

Linear Optimization Problem Set

Alberto Quaini

Exercise 3

$$\begin{aligned}
 &\text{maximize} && 4b + 3j \\
 &\text{subject to} && 15b + 10j \leq 1800 \\
 &&& 2b + 2j \leq 300 \\
 &&& j \leq 200 \\
 &&& b, j \geq 0
 \end{aligned}$$

Exercise 4

$$\begin{aligned}
 &\text{maximize} && 2x_{AB} + 5x_{AD} + 5x_{BC} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 2x_{CF} + 4x_{DE} + 3x_{EF} \\
 &\text{subject to} && x_{AB} + x_{AD} = 10 \\
 &&& x_{BC} + x_{BD} + x_{BE} + x_{BF} - x_{AB} = 1 \\
 &&& x_{CF} - x_{BC} = -2 \\
 &&& x_{DE} - x_{AD} - x_{BD} = -3 \\
 &&& x_{EF} - x_{BE} - x_{DE} = 4 \\
 &&& -x_{BF} - x_{CF} - x_{EF} = -10 \\
 &&& 0 \leq x_{AB}, x_{AD}, x_{BC}, x_{BD}, x_{BE}, x_{BF}, x_{CF}, x_{DE}, x_{EF} \leq 6
 \end{aligned}$$

Exercise 5

(i)

$$\begin{aligned}
 &\text{maximize} && 3x_1 + x_2 \\
 &\text{subject to} && x_1 + 3x_2 + w_1 = 15 \\
 &&& 2x_1 + 3x_2 + w_2 = 18 \\
 &&& x_1 - x_2 + w_3 = 4 \\
 &&& x_1, x_2, w_1, w_2, w_3 \geq 0
 \end{aligned}$$

ζ	$=$		$3x_1$	$+$	x_2	
<hr/>						
w_1	$=$	15	$-$	x_1	$-$	$3x_2$
w_2	$=$	18	$-$	$2x_1$	$-$	$3x_2$
w_3	$=$	4	$-$	x_1	$+$	x_2
<hr/>						
ζ	$=$	12	$+$	$4x_2$	$-$	$3w_3$
<hr/>						
w_1	$=$	11	$-$	$4x_2$	$+$	w_3
w_2	$=$	10	$-$	$5x_2$	$+$	$2w_3$
x_1	$=$	4	$+$	x_2	$-$	w_3
<hr/>						
ζ	$=$	20	$-$	$\frac{4}{5}w_2$	$-$	$\frac{7}{5}w_3$
<hr/>						
w_1	$=$	3	$+$	$\frac{4}{5}w_2$	$-$	$\frac{3}{5}w_3$
x_2	$=$	2	$-$	$\frac{1}{5}w_2$	$+$	$\frac{2}{5}w_3$
x_1	$=$	6	$-$	$\frac{1}{5}w_2$	$-$	$\frac{3}{5}w_3$
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Optimizer: (6, 2)
 Optimum value: 20

(ii)

$$\begin{aligned}
 &\text{maximize} && 4x + 6y \\
 &\text{subject to} && -x + 3x_2 + w_1 = 11 \\
 &&& x + y + w_2 = 27 \\
 &&& 2x + 5y + w_3 = 90 \\
 &&& x, y, w_1, w_2, w_3 \geq 0
 \end{aligned}$$

ζ	=		$4x$	+	$6y$	
w_1	=	11	+	x	- y	
w_2	=	27	-	x	- y	
w_3	=	90	-	$2x$	- $5y$	
<hr/>						
ζ	=	66	+	$10x$	- $6w_1$	
y	=	11	+	x	- w_1	
w_2	=	16	-	$2x$	+	w_1
w_3	=	35	-	$7x$	+	$5w_1$
<hr/>						
ζ	=	116	+	$\frac{8}{7}w_1$	-	$\frac{10}{7}w_3$
y	=	16	-	$\frac{2}{7}w_1$	-	$\frac{1}{7}w_3$
w_2	=	6	-	$\frac{3}{7}w_1$	+	$\frac{2}{7}w_3$
x	=	5	+	$\frac{5}{7}w_1$	-	$\frac{1}{7}w_3$
<hr/>						
ζ	=	132	-	$\frac{8}{3}w_2$	-	$\frac{2}{7}w_3$
y	=	12	+	$\frac{2}{3}w_2$	-	$\frac{1}{3}w_3$
w_1	=	14	-	$\frac{7}{3}w_2$	+	$\frac{2}{3}w_3$
x	=	15	-	$\frac{5}{3}w_2$	+	$\frac{1}{3}w_3$

