-1}(x, g(f(z, y))) \tag{5.2}$$

for some functions $g: R^* \to R^*$ and $f: R \times R^* \to R^*$ where $f(z, y)$ is one-one and onto in y for each z and where $f^{-1}(x, u)$ is the unique solution z of $u = f(x, z)$. Then each function $T_n(x^n, y_0)$ depends only on (x_0, x_n, y_0). If g is the identity function, then $T_n(x^n, y_0) = T(x_0, x_n, y_0)$.

Conversely, if $T_2(x^2, y_0)$ depends only on x_0, x_2 and y_0, and if $T(x, z, y)$ is strictly increasing and continuous in y for each x and z, then T satisfies (5.2) *with g strictly increasing and continuous. If $T_2(x^2, y_0) = T(x_0, x_2, y_0)$, we may take g to be the identity.*

We conclude this section by establishing a criterion for the relation pT^*q to hold. Call a transition function p *deterministic* if there is a function $f: R \to R$ such that $p(x, y) = 1$ if $y \geqq f(x)$, $= 0$ otherwise. Clearly the transition functions form a convex set whose extreme points are the deterministic functions. For any transition function p, let

$$f(x; p, \alpha) = \inf\{y \in R; p(x, y) \geqq \alpha\} \tag{5.3}$$

for $x \in R$ and $\alpha \in (0, 1)$. Then associated with p is the family of deterministic transition functions $\{p_\alpha, \alpha \in (0, 1)\}$ given by $p_\alpha(x, y) = 0$ if $y < f(x; p, \alpha)$, $= 1$ otherwise, and

$$p(x, y) = \textstyle\int_0^1 p_\alpha(x, y)\, d\alpha\ . \tag{5.4}$$

THEOREM 7. *For transition functions p and q and comparison function T, pT^*q if and only if $p_\alpha T^* q_\alpha$ for all $\alpha \in (0, 1)$.*

PROOF. Assume pT^*q, and take $\alpha \in (0, 1)$, $x, y, z \in R$. If $p(x, z) < \alpha$,

$p_\alpha(x, z) = 0$, while $p(x, z) \geqq \alpha$ implies $q(y, T(x, z, y)) \geqq \alpha$ and hence $q_\alpha(y, T(x, z, y)) = 1$. In either case $p_\alpha T^* q_\alpha$.

The converse follows from (5.4) and the definition.

6. Reflexive transition functions. For a given comparison function T a transition function p is *reflexive* if pT^*p. We use $R(T)$ to denote the class of reflexive transition functions. For the comparison function used in the proof of Theorem 5, $R(T)$ consists of the class of stochastically monotonic transition functions studied in Daley (1968) and O'Brien (1972). They made use of the fact that if $p \leqq q$ and either p or q is in $R(T)$, then pT^*q. This fact holds for general T and leads us to study the nature of $R(T)$.

Suppose T is a comparison function such that $T(x, z, x) \leqq z$ (all $x, z \in R$). If pT^*q, then $p(x, z) \leqq q(x, T(x, z, x)) \leqq q(x, z)$. Thus T^* is anti-symmetric and transitive. The restriction of T^* to $R(T)$ is an order relation (T^* being reflexive on $R(T)$) and $R(T)$ is a distributive lattice under T^*.

It is almost trivial that $R(\min(S, T)) = R(S) \cap R(T)$ and that if $S \leqq T$, then $R(S) \subseteqq R(T)$. Applying Theorem 7 we obtain the first part of

THEOREM 8. *For any comparison function T, $p \in R(T)$ if and only if $f(y; p, \alpha) \leqq T(x, z, y)$ whenever $f(x; p, \alpha) \leqq z$. Hence $R(T)$ is nonempty if and only if there is a function $h: R \to R$ such that $h(y) \leqq T(x, z, y)$ whenever $h(x) \leqq z$.*

PROOF. For $R(T)$ nonempty and $p \in R(T)$, let $h(x) = f(x; p, .5)$ and apply the first statement of the theorem. Conversely, when such a function h exists, define p by $p(x, y) = 1$ if $y \geqq h(x)$, $= 0$ otherwise. Then $p \in R(T)$.

