On The Stability of Approximate Message Passing with Independent Measurement Ensembles (Supplementary Material)

APPENDIX A
PSEUDO-LIPSCHITZ FUNCTION

***Definition* 1.** *For a $k>1$, a function $f:\mathbb{R}^{n\times 1}\to\mathbb{R}$ is said pseudo-Lipschitz of order k if there exists a constant $L>0$ such that $|f(\mathbf{x})-f(\mathbf{y})|\le L(1+\|\mathbf{x}\|^{k-1}+\|\mathbf{y}\|^{k-1})\|\mathbf{x}-\mathbf{y}\|$, for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^{n\times 1}$; the first order derivative of f is bounded by a polynomial of order $(k-1)$, i.e., polynomial smoothness.*

APPENDIX B
PROOF OF PROPOSITION 1

This follows from $\sum_{m=1}^{n}\mathbb{E}\big[X_{n,m}^2;|X_{n,m}|>\epsilon\big]\le\sum_{m=1}^{n}\mathbb{E}\Big[\frac{X_{n,m}^{2+2\alpha}}{\epsilon^{2\alpha}}\Big]$, where the right-hand side converges to zero because $\frac{n}{\epsilon^{2\alpha}}o(n^{-1})\to 0$ as $n\to\infty$.

APPENDIX C
PROOF OF PROPOSITION 3

Recalling the following subspace decomposition $\mathbf{A}=\mathbf{P}_{\mathbf{M}}^{\perp}\mathbf{A}\mathbf{P}_{\mathbf{Q}}^{\perp}+\mathbf{A}\mathbf{P}_{\mathbf{Q}}+\mathbf{P}_{\mathbf{M}}\mathbf{A}-\mathbf{P}_{\mathbf{M}}\mathbf{A}\mathbf{P}_{\mathbf{Q}}$, the orthogonal projection of $\mathbf{A}$ onto $\mathcal{P}_{\mathbf{X},\mathbf{Y}}$ is given by $\mathbf{A}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}}=\mathbf{P}_{\mathbf{M}}^{\perp}\mathbf{A}\mathbf{P}_{\mathbf{Q}}^{\perp}+\mathbf{B}$. For any integrable function ψ, $\mathbb{E}\left[\psi(\mathbf{A}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}})\right]=\mathbb{E}\left[\psi(\mathbf{P}_{\mathbf{M}}^{\perp}\mathbf{A}\mathbf{P}_{\mathbf{Q}}^{\perp}+\mathbf{B})\right]=\mathbb{E}\left[\psi(\mathbf{P}_{\mathbf{M}}^{\perp}\widetilde{\mathbf{A}}\mathbf{P}_{\mathbf{Q}}^{\perp}+\mathbf{B})\right]=\mathbb{E}\left[\psi(\widetilde{\mathbf{A}}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}})\right]$, where the second equality follows from the fact that $\mathbf{A}\overset{d}{=}\widetilde{\mathbf{A}}$. Hence, $\mathbf{A}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}}\overset{d}{=}\mathbf{P}_{\mathbf{M}}^{\perp}\widetilde{\mathbf{A}}\mathbf{P}_{\mathbf{Q}}^{\perp}+\mathbf{B}$, which completes the proof.

APPENDIX D
PROOF OF PROPOSITION 4

Denoting $B=\sqrt{n}A(n)\sim\mu$ yields $\mathbb{E}\big[A^{2+2\alpha}(n)\big]=\mathbb{E}\big[B^{2(1+\alpha)}\big]n^{-(1+\alpha)}=o(n^{-2})$, where the last step uses the facts that $\mathbb{E}\big[B^{2(1+\alpha)}\big]$ is independent of n and $\alpha>1$.