Optimizer: (15, 12)
 Optimum value: 132

Exercise 6

$$\begin{aligned}
 &\text{maximize} && 4b + 3j \\
 &\text{subject to} && 15b + 10j + w_1 = 1800 \\
 &&& 2b + 2j + w_2 = 300 \\
 &&& j + w_3 = 200 \\
 &&& b, j, w_1, w_2, w_3 \geq 0
 \end{aligned}$$

ζ	=			$4b$	+	$3j$
w_1	=	1800	-	$15b$	-	$10j$
w_2	=	300	-	$2b$	-	$2j$
w_3	=	200	-	j		
<hr/>						
ζ	=	450	+	b	-	$\frac{3}{2}w_2$
w_1	=	300	-	$5b$	+	$5w_2$
j	=	150	-	b	-	$\frac{1}{2}w_2$
w_3	=	50	+	b	+	$\frac{1}{2}w_2$
<hr/>						
ζ	=	510	-	$\frac{1}{5}w_1$	-	$\frac{1}{2}w_2$
b	=	60	-	$\frac{1}{5}w_1$	+	w_2
j	=	90	+	$\frac{1}{5}w_1$	-	$\frac{3}{2}w_2$
w_3	=	110	-	$\frac{1}{5}w_1$	+	$\frac{3}{2}w_2$

Optimal choice: 60 GI Barb soldiers, 90 Joey dolls

Maximal profit: \$510

Exercise 7

(i)

$$\begin{aligned}
 &\text{maximize} && x_1 + 2x_2 \\
 &\text{subject to} && -4x_1 - 2x_2 + x_3 = -8 \\
 &&& -2x_1 + 3x_2 + x_4 = 6 \\
 &&& x_1 + x_5 = 3 \\
 &&& x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

Auxiliary problem:

$$\begin{aligned}
 &\text{maximize} && -x_0 \\
 &\text{subject to} && -4x_1 - 2x_2 + x_3 - x_0 = -8 \\
 &&& -2x_1 + 3x_2 + x_4 - x_0 = 6 \\
 &&& x_1 + x_5 - x_0 = 3 \\
 &&& x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

ζ	=					-	x_0	
x_3	=	-8	+	$4x_1$	+	$2x_2$	+	x_0
x_4	=	6	+	$2x_1$	-	$3x_2$	+	x_0
x_5	=	3	-	x_1			+	x_0
ζ	=	-8	+	$4x_1$	+	$2x_2$	-	x_3
x_0	=	8	-	$4x_1$	-	$2x_2$	+	x_3
x_4	=	14	-	$2x_1$	-	$5x_2$	+	x_3
x_5	=	11	-	$5x_1$	-	$2x_2$	+	x_3
ζ	=						-	x_0
x_1	=	2	-	$\frac{1}{2}x_2$	+	$\frac{1}{4}x_3$	-	$\frac{1}{4}x_0$
x_4	=	10	-	$4x_2$	+	$\frac{1}{2}x_3$	+	$\frac{1}{2}x_0$
x_5	=	1	+	$\frac{1}{2}x_2$	-	$\frac{1}{4}x_3$	+	$\frac{5}{4}x_0$
ζ	=	2	+	$\frac{3}{2}x_2$	+	$\frac{1}{4}x_3$		
x_1	=	2	-	$\frac{1}{2}x_2$	+	$\frac{1}{4}x_3$		
x_4	=	10	-	$4x_2$	+	$\frac{1}{2}x_3$		
x_5	=	1	+	$\frac{1}{2}x_2$	-	$\frac{1}{4}x_3$		
ζ	=	3	+	$2x_2$	-	x_5		
x_1	=	3			-	x_5		
x_4	=	12	-	$3x_2$	-	$2x_5$		
x_3	=	4	+	$2x_2$	-	$4x_5$		
ζ	=	11	-	$\frac{2}{3}x_4$	-	$\frac{7}{3}x_5$		
x_1	=	3			-	x_5		
x_2	=	4	-	$\frac{1}{3}x_4$	-	$\frac{2}{3}x_5$		
x_3	=	4	-	$\frac{2}{3}x_4$	-	$\frac{16}{3}x_5$		

