# Hedging Portfolios of Options with Real Data

Quan Tran*

January 2, 2025

**Abstract**

This is Quan Tran's report for the **Hedging Assignment** of the course TU-E2211 - Financial Risk Management with Derivatives 1 taken at Aalto University. This study explores the effectiveness of delta and delta-vega hedging strategies in mitigating price and volatility risks associated with S&P 500 options. Using real-world data, the analysis evaluates these strategies under varying re-hedging frequencies and incorporates the impact of transaction costs. The findings demonstrate that delta-vega hedging consistently outperforms delta hedging in both single-option and portfolio contexts, offering higher mean effectiveness and reduced sensitivity to extreme market movements. Moreover, delta-vega hedging exhibits greater resilience to transaction costs, maintaining robustness even under significant fees. These results highlight the practical applications of these hedging strategies in financial risk management and emphasize the potential for future improvements through advanced modeling techniques, such as stochastic volatility models and machine learning-based parameter estimation. The insights gained provide a foundation for refining hedging approaches to meet the demands of increasingly complex financial markets.

*100644052; quan.tran@aalto.fi;

# Contents

# 1. Introduction

The dynamic nature of financial markets necessitates robust strategies to mitigate risks associated with price and volatility fluctuations of financial instruments. Hedging, a fundamental aspect of financial risk management, aims to protect portfolios from adverse market movements by employing derivative instruments and underlying assets. Among various strategies, delta hedging and delta-vega hedging are widely recognized for their effectiveness in managing directional and volatility risks.

This report examines the application of delta and delta-vega hedging strategies using real-world data from Standard & Poor's 500 (S&P 500) options. The analysis investigates the effectiveness of these strategies under varying re-hedging frequencies and explores their sensitivity to transaction costs. Furthermore, it assesses the performance of these strategies when applied to single options and portfolios, providing insights into their practical implications in financial risk management. The findings are expected to contribute to the broader discourse on optimizing hedging strategies and addressing real-world challenges in financial markets.

The report is organized as follows: Section 2 focuses on hedging single options, beginning with delta hedging and progressing to delta-vega hedging, examining their effectiveness and sensitivity to various re-hedging frequencies. It also explores the impact of transaction costs on the performance of these strategies. Section 3 expands the analysis to portfolios, specifically a strangle position combining call and put options. This section evaluates the application of delta and delta-vega hedging in portfolio management, comparing their accuracy and resilience to transaction costs. Finally, the report concludes by synthesizing the findings, discussing their implications for financial risk management, and identifying potential areas for further investigation.

# 2. Single Options

This section explores the effectiveness of two hedging strategies—Delta hedging and Delta-Vega hedging—in hedging a portfolio of a single S&P 500 call option with a chosen strike price of $4525 against price changes.

### Dataset

To ensure reliable statistical estimates of the performance of hedging strategies across various re-hedging frequencies, the two strategies were tested on thirteen Standard & Poor's 500 (S&P 500) call options. These options shared the same underlying asset, strike price, and time to maturity but had different expiration dates.: 20-08-2021, 17-09-2021, 15-10-2021, 18-02-2022, 18-03-2022, 14-04-2022, 20-05-2022, 17-03-2023, 21-07-2023, 18-08-2023, 15-09-2023, 20-10-2023, and 17-11-2023. This dataset was comprehensive, as it included all sold call options with a strike price of $4525 that allowed for the calculation of Implied Volatility, Delta $\Delta$, and Vega $\kappa$ over the 45-day period prior to expiration (more details in the feature engineering section below).

For each option, the data about its settlement, bid, and ask prices, reference Delta's $\Delta$ and Vega's $\kappa$, and time to maturity were fetched from Refinitiv's workspace. Please note that the reference Delta's $\Delta$ and Vega's $\kappa$ provided by Refinitiv were only available for only few couples of days, typically 5–7 days within the expiration dates. On the other hand, the underlying prices were obtained from (Yahoo Finance, 2024), as Refinitiv did not provide access to S&P 500 data. Additionally, the reference risk-free rates were sourced from the (U.S. Department of the Treasury, 2024), specifically the daily Treasury par yield curve rates for three-month securities.

**Feature Engineering**

In the obtained datasets, missing spot prices (NaN) were replaced with the arithmetic mean of the bid and ask prices. The times to maturity $t$'s, $0 < t < 45$, are annualized, i.e., they were divided by 365. Using the collected data on spot prices, risk-free rates, and time to maturity, the Implied Volatilities $\sigma_t$ for each option were calculated at each timestamp $t$, $t \in (0, \frac{45}{365})$. These calculations utilized the following Black-Scholes model over the 45-day period leading to each option's expiration:

$$\begin{cases} C_t = \mathcal{N}(d_+)S_t - \mathcal{N}(d_-)Ke^{-r_t t}, \\ d_+ = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r_t + \frac{\sigma_t^2}{2}\right)t}{\sigma_t\sqrt{t}}, \\ d_- = d_+ - \sigma\sqrt{t}. \end{cases}$$

where $t$ is the time to maturity and $C_t$ is the respective call option price at timestamp $t$. $K$ is the strike price of the option, while $S_t$ and $\sigma$ are the spot price (settlement price) and the volatility of the underlying asset (here, S&P 500) at timestamp $t$. $r$ is the risk-free rate and $\mathcal{N}$ is the Gaussian integral. The calculations were done in R with the help of the `library(fOptions)` and its `GBSVolatility()` function.

On the other hand, the values $\Delta_t$ and $\kappa_t$, $0 < t < \frac{45}{365}$, the option Greeks Delta and Vega at timestamp t of each option were calculated using the `calculate_greeks()` function, as defined below.

Definition 2.1. (Delta of an Option)

The Delta $\Delta$ is the first partial derivative of Black-Scholes price with respect to the underlying asset, measuring the rate of change of the option price with respect to the underlying asset's price:

$$\Delta_t := \frac{\partial C_t}{\partial S} = \mathcal{N}(d_+),$$

where $d_+ = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r_t + \frac{\sigma_t^2}{2}\right)t}{\sigma_t\sqrt{t}}$ and $\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\,dy$ the Gaussian integral. Therefore, Delta is an increasing function of the underlying asset.

Definition 2.2. (Vega of an Option)

The Vega $\kappa$ is the first partial derivative of Black-Scholes price with respect to the Implied Volatility $\sigma_t$ of the underlying asset, measuring the rate of change of the option price with respect to the Implied Volatility:

$$\kappa_t := \frac{\partial C_t}{\partial \sigma} = S_t \sqrt{t} \mathcal{N}'(d_+).$$

Therefore, the Vega quantifies price uncertainty due to volatility estimates.

## 2.1. Delta Hedging a Single Option

Delta hedging, in this context, was a hedging strategy aiming to hedge a portfolio of a single call option against the price movements by reaching delta neutrality (Corporate Finance Institute, 2024a). Investors or traders employed delta hedging to lock in profits or reduce losses from sudden price movements, focusing instead on other factors that may impact the portfolio, such as time decay (Theta $\Theta$) or volatility (Vega $\kappa$). The neutrality was achieved by additionally taking a position in $x_t$ units of the underlying asset. Denote $\Pi$ the aggregate portfolio consisted of the original call option and the underlying asset. One could find $x_t$ by solving the following equation:

$$\begin{aligned} &\Delta(\Pi) = x_t \Delta(S_t) + \Delta(C_t) = 0 \\ &\Leftrightarrow x_t = -\frac{\Delta(C_t)}{\Delta(S_t)} \\ &\Leftrightarrow x_t = -\Delta(C_t) \qquad (\Delta(S_t) = 1) \end{aligned}$$

Therefore, in order to reach the delta-neutral state, we shorted $\Delta(C_t)$ units of the underlying assets. However, this model assumes that $x_t$ is a continuous function of time to maturity ($t$), meaning the underlying asset's allocation is continuously adjusted. However, such adjustments are not feasible in practical scenarios. Therefore, besides the error and the effectiveness of the hedge, this section also analyzed how the re-hedging frequency affected those. The (unbiased) mean squared error (MSE) and the effectiveness of the hedging strategy were defined below:

Definition 2.3. ((Unbiased) Mean Squared Error)

The mean squared error $E$ is defined as the unbiased estimate of the mean of the differences between the changes in the price of the original call option and the changes in the price of the replicating portfolio:

$$E := \frac{1}{n-1} \sum_{i=1}^{n-1} \left[ (C_{i+1} - C_i) - \Delta_i \left( S_{i+1} - S_i \right) \right]^2,$$

where $n$ is the number of trading days in the 45-day period ending on the expiration date of the call option. $C_i$ and $S_i$ are the prices of the option and the underlying asset, respectively, while $\Delta_i$ is the amount of the shorted underlying asset on the $i^{\text{th}}$ day.

Definition 2.4. (Effectiveness of the Hedging Strategy)

The effectiveness $F$ of the hedging strategy is defined as, in this context, the ratio of the mean squared errors under hedge and without hedge:

$$F := \left(1 - \frac{E}{E_{\text{no hedge}}}\right) \times 100\%,$$

where $E_{\text{no hedge}} = \frac{1}{n-1}\sum_{i=1}^{n-1}(C_{i+1} - C_i)^2$. If $F = 100\%$, we say the hedge is a "perfect match." This quantity was of interest because, in the repeated experiments, not every option was ATM; some of them were even relatively deep in-the-money (ITM) or out-the-money (OTM). Effectiveness could help remove the absolute scale bias.

An initial test was conducted on an at-the-money (ATM) call option expiring on 20-05-2022. This choice was motivated by the superior sensitivity to price changes and higher liquidity typically exhibited by ATM options. In this section, it was assumed that there were no market frictions, such as transaction fees. The results are depicted in Table 2.1 and Figure 2.1.

| **Frequency (re-hedge(s)/calendar day)** | **MSE ($\$^2$)** | **Effectiveness (%)** |
|---|---|---|
| 1 | 23.35 | 86.7 |
| 1/2 | 39.07 | 77.8 |
| 1/3 | 25.47 | 85.5 |
| 1/4 | 18.29 | 89.6 |
| 1/5 | 49.47 | 71.8 |
| 1/6 | 26.10 | 85.1 |
| 1/7 | 88.54 | 49.6 |
| 1/8 | 84.61 | 51.8 |
| 1/9 | 26.92 | 84.7 |
| 1/10 | 42.52 | 75.8 |

Table 2.1: MSE and hedging effectiveness for various frequencies over the 45-day period: Delta hedging of a single call option.