APPENDIX E
PROOF OF PROPOSITION 5

Denoting $X_{N,ij}=A_{ij}(N)v_j(N)$, then $\{X_{N,ij}\colon 1\le j\le N\}$ is an independent zero-mean triangular array, for $i=1,2,\dots,n$. We claim that $\{X_{N,ij}\colon 1\le j\le N\}$ satisfies two conditions in Proposition 2, $\forall i$. First, we note that $\sum_{j=1}^{N}\mathbb{E}[X_{N,ij}^2]=\sum_{j=1}^{N}\mathbb{E}[A_{ij}^2(N)]v_j^2(N)=\frac{1}{n}\sum_{j=1}^{N}v_j^2(N)=\frac{1}{\rho}\langle\mathbf{v}^2(N)\rangle\to\frac{s_0^2}{\rho}$ as $n\to\infty$. Second, applying Proposition 4 to $A_{ij}(N)$ gives $\mathbb{E}[A_{ij}^{2+2\alpha}(N)]=o(n^{-2})$, leading to $\mathbb{E}[X_{N,ij}^{2+2\alpha}]=\mathbb{E}[A_{ij}^{2+2\alpha}(N)]|v_j(N)|^{2+2\alpha}\le o(n^{-2})\frac{n}{\rho}\frac{1}{N}\|\mathbf{v}(N)\|_{2+2\alpha}^{2+2\alpha}=o(n^{-1})$, where the last equality holds because $\limsup_{N\to\infty}\frac{1}{N}\|\mathbf{v}(N)\|_{2+2\alpha}^{2+2\alpha}<\infty$ and ρ is a constant. Applying Proposition 2 to $\{X_{N,ij}:1\le j\le N\}$ leads to $[\mathbf{A}(N)\mathbf{v}(N)]_i=\sum_{j=1}^{N}X_{N,ij}\overset{d}{\Rightarrow}\mathcal{N}\left(0,\frac{s_0^2}{\rho}\right)$ as $N\to\infty$, $\forall i$. Hence, $\widehat{\mathbf{A}(N)\mathbf{v}(N)}\overset{d}{\Rightarrow}\mathcal{N}\left(0,\frac{s_0^2}{\rho}\right)$ as $N\to\infty$.

APPENDIX F
WELL-KNOWN LEMMAS

***Lemma* 2.** *(Stein's Lemma [38]) For jointly zero-mean Gaussian random variables Z_1 and Z_2, and any function $\phi:\mathbb{R}\to\mathbb{R}$, where $\mathbb{E}[\phi'(Z_2)]$ and $\mathbb{E}[Z_1\phi(Z_2)]$ exist, the following holds $\mathbb{E}[Z_1\phi(Z_2)]=\mathrm{Cov}(Z_1,Z_2)\mathbb{E}[\phi'(Z_2)]$, where $\mathrm{Cov}(Z_1,Z_2)$ is the covariance between Z_1 and Z_2.*

***Lemma* 3.** *(Strong Law of Large Number [39]) Let $\{X_{n,m}:1\le m\le n\}$ be a triangular array of random variables with $(X_{n,1},X_{n,2},\dots,X_{n,n})$ mutually independent with zero-mean for each n and $\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}[|X_{n,m}|^{2+\kappa}]\le cn^{\kappa/2}$ for some $0<\kappa<1$ and $c<\infty$. Then $\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}X_{n,m}\overset{a.s.}{=}0$.*

***Lemma* 4.** *(Holder's inequality [40]) For random variables X and Y, $\mathbb{E}[|X+Y|^r]\le c_r(\mathbb{E}[|X|^r]+\mathbb{E}[|Y|^r])$, where $c_r=1$ if $0<r\le 1$ and $c_r=2^{r-1}$ otherwise. In particular, the inequality becomes $\mathbb{E}[|X+y|^r]\le c_r(\mathbb{E}[|X|^r]+|y|^r])$ when $Y=y$ being a constant.*

***Lemma* 5.** *(Lyapunov's inequality [40]) Suppose a random variable X and a constant κ with $0<\kappa<1$, then $|\mathbb{E}[X]|^{2+\kappa}\le\mathbb{E}[|X|^{2+\kappa}]$.*

***Proposition* 7.** *Suppose the $\mathbf{P}_{\mathbf{M}(n)}=\left(\frac{1}{\sqrt{n}}\mathbf{V}(n)\right)\left(\frac{1}{\sqrt{n}}\mathbf{V}(n)\right)^*$, where $\mathbf{M}(n)\in\mathbb{R}^{n\times t}$ $(t\le n)$, t is a fixed constant, and $\mathbf{V}(n)=[\mathbf{v}_1(n),\mathbf{v}_2(n),\dots,\mathbf{v}_t(n)]\in\mathbb{R}^{n\times t}$ is an orthogonal basis of $\mathbf{M}(n)$ such that $\mathbf{V}^*(n)\mathbf{V}(n)=n\mathbf{I}$. If we let $\mathbf{a}(n)\in\mathbb{R}^{n\times 1}$ be a random vector with independent entries, which have zero mean and finite variance σ_a^2, then $\lim_{n\to\infty}\mathbf{P}_{\mathbf{M}}(n)\mathbf{a}(n)\overset{a.s.}{=}\mathbf{0}_n$, where $\mathbf{0}_n$ is the $n\times 1$ all-zero vector.*

Proof. Denoting $\widetilde{\mathbf{a}}(n)=\frac{\mathbf{a}(n)}{\|\mathbf{a}(n)\|_2}$ yields $\mathbf{P}_{\mathbf{M}(n)}\mathbf{a}(n)=\mathbf{V}(n)\frac{\|\mathbf{a}(n)\|_2}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\mathbf{V}^*(n)\right)\widetilde{\mathbf{a}}(n)$. The proposition follows from the fact that $\frac{\|\mathbf{a}(n)\|_2}{\sqrt{n}}\overset{a.s.}{=}\sigma_a$ and $\left(\frac{1}{\sqrt{n}}\mathbf{V}^*(n)\right)\widetilde{\mathbf{a}}(n)\overset{a.s.}{=}\mathbf{0}_n$ as $n\to\infty$. ☐

