EMBEDDINGS INTO SIMPLE ASSOCIATIVE ALGEBRAS

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INTRODUCTION

In a recent paper [1], Hall showed that if $K_1, \ldots, K_4$ are any nontrivial groups and the group L satisfies the condition $|L| \leqslant |K_1 * \ldots * K_4|$, then L can be embedded in a simple group S containing $K_1, \ldots, K_4$ and generated by them: $S = \langle K_1, \ldots, K_4 \rangle$. We will consider the analogous question for associative rings and algebras. Let us first mention that in the [2] it was proved that any associative algebra can be embedded in a simple associative algebra. Regarding associative rings, a ring R can be embedded in a simple ring if and only if it has a characteristic, i.e., either its additive group is torsion-free or there exists a prime p such that $px = 0, x \in R$. In the first case, R can be embedded in an algebra over the field of rational numbers; in the second, R is an algebra over the field with p elements. Thus the problem for rings reduces to the problem for algebras over an at most countable field.

THEOREM 1. Suppose k is an at most countable field and A, K_1, K_2, K_3 are nonzero associative algebras over k such that $|A| \leqslant |K_1 * K_2 * K_3|$. Then A can be embedded in a simple associative algebra $\mathfrak{A}$ generated by subalgebras K_1, K_2, K_3.

It follows, of course, that if A, K_1, K_2, K_3 are associative rings of characteristic $p \geqslant 0$ and $|A| \leqslant |K_1 * K_2 * K_3|$, then A can be embedded in a simple ring $\mathfrak{A}$ generated by K_1, K_2, K_3. The converse is also true: if $A \subset \mathfrak{A} = \langle K_1, K_2, K_3 \rangle$ and $\mathfrak{A}$ is a simple ring, then all four rings A, K_1, K_2, K_3 have the same characteristic $p \geqslant 0$.

In the general case we have

THEOREM 1'. Suppose k is any field and A, K_1, K_2, K_3 are nonzero associative algebras over k such that $|A| \leqslant |K_1 * K_2 * K_3|$ and $\dim K_1 * K_2 * K_3 \geqslant |k|$. Then A can be embedded in a simple associative algebra $\mathfrak{A}$ generated by subalgebras K_1, K_2, K_3.

Thus, in Theorem 1' there is one additional restriction ($\dim K_1 * K_2 * K_3 \geqslant |k|$), which, evidently, is also necessary.

COROLLARY. Any countable associative algebra can be embedded in a simple associative algebra with three generators.

We now consider the question of when an algebra $\mathfrak{A}$ is a sum of subalgebras K_i. In this case we consider, instead of three, four subalgebras $K_1, \ldots, K_4$ of infinite dimension over k.

In order to formulate the theorem we require a certain condition on the algebras K_i.

Definition. Suppose K is an associative algebra over a field k, $\dim_k K = \alpha \geqslant \aleph_0$. We say that K satisfies condition (*) if K contains a countable series

$$0 < K^{(1)} < K^{(2)} < \ldots < K^{(n)} < \ldots \tag{*}$$

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of subalgebras, the union of which is K, such that the dimension of each subalgebra in (*) and its codimension in the next algebra are equal to α:

$$\dim K^{(n)} = \dim K^{(n+1)}/K^{(n)} = \alpha , \quad n \geqslant 1.$$

THEOREM 2. Suppose k is a field, $K_1, \ldots, K_4$ are algebras over k of infinite dimension satisfying condition (*), and $\alpha \geqslant |k|$. Then any algebra A of dimension $\leqslant \alpha$ can be embedded in a simple associative algebra $\mathfrak{A}$ which is the sum of subalgebras $K_1, \ldots, K_4$:

$$\mathfrak{A} = K_1 + \ldots + K_4 .$$

Examples of algebras satisfying condition (*) are any reduced free algebras (i.e., free in certain varieties) of rank α. Thus, Theorem 2 asserts, in particular, that a simple algebra $\mathfrak{A}$ can be represented as a sum of four nilpotent (and even with zero multiplication) subalgebras.

The author [4] has raised the following question: Is an associative ring which is the sum of three nilpotent subrings itself nilpotent? Earlier [5] it was proved that an associative ring which is the sum of two nilpotent subrings is nilpotent. A negative answer to the previous question is provided by

THEOREM 3. An arbitrary associative algebra can be embedded in a simple associative algebra which is the sum of three nilpotent subalgebras (each of which is a free nilpotent algebra of index 3).

To prove the above theorems we use the method of composition of elements of a free associative algebra, which is due to A. I. Shirshov [3].

§1. COMPOSITION OF ELEMENTS OF A FREE ASSOCIATIVE ALGEBRA

1. Suppose k is any field, X is a set, and $F = k\langle X\rangle$ is the free associative algebra (without unity) over k freely generated by X. We assume that the set of all words in X is linearly ordered, and that this order $\leqslant$ satisfies the minimum condition and is compatible with the multiplication of words, i.e., the free semigroup $\langle X\rangle$ is an ordered semigroup satisfying (as a partially ordered set) the descending chain condition. For most applications it suffices to consider orderings of words induced by a degree function $d(x)$ on $k\langle X\rangle$ such that $d(x) > 0$, $x \in X$, and by a total order on X compatible with $d(x)$ (i.e., $x > y$, if $d(x) > d(y)$, $x, y \in X$). Namely, words of higher degree are considered to be larger than words of smaller degree, and words of the same degree are ordered lexicographically. Such an ordering will be called standard relative to the degree function $d(x)$. If as $d(x)$ we take the function such that $d(x) = 1$, $x \in X$, then the corresponding ordering will simply be called standard.

If f is a nonzero element of F, then the leading word occurring in f with nonzero coefficient will be denoted by $\bar{f}$; clearly, $\overline{uv} = \bar{u}\bar{v}$.

We introduce the concept of the composition of elements f, g of the algebra F relative to a word w. Suppose

$$w = \bar{f}a = b\bar{g}$$

and the distinguished subwords $\bar{f}, \bar{g}$ of w intersect. Then the polynomial

$$(f, g)_w = \alpha^{-1} f a - \beta^{-1} b g, \tag{1}$$

where $f=\alpha\bar{f}+\dots, g=\beta\bar{g}+\dots$, is called the composition of f, g relative to w. Clearly, either $(f,g)_w = 0$, or $\overline{(f,g)_w} < w$.

Suppose S is some subset of elements of F. We say that S is closed under compositions if no leading word $\bar{s}, s \in S$ contains the leading word of another element of S as a subword, and for any elements $f, g \in S$ and any composition $(f,g)_w$ in F we have

$$(f,g)_w = \sum \alpha_i a_i f_i b_i, \tag{2}$$

where $f_i \in S, \alpha_i \neq 0$, a_i, b_i are words and $a_i \bar{f}_i b_i < w$ [i.e., the leading word $a_i \bar{f}_i b_i$ of any summand $a_i f_i b_i$ of (2) is strictly less than the word w relative to which the composition is considered]. Henceforth, in calculating a composition $(f,g)_w$ we will omit expressions of the form $\alpha_i a_i f_i b_i$, $f_i \in S$, $a_i \bar{f}_i b_i < w$ and replace the equality sign by $\equiv$; in this notation the closure condition is written:

$$(f,g)_w \equiv 0.$$

The following assertion is an analogue of a lemma of Shirshov [3] (see also [7], §1, Shirshov's Lemma).

Proposition 1. Suppose S is a closed set of elements of the free algebra $F=k\langle X\rangle$, and J is the ideal of F generated by S. If $u \in J$, then the leading word $\bar{u}$ of u contains as a subword some leading word $\bar{s}$, where $s \in S$.