Optimal point: (3, 4)

Optimal value: 11

(ii)

$$\begin{aligned}
& \text{maximize} && 5x_1 + 2x_2 \\
& \text{subject to} && 5x_1 + 3x_2 + x_3 = 15 \\
& && 3x_1 + 5x_2 + x_4 = 15 \\
& && 4x_1 - 3x_2 + x_5 = -12 \\
& && x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

Auxiliary problem:

$$\begin{aligned}
& \text{maximize} && -x_0 \\
& \text{subject to} && 5x_1 + 3x_2 + x_3 - x_0 = 15 \\
& && 3x_1 + 5x_2 + x_4 - x_0 = 15 \\
& && 4x_1 - 3x_2 + x_5 - x_0 = -12 \\
& && x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

ζ	$=$						$-$	x_0
<hr/>								
x_3	$=$	15	$-$	$5x_1$	$-$	$3x_2$	$+$	x_0
x_4	$=$	15	$-$	$3x_1$	$-$	$5x_2$	$+$	x_0
x_5	$=$	-12	$-$	$4x_1$	$+$	$3x_2$	$+$	x_0
<hr/>								
ζ	$=$	-12	$-$	$4x_1$	$+$	$3x_2$	$-$	x_5
<hr/>								
x_3	$=$	27	$-$	x_1	$-$	$6x_2$	$+$	x_5
x_4	$=$	27	$+$	x_1	$-$	$8x_2$	$+$	x_5
x_0	$=$	12	$+$	$4x_1$	$-$	$3x_2$	$+$	x_5
<hr/>								
ζ	$=$	$-\frac{15}{8}$	$-$	$\frac{29}{8}x_1$	$-$	$\frac{3}{8}x_4$	$-$	$\frac{5}{8}x_5$
x_3	$=$	$\frac{27}{4}$	$-$	$\frac{7}{4}x_1$	$+$	$\frac{3}{4}x_4$	$+$	$\frac{1}{4}x_5$
x_2	$=$	$\frac{27}{8}$	$+$	$\frac{1}{8}x_1$	$-$	$\frac{1}{8}x_4$	$+$	$\frac{1}{8}x_5$
x_0	$=$	$\frac{15}{8}$	$+$	$\frac{29}{8}x_1$	$+$	$\frac{3}{8}x_4$	$+$	$\frac{5}{8}x_5$

The original problem has no feasible solutions.

(iii)

$$\begin{aligned}
& \text{maximize} && -3x_1 + x_2 \\
& \text{subject to} && x_2 + x_3 = 4 \\
& && -2x_1 + 3x_2 + x_4 = 6 \\
& && x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

ζ	$=$			$-$	$3x_1$	$+$	x_2
<hr/>							
x_3	$=$	4				$-$	x_2
x_4	$=$	6	$+$	$2x_1$	$-$	$3x_2$	
<hr/>							
ζ	$=$	2	$-$	$\frac{7}{3}x_1$	$-$	$\frac{1}{3}x_4$	
x_3	$=$	2	$-$	$\frac{2}{3}x_1$	$+$	$\frac{1}{3}x_4$	
x_2	$=$	2	$+$	$\frac{2}{3}x_1$	$-$	$\frac{1}{3}x_4$	

Optimal point: $(0, 2)$

Optimal value: 2

Exercise 8

Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded, but where the objective function still has a unique feasible maximizer.

$$\begin{array}{ll}\text{maximize} & -x - y - z \\ \text{subject to} & x, y, z \geq 0\end{array}$$

Maximizer: $(0, 0, 0)$

Exercise 9

Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded and where the objective function has no maximizer.

$$\begin{array}{ll}\text{maximize} & x + y + z \\ \text{subject to} & x, y, z \geq 0\end{array}$$

Exercise 10

Give an example of a three-dimensional linear problem where the feasible region is empty.