For the rest of this section, we assume T is a comparison function of the form $T(x, z, y) = z + g(y - x)$ for some non-decreasing function $g: R^* \to R^*$. This special case includes the comparison function used in Theorem 5 and all those discussed in Sections 7 and 8. If

$$f(y; p, \alpha) - f(x; p, \alpha) \leqq g(y - x) \qquad \text{(all } x, y, \alpha) \tag{6.1}$$

then $p \in R(T)$ by Theorem 8. Conversely, if $p \in R(T)$, fix x, y and α and let $z = f(x; p, \alpha)$; then (6.1) holds by Theorem 8. Similarly, $R(T)$ is nonempty if and only if there exists a function $h: R \to R$ such that

$$h(y) - h(x) \leqq g(y - x) \qquad \text{(all } x, y \in R). \tag{6.2}$$

The last condition leads to the following.

THEOREM 9. *For $T(x, z, y) = z + g(y - x)$, $R(T)$ is nonempty if and only if $g(x) > -\infty$ (all $x \in R$) and $A \equiv \inf(g(x_1) + \cdots + g(x_n)) > -\infty$ where the infimum is over all $n \in N$ and $x_1, \cdots, x_n \in R$ for which $x_1 + \cdots + x_n = -1$.*

PROOF. For $R(T)$ nonempty, (6.2) implies $g(x) > -\infty$ for all x and, if $x_1 + \cdots + x_n = -1$, $g(x_1) + \cdots + g(x_n) \geqq h(-1) - h(0)$, whence $A > -\infty$. To prove the converse, it suffices to find $h: R \to R$ for which (6.2) holds.

First assume that $g(a) < \infty$ for some $a > 0$. Define $h: R \to R^*$ by $h(x) = \inf[g(u_1) + \cdots + g(u_n)]$, where the infimum is over all n and $u_1, \cdots, u_n$ such that $u_1 + \cdots + u_n = x$. Then h is finite-valued, since $A = h(-1) \leqq h(x) + h(-x-1)$, and (6.2) holds. Now assume $g(0+) = \infty$. Since $A > -\infty$ and g is non-decreasing, there is an $\varepsilon > 0$ and a finite constant $K > 0$ such that $g(x) \geqq Kx$ for $x \in (-\varepsilon, 0)$. Consequently there is a function $f\colon (-\infty, 0] \to R$ which is concave, increasing and continuous, satisfies $f(0) = 0$ and satisfies $f(x) \leqq g(x)$ for all $x \leqq 0$. Define $h\colon R \to R$ by $h(x) = \frac{1}{2}f(2x)$ for $x \leqq 0$, $= -\frac{1}{2}f(-2x)$ for $x > 0$. Then h satisfies (6.2), which completes the proof.

COROLLARY 9.1. *If T is as in the theorem and $R(T)$ is nonempty, then $g(x) + g(-x) \geqq 0$ for all x. In particular $g(0) \geqq 0$. If $g(x) + g(-x) = 0$ for all x, then $g(x) = cx$ for some constant c.*

7. Examples. We give several simple examples related to Theorem 3. In each case T has the property that $T_n(x^n, y_0) = T(x_0, x_n, y_0)$. Moreover, T has the form $T(x, z, y) = z + g(y-x)$.

EXAMPLE 7.1. Let $T(x, z, y) = z$. Then pT^*q if and only if $p(x, z) \leqq q(y, z)$ (all x, y, z); hence pT^*p if and only if $p(x, z)$ is independent of z. Jacobs and Schach (1972) proved our Theorem 3 for this T under the assumption that $p, q \in R(T)$, i.e., the processes concerned were sequences of independent random variables.

EXAMPLE 7.2. Let $T(x, z, y) = z + y - x$. Then pT^*q if and only if $p(x, x + z) \leqq q(y, y + z)$ (all x, y, z). For such p and q and any df's F and G, there are processes $(\{X_n\}, \Omega, p, F)$ and $(\{Y_n\}, \Omega, q, G)$ such that $Y_n - Y_0 \leqq X_n - X_0$ (all $n \in N$). Defining $S(x, z, y) = T(x, z, y)$ if $y \leqq x$, $= \infty$ otherwise, pS^*q if and only if $p(x, x + z) \leqq q(y, y + z)$ whenever $y \leqq x$, and we must then have $F \leqq G$ in order to guarantee $Y_n - Y_0 \leqq X_n - X_0$. Note that pT^*p if and only if $p(x, x + z)$ is independent of x, while pS^*p if and only if $p(x, x + z)$ is non-increasing in x for each z.