Proof. Suppose

$$u = \sum_{i=1}^{n} \alpha_i c_i f_i d_i, \tag{3}$$

where $\alpha_i \neq 0$, c_i, d_i are words. Consider the leading word among the words of the form

$$c_i \bar{f}_i d_i. \tag{4}$$

Suppose this word is $c_1 \bar{f}_1 d_1$. If there are no similar words in the right-hand side of (3), then $\bar{u} = c_1 \bar{f}_1 d_1$ and the proposition is proved. Otherwise, we have

$$c_1 \bar{f}_1 d_1 = c_i \bar{f}_i d_i, \quad i > 1. \tag{5}$$

Let us assume, for simplicity, that the words $\bar{f}_1, \bar{f}_i$ occur in the polynomials f_1, f_i with coefficients 1. We consider two cases.

1) The distinguished subword $\bar{f}_1$ is a subword of c_i, or the distinguished subword $\bar{f}_i$ is a subword of c_1. Suppose, for example, the first possibility is realized:

$$c_i = c_1 \bar{f}_1 a, \quad d_1 = a \bar{f}_i d_i.$$

We rewrite the sum of the first and i-th summands in (3) in the form

$$\alpha_1 c_1 f_1 d_1 + \alpha_i c_i f_i d_i = \alpha_1 c_1 f_1 a \bar{f}_i d_i + \alpha_i c_1 f_1 a f_i d_i - \alpha_i c_1 (f_1 - \bar{f}_1) a f_i d_i \equiv (\alpha_1 + \alpha_i) c_1 f_1 d_1.$$

As a result, the right-hand side of (3) is replaced by an expression of the same form, in which the number of summands with leading word $c_1 \bar{f}_1 d_1$ is decreased [or, in general, the leading word among the words (4) is decreased].

2) Suppose we are not in the previous situation. Then the distinguished words $\bar{f}_1$ and $\bar{f}_i$ intersect. Since they cannot be subwords of one another, either a beginning of $\bar{f}_i$ coincides with an end of $\bar{f}_1$, or a beginning of $\bar{f}_1$ with an end of $\bar{f}_i$. Suppose, for example, the first possibility is realized. Then

$$c_i = c_1 b, \quad d_1 = a d_i, \quad \bar{f}_1 a = b \bar{f}_i = w.$$

We rewrite the sum of the terms under consideration in the form

$$\alpha_1 c_1 f_1 d_1 + \alpha_i c_i f_i d_i = \alpha_1 c_1 f_1 a d_i + \alpha_i c_1 b f_i d_i =$$
$$= (\alpha_1 + \alpha_i) c_1 f_1 a d_i - \alpha_i c_1 (f_1 a - b f_i) d_i =$$
$$= (\alpha_1 + \alpha_i) c_1 f_1 a d_i - \alpha_i c_1 (f_1, f_i)_w d_i \equiv (\alpha_1 + \alpha_i) c_1 f_1 d_1;$$

the congruence at the end holds because S is closed under compositions. As a result of these transformations, we have again decreased the number of summands in (3) with leading word $c_1 \bar{f}_1 d_1$.

Thus, after a finite number of steps we arrive at a situation where the leading word among the words (4) has no similar counterparts in the right-hand side of (3). The proposition is proved.

COROLLARY 1. A basis of the algebra $A = \langle X; S \rangle$, presented by the generators X and the set of defining relations $s = 0$, $s \in S$, is formed by all words in X containing no subwords $\bar{s}$, $s \in S$.

Proof. Clearly, any element of A can be represented as a linear combination of words containing no subwords $\bar{s}$, $s \in S$. The linear independence of these words follows from Proposition 1.

2. As the first application of Proposition 1 we offer the following proof of the Birkhoff–Witt theorem. Suppose L is a Lie algebra over the field k, $\{a_i, i \in I\}$ is a basis of L, and $a_i a_j - [a_i a_j] = 0$, $i > j$, is the multiplication table (here and below, the set of indices of basis elements or or generating elements is always assumed to be totally ordered). Consider the associative algebra

$$BW(L) = \langle \{a_i, i \in I\}; S = \{a_i a_j - a_j a_i - [a_i a_j], i > j\} \rangle.$$

We introduce the standard ordering of the words of the free algebra $F = k\langle \{a_i, i \in I\} \rangle$.

LEMMA 1. The set S is closed under compositions.

Proof. Consider the only possible type of composition of elements of S:

$$(a_i a_j - a_j a_i - [a_i a_j]) a_k - a_i (a_j a_k - a_k a_j - [a_j a_k]) \; (i > j > k) =$$
$$= a_i a_k a_j - a_j a_i a_k - [a_i a_j] a_k + a_i [a_j a_k] \equiv a_k a_j a_i +$$
$$+ [a_i a_k] a_j + a_k [a_i a_j] - a_k a_j a_i - a_j [a_i a_k] - [a_j a_k] a_i -$$
$$- [a_i a_j] a_k + a_i [a_j a_k] \equiv [[a_i a_k] a_j] - [[a_i a_j] a_k] + [a_i [a_j a_k]] \equiv 0.$$

The lemma is proved.

This lemma and Corollary 1 show that a basis of the algebra $BW(L)$ is formed by all words containing no subwords $a_i a_j$, $i > j$. This is the Birkhoff–Witt theorem.

Remark. Recently, Bergman [6] has begun a systematic study of the equality problem in certain algebras presented by generators and defining relations, using the so-called "diamond" lemma. Even though the "diamond" lemma and Proposition 1 are formulated in different terms, they are essentially equivalent (in our opinion, the language of compositions is simpler than the language of transformations used in the "diamond" lemma, but this, no doubt, is force of habit).

§2. PROOF OF THEOREMS 1 AND 1'

1. We now turn to the proof of Theorems 1 and 1' stated in the introduction. It is well known (and follows from Proposition 1) that a basis of the free product $A*B$ of (nonzero) associative algebras A, B is formed by certain words generated by fixed bases of A and B, viz., those words which do not contain two elements of the same basis side by side. It follows that $\dim A*B = \max\{\dim A, \dim B, \aleph_0\}$, $|A*B| = \max\{|k|, \dim A*B\}$.

We distinguish in Theorem 1 the case where all of the algebras K_1, K_2, K_3 are finite-dimensional. In this case, $|A| \leqslant \aleph_0$, $\dim A \leqslant \aleph_0$. Let us assume, for the sake of definiteness, that A is a countable-dimensional algebra. We fix in the algebras A, K_1, K_2, K_3 bases $\{a_i,\ i \in N_0\}$, $\{u_i, i \in N_1\}$, $\{v_i, i \in N_2\}$, $\{w_i, i \in N_3\}$, respectively, and present these algebras by their multiplication tables:

$$\begin{gathered} a_i a_j - [a_i a_j] = 0, \quad i, j \in N_0; \quad u_i u_j - [u_i u_j] = 0, \quad i, j \in N_1; \\ v_i v_j - [v_i v_j] = 0, \quad i, j \in N_2; \quad w_i w_j - [w_i w_j] = 0, \quad i, j \in N_3. \end{gathered} \tag{1}$$

Put $u_1 = u$, $v_1 = v$, $w_1 = w$ and consider the set of polynomials

$$I = \{uvw(uw)^{n-1}v + u(vw)^{n-1}vuw + v(uw)^n v + vw(uv)^{n-1}uw,\ n \geqslant 2\}. \tag{2}$$

We now define and algebra $\mathfrak{A}$, which will turn out to be the desired one. As generators of this algebra we take the set

$$X = \{a_i\} \cup \{u_i\} \cup \{v_i\} \cup \{w_i\} \cup \{x_m^{(n)}, y_m^{(n)}, m, n \geqslant 1\}. \tag{3}$$

In the system of defining relations of $\mathfrak{A}$ we include, first of all, the relations (1). In addition to relations (1), we impose on X relations of second and third types. The relations of second type will guarantee the representation of all generators (3) of $\mathfrak{A}$ in the form of polynomials in $\{u_i\} \cup \{v_i\} \cup \{w_i\}$ and the relations of third type will have the form $x_m^{(n)} a y_m^{(n)} = b$, where a, b are polynomials containing no occurrences $x_m^{(i)}, y_m^{(i)}$, $i \geqslant n$. Here we will have to worry about whether or not a is a nonzero element of $\mathfrak{A}$.