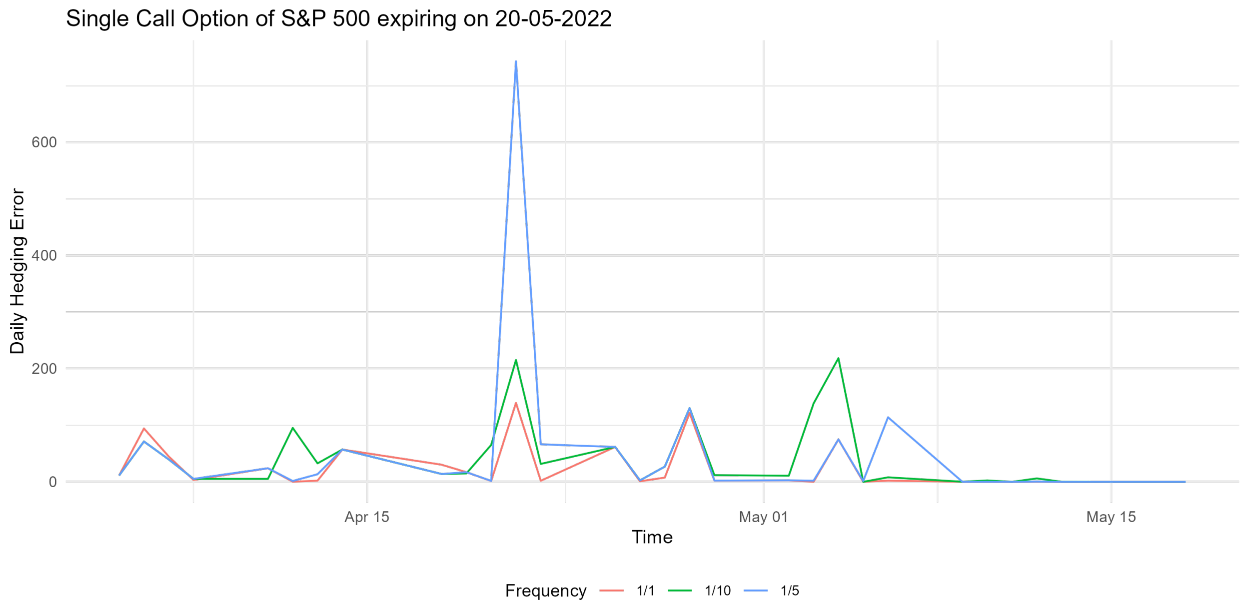


Figure 2.1: Daily hedging error for various re-hedging frequencies: Delta hedging of a single call option.

Table 2.1 reveals a surprising observation: a decrease in frequency does not consistently lead to higher MSE or reduced effectiveness. On the other hand, Figure 2.1 shows that frequency 1/1 generally had better daily errors than higher frequencies 1/5 and 1/10 did. It is also worth noting that there were relatively many spikes in the error, which can be divided into two groups: high spikes and low spikes. While low spikes occurred occasionally throughout the 45-day period, high spikes were less frequent, but there was one significantly high spike appearing in April 20$^{\text{th}}$ - April 21$^{\text{th}}$. To get better statistical estimates, the experiment was repeated with 13 other call options having the same strike price ($4525) and the underlying asset (S&P 500) but different expiration dates: 20-08-2021, 17-09-2021, 15-10-2021, 18-02-2022, 18-03-2022, 14-04-2022, 20-05-2022, 17-03-2023, 21-07-2023, 18-08-2023, 15-09-2023, 20-10-2023, and 17-11-2023.

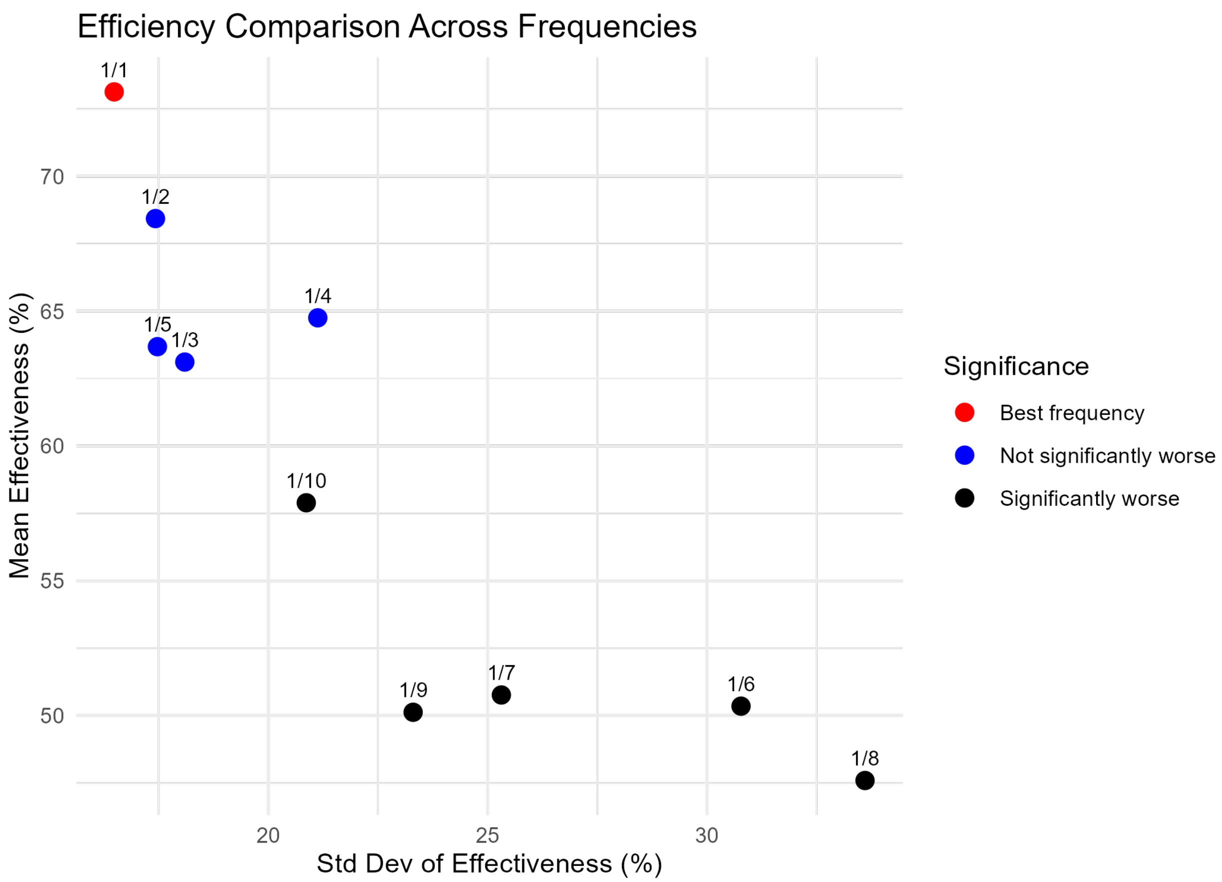


Figure 2.2: Means and standard deviations of the effectiveness for various frequencies over the 45-day period: Delta hedging of a single call option.

As illustrated in Figure 2.2, in general, higher frequencies generally offered better performances, i.e., better average effectiveness with a lower standard deviation compared to lower frequencies. Particularly, the lowest frequency 1/1 provided the best results with an average effectiveness of 73.13% and a standard deviation of 16.48%. Moreover, a one-tailed Welch t-test was conducted for each pair of the best frequency and another frequency $j$ for $j \in \{1/2, 1/3, \ldots, 1/10\}$:

- **Null Hypothesis** ($\mathbf{H}_0$): Best frequency is not better than the frequency $j$ (i.e., $\mu_{\text{best frequency}} \leq \mu_{\text{frequency j}}$).
- **Alternative Hypothesis** ($\mathbf{H}_A$): Best frequency is better than the frequency $j$ (i.e., $\mu_{\text{best frequency}} > \mu_{\text{frequency j}}$).

It turned out that the frequencies $1/6, 1/7, \ldots, 1/10$ are significantly worse than the best frequency.

## 2.2. Delta-Vega Hedging a Single Option

Delta-vega hedging, in this context, was a hedging strategy aiming to hedge a portfolio of a single call option against the underlying's volatility changes in addition to the option's price movements by reaching both delta neutrality and vega neutrality (Corporate Finance Institute, 2024b). Investors or traders employed delta-vega hedging to minimize exposure to both the directional risk (Delta $\Delta$) and the volatility risk (Vega $\kappa$), focusing instead on other factors that may impact the portfolio, such as the time decay (Theta $\Theta$) and Gamma $\Gamma$). This is achieved by constructing a replicating portfolio comprising $x_t$ units of the underlying asset and $y_t$ units of replicating option. Denote $\Pi$ the aggregate portfolio consisted of the original call option and the replicating portfolio, $S_t$ the underlying asset, $C_t^O$ and $C_t^R$ the original call option and the replicating option, respectively. Then one could find $x_t$ and $y_t$ by solving the following system of equations:

$$
\begin{aligned}
&\begin{cases} \Delta(\Pi) & = 0 = -x_t\Delta(S_t) - y_t\Delta(C_t^R) + \Delta(C_t^O) \\ \kappa(\Pi) & = 0 = -x_t\kappa(S_t) - y_t\kappa(C_t^R) + \kappa(C_t^O) \end{cases} \\
\Leftrightarrow &\begin{cases} x_t = -y_t\Delta(C_t^R) + \Delta(C_t^O) \\ y_t = \dfrac{\kappa(C_t^O)}{\kappa(C_t^R)} \end{cases} \\
\Leftrightarrow &\begin{cases} x_t = \Delta(C_t^O) - \dfrac{\kappa(C_t^O)}{\kappa(C_t^R)}\Delta(C_t^R) \\ y_t = \dfrac{\kappa(C_t^O)}{\kappa(C_t^R)} \end{cases} \\
\Leftrightarrow &\begin{cases} x_t = \dfrac{\partial C_t^O}{\partial S} - \dfrac{\partial C_t^O/\partial\sigma}{\partial C_t^R/\partial\sigma}\dfrac{\partial C_t^R}{\partial S} \\ y_t = \dfrac{\partial C_t^O/\partial\sigma}{\partial C_t^R/\partial\sigma} \end{cases}
\end{aligned}
$$

As previously mentioned in Section 2.1, trading was done discretely. Therefore, the metrics MSE and effectiveness were also analyzed in this case. However, the MSE needed to be appropriately adjusted as follows:

Definition 2.5. ((Unbiased) Mean Squared Error)

The mean squared error $E$ is defined as the unbiased estimate of the mean of the differences between the changes in the price of the original call option and the changes in the price of the replicating portfolio:

$$E := \frac{1}{n-1}\sum_{i=1}^{n-1}\left[\left(C_{i+1}^O - C_i^O\right) - \alpha_i\left(S_{i+1} - S_i\right) - \eta_i\left(C_{i+1}^R - C_i^R\right)\right]^2,$$

where $n$ is the number of trading days in the 45-day period ending on the expiration date of the call option. $C_i$ and $S_i$ are the prices of the option and the underlying asset, respectively, while $\alpha_i$ and $\eta_i$ are the units of the longed underlying asset and replicating option, respectively, on the $i^{\text{th}}$ day.

