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Note

The Size of Uniquely Colorable Graphs

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The graphs considered here are finite, simple, undirected, and loopless. As usual, let G be a graph, $V(G)$ and $E(G)$ its vertex set and edge set, respectively. If $V' \subset V$, then $G[V']$ denotes the subgraph of G induced by V'. A k-coloring of G is the partitioning of $V(G)$ into k color classes so that no two vertices of the same class are adjacent. If every k-coloring of G induces the same partition of $V(G)$, then we say G is uniquely k-colorable. Now let G be a uniquely k-colorable graph of order n. What is the lower bound of its size m? The question was raised in [1]. In this note we shall discuss this problem.

The graphs called q-trees are defined by recursion: the smallest q-tree is the complete graph K_{q+1} and a q-tree with $n+1$ vertices, where $n>q$ is obtained from a q-tree with n vertices and K_{q+1} by overlapping in Kq. It is easy to show that the q-tree is uniquely $(q+1)$-colorable.

THEOREM. *Let G be a uniquely k-colorable graph with order n and size m; then*

$$m \geqslant (k-1)n - k(k-1)/2$$

and the bound is the best possible.

Proof. Suppose that the color classes are $V_1, ..., V_k$. By the Theorem in [2], $G[V_i \cup V_j]$ is connected for $i, j = 1, ..., k$, $i \neq j$. So

$$|E(G[V_i \cup V_j])| \geqslant |V_i \cup V_j| - 1;$$

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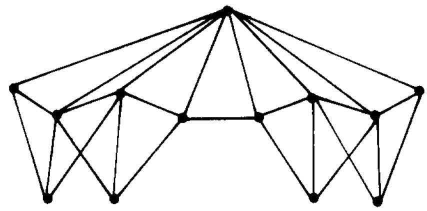

FIGURE 1

thus

$$\begin{aligned}|E(G)| &= \sum_{i \neq j} |E(G[V_i \cup V_j]) \\ &\geqslant \sum_{i \neq j} |V_i \cup V_j| - k(k-1)/2 \\ &= (k-1) \sum |V_i| - k(k-1)/2 \\ &= (k-1)n - k(k-1)/2.\end{aligned}$$

Because the size of a $(k-1)$-tree with order n is $(k-1)n-k(k-1)/2$, the proof is completed.

We can construct many uniquely k-colorable graphs with order n and size $(k-1)\,n-k(k-1)/2$. Let G be a uniquely k-colorable graph with order $n-1$ and size $(k-1)(n-1)-k(k-1)/2$. We take $k-1$ vertices from $k-1$ color classes and add a vertex adjacent to them. Then the resultant graph is a uniquely k-colorable graph with order n and size $(k-1)n-k(k-1)/2$.

Now we ask the following question: if G is a uniquely k-colorable graph with order n and size $(k-1)\,n-k(k-1)/2$, does the equality $\delta(G)=k-1$ hold? The answer is no, as shown in Fig. 1. But for such a graph G it is easy to get the inequality $\delta(G) \leqslant 2k-3$. Can it be improved?

At last we raise the following

Conjecture. If G is uniquely k-colorable graph with order n and size $(k-1)\,n-k(k-1)/2$, then G contains a K_k as its subgraph.

REFERENCES

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2. D. CARTWRIGHT AND F. HARARY, On colorings of signed graphs, *Elem. Math.* **23** (1968), 85–89.