Consider the chain of free algebras

$$F_0 = k\langle X_0\rangle \subset F_1 = k\langle X_1\rangle \subset \ldots \subset F_n = k\langle X_n\rangle \subset \ldots,$$

where

$$X_0 = \{a_i\} \cup \{u_i\} \cup \{v_i\} \cup \{w_i\},\ X_1 = X_0 \cup \{x_m^{(1)}, y_m^{(1)}, m \geqslant 1\}, \ldots, X_{n+1} = X_n \cup \{x_m^{(n+1)}, y_m^{(n+1)}, m \geqslant 1\}, \ldots.$$

Let $F = \bigcup F_n = k\langle X\rangle$. Fix certain one-to-one correspondences

$$(F_n^*, F_n^*) \xleftrightarrow{e_n} \{(x_m^{(n+1)}, y_m^{(n+1)}),\ m \geqslant 1\}, \tag{4}$$

where $n \geqslant 0$, $F_n^* = F_n \setminus \{0\}$. We define a degree function $d(x)$ on the algebra $F = k(X)$. On the subalgebra F_0 this is the usual degree function [$d(x) = 1$ if $x \in X_0$]. We now put

$$d(x_m^{(1)}) = d(y_m^{(1)}) = d(b), \text{ where } (x_m^{(1)}, y_m^{(1)}) \xleftrightarrow{\varepsilon_0} (a, b) \in (F_0^*, F_0^*),$$
$$\cdots \qquad \vdots$$
$$d(x_m^{(n+1)}) = d(y_m^{(n+1)}) = d(b), \text{ where } (x_m^{(n+1)}, y_m^{(n+1)}) \xleftrightarrow{\varepsilon_n} (a, b) \in (F_n^*, F_n^*),$$
$$\cdots$$

This gives us an inductive definition of the function $d(x)$ on the algebra F. Consider the standard ordering of the words of F relative to $d(x)$, assuming that $a_{i_1} < w_{i_2} < v_{i_3} < u_{i_4}$ (for all i_1, i_2, i_3, i_4) and the rest of the order on X is arbitrary (but compatible with $d(x)$, i.e., $x > y$ if $d(x) > d(y)$, $x, y \in X$).

The relations of second type of the algebra $\mathfrak{A}$ are those of the form

$$uvw(uw)^{n-1}v + u(vw)^{n-1}vuw + v(uw)^n v + vw(uv)^{n-1}uw - a = 0, \tag{5}$$

where $a \in \{a_i\} \cup \{x_m^{(n)}, y_m^{(n)}, n, m \geqslant 1\}$, $n \geqslant max\,\{d(a), 2\}$ and we assume that for distinct elements a the number n are distinct. Note that the leading word of the polynomial forming the left-hand side of (5) is $uvw(uw)^{n-1}v$. Let us immediately list the consequences of (1) and (5) which we will need (we always start with the leading term of the corresponding polynomial):

$$u_i v(uw)^n v + u_i vw(uv)^{n-1}uw + [u_i u]vw(uw)^{n-1}v + [u_i u](vw)^{n-1}vuw - u_i a = 0,$$
$$u(vw)^{n-1}vuwv_j + vw(uv)^{n-1}uwv_j + uvw(uw)^{n-1}[vv_j] + v(uw)^n[vv_j] - av_j = 0, \tag{6}$$
$$u_i vw(uv)^{n-1}uwv_j + u_i v(uw)^n[vv_j] + [u_i u]vw(uw)^{n-1}[vv_j] + [u_i u](vw)^{n-1}vuwv_j - u_i a v_j = 0,$$

where u_i, v_j are arbitrary elements of the sets $\{u_i, i \in N_1\}, \{v_j, j \in N_2\}$.

Before indicating the relations of third type of the algebra $\mathfrak{A}$, we define the concept of a canonical word of the algebra F_n, $n \geqslant 0$. The canonical words of F_0 are the words in X_0 containing no leading words of the left-hand sides of (1), (5), and (6). The canonical words of F_1 are the words in X_1 which do not contain, besides the previous subwords [i.e., the leading words of (1), (5), and (6)], subwords of the form

$$x_m^{(1)} \bar{a} y_m^{(1)},$$

where $(x_m^{(1)}, y_m^{(1)}) \xleftrightarrow{\varepsilon} (a, b) \in (F_0^*, F_0^*)$, and a, b are linear combinations of canonical words of F_0. The canonical words of F_2 are the words in X_2 which do not contain, besides the previous subwords, subwords of the form

$$x_m^{(2)} \bar{a} y_m^{(2)},$$

where $(x_m^{(2)}, y_m^{(2)}) \xleftrightarrow{\varepsilon} (a, b) \in (F_1^*, F_1^*)$ and a, b are linear combinations of canonical words of F_1, and so on.

The relations of third type of the algebra $\mathfrak{A}$ are all relations of the form

$$x_m^{(n)} a y_m^{(n)} - b = 0, \tag{7}$$

where $n \geqslant 1$, $(x_m^{(n)}, y_m^{(n)}) \xleftrightarrow{\varepsilon} (a, b) \in (F^*_{n-1}, F^*_{n-1})$, and a, b are linear combinations of canonical words [recall that $d(x_m^{(n)}) = d(y_m^{(n)}) = d(b)$].

This completes the construction of the algebra $\mathfrak{A}$.

LEMMA 2. Each word in the generating set X is equal in the algebra $\mathfrak{A}$ to a linear combination of canonical words.

Lemma 2 follows from the fact that if a word in X contains an occurrence of the leading word of some relation (1), (5), (6), or (7), then this word is equal in $\mathfrak{A}$ to a linear combination of smaller words (i.e., either words of smaller degree or words smaller in the lexicographic sense).

Let S denote the set of left-hand sides of relations (1) and (5)-(7).

LEMMA 3. The set S is closed under compositions.

Proof. Consider all possible compositions of elements of S. First of all, note that no leading word of S contains the leading word of another element of this set as a subword. The compositions of the elements (1) [i.e., the left-hand sides of (1)] with one another are congruent to zero by the associative law. The elements (7) do not form compositions with any elements of S. The elements (5) and (6) do not form compositions with each other. Thus, it remains to consider all compositions of the elements (5) and (6) with the elements (1). Let us look at several of these:

$$
\begin{gathered}
1)\ u_i\left(uvw(uw)^{n-1}v + u(vw)^{n-1}vuw + v(uw)^n v + vw(uv)^{n-1}uw - a\right) - \\
- (u_i u - [u_i u])\, vw(uw)^{n-1}v \equiv u_i v(uw)^n v + u_i vw(uv)^{n-1}uw + \\
+ [u_i u]\, vw(uw)^{n-1}v + [u_i u](vw)^{n-1}vuw - u_i a \equiv 0;
\end{gathered}
$$

$$
\begin{gathered}
2)\ u_j\left(u_i v(uw)^n v + u_i vw(uv)^{n-1}uw + [u_i u]\, vw(uw)^{n-1}v + \right. \\
\left. + [u_i u](vw)^{n-1}vuw - u_i a\right) - (u_j u_i - [u_j u_i])\, v(uw)^n v \equiv [u_j u_i]\, vw \times \\
\times (uv)^{n-1}uw + [u_j u_i u]\, vw(uw)^{n-1}v + [u_j u_i u](vw)^{n-1}vuw + [u_j u_i]\, v(uw)^n v - [u_j u_i]\, a \equiv 0;
\end{gathered}
$$

$$
\begin{gathered}
3)\ u_i\left(u(vw)^{n-1}vuwv_j + vw(uv)^{n-1}uwv_j + uvw(uw)^{n-1}[vv_j] + \right. \\
\left. + v(uw)^n[vv_j] - av_j\right) - (u_i u - [u_i u])(vw)^{n-1}vuwv_j \equiv u_i vw(uv)^{n-1} \times \\
\times uwv_j + u_i v(uw)^n[vv_j] + [u_i u]\, vw(uw)^{n-1}[vv_j] + [u_i u](vw)^{n-1} \times \\
\times vuwv_j - u_i a v_j \equiv 0.
\end{gathered}
$$

The remaining compositions are handled in a similar way.