APPENDIX G
PROOF OF SE IN (7) THEOREM 1

Substituting $\phi_b(\mathbf{u}_i^t)=(b_i^t)^2$ into (16) gives $\lim_{n\to\infty}\langle\mathbf{b}^t,\mathbf{b}^t\rangle\overset{a.s}{=}\mathbb{E}\left[\sigma_t^2\widetilde{Z}_t^2\right]=\sigma_t^2$. Using (14) with $t_1=t_2=t$ yields $\lim_{N\to\infty}\frac{1}{\rho}\langle\mathbf{q}^t,\mathbf{q}^t\rangle=\sigma_t^2$. Then substituting $\phi_h(\mathbf{v}_i^{t-1})=f_t^2(h_i^t,x_{0i})=(q_i^t)^2$ into (17) leads to $\lim_{N\to\infty}\langle\mathbf{q}^t,\mathbf{q}^t\rangle\overset{a.s}{=}\mathbb{E}\left[f_t^2(\tau_{t-1}Z_{t-1},X_0)\right]$, resulting in $\sigma_t^2=\frac{1}{\rho}\mathbb{E}\left[f_t^2(\tau_{t-1}Z_{t-1},X_0)\right]$. Showing the rest half $\tau_t^2=\mathbb{E}\left[g_t^2(\sigma_tZ,W)\right]$ of the SE in (7) follows from the exactly same procedure as the above. Setting $\phi_h(\mathbf{v}_i^t)=(h_i^{t+1})^2$ in (17) gives $\lim_{N\to\infty}\langle\mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle=\tau_t^2$. Using (15) with $t_1=t_2=t$ yields $\lim_{N\to\infty}\langle\mathbf{m}^t,\mathbf{m}^t\rangle=\tau_t^2$. Applying (16) to $\phi_b(\mathbf{u}_i^t)=g_t^2(b_i^t,w_i)$ yields $\lim_{N\to\infty}\langle\mathbf{m}^t,\mathbf{m}^t\rangle=\mathbb{E}\left[g_t^2(\sigma_tZ,W)\right]$. Therefore, $\tau_t^2=\mathbb{E}\left[g_t^2(\sigma_tZ,W)\right]$, concluding the proof.

APPENDIX H
PROOF OF THEOREM 1: STEPS 3 AND 4

A. Step 3: We show a), b), c), and d) of Theorem 1 conditioning on $\mathcal{F}_{t,t}=\{\mathbf{b}^0,\dots,\mathbf{b}^{t-1},\mathbf{m}^0,\dots,\mathbf{m}^{t-1},\mathbf{h}^1,\dots,\mathbf{h}^t,\mathbf{q}^0,\dots,\mathbf{q}^t,\mathbf{x}_0,\mathbf{w}\}$.

a) Note that conditioning on $\mathcal{F}_{t,t}$ is equivalent to conditioning on $\mathcal{P}_{\mathbf{X}_t,\mathbf{Y}_t}=\{\mathbf{A}|\mathbf{A}^*\mathbf{M}_t=\mathbf{X}_t,\mathbf{A}\mathbf{Q}_t=\mathbf{Y}_t\}$. Applying Proposition 3 to obtain the conditional distribution $\mathbf{A}|_{\mathcal{F}_{t,t}}$

and following the same procedure as in [26, Lemma 1a], the conditional distribution of $\mathbf{b}^t$ on $\mathcal{F}_{t,t}$ is expressed as

$$\mathbf{b}^t|_{\mathcal{F}_{t,t}} \stackrel{d}{=} \sum_{j=0}^{t-1} \beta_j \mathbf{b}^j + \widetilde{\mathbf{A}}\mathbf{q}_\perp^t - \mathbf{P}_{\mathbf{M}_t}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t + \mathbf{M}_t \vec{\mathbf{o}}_t(1). \quad (29)$$

