Introduction to Machine Learning

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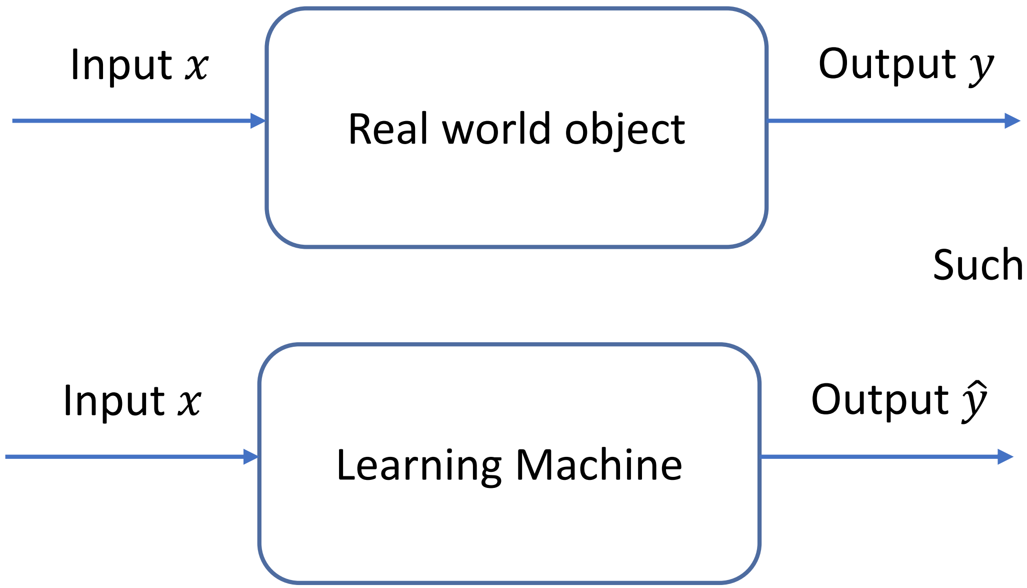
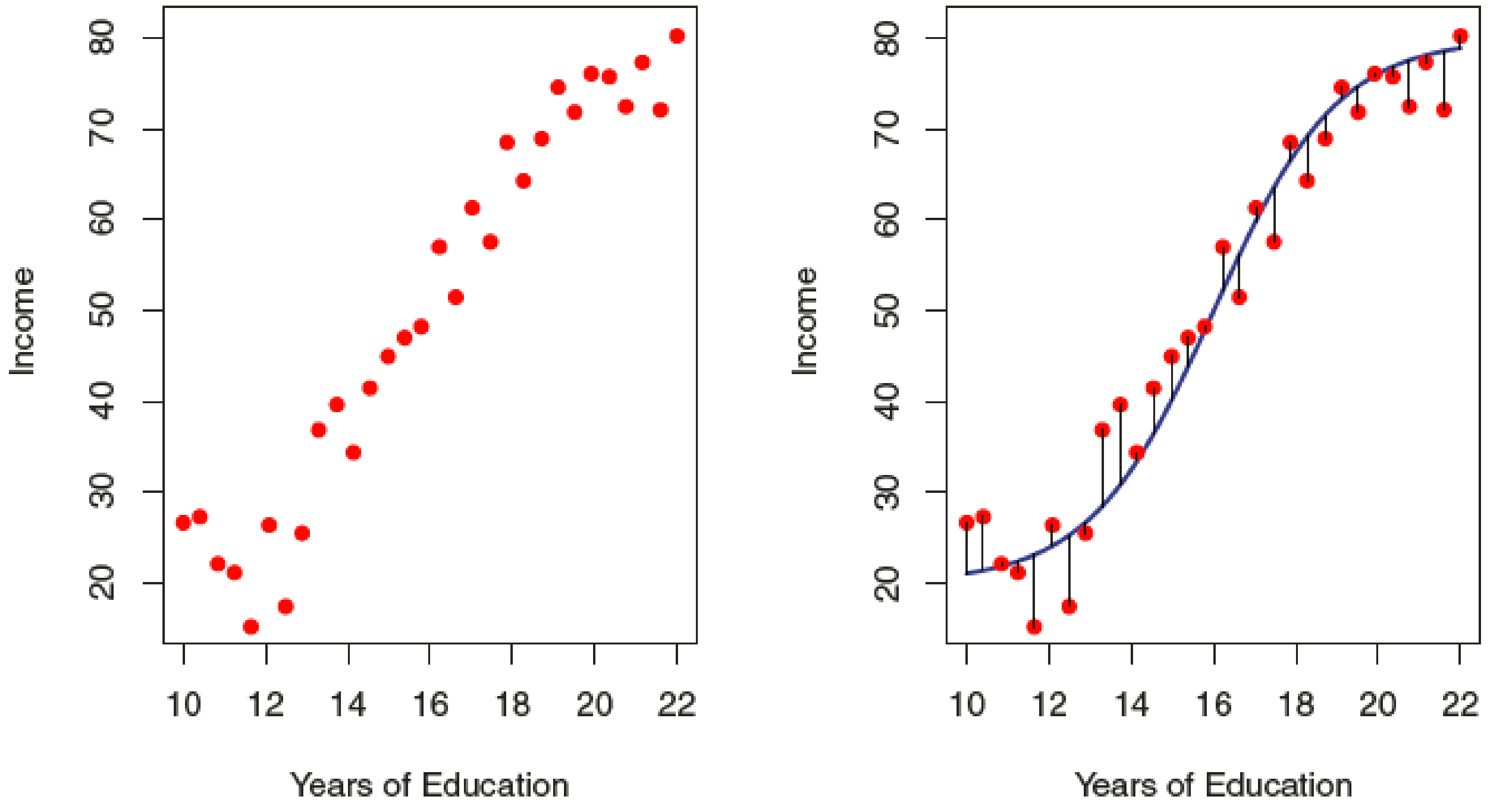
What is Machine Learning?