The lemma is proved.

We now complete the proof of Theorem 1 in the case where K_1, K_2, K_3 are finite-dimensional. Lemmas 2 and 3 and Proposition 1 show that the set of all canonical words in X is a basis of $\mathfrak{A}$. Thus, $\mathfrak{A}$ contains A, K_1, K_2, K_3 as subalgebras. In view of (5), $\mathfrak{A}$ is generated by the subalgebras K_1, K_2, K_3, and, in view of (7), $\mathfrak{A}$ is simple.

2. We now consider the second part of Theorem 1, i.e., the case where the dimension of at least one of the algebras K_1, K_2, K_3 is infinite. Suppose $\max\{\dim K_i,\ 1 \leqslant i \leqslant 3\} = \alpha \geqslant \aleph_0$. By the observation made at the very beginning of the previous subsection, $\dim K_1 * K_2 * K_3 = \alpha$, and, since the field k is at most countable, $|K_1 * K_2 * K_3| = \alpha$. We assume, for the sake of

definiteness, that $\dim_k A = \alpha$, and then $|A| = \alpha$.

We will prove this part of Theorem 1 together with Theorem 1'. Suppose in Theorem 1' that $\dim K_1 * K_2 * K_3 = \alpha$ and α is an uncountable cardinal. Since, by hypothesis, $\alpha \geqslant |k|$ we have $|K_1 * K_2 * K_3| = \alpha$ and $\max\{\dim K_i, 1 \leqslant i \leqslant 3\} = \alpha$. We may again assume that $\dim A = |A| = \alpha$. Thus, Theorem 1 and Theorem 1' will be completely proved if we can prove

Proposition 2. Suppose A, K_1, K_2, K_3 are algebras over k, where

$$\dim_k A = \dim_k K_3 = |A| = |K_3| = \alpha \geqslant |K_2|, |K_3|$$

and $\alpha \geqslant \aleph_0$. Then A can be embedded in a simple algebra $\mathfrak{A}$ generated by subalgebras K_1, K_2, K_3.

Proof. The proof of this proposition will be parallel to the one contained in the previous subsection. We identify the cardinal α with the first ordinal of cardinality α. We present the algebras A, K_1, K_2, K_3 by bases and the multiplication tables (1). Put $u_1 = u$, $v_1 = v$, $w_1 = w$ and in the role of (2) take the set

$$I = \{uvw_\beta (uw)^{n-1} v + uvw_\beta (vw)^{n-2} vuw + vuw_\beta (uw)^{n-1} v + vw_\beta (uv)^{n-1} uw,\ n \geqslant 2,\ 1 \leqslant \beta < \alpha\}. \quad (2')$$

As the system of generators of the desired algebra $\mathfrak{A}$ take the set

$$X = \{a_i\} \cup \{u_i\} \cup \{v_i\} \cup \{w_i\} \cup \{x_m^{(n)}, y_m^{(n)},\ 1 \leqslant n < \omega,\ 1 \leqslant m < \alpha\}. \quad (3')$$

Consider again the chain of free algebras F_n, $n \geqslant 0$, $|F_n| = \alpha$, and the one-to-one correspondences

$$(F_n^*, F_n^*) \xleftrightarrow{\varepsilon_n} \{(x_m^{(n+1)}, y_m^{(n+1)},\ 1 \leqslant m < \alpha\}. \quad (4')$$

We define, as in the first subsection, a degree function $d(x)$ on F and consider the standard ordering of words of F relative to $d(x)$.

The relations of second type of the algebra $\mathfrak{A}$ have the form

$$uvw_\beta (uw)^{n-2} v + uvw_\beta (vw)^{n-2} vuw + vuw_\beta (uw)^{n-1} v + vw_\beta (uv)^{n-1} uw - a = 0, \quad (5')$$

where $a \in \{a_i\} \cup \{x_m^{(n)}, y_m^{(n)},\ 1 \leqslant n < \omega,\ 1 \leqslant m < \alpha,\}$, $n \geqslant \max(d(a), 2)$, and we assume that for distinct elements a the pairs (β, n) are distinct. We next write down the relations (6') [consequences of (1) and (5')] obtained from (6) by replacing the first letter w by w_β. We define the canonical words in X exactly as in the first subsection. The relations of third type of the algebra $\mathfrak{A}$ have the form

$$x_m^{(n)} a y_m^{(n)} - b = 0, \quad (7')$$

where $n \geqslant 1$, $(x_m^{(n)}, y_m^{(n)}) \xleftrightarrow{\varepsilon} (a, b) \in (F_{n-1}^*, F_{n-1}^*)$, and a, b are linear combinations of canonical words.

Let S be the set of left-hand sides of relations (1) and (5')-(7'). Exactly as above, Lemmas 2 and 3 hold, and these show that $\mathfrak{A}$ is the desired algebra.

Proposition 2 is proved.

§2. PROOF OF THEOREM 2

We may assume, without loss of generality, that $\dim A = \alpha$. We choose in A an arbitrary basis $\{a_i, 1 \leqslant i < \alpha\}$ (we have identified α with the first ordinal of cardinality α) with multiplication table

$$a_i a_j - [a_i a_j] = 0, \quad 1 \leqslant i < \alpha. \tag{1}$$

In view of condition (*), the algebras K_ℓ, $1 \leqslant \ell \leqslant 4$, contain bases

$$\begin{aligned} U &= \bigcup_{1 \leqslant n < \omega} \{u_i^{(n)}, 1 \leqslant i < \alpha\}, \quad V = \bigcup_{1 \leqslant n < \omega} \{v_i^{(n)}, 1 \leqslant i < \alpha\}, \\ W &= \bigcup_{1 \leqslant n < \omega} \{w_i^{(n)}, 1 \leqslant i < \alpha\}, \quad S = \bigcup_{1 \leqslant n < \omega} \{s_i^{(n)}, 1 \leqslant i < \alpha\}, \end{aligned} \tag{2}$$

respectively (each is the union of nonintersecting subsets of the form indicated). In these bases the multiplication tables of the algebras $K_1, \ldots, K_4$ have the form

$$u_i^{(n)} u_j^{(m)} - [u_i^{(n)} u_j^{(m)}] = 0, \; v_i^{(n)} v_j^{(m)} - [v_i^{(n)} v_j^{(m)}] = 0, \; w_i^{(n)} w_j^{(m)} - [w_i^{(n)} w_j^{(m)}] = 0, \tag{3}$$

$$s_i^{(n)} s_j^{(m)} - [s_i^{(n)} s_j^{(m)}] = 0, \tag{4}$$

respectively, where, for example, the element $[u_i^{(n)} u_j^{(m)}]$ is a linear combination of elements of the form $u_k^{(n_k)}$, where $n_k \leqslant \max(n,m)$.