By Proposition 7 in Appendix F, the third term on the right-hand side of (29) converges to $\lim_{n\to\infty} \mathbf{P}_{\mathbf{M}_t}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t \stackrel{a.s.}{=} \mathbf{0}_n$. Similar to part a) of Step 2, we verify the convergence $\lim_{n\to\infty} \mathbf{M}_t \vec{\mathbf{o}}_t(1) \stackrel{a.s.}{=} \mathbf{0}_n$ by characterizing the expectation and variance of its empirical distribution $\widehat{\mathbf{M}_t \vec{\mathbf{o}}_t(1)}$ as $n \to \infty$. Indeed, $\lim_{n\to\infty} |\langle \mathbf{M}_t \vec{\mathbf{o}}_t(1)\rangle| \le \lim_{n\to\infty} |o(1)| \frac{1}{n}\sum_{i=1}^n \left|\sum_{j=0}^{t-1} m_i^j\right|$ $\le \lim_{n\to\infty} |o(1)| \sum_{j=0}^{t-1} \frac{1}{n}\sum_{i=1}^n |m_i^j| \stackrel{a.s.}{=} 0$, where the last equality holds because applying $\phi_b(b_i^j, w_i) = g_j(b_i^j, w_i)$ to the induction hypothesis of (16), for $j < t$, leads to $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |m_i^j| \stackrel{a.s.}{=} \mathbb{E}[|g_j(\sigma_j \widetilde{Z}_j, W)|] < \infty$. Hence, $\lim_{n\to\infty} \langle \mathbf{M}_t \vec{\mathbf{o}}_t(1)\rangle \stackrel{a.s.}{=} 0$. For the variance of $\widehat{\mathbf{M}_t \vec{\mathbf{o}}_t(1)}$, we get

$$\begin{aligned}\lim_{n\to\infty} \langle \mathbf{M}_t \vec{\mathbf{o}}_t(1)\rangle_2 &= \lim_{n\to\infty} \frac{1}{n}[o(1)]^2 \sum_{i=1}^n \left(\sum_{j=0}^{t-1} m_i^j\right)^2, \\ &\le \lim_{n\to\infty} [o(1)]^2 t \frac{1}{n}\sum_{i=1}^n \sum_{j=0}^{t-1} (m_i^j)^2, \quad (30a) \\ &= \lim_{n\to\infty} [o(1)]^2 t \sum_{j=0}^{t-1} \langle \mathbf{m}^j, \mathbf{m}^j\rangle \stackrel{a.s.}{=} 0, \quad (30b)\end{aligned}$$

where (30a) follows from the Cauchy-Schwarz inequality and (30b) holds because $\lim_{n\to\infty} \langle \mathbf{m}^j, \mathbf{m}^j\rangle \stackrel{a.s.}{=} \mathbb{E}[g_j^2(\sigma_j \widetilde{Z}_j, W)] < \infty$, $\forall j$, which is a direct consequence of the induction hypothesis of (16) with $\phi_b(b_i^j, w_i) = g_j^2(b_i^j, w_i)$. Hence, (30b) is equivalent to $\lim_{n\to\infty} \langle \mathbf{M}_t \vec{\mathbf{o}}_t(1)\rangle_2 \stackrel{a.s.}{=} 0$. Therefore, $\lim_{n\to\infty} \mathbf{M}_t \vec{\mathbf{o}}_t(1) \stackrel{a.s.}{=} \mathbf{0}_n$, implying

$$\mathbf{b}^t|_{\mathcal{F}_{t,t}} \stackrel{d}{\Rightarrow} \sum_{j=0}^{t-1} \beta_j \mathbf{b}^j + \widetilde{\mathbf{A}}\mathbf{q}_\perp^t. \quad (31)$$

b) Note that by the induction hypothesis of (17) for $\phi_h(h_i^t, x_{0i}) = f_t^{1+\alpha}(h_i^t, x_{0i})$, we get $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N |q_i^t|^{2+2\alpha} \stackrel{a.s.}{=} \mathbb{E}[f_t^{2+2\alpha}(\tau_{t-1} Z_{t-1}, X_0)] < \infty$. On the other hand, $\sum_{i=1}^N |q_{\perp i}^t|^{2+2\alpha} < \sum_{i=1}^N |q_i^t|^{2+2\alpha}$. Thus, we have $\limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^N |q_{\perp i}^t|^{2+2\alpha} < \infty$, which concludes (12).

c) For $t_1 < t$ and $t_2 = t$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t\rangle &\stackrel{d}{=} \lim_{n\to\infty} \sum_{j=0}^{t-1} \beta_j \langle \mathbf{b}^{t_1}, \mathbf{b}^j\rangle + \lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle, \quad (32a) \\ &\stackrel{a.s.}{=} \sum_{j=0}^{t-1} \beta_j \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^j\rangle}{\rho} + \lim_{n\to\infty} \frac{\mathbf{b}^{t_1 *}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}{n}, \quad (32b)\end{aligned}$$