Such that $\|\hat{y} - y\|$ is very small

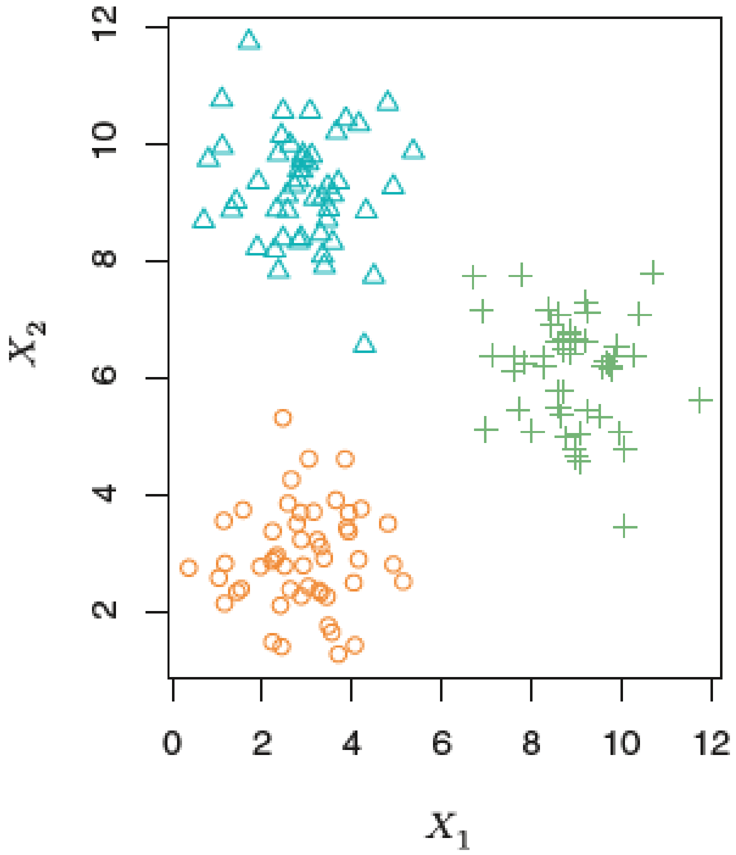
This is called supervised learning
The Machine learns the internal (invisible) states of the real world object through the observation of the behaviors of the object

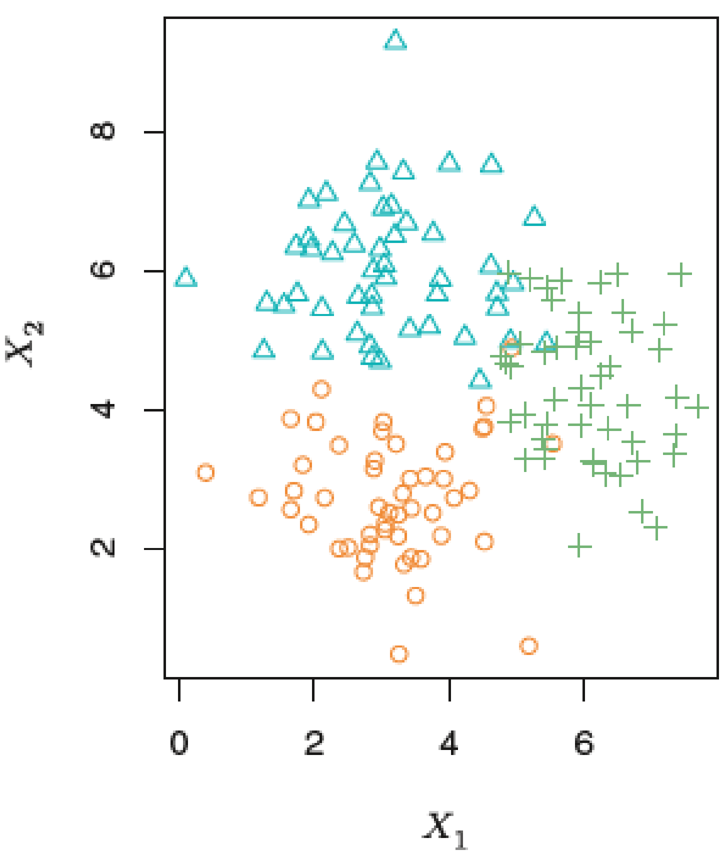
What is Machine Learning?



Why do we need to find the function?

What is Machine Learning?



