

APPENDIX A
PSEUDO-LIPSCHITZ FUNCTION

Definition 1. For a $k > 1$, a function $f : \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}$ is said pseudo-Lipschitz of order k if there exists a constant $L > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq L(1 + \|\mathbf{x}\|^{k-1} + \|\mathbf{y}\|^{k-1})\|\mathbf{x} - \mathbf{y}\|$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n \times 1}$; the first order derivative of f is bounded by a polynomial of order $(k-1)$, i.e., polynomial smoothness.

APPENDIX B
PROOF OF PROPOSITION 1

This follows from $\sum_{m=1}^n \mathbb{E}[X_{n,m}^2; |X_{n,m}| > \epsilon] \leq \sum_{m=1}^n \mathbb{E}\left[\frac{X_{n,m}^{2+2\alpha}}{\epsilon^{2\alpha}}\right]$, where the right-hand side converges to zero because $\frac{n}{\epsilon^{2\alpha}} o(n^{-1}) \rightarrow 0$ as $n \rightarrow \infty$.

APPENDIX C
PROOF OF PROPOSITION 3

Recalling the following subspace decomposition $\mathbf{A} = \mathbf{P}_M^\perp \mathbf{A} \mathbf{P}_Q^\perp + \mathbf{A} \mathbf{P}_Q + \mathbf{P}_M \mathbf{A} - \mathbf{P}_M \mathbf{A} \mathbf{P}_Q$, the orthogonal projection of \mathbf{A} onto $\mathcal{P}_{\mathbf{X}, \mathbf{Y}}$ is given by $\mathbf{A}|_{\mathcal{P}_{\mathbf{X}, \mathbf{Y}}} = \mathbf{P}_M^\perp \mathbf{A} \mathbf{P}_Q^\perp + \mathbf{B}$. For any integrable function ψ , $\mathbb{E}[\psi(\mathbf{A}|_{\mathcal{P}_{\mathbf{X}, \mathbf{Y}}})] = \mathbb{E}[\psi(\mathbf{P}_M^\perp \mathbf{A} \mathbf{P}_Q^\perp + \mathbf{B})] = \mathbb{E}[\psi(\mathbf{P}_M^\perp \tilde{\mathbf{A}} \mathbf{P}_Q^\perp + \mathbf{B})] = \mathbb{E}[\psi(\tilde{\mathbf{A}}|_{\mathcal{P}_{\mathbf{X}, \mathbf{Y}}})]$, where the second equality follows from the fact that $\mathbf{A} \stackrel{d}{=} \tilde{\mathbf{A}}$. Hence, $\mathbf{A}|_{\mathcal{P}_{\mathbf{X}, \mathbf{Y}}} \stackrel{d}{=} \mathbf{P}_M^\perp \tilde{\mathbf{A}} \mathbf{P}_Q^\perp + \mathbf{B}$, which completes the proof.

APPENDIX D
PROOF OF PROPOSITION 4

Denoting $B = \sqrt{n}A(n) \sim \mu$ yields $\mathbb{E}[A^{2+2\alpha}(n)] = \mathbb{E}[B^{2(1+\alpha)}]n^{-(1+\alpha)} = o(n^{-2})$, where the last step uses the facts that $\mathbb{E}[B^{2(1+\alpha)}]$ is independent of n and $\alpha > 1$.

APPENDIX E
PROOF OF PROPOSITION 5

Denoting $X_{N,ij} = A_{ij}(N)v_j(N)$, then $\{X_{N,ij} : 1 \leq j \leq N\}$ is an independent zero-mean triangular array, for $i = 1, 2, \dots, n$. We claim that $\{X_{N,ij} : 1 \leq j \leq N\}$ satisfies two conditions in Proposition 2, $\forall i$. First, we note that $\sum_{j=1}^N \mathbb{E}[X_{N,ij}^2] = \sum_{j=1}^N \mathbb{E}[A_{ij}^2(N)]v_j^2(N) = \frac{1}{n} \sum_{j=1}^N v_j^2(N) = \frac{1}{\rho} \langle \mathbf{v}^2(N) \rangle \rightarrow \frac{s_0^2}{\rho}$ as $n \rightarrow \infty$. Second, applying Proposition 4 to $A_{ij}(N)$ gives $\mathbb{E}[A_{ij}^{2+2\alpha}(N)] = o(n^{-2})$, leading to $\mathbb{E}[X_{N,ij}^{2+2\alpha}] = \mathbb{E}[A_{ij}^{2+2\alpha}(N)]|v_j(N)|^{2+2\alpha} \leq o(n^{-2}) \frac{n}{\rho} \|\mathbf{v}(N)\|_{2+2\alpha}^{2+2\alpha} = o(n^{-1})$, where the last equality holds because $\limsup_{N \rightarrow \infty} \frac{1}{N} \|\mathbf{v}(N)\|_{2+2\alpha}^{2+2\alpha} < \infty$ and ρ is a constant. Applying Proposition 2 to $\{X_{N,ij} : 1 \leq j \leq N\}$ leads to $[\mathbf{A}(N)\mathbf{v}(N)]_i = \sum_{j=1}^N X_{N,ij} \xrightarrow{d} \mathcal{N}\left(0, \frac{s_0^2}{\rho}\right)$ as $N \rightarrow \infty$, $\forall i$. Hence, $\mathbf{A}(\widehat{N})\mathbf{v}(N) \xrightarrow{d} \mathcal{N}\left(0, \frac{s_0^2}{\rho}\right)$ as $N \rightarrow \infty$.

APPENDIX F
WELL-KNOWN LEMMAS

Lemma 2. (Stein's Lemma [38]) For jointly zero-mean Gaussian random variables Z_1 and Z_2 , and any function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, where $\mathbb{E}[\phi'(Z_2)]$ and $\mathbb{E}[Z_1\phi(Z_2)]$ exist, the following holds $\mathbb{E}[Z_1\phi(Z_2)] = \text{Cov}(Z_1, Z_2)\mathbb{E}[\phi'(Z_2)]$, where $\text{Cov}(Z_1, Z_2)$ is the covariance between Z_1 and Z_2 .

Lemma 3. (Strong Law of Large Number [39]) Let $\{X_{n,m} : 1 \leq m \leq n\}$ be a triangular array of random variables with $(X_{n,1}, X_{n,2}, \dots, X_{n,n})$ mutually independent with zero-mean for each n and $\frac{1}{n} \sum_{m=1}^n \mathbb{E}[|X_{n,m}|^{2+\kappa}] \leq cn^{\kappa/2}$ for some $0 < \kappa < 1$ and $c < \infty$. Then $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n X_{n,m} \stackrel{a.s.}{=} 0$.