$$\begin{array}{ll}\text{maximize} & x + y + z \\ \text{subject to} & x + y + z \leq -1 \\ & x, y, z \geq 0\end{array}$$

Exercise 11

Give an example of a three-dimensional linear problem where the feasible region is nonempty, closed, and bounded, but $(0, 0, 0)$ is not feasible.

$$\begin{array}{ll}\text{maximize} & x + y + z \\ \text{subject to} & x + y + z \geq 1 \\ & x + y + z \leq 4 \\ & x, y, z \geq 0\end{array}$$

Auxiliary problem:

$$\begin{array}{ll}\text{maximize} & -w \\ \text{subject to} & -x - y - z - w \leq -1 \\ & x + y + z - w \leq 4 \\ & x, y, z, w \geq 0\end{array}$$

Exercise 12

$$\begin{array}{ll}\text{maximize} & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0 \\ & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0 \\ & x_1 + x_7 = 0 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0\end{array}$$

ζ	=		$10x_1$	-	$57x_2$	-	$9x_3$	-	$24x_4$
x_5	=		$-0.5x_1$	+	$1.5x_2$	+	$0.5x_3$	-	x_4
x_6	=		$-0.5x_1$	+	$5.5x_2$	+	$2.5x_3$	-	$9x_4$
x_7	=	1	$-x_1$						
<hr/>									
ζ	=		$-27x_2$	+	x_3	-	$44x_4$	-	$20x_5$
x_1	=		$3x_2$	+	x_3	-	$2x_4$	-	$2x_5$
x_6	=		$4x_2$	+	$2x_3$	-	$8x_4$	+	x_5
x_7	=	1	$-3x_2$	-	x_3	+	$2x_4$	+	$2x_5$
<hr/>									
ζ	=	1	$-30x_2$	-	$42x_4$	-	$18x_5$	-	x_7
x_1	=	1						-	x_7
x_6	=	2	$-2x_2$	-	$4x_4$	+	$5x_5$	-	$2x_7$
x_3	=	1	$-3x_2$	+	$2x_4$	+	$2x_5$	-	x_7

Optimal point: $(1, 0, 1, 0)$ Optimum value: 1

Exercise 15

If $x \in \mathbb{R}^n$ is feasible for the primal and $y \in \mathbb{R}^m$ is feasible for the dual, then $c^T x \leq b^T y$.

Proof:

Notice that

$$\begin{aligned}
 c^T x &= x^T c \\
 &\leq x^T A^T y \\
 &= y^T A x \\
 &\leq y^T b \\
 &= b^T y.
 \end{aligned}$$

Exercise 17

For a linear optimization problem in standard form, the dual of the dual optimization problem is again the primal problem.

Proof:

Consider the primal problem:

$$\begin{aligned}
 &\text{maximize } c^T x \\
 &\text{subject to } Ax \preceq b \\
 &\quad \mathbf{x} \succeq 0
 \end{aligned}$$

with the dual problem:

$$\begin{aligned}
 &\text{minimize } b^T y \\
 &\text{subject to } A^T y \succeq c \\
 &\quad y \succeq 0
 \end{aligned}$$

The dual in standard form becomes:

$$\begin{aligned} & \text{maximize } (-b^T)y \\ & \text{subject to } (-A)^T y \preceq -c \\ & \quad y \succeq 0 \end{aligned}$$

and the dual of the dual is:

$$\begin{aligned} & \text{minimize } (-c^T)\mathbf{x} \\ & \text{subject to } ((-A)^T)^T x \succeq -b \\ & \quad x \succeq 0 \end{aligned}$$

In standard form, this is:

$$\begin{aligned} & \text{maximize } c^T x \\ & \text{subject to } Ax \preceq b \\ & \quad x \succeq 0 \end{aligned}$$

which is the original primal problem.