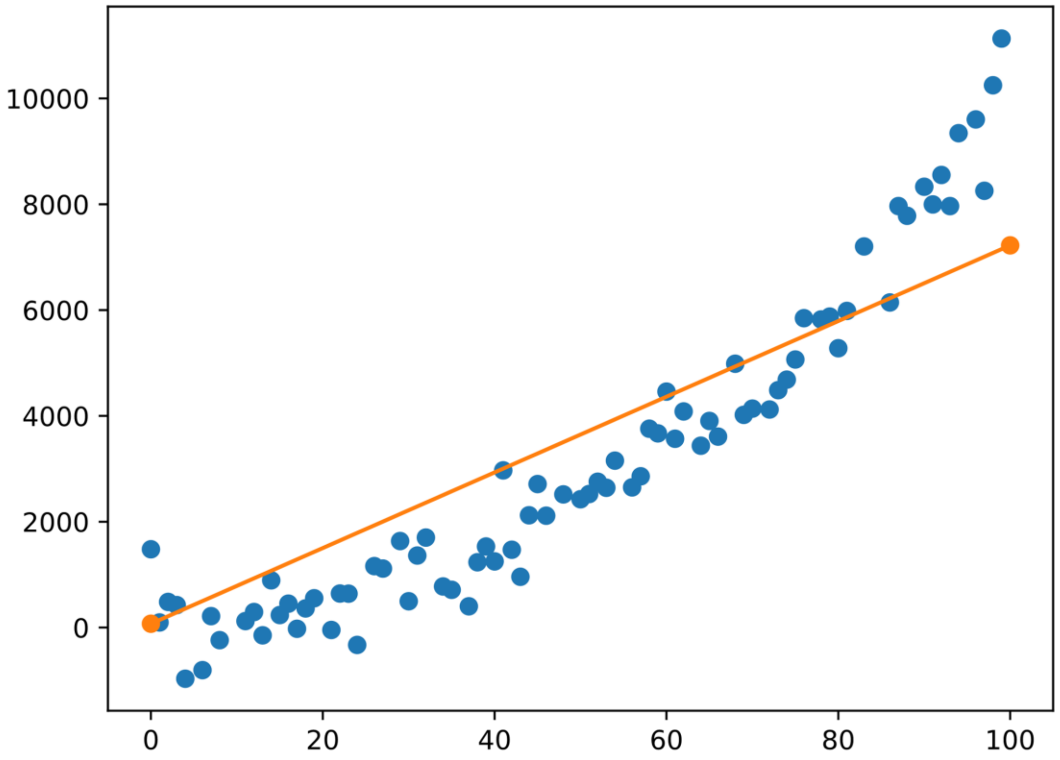


No previous outcome is given. The machine needs to learn the structure by themselves: Unsupervised learning

Machine Learning Phases

- It is composed of two phases
  - Training
  - Validation
- Training
  - Inputs are fed into the learning machine
  - The parameters of the learning machine are tune so that the predicted outputs are close to the target outputs
  - Similar to homework practicing for students
- Validation
  - Some of the data samples are retained and not used during training
  - Use the retained samples to test if the learned machine can predict close to the target values of the retained samples when fed with the retained input samples
  - Can be thought as exams for students

Linear Regression



Linear Regression

- Suppose that we have the following linear hypothesis function:

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

suppose that $x_0 = 1$, we can rewrite the function as:

$$h_\theta(x) = \sum_{i=0}^{n} \theta_i x_i = \theta^T x$$

with $x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$

Loss Function

- Suppose that we have m samples x^j with $j = \{1 \rightarrow m\}$
- Now we compute the predicted $h_\theta$value for each x^j
- We want this value $\sum_{j=1}^{m}\left(h_\theta\left(x^j\right) - y^j\right)^2$ to be minimized
  - What can we change so the value becomes minimized?
- Because x^j and y^j are collected data, so they are given fact. As a result, they cannot be changed
  - We can only change θ

Loss Function

- We define the following loss function as a function of θ:

$$L(\theta) = \sum_{j=1}^{m} \left(h_\theta(x^j) - y^j\right)^2$$

- We will find θ^* such that $L(\theta^*)$ is smallest:

$$\theta^* = \arg\min_\theta L(\theta)$$

- How do we find such θ^*?

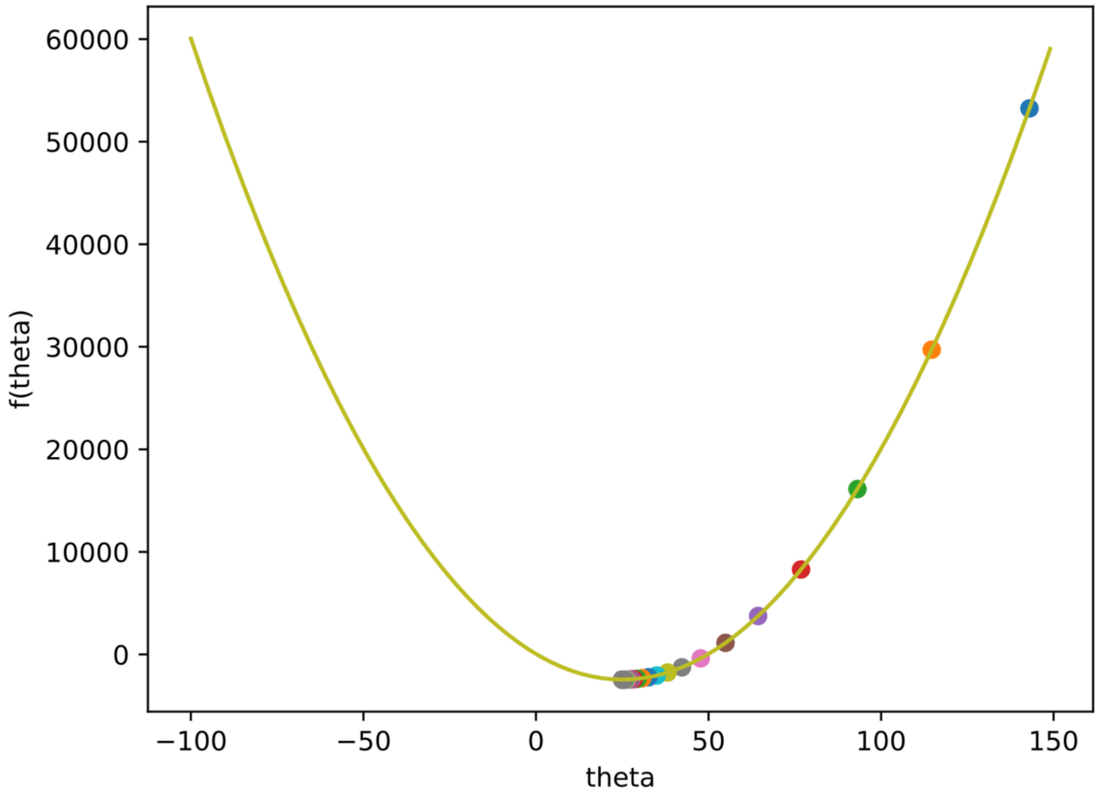
Gradient Descent

- Let us take an example: Find θ such that $\mathrm{f}(\theta) = 4\theta^2 - 200\theta + 50$ is smallest:

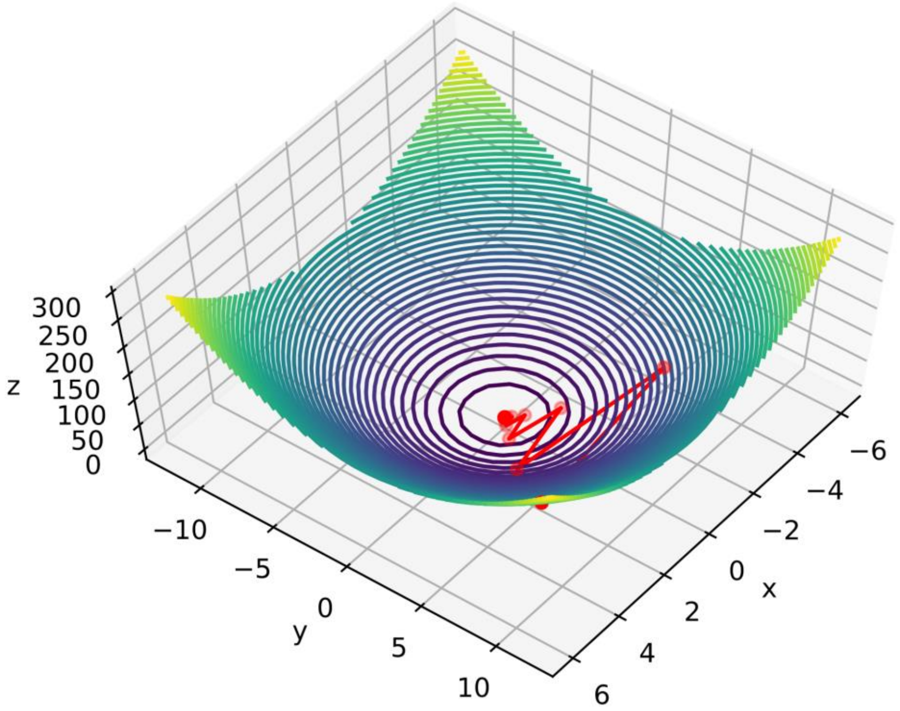
$$\nabla \mathrm{f}(\theta) = 8\theta - 200$$
$$\theta = \theta \; - \; \alpha * \nabla f(\theta)$$

- α is called the learning rate
- When do we stop?

Gradient Descent Coding



Multivariate Gradient Descent



Gradient Descent for Linear Regression Loss Function

$$\nabla L(\theta) = \frac{\partial \sum_{j=1}^{m} \left(h_\theta\left(x^j\right) - y^j\right)^2}{\partial \theta}$$

- Because this is a function of θ, we have:

$$\nabla L(\theta) = \sum_{j=1}^{m} \frac{\partial \left(h_\theta\left(x^j\right) - y^j\right)^2}{\partial \theta}$$

- According to the matrix derivation we have:

$$\frac{\partial \left(h_\theta\left(x^j\right) - y^j\right)^2}{\partial \theta} = \begin{bmatrix} \frac{\partial \left(h_\theta\left(x^j\right) - y^j\right)^2}{\partial \theta_0} \\ \frac{\partial \left(h_\theta\left(x^j\right) - y^j\right)^2}{\partial \theta_1} \\ \vdots \\ \frac{\partial \left(h_\theta\left(x^j\right) - y^j\right)^2}{\partial \theta_n} \end{bmatrix}$$

Gradient Descent for Linear Regression Loss Function

$$\frac{\partial\left(h_\theta(x^j) - y^j\right)^2}{\partial\theta_i} = 2 * \left(h_\theta(x^j) - y^j\right) * \frac{\partial\left(h_\theta(x^j) - y^j\right)}{\partial\theta_i}$$

$$= 2 * \left(h_\theta(x^j) - y^j\right) * \frac{\partial \sum_{k=0}^{n} \theta_k x_k^j}{\partial\theta_i}$$

$$= 2 * \left(h_\theta(x^j) - y^j\right) * x_i^j$$

- So we have:

$$\frac{\partial\left(h_\theta(x^j) - y^j\right)^2}{\partial\theta} = 2 * \begin{bmatrix} \left(h_\theta(x^j) - y^j\right) * x_0^j \\ \left(h_\theta(x^j) - y^j\right) * x_1^j \\ \vdots \\ \left(h_\theta(x^j) - y^j\right) * x_n^j \end{bmatrix} = 2 * \left(h_\theta(x^j) - y^j\right) * \begin{bmatrix} x_0^j \\ x_1^j \\ \vdots \\ x_n^j \end{bmatrix}$$

