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# Measure theory exercises

## Section 1

**Exercise 1.3:**

1. Let $a \in \mathbb{R}$ and define $A_1 := (-\infty, a)$. Clearly $A_1 \in \mathcal{G}_1$, however its complement, $A_1^c = [a, +\infty)$, is not in $\mathcal{G}_1$. Therefore $\mathcal{G}_1$ is not an algebra.

2. $\mathcal{G}_2 := \{A : A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], (a, \infty)\}$ is an algebra, but not a $\sigma-$algebra. Clearly, $\mathcal{G}_2$ contains the empty set. Also, $\mathcal{G}_2$ is closed under complements because it contains the complements of the three basic intervals $(a, b]$, $(-\infty, b]$ and $(a, \infty)$ and, by the properties of complements, it contains the complements of any finite union of the basic intervals. Finally, $\mathcal{G}_2$ is closed under finite union as the finite union of finite unions of the three basic intervals is still a finite union of these basic intervals. However, $\mathcal{G}_2$ is not a $\sigma-$algebra since it clearly does not contain an infinite union of the three basic interval.

3. $\mathcal{G}_3 := \{A : A \text{ is a countable union of intervals of the form } (a, b], (-\infty, b], (a, \infty)\}$ is a $\sigma-$algebra, hence also an algebra. Everything discussed for $\mathcal{G}_2$ holds except for the fact that infinite unions of the basic intervals belong to $\mathcal{G}_3$.

**Exercise 1.7:** By definition, any $\sigma-$algebra contains $\emptyset$. Also, it contains $X$, since it must be closed under complements. Therefore, $\{\emptyset, X\}$ is contained in any $\sigma-$algebra and is thus the smallest $\sigma-$algebra on $X$.

On the other hand, by definition, any $\sigma-$algebra on $X$ is a set of subsets of $X$, therefore it is contained in the power set of $X$, which is the set of all subsets of $X$.

**Exercise 1.10:** Let $\mathcal{N} = \underset{\alpha}{\cap} \mathcal{S}_\alpha$. $\emptyset \in \mathcal{N}$ because $\emptyset \in \mathcal{S}_\alpha$, for every $\alpha$. Also, if $A \in \mathcal{N}$, we have that $A$ is in every $\mathcal{S}_\alpha$, and since these are closed under complements, $A^c \in \mathcal{N}$. Finally, if $A_1, A_2, \ldots \in \mathcal{N}$, they belong to each $\mathcal{S}_\alpha$ and so does $\cup_{n=1}^\infty A_n$, and we have that it also belongs to $\mathcal{N}$. In conclusion, $\mathcal{N}$ is a $\sigma-$algebra.

**Exercise 1.17:**

1. Take $A, B \in \mathcal{S}$, with $A \subset B$. Notice that $B$ can be written as the union of two disjoint sets in the following way: $B = (A \cap B) \cup (A^c \cap B)$. Then, $\mu(B) = \mu(A \cap B) + \mu(A^c \cap B) = \mu(A) + \mu(A^c \cap B)$. Since a measure is nonnegative, $\mu(B) \geq \mu(A)$. Therefore $\mu$ is monotone.

2. Let $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ and define the following sets: $A := \cup_{n \in \mathbb{N}} A_n$, $B_1 := A_1$, $B_2 := A_2 - A_1$, $B_3 := A_3 - (A_1 \cup A_2)$, and so on. Then, $A = \cup_{n \in \mathbb{N}} B_n$. By monotonicity, for each $n \in \mathbb{N}$, $\mu(B_n) \leq \mu(A_n)$ since $B_n \subset A_n$. Therefore we obtain $\mu(A) = \sum_{n \in \mathbb{N}} \mu(B_n) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$.