Similar to what was done in Section 2.1, the same at-the-money (ATM) call option expiring on 20-05-2022 was initially tested, together with a call option expiring on 17-06-2022 as the replicating option. Likewise, in this section, it was assumed that there were no market frictions, such as transaction fees. The results are illustrated in Table 2.2 and Figure 2.3.

| Frequency (re-hedge(s)/calendar day) | MSE ($\$^2$) | Effectiveness (%) |
|---|---|---|
| 1 | 25.35 | 85.6 |
| 1/2 | 30.42 | 82.7 |
| 1/3 | 29.26 | 83.3 |
| 1/4 | 28.55 | 83.7 |
| 1/5 | 32.86 | 81.3 |
| 1/6 | 28.25 | 83.9 |
| 1/7 | 38.69 | 78.0 |
| 1/8 | 32.36 | 81.6 |
| 1/9 | 23.95 | 86.4 |
| 1/10 | 24.43 | 86.1 |

Table 2.2: MSE and hedging effectiveness for various frequencies over the 45-day period: Delta-vega hedging of a single call option.

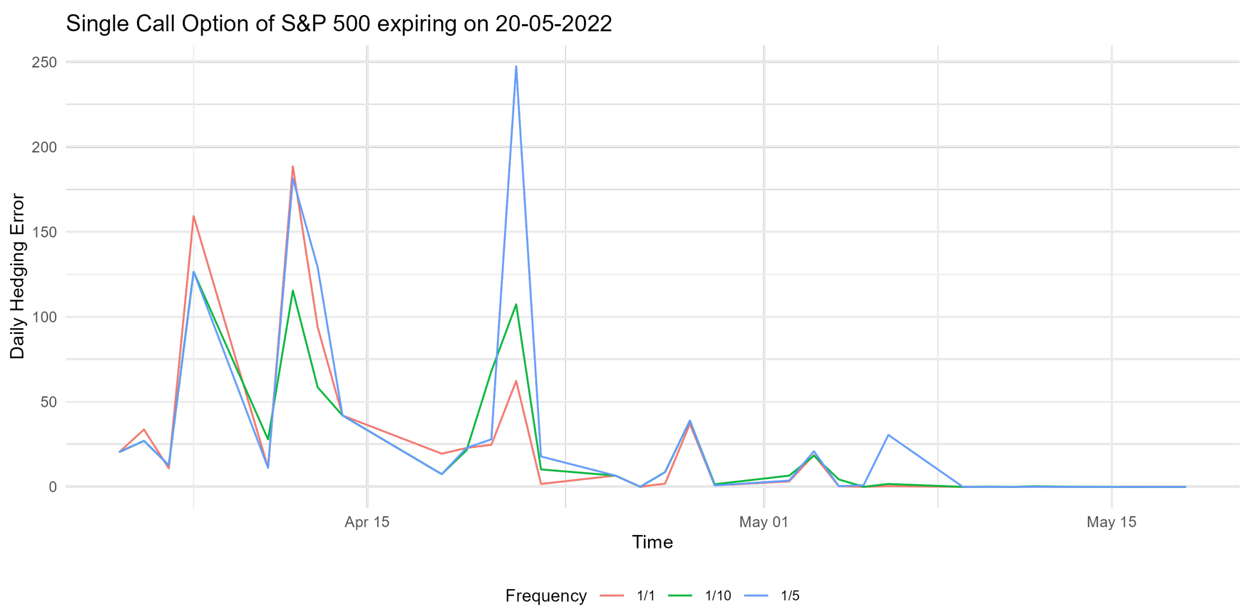


Figure 2.3: Daily hedging error for various re-hedging frequencies: Delta-vega hedging of a single call option.

As can be seen in Table 2.1, the effectiveness of one frequency is relatively comparable to the others, maintaining a rate of approximately 80%. On the other hand, similar to what was shown in delta-hedging, Figure 2.3 shows that frequency 1/1 generally had better daily errors than higher frequencies 1/5 and 1/10 did. However, it is noteworthy that there were considerably fewer spikes than delta hedging as shown in Figure 2.1, and those spikes were also considerably less extreme. The spikes now could be grouped into two groups: high spikes and low spikes. While high spikes occurred at the beginning of the 45-day period, low spikes clustered at the end. Together, these two results suggested that delta hedging helped reduce volatility risk. In a manner analogous to the approach taken previously, this experiment was also repeated with 11 other call options having the same strike price ($4525) and the underlying asset (S&P 500) of different expiration dates: 20-08-2021, 17-09-2021, 18-02-2022, 18-03-2022, 14-04-2022, 20-05-2022, 17-03-2023, 21-07-2023, 18-08-2023, 15-09-2023, and 20-10-2023.

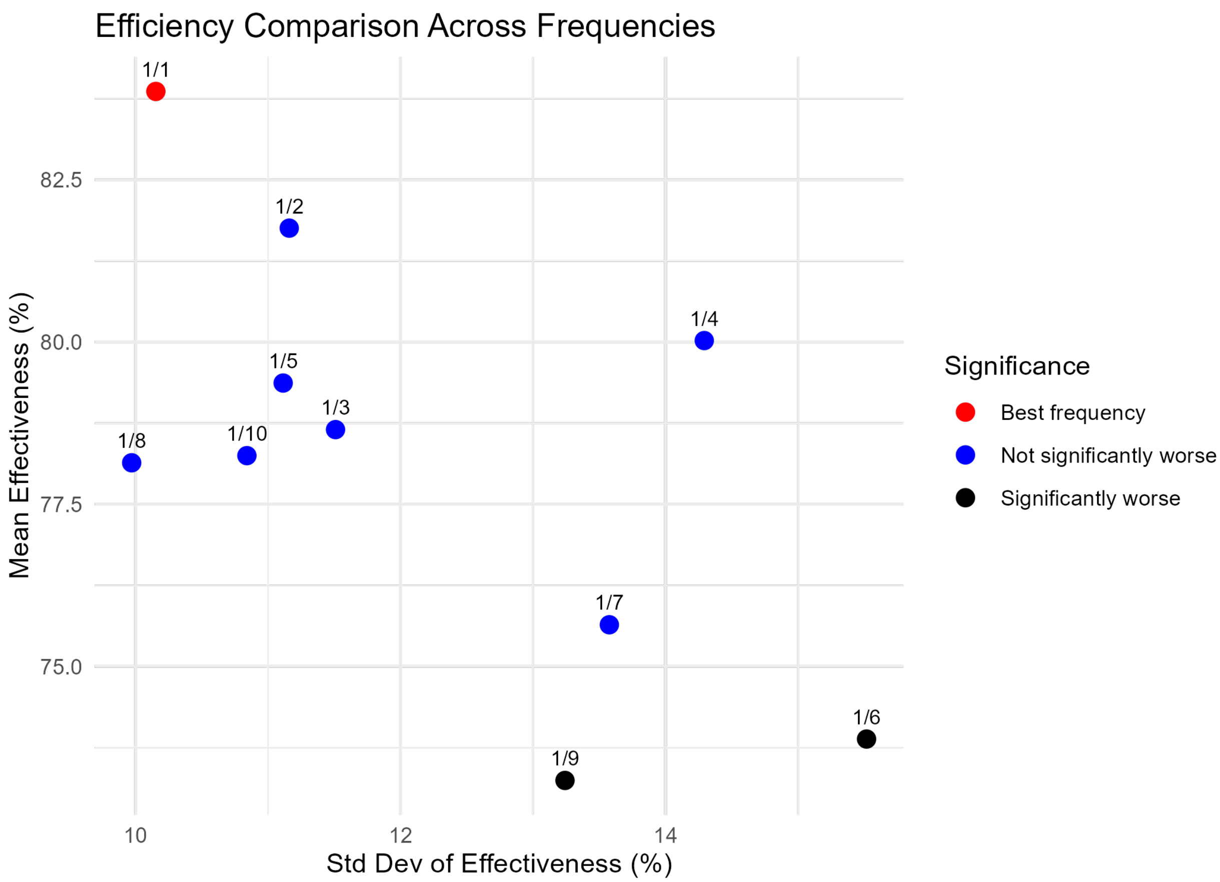


Figure 2.4: Means and standard deviations of the effectiveness for various frequencies over the 45-day period: Delta-vega hedging of a single call option.

As illustrated in Figure 2.4, in general, higher frequencies also generally offered better performances, i.e., better average effectiveness with a lower standard deviation compared to lower frequencies; however, the gap between high frequencies and low frequencies was not as evident as in delta hedging. Particularly, the frequencies 1/1 provided the best performance, with a mean effectiveness of 83.86% and a standard deviation of 10.15%. In the same way as it was carried out in Section 2.1, a one-tailed Welch t-test was conducted for each pair of the best frequency and another frequency $j$ for $j \in \{1/2, 1/3, \dots, 1/10\}$. It turned out that the frequencies 1/6 and 1/9 are significantly worse than the best frequency.

## 2.3. Hedging Accuracy Compared

By comparing Figure 2.1 and Figure 2.3, the previous section already pointed out that there were considerably fewer spikes in delta-vega hedging than delta hedging, and those spikes were also considerably less extreme. Similarly, by comparing Table 2.1 and Table 2.2, one can easily notice that delta-vega hedging consistently maintained a fair effectiveness level, while delta hedging did not. Figure 2.5 visualizes a line plot with error bars, where the lines represent the average effectiveness of the two hedging strategies and the error bars illustrate the standard deviation of the effectiveness at each frequency.