Gradient Descent for Linear Regression Loss Function

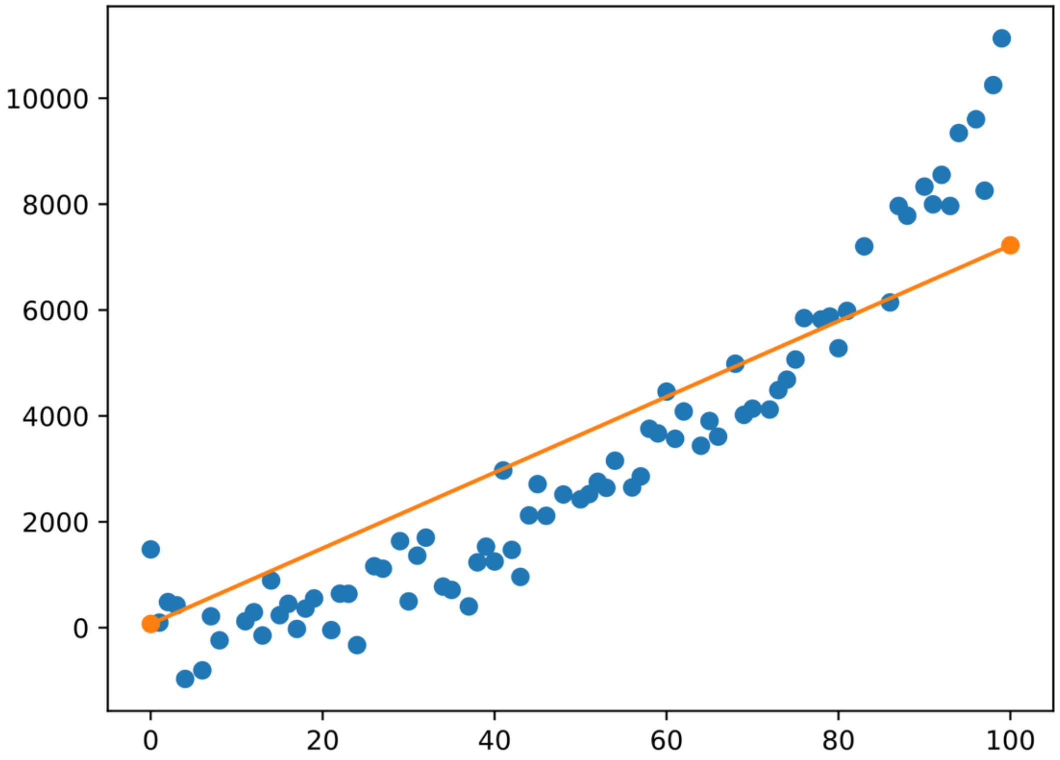
- Finally

$$\nabla L(\theta) = \frac{2}{m} * \sum_{j=1}^{m} \left(h_\theta\left(x^j\right) - y^j\right) * \begin{bmatrix} x_0^j \\ x_1^j \\ \vdots \\ x_n^j \end{bmatrix}$$

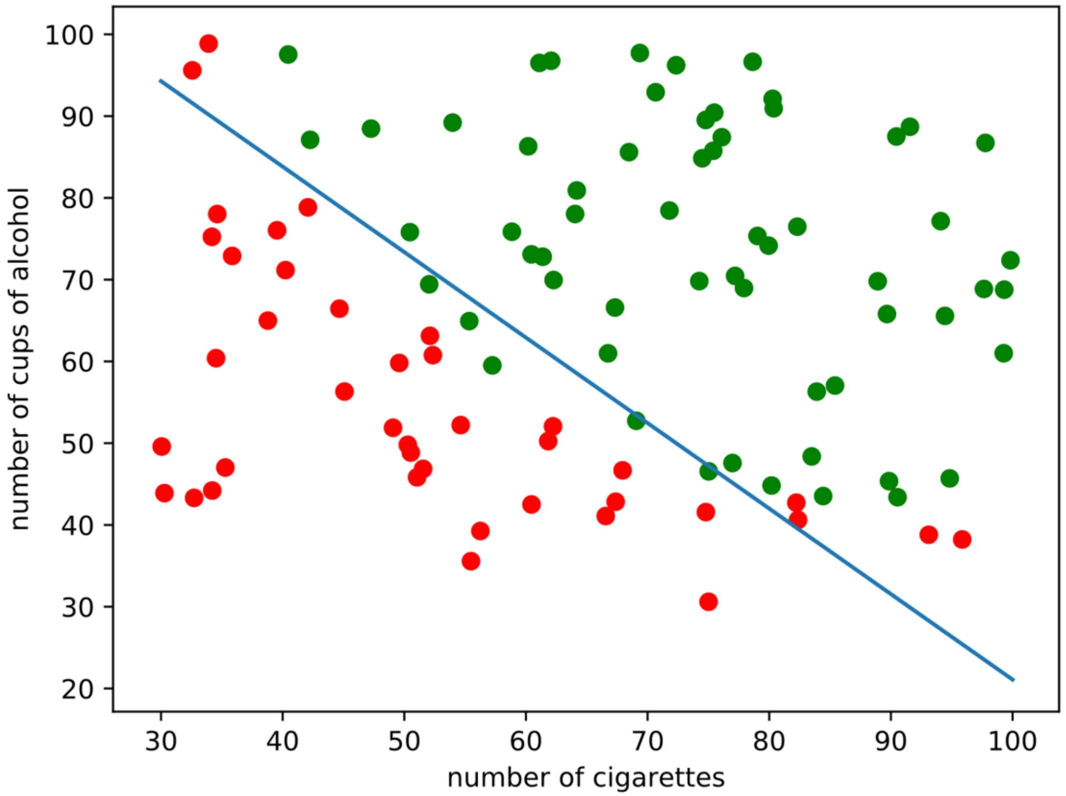
- We then use the same update method:

$$\theta = \theta - \alpha \nabla L(\theta)$$

Linear Regression Coding



Binary Classification



Now $y \in \{0,1\}$

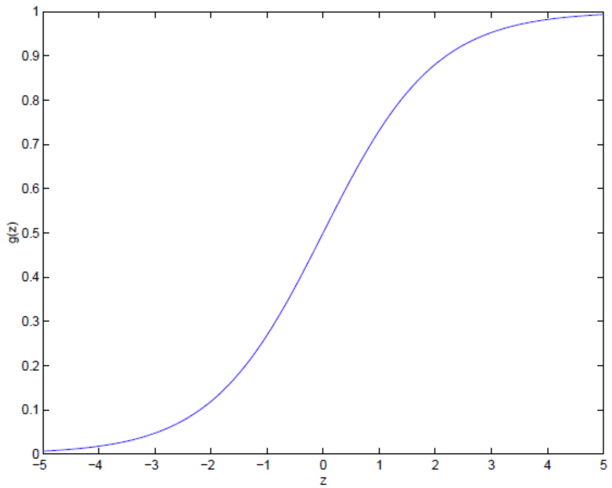
Logistic Regression

- Now $y \in \{0,1\}$, we need to modify hypothesis function $h_\theta(x)$ so that it takes values between 1 and 0
- We choose the following function

$$h_\theta(x) = g(\theta^T x) = \frac{1}{1+e^{-\theta^T x}}$$

where

$$g(z) = \frac{1}{1+e^{-z}}$$



Is called logistic function or sigmoid function

- $g(z)$ takes value from 0 to 1

Logistic Regression

- Because now $h_\theta(x)$ takes value from 0 to 1, we assume that it is the probability of y equal to 1

$$P(y=1|x;\theta) = h_\theta(x)$$
$$P(y=0|x;\theta) = 1 - h_\theta(x)$$

equivalently

$$p(y|x;\theta) = (h_\theta(x))^y(1-h_\theta(x))^{1-y}$$

- Suppose that we have m training examples $\{\vec{x}, \vec{y}\}$ generated independently, we then have the likelihood function:

$$\begin{aligned} L(\theta) &= p(\vec{y}|\vec{x};\theta) \\ &= \prod_{j=1}^{m} p(y^j|x^j;\theta) \\ &= \prod_{j=1}^{m} (h_\theta(x^j))^{y^j}(1-h_\theta(x^j))^{1-y^j} \end{aligned}$$

Logistic Regression

- To avoid roundoff error of multiplication, we take the log of the likelihood function:

$$l(\theta) = L(\theta)$$

$$= \sum_{j=1}^{m} y^j \log h_\theta\left(x^j\right) + (1 - y^j)\ \log(1 - h_\theta(x^j))$$

- How do we maximize the likelihood function? We use the same gradient descent method to find θ.

Gradient Descent of Logistic Regression

$$\frac{\partial l(\theta)}{\partial \theta_i} = \frac{\partial \sum_{j=1}^{m} y^j \log h_\theta\left(x^j\right) + (1 - y^j)\ \log(1 - h_\theta(x^j))}{\partial \theta_i}$$

$$= \sum_{j=1}^{m} \frac{\partial\left(y^j \log h_\theta(x^j) + (1-y^j)\ \log\left(1-h_\theta(x^j)\right)\right)}{\partial \theta_i}$$

$$= \sum_{j=1}^{m} y^j \frac{\partial(\log h_\theta(x^j)\,)}{\partial \theta_i} + (1 - y^j) \frac{\partial\left(\log\left(1-h_\theta(x^j)\right)\right)}{\partial \theta_i}$$

$$= \sum_{j=1}^{m} y^j \frac{1}{h_\theta(x^j)} \frac{\partial\left(h_\theta(x^j)\right)}{\partial \theta_i} + (1 - y^j) \frac{1}{1-h_\theta(x^j)} \frac{\partial\left(1-h_\theta(x^j)\right)}{\partial \theta_i}$$

$$= \sum_{j=1}^{m} \left(y^j \frac{1}{h_\theta(x^j)} - \ (1 - y^j) \frac{1}{1-h_\theta(x^j)}\right) \frac{\partial\left(h_\theta(x^j)\right)}{\partial \theta_i}$$

$$= \sum_{j=1}^{m} \left(y^j \frac{1}{h_\theta(x^j)} - \ (1 - y^j) \frac{1}{1-h_\theta(x^j)}\right) \frac{\partial\left(g(\theta x^j)\right)}{\partial \theta_i}$$

$$= \sum_{j=1}^{m} \left(y^j \frac{1}{h_\theta(x^j)} - \ (1 - y^j) \frac{1}{1-h_\theta(x^j)}\right) \frac{\partial\left(g(\theta x^j)\right)}{\partial (\theta x^j)} \frac{\partial(\theta x^j)}{\partial \theta_i}$$

$$= \sum_{j=1}^{m} \left(y^j \frac{1}{h_\theta(x^j)} - \ (1 - y^j) \frac{1}{1-h_\theta(x^j)}\right) \frac{\partial\left(g(\theta x^j)\right)}{\partial (\theta x^j)} x_i^j$$

Gradient Descent of Logistic Regression

- Note that

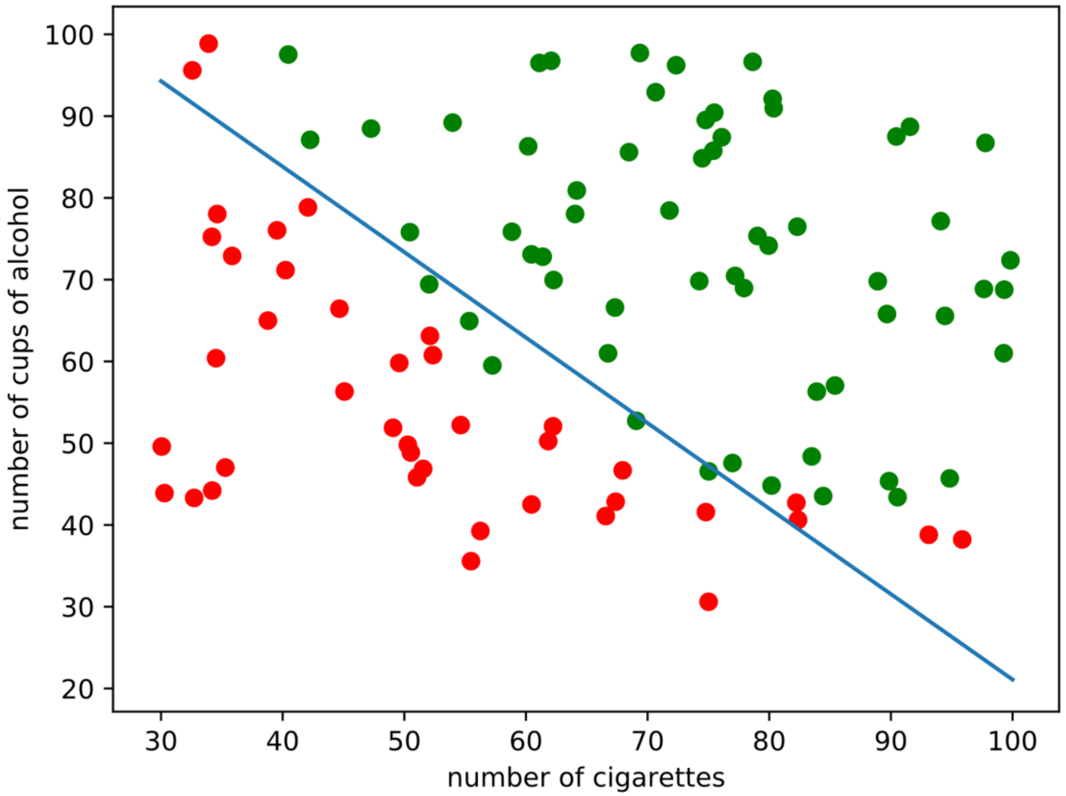
$$\frac{\partial g(z)}{\partial z} = \frac{1}{(1+e^{-z})^2} e^{-z}$$
$$= \frac{1}{(1+e^{-z})}\left(1 - \frac{1}{(1+e^{-z})}\right)$$
$$= g(z)(1-g(z))$$

- So we have

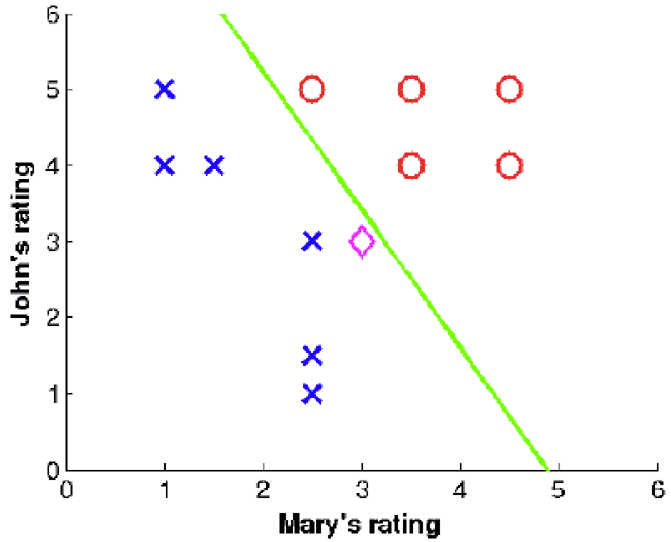
$$\frac{\partial l(\theta)}{\partial \theta_i} = \sum_{j=1}^{m}\left(y^j \frac{1}{h_\theta(x^j)} - \ (1-y^j)\frac{1}{1-h_\theta(x^j)}\right)\left(h_\theta\left(x^j\right)\right)(1-h_\theta x^j)x_i^j$$

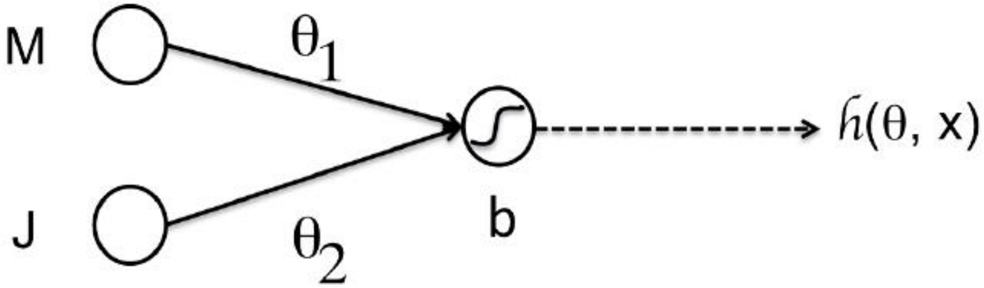
$$= \sum_{j=1}^{m}\left(y^j - h_\theta\left(x^j\right)\right)x_i^j$$

