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Exercise 3

$D : V \to V$ can be represented in matrix form as

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which is upper triangular with all diagonal elements zero. Thus, all eigenvalues are 0, with algebraic multiplicity 3. However, if x is an eigenvector of D corresponding to $\lambda = 0$, then $Dx = 0$. Given the form of D we conclude that $x_2 = x_3 = 0$ and so the eigenspace of $\lambda = 0$ is span$\{1\}$. Therefore, $\lambda = 0$ has geometric multiplicity 1.

Exercise 4

(i) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since A is hermitian, we know that $a = \bar{a}$, $d = \bar{d}$ and $b = \bar{c}$. Then

$$\begin{gathered} \det A = ad - cb = ad - c\bar{c} = ad - ||c||^2 = \\ \bar{a}\bar{d} - ||c||^2 = \overline{ad - ||c||^2} = \overline{\det A} \end{gathered}$$

and

$$\mathrm{Tr}(A) = a + d = \bar{a} + \bar{d} = \overline{a + d} = \overline{\mathrm{Tr}(A)}.$$

Thus both the determinant and the trace of A are real. Notice that usign Exercise 3 we have

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - ||c||^2)}}{2}$$

and the discriminant becomes $(a-d)^2 + 4||c||^2$, which is real and nonnegative, therefore A has only real eigenvalues.

(ii) If A is skew-symmetric, then $a = -\bar{a}$ and $d = -\bar{d}$, so they are imaginary, and $b = -\bar{c}$. Thus $bc = -||c||^2$, and

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a-d)^2 - 4||c||^2}}{2}.$$

Let $a = \alpha i$ and $d = \beta i$. Then $(a-d)^2 = i^2(\alpha - \beta)^2$ is clearly negative. Therefore the discriminant is negative and the eigenvalues are all imaginary.

Exercise 6

Let $R \in \mathbb{M}_n(\mathbb{F})$ be an upper-triangular matrix with diagonal entries r_{ii}. Then $\lambda I - R$ is also upper-triangular and so $\det R = \Pi_{i=1}^n(\lambda_i - r_{ii})$. Since r_{ii} are the roots of the characteristic polynomials, $\lambda_i = r_{ii}$.

Exercise 8

(i) We know that V is the span of S. If the vectors in S are linearly independent, then S is a basis for V. From Problem Set 2 we noticed that the vectors in S are orthonormal under the inner product $< a, b >:= \frac{1}{\pi}\int_{-\pi}^{\pi} a(x)b(x)dx$. Therefore they are independent and are thus a basis of V.

(ii) Since $d\sin(x)/dx = cos(x)$, $d\cos(x)/dx = -\sin(x)$, $d\sin(2x)/dx = 2\cos(2x)$ and $d\cos(2x)/dx = -2\sin(2x)$, we have that

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}.$$

(iii) Two complementary $D-$invariant subspaces are $\text{span}\{\sin(x), \cos(x)\}$ and $\text{span}\{\sin(2x), \cos(2x)\}$.

Exercise 13

Since $\det(\lambda I - A) = \lambda^2 - 1.4\lambda + 0.4$, the eigenvalues are 1 and 0.4, with corresponding eigenvectors $(2, 1)$, and $(1, -1)$. Therefore

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Exercise 15

A is semisimple, thus there exist matrices Λ and P such that $A = P\Lambda P^{-1}$. Then

$$f(A) = a_0PP^{-1} + a_1P\Lambda P^{-1} + \ldots + a_nP\Lambda^nP^{-1} = Pf(\Lambda)P^{-1},$$

where $f(\Lambda)$ is diagonal with elements $(f(\lambda_i))_{i=1}^n$. Since $f(A)$ is similar to $f(\Lambda)$, they have the same eigenvalues.

Exercise 16

(i) By Proposition 4.3.10,

$$A^k = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.4^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Consider the matrix

$$B = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

Their difference is

$$A^k - B = \frac{1}{3}\begin{bmatrix} 0.4^k & -2 \times 0.4^k \\ -0.4^k & 2 \times 0.4^k \end{bmatrix},$$

and its $1-$norm is $4/3 \times 0.4^k$, which converges to 0.

(ii) The $\infty-$norm of $A^k - B$ is 0.4^k, whereas the Frobenius norm is

$$\sqrt{\operatorname{tr}\left((A^k - B)^T (A^k - B)\right)} = \sqrt{10 \times 0.4^{2k}}$$

and both of them converge to zero.