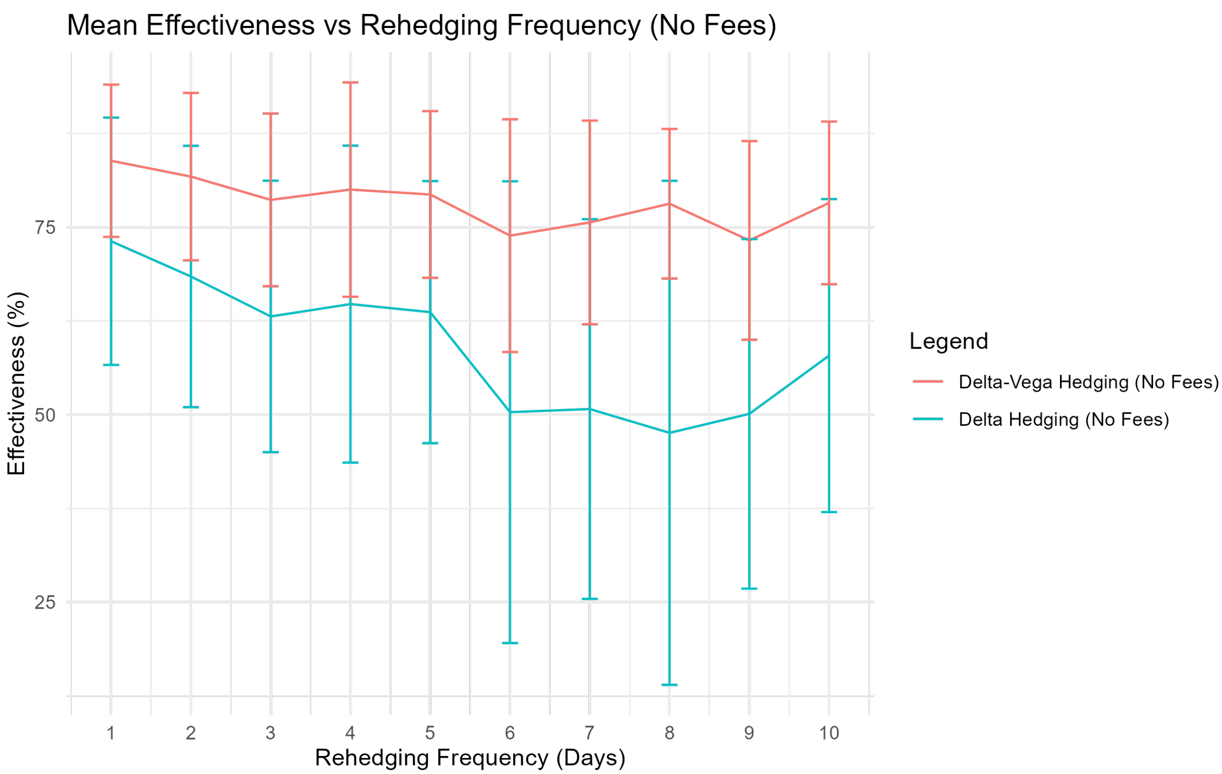


Figure 2.5: Mean effectiveness by re-hedging frequencies for two hedging strategies applied to a single call option.

At first glance, delta-vega hedging demonstrated higher mean effectiveness and lower standard deviation compared to delta hedging across all frequencies. To truly tell whether the delta-vega hedging strategy was indeed (statistically) better than the other, a one-tailed Welch t-test was conducted for each frequency:

- **Null Hypothesis ($\mathbf{H}_0$)**: Delta-vega hedging is not better than delta hedging (i.e., $\mu_{\text{delta-vega}} \leq \mu_{\text{delta}}$).
- **Alternative Hypothesis ($\mathbf{H}_A$)**: Delta-vega hedging is better than delta hedging (i.e., $\mu_{\text{delta-vega}} > \mu_{\text{delta}}$).

The results of the test are presented below in Table 2.3, with a significance level $\alpha$ of 0.05.

| **Frequency (re-hedge(s)/calendar day)** | **Result** |
|---|---|
| 1 | Null Hypothesis Rejected |
| 1/2 | Null Hypothesis Rejected |
| 1/3 | Null Hypothesis Rejected |
| 1/4 | Null Hypothesis Rejected |
| 1/5 | Null Hypothesis Rejected |
| 1/6 | Null Hypothesis Rejected |
| 1/7 | Null Hypothesis Rejected |
| 1/8 | Null Hypothesis Rejected |
| 1/9 | Null Hypothesis Rejected |
| 1/10 | Null Hypothesis Rejected |

Table 2.3: Welch's t-test results for various frequencies with significance level $\alpha = 0.05$.

As can be seen in Table 2.3, for all frequencies, delta-vega hedging performed significantly better than delta hedging.

## 2.4. Effect of Adding Transaction Costs

This section explores how fees degrade hedging effectiveness, with the linear notional-based fee as the studied kind of fee. For instance, if the fee rate = 0.02 (i.e., 2%), the investor is paying 2% of the notional he trades each time he rebalances. The MSE was adjusted such that they incorporated the transaction fee in this case:

$$E := \frac{1}{n-1} \sum_{i=1}^{n-1} \left[ \left| (\mathrm{OP}_{i+1} - \mathrm{OP}_i) - (\mathrm{RE}_{i+1} - \mathrm{RE}_i) \right| + \mathrm{fee}_i \right]^2 ,$$

where $n$ is the number of trading days in the 45-day period ending on the expiration date of the call option. $\mathrm{OP}_i$ and $\mathrm{RE}_i$ are the values of the original and replicating portfolio, respectively, while $\mathrm{fee}_i$ is the transaction fee paid on the $i^{\text{th}}$ day.

Figure 2.6 and Figure 2.7 below illustrate how the effectiveness varies with respect to the fee rate for various frequencies for delta hedging and delta-vega hedging, respectively. The data indicate a clear trend: effectiveness decreases as the fee rate rises, with the decline exhibiting a non-linear pattern. At lower fee rates (0 to 0.01), the decay is gentler. Beyond this point, the decay becomes steeper, suggesting a higher sensitivity of effectiveness to fee rates as costs increase. Moreover, it seemed that delta hedging was more sensitive to notional-based fee than delta-vega hedging, ultimately reaching negative effectiveness when the fee rate reached 0.02. On the other hand, delta-vega hedging maintained a fair effectiveness across all fee rates, with the effectiveness arriving at around 63% level when the fee rate reached 0.02.

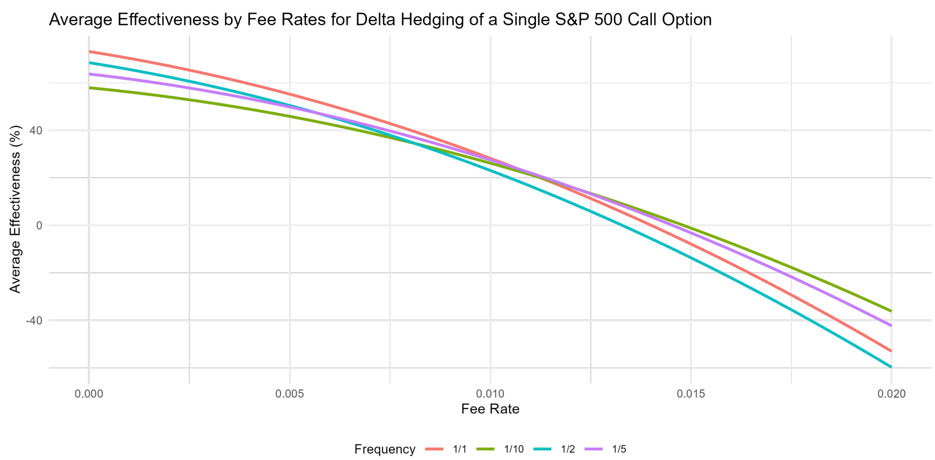


Figure 2.6: Average Effectiveness by Fee Rates for Delta-Vega Hedging of a Single S&P 500 Call Option

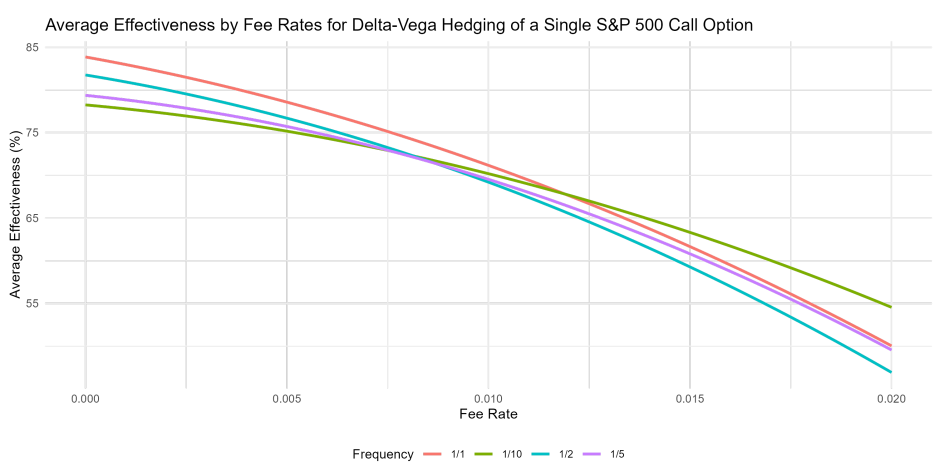


Figure 2.7: Average Effectiveness by Fee Rates for Delta-Vega Hedging of a Single S&P 500 Call Option

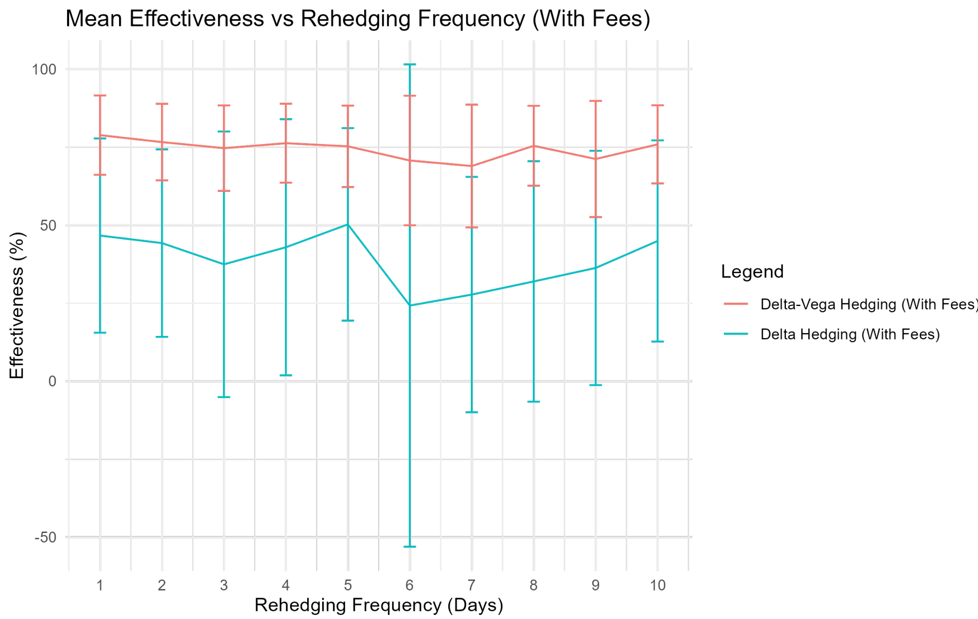


Figure 2.8: Mean effectiveness by re-hedging frequencies for two hedging strategies applied to a single call option with a fee rate = 0.01.