where (32a) follows from (31) and (32b) results from the induction hypothesis (14) for $t_1 < t$ and $t_2 = j < t$. Now, using Proposition 6, we get $\frac{\mathbf{b}^{t_1 *}}{\|\mathbf{b}^{t_1}\|_2}\widetilde{\mathbf{A}}\frac{\mathbf{q}_\perp^t}{\|\mathbf{q}_\perp^t\|_2} \stackrel{d}{=} \frac{Z}{\sqrt{n}}$, where $Z \sim \mathcal{N}(0,1)$. Hence, $\frac{\mathbf{b}^{t_1 *}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}{n} \stackrel{d}{=} \frac{\|\mathbf{b}^{t_1}\|_2}{\sqrt{n}} \frac{\|\mathbf{q}_\perp^t\|_2}{\sqrt{N}} \frac{1}{\sqrt{\rho}} \frac{Z}{\sqrt{n}}$, i.e.,

$$\frac{\mathbf{b}^{t_1 *}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}{n} \stackrel{d}{=} \frac{1}{\sqrt{\rho}} \sqrt{\langle \mathbf{b}^{t_1}, \mathbf{b}^{t_1}\rangle \langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle} \frac{Z}{\sqrt{n}}. \quad (33)$$

By the induction hypothesis of (14), we have $\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^{t_1}\rangle = \lim_{n\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^{t_1}\rangle}{\rho} < \infty$. Moreover, the $\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle$ converges to $\lim_{N\to\infty} \langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle < \lim_{N\to\infty} \langle \mathbf{q}^t, \mathbf{q}^t\rangle < \infty$ because using the induction hypothesis (17) we have $\langle \mathbf{q}^t, \mathbf{q}^t\rangle = \frac{1}{N}\sum_{i=1}^N f_t^2(h_i^t, x_{0i}) \stackrel{a.s.}{=} \mathbb{E}[f_t^2(\tau_{t-1} Z_{t-1}, X_0)] < \infty$. Thus, for $t_1 < t$,

$$\lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \mathbf{b}^{t_1}\rangle \stackrel{a.s.}{=} 0. \quad (34)$$

Substituting (34) into (32b) gives $\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t\rangle \stackrel{a.s.}{=} \sum_{j=0}^{t-1} \beta_j \lim_{n\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^j\rangle}{\rho} = \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_{||}^t\rangle}{\rho}$ due to (8), implying

$$\begin{aligned}\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t\rangle &\stackrel{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_{||}^t\rangle}{\rho} + \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_\perp^t\rangle}{\rho}, \quad (35a) \\ &= \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^t\rangle}{\rho}, \quad (35b)\end{aligned}$$

where (35a) follows from the fact that $\mathbf{q}^j$ is orthogonal to $\mathbf{q}_\perp^t$, for $j < t$, and (35b) holds due to (9), concluding (14) when $t_1 < t$ and $t_2 = t$.

For the case of $t_1 = t_2 = t$, it is similarly given by $\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle \stackrel{d}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{n\to\infty} \langle \mathbf{b}^i, \mathbf{b}^j\rangle + 2\sum_{i=0}^{t-1} \beta_i \lim_{n\to\infty} \langle \mathbf{b}^i, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle + \lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle$ due to (31). Then, by (34), the following holds

$$\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{n\to\infty} \langle \mathbf{b}^i, \mathbf{b}^j\rangle + \lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle. \quad (36)$$

Using Proposition 5, $\widehat{\widetilde{\mathbf{A}}\mathbf{q}_\perp^t} \stackrel{d}{\Rightarrow} \mathcal{N}\left(0, \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}\right)$. Thus, the second moment of $\widehat{\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}$ is

$$\lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle \stackrel{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}. \quad (37)$$

Now, incorporating (37) in (36), $\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{n\to\infty} \langle \mathbf{b}^i, \mathbf{b}^j\rangle + \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}$, resulting in

$$\begin{aligned}\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle &\stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i \beta_j \lim_{N\to\infty} \frac{\langle \mathbf{q}^i, \mathbf{q}^j\rangle}{\rho} + \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}, \quad (38a) \\ &\stackrel{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}_{||}^t, \mathbf{q}_{||}^t\rangle}{\rho} + \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}, \\ &\stackrel{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}^t, \mathbf{q}^t\rangle}{\rho},\end{aligned}$$

where (38a) is due to the induction hypothesis (14) for $0 \le t_1 = i, t_2 = j \le t-1$. This completes the proof of (14) at the tth iteration.

