Alberto Quaini

Inner Product Spaces Exercises

0.1 Exercise 1

(i)

$$\begin{aligned}&\left(||x+y||^2-||x-y||^2\right)/4=\\&\left(<x,x>+<y,y>+2<x,y>-<x,x>-<y,y>+2<x,y>\right)/4=\\&<x,y>.\end{aligned}$$

(ii)

$$\begin{aligned}&\left(||x+y||^2+||x-y||^2\right)/4=\\&\left(<x,x>+<y,y>+2<x,y>+<x,x>+<y,y>-2<x,y>\right)/2=\\&<x,x>+<y,y>.\end{aligned}$$

0.2 Exercise 2

$$\begin{aligned}&(||x+y||^2-||x-y||^2+i||x-iy||^2-i||x+iy||^2)/4=\\&(<x+y,x+y>-<x-y,x-y>+i<x-iy,y-iy>-i<x+iy,x+iy>)/4=\\&(2<x,y>+2<y,x>-2<x,y>+2<y,x>)/4=\\&<x,y>.\end{aligned}$$

0.3 Exercise 3

$<x,x^5>=\int_0^1 x^6dx=x^7/7|_0^1=1/7$, $||x||=\int_0^1 x^2dx=x^3/3|_0^1=1/3$ and $||x^5||=\int_0^1 x^10dx=x^11/11|_0^1=1/11$. Therefore $\cos\theta=\sqrt{33}/7$ implies $\theta=34.5$.

0.4 Exercise 4

(i)

$$||\cos(t)||=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(t)dt=\frac{1}{\pi}\left.\frac{\cos(x)\sin(x)-x}{2}\right|_{-\pi}^{p}i=\frac{\pi}{\pi}=1,$$

and similarly $||\sin(t)||=1$. Also

$$||\cos(2t)||=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(2t)dt=\frac{1}{\pi}\left.\frac{\sin(4t)+4t}{8}\right|_{-\pi}^{p}i=\frac{\pi}{\pi}=1,$$

and similarly $||\sin(2t)||=1$. Therefore the basis is normalized.

The following integrals:

$$< \cos(t), \sin(t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(t)dt = \frac{1}{\pi}\left.\frac{\sin^2(x)}{x}\right|_{-\pi}^{p} i = 0,$$

$$< \cos(t), \cos(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(2t)dt = \frac{1}{\pi}\ frac 3\sin(t) - 2\sin^3(t)3|_{-\pi}^{p} i = 0,$$

$$< \cos(t), \sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(2t)dt = \frac{1}{\pi}\ frac{-}2\cos^3(t)3|_{-\pi}^{p} i = 0,$$

$$< \cos(2t), \sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t)\sin(2t)dt = \frac{1}{\pi}\ frac{-}\cos^2(2t)4|_{-\pi}^{p} i = 0,$$

and so on, shows that S is an orthonormal basis.

(ii)

$$||t|| = \frac{1}{\pi}\int_{-\pi}^{\pi} tdt = \frac{1}{\pi}\left.\frac{t^2}{2}\right|_{-\pi}^{\pi} = 0.$$