Logistic Regression Coding

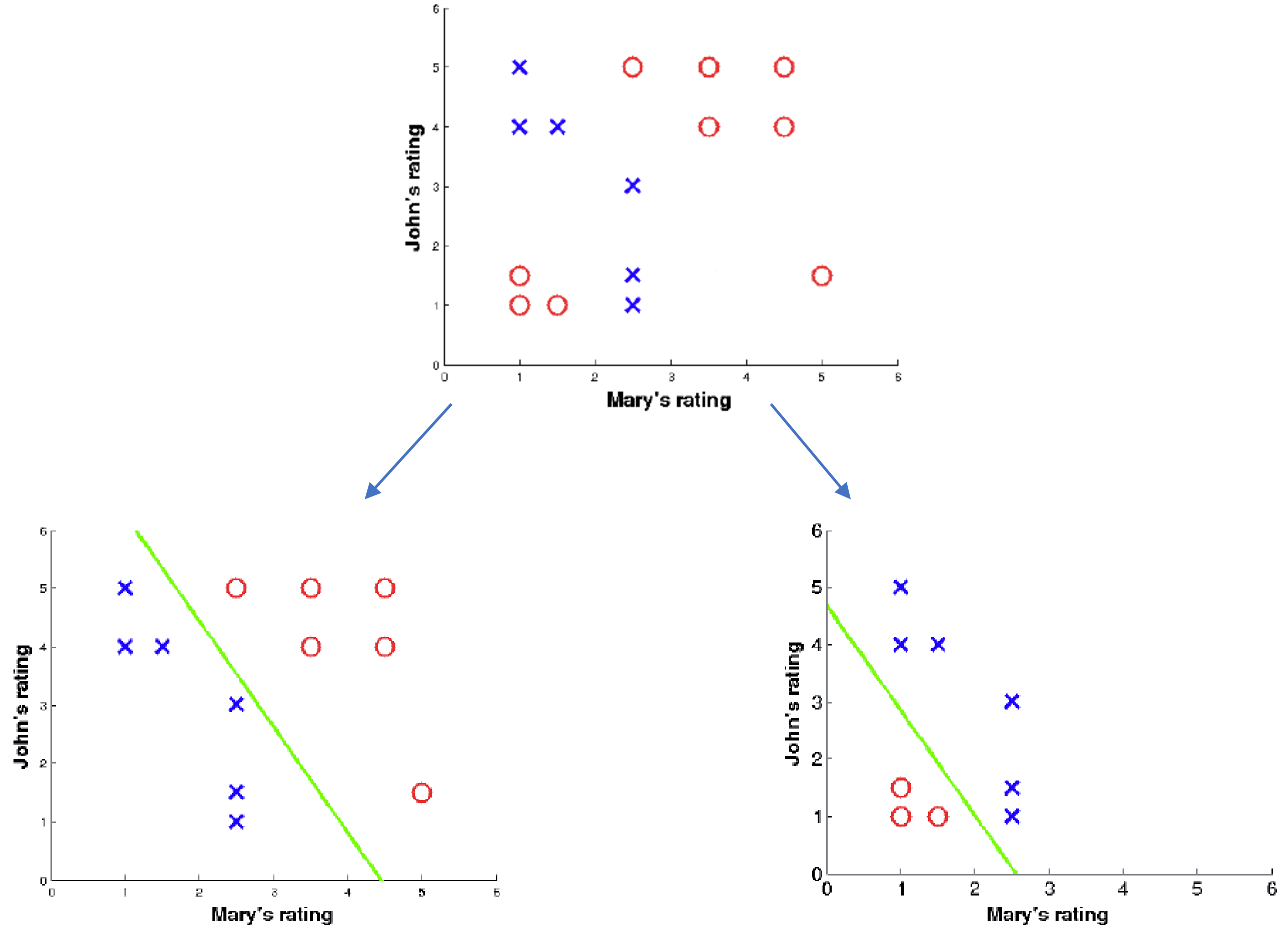


Logistic Regression

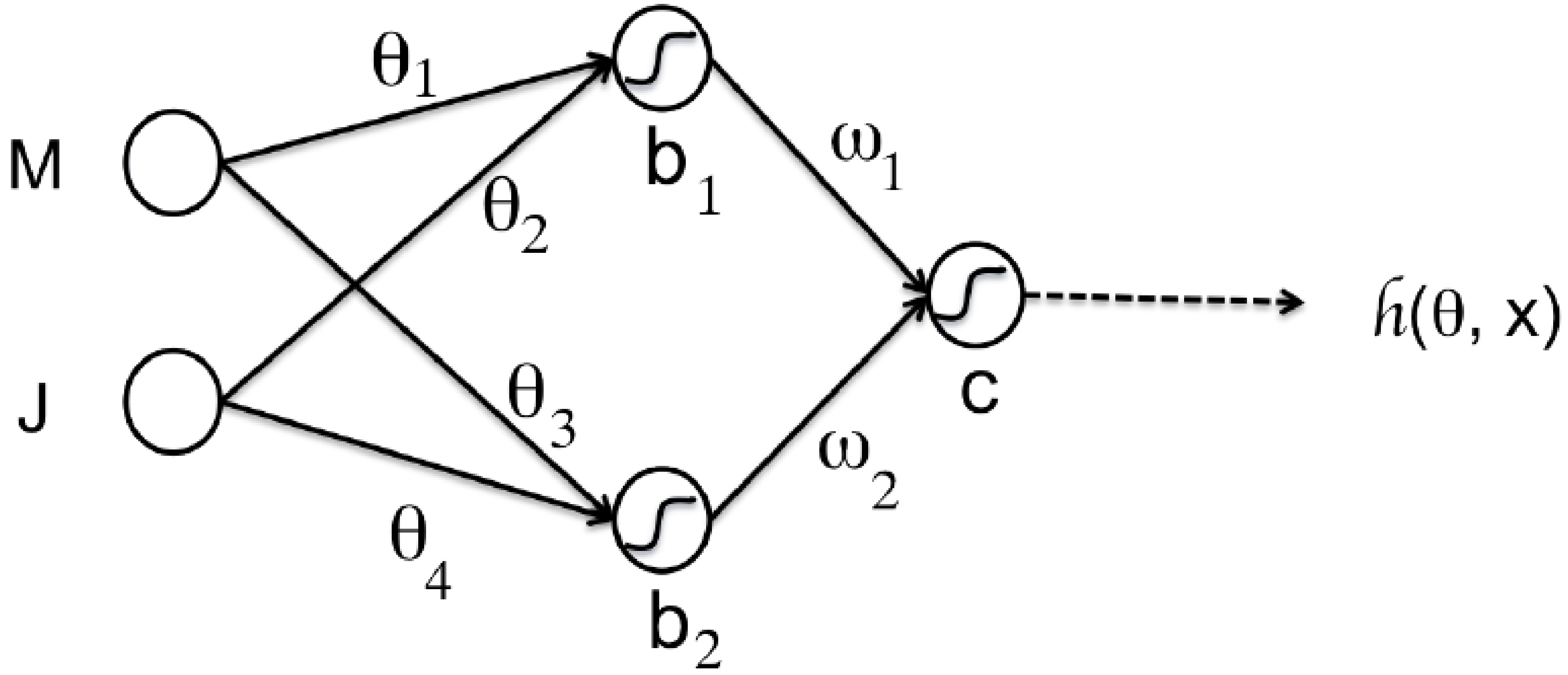




Limitation of Logistic Regression



Neural Networks



Next Class: Neural Networks Basics