d) Defining $\lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t \rangle}{\rho} \overset{a.s.}{=} \gamma_t^2$, we can write by (37) that $\widehat{\widetilde{\mathbf{A}}\mathbf{q}_\perp^t} \overset{d}{\Rightarrow} \mathcal{N}(0, \gamma_t^2)$. Using (31) in conjunction with the latter, we get

$$b_i^t|_{\mathcal{F}_{t,t}} \overset{d}{\Rightarrow} \sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z, \text{ for } i = 1, 2, \ldots, n, \tag{39}$$

where $Z \sim \mathcal{N}(0, 1)$. Similar to Step 1d), using (39) $\mathbf{u}_i^t \overset{d}{\Rightarrow} \widetilde{\mathbf{u}}_i^t$, where $\mathbf{u}_i^t = (b_i^0 \ldots, b_i^t, w_i)$ and $\widetilde{\mathbf{u}}_i^t = (b_i^0, \ldots, b_i^{t-1}, \sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z, w_i)$, $\forall i$. To prove (16), we first claim that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \phi_b(\widetilde{\mathbf{u}}_i^t) - \mathbb{E}\Big[\phi_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_t \widetilde{Z}_t, W)\Big] \overset{a.s.}{=} 0$. By the triangular inequality, $\Big|\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \phi_b(\widetilde{\mathbf{u}}_i^t) - \mathbb{E}\Big[\phi_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_t \widetilde{Z}_t, W)\Big]\Big| \leq X_1^t + X_2^t$, where $X_1^t = \Big|\frac{1}{n} \sum_{i=1}^n \Big(\phi_b(\widetilde{\mathbf{u}}_i^t) - \widetilde{\phi}_b(\mathbf{u}_i^{t-1})\Big)\Big|$, $X_2^t = \Big|\frac{1}{n} \sum_{i=1}^n \widetilde{\phi}_b(\mathbf{u}_i^{t-1}) - \mathbb{E}[\phi_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_t \widetilde{Z}_t, W)]\Big|$, and $\widetilde{\phi}_b(\mathbf{u}_i^{t-1}) = \mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]$. Similar to Step 1d), we verify $\lim_{n\to\infty} X_1^t \overset{a.s.}{=} 0$ and $\lim_{n\to\infty} X_2^t \overset{a.s.}{=} 0$.

First showing $\lim_{n\to\infty} X_1^t \overset{a.s.}{=} 0$ is of interest. By (5), $|\phi_b(\widetilde{\mathbf{u}}_i^t)| \leq c_1^t \exp\Big[c_2^t \Big(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + \gamma_t^\lambda |Z|^\lambda + |w_i|^\lambda\Big)\Big]$, where $c_1^t > 0$, $c_2^t > 0$, and $1 \leq \lambda < 2$ are constant. Hence, $\mathbb{E}_Z[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}] \leq c_3^t \exp\Big[c_4^t \Big(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + |w_i|^\lambda\Big)\Big] \mathbb{E}_Z\left[\exp(c_4^t \gamma_t^\lambda |Z|^\lambda)\right]$, where $0 < \kappa < 1$, $c_3^t = (c_1^t)^{2+\kappa}$, $c_4^t = c_2^t(2+\kappa)$, resulting in

$$\mathbb{E}_Z[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}] \leq c_5^t \exp\left[c_4^t \left(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + |w_i|^\lambda\right)\right], \tag{40}$$

and $c_5^t = c_3^t \mathbb{E}_Z\left[\exp(c_4^t \gamma_t^\lambda |Z|^\lambda)\right]$ is constant. We define $X_{n,i}^t = \phi_b(\widetilde{\mathbf{u}}_i^t) - \widetilde{\phi}_b(\mathbf{u}_i^{t-1}) = \phi_b(\widetilde{\mathbf{u}}_i^t) - \mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]$ such that $X_1^t = |\frac{1}{n} \sum_{i=1}^n X_{n,i}^t|$. To prove $\lim_{n\to\infty} X_1^t \overset{a.s.}{=} 0$, we show that $\{X_{n,i}^t\}_{i=1}^n$ satisfy Lemma 3 in Appendix F. Indeed, $\mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}]$ is upper bouned as follows,

$$\begin{aligned}
\mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}] &\leq 2^{1+\kappa}\Big[\mathbb{E}_Z\left[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}\right] + |\mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]|^{2+\kappa}\Big], && \text{(41a)}\\
&\leq 2^{1+\kappa}\left[\mathbb{E}_Z\big[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}\big] + \mathbb{E}_Z\big[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}\big]\right], && \text{(41b)}\\
&\leq 2^{2+\kappa}\left[\mathbb{E}_Z\left[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}\right]\right],\\
&\leq c_6^t \exp\left[c_4^t \left(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + |w_i|^\lambda\right)\right], && \text{(41c)}
\end{aligned}$$