Figure 2.8 visualizes a line plot with error bars, where the lines represent the average effectiveness of the two hedging strategies and the error bars illustrate the standard deviation of the effectiveness at each frequency when fee rate was 0.01. In line with what was done previously in Section 2.3, a one-tailed Welch t-test was conducted for each frequency. The results of the test are presented below in Table 2.4, with a significance level $\alpha$ of 0.05.

| **Frequency (re-hedge(s)/calendar day)** | **Result** |
|---|---|
| 1 | Null Hypothesis Rejected |
| 1/2 | Null Hypothesis Rejected |
| 1/3 | Null Hypothesis Rejected |
| 1/4 | Null Hypothesis Rejected |
| 1/5 | Null Hypothesis Rejected |
| 1/6 | Null Hypothesis Rejected |
| 1/7 | Null Hypothesis Rejected |
| 1/8 | Null Hypothesis Rejected |
| 1/9 | Null Hypothesis Rejected |
| 1/10 | Null Hypothesis Rejected |

Table 2.4: Welch's t-test results for various frequencies with significance level $\alpha = 0.05$.

As can be seen in Table 2.4, for all frequencies, delta-vega hedging still performed significantly better than delta hedging when the fee rate was 0.01.

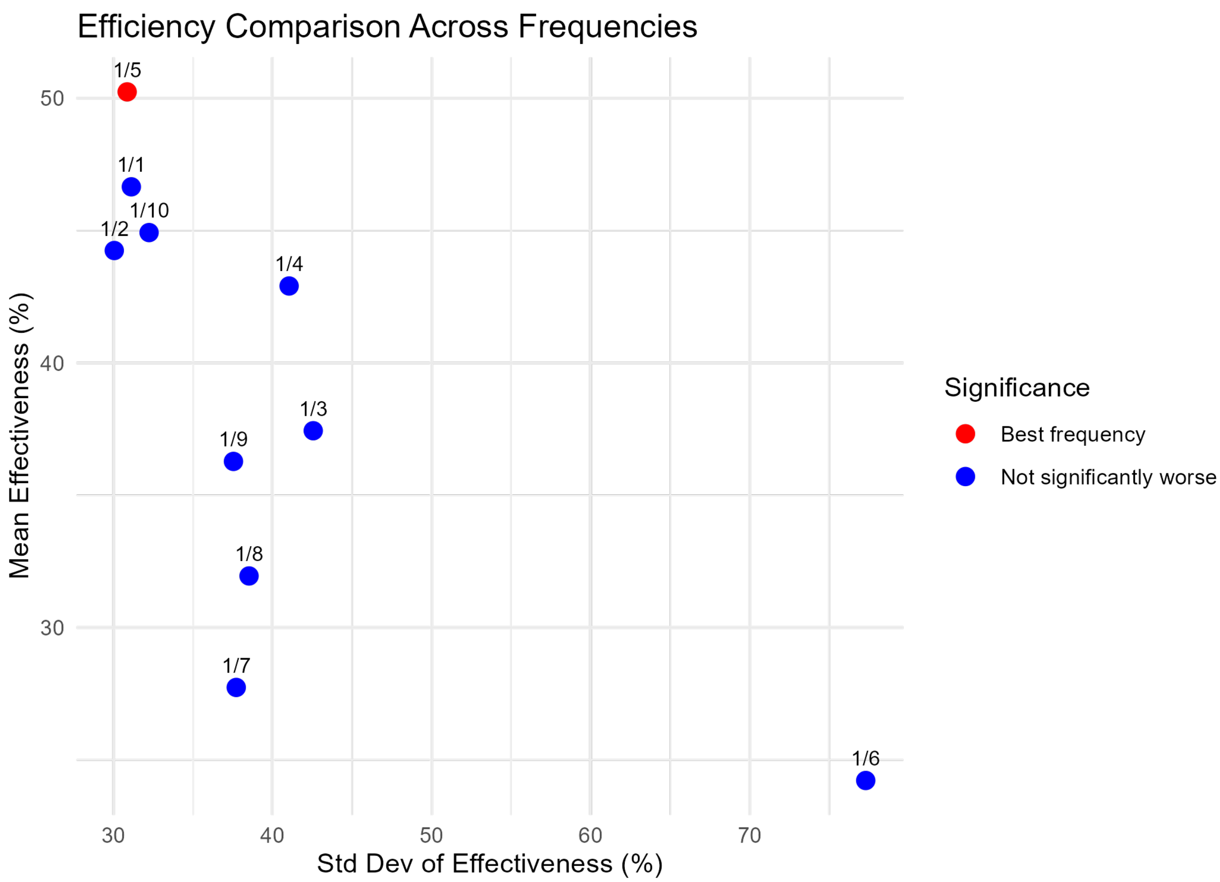


Figure 2.9: Means and standard deviations of the effectiveness for various frequencies over the 45-day period: Delta hedging of a single call option with fee rate = 0.01.

As illustrated in Figure 2.9, when the fee rate was 0.01, the frequency 1/5, with a mean of 50.23% and a standard deviation of 30.86%, replaced the frequency 1/1 in Figure 2.2 as the best frequency. This is sensible. If there is some market friction (here, transaction fee), frequent re-hedging strategies are no longer guaranteed to be optimal. However, higher frequencies were still generally better than lower ones.

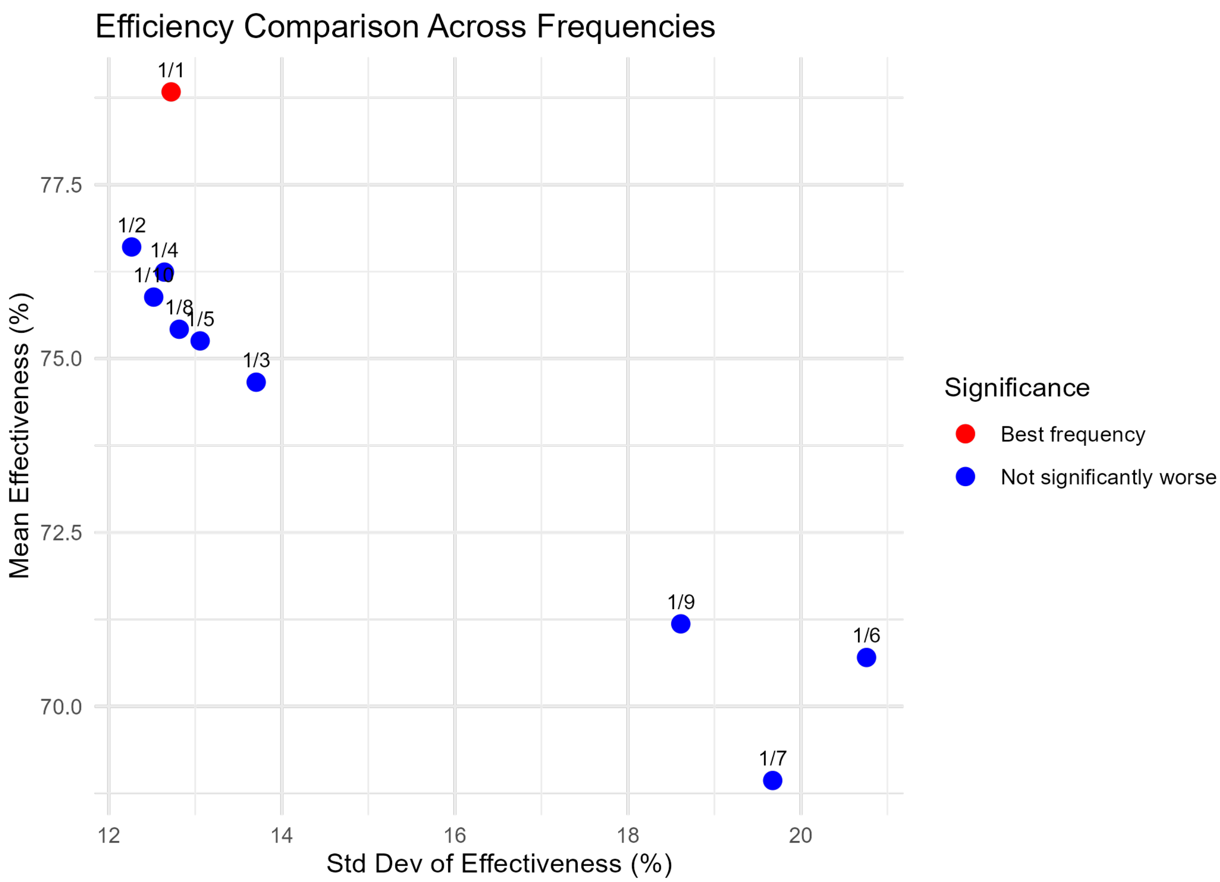


Figure 2.10: Means and standard deviations of the effectiveness for various frequencies over the 45-day period: Delta-vega hedging of a single call option with fee rate = 0.01.

Figure 2.10 demonstrates that at a fee rate of 0.01, the frequency of 1/1 remained the most effective, consistent with earlier findings (Figure 2.2). Additionally, higher rebalancing frequencies generally outperformed lower ones. This suggests that delta-vega hedging was less dependent on frequent adjustments, making it more resilient to market frictions such as transaction fees.

# 3. Portfolio of Options

Similar to Section 2, this section explores the effectiveness of two hedging strategies—Delta hedging and Delta-Vega hedging. However, instead of a portfolio of a single S&P 500 call option, it studies hedging a portfolio taking a strangle position with a call option with a strike price of $4525 and a put option with a strike price of $4450.

**Dataset**

Eleven pairs of S&P 500 call and put options expiring on the same day were tested. The expiration dates were as follows: 21-05-2021, 18-06-2021, 15-07-2021, 19-08-2022, 16-12-2022, 20-01-2023, 17-02-2023, 17-03-2023, 21-04-2023, 19-05-2023, and 16-06-2023. Likewise, this dataset was comprehensive, as it included all sold call and put options with the aforementioned strike prices that allowed for the calculation of Implied Volatility, Delta $\Delta$, and Vega $\kappa$ over the 45-day period prior to expiration. All of the sources for the data remained the same as in Section 2.

**Feature Engineering**

As was done before, missing spot prices (NaN) were filled with the arithmetic mean of the bid and ask prices. The Implied Volatilities $\sigma_t$, Delta $\Delta_t$, and $\kappa_t$ for each (call or put) option were calculated at each timestamp $t$, $t \in \left(0, \frac{45}{365}\right)$, in R with functions `GBSVolatility()` and `calculate_greeks()`.

