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Measure theory exercises

Section 1

Exercise 1.3:

1. $\mathcal{G}_1 := \{A : A \subset \mathbb{R}, A \text{ is open}\}$ is not an algebra on $\mathbb{R}$, hence not a $\sigma-$algebra. Let $a \in \mathbb{R}$ and define $A_1 := (-\infty, a)$. Clearly $A_1 \in \mathcal{G}_1$, however its complement, $A_1^c = [a, +\infty)$, is not in $\mathcal{G}_1$.

2. $\mathcal{G}_2 := \{A : A \text{ is a finite union of intervals of the form } (a,b], (-\infty,b], (a,\infty)\}$ is an algebra, but not a $\sigma-$algebra. Clearly, $\mathcal{G}_2$ contains the empty set. Also, $\mathcal{G}_2$ is closed under complements because it contains the complements of the three basic intervals $(a,b]$, $(-\infty,b]$ and (a,∞) and, by the properties of complements, it contains the complements of any finite union of the basic intervals. Finally, $\mathcal{G}_2$ is closed under finite union as the finite union of finite unions of the three basic intervals is still a finite union of these basic intervals. However, $\mathcal{G}_2$ is not a $\sigma-$algebra since it clearly does not contain an infinite union of the three basic interval.

3. $\mathcal{G}_3 := \{A : A \text{ is a countable union of intervals of the form } (a,b], (-\infty,b], (a,\infty)\}$ is a $\sigma-$algebra, hence also an algebra. Everything discussed for $\mathcal{G}_2$ holds except for the fact that infinite unions of the basic intervals belong to $\mathcal{G}_3$.

Exercise 1.7: By definition, any $\sigma-$algebra contains $\emptyset$. Also, it contains X, since it must be closed under complements. Therefore, $\{\emptyset, X\}$ is contained in any $\sigma-$algebra and is thus the smallest $\sigma-$algebra on X.

On the other hand, by definition, any $\sigma-$algebra on X is a set of subsets of X, therefore it is contained in the power set of X, which is the set of all subsets of X.

Exercise 1.10: Let $\mathcal{N} = \underset{\alpha}{\cap}\mathcal{S}_\alpha$. $\emptyset \in \mathcal{N}$ because $\emptyset \in \mathcal{S}_\alpha$, for every α. Also, if $A \in \mathcal{N}$, we have that A is in every $\mathcal{S}_\alpha$, and since these are closed under complements, $A^c \in \mathcal{N}$. Finally, if $A_1, A_2, \ldots \in \mathcal{N}$, they belong to each $\mathcal{S}_\alpha$ and so does $\cup_{n=1}^\infty A_n$, and we have that it also belongs to $\mathcal{N}$. In conclusion, $\mathcal{N}$ is a $\sigma-$algebra.

Exercise 1.17:

1. Take $A, B \in \mathcal{S}$, with $A \subset B$. Notice that B can be written as the union of two disjoint sets in the following way: $B = (A \cap B) \cup (A^c \cap B)$. Then, $\mu(B) = \mu(A \cap B) + \mu(A^c \cap B) = \mu(A) + \mu(A^c \cap B)$. Since a measure is nonnegative, $\mu(B) \geq \mu(A)$. Therefore μ is monotone.

2. Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ and define the following sets: $A := \cup_{n\in\mathbb{N}} A_n$, $B_1 := A_1$, $B_2 := A_2 - A_1$, $B_3 := A_3 - (A_1 \cup A_2)$, and so on. Then, $A = \cup_{n\in\mathbb{N}} B_n$. By monotonicity, for each $n \in \mathbb{N}$, $\mu(B_n) \leq \mu(A_n)$ since $B_n \subset A_n$. Therefore we obtain $\mu(A) = \sum_{n\in\mathbb{N}} \mu(B_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$.

Exercise 1.18: λ is a measure because (i) $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$ and (ii) for any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ with A_n's paiwise disjoint we have $\lambda(\cup_{n\in\mathbb{N}} A_n) = \mu\left((\cup_{n\in\mathbb{N}} A_n) \cap B\right) = \mu\left(\cup_{n\in\mathbb{N}}(A_n \cap B)\right) = \sum_{n\in\mathbb{N}} \mu(A_n \cap B) = \sum_{n\in\mathbb{N}} \lambda(A_n)$.

Exercise 1.20: Since $\mu(A_1) < \infty$, by monotonicity $\mu(A_i) < \infty$ for each $n \in \mathbb{N}$. Consider the increasing sequence $\{A_1 - A_n\}_{n\in\mathbb{N}}$, define $A = \cap_{n\in\mathbb{N}} A_n$ and note that $\lim_{n\to\infty}(A_1 - A_n) = A_1 - \lim_{n\to\infty} A_n = A_1 - A$.

Since μ is continuous from below,

$$\begin{aligned}\mu(\cap_{i=1}^{\infty} A_i) &= \mu[A_1 - \cup_{i=1}^{\infty}(A_1 - A_i)] = \mu(A_1) - \mu(\cup_{i=1}^{\infty}(A_1 - A_n)) \\ &= \mu(A_1) - \lim_{n\to\infty} \mu(A_1 - A_n) = \mu(A_1) - \lim_{n\to\infty} [\mu(A_1) - \mu(A_n)] = \lim_{n\to\infty} \mu(A_n)\end{aligned}$$

Therefore $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

Section 2

Exercise 2.10: Clearly, $B = [(B \cap E) \cup (B \cap E^c)] =: F$. In particular, $B \subset F$ and by monotonicity and countable subadditivity we have $\mu^*(B) \leq \mu^*(F) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Therefore requiring $(^*)$ is the same as requiring $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$