where (41a) follows from Lemma 4 (Holder's inequality) in Appendix F, (41b) follows from Lemma 5 (Lyapunov's inequality) in Appendix F, and (41c) follows from (40) with $c_6^t = 2^{2+\kappa} c_5^t$. We denote the last term of (41c) as $\psi_b(\mathbf{u}_i^{t-1}) = c_6^t \exp\Big[c_4^t \Big(\sum_{j=0}^{t-1} |\beta_j b_i^j|^\lambda + |w_i|^\lambda\Big)\Big]$. Then, $\psi_b(\mathbf{u}_i^{t-1})$ is a controlled function. From (41c), we get, for n is sufficiently large,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n \mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}] &\leq \frac{1}{n}\sum_{i=1}^n \psi_b(\mathbf{u}_i^{t-1}),\\
&\overset{a.s.}{=} \mathbb{E}[\psi_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_{t-1} \widetilde{Z}_{t-1}, W)], && \text{(42a)}\\
&< cn^{\kappa/2}, && \text{(42b)}
\end{aligned}$$

where c is a positive constant, (42a) is due to the induction hypothesis (16), and (42b) holds because $\mathbb{E}[\psi_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_{t-1} \widetilde{Z}_{t-1}, W)] = c_7^t < \infty$ and there exists n_t, a positive constant, such that $c_7^t < cn^{\kappa/2}$ for $n > n_t$. By Lemma 3 in Appendix F, we get $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_{n,i}^t \overset{a.s.}{=} 0$, implying

$$\frac{1}{n}\sum_{i=1}^n \Big(\phi_b(\widetilde{\mathbf{u}}_i^t) - \widetilde{\phi}_b(\mathbf{u}_i^{t-1})\Big) \overset{a.s.}{=} 0, \tag{43}$$

which proving $\lim_{n\to\infty} X_1^t \overset{a.s.}{=} 0$.

Now, showing $\lim_{n\to\infty} X_2^t \overset{a.s.}{=} 0$ is of interest. By the induction hypothesis (16), $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \widetilde{\phi}_b(\mathbf{u}_i^{t-1}) \overset{a.s.}{=} \mathbb{E}[\widetilde{\phi}_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_{t-1} \widetilde{Z}_{t-1}, W)]$, resulting in

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \widetilde{\phi}_b(\mathbf{u}_i^{t-1})\\
&= \mathbb{E}\left[\mathbb{E}_Z\left[\phi_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_{t-1} \widetilde{Z}_{t-1}, \sum_{j=0}^{t-1} \beta_j \sigma_j \widetilde{Z}_j + \gamma_t Z, W)\right]\right],\\
&= \mathbb{E}\left[\phi_b(\sigma_0 \widetilde{Z}_0, \ldots, \sigma_{t-1} \widetilde{Z}_{t-1}, \sum_{j=0}^{t-1} \beta_j \sigma_j \widetilde{Z}_j + \gamma_t Z, W)\right],
\end{aligned} \tag{44}$$

where (44) follows from the substitution $\widetilde{\phi}_b(\mathbf{u}_i^{t-1}) = \mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]$. Therefore, showing $\lim_{n\to\infty} X_2^t \overset{a.s.}{=} 0$ is equivalent to proving $\sum_{j=0}^{t-1} \beta_j \sigma_j \widetilde{Z}_j + \gamma_t Z = \sigma_t \widetilde{Z}_t$, where $\widetilde{Z}_t \sim \mathcal{N}(0, 1)$ and σ_t is defined in (7).

In particular, for $\phi_b(\mathbf{u}_i^t) = (b_i^t)^2$, we get $\phi_b(\widetilde{\mathbf{u}}_i^t) = \Big(\sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z\Big)^2$ because $\widetilde{\mathbf{u}}_i^t = (b_i^0, \ldots, b_i^{t-1}, \sum_{j=0}^{t-1} \beta_j b_i^j + \gamma_t Z, w_i)$. Combining (43) and (44),

$$\begin{aligned}
&\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \phi_b(\mathbf{u}_i^t)\\
&\overset{a.s.}{=} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \phi_b(\widetilde{\mathbf{u}}_i^t) \overset{a.s.}{=} \mathbb{E}\left[\left(\sum_{j=0}^{t-1} \beta_j \sigma_j \widetilde{Z}_j + \gamma_t Z\right)^2\right].
\end{aligned} \tag{45}$$

Using (14), $\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t \rangle \overset{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}^t, \mathbf{q}^t \rangle}{\rho} = \sigma_t^2$, where the last equality holds because by the induction hypothesis (17) for $\phi_h(\mathbf{v}_i^{t-1}) = f_t^2(h_i^t, x_{0i})$ in (17), $\frac{1}{\rho} \lim_{N\to\infty} \langle \mathbf{q}^t, \mathbf{q}^t \rangle \overset{a.s.}{=} \frac{1}{\rho} \mathbb{E}[f_t^2(\tau_{t-1} Z, X_0)] = \sigma_t^2$. Hence, $\mathbb{E}\left[\Big(\sum_{j=0}^{t-1} \beta_j \sigma_j \widetilde{Z}_j + \gamma_t Z\Big)^2\right] \overset{a.s.}{=} \sigma_t^2$, implying $\sum_{j=0}^{t-1} \beta_j \sigma_j \widetilde{Z}_j + \gamma_t Z = \sigma_t \widetilde{Z}_t$ due to (45), verifying that $\lim_{n\to\infty} X_2^t \overset{a.s.}{=} 0$, which completes the proof of (16).