## 3.1. Delta Hedging a Portfolio of Options

Following the idea of reaching a delta-neutral state, the delta hedging strategy was conducted by additionally taking a position in $x_t$ units of the underlying asset. Denote $\Pi$ the aggregate portfolio consisted of the original call ($C_{\mathrm{Ca},t}$) and put ($C_{\mathrm{P},t}$) options and the underlying asset. The delta neutrality was expressed by the following equation:

$$\begin{aligned}
&\Delta(\Pi) = x_t \Delta(S_t) + \Delta(C_{\mathrm{Ca},t}) + \Delta(C_{\mathrm{P},t}) = 0 \\
&\Leftrightarrow x_t = -\left[\Delta(C_{\mathrm{Ca},t}) + \Delta(C_{\mathrm{P},t})\right]
\end{aligned}$$

Therefore, in order to reach the delta-neutral state, we shorted $\Delta(C_{\mathrm{Ca},t})+\Delta(C_{\mathrm{P},t})$ (or longed, if $\Delta(C_{\mathrm{Ca},t}) + \Delta(C_{\mathrm{P},t}) > 0$) units of the underlying assets. The MSE and the effectiveness of the strategy were also studied in the same way as before.

An initial test was conducted on a call option expiring on 19-05-2023, together with its paired put option. In this section, it was also assumed that there were no market frictions, such as transaction fees. The results are illustrated in Table 3.1 and Figure 2.1.

| Frequency (re-hedge(s)/calendar day) | MSE ($^2$) | Effectiveness (%) |
|---|---|---|
| 1/1 | 173.52 | 80.9 |
| 1/2 | 172.71 | 81.0 |
| 1/3 | 171.01 | 81.1 |
| 1/4 | 172.41 | 81.0 |
| 1/5 | 171.12 | 81.1 |
| 1/6 | 169.55 | 81.3 |
| 1/7 | 173.20 | 80.9 |
| 1/8 | 171.17 | 81.1 |
| 1/9 | 170.80 | 81.2 |
| 1/10 | 173.08 | 80.9 |

Table 3.1: MSE and hedging effectiveness for various frequencies over the 45-day period: Delta hedging of a strangle position.

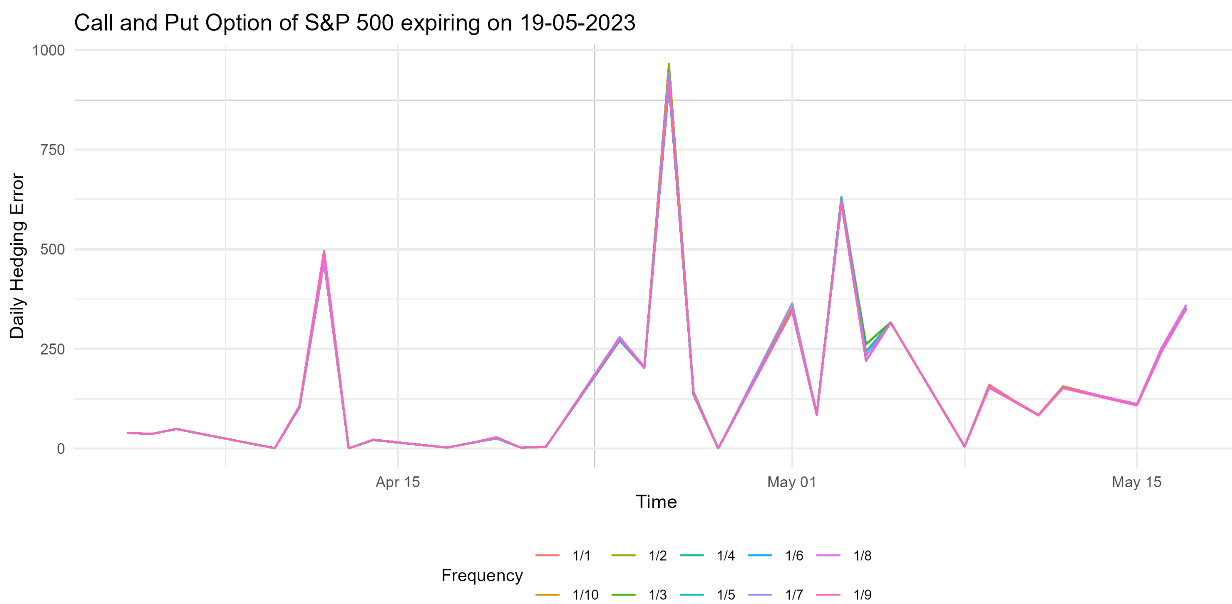


Figure 3.1: Daily hedging error for various re-hedging frequencies: Delta hedging of a strangle position.

As one may notice in Table 3.1 and Figure 3.1, the effectiveness and the error of one frequency closely matched the others. This suggests that effectiveness does not vary as much as the frequency. In the same manner, to get better statistical estimates, this experiment was repeated with ten other aforementioned pairs of put and call options.

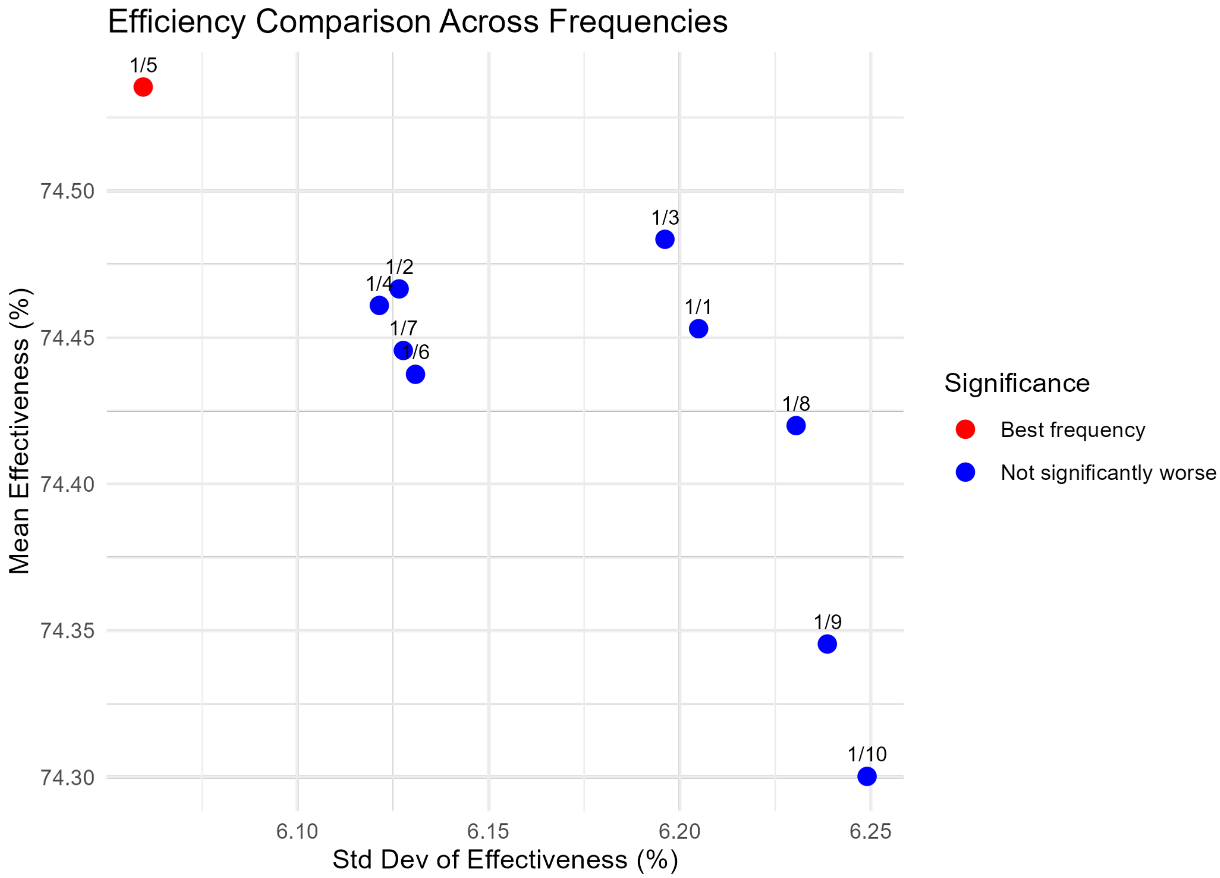


Figure 3.2: Means and standard deviations of the effectiveness for various frequencies over the 45-day period: Delta hedging of a strangle position.

The results in Figure 3.2 agree with earlier results introduced in Figure 3.1 that the performance of one frequency is rather analogous to the others. Particularly, the frequencies 1/5 provided the best performance, with a mean effectiveness of 74.53% and a standard deviation of 6.06%. In the same way as it was performed previously, a one-tailed Welch t-test was conducted for each pair of the best frequency and another frequency $j$ for $j \in \{1/1, 1/2, \dots, 1/4, 1/6, \dots, 1/10\}$. According to the test, there were no significantly worse frequencies.

## 3.2. Delta-Vega Hedging a Portfolio of Options

In accordance with the principle of delta and vega neutrality, the delta-vega hedging strategy was conducted by additionally taking a position in $x_t$ units of the underlying asset and $y_t$ units of a replicating option. In contrast to Section 2.2, the replicating option was chosen to be the option having a strike price, either \$4450 for the put option or \$4525 for the call option, closest to the price of the underlying asset. This choice was motivated by the fact that options that are either relatively deep ITM or ATM are not sensitive to price and implied volatility changes. Denote $\Pi$ the aggregate portfolio consisted of the original call ($C^O_{\mathrm{Ca},t}$) and put ($C^O_{\mathrm{P},t}$) options and the underlying asset and $C^R_t$. The delta neutrality was expressed by the following equation:

$$\begin{cases} \Delta(\Pi) & = 0 = -x_t\Delta(S_t) - y_t\Delta(C^R_t) + \Delta(C^O_{\mathrm{Ca},t}) + \Delta(C^O_{\mathrm{P},t}) \\ \kappa(\Pi) & = 0 = -x_t\kappa(S_t) - y_t\kappa(C^R_t) + \kappa(C^O_{\mathrm{Ca},t}) + \kappa(C^O_{\mathrm{P},t}) \end{cases}$$

$$\Leftrightarrow \begin{cases} x_t = -y_t\Delta(C^R_t) + \Delta(C^O_{\mathrm{Ca},t}) + \Delta(C^O_{\mathrm{P},t}) \\ y_t = \dfrac{\kappa(C^O_{\mathrm{Ca},t}) + \kappa(C^O_{\mathrm{P},t})}{\kappa(C^R_t)} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_t = \Delta(C^O_{\mathrm{Ca},t}) + \Delta(C^O_{\mathrm{P},t}) - \dfrac{\kappa(C^O_{\mathrm{Ca},t}) + \kappa(C^O_{\mathrm{P},t})}{\kappa(C^R_t)}\Delta(C^R_t) \\ y_t = \dfrac{\kappa(C^O_{\mathrm{Ca},t}) + \kappa(C^O_{\mathrm{P},t})}{\kappa(C^R_t)} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_t = \dfrac{\partial C^O_{\mathrm{Ca},t}}{\partial S} + \dfrac{\partial C^O_{\mathrm{P},t}}{\partial S} - \dfrac{\partial C^O_{\mathrm{Ca},t}/\partial\sigma + \partial C^O_{\mathrm{P},t}/\partial\sigma}{\partial C^R_t/\partial\sigma}\dfrac{\partial C^R_t}{\partial S} \\ y_t = \dfrac{\partial C^O_{\mathrm{Ca},t}/\partial\sigma + \partial C^O_{\mathrm{P},t}/\partial\sigma}{\partial C^R_t/\partial\sigma} \end{cases}$$