Lemma 4. (Holder's inequality [40]) For random variables X and Y , $\mathbb{E}[|X + Y|^r] \leq c_r(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r])$, where $c_r = 1$ if $0 < r \leq 1$ and $c_r = 2^{r-1}$ otherwise. In particular, the inequality becomes $\mathbb{E}[|X + y|^r] \leq c_r(\mathbb{E}[|X|^r] + |y|^r)$ when $Y = y$ being a constant.

Lemma 5. (Lyapunov's inequality [40]) Suppose a random variable X and a constant κ with $0 < \kappa < 1$, then $|\mathbb{E}[X]|^{2+\kappa} \leq \mathbb{E}[|X|^{2+\kappa}]$.

Proposition 7. Suppose the $\mathbf{P}_{M(n)} = (\frac{1}{\sqrt{n}}\mathbf{V}(n))(\frac{1}{\sqrt{n}}\mathbf{V}(n))^*$, where $\mathbf{M}(n) \in \mathbb{R}^{n \times t}$ ($t \leq n$), t is a fixed constant, and $\mathbf{V}(n) = [\mathbf{v}_1(n), \mathbf{v}_2(n), \dots, \mathbf{v}_t(n)] \in \mathbb{R}^{n \times t}$ is an orthogonal basis of $\mathbf{M}(n)$ such that $\mathbf{V}^*(n)\mathbf{V}(n) = n\mathbf{I}$. If we let $\mathbf{a}(n) \in \mathbb{R}^{n \times 1}$ be a random vector with independent entries, which have zero mean and finite variance σ_a^2 , then $\lim_{n \rightarrow \infty} \mathbf{P}_{M(n)}\mathbf{a}(n) \stackrel{a.s.}{=} \mathbf{0}_n$, where $\mathbf{0}_n$ is the $n \times 1$ all-zero vector.

Proof. Denoting $\tilde{\mathbf{a}}(n) = \frac{\mathbf{a}(n)}{\|\mathbf{a}(n)\|_2}$ yields $\mathbf{P}_{M(n)}\mathbf{a}(n) = \mathbf{V}(n) \frac{\|\mathbf{a}(n)\|_2}{\sqrt{n}} \left(\frac{1}{\sqrt{n}}\mathbf{V}^*(n)\right) \tilde{\mathbf{a}}(n)$. The proposition follows from the fact that $\frac{\|\mathbf{a}(n)\|_2}{\sqrt{n}} \stackrel{a.s.}{=} \sigma_a$ and $\left(\frac{1}{\sqrt{n}}\mathbf{V}^*(n)\right) \tilde{\mathbf{a}}(n) \stackrel{a.s.}{=} \mathbf{0}_n$ as $n \rightarrow \infty$. \square

APPENDIX G
PROOF OF SE IN (7) THEOREM 1

Substituting $\phi_b(\mathbf{u}_i^t) = (b_i^t)^2$ into (16) gives $\lim_{n \rightarrow \infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle \stackrel{a.s.}{=} \mathbb{E}[\sigma_t^2 \tilde{Z}_t^2] = \sigma_t^2$. Using (14) with $t_1 = t_2 = t$ yields $\lim_{N \rightarrow \infty} \frac{1}{\rho} \langle \mathbf{q}^t, \mathbf{q}^t \rangle = \sigma_t^2$. Then substituting $\phi_h(\mathbf{v}_i^{t-1}) = f_t^2(h_i^t, x_{0i}) = (q_i^t)^2$ into (17) leads to $\lim_{N \rightarrow \infty} \langle \mathbf{q}^t, \mathbf{q}^t \rangle \stackrel{a.s.}{=} \mathbb{E}[f_t^2(\tau_{t-1}Z_{t-1}, X_0)]$, resulting in $\sigma_t^2 = \frac{1}{\rho} \mathbb{E}[f_t^2(\tau_{t-1}Z_{t-1}, X_0)]$. Showing the rest half $\tau_t^2 = \mathbb{E}[g_t^2(\sigma_t Z, W)]$ of the SE in (7) follows from the exactly same procedure as the above. Setting $\phi_h(\mathbf{v}_i^t) = (h_i^{t+1})^2$ in (17) gives $\lim_{N \rightarrow \infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1} \rangle = \tau_t^2$. Using (15) with $t_1 = t_2 = t$ yields $\lim_{N \rightarrow \infty} \langle \mathbf{m}^t, \mathbf{m}^t \rangle = \tau_t^2$. Applying (16) to $\phi_b(\mathbf{u}_i^t) = g_t^2(b_i^t, w_i)$ yields $\lim_{N \rightarrow \infty} \langle \mathbf{m}^t, \mathbf{m}^t \rangle = \mathbb{E}[g_t^2(\sigma_t Z, W)]$. Therefore, $\tau_t^2 = \mathbb{E}[g_t^2(\sigma_t Z, W)]$, concluding the proof.

APPENDIX H
PROOF OF THEOREM 1: STEPS 3 AND 4

A. Step 3: We show a), b), c), and d) of Theorem 1 conditioning on $\mathcal{F}_{t,t} = \{\mathbf{b}^0, \dots, \mathbf{b}^{t-1}, \mathbf{m}^0, \dots, \mathbf{m}^{t-1}, \mathbf{h}^1, \dots, \mathbf{h}^t, \mathbf{q}^0, \dots, \mathbf{q}^t, \mathbf{x}_0, \mathbf{w}\}$.

a) Note that conditioning on $\mathcal{F}_{t,t}$ is equivalent to conditioning on $\mathcal{P}_{\mathbf{X}_t, \mathbf{Y}_t} = \{\mathbf{A}|\mathbf{A}^*\mathbf{M}_t = \mathbf{X}_t, \mathbf{A}\mathbf{Q}_t = \mathbf{Y}_t\}$. Applying Proposition 3 to obtain the conditional distribution $\mathbf{A}|_{\mathcal{F}_{t,t}}$

and following the same procedure as in [26, Lemma 1a], the conditional distribution of \mathbf{b}^t on $\mathcal{F}_{t,t}$ is expressed as

$$\mathbf{b}^t|_{\mathcal{F}_{t,t}} \stackrel{d}{=} \sum_{j=0}^{t-1} \beta_j \mathbf{b}^j + \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t - \mathbf{P}_{\mathbf{M}_t} \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t + \mathbf{M}_t \vec{\sigma}_t(1). \quad (29)$$

By Proposition 7 in Appendix F, the third term on the right-hand side of (29) converges to $\lim_{n \rightarrow \infty} \mathbf{P}_{\mathbf{M}_t} \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t \stackrel{a.s.}{=} \mathbf{0}_n$. Similar to part a) of Step 2, we verify the convergence $\lim_{n \rightarrow \infty} \mathbf{M}_t \vec{\sigma}_t(1) \stackrel{a.s.}{=} \mathbf{0}_n$ by characterizing the expectation and variance of its empirical distribution $\widehat{\mathbf{M}_t \vec{\sigma}_t(1)}$ as $n \rightarrow \infty$. Indeed, $\lim_{n \rightarrow \infty} |\langle \mathbf{M}_t \vec{\sigma}_t(1) \rangle| \leq \lim_{n \rightarrow \infty} |o(1)| \frac{1}{n} \sum_{i=1}^n \left| \sum_{j=0}^{t-1} m_i^j \right| \stackrel{a.s.}{=} 0$, where the last equality holds because applying $\phi_b(b_i^j, w_i) = g_j(b_i^j, w_i)$ to the induction hypothesis of (16), for $j < t$, leads to $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n |m_i^j| \stackrel{a.s.}{=} \mathbb{E}[|g_j(\sigma_j \tilde{Z}_j, W)|] < \infty$. Hence, $\lim_{n \rightarrow \infty} \langle \mathbf{M}_t \vec{\sigma}_t(1) \rangle \stackrel{a.s.}{=} 0$. For the variance of $\widehat{\mathbf{M}_t \vec{\sigma}_t(1)}$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle \mathbf{M}_t \vec{\sigma}_t(1) \rangle_2 &= \lim_{n \rightarrow \infty} \frac{1}{n} [o(1)]^2 \sum_{i=1}^n \left(\sum_{j=0}^{t-1} m_i^j \right)^2, \\ &\leq \lim_{n \rightarrow \infty} [o(1)]^2 t \frac{1}{n} \sum_{i=1}^n \sum_{j=0}^{t-1} (m_i^j)^2, \quad (30a) \\ &= \lim_{n \rightarrow \infty} [o(1)]^2 t \sum_{j=0}^{t-1} \langle \mathbf{m}^j, \mathbf{m}^j \rangle \stackrel{a.s.}{=} 0, \quad (30b) \end{aligned}$$

where (30a) follows from the Cauchy-Schwarz inequality and (30b) holds because $\lim_{n \rightarrow \infty} \langle \mathbf{m}^j, \mathbf{m}^j \rangle \stackrel{a.s.}{=} \mathbb{E}[g_j^2(\sigma_j \tilde{Z}_j, W)] < \infty$, $\forall j$, which is a direct consequence of the induction hypothesis of (16) with $\phi_b(b_i^j, w_i) = g_j^2(b_i^j, w_i)$. Hence, (30b) is equivalent to $\lim_{n \rightarrow \infty} \langle \mathbf{M}_t \vec{\sigma}_t(1) \rangle_2 \stackrel{a.s.}{=} 0$. Therefore, $\lim_{n \rightarrow \infty} \mathbf{M}_t \vec{\sigma}_t(1) \stackrel{a.s.}{=} \mathbf{0}_n$, implying

$$\mathbf{b}^t|_{\mathcal{F}_{t,t}} \stackrel{d}{=} \sum_{j=0}^{t-1} \beta_j \mathbf{b}^j + \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t. \quad (31)$$

b) Note that by the induction hypothesis of (17) for $\phi_b(h_i^t, x_{0i}) = f_t^{1+\alpha}(h_i^t, x_{0i})$, we get $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N |q_i^t|^{2+2\alpha} \stackrel{a.s.}{=} \mathbb{E}[f_t^{2+2\alpha}(\tau_{t-1} Z_{t-1}, X_0)] < \infty$. On the other hand, $\sum_{i=1}^N |q_{\perp i}^t|^{2+2\alpha} < \sum_{i=1}^N |q_i^t|^{2+2\alpha}$. Thus, we have $\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N |q_{\perp i}^t|^{2+2\alpha} < \infty$, which concludes (12).

c) For $t_1 < t$ and $t_2 = t$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t \rangle &\stackrel{d}{=} \lim_{n \rightarrow \infty} \sum_{j=0}^{t-1} \beta_j \langle \mathbf{b}^{t_1}, \mathbf{b}^j \rangle + \lim_{n \rightarrow \infty} \langle \mathbf{b}^{t_1}, \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t \rangle, \quad (32a) \\ &\stackrel{a.s.}{=} \sum_{j=0}^{t-1} \beta_j \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^j \rangle}{\rho} + \lim_{n \rightarrow \infty} \frac{\mathbf{b}^{t_1*} \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t}{n}, \quad (32b) \end{aligned}$$

where (32a) follows from (31) and (32b) results from the induction hypothesis (14) for $t_1 < t$ and $t_2 = j < t$. Now,

using Proposition 6, we get $\frac{\mathbf{b}^{t_1*} \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t}{n} \stackrel{d}{=} \frac{\|\mathbf{b}^{t_1}\|_2 \|\tilde{\mathbf{A}} \mathbf{q}_{\perp}^t\|_2}{\sqrt{n} \|\mathbf{q}_{\perp}^t\|_2} \stackrel{d}{=} \frac{Z}{\sqrt{n}}$, where $Z \sim \mathcal{N}(0, 1)$. Hence, $\frac{\mathbf{b}^{t_1*} \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t}{n} \stackrel{d}{=} \frac{1}{\sqrt{\rho}} \sqrt{\langle \mathbf{b}^{t_1}, \mathbf{b}^{t_1} \rangle \langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle} \frac{Z}{\sqrt{n}}$, i.e.,

$$\frac{\mathbf{b}^{t_1*} \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t}{n} \stackrel{d}{=} \frac{1}{\sqrt{\rho}} \sqrt{\langle \mathbf{b}^{t_1}, \mathbf{b}^{t_1} \rangle \langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle} \frac{Z}{\sqrt{n}}. \quad (33)$$

By the induction hypothesis of (14), we have $\lim_{n \rightarrow \infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^{t_1} \rangle = \lim_{n \rightarrow \infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^{t_1} \rangle}{\rho} < \infty$. Moreover, the $\langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle$ converges to $\lim_{N \rightarrow \infty} \langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle < \infty$ because using the induction hypothesis (17) we have $\langle \mathbf{q}^t, \mathbf{q}^t \rangle = \frac{1}{N} \sum_{i=1}^N f_t^2(h_i^t, x_{0i}) \stackrel{a.s.}{=} \mathbb{E}[f_t^2(\tau_{t-1} Z_{t-1}, X_0)] < \infty$. Thus, for $t_1 < t$,

$$\lim_{n \rightarrow \infty} \langle \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t, \mathbf{b}^{t_1} \rangle \stackrel{a.s.}{=} 0. \quad (34)$$

Substituting (34) into (32b) gives $\lim_{n \rightarrow \infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t \rangle \stackrel{a.s.}{=} \sum_{j=0}^{t-1} \beta_j \lim_{n \rightarrow \infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^j \rangle}{\rho} = \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_{\perp}^t \rangle}{\rho}$ due to (8), implying

$$\lim_{n \rightarrow \infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t \rangle \stackrel{a.s.}{=} \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_{\perp}^t \rangle}{\rho} + \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_{\perp}^t \rangle}{\rho}, \quad (35a)$$

$$= \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^t \rangle}{\rho}, \quad (35b)$$

where (35a) follows from the fact that \mathbf{q}^j is orthogonal to \mathbf{q}_{\perp}^t , for $j < t$, and (35b) holds due to (9), concluding (14) when $t_1 < t$ and $t_2 = t$.

For the case of $t_1 = t_2 = t$, it is similarly given by $\lim_{n \rightarrow \infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle \stackrel{d}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{n \rightarrow \infty} \langle \mathbf{b}^i, \mathbf{b}^j \rangle + 2 \sum_{i=0}^{t-1} \beta_i \lim_{n \rightarrow \infty} \langle \mathbf{b}^i, \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t \rangle + \lim_{n \rightarrow \infty} \langle \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t, \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t \rangle$ due to (31). Then, by (34), the following holds

$$\lim_{n \rightarrow \infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{n \rightarrow \infty} \langle \mathbf{b}^i, \mathbf{b}^j \rangle + \lim_{n \rightarrow \infty} \langle \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t, \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t \rangle. \quad (36)$$

Using Proposition 5, $\widehat{\tilde{\mathbf{A}} \mathbf{q}_{\perp}^t} \stackrel{d}{=} \mathcal{N}\left(0, \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle}{\rho}\right)$.

Thus, the second moment of $\widehat{\tilde{\mathbf{A}} \mathbf{q}_{\perp}^t}$ is

$$\lim_{n \rightarrow \infty} \langle \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t, \tilde{\mathbf{A}} \mathbf{q}_{\perp}^t \rangle \stackrel{a.s.}{=} \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle}{\rho}. \quad (37)$$

Now, incorporating (37) in (36), $\lim_{n \rightarrow \infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{n \rightarrow \infty} \langle \mathbf{b}^i, \mathbf{b}^j \rangle + \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle}{\rho}$, resulting in

$$\lim_{n \rightarrow \infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^i, \mathbf{q}^j \rangle}{\rho} + \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle}{\rho}, \quad (38a)$$

$$\begin{aligned} &\stackrel{a.s.}{=} \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle}{\rho} + \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}_{\perp}^t, \mathbf{q}_{\perp}^t \rangle}{\rho}, \\ &\stackrel{a.s.}{=} \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^t, \mathbf{q}^t \rangle}{\rho}, \end{aligned}$$

where (38a) is due to the induction hypothesis (14) for $0 \leq t_1 = i, t_2 = j \leq t-1$. This completes the proof of (14) at the t th iteration.

d) Defining $\lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t \rangle}{\rho} \stackrel{a.s.}{=} \gamma_t^2$, we can write by (37) that $\widehat{\mathbf{A}\mathbf{q}_\perp^t} \stackrel{d}{\Rightarrow} \mathcal{N}(0, \gamma_t^2)$. Using (31) in conjunction with the latter, we get

$$b_i^t|_{\mathcal{F}_{t,t}} \stackrel{d}{\Rightarrow} \sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z, \text{ for } i = 1, 2, \dots, n, \quad (39)$$

where $Z \sim \mathcal{N}(0, 1)$. Similar to Step 1d), using (39) $\mathbf{u}_i^t \stackrel{d}{\Rightarrow} \tilde{\mathbf{u}}_i^t$, where $\mathbf{u}_i^t = (b_i^0, \dots, b_i^t, w_i)$ and $\tilde{\mathbf{u}}_i^t = (b_i^0, \dots, b_i^{t-1}, \sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z, w_i)$, $\forall i$. To prove (16), we first claim that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \phi_b(\tilde{\mathbf{u}}_i^t) - \mathbb{E}[\phi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_t \tilde{Z}_t, W)] \stackrel{a.s.}{=} 0$. By the triangular inequality, $\left| \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \phi_b(\tilde{\mathbf{u}}_i^t) - \mathbb{E}[\phi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_t \tilde{Z}_t, W)] \right| \leq X_1^t + X_2^t$, where $X_1^t = \left| \frac{1}{n} \sum_{i=1}^n (\phi_b(\tilde{\mathbf{u}}_i^t) - \tilde{\phi}_b(\mathbf{u}_i^{t-1})) \right|$, $X_2^t = \left| \frac{1}{n} \sum_{i=1}^n \tilde{\phi}_b(\mathbf{u}_i^{t-1}) - \mathbb{E}[\phi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_t \tilde{Z}_t, W)] \right|$, and $\tilde{\phi}_b(\mathbf{u}_i^{t-1}) = \mathbb{E}_Z[\phi_b(\tilde{\mathbf{u}}_i^t)]$. Similar to Step 1d), we verify $\lim_{n \rightarrow \infty} X_1^t \stackrel{a.s.}{=} 0$ and $\lim_{n \rightarrow \infty} X_2^t \stackrel{a.s.}{=} 0$.

First showing $\lim_{n \rightarrow \infty} X_1^t \stackrel{a.s.}{=} 0$ is of interest. By (5), $|\phi_b(\tilde{\mathbf{u}}_i^t)| \leq c_1^t \exp \left[c_2^t \left(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + \gamma_t^\lambda |Z|^\lambda + |w_i|^\lambda \right) \right]$, where $c_1^t > 0$, $c_2^t > 0$, and $1 \leq \lambda < 2$ are constant. Hence, $\mathbb{E}_Z[|\phi_b(\tilde{\mathbf{u}}_i^t)|^{2+\kappa}] \leq c_3^t \exp \left[c_4^t \left(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + |w_i|^\lambda \right) \right] \mathbb{E}_Z[\exp(c_4^t \gamma_t^\lambda |Z|^\lambda)]$, where $0 < \kappa < 1$, $c_3^t = (c_1^t)^{2+\kappa}$, $c_4^t = c_2^t(2 + \kappa)$, resulting in

$$\mathbb{E}_Z[|\phi_b(\tilde{\mathbf{u}}_i^t)|^{2+\kappa}] \leq c_5^t \exp \left[c_4^t \left(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + |w_i|^\lambda \right) \right], \quad (40)$$

and $c_5^t = c_3^t \mathbb{E}_Z[\exp(c_4^t \gamma_t^\lambda |Z|^\lambda)]$ is constant. We define $X_{n,i}^t = \phi_b(\tilde{\mathbf{u}}_i^t) - \tilde{\phi}_b(\mathbf{u}_i^{t-1}) = \phi_b(\tilde{\mathbf{u}}_i^t) - \mathbb{E}_Z[\phi_b(\tilde{\mathbf{u}}_i^t)]$ such that $X_1^t = \left| \frac{1}{n} \sum_{i=1}^n X_{n,i}^t \right|$. To prove $\lim_{n \rightarrow \infty} X_1^t \stackrel{a.s.}{=} 0$, we show that $\{X_{n,i}^t\}_{i=1}^n$ satisfy Lemma 3 in Appendix F. Indeed, $\mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}]$ is upper bounded as follows,

$$\mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}] \leq 2^{1+\kappa} \left[\mathbb{E}_Z[|\phi_b(\tilde{\mathbf{u}}_i^t)|^{2+\kappa}] + |\mathbb{E}_Z[\phi_b(\tilde{\mathbf{u}}_i^t)]|^{2+\kappa} \right], \quad (41a)$$

$$\leq 2^{1+\kappa} \left[\mathbb{E}_Z[|\phi_b(\tilde{\mathbf{u}}_i^t)|^{2+\kappa}] + \mathbb{E}_Z[|\phi_b(\tilde{\mathbf{u}}_i^t)|^{2+\kappa}] \right], \quad (41b)$$

$$\leq 2^{2+\kappa} \left[\mathbb{E}_Z[|\phi_b(\tilde{\mathbf{u}}_i^t)|^{2+\kappa}] \right], \quad (41c)$$

where (41a) follows from Lemma 4 (Holder's inequality) in Appendix F, (41b) follows from Lemma 5 (Lyapunov's inequality) in Appendix F, and (41c) follows from (40) with $c_6^t = 2^{2+\kappa} c_5^t$. We denote the last term of (41c) as $\psi_b(\mathbf{u}_i^{t-1}) = c_6^t \exp \left[c_4^t \left(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + |w_i|^\lambda \right) \right]$. Then, $\psi_b(\mathbf{u}_i^{t-1})$ is a controlled function. From (41c), we get, for n is sufficiently large,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}] &\leq \frac{1}{n} \sum_{i=1}^n \psi_b(\mathbf{u}_i^{t-1}), \\ &\stackrel{a.s.}{=} \mathbb{E}[\psi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_{t-1} \tilde{Z}_{t-1}, W)], \quad (42a) \\ &< cn^{\kappa/2}, \quad (42b) \end{aligned}$$

where c is a positive constant, (42a) is due to the induction hypothesis (16), and (42b) holds because $\mathbb{E}[\psi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_{t-1} \tilde{Z}_{t-1}, W)] = c_7^t < \infty$ and there exists n_t , a positive constant, such that $c_7^t < cn^{\kappa/2}$ for $n > n_t$. By Lemma 3 in Appendix F, we get $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_{n,i}^t \stackrel{a.s.}{=} 0$, implying

$$\frac{1}{n} \sum_{i=1}^n (\phi_b(\tilde{\mathbf{u}}_i^t) - \tilde{\phi}_b(\mathbf{u}_i^{t-1})) \stackrel{a.s.}{=} 0, \quad (43)$$

which proving $\lim_{n \rightarrow \infty} X_1^t \stackrel{a.s.}{=} 0$.

Now, showing $\lim_{n \rightarrow \infty} X_2^t \stackrel{a.s.}{=} 0$ is of interest. By the induction hypothesis (16), $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \tilde{\phi}_b(\mathbf{u}_i^{t-1}) \stackrel{a.s.}{=} \mathbb{E}[\phi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_{t-1} \tilde{Z}_{t-1}, W)]$, resulting in

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \tilde{\phi}_b(\mathbf{u}_i^{t-1}) \\ &= \mathbb{E} \left[\mathbb{E}_Z \left[\phi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_{t-1} \tilde{Z}_{t-1}, \sum_{j=0}^{t-1} \beta_j \sigma_j \tilde{Z}_j + \gamma_t Z, W) \right] \right], \\ &= \mathbb{E} \left[\phi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_{t-1} \tilde{Z}_{t-1}, \sum_{j=0}^{t-1} \beta_j \sigma_j \tilde{Z}_j + \gamma_t Z, W) \right], \quad (44) \end{aligned}$$

where (44) follows from the substitution $\tilde{\phi}_b(\mathbf{u}_i^{t-1}) = \mathbb{E}_Z[\phi_b(\tilde{\mathbf{u}}_i^t)]$. Therefore, showing $\lim_{n \rightarrow \infty} X_2^t \stackrel{a.s.}{=} 0$ is equivalent to proving $\sum_{j=0}^{t-1} \beta_j \sigma_j \tilde{Z}_j + \gamma_t Z = \sigma_t \tilde{Z}_t$, where $\tilde{Z}_t \sim \mathcal{N}(0, 1)$ and σ_t is defined in (7).

In particular, for $\phi_b(\mathbf{u}_i^t) = (b_i^t)^2$, we get $\phi_b(\tilde{\mathbf{u}}_i^t) = \left(\sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z \right)^2$ because $\tilde{\mathbf{u}}_i^t = (b_i^0, \dots, b_i^{t-1}, \sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z, w_i)$. Combining (43) and (44),

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \phi_b(\mathbf{u}_i^t) \\ &\stackrel{a.s.}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \phi_b(\tilde{\mathbf{u}}_i^t) \stackrel{a.s.}{=} \mathbb{E} \left[\left(\sum_{j=0}^{t-1} \beta_j \sigma_j \tilde{Z}_j + \gamma_t Z \right)^2 \right]. \quad (45) \end{aligned}$$

Using (14), $\lim_{n \rightarrow \infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle \stackrel{a.s.}{=} \lim_{N \rightarrow \infty} \frac{\langle \mathbf{q}^t, \mathbf{q}^t \rangle}{\rho} = \sigma_t^2$, where the last equality holds because by the induction hypothesis (17) for $\phi_h(\mathbf{v}_i^{t-1}) = f_t^2(h_i^t, x_{0i})$ in (17), $\frac{1}{\rho} \lim_{N \rightarrow \infty} \langle \mathbf{q}^t, \mathbf{q}^t \rangle \stackrel{a.s.}{=} \frac{1}{\rho} \mathbb{E}[f_t^2(\tau_{t-1} Z, X_0)] = \sigma_t^2$.

Hence, $\mathbb{E} \left[\left(\sum_{j=0}^{t-1} \beta_j \sigma_j \tilde{Z}_j + \gamma_t Z \right)^2 \right] \stackrel{a.s.}{=} \sigma_t^2$, implying $\sum_{j=0}^{t-1} \beta_j \sigma_j \tilde{Z}_j + \gamma_t Z = \sigma_t \tilde{Z}_t$ due to (45), verifying that $\lim_{n \rightarrow \infty} X_2^t \stackrel{a.s.}{=} 0$, which completes the proof of (16).

B. Step 4: We show a), b), c), and d) of Theorem 1 conditioning on $\mathcal{F}_{t+1,t} = \{\mathbf{b}^0, \dots, \mathbf{b}^t, \mathbf{m}^0, \dots, \mathbf{m}^t, \mathbf{h}^1, \dots, \mathbf{h}^t, \mathbf{q}^0, \dots, \mathbf{q}^t, \mathbf{x}_0, \mathbf{w}\}$.

The proof of Step 4 is similar to the proof of Step 3. Thus, we only present the features that are unique in Step 4.

a) Similar to Step 3a), using Proposition 3 to characterize $\mathbf{A}|_{\mathcal{F}_{t+1,t}}$ and following the same procedure as in [26, Lemma 1a], the $\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}}$ is

$$\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{=} \sum_{j=0}^{t-1} \alpha_j \mathbf{h}^{j+1} + \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t - \mathbf{P}_{\mathbf{Q}_{t+1}} \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t + \mathbf{Q}_{t+1} \vec{\sigma}_t(1). \quad (46)$$

By Proposition 7 in Appendix F, $\lim_{N \rightarrow \infty} \mathbf{P}_{\mathbf{Q}_{t+1}} \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \stackrel{a.s.}{=} \mathbf{0}_N$. Similar to Step 3a), we verify that $\lim_{N \rightarrow \infty} \mathbf{Q}_{t+1} \vec{\sigma}_t(1) \stackrel{a.s.}{=} \mathbf{0}_N$ by characterizing (i) the expectation of the empirical distribution $\mathbf{Q}_{t+1} \vec{\sigma}_t(1)$ is bounded as $\lim_{N \rightarrow \infty} |\langle \mathbf{Q}_{t+1} \vec{\sigma}_t(1) \rangle| \leq \lim_{N \rightarrow \infty} |o(1)| \sum_{j=0}^t \frac{1}{N} \sum_{i=1}^N |q_i^j| \stackrel{a.s.}{=} 0$ and (ii) the empirical variance of $\mathbf{Q}_{t+1} \vec{\sigma}_t(1)$ is bounded and converges to $\lim_{N \rightarrow \infty} \langle \mathbf{Q}_{t+1} \vec{\sigma}_t(1) \rangle_2 \leq \lim_{N \rightarrow \infty} [o(1)]^2 (t+1) \sum_{j=0}^t \langle \mathbf{q}^j, \mathbf{q}^j \rangle \stackrel{a.s.}{=} 0$. Therefore, using $\lim_{N \rightarrow \infty} \mathbf{Q}_{t+1} \vec{\sigma}_t(1) \stackrel{a.s.}{=} \mathbf{0}_N$, we get

$$\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{=} \sum_{j=0}^{t-1} \alpha_j \mathbf{h}^{j+1} + \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \quad (47)$$

which completes the proof of (11).

b) Using (16) for $\phi_b(b_i^t, w_i) = g_t^{1+\alpha}(b_i^t, w_i)$, we get $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n |m_i^t|^{2+2\alpha} \stackrel{a.s.}{=} \mathbb{E}[g_t^{2+2\alpha}(\sigma_t Z_t, W)] < \infty$. Because $\sum_{i=1}^n |m_{\perp i}^t|^{2+2\alpha} < \sum_{i=1}^n |m_i^t|^{2+2\alpha}$, the following holds $\limsup_{n \rightarrow \infty} \sum_{i=1}^n |m_{\perp i}^t|^{2+2\alpha} < \infty$, which concludes (13).

c) For $t_1 < t$ and $t_2 = t$, we have $\lim_{N \rightarrow \infty} \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t+1} \rangle \stackrel{d}{=} \lim_{N \rightarrow \infty} \sum_{j=0}^{t-1} \alpha_j \langle \mathbf{h}^{t_1+1}, \mathbf{h}^j \rangle + \lim_{N \rightarrow \infty} \langle \mathbf{h}^{t_1+1}, \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \rangle$ due to (47), resulting in

$$\lim_{N \rightarrow \infty} \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t+1} \rangle \stackrel{a.s.}{=} \sum_{j=0}^{t-1} \alpha_j \lim_{n \rightarrow \infty} \langle \mathbf{m}^{t_1}, \mathbf{m}^j \rangle + \lim_{N \rightarrow \infty} \frac{\mathbf{m}_\perp^{t*} \tilde{\mathbf{A}} \mathbf{h}^{t_1+1}}{N}, \quad (48)$$

where (48) is by the induction hypothesis (15). Note that $\frac{\mathbf{m}_\perp^{t*}}{\|\mathbf{m}_\perp^t\|_2} \tilde{\mathbf{A}} \frac{\mathbf{h}^{t_1+1}}{\|\mathbf{h}^{t_1+1}\|_2} \stackrel{d}{=} \frac{Z}{\sqrt{n}}$ due to Proposition 6. The second term in (48) is represented as

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\mathbf{m}_\perp^{t*} \tilde{\mathbf{A}} \mathbf{h}^{t_1+1}}{N} &\stackrel{d}{=} \lim_{N \rightarrow \infty} \frac{\|\mathbf{m}_\perp^t\|_2}{\sqrt{n}} \frac{\|\mathbf{h}^{t_1+1}\|_2}{\sqrt{N}} \frac{\sqrt{n}}{\sqrt{N}} \frac{Z}{\sqrt{n}}, \\ &= \sqrt{\rho} \lim_{N \rightarrow \infty} \sqrt{\langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t \rangle \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t_1+1} \rangle} \frac{Z}{\sqrt{n}} \stackrel{a.s.}{=} 0, \end{aligned} \quad (49)$$

Substituting (49) into (48) yields

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t+1} \rangle &= \lim_{n \rightarrow \infty} \langle \mathbf{m}^{t_1}, \sum_{j=0}^{t-1} \alpha_j \mathbf{m}^j \rangle = \lim_{n \rightarrow \infty} \langle \mathbf{m}^{t_1}, \mathbf{m}_\parallel^t \rangle, \\ &= \lim_{n \rightarrow \infty} \langle \mathbf{m}^{t_1}, \mathbf{m}_\parallel^t \rangle + \lim_{n \rightarrow \infty} \langle \mathbf{m}^{t_1}, \mathbf{m}_\perp^t \rangle = \lim_{n \rightarrow \infty} \langle \mathbf{m}^{t_1}, \mathbf{m}^t \rangle, \end{aligned}$$

concluding (15) when $t_1 < t$ and $t_2 = t$.

For $t_1 = t_2 = t$, by (47),

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1} \rangle &\stackrel{d}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{N \rightarrow \infty} \langle \mathbf{h}^{i+1}, \mathbf{h}^{j+1} \rangle \\ &+ 2 \sum_{i=0}^{t-1} \alpha_i \lim_{N \rightarrow \infty} \langle \mathbf{h}^{i+1}, \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \rangle + \lim_{N \rightarrow \infty} \langle \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \rangle. \end{aligned} \quad (50)$$

Then, by (49), the following holds

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1} \rangle &\stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{N \rightarrow \infty} \langle \mathbf{h}^{i+1}, \mathbf{h}^{j+1} \rangle + \\ &\lim_{N \rightarrow \infty} \langle \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \rangle. \end{aligned} \quad (51)$$

By Proposition 5, the empirical distribution of $\tilde{\mathbf{A}}^* \mathbf{m}_\perp^t$ converges to $\tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \stackrel{d}{=} \mathcal{N}(0, \lim_{n \rightarrow \infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t \rangle)$. Hence, the second moment of $\tilde{\mathbf{A}}^* \mathbf{m}_\perp^t$ converges to

$$\lim_{N \rightarrow \infty} \langle \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \rangle \stackrel{a.s.}{=} \lim_{n \rightarrow \infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t \rangle. \quad (52)$$

Substituting (52) into (51) leads to $\lim_{N \rightarrow \infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1} \rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{N \rightarrow \infty} \langle \mathbf{h}^{i+1}, \mathbf{h}^{j+1} \rangle + \lim_{n \rightarrow \infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t \rangle$, implying

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1} \rangle &\stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{n \rightarrow \infty} \langle \mathbf{m}^i, \mathbf{m}^j \rangle + \lim_{n \rightarrow \infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t \rangle, \\ &= \lim_{n \rightarrow \infty} \langle \mathbf{m}_\parallel^t, \mathbf{m}_\parallel^t \rangle + \lim_{n \rightarrow \infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t \rangle = \lim_{n \rightarrow \infty} \langle \mathbf{m}^t, \mathbf{m}^t \rangle. \end{aligned}$$

Therefore, (15) also holds for $t_1 = t_2 = t$, which completes the proof.

d) Defining $\lim_{n \rightarrow \infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t \rangle \stackrel{a.s.}{=} \Gamma_t^2$, we can write

$$\tilde{\mathbf{A}}^* \mathbf{m}_\perp^t \stackrel{d}{=} \mathcal{N}(0, \Gamma_t^2). \quad (53)$$

Using (53) and (47), the following convergence holds

$$\mathbf{h}_i^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{=} \sum_{j=0}^{t-1} \alpha_j \mathbf{h}_i^{j+1} + \Gamma_t Z, \quad (54)$$

where $Z \sim \mathcal{N}(0, 1)$. Similar to Step 3d), we can write, using (54), $\mathbf{v}_i^t \stackrel{d}{=} \tilde{\mathbf{v}}_i^t$, where $\mathbf{v}_i^t = (h_i^1, \dots, h_i^{t+1}, x_{0i})$ and $\tilde{\mathbf{v}}_i^t = (h_i^1, \dots, h_i^t, \sum_{j=0}^{t-1} \alpha_j \mathbf{h}_i^{j+1} + \Gamma_t Z, x_{0i})$. Hence, to prove (17) we first claim that $\left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi_h(\tilde{\mathbf{v}}_i^t) - \mathbb{E}[\phi_h(\tau_0 Z_0, \dots, \tau_t Z_t, X_0)] \right| \stackrel{a.s.}{=} 0$. Similar to Step 2d), using the triangular inequality, we verify that $\lim_{N \rightarrow \infty} Y_1^t \stackrel{a.s.}{=} 0$ and $\lim_{N \rightarrow \infty} Y_2^t \stackrel{a.s.}{=} 0$, where $Y_1^t = \left| \frac{1}{N} \sum_{i=1}^N (\phi_h(\tilde{\mathbf{v}}_i^t) - \tilde{\phi}_h(\mathbf{v}_i^{t-1})) \right|$ and $Y_2^t = \left| \frac{1}{N} \sum_{i=1}^N \tilde{\phi}_h(\mathbf{v}_i^{t-1}) - \mathbb{E}[\phi_h(\tau_0 Z_0, \dots, \tau_t Z_t, X_0)] \right|$, and $\tilde{\phi}_h(\mathbf{v}_i^{t-1}) = \mathbb{E}_Z[\phi_h(\tilde{\mathbf{v}}_i^t)], \forall i$.

First showing $\lim_{N \rightarrow \infty} Y_1^t$ is of interest. By (5), $|\phi_h(\tilde{\mathbf{v}}_i^t)| \leq d_1^t \exp \left[d_2^t \left(\sum_{j=0}^{t-1} |\alpha_j \mathbf{h}_i^{j+1}|^\lambda + \Gamma_t^\lambda |Z|^\lambda + |x_{0i}|^\lambda \right) \right]$, where $d_1^t > 0$, $d_2^t > 0$, and $1 \leq \lambda < 2$ are constant. Hence,

$$\mathbb{E}_Z \left[|\phi_h(\tilde{\mathbf{v}}_i^t)|^{2+\kappa} \right] \leq d_5^t \exp \left[d_4^t \left(\sum_{j=0}^{t-1} |\alpha_j \mathbf{h}_i^{j+1}|^\lambda + |x_{0i}|^\lambda \right) \right],$$

where $0 < \kappa < 1$, $d_4^t = d_2^t(2 + \kappa)$, and $d_5^t = (d_1^t)^{2+\kappa} \mathbb{E}_Z \left[\exp(d_4^t \Gamma_t^\lambda |Z|^\lambda) \right]$ are constants. Define $Y_{N,i}^t = \phi_h(\tilde{\mathbf{v}}_i^t) - \mathbb{E}_Z[\phi_h(\tilde{\mathbf{v}}_i^t)], \forall i$. To prove the convergence $\lim_{n \rightarrow \infty} Y_1^t \stackrel{a.s.}{=} 0$, we will show that $\{Y_{N,i}^t\}_{i=1}^N$ satisfy

the condition in Lemma 3 in Appendix F. Indeed, the $\mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}]$ is upper bounded as follows.