(iii) By theorem 4.3.12, the eigenvalues of $3I + 5A + A^3$ are given by $f(\lambda_i) = 3 + 5\lambda_i + \lambda_i^3$, where λ_i's are the eigenvalues of A. So the eigenvalues are $f(1) = 9$ and $f(0.4) = 5.064$.

Exercise 18

Let λ be an eigenvalue of A, then it is also an eigenvalue of A^T. Then there exists a nonzero vector x such that $A^T x = \lambda x$. Transposing both the RHS and the LHS we get the desired result.

Exercise 20

Since A is orthonormally similar to B, we know that there exists an orthonormal P such that $B = PAP^H$. Since A is hermitian,

$$B^H = (PAP^H)^H = PA^H P^H = PAP^H = B.$$

Exercise 24

First notice that the denominator is real nonnegative. Also, notice that if A is hermitian, then

$$x^H Ax = x^H A^H x = (x^H Ax)^H = \overline{x^H Ax}.$$

Thus $< x, Ax >= \overline{< x, Ax >}$, and so it is real. On the other hand, if A is skew-hermitian, then

$$x^H Ax = -x^H A^H x = -(x^H Ax)^H = -\overline{x^H Ax}.$$

Thus $< x, Ax >= -\overline{< x, Ax >}$, and is therefore imaginary.

Exercise 25

(i) Take an arbitrary vector x in $\mathbb{C}^n$, then there exist coefficients a_i's such that $x = \sum_i a_i x_i$, since $\{x_i\}_i$ is a basis. Then

$$\left(\sum_j x_j x_j^H\right) \sum_i a_i x_i = \sum_j x_j x_j^H a_j x_j + \sum_j \sum_{i \neq j} x_j x_j^H a_i x_i = \sum_j a_j x_j$$

because $x_j^H x_j = 1$ for any j and $x_j^H x_i = 0$ for any $i \neq j$. Thus $(\sum_j x_j x_j^H)x = x$ for any x in $\mathbb{C}^n$. It must then be that $\sum_j x_j x_j^H = I$.

(ii) Notice that

$$Ax = \sum_j A a_j x_j = \sum_j a_j \lambda_j x_j$$

and

$$\left(\sum_j \lambda_j x_j x_j^H\right)\left(\sum_i a_i x_i\right) = \sum_j \lambda_j x_j x_j^H a_j x_j + \sum_j \sum_{i \neq j} \lambda_j x_j x_j^H a_i x_i = \sum_j a_j \lambda_j x_j,$$

shows that $A = \sum_j \lambda_j x_j x_j^H$.

Exercise 27

Since A is positive definite, it is hermitian, hence its diagonal elements are reals. Also, let e_i denote i^{th} standard basis vector of $\mathbb{F}^n$. Then we have that, for any i:

$$a_{ii} = e_i^H A e_i = \langle e_i, A e_i \rangle > 0.$$

Exercise 28

By proposition 4.5.7, There exist matrices S_A and S_B such that $A = S_A^H A_A$ and $B = S_B^H S_B$. Then

$$\operatorname{Tr}(AB) = \operatorname{Tr}(S_A^H S_A S_B^H S_B) = \operatorname{Tr}(S_B S_A^H S_A S_B^H) = \operatorname{Tr}((S_A S_B^H)^H S_A S_B^H) = ||S_A S_B^H||_F^2 \geq 0.$$

By Proposition 4.5.6 $A = Q_A D_A Q_A^H$ and $B = Q_B D_B Q_D^H$, where Q_A and Q_B are orthonormal and D_A, D_B are diagonal matrices containing the eigenvalues of A and B respectively. Since the trace is invariant under orthonormal transformations we have

$$\operatorname{Tr}(AB) = \operatorname{Tr}(D_A D_B) = \sum_i \lambda_i^A \lambda_i^B \leq \left(\sum_i \lambda_i^A\right)\left(\sum_i \lambda_i^B\right) = \operatorname{Tr}(A)\operatorname{Tr}(B),$$

which concludes the proof.

