Image Transform Solution

Thursday, 21 January 2021 12:02 PM

Let rotate, scale, translate T

$$T(\breve{x};\theta,c,a,b) = c\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}a\\ b\end{pmatrix}$$

$$= \begin{pmatrix} cx\cos\theta - cy\sin\theta + a \\ cx\sin\theta + cy\cos\theta + b\end{pmatrix} = \begin{pmatrix}T_x(\breve{x})\\ T_y(\breve{x})\end{pmatrix}$$

Let $\breve{x}_n$ be position of n-th feature

" $\breve{x}_n^*$ " desired position

Objective $\Phi = \sum_n (T(\breve{x}_n) - \breve{x}_n^*)^T (T(\breve{x}_n) - \breve{x}_n^*)$

$$= \sum_n (T_x(\breve{x}_n) - x_n^*)^2 + (T_y(\breve{x}_n) - y_n^*)^2$$

Minimize Φ : $\partial\Phi/\partial a = \partial\Phi/\partial b = \partial\Phi/\partial c = \partial\Phi/\partial\theta = 0$

① $\partial\Phi/\partial a = 0 = 2\sum_n (T_x(\breve{x}_n) - x_n^*)$

$$= 2\sum_n cx_n\cos\theta - cy_n\sin\theta + a - x_n^*$$

$$\rightarrow \quad a = -c\hat{x}\cos\theta + c\hat{y}\sin\theta + \hat{x}^*$$

② $\partial\Phi/\partial b = 0$

$$\rightarrow \quad b = -c\hat{x}\sin\theta - c\hat{y}\cos\theta + \hat{y}^*$$

Mass center of transformed feats is same of target feats

③ $\partial\Phi/\partial\theta = 0 = 2c\sum_n \Big[(\cancel{cx_n\cos\theta} - \cancel{cy_n\sin\theta} + a - x_n^*)(-x_n\sin\theta - y_n\cos\theta)$

$$+ (\cancel{cx_n\sin\theta} + \cancel{cy_n\cos\theta} + b - y_n^*)(x_n\cos\theta - y_n\sin\theta)\Big]$$

c-terms cancel out

inject ① & ②

$$= 2c\sum_n (-c\hat{x}\cos\theta + c\hat{y}\sin\theta + \hat{x}^* - x_n^*)(-x_n\sin\theta - y_n\cos\theta)$$
$$+ (-c\hat{x}\sin\theta - c\hat{y}\cos\theta + \hat{y}^* - y_n^*)(x_n\cos\theta - y_n\sin\theta)$$

$$\begin{aligned}= 2c\sum_n \Big[\ & \cancel{c\hat{x}x_n\cos\theta\sin\theta} - c\hat{y}x_n\sin^2\theta - \hat{x}^*x_n\sin\theta + x_nx_n^*\sin\theta\\ &+ c\hat{x}y_n\cos^2\theta - c\hat{y}\cancel{y_n\cos\theta\sin\theta} - \hat{x}^*y_n\cos\theta + y_nx_n^*\cos\theta\\ &- \cancel{c\hat{x}x_n\cos\theta\sin\theta} - c\hat{y}x_n\cos^2\theta + \hat{y}^*x_n\cos\theta - x_ny_n^*\cos\theta\\ &+ c\hat{x}y_n\sin^2\theta + c\hat{y}\cancel{y_n\cos\theta}\sin\theta - \hat{y}^*y_n\sin\theta + y_ny_n^*\sin\theta\ \Big]\end{aligned}$$

$$= 2c\sum_n \underbrace{c\hat{x}y_n - c\hat{y}x_n}_{\text{Cancel out upon Summation}} + (-\hat{x}^*x_n - \hat{y}^*y_n + x_nx_n^* + y_ny_n^*)\sin\theta + (-\hat{x}^*y_n + \hat{y}^*x_n + y_nx_n^* - x_ny_n^*)\cos\theta\ \Big]$$

$$= 0$$

$$0 = p\sin\theta + q\cos\theta \Rightarrow -q/p = \sin\theta/\cos\theta = \tan\theta$$

$$\theta = \operatorname{atan}(-q/p)$$

④ $\partial\Phi/\partial c = 0 = 2\sum_n \Big[(cx_n\cos\theta - cy_n\sin\theta + a - x_n^*)(x_n\cos\theta - y_n\sin\theta)$

$$+ (cx_n\sin\theta + cy_n\cos\theta + b - y_n^*)(x_n\sin\theta + y_n\cos\theta)$$

$$\begin{aligned}= 2\sum_n \Big[\ & cx_n^2\cos^2\theta - \cancel{cx_ny_n\cos\theta\sin\theta} + ax_n\cos\theta - x_nx_n^*\cos\theta\\ &- \cancel{cx_ny_n\cos\theta\sin\theta} + cy_n^2\sin^2\theta - ay_n\sin\theta + y_nx_n^*\sin\theta\\ &+ cx_n^2\sin^2\theta + \cancel{cx_ny_n\cos\theta\sin\theta} + bx_n\sin\theta - x_ny_n^*\sin\theta\\ &+ \cancel{cx_ny_n\cos\theta\sin\theta} + cy_n^2\cos^2\theta + by_n\cos\theta - y_ny_n^*\cos\theta\ \Big]\end{aligned}$$

$$\begin{aligned}= 2\sum_n \Big[\ cx_n^2 + cy_n^2\ & - c\hat{x}x_n\cos^2\theta + c\hat{y}\cancel{x_n\cos\theta}\sin\theta + \hat{x}^*x_n\cos\theta - x_nx_n^*\cos\theta\\ & + \cancel{c\hat{x}y_n\cos\theta\sin\theta} - c\hat{y}y_n\sin^2\theta - \hat{x}^*y_n\sin\theta + y_nx_n^*\sin\theta\\ & - c\hat{x}x_n\sin^2\theta - c\hat{y}\cancel{x_n\cos\theta}\sin\theta + \hat{y}^*x_n\sin\theta - x_ny_n^*\sin\theta\end{aligned}$$

$$
\begin{aligned}
= 2\sum_n \Big[\; & c x_n^2 + c y_n^2 - c\hat{x} x_n \cos^2\theta + \cancel{c\hat{y} x_n \cos\theta} \sin\theta + \hat{x}^* x_n \cos\theta - x_n x_n^* \cos\theta \\
& + \cancel{c\hat{x} y_n \cos\theta \sin\theta} - c\hat{y} y_n \sin^2\theta - \hat{x}^* y_n \sin\theta + y_n x_n^* \sin\theta \\
& - c\hat{x} x_n \sin^2\theta - \cancel{c\hat{y} x_n \cos\theta} \sin\theta + \hat{y}^* x_n \sin\theta - x_n y_n^* \sin\theta \\
& - \cancel{c\hat{x} y_n \cos\theta \sin\theta} - c\hat{y} y_n \cos^2\theta + \hat{y}^* y_n \cos\theta - y_n y_n^* \cos\theta
\end{aligned}
$$

$$
\begin{aligned}
= 2\sum_n c\,(x_n^2 + y_n^2 - \hat{x} x_n - \hat{y} y_n) & + \left[(\hat{x}^* - x_n^*)\, x_n + (\hat{y}^* - y_n^*)\, y_n \right] \cos\theta \\
& + \left[-(\hat{x}^* - x_n^*)\, y_n + (\hat{y}^* - y_n^*)\, x_n \right] \sin\theta \,\Big]
\end{aligned}
$$

$$
= 2\,(\alpha c + \beta\cos\theta + \gamma\sin\theta)
$$

$$
c = -\frac{\beta\cos\theta + \gamma\sin\theta}{\alpha}
$$