$$\begin{aligned}\mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}] &\leq 2^{1+\kappa} \left[\mathbb{E}_Z[|\phi_h(\tilde{\mathbf{v}}_i^t)|^{2+\kappa}] + |\mathbb{E}_Z[\phi_h(\tilde{\mathbf{v}}_i^t)]|^{2+\kappa} \right], \\ &\leq 2^{1+\kappa} \left[\mathbb{E}_Z[|\phi_h(\tilde{\mathbf{v}}_i^t)|^{2+\kappa}] + \mathbb{E}_Z[|\phi_h(\tilde{\mathbf{v}}_i^t)|^{2+\kappa}] \right], \\ &\leq 2^{2+\kappa} \left[\mathbb{E}_Z[|\phi_h(\tilde{\mathbf{v}}_i^t)|^{2+\kappa}] \right] \\ &\leq d_6^t \exp \left[d_4^t \left(\sum_{j=0}^{t-1} |\alpha_j h_i^{j+1}|^\lambda + |x_{0i}|^\lambda \right) \right], \quad (55a)\end{aligned}$$

Denoting the last term of (55a) as $\psi_h(\mathbf{v}_i^{t-1}) = d_6^t \exp \left[d_4^t \left(\sum_{j=0}^{t-1} |\alpha_j h_i^{j+1}|^\lambda + |x_{0i}|^\lambda \right) \right]$, then $\psi_h(\mathbf{v}_i^{t-1})$ is a controlled function. From (55a), we get, for N is sufficiently large,

$$\begin{aligned}\frac{1}{N} \sum_{i=1}^n \mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}] &\leq \frac{1}{N} \sum_{i=1}^n \psi_h(\mathbf{v}_i^{t-1}), \\ &\stackrel{a.s.}{=} \mathbb{E}[\psi_h(\tau_0 Z_0, \dots, \tau_{t-1} Z_{t-1}, X_0)], \\ &< cN^{\kappa/2}, \quad (56a)\end{aligned}$$

where c is a positive constant and (56a) holds because $\mathbb{E}[\psi_b(\sigma_0 \tilde{Z}_0, \dots, \sigma_{t-1} \tilde{Z}_{t-1}, W)] = d_7^t < \infty$ and there exists N_t , a positive constant, such that $d_7^t < cN^{\kappa/2}$ for $N > N_t$. Using Lemma 3 in Appendix F, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N Y_{N,i}^t \stackrel{a.s.}{=} 0$, implying $\lim_{N \rightarrow \infty} Y_1^t \stackrel{a.s.}{=} 0$.

We are now ready to verify the convergence $\lim_{N \rightarrow \infty} Y_2^t \stackrel{a.s.}{=} 0$. Applying the induction hypothesis (17) for $\tilde{\phi}_b(\mathbf{v}_i^{t-1})$ gives

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \tilde{\phi}_h(\mathbf{v}_i^{t-1}) &\stackrel{a.s.}{=} \mathbb{E}[\tilde{\phi}_h(\tau_0 Z_0, \dots, \tau_{t-1} Z_{t-1}, X_0)], \\ &= \mathbb{E} \left[\mathbb{E}_Z[\phi_h(\tau_0 Z_0, \dots, \tau_{t-1} Z_{t-1}, \sum_{j=0}^{t-1} \alpha_j \tau_j Z_j + \Gamma_t Z, X_0)] \right], \\ &= \mathbb{E} \left[\phi_h(\tau_0 Z_0, \dots, \tau_{t-1} Z_{t-1}, \sum_{j=0}^{t-1} \alpha_j \tau_j Z_j + \Gamma_t Z, X_0) \right].\end{aligned}$$

Therefore, showing $\lim_{N \rightarrow \infty} Y_2^t \stackrel{a.s.}{=} 0$ is equivalent to proving $\sum_{j=0}^{t-1} \alpha_j \tau_j Z_j + \Gamma_t Z = \tau_t Z_t$, where $Z_t \sim \mathcal{N}(0, 1)$ and τ_t is defined in (7). Similar to the proof of $\lim_{n \rightarrow \infty} X_2^t \stackrel{a.s.}{=} 0$ in Step 3d), setting $\phi_h(\mathbf{v}_i^t) = (h_i^t)^2$, i.e., $\phi_h(\tilde{\mathbf{v}}_i^t) = \left(\sum_{j=0}^{t-1} \alpha_j h_i^{j+1} + \Gamma_t Z \right)^2$, we get

$$\begin{aligned}\lim_{N \rightarrow \infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1} \rangle &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi_h(\mathbf{v}_i^t) \\ &\stackrel{a.s.}{=} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi_h(\tilde{\mathbf{v}}_i^t) \stackrel{a.s.}{=} \mathbb{E} \left[\left(\sum_{j=0}^{t-1} \alpha_j \tau_j Z_j + \Gamma_t Z \right)^2 \right].\end{aligned}$$

Using (15), we get $\lim_{N \rightarrow \infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1} \rangle \stackrel{a.s.}{=} \lim_{n \rightarrow \infty} \langle \mathbf{m}^t, \mathbf{m}^t \rangle = \tau_t^2$, where the last equality holds

by the induction hypothesis (16) for $\phi_b(\mathbf{u}_i^t) = g_t^2(b_i^t, w_i)$, resulting in $\lim_{n \rightarrow \infty} \langle \mathbf{m}^t, \mathbf{m}^t \rangle \stackrel{a.s.}{=} \mathbb{E}[g_t^2(\sigma_t \tilde{Z}_t, W)] = \tau_t^2$. Hence, $\mathbb{E} \left[\left(\sum_{j=0}^{t-1} \alpha_j \tau_j Z_j + \Gamma_t Z \right)^2 \right] \stackrel{a.s.}{=} \tau_t^2$, which implies $\sum_{j=0}^{t-1} \alpha_j \tau_j Z_j + \Gamma_t Z = \tau_t Z_t$. Thus, it is verified that $\lim_{N \rightarrow \infty} Y_2^t \stackrel{a.s.}{=} 0$, which completes the proof of (17).