Exercise 18

Primal:

$$\begin{aligned} & \text{maximize } x_1 + x_2 \\ & \text{subject to } 2x_1 + x_2 + w_1 = 3 \\ & \quad x_1 + 3x_2 + w_2 = 5 \\ & \quad 2x_1 + 3x_2 + w_3 = 4 \\ & \quad x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

ζ	$=$		x_1	$+$	x_2	
w_1	$=$	3	$-$	$2x_1$	$-$	x_2
x_2	$=$	5	$-$	x_1	$-$	$3x_2$
x_5	$=$	4	$-$	$2x_1$	$-$	$3x_2$
<hr/>						
ζ	$=$	$\frac{3}{2}$	$+$	$\frac{1}{2}x_2$	$-$	$\frac{1}{2}w_1$
x_1	$=$	$\frac{3}{2}$	$-$	$\frac{1}{2}x_2$	$-$	$\frac{1}{2}w_1$
w_2	$=$	$\frac{7}{2}$	$-$	$\frac{5}{2}x_2$	$+$	$\frac{1}{2}w_1$
w_3	$=$	1	$-$	$2x_2$	$+$	w_1
<hr/>						
ζ	$=$	$\frac{7}{4}$	$-$	$\frac{1}{4}w_1$	$-$	$\frac{1}{4}w_3$
x_1	$=$	$\frac{5}{4}$	$-$	$\frac{3}{4}w_1$	$+$	$\frac{1}{4}w_3$
w_2	$=$	$\frac{9}{4}$	$-$	$\frac{3}{4}w_1$	$+$	$\frac{5}{4}w_3$
x_2	$=$	$\frac{1}{2}$	$+$	$\frac{1}{2}w_1$	$-$	$\frac{1}{2}w_3$

Optimal point: $(\frac{5}{4}, \frac{1}{2})$

Optimum value: $\frac{7}{4}$

Dual:

$$\begin{aligned} & \text{minimize} && 3y_1 + 5y_2 + 4y_3 \\ & \text{subject to} && 2y_1 + y_2 + 2y_3 \geq 1 \\ & && y_1 + 3y_2 + 3y_3 \geq 1 \\ & && y_1 + 3y_2 + 3y_3 \geq 1 \\ & && y_1, y_2, y_3 \geq 0 \end{aligned}$$

Dual in standard form:

$$\begin{aligned} & \text{maximize} && -3y_1 - 5y_2 - 4y_3 \\ & \text{subject to} && -2y_1 - y_2 - 2y_3 + v_1 - v_0 = -1 \\ & && -y_1 - 3y_2 - 3y_3 + v_2 - v_0 = -1 \\ & && y_1, y_2, y_3, v_1, v_2 \geq 0 \end{aligned}$$

ζ	=											-	v_0	
v_1	=	-1	+	$2y_1$	+	y_2	+	$2y_3$	+					v_0
v_2	=	-1	+	y_1	+	$3y_2$	+	$3y_3$	+					v_0
ζ	=	-1	+	$2y_1$	+	y_2	+	$2y_3$	-					v_1
v_0	=	1	-	$2y_1$	-	y_2	-	$2y_3$	+					v_1
v_2	=		-	y_1	+	$2y_2$	+	y_3	+					v_1
ζ	=											-	v_0	
y_2	=	1	-	$2y_1$	-	$2y_3$	+	v_1	-					v_0
v_2	=	2	-	$5y_1$	-	$3y_3$	+	$3v_1$	-					$2v_0$
ζ	=	-2	+	y_1	-	$3y_2$	-	$2v_1$						
y_3	=	$\frac{1}{2}$	-	y_1	-	$\frac{1}{2}y_2$	+	$\frac{1}{2}v_1$						
v_2	=	$\frac{1}{2}$	-	$2y_1$	+	$\frac{3}{2}y_2$	+	$\frac{3}{2}v_1$						
ζ	=	$-\frac{7}{4}$	-	$\frac{3}{2}y_2$	-	$\frac{5}{4}v_1$	-	$\frac{1}{2}v_2$						
y_3	=	$\frac{1}{4}$	-	$2\frac{3}{2}y_2$	-	$\frac{1}{4}v_1$	+	$\frac{1}{2}v_2$						
y_1	=	$\frac{1}{4}$	+	$\frac{3}{2}y_2$	+	$\frac{3}{4}v_1$	-	$\frac{1}{2}v_2$						

Optimal point: $(\frac{1}{4}, 0, \frac{1}{4})$

Optimum value: $\frac{7}{4}$