Exercise 31

(i) Let $B = A^H A$, then B is hermitian. Then by Corollary 4.4.8 B has an orthonormal eigenbasis, say $\{b_i\}_{i=1}^n$, which spans $\mathbb{F}^n$, and real eigenvalues $\{\sigma_i\}_{i=1}^n$. Take an arbitrary $x \in \mathbb{F}^n$ and real $(a_i)_{i=1}^n$ such that $x = \sum_i a_i b_i$. We have

$$||x||_2 =< \sum_i a_i b_i, \sum_i a_i b_i >^{1/2}= \sqrt{\sum_i a_i^2}$$

since the b_i's are orthonormal. Also

$$Bx = B\left(\sum_i a_i b_i\right) = \sum_i a_i \sigma_i b_i.$$

Let σ_1 be the largest eigenvalue of B. Then

$$\begin{aligned} ||Ax|| = &< Ax, Ax >=< x, A^H Ax >=< x, Bx >= \\ &< \sum_i a_i b_i, \sum_i \sigma_i a_i b_i >= \sqrt{\sum_i a_i \sigma_i \bar{a}_i} \leq ||x|| \max_i \sqrt{|\sigma_i|}. \end{aligned}$$

So $||A|| = \sup\{||Ax|| \ : \ ||x|| = 1\} \leq \max_i \sqrt{|\sigma_i|}$.

If we pick $x = b_1$, then

$$||A|| \geq< b_1, Bb_1 >=< b_1, \sigma_1 b_1 >= \sqrt{|\sigma_1|},$$

which proves the result.

(ii) Let $A = U\Sigma V^H$ be the singular value decomposition of A. Then, $A^{-1} = (V^H)^{-1}\Sigma^{-1}U^{-1}$. This is also a singular value decomposition, since $(V^H)^{-1} = V$ and $U^{-1} = U^H$, because U and V are orthonormal matrices, and Σ^{-1} is diagonal. The largest singular value of Σ^{-1} is $\frac{1}{\sigma_n}$, where σ_n is the smallest one of A. Therefore, by (i) we have that $\left\|A^{-1}\right\|_2 = \sigma_n^{-1}$.

(iii) Let $A = U\Sigma V^H$. Then,

$$A^H = V\Sigma^H U^H A^T = \overline{V}\Sigma U^T A^H A = (V\Sigma^H U^H)(U\Sigma V^H) = V(\Sigma^H\Sigma)V^H$$

All of these are singular value decompositions. Now, consider the singular values of each of these decompositions (where $\Sigma^H = \Sigma$ because singular values are reals). Thus, A^H, A^T and A all have the same singular values. So by (i) we have that $\left\|A^H\right\|_2^2 = \left\|A^T\right\|_2^2 = \|A\|_2^2 = \sigma_1^2$. Also, notice that the diagonal elements of $(\Sigma^H\Sigma)$ are the singular values squared. We conclude that $\left\|A^H A\right\|_2 = \sigma_1^2$.

Exercise 36

$-I$ has both eigenvalues -1 and both singular values 1.

Exercise 38

I will prove (1) and (5), the other proofs follow a similar logic.

(1)

Let $A = U_1 \Sigma_1 V_1^H$ be the compact SVD of A. The Moore-Penrose pseudoinverse of A is $A^\dagger = V_1 \Sigma_1^{-1} U_1^H$. Notice that,

$$\begin{aligned} AA^\dagger A &= (U_1\Sigma_1 V_1^H)(V_1\Sigma_1^{-1}U_1^H)(U_1\Sigma_1 V_1^H) = U_1\Sigma_1(V_1^H V_1)\Sigma_1^{-1}(U_1^H U_1)\Sigma_1 V_1^H \\ &= U_1\Sigma_1\Sigma_1^{-1}\Sigma_1 V_1^H = U_1\Sigma_1 V_1^H = A \end{aligned}$$

Therefore, $AA^\dagger A = A$.

(5)

First, notice that by (1), $AA^\dagger AA^\dagger = AA^\dagger$, so that $AA^\dagger$ is idempotent. Then, notice that $AA^\dagger = U_1 U_1^H$ where U_1 is an orthonormal basis for the range of A. Let u_i denote the $m-$ dimensional i^{th} column of U_1. Then

$$U_1 U_1^H x = U_1(u_1^H x, \ldots, u_r^H x)^T = \sum_i u_i u_i^H x = \sum_i u_i^H x u_i = \sum_i < u_i, x > u_i,$$

which is the projection of x on the range of A.