B. Step 4: We show a), b), c), and d) of Theorem 1 conditioning on $\mathcal{F}_{t+1,t} = \{\mathbf{b}^0, \ldots, \mathbf{b}^t, \mathbf{m}^0, \ldots, \mathbf{m}^t, \mathbf{h}^1, \ldots, \mathbf{h}^t, \mathbf{q}^0, \ldots, \mathbf{q}^t, \mathbf{x}_0, \mathbf{w}\}$.

The proof of Step 4 is similar to the proof of Step 3. Thus, we only present the features that are unique in Step 4.

a) Similar to Step 3a), using Proposition 3 to characterize $\mathbf{A}|_{\mathcal{F}_{t+1,t}}$ and following the same procedure as in [26, Lemma 1a], the $\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}}$ is

$$\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{=} \sum_{j=0}^{t-1} \alpha_j \mathbf{h}^{j+1} + \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t - \mathbf{P}_{\mathbf{Q}_{t+1}} \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t + \mathbf{Q}_{t+1} \overrightarrow{\mathbf{o}}_t(1). \tag{46}$$

By Proposition 7 in Appendix F, $\lim_{N\to\infty} \mathbf{P}_{\mathbf{Q}_{t+1}} \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t \stackrel{a.s.}{=} \mathbf{0}_N$. Similar to Step 3a), we verify that $\lim_{N\to\infty} \mathbf{Q}_{t+1} \overrightarrow{\mathbf{o}}_t(1) \stackrel{a.s.}{=} \mathbf{0}_N$ by characterizing (i) the expectation of the empirical distribution $\widehat{\mathbf{Q}_{t+1} \overrightarrow{\mathbf{o}}_t(1)}$ is bounded as $\lim_{N\to\infty} |\langle \mathbf{Q}_{t+1} \overrightarrow{\mathbf{o}}_t(1)\rangle| \leq \lim_{N\to\infty} |o(1)| \sum_{j=0}^{t} \frac{1}{N} \sum_{i=1}^{N} |q_i^j| \stackrel{a.s.}{=} 0$ and (ii) the empirical variance of $\widehat{\mathbf{Q}_{t+1} \overrightarrow{\mathbf{o}}_t(1)}$ is bounded an converges to $\lim_{N\to\infty} \langle \mathbf{Q}_{t+1} \overrightarrow{\mathbf{o}}_t(1)\rangle_2 \leq \lim_{N\to\infty} [o(1)]^2 (t+1) \sum_{j=0}^{t} \langle \mathbf{q}^j, \mathbf{q}^j\rangle \stackrel{a.s.}{=} 0$. Therefore, using $\lim_{N\to\infty} \mathbf{Q}_{t+1} \overrightarrow{\mathbf{o}}_t(1) \stackrel{a.s.}{=} \mathbf{0}_N$, we get

$$\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{\Rightarrow} \sum_{j=0}^{t-1} \alpha_j \mathbf{h}^{j+1} + \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \tag{47}$$

which completes the proof of (11).

b) Using (16) for $\phi_b(b_i^t, w_i) = g_t^{1+\alpha}(b_i^t, w_i)$, we get $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |m_i^t|^{2+2\alpha} \stackrel{a.s.}{=} \mathbb{E}[g_t^{2+2\alpha}(\sigma_t \widetilde{Z}_t, W)] < \infty$. Because $\sum_{i=1}^{n} |m_{\perp i}^t|^{2+2\alpha} < \sum_{i=1}^{n} |m_i^t|^{2+2\alpha}$, the following holds $\limsup_{n\to\infty} \sum_{i=1}^{n} |m_{\perp i}^t|^{2+2\alpha} < \infty$, which concludes (13).

c) For $t_1 < t$ and $t_2 = t$, we have $\lim_{N\to\infty} \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t+1}\rangle \stackrel{d}{=} \lim_{N\to\infty} \sum_{j=0}^{t-1} \alpha_j \langle \mathbf{h}^{t_1+1}, \mathbf{h}^j\rangle + \lim_{N\to\infty} \langle \mathbf{h}^{t_1+1}, \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t\rangle$ due to (47), resulting in

$$\lim_{N\to\infty} \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t+1}\rangle \stackrel{a.s.}{=} \sum_{j=0}^{t-1} \alpha_j \lim_{n\to\infty} \langle \mathbf{m}^{t_1}, \mathbf{m}^j\rangