

## Inner Product Spaces Exercises

### Exercise 1

(i)

$$\begin{aligned} & (||x+y||^2 - ||x-y||^2) / 4 = \\ & (< x, x > + < y, y > + 2 < x, y > - < x, x > - < y, y > + 2 < x, y >) / 4 = \\ & < x, y > . \end{aligned}$$

(ii)

$$\begin{aligned} & (||x+y||^2 + ||x-y||^2) / 4 = \\ & (< x, x > + < y, y > + 2 < x, y > + < x, x > + < y, y > - 2 < x, y >) / 2 = \\ & < x, x > + < y, y > . \end{aligned}$$

### Exercise 2

$$\begin{aligned} & (||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2) / 4 = \\ & (< x+y, x+y > - < x-y, x-y > + i < x-iy, y-iy > - i < x+iy, x+iy >) / 4 = \\ & (2 < x, y > + 2 < y, x > - 2 < x, y > + 2 < y, x >) / 4 = \\ & < x, y > . \end{aligned}$$

### Exercise 3

$< x, x^5 > = \int_0^1 x^6 dx = x^7/7|_0^1 = 1/7$ ,  $||x|| = \int_0^1 x^2 dx = x^3/3|_0^1 = 1/3$  and  $||x^5|| = \int_0^1 x^10 dx = x^11/11|_0^1 = 1/11$ . Therefore  $\cos \theta = \sqrt{33}/7$  implies  $\theta = 34.5$ .

### Exercise 8

(i)

$$||\cos(t)|| = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(t) dt = \frac{1}{\pi} \left. \frac{\cos(x) \sin(x) - x}{2} \right|_{-\pi}^{\pi} \quad i = \frac{\pi}{\pi} = 1,$$

and similarly  $||\sin(t)|| = 1$ . Also

$$||\cos(2t)|| = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(2t) dt = \frac{1}{\pi} \left. \frac{\sin(4t) + 4t}{8} \right|_{-\pi}^{\pi} \quad i = \frac{\pi}{\pi} = 1,$$

and similarly  $||\sin(2t)|| = 1$ . Therefore the basis is normalized.

The following integrals:

$$\langle \cos(t), \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \sin(t) dt = \frac{1}{\pi} \left. \frac{\sin^2(x)}{x} \right|_{-\pi}^{\pi} = 0,$$

$$\langle \cos(t), \cos(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \cos(2t) dt = \frac{1}{\pi} \left. \frac{3 \sin(t) - 2 \sin^3(t)}{3} \right|_{-\pi}^{\pi} = 0,$$

$$\langle \cos(t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \sin(2t) dt = \frac{1}{\pi} \left. \frac{-2 \cos^3(t)}{3} \right|_{-\pi}^{\pi} = 0,$$

$$\langle \cos(2t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t) \sin(2t) dt = \frac{1}{\pi} \left. \frac{-\cos^2(2t)}{2} \right|_{-\pi}^{\pi} = 0,$$

and so on, shows that  $S$  is an orthonormal basis.

(ii)

$$\|t\| = \frac{1}{\pi} \int_{-\pi}^{\pi} t dt = \frac{1}{\pi} \left. \frac{t^2}{2} \right|_{-\pi}^{\pi} = 0.$$

(iii) Since  $\langle x, \cos(3x) \rangle = 0$  for any  $x \in S$ ,  $\text{proj}_X(\cos(3x)) = 0$ .

(i)

$$\langle \sin(t), t \rangle = \sin(t) - t \cos(t) \Big|_{-\pi}^{\pi} = 2\pi,$$

$$\langle \cos(t), t \rangle = t \sin(t) - \cos(t) \Big|_{-\pi}^{\pi} = 0,$$

$$\langle \cos(2t), t \rangle = (2t \sin(2t) + \cos(2t)) / 4 \Big|_{-\pi}^{\pi} = 0, \text{ and finally}$$

$$\langle \sin(2t), t \rangle = \sin(2x) - 2x \cos(2x) / 4 \Big|_{-\pi}^{\pi} = -\pi.$$

Therefore,  $\text{proj}_X(t) = 2\pi \sin(t) - \pi \sin(2t)$

## Exercise 9

A rotation of angle  $\theta$  in  $\mathbb{R}^2$  represented as a matrix  $R$  in the standard basis is an orthonormal transformation since

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^T \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly, one shows that  $RR^T = I$ . Therefore, a rotation in  $\mathbb{R}^2$  is an orthonormal transformation.

## Exercise 10

(i) Suppose  $Q$  represents an orthonormal operator on  $\mathbb{F}^n$ . Then  $\langle x, y \rangle = \langle Q(x), Q(y) \rangle$  for each  $x, y \in \mathbb{F}^n$ . Since  $\langle Q(x), Q(y) \rangle = (Qx)^H(Qy) = x^H Q^H Q y$ , it equals  $x^H y$  for all  $x, y \in \mathbb{F}^n$  only if  $Q^H Q = I$ . On the other hand if  $Q^H Q = Q Q^H = I$ , then  $\langle Q(x), Q(y) \rangle = (Qx)^H(Qy) = x^H Q^H Q y = x^H y = \langle x, y \rangle$ .

(ii)

$$\|Qx\| = \sqrt{\langle Qx, Qx \rangle} = \sqrt{x^H Q^H Q x} = \sqrt{\langle x, x \rangle} = \|x\|.$$

(iii) If  $Q^H Q = Q Q^H = I$ , then  $Q^{-1} = Q^H$ . Since  $(Q^H)^H = Q$ ,  $Q^H$  is also orthonormal:

$$(Q^H)^H Q^H = Q Q^H = I = Q^H Q = Q^H (Q^H)^H.$$

(iv) Let  $q_i$  denote the  $i^{\text{th}}$  column of  $Q$ . Since  $Q$  is orthonormal,  $(Q^H Q)_{ij} = q_i^H q_j = \langle q_i, q_j \rangle$  is 1 if  $i = j$  and 0 if  $i \neq j$ . Thus, the columns of  $Q$  are orthonormal.

(v) The matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

shows that the converse is not true.

(vi)

$$(Q_1 Q_2)^H Q_1 Q_2 = Q_2^H Q_1^H Q_1 Q_2 = Q_2^H Q_2 = I$$

and

$$Q_1 Q_2 (Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H = Q_1 Q_1^H = I.$$

Therefore,  $Q_1 Q_2$  is orthonormal.

## Exercise 11

Fix  $N \in \mathbb{N}$ ,  $N > 0$ , and suppose  $\{x_i\}_{i=1}^N$  is a set of linearly dependent vectors in  $V$ . Also, suppose, without loss of generality, that for  $2 < k < N$ ,  $\{x_i\}_{i=1}^{k-1}$  is a linearly independent set and  $\{x_i\}_{i=1}^k$  is a linearly dependent set. Then  $\{q_i\}_{i=1}^{k-1}$  (as they are defined in the book) is also a linearly independent set. However, since  $x_k \in \text{span}(\{x_i\}_{i=1}^{k-1})$ , we have that  $q_k = 0$ . Therefore the Gram-Schmidt orthonormalization process brakes down.

## Exercise 16

(i) Let  $A \in \mathbb{M}_{m \times n}$  where  $\text{rank}(A) = n \leq m$ . Then there exist orthonormal  $Q \in \mathbb{M}_{m \times m}$  and upper triangular  $R \in \mathbb{M}_{m \times n}$  such that  $A = QR$ . Since  $\tilde{Q} = -Q$  is still orthonormal  $(-Q)(-Q)^H = -Q(-Q^H) = QQ^H = I$  and similarly one shows  $(-Q)^H(-Q) = I$  and  $\tilde{R} = -R$  is still upper triangular,  $A = QR = \tilde{Q}\tilde{R}$ . Therefore QR-decomposition is not unique.

(ii) Now take a reduced QR-decomposition  $A = \hat{Q}\hat{R}$ , where  $\hat{Q} \in \mathbb{M}_{m \times n}$  is orthonormal and  $\hat{R} \in \mathbb{M}_{n \times n}$  is

upper triangular. Since  $A$  has full column rank,  $\hat{R}$  has full rank and is therefore nonsingular. Then,

$$\begin{aligned} A^H A x &= A^H b \implies \\ (\hat{Q} \hat{R})^H \hat{Q} \hat{R} x &= (\hat{Q} \hat{R})^H b \implies \\ \hat{R}^H \hat{Q}^H \hat{Q} \hat{R} x &= \hat{R}^H \hat{Q}^H b, \end{aligned}$$

and premultiplying both LHS and RHS of the last equation by  $\hat{R}^{-1}$  gives  $\hat{R} x = \hat{Q}^H b$ .

### Exercise 23

Let  $x, y \in V$  and define  $v := -y$ . Since a norm is nonnegative and satisfies the triangular property,  $\|x\| - \|v\| \leq \|x\| + \|v\| \leq \|x + v\|$ . Then our definition of  $v$  implies  $\|x\| - \|y\| = \|x\| - \|-y\| \leq \|x - y\|$ . Interchanging the role of  $x$  and  $y$  and using the homogeneity property of norms we have  $\|y\| - \|x\| \leq \|y - x\| = \|-(x - y)\| = \|x - y\|$ , and the result follows.

### Exercise 24

(i) Since  $|f(t)| \geq 0$  for every  $t$ , so is  $\int_a^b |f(t)| dt$ . In addition, if  $f = 0$ , then  $\int_a^b |f(t)| dt = 0$ . On the other hand, if  $\int_a^b |f(t)| dt = 0$  and  $|f(t)| \geq 0$ , it must be that  $|f(t)| = 0$  for all  $t$ , implying that  $f = 0$ . Now take a constant  $c \in \mathbb{F}$ , then  $\int_a^b |cf(t)| dt = \int_a^b |c| |f(t)| dt = |c| \int_a^b |f(t)| dt$ , since  $c$  does not depend on  $t$ . Finally, take  $g \in C([a, b]; \mathbb{F})$ . Since  $|f(t) + g(t)| \leq |f(t)| + |g(t)|$  for all  $t$  and the integral is a linear operator, we have that  $\int_a^b |f(t) + g(t)| dt \leq \int_a^b |f(t)| dt + \int_a^b |g(t)| dt$ .

(ii) Since  $|f(t)|^2 \geq 0$  for every  $t$ , so is  $\int_a^b |f(t)|^2 dt$  and its square root. In addition, if  $f = 0$ , then  $|f(t)|^2 = 0$  for all  $t$  and  $\sqrt{\int_a^b |f(t)|^2 dt} = 0$ . On the other hand, if  $\sqrt{\int_a^b |f(t)|^2 dt} = 0$ , then  $\int_a^b |f(t)|^2 dt = 0$  and since  $|f(t)|^2 \geq 0$  for all  $t$ , it must be that  $|f(t)|^2 = 0$  for all  $t$ , implying that  $f = 0$ . Now take a constant  $c \in \mathbb{F}$ , then  $\sqrt{\int_a^b |cf(t)|^2 dt} = \sqrt{\int_a^b |c|^2 |f(t)|^2 dt} = |c| \sqrt{\int_a^b |f(t)|^2 dt}$ , since  $c$  does not depend on  $t$ . Finally, take  $g \in C([a, b]; \mathbb{F})$ . Since  $|f(t) + g(t)| \leq |f(t)| + |g(t)|$  for all  $t$ ,  $x \mapsto x^2$  and  $x \mapsto \sqrt{x}$  are monotonically increasing for nonnegative  $x$  and the integral is a linear operator, we have that  $\sqrt{\int_a^b |f(t) + g(t)|^2 dt} \leq \sqrt{\int_a^b |f(t)|^2 dt + \int_a^b |g(t)|^2 dt} \leq \|f\|_{L^2} + \|g\|_{L^2}$ .

(iii) Since  $|f(x)| \geq 0$  for all  $x$ , so is the  $\sup_{x \in [a, b]} |f(x)|$ . In addition, if  $f = 0$ , then  $\sup_{x \in [a, b]} |f(x)|$  is also zero. On the other hand, since  $|f(x)| \geq 0$  for all  $x$ ,  $0 \leq \sup_{x \in [a, b]} |f(x)| = 0$  implies that we must have  $f = 0$ . Now take a constant  $c \in \mathbb{F}$ , then  $\sup_{x \in [a, b]} |cf(x)| = \sup_{x \in [a, b]} |c| |f(x)| = |c| \sup_{x \in [a, b]} |f(x)|$ . Finally, take  $g \in C([a, b]; \mathbb{F})$ . Since  $|f(x) + g(x)| \leq |f(x)| + |g(x)|$  for all  $x$ , we have that  $\sup_{x \in [a, b]} |f(x) + g(x)| \leq \sup_{x \in [a, b]} \{|f(x)| + |g(x)|\} \leq \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |g(x)|$ .

### Exercise 26

We show that topological equivalence is an equivalence relation. Let  $\|\cdot\|_r$  be a norm on  $X$  for  $r \in \{a, b, c\}$ . Clearly  $\|\cdot\|_r$  is in topologically equivalent with itself, just pick any  $0 < m \leq 1$  and any  $M \geq 1$  to show this. Also, suppose that  $\|\cdot\|_a$  is topologically equivalent to  $\|\cdot\|_b$  with constants  $0 < m \leq M$ . Then,  $\|\cdot\|_b$  is topologically equivalent to  $\|\cdot\|_a$  with constants  $0 < 1/M' \leq 1/m'$ . Finally, if  $\|\cdot\|_a$  is topologically equivalent to  $\|\cdot\|_b$  with constants  $0 < m \leq M$  and so is  $\|\cdot\|_b$  with  $\|\cdot\|_c$  with constants  $0 < m' \leq M'$ , then  $\|\cdot\|_a$  is topologically equivalent to  $\|\cdot\|_b$  with constants  $0 < mm' \leq MM'$ .

Take  $x \in \mathbb{R}^n$  Notice that

$$\sum_{i=1}^n |x_i|^2 \leq \left( \sum_{i=1}^n |x_i|^2 + 2 \sum_{i \neq j} |x_i| |x_j| \right) = \left( \sum_{i=1}^n |x_i| \right)^2$$

and that

$$\sum_{i=1}^n |x_i| \cdot 1 \leq \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} \left( \sum_{i=1}^n 1^2 \right)^{1/2} = \sqrt{n} \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2}$$

prove that  $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$ .

Also notice that

$$\max_i |x_i| = \left( \max_i |x_i|^2 \right)^{1/2} \leq \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} =$$

and

$$\sum_{i=1}^n |x_i|^2 \leq n \cdot \max_i |x_i|^2$$

prove that  $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$ .

## Exercise 28

(i) Notice that (applying the results of the previous exercise)

$$\sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq \sqrt{n} \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

and

$$\sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \geq \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_1} \geq \frac{1}{\sqrt{n}} \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

imply that  $\frac{1}{\sqrt{n}} \|A\|_2 \leq \|A\|_1 \leq \|A\|_2$ .

(ii) Notice that

$$\sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sup_{x \neq 0} \frac{\sqrt{n} \|Ax\|_\infty}{\|x\|_\infty},$$

and

$$\sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \geq \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\sqrt{n} \|x\|_\infty}.$$

## Exercise 29

Take an arbitrary  $x \neq 0$  and suppose  $\|\cdot\|$  is an inner product induced norm. Since

$$\|Qx\| = (\langle Qx, Qx \rangle)^{1/2} = (\langle Q^H Qx, x \rangle)^{1/2} = (\langle x, x \rangle)^{1/2} = \|x\|,$$

then

$$\|Q\| = \sup_{x \neq 0} \frac{\|Qx\|}{\|x\|} = 1.$$

Now let  $R_x : \mathbb{M}_n(\mathbb{F}) \rightarrow \mathbb{F}^n, A \mapsto Ax$  for every  $x \in \mathbb{F}^n$ . Notice that

$$\|R_x\| = \sup_{A \neq 0} \frac{\|Ax\|}{\|A\|} = \sup_{A \neq 0} \frac{\|Ax\| \|x\|}{\|A\| \|x\|} \leq \sup_{A \neq 0} \left( \frac{\|Ax\| \|x\|}{\|Ax\|} \right) = \|x\|.$$

### Exercise 30

Take an arbitrary  $A \in \mathbb{M}_n(\mathbb{F})$ . First,  $\|A\|_S = \|SAS^{-1}\| \geq 0$  for any  $A$  because  $\|\cdot\|$  is a norm on  $\mathbb{M}_n(\mathbb{F})$  and  $SAS^{-1} \in \mathbb{M}_n(\mathbb{F})$ . In addition,  $\|0\|_S = \|S0S^{-1}\| = \|0\| = 0$  and if  $0 = \|A\|_S = \|SAS^{-1}\|$ , then  $SAS^{-1} = 0$  which implies  $A = 0$ . Second, take  $a \in \mathbb{F}$ , then

$$\|aA\|_S = \|SaAS^{-1}\| = \|aSAS^{-1}\| = |a| \|SAS^{-1}\| = |a| \|A\|_S.$$

Finally, let  $B \in \mathbb{M}_n(\mathbb{F})$  and notice that

$$\|A + B\|_S = \|S(A + B)S^{-1}\| = \|SAS^{-1} + SBS^{-1}\| \leq \|SAS^{-1}\| + \|SBS^{-1}\| = \|A\|_S + \|B\|_S.$$

Therefore  $\|\cdot\|_S$  is a norm on  $\mathbb{M}_n(\mathbb{F})$ .