**Exercise 1.18:** $\lambda$ is a measure because (i) $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$ and (ii) for any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ with $A_n$'s paiwise disjoint we have $\lambda(\cup_{n \in \mathbb{N}} A_n) = \mu\left((\cup_{n \in \mathbb{N}} A_n) \cap B\right) = \mu\left(\cup_{n \in \mathbb{N}} (A_n \cap B)\right) = \sum_{n \in \mathbb{N}} \mu(A_n \cap B) = \sum_{n \in \mathbb{N}} \lambda(A_n)$.

**Exercise 1.20:** Since $\mu(A_1) < \infty$, by monotonicity $\mu(A_i) < \infty$ for each $n \in \mathbb{N}$. Consider the increasing sequence $\{A_1 - A_n\}_{n \in \mathbb{N}}$, define $A = \cap_{n \in \mathbb{N}} A_n$ and note that $\lim_{n \to \infty}(A_1 - A_n) = A_1 - \lim_{n \to \infty} A_n = A_1 - A$.

Since $\mu$ is continuous from below,

$$\begin{aligned}\mu(\cap_{i=1}^{\infty} A_i) &= \mu[A_1 - \cup_{i=1}^{\infty}(A_1 - A_i)] = \mu(A_1) - \mu(\cup_{i=1}^{\infty}(A_1 - A_n)) \\ &= \mu(A_1) - \lim_{n\to\infty} \mu(A_1 - A_n) = \mu(A_1) - \lim_{n\to\infty}[\mu(A_1) - \mu(A_n)] = \lim_{n\to\infty}\mu(A_n)\end{aligned}$$

Therefore $\mu(A) = \lim_{n\to\infty}\mu(A_n)$.

### Section 2

**Exercise 2.10:** Clearly, $B = [(B \cap E) \cup (B \cap E^c)] =: F$. In particular, $B \subset F$ and by monotonicity and countable subadditivity we have $\mu^*(B) \leq \mu^*(F) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Therefore requiring (*) is the same as requiring $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

### Exercise 2.14:

In order to show that $\sigma(\mathcal{B}) \subset \mathcal{M}$ we first prove that $\sigma(\mathcal{A}) = \sigma(\mathcal{O}) = \sigma(\mathcal{B})$ by showing that $\sigma(\mathcal{A})$ can generate open intervals and that $\sigma(\mathcal{O})$ can generate the three basic intervals of $\sigma(\mathcal{A})$, then we use Carateódory Extension Theorem shows that $\sigma(\mathcal{B}) \subset \mathcal{M}$.

First, notice that given two reals $a$ and $b$, $(a,b) = \bigcup_{n\in\mathbb{N}}(a, b-1/n]$. Thus $\sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. On the other hand, $(a,b] = \bigcap_{n\in\mathbb{N}}(a, b+1/n)$, $(a,\infty) = \bigcup_{n\in\mathbb{N}}(a,n)$ and $(-\infty, b] = \bigcup_{n\in\mathbb{N}}(-n, b]$ (we now can use intervals of the type $(-a, b]$ since we showed that they can be generated by $\sigma(\mathcal{O})$).

## Section 3

### Exercise 3.1:

Let $A := \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$. Also, fix $\epsilon > 0$ and define $A_n := (a_n - 2^{-n}\epsilon, a_n + 2^{-n}\epsilon)$ for every $n \in \mathbb{N}$. Notice that $A \subset \bigcup_{n\in\mathbb{N}} A_n$ and $\mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} 2^{1-n} = 2\epsilon$. Since this holds for any $\epsilon > 0$, by the definition of Lebesgue measure $\mu(A) = 0$.

### Exercise 3.4:

Since $\mathcal{M}$ is a $\sigma-$algebra, if $\{x \in X : f(x) < a\}$ is measurable, so are $\{x \in X : f(x) \leq a\} = \bigcap_{n\in\mathbb{N}}\{x \in X : f(x) < a + 1/n\}$ and their respective complements $\{x \in X : f(x) \geq a\}$ and $\{x \in X : f(x) > a\}$.