Consider the ascending series of free algebras

$$F_0 = k\langle X_0 \rangle \subset F_1 = k\langle X_1 \rangle \subset \ldots \subset F_n = k\langle X_n \rangle \subset \ldots,$$

where

$$\begin{aligned} X_0 &= \{a_i\}, \; X_1 = X_0 \cup \{u_i^{(1)}, v_i^{(1)}, w_i^{(1)}, x_i^{(1)}, y_i^{(1)}\}, \ldots, \\ X_n &= X_{n-1} \cup \{u_i^{(n)}, v_i^{(n)}, w_i^{(n)}, x_i^{(n)}, y_i^{(n)}\}, \ldots, \end{aligned} \tag{5}$$

here, $1 \leqslant i < \alpha$, $\{x_i^{(n)}, y_i^{(n)}, 1 \leqslant i < \alpha, 1 \leqslant n < \omega\}$ is a new set of elements. Let $\langle X_n \rangle_2$ denote the set of all words in X_n of length 2 which do not contain the subwords aa, uu, vv, ww; here and in what follows, for simplicity of notation, we omit the upper and lower indices in the symbols a_i, $u_i^{(n)}$, $v_i^{(n)}$, $w_i^{(n)}$. Fix the following one-to-one correspondences:

$$X_0 \xleftrightarrow{\varepsilon_0} \{(u_i^{(1)}, v_i^{(1)}, w_i^{(1)})\}, \quad (F_0^*, F_0^*) \xleftrightarrow{\delta_0} \{(x_i^{(1)}, y_i^{(1)})\},$$

$$\langle X_1 \rangle_2 \xleftrightarrow{\varepsilon_1} \{(u_i^{(2)}, v_i^{(2)}, w_i^{(2)})\}, \quad (F_1^*, F_1^*) \xleftrightarrow{\delta_1} \{(x_i^{(2)}, y_i^{(2)})\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\langle X_n \rangle_2 \xleftrightarrow{\varepsilon_n} \{(u_i^{(n+1)}, v_i^{(n+1)}, w_i^{(n+1)})\}, \quad (F_n^*, F_n^*) \xleftrightarrow{\delta_n} \{(x_i^{(n+1)}, y_i^{(n+1)})\}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We define a degree function $d(x)$ on the algebra $F = \bigcup F_n = k\langle X\rangle$. Put

$$d(a_i)=1,\ d(u_i^{(n)})=d(v_i^{(n)})=d(w_i^{(n)})=2^{2n}.$$

Also, if $(x_i^{(n+1)}, y_i^{(n+1)}) \overset{\delta}{\longleftrightarrow} (a,b), n \geq 0$, then put

$$d(x_i^{(n+1)})=d(y_i^{(n+1)})=d(b).$$

This completely defines $d(x)$. We now construct an algebra $\mathfrak{A}$, which will turn out to be the desired one. As the system of generators of $\mathfrak{A}$ we take the set $X=\bigcup X_n$, $n \geq 0$. In the system of defining relations of $\mathfrak{A}$ we include, first of all, the relations (3). We introduce the abbreviation

$$s_i^{(n)} = u_i^{(n)} + v_i^{(n)} + w_i^{(n)} + a_i^{(n)}, \tag{6}$$

where $a_i^{(n)} \overset{\varepsilon}{\longleftrightarrow} (u_i^{(n)}, v_i^{(n)}, w_i^{(n)})$, $1 \leq i < \alpha$, $n \geq 0$ (there are no $s_i^{(n)}$ among the generators of $\mathfrak{A}$). If

$$[s_i^{(n)} s_j^{(m)}] = \sum \alpha_{ijk}^{(n_k)} s_k^{(n_k)}$$

in K_4, then in $\mathfrak{A}$ we put, by definition.

$$[s_i^{(n)} s_j^{(m)}] = \sum \alpha_{ijk}^{(n_k)} (u_k^{(n_k)} + v_k^{(n_k)} + w_k^{(n_k)} + a_k^{(n_k)}).$$

To the multiplication table (4) of K_4 there corresponds in $\mathfrak{A}$ the system of relations

$$(u_i^{(n)}+v_i^{(n)}+w_i^{(n)}+a_i^{(n)})(u_j^{(m)}+v_j^{(m)}+w_j^{(m)}+a_j^{(m)})-[s_i^{(n)} s_j^{(m)}]=0. \tag{7}$$

Before going any further, let us make one simplification in writing relations (3) and (7). These relations are homogeneous in form: in all expressions occurring in these relations (i.e., in the words $u_i^{(n)} u_j^{(m)}$, etc., and in the functional symbols $[u_i^{(n)} u_j^{(m)}], \ldots, [s_i^{(n)} s_j^{(m)}]$) there appear in the first and second places elements u, v, w, s, a with the same respective indices. For the sake of brevity, we will write these relations in the form

$$uu - [uu] = 0, \quad vv - [vv] = 0, \quad ww - [ww] = 0, \tag{3'}$$

$$(u+v+w+a)(u+v+w+a) - [ss] = 0. \tag{7'}$$

For (7') we will use the matrix form

$$\begin{bmatrix} [uu] & uv & uw & ua \\ vu & [vv] & vw & va \\ wu & wv & [ww] & wa \\ au & av & aw & aa \end{bmatrix} - [ss] = 0; \tag{7''}$$

thus, matrix brackets denote the sum of the entries of the matrix.

We define an ordering of the set X by saying that $a < w^{(n)} < v^{(n)} < u^{(n)}$, and the rest is arbitrary but compatible with $d(x)$; in particular, $u_i^{(n)} < u_j^{(n+1)}$, $v_i^{(n)} < v_j^{(n+1)}$, $w_i^{(n)} < w_j^{(n+1)}$. We introduce the standard ordering of the words of F relative to $d(x)$. In particular, $u_i^{(n)} > a_i^{(n)}$, since $a_i^{(n)} \in \langle X_{n-1} \rangle_2$ and $d(u_i^{(n)}) = 2^{2n}$, $d(a_i^{(n)}) \leq 2^{2n-1}$. Let us now distinguish the leading words of (3) and (7). In the relations (3) these are the words uu, vv, ww, respectively. The leading word of (7) is uv.

We now write down the consequences of relations (3) and (7) which we will need later [each of these is obtained from (7) by multiplying it on the left by an arbitrary element $u = u_k^{(l)}$ and then applying (3)]:

$$\begin{bmatrix} [uu]v & u[vv] \\ uwv & uav \end{bmatrix} - \begin{bmatrix} vu & [vv] & vw & va \\ wu & wv & [ww] & wa \\ au & av & aw & [aa] \end{bmatrix} u -$$

$$- \begin{bmatrix} vu & [vv] & vw & va \\ wu & wv & [ww] & wa \\ au & av & aw & [aa] \end{bmatrix} w - \begin{bmatrix} vu & [vv] & vw & va \\ wu & wv & [ww] & wa \\ au & av & aw & [aa] \end{bmatrix} a +$$

$$+ [ss]u + [ss]w + [ss]a - u[ss] = 0 \tag{8}$$

(the unabbreviated form of this relation is obtained from it by a placement of arbitrary indices, the same indices in the first, second, and third places, respectively). The leading word of the left-hand side of (8) is uwv.

We define the canonical words in X_n, $n \geq 0$. The canonical words in X_0 are the words in X_0 which do not contain the subwords $a_i a_j$, i.e., the leading words of (1). The canonical words in X_1 are the words in X_1 which do not contain, besides the previous subwords, the leading words of (3), (7), and (8), and also the following subwords:

$$x_i^{(1)} \bar{a} y_i^{(1)}, \tag{9}$$

where $1 \leq i < \infty$, $(x_i^{(1)}, y_i^{(1)}) \xleftrightarrow{\delta} (a, b) \in (F_0^*, F_0^*)$ and a, b are linear combinations of canonical words of F_0. The canonical words in X_2 do not contain, besides the leading words of (1), (3), (7), and (8) and the subwords (9), the subwords

$$x_i^{(2)} \bar{a} y_i^{(2)},$$

where $1 \leq i < \infty$, $(x_i^{(2)}, y_i^{(2)}) \xleftrightarrow{\delta} (a, b) \in (F_1^*, F_1^*)$, and a, b are linear combinations of canonical words in X_1; and so on.

We now consider the last type of relation of the algebra $\mathfrak{A}$:

$$x_i^{(n)} a y_i^{(n)} - b = 0, \tag{10}$$

where $n \geq 1$, $1 \leq i < \infty$, $(x_i^{(n)}, y_i^{(n)}) \xleftrightarrow{\delta} (a, b) \in (F_{n-1}^*, F_{n-1}^*)$, and a, b are linear combinations of canonical words. We have $d(x_i^{(n)} a y_i^{(n)}) > d(b)$, i.e., $x_i^{(n)} a y_i^{(n)}$ is the leading word of (10). Clearly, we have

LEMMA 4. Each word of the algebra $\mathfrak{A}$ is equal to a linear combination of canonical words.

Let $\mathcal{S}$ denote the set of left-hand sides of relations (1), (3), (7), (8), and (10).

LEMMA 5. The set $\mathcal{S}$ is closed under compositions.

Proof. Let us consider all possible compositions of elements of $\mathcal{S}$. As in Lemma 3, the elements (10) do not form compositions with the remaining elements of $\mathcal{S}$, the elements (7) and and (8) do not form compositions with each other, and the compositions of the elements (1) and (3) with each other are easily calculated. There remain four compositions between the elements (3) and (7), (8). We consider all of these:

1)
$$u\begin{bmatrix}[uu] & uv & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} - u[ss] - (uu-[uu])v \equiv$$

$$\equiv \begin{bmatrix}[uuu] & [uu]v & [uu]w & [uu]a\\ & u[vv] & & \\ uwu & uwv & u[ww] & uwa\\ uau & uav & uaw & u[aa]\end{bmatrix} - u[ss] - \begin{bmatrix}[uu] & & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} u + [ss]u -$$

$$-\begin{bmatrix}[uu] & & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} w + [ss]w - \begin{bmatrix}[uu] & & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} a + [ss]a \equiv$$

$$\equiv \begin{bmatrix}[uu]v & u[vv]\\ uwv & uav\end{bmatrix} - \begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} u - \begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} w -$$

$$-\begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} a + [ss]u + [ss]w + [ss]a - u[ss] \equiv 0 \quad \text{(in view of (8))}.$$

2)
$$\begin{bmatrix}[uu] & uv & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} v - [ss]v - u(vv-[vv]) \equiv \begin{bmatrix}[uu]v & u[vv] & & uav\\ & [vvv] & vwv & vav\\ & w[vv] & [ww]v & wav\\ & a[vv] & awv & [aa]v\end{bmatrix} -$$

$$-[ss]v - \begin{bmatrix}[uu]v & u[vv]\\ & uav\end{bmatrix} + \begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} u + \begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} w +$$

$$+\begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ aw & av & aw & [aa]\end{bmatrix} a - [ss]u - [ss]w - [ss]a + u[ss] - v\begin{bmatrix}[uu] & & & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} +$$

$$+v[ss] - w\begin{bmatrix}[uu] & & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} + w[ss] - a\begin{bmatrix}[uu] & & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} + a[ss] \equiv$$

$$\equiv (u+v+w+a)[ss] - [ss](u+v+w+a) \equiv [s[ss]] - [[ss]s] \equiv 0.$$

3)

$$u\begin{bmatrix}[uu]v & u[vv]\\ uwv & uav\end{bmatrix}-u\begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}u-u\begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}w-$$

$$-u\begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}a+u[ss]u+u[ss]w+u[ss]a-[uu][ss]-(uu-[uu])wv\equiv$$

$$\equiv\begin{bmatrix}[uuu]v & [uu][vv]\\ [uu]wv & [uu]av\end{bmatrix}+\begin{bmatrix}u[ss]u & u[ss]w\\ u[ss]a & -[uu][ss]\end{bmatrix}-u\begin{bmatrix}wu & [vv] & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}u-$$

$$-u\begin{bmatrix}wu & [vv] & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}w-u\begin{bmatrix}wu & [vv] & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}a+$$

$$+\begin{bmatrix}[uu]v & u[vv]\\ & uav\end{bmatrix}u-P[uu]-Pwu-Pau+\begin{bmatrix}[ss]u & [ss]w\\ [ss]a & -u[ss]\end{bmatrix}u+$$

$$\left(\text{here,}\quad P=\begin{bmatrix}vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}\right)$$

$$+\begin{bmatrix}[uu]v & u[vv]\\ & uav\end{bmatrix}w-Puw-P[ww]-Paw+\begin{bmatrix}[ss]u & [ss]w\\ [ss]a & -u[ss]\end{bmatrix}w+$$

$$+\begin{bmatrix}[uu]v & u[vv]\\ & uav\end{bmatrix}a-Pua-Pwa-P[aa]+\begin{bmatrix}[ss]u & [ss]w\\ [ss]a & -u[ss]\end{bmatrix}a+$$

$$+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}[uu]-[ss][uu]+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}wu-[ss]wu+$$

$$+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}au-[ss]au+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}uw-[ss]uw+$$

$$+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}[ww]-[ss][ww]+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}aw-[ss]aw+$$

$$+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}ua-[ss]ua+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}wa-[ss]wa+$$

$$+\begin{bmatrix}[uu] & uw & ua\\ & P & \end{bmatrix}[aa]-[ss][aa]\equiv$$

$$\equiv\begin{bmatrix}[uuu]v & [uu][vv]\\ [uu]wv & [uu]av\end{bmatrix}+\begin{bmatrix}u[ss]u & u[ss]w\\ u[ss]a & -[uu][ss]\end{bmatrix}-u[vv]u-u[vv]w-u[vv]a-$$

$$-uavu-uavw-uava+\begin{bmatrix}[uu]v & u[vv]\\ & uav\end{bmatrix}u+\begin{bmatrix}[ss]u & [ss]w\\ [ss]a & -u[ss]\end{bmatrix}u+\begin{bmatrix}[uu]v & u[vv]\\ & uav\end{bmatrix}w+$$

$$+\begin{bmatrix}[ss]u & [ss]w\\ [ss]a & -u[ss]\end{bmatrix}w+\begin{bmatrix}[uu]v & u[vv]\\ & uav\end{bmatrix}a+\begin{bmatrix}[ss]u & [ss]w\\ [ss]a & -u[ss]\end{bmatrix}a-$$

$$-[ss]\begin{bmatrix}[uu] & uw & ua\\ wu & [ww] & wa\\ au & aw & [aa]\end{bmatrix} + [uu]\begin{bmatrix}[uu] & uw & ua\\ wu & [ww] & wa\\ au & aw & [aa]\end{bmatrix} = -[uu][ss] +$$

$$+[uu]\begin{bmatrix}[uu] & uv & uw & ua\\ vu & [vv] & vw & va\\ wu & wv & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix} \equiv -[uu][ss] + [uu][ss] \equiv 0.$$