EXAMPLE 7.3. Our final example is related to a result of Lamperti (1959). Fix $M > 0$ and define $T(x, z, y) = z + 2M$ if $y \leqq x + 2M$, $= \infty$ otherwise. Then pT^*q if and only if $p(x, z) \leqq q(y, z + 2M)$ whenever $y \leqq x + 2M$. For such p and q and df's F and G with $F(z) \leqq G(z + 2M)$, there exist processes $(\{X_n\}, \Omega, p, F)$ and $(\{Y_n\}, \Omega, q, G)$ such that $P(Y_n \leqq X_n + 2M,\ n \in N) = 1$. Let p be such that $p(x, x - M) = 0$, $p(x, x + M) = 1$, and $p(x, x + z)$ is non-decreasing in x for each z. Then (cf. (5.3)) $x - M \leqq f(x; p, \alpha) \leqq x + M$, and so $f(y; p, \alpha) - f(x; p, \alpha) \leqq 2M$, if $y - x \leqq 2M$, whence by (6.1), $p \in R(T)$. In particular, $p \in R(T)$ if p is a transition function for sums of independent random variables with each term bounded by M.

8. Random walks. We give in Theorem 10 below a law of the iterated logarithm for a class of simple random walks on N_0 with a reflecting barrier at 0. To this end, we need to modify Theorem 3 in an obvious manner to cover the case where the state space of the process (and hence the domain of definition of the transition functions) is a subset of R. In the present context, a simple random walk is a Markov process such that $p(0, 0) = 0$, and for $(i, j) \in N_0 \times N_0$, $p(i, j) = 1$ if $j > i$, $= 0$ if $j < i - 1$, and $= p(i, j - 1)$ if $j = i$. Thus, p is determined by $\{p(i, i), i \in N\}$.

THEOREM 10. *Let* $(\{Y_n\}, \Omega, q, G)$ *be a simple random walk for which* q *satisfies*

$$q(i, i) \geqq (i - m - \nu)/(2(i - m)) , \qquad i > m , \tag{8.1}$$

for some $m \in N_0$ *and some* $\nu < 1$. *Then*

$$P(\limsup_{n\to\infty} Y_n/(2n \log \log n)^{\frac{1}{2}} \leqq 1) = 1 . \tag{8.2}$$

PROOF. Let p be a transition function for a simple random walk with $p(i, i) = (i - \nu)/(2i)$ for $i \in N$. Let T be the comparison function $T(i, k, j) = k + m + 1$ for $j \leqq i + m + 1$, $= \infty$ otherwise. It is not difficult to check that pT^*q, and since $T_n(i^n, j) = T(i_0, i_n, j)$, it follows that there exist random walks $(\{X_n\}, \Omega', p, G)$ and $(\{Z_n\}, \Omega', q, G)$ such that $P(Z_n \leqq X_n + m + 1$, all $n) = 1$. Now Brezis, Rosenkrantz and Singer (1971) showed that (8.2) holds for $\{X_n\}$, so it holds for $\{Z_n\}$ and hence for $\{Y_n\}$.

The comparison technique can also be used to get a lower bound for certain simple random walks. We give a shorter proof of the following result due to Brezis *et al.* (1971).

THEOREM 11. *Let* $(\{Y_n\}, \Omega, q, G)$ *be a simple random walk such that* $q(i, i) \leqq \frac{1}{2}$ *for all* $i \in N_0$. *Then*

$$P(\limsup_{n\to\infty} Y_n/(2n \log \log n)^{\frac{1}{2}} \geqq 1) = 1 . \tag{8.3}$$

PROOF. Assume $G(0) = 1$ for convenience, and let $T(i, k, j) = k$ if $j \leqq i$, $= \infty$ otherwise. Let p be a transition function for a simple random walk with $p(i, i) = \frac{1}{2}$ $(i \in N)$. Then in the proof of Theorem 3 with $F = G$, we require only that $p(i, k) \leqq q(j, T(i, k, j))$ for i and j both even or both odd, and this does in fact hold. Therefore there exist simple random walks $(\{X_n\}, \Omega', p, G)$ and $(\{Z_n\}, \Omega', q, G)$ with $X_n \leqq Z_n$ a.s. By the law of the iterated logarithm for X_n, (8.3) holds for Z_n, and hence for Y_n.

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