In the similar way, the same call option expiring on 19-05-2023, together with its paired put option, were firstly tested, coupled with a put option expiring on 16-06-2023 as the replicating option. Likewise, in this section, it was assumed that there were no market frictions, such as transaction fees. The results are demonstrated in Table 3.2 and Figure 3.3.

| **Frequency (re-hedge(s)/calendar day)** | **MSE** | **Effectiveness (%)** |
|---|---|---|
| 1/1 | 53.94 | 94.1 |
| 1/2 | 53.60 | 94.1 |
| 1/3 | 51.36 | 94.3 |
| 1/4 | 53.58 | 94.1 |
| 1/5 | 53.90 | 94.1 |
| 1/6 | 51.53 | 94.3 |
| 1/7 | 53.32 | 94.1 |
| 1/8 | 51.25 | 94.3 |
| 1/9 | 50.13 | 94.5 |
| 1/10 | 48.84 | 94.6 |

Table 3.2: MSE and hedging effectiveness for various frequencies over the 45-day period: Delta-vega hedging of a strangle position.

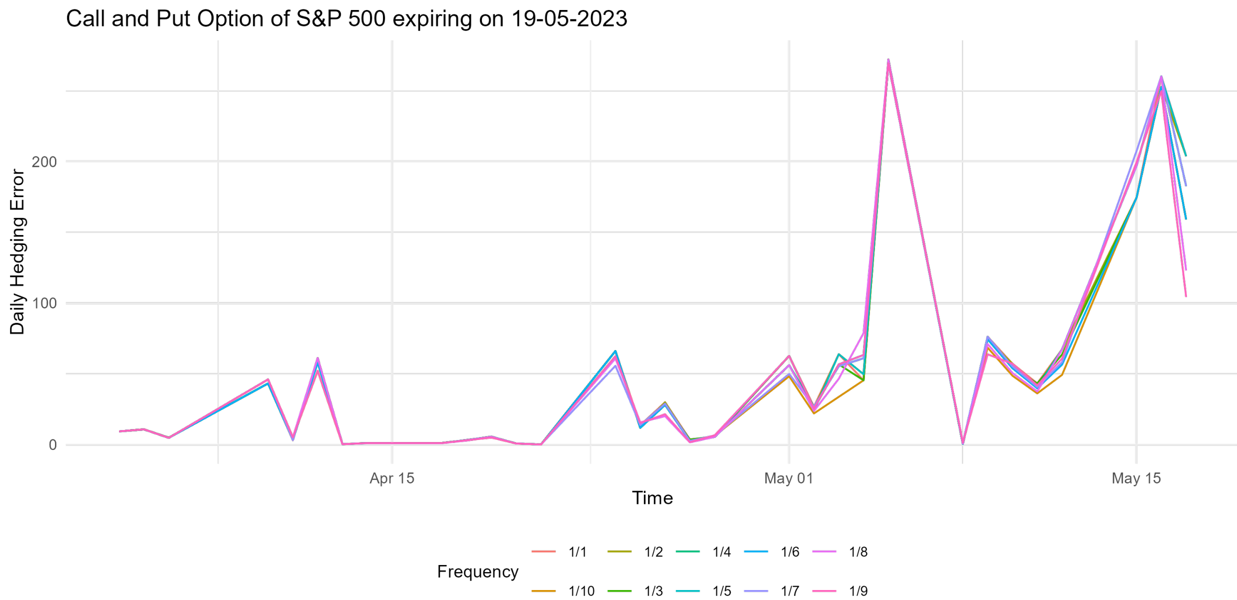


Figure 3.3: Daily hedging error for various re-hedging frequencies: Delta-vega hedging of a strangle position.

Table 3.2 and Figure 3.3 show that the performances of the frequencies are relatively identical, consistent with earlier findings (Table 3.1 and Figure 3.1). It is important to highlight that the spikes in Figure 3.3 were far less extreme than those in Figure 3.1. In the same fashion, this experiment was also repeated with 7 other pairs of put and call options of the following expiration dates (and replication option's expiration dates): 21-05-2021 (2021-06-18), 18-06-2021 (16-07-2021), 16-12-2022 (20-01-2023), 20-01-2023 (17-02-2023), 17-02-2023 (17-03-2023), 17-03-2023 (21-04-2023), and 21-04-2023 (16-06-2023).

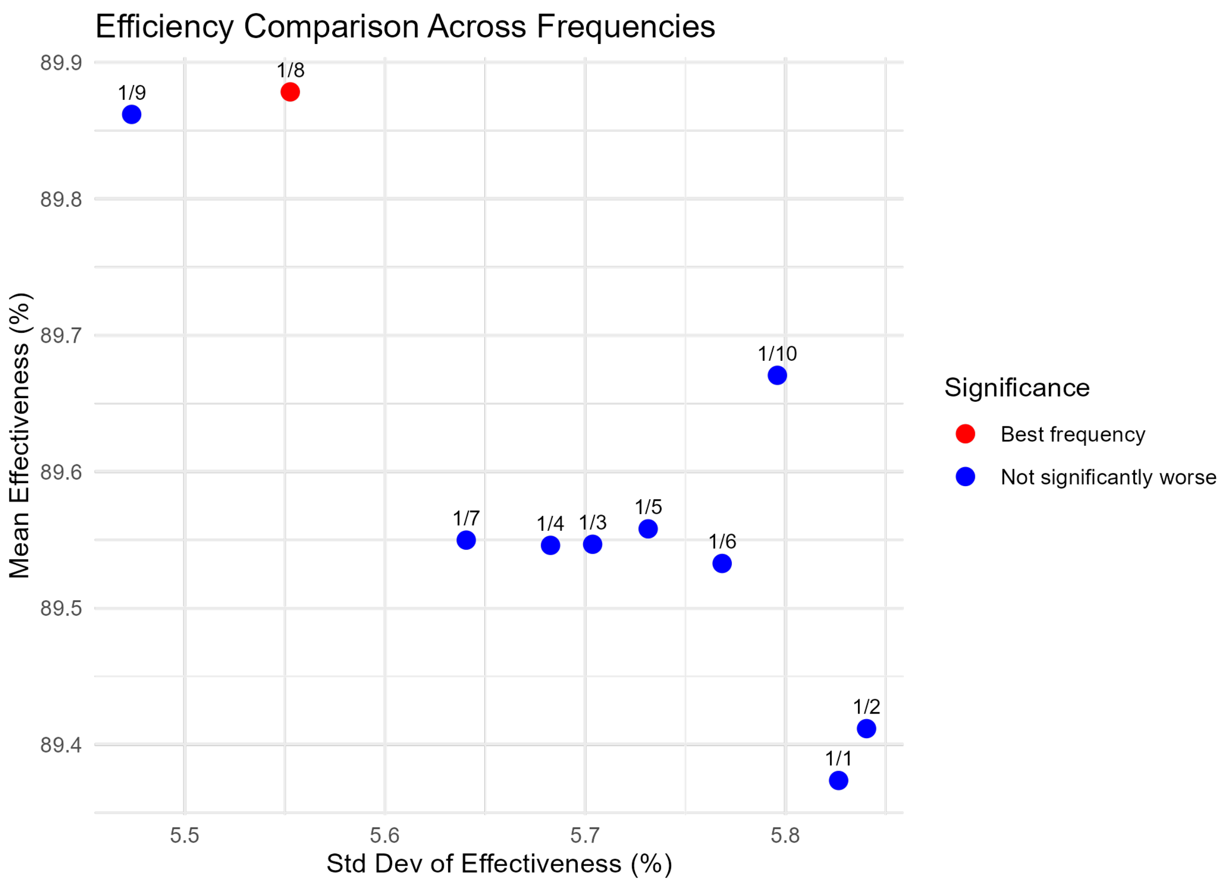


Figure 3.4: Means and standard deviations of the effectiveness for various frequencies over the 45-day period: Delta-vega hedging of a strangle position.

Similarly, as depicted in Figure 3.4, the results agree with earlier results introduced in Figure 3.1 that the performance of one frequency is rather analogous to the others. Particularly, the frequencies 1/8 provided the best performance, with a mean effectiveness of 89.88% and a standard deviation of 5.55%. However, the Figure also Unveils an unexpected finding: lower frequencies seem to perform slightly better than their higher counterparts. Moreover, in the same way as it was performed previously, a one-tailed Welch t-test was conducted for each pair of the best frequency and another frequency $j$ for $j \in 1/1, \ldots, 1/7, 1/9, 1/10$ . According to the test, there were no significantly worse frequencies.

## 3.3. Hedging Accuracy Compared

By comparing Figure 3.1 and Figure 3.3, the previous section already pointed out that there were considerably fewer extreme spikes in delta-vega hedging than delta hedging. Similarly, by comparing Table 3.1 and Table 3.2, one can easily notice that delta-vega hedging consistently maintained a better effectiveness level than delta hedging did. Furthermore, the following Figure 3.5 visualizes a line plot with error bars, where the lines represent the average effectiveness of the two hedging strategies and the error bars illustrate the standard deviation of the effectiveness at each frequency.