### Exercise 3.7:

Since $+$ and $\cdot$ are continuous binary functions and absolute values is a continuous unary function, they are special cases of 4. As for $\max\{f, g\}$ and $\min\{f, g\}$, these can be obtained via 2. by defining $\{f_n\}_{n\in\mathbb{N}}$ so that $f_n = f$ for $n$ even and $f_n = g$ for $n$ odd.

### Exercise 3.14:

Fix an $\epsilon > 0$. Since $f$ is bounded, there is an $M \in \mathbb{R}$ such that $|f| < M$ everywhere. so $X \subset E_i^M$ for some $i$. Note that there is an $N \geq M$ such that $\frac{1}{2^N} < \epsilon$. Then for any $n \geq N$, $||f(x) - s_n(x)|| < \epsilon$, so we have uniform convergence.

## Section 4

### Exercise 4.13:

Since $0 \leq ||f|| < M$, we can apply Proposition 4.5 to obtain $0 \leq \int_E ||f|| d\mu \leq M\mu(E) < \infty$. Therefore $f \in \mathcal{L}^1(\mu, E)$.

### Exercise 4.14:

Proof by contrapositive. Suppose there exists a measurable set $E' \subset E$ with positive $\mu-$measure such that $f(E') = \{\infty\}$ (we consider just $\infty$ without loss of generality). Then $\infty = \int_{E'} f d\mu \leq \int_E f d\mu \leq \int_E ||f|| d\mu$ (the proof of the first inequality can be found in the proof of Exercise 4.16). Therefore $f$ is not in $\mathcal{L}^1(\mu, E)$.

### Exercise 4.15:

Define $B(f) := \{s : 0 \leq s \leq f, s \text{ measurable and simple}\}$. Since $f \leq g$, $f^+ \leq g^+$ and $f^- \geq g^-$. Then $B(f^+) \subset B(g^+)$, which implies that $\int_E f^+ d\mu \leq \int_E g^+ d\mu$, and $B(g^-) \subset B(f^-)$, which implies that $\int_E f^- d\mu \geq \int_E g^- d\mu$. Therefore

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \leq \int_E g^+ d\mu - \int_E g^- d\mu = \int_E g d\mu.$$

### Exercise 4.16:

Fix an arbitrary measurable simple function $s(x) := \sum_{i=1}^N c_i \chi_{E_i}$ (definition from the lecture notes). Since $A \subset E$, $\mu(A \cap E_i) \leq \mu(E \cap E_i)$ for each $i$. Then $\int_A s d\mu := \sum_{i=1}^N c_i \mu(A \cap E_i) \leq \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu$. Since the choice of $s$ was arbitrary,

$$\int_A ||f|| d\mu = \sup \left\{ \int_A s d\mu : 0 \leq s \leq ||f||, s \text{ simple}, s \text{ measurable} \right\}$$

is less than or equal to

$$\int_E ||f|| d\mu = \sup \left\{ \int_E s d\mu : 0 \leq s \leq ||f||, s \text{ simple}, s \text{ measurable} \right\}.$$

### Exercise 4.21:

Define $\lambda_1(A) := \int_A f^+ d\mu$ and $\lambda_2(A) := \int_A f^- d\mu$, then $\int_A f d\mu = \lambda_1(A) - \lambda_2(A)$. Since $A = (A - B) \cup B$ and $\lambda_i$ is a measure for $i = 1, 2$ (Theorem 4.6), $\lambda_i(A) = \lambda_i(A - B) + \lambda_i(B)$ for $i = 1, 2$. However, by

Proposition 4.6 we have $\lambda_i(A - B) = 0$ for $i = 1, 2$. Therefore, $\lambda_i(A) = \lambda_i(B)$ for $i = 1, 2$. This implies that $\int_A f d\mu = \lambda_1(B) - \lambda_2(B) = \int_B f d\mu$, which implies the result of the corollary.