4)

$$\begin{bmatrix}[uu]v & u[vv]\\ uwv & uav\end{bmatrix} v - P_{uw} - P_{wv} - P_{av} + \begin{bmatrix}[ss]uv & [ss]wv\\ [ss]av & -u[ss]v\end{bmatrix} -$$

$$-uw\,(vv - [vv]) \equiv \begin{bmatrix}[uu][vv] & u[vvv]\\ uw[vv] & ua[vv]\end{bmatrix} +$$

$$+ PQ - P[ss] - [ss]Q + [ss][ss] - u[ss]v \equiv$$

(here and below, $Q = \begin{bmatrix}[uu] & & uw & ua\\ vu & [vv] & vw & va\\ wu & & [ww] & wa\\ au & & aw & [aa]\end{bmatrix}$, $T = \begin{bmatrix}[uu] & & uw & ua\\ vu & [vv] & vw & va\\ wu & wu & [ww] & wa\\ au & av & aw & [aa]\end{bmatrix}$)

$$\equiv [ss][ss] - (v+w+a)s[ss] - [ss]s(u+w+a) + \begin{bmatrix}[uu][vv] & u[vvv]\\ uw[vv] & ua[vv]\end{bmatrix} - \begin{bmatrix}u[ss]v & \\ [ss] & [vv]\end{bmatrix} +$$

$$+vu\begin{bmatrix}[uu] & & uw & ua\\ & [vv] & & \\ wu & & [ww] & wa\\ au & & aw & [aa]\end{bmatrix} + wu\begin{bmatrix}[uu] & & uw & ua\\ & [vv] & & \\ wu & & [ww] & wa\\ au & & aw & [aa]\end{bmatrix} + au\begin{bmatrix}[uu] & & uw & ua\\ & [vv] & & \\ wu & & [ww] & wa\\ au & & aw & [aa]\end{bmatrix} +$$

$$+ [vv]Q + wvQ + avQ + vwQ + [ww]Q + awQ + vaQ + waQ + [aa]Q -$$

$$- vTu + v[ss]u - vTw + v[ss]w - vTa + v[ss]a - wTu + w[ss]u -$$

$$- wTw + w[ss]w - wTa + w[ss]a - aTu + a[ss]u - aTw + a[ss]w - aTa +$$

$$+ a[ss]a \equiv [ss][ss] - (v+w+a)s[ss] - [ss]s(u+w+a) + su[vv] + sv[vv] +$$

$$+ sw[vv] + sa[vv] + (v+w+a)[ss](u+w+a) - u[ss]v - [ss][vv] \equiv$$

$$\equiv [ss][ss] - (v+w+a)[ss]s - s[ss](u+w+a) + [ss][vv] + (v+w+a)[ss]\times$$
$$\times (u+w+a) - u[ss]v - [ss][vv] \equiv [ss][ss] - (v+w+a)[ss]v - u[ss]v -$$
$$- s[ss](u+w+a) \equiv [ss][ss] - s[ss]s \equiv 0.$$

Lemma 5 is proved.

It follows from Lemmas 4 and 5 that a basis of the algebra $\mathfrak{A}$ is formed by the set of all canonical words in X. This shows that $\mathfrak{A}$ contains subalgebras $A, K_1, \dots, K_4$. In view of (7), $\mathfrak{A}$ coincides with the sum of $K_1, \dots, K_4$, and, in view of (9), $\mathfrak{A}$ is simple.

§3. PROOF OF THEOREM 3

Suppose A is an associative algebra over the field k with basis $\{a_i,\ 1\leq i<\alpha\}$ and multiplication table

$$a_i a_j - [a_i a_j] = 0,\quad 1\leq i, j<\alpha. \tag{1}$$

For simplicity, we will assume that $\alpha \geq max\{|k|, \aleph_0\}$. Consider the ascending series of free subalgebras

$$F_0 = k<X_0> \subset F_1 = k<X_1> \subset \ldots \subset F_n = k<X_n> \subset \ldots, \tag{2}$$

where

$$\begin{aligned} X_0 &= \{a_i\},\quad X_1 = X_0 \cup \{u_i^{(1)}, v_i^{(1)}, x_i^{(1)}, y_i^{(1)}\}, \ldots, \\ X_n &= X_{n-1} \cup \{u_i^{(n)}, v_i^{(n)}, x_i^{(n)}, y_i^{(n)}\}, \ldots; \end{aligned} \tag{3}$$

here, $1\leq i<\alpha$, $\{u_i^{(n)}, v_i^{(n)}, x_i^{(n)}, y_i^{(n)},\ 1\leq i<\alpha,\ 1\leq n<\omega\}$ is a set of variables. Let $<X_n>_2$ denote the set of all words in X_n of length 2 which do not contain subwords $a_i a_j$. Fix the following one-to-one correspondences:

$$\begin{aligned} &X_0 \xleftrightarrow{\varepsilon_0} \{(u_i^{(1)}, v_i^{(1)})\},\quad (F_0^*, F_0^*) \xleftrightarrow{\delta_0} \{(x_i^{(1)}, y_i^{(1)})\}, \\ &<X_1>_2 \xleftrightarrow{\varepsilon_1} \{(u_i^{(2)}, v_i^{(2)})\},\quad (F_1^*, F_1^*) \xleftrightarrow{\delta_1} \{(x_i^{(2)}, y_i^{(2)})\}, \\ &\qquad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ &<X_n>_2 \xleftrightarrow{\varepsilon_n} \{(u_i^{(n)}, v_i^{(n)})\},\quad (F_n^*, F_n^*) \xleftrightarrow{\delta_n} \{(x_i^{(n+1)}, y_i^{(n+1)})\} \\ &\qquad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \end{aligned} \tag{4}$$

We define (inductively) a degree function $d(x)$ on the algebra $F = \cup F_n = k<X>$ in exactly the same way as in §2, i.e.,

$$d(a_i) = 1,\quad d(u_i^{(n)}) = d(v_i^{(n)}) = 2^{2n},\quad d(x_i^{(n)}) = d(y_i^{(n)}) = d(b),$$

where $(x_i^{(n)}, y_i^{(n)}) \xleftrightarrow{\delta_{n-1}} (a, b) \in (F_{n-1}^*, F_{n-1}^*)$ (the function F_{n-1}^* is defined on $d(x)$ in the previous step). We order the set X starting from $a_i < v^{(n)} < u^{(n)}$ and in accordance with $d(x)$.

As the system of generators of the desired algebra $\mathfrak{A}$ we take the set $X = \cup X_n$. We introduce the abbreviation

$$w_i^{(n)} = u_i^{(n)} + v_i^{(n)} + a_i^{(n)},$$

where $a_i^{(n)} \xleftrightarrow{\varepsilon_{n-1}} (u_i^{(n)}, v_i^{(n)})$, $a_i^{(n)} \in <X_{n-1}>_2$. We include in the system of defining relations of $\mathfrak{A}$ the relations (1) and also the relations

$$uuu = 0,\quad vvv = 0,\quad www = 0 \tag{5}$$

(the placement of indices is assumed to be arbitrary). We rewrite the last of the relations (5) in the form

$$uuv + P = 0, \tag{6}$$

where

$$P = \begin{bmatrix} & & uvu & uvv & uua & uva & uau & uav & uaa \\ vuu & vuv & vvu & & vua & vva & vau & vav & vaa \\ auu & auv & avu & avv & aua & ava & aau & aav & [aaa] \end{bmatrix};$$

here and in what follows, as in §2, matrix brackets replace the summation symbol and the placement of indices is arbitrary, but with the same indices for the first, second, and third places, respectively. From (5) and (6), multiplying (6) on the right by vv, we obtain

$$uvuvv + P''vv - vP'v - aP'v + vvP + vaP + avP + aaP = 0, \tag{7}$$

where

$$P' = \begin{bmatrix} & uvu & uvv & uua & uva & uau & uav & uaa \\ vuv & vvu & & vua & vva & vau & vav & vaa \\ auv & avu & avv & aua & ava & aau & aav & [aaa] \end{bmatrix},$$

$$P'' = \begin{bmatrix} & & uvv & uua & uva & uau & uav & uaa \\ vuv & vvu & & vua & vva & vau & vav & vaa \\ auv & avu & avv & aua & ava & aau & aav & [aaa] \end{bmatrix}.$$

Relations (6) and (7) begin with the leading words. Let us define the canonical words in X_n, $n \geq 0$. The canonical words in X_0 are the words in X_0 containing no occurrences of the leading words $a_i a_j$ of (1). The canonical words in X_1 are the words in X_1 which do not contain the leading words of (1), (5), (6), and (7), and which do not contain the words

$$x_i^{(1)} \bar{a}\, y_i^{(1)},$$

where $(x_i^{(1)}, y_i^{(1)}) \longleftrightarrow (a, b) \in (F_0^*, F_0^*)$ and a, b are linear combinations of canonical words of F_0, and so on.

The last type of relation of the algebra $\mathfrak{A}$ has the form

$$x_i^{(n)} a\, y_i^{(n)} - b = 0, \tag{8}$$

where $n \geq 1$, $(x_i^{(n)}, y_i^{(n)}) \xrightarrow{\varphi_n} (a.b) \in (F_{n-1}^*, F_{n-1}^*)$, and a, b are linear combinations of canonical words. Let S be the set of left-hand sides of relations (1), (5), (6), (7), and (8) of $\mathfrak{A}$.

<u>LEMMA 6.</u> Each word of the algebra $\mathfrak{A}$ is equal to a linear combination of canonical words.

<u>LEMMA 7.</u> The set S is closed under compositions.

<u>Proof.</u> Consider all possible compositions of the elements (5), (6), and (7):

$$1)\ u(uuv + P) - uuuv \equiv u \begin{bmatrix} & & & & & uau & uav & u[aa] \\ vuu & vuv & vvu & vua & vva & vau & vav & v[aa] \\ auu & auv & avu & avv\, aua & ava & aau & aav & [aaa] \end{bmatrix}$$

$$- Pu - P'v - Pa + vP + aP \equiv 0.$$

$$2)\ (uuv + P)vv - uuvvv \equiv uvuvv + P''vv - vP'v - aP'v +$$

$$+ vvP + vaP + avP + aaP \equiv 0 \quad \text{(in view of (7))}.$$

$$3)\ (uvuvv + P''vv - vP'v - aP'v + vvP + vaP + avP + aaP)v -$$
$$- uvuvvv \equiv - vP'vv - aP'vv + vvPv + vaPv + avPv + aaPv \equiv$$
$$\equiv -vP''vv + vP''vv - vvP'v - vaP'v + vvaP + vavP + vaaP -$$
$$- aP''vv + aP''vv - avP'v - aaP'v + avvP + avaP + aavP +$$
$$+ aaaP + vvP'v - vvaP + vaP'v - vavP - vaaP + avP'v - avvP - avaP + aaP'v - aaP \equiv 0.$$

$$4)\ u(uvuvv + P''vv - vP'v - aP'v + vvP + vaP + avP + aaP) -$$
$$-(uuv + P)uvv \equiv \cancel{uP''vv} - uvP'v - uaP'v + uvvP + uvaP + auvP +$$
$$+ uaaP - \begin{bmatrix} & & \cancel{uvu} & uvv & uua & uva & \cancel{uau} & uav & uaa \\ vuu & \cancel{vuv} & \cancel{vvu} & & vua & vva & \cancel{vau} & vav & vaa \\ auu & \cancel{auv} & \cancel{avu} & avv & aua & ava & \cancel{aau} & aav & aaa \end{bmatrix} uvv +$$
$$+ uv(P'v - vP - aP) + \cancel{vP''vv} - vvP'v - vaP'v + vvaP + vavP +$$
$$+ vaaP + vv(P'v - vP - aP) + av(P'v - vP - aP) + ua(P'v - vP - aP) +$$
$$+ va(P'v - vP - aP) + aa(P'v - vP - aP) + \cancel{aP''vv} - avP'v +$$
$$+ aaP'v + avvP + avaP + aavP + aaaP +$$
$$+ u\begin{bmatrix} & & uvv & uua & uva & uau & uav & uaa \\ vuv & vvu & & vua & vva & vau & vav & vaa \\ auv & avu & avv & aua & ava & aau & aav & aaa \end{bmatrix} vv -$$
$$- \begin{bmatrix} & & uvu & uvv & uua & uva & uau & uav & uaa \\ vuu & vuv & vvu & & vua & vva & vau & vav & vaa \\ auu & auv & avu & avv & aua & ava & aau & aav & aaa \end{bmatrix} avv +$$
$$+ a\begin{bmatrix} & & uvv & uua & uva & uau & uav & uaa \\ vuv & vvu & & vua & vva & vau & vav & vaa \\ auv & avu & avv & aua & ava & aau & aav & aaa \end{bmatrix} vv +$$
$$+ v\begin{bmatrix} & & uvv & uua & uva & uau & uav & uaa \\ vuv & vvu & & vua & vva & vau & vav & vaa \\ auv & avu & avv & aua & ava & aau & aau & aaa \end{bmatrix} vv \equiv 0.$$

$$5)\ uu(uvuvv + P''vv - vP'v - aP'v + vvP + vaP + avP + aaP) -$$
$$- uuuvuvv \equiv uuP''vv - uuvP'v - uuaP'v + uuvvP + uuvaP +$$
$$+ uuavP + uuaaP \equiv uPuvv \text{ (in view of 4)} \equiv 0 \text{ (in view of 1)}.$$

The lemma is proved.

It follows from this lemma (in view of §1) that the set of all canonical words in X forms a basis of the algebra $\mathfrak{A}$. Consequently, $\mathfrak{A} \supseteq A$ and the subalgebras $U = \langle\{u_i^{(n)}\}\rangle$, $V = \langle\{v_i^{(n)}\}\rangle$, $W = \langle\{w_i^{(n)}\}\rangle$ are free nilpotent of class 3. It follows from (4) and (8) that $\mathfrak{A}$ is a simple algebra which is the sum of the subalgebras U, V, W.

Theorem 3 is proved.

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VARIETIES OF GENERALIZED STANDARD AND GENERALIZED ACCESSIBLE ALGEBRAS

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In this paper we shall consider algebras over an associative and commutative ring Φ with unity and containing 1/6.

For the variety of generalized accessible algebras and certain of its subvarieties, considered in the present paper, we shall use the following notation:

$\mathcal{GAcc}$ - variety of generalized accessible algebras,

$\mathcal{Acc}$ - variety of accessible algebras,

$\mathcal{GSt}$ - variety of generalized standard algebras,

$\mathcal{St}$ - variety of standard algebras,

$\mathcal{Comm}$ - variety of commutative algebras,

$\mathcal{Alt}$ - variety of alternative algebras,

$\mathcal{Jord}$ - variety of Jordan algebras,

$\mathcal{Ass}$ - variety of associative algebras,

$\mathcal{AssComm}$ - variety of commutative associative algebras.

Variety $\mathcal{GAcc}$ was defined in 1969 in [1], variety $\mathcal{Acc}$ in 1956 in [2], variety $\mathcal{GSt}$ in 1968 in [3], and variety St in 1948 in [4].

These varieties are related by the set-theoretical inclusions shown in the diagram on the next page (Fig. 1).

Moreover, the ordered set φ of the varieties shown on this diagram is a sub-semilattice of the lattice of all varieties of algebras relative to the operation of intersection.

The basic result of the present paper is the proof of the assertion that this ordered set forms a sublattice in the lattice of all varieties of algebras relative to the operations of union and intersection. In particular, there hold the equations

$$\mathcal{GAcc} = \mathcal{Comm} + \mathcal{Alt}, \quad \mathcal{GSt} = \mathcal{Jord} + \mathcal{Alt}.$$

We mention that for variety $\mathcal{Acc}$ and its subvarieties, the corresponding assertion was proven in [5].

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