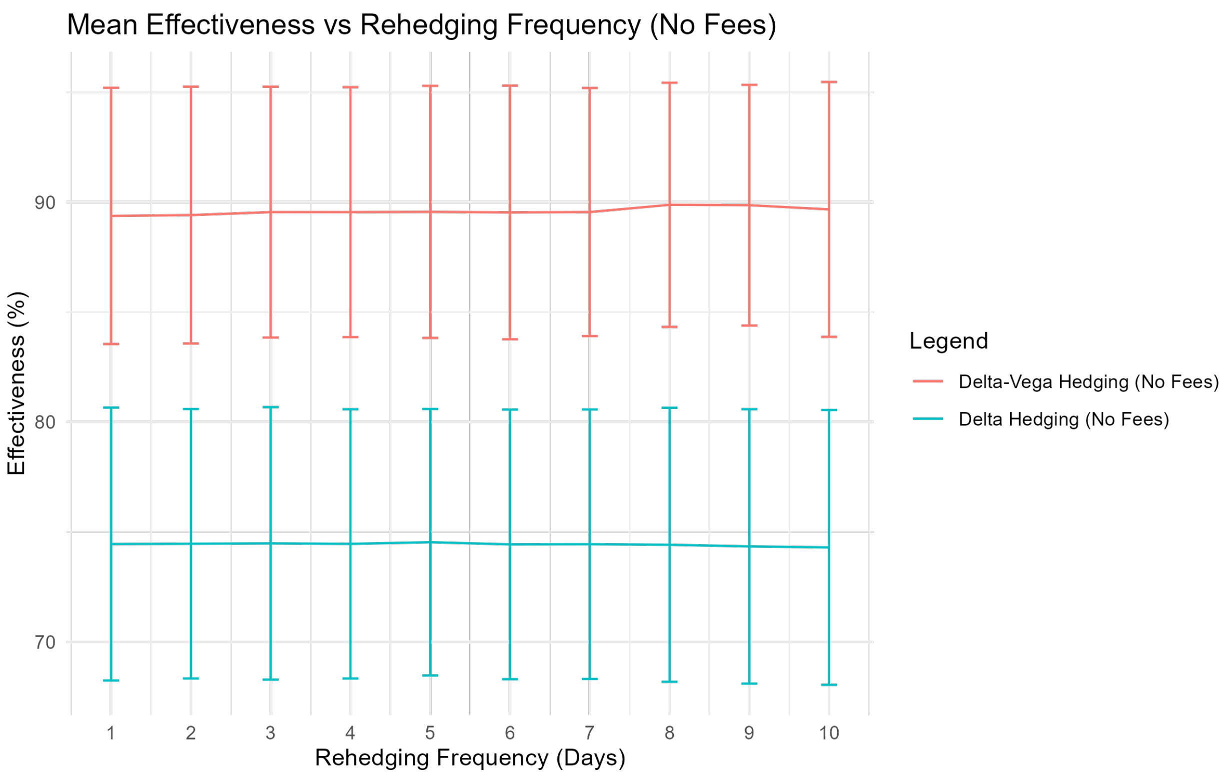


Figure 3.5: Mean effectiveness by re-hedging frequencies for two hedging strategies applied to a strangle position.

As is obvious in Figure 3.5, the delta-vega hedging was apparently better than delta hedging when applied to the strangle. This is confirmed with ten one-tailed Welch t-tests, one for each frequency. The results of the test are presented below in Table 3.3, with a significance level $\alpha$ of 0.05.

| Frequency (re-hedge(s)/calendar day) | Result |
|---|---|
| 1 | Null Hypothesis Rejected |
| 1/2 | Null Hypothesis Rejected |
| 1/3 | Null Hypothesis Rejected |
| 1/4 | Null Hypothesis Rejected |
| 1/5 | Null Hypothesis Rejected |
| 1/6 | Null Hypothesis Rejected |
| 1/7 | Null Hypothesis Rejected |
| 1/8 | Null Hypothesis Rejected |
| 1/9 | Null Hypothesis Rejected |
| 1/10 | Null Hypothesis Rejected |

Table 3.3: Welch's t-test results for various frequencies with significance level $\alpha = 0.05$.

## 3.4. Effect of Adding Transaction Costs

Similar to Section 2.4, this section studies how fees degrade hedging effectiveness, with the linear notional-based fee as the studied kind of fee. Figure 3.6 and Figure 3.7 below illustrate how the effectiveness varies with respect to the fee rate for various frequencies for delta hedging and delta-vega hedging when applied to a strangle position, respectively.

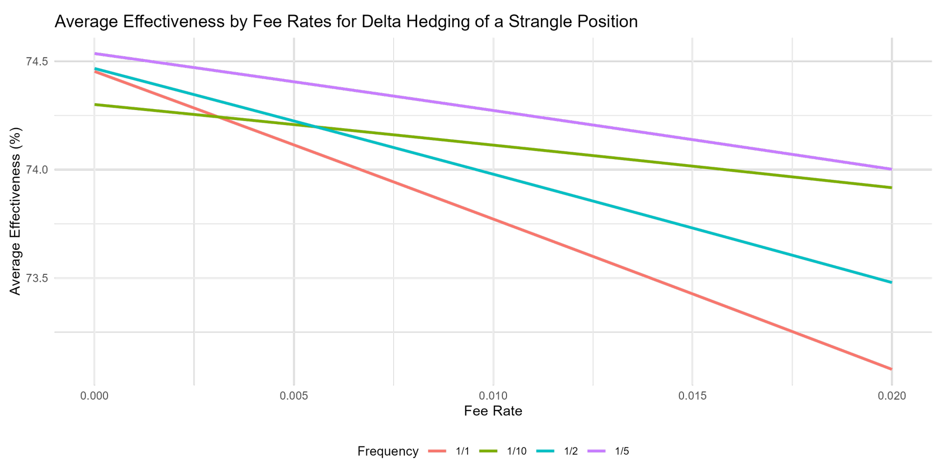


Figure 3.6: Average Effectiveness by Fee Rates for Delta Hedging of a Strangle Position.

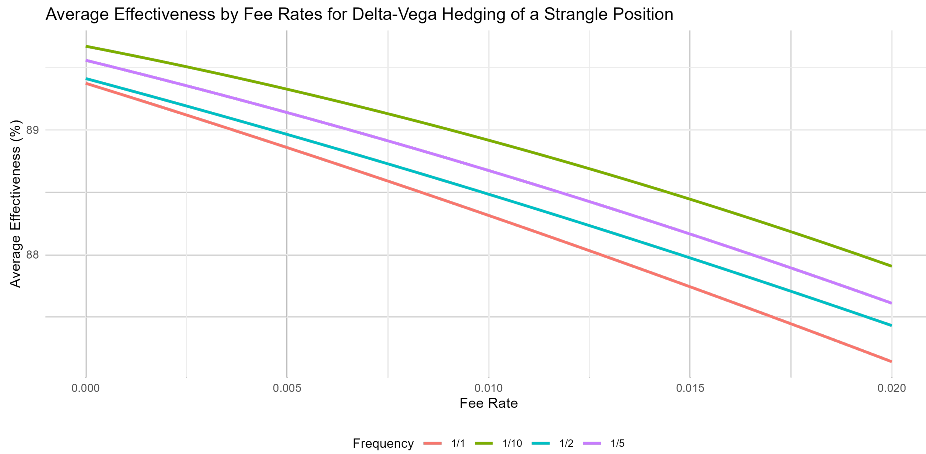


Figure 3.7: Average Effectiveness by Fee Rates for Delta-Vega Hedging of a Strangle Position.

The data revealed a trend similar to that unraveled by Section 2.4: effectiveness decreases as the fee rate rises, with the decline exhibiting a non-linear pattern. At lower fee rates (0 to 0.01), the decay is gentler. Beyond this point, the decay becomes steeper, suggesting a higher sensitivity of effectiveness to fee rates as costs increase. Both of the strategies are relatively insensitive to transaction fee, as their effectiveness does not deteriorate much when the fee rate increases.

# 4. Conclusion

The analysis conducted in this report demonstrates the effectiveness of hedging strategies, particularly delta and delta-vega hedging, in managing the risks associated with S&P 500 options. Through thorough experimentation and statistical evaluation, it has been shown that delta-vega hedging consistently outperforms delta hedging in both single option and portfolio contexts. The insights derived underscore the practical applications of these strategies and provide a solid foundation for further exploration in financial risk management.

## 4.1. Discussion

The findings of this study highlight several critical points about the application and limitations of hedging strategies. Delta-vega hedging exhibited higher mean effectiveness and lower sensitivity to market fluctuations compared to delta hedging. This result aligns with theoretical expectations, as delta-vega hedging accounts for both price and volatility changes, whereas delta hedging addresses only price risk. This advantage was evident in both single option scenarios and portfolio contexts, particularly when handling complex instruments like strangles.

Incorporating transaction costs into the analysis revealed an important trade-off between hedging frequency and cost-efficiency. While high-frequency adjustments offered better error control in frictionless markets, their effectiveness diminished significantly in the presence of transaction fees. In contrast, delta-vega hedging demonstrated greater resilience to transaction costs, maintaining reasonable levels of effectiveness even at higher fee rates. This highlights its robustness and potential suitability for real-world applications.

The findings also suggest that portfolio hedging is less sensitive to frequency variation, with delta-vega hedging consistently outperforming delta hedging in terms of effectiveness. Its ability to mitigate extreme market fluctuations further supports its value for managing risks in more complex financial scenarios. However, limitations remain, including the reliance on the Black-Scholes model, which assumes constant volatility and interest rates—assumptions that may not reflect real-world dynamics. Moreover, the dataset used for this analysis was limited to historical S&P 500 data, which, while robust, does not fully represent the diversity of instruments or conditions present in global markets.

## 4.2. Further Improvement

Building on the insights from this study, future research could incorporate more advanced modeling techniques to address the limitations of the Black-Scholes framework. Stochastic volatility models, such as Heston or SABR, could provide more accurate reflections of market dynamics and enhance the precision of hedging strategies. Additionally, integrating machine learning approaches for parameter estimation, including delta, vega, and implied volatility, could further improve the effectiveness of these strategies, particularly in highly volatile or uncertain markets.

Another promising avenue for exploration involves extending the analysis to additional hedging strategies. For example, delta-gamma hedging or multi-greek approaches could address higher-order risks, providing a more comprehensive risk management framework. Expanding the scope of experimentation to include diverse market scenarios, such as those characterized by extreme volatility or low liquidity, would yield deeper insights into the robustness and adaptability of these strategies.

Furthermore, efforts to optimize cost-effective hedging techniques could enhance practical applicability. This might include adaptive hedging frequencies that respond dynamically to prevailing market conditions, thereby reducing the impact of transaction costs without compromising hedging performance. By broadening the scope of analysis and integrating innovative methodologies, future research can advance the field of financial risk management, offering more sophisticated and reliable tools for navigating the complexities of modern financial markets.

In conclusion, while delta and delta-vega hedging are foundational strategies for mitigating risk in options trading, their efficacy can be significantly increased through continuous refinement. The insights provided by this study lay the foundation for future exploration and innovation, ensuring that these strategies remain relevant and effective in increasingly complex and dynamic financial environments.

# References

Corporate Finance Institute. (2024a). Delta Hedging - Overview, How It Works, Pros and Cons.

Corporate Finance Institute. (2024b). Vega Neutral.

U.S. Department of the Treasury. (2024). *Daily Treasury Yield Curve Rates*.

Yahoo Finance. (2024). *S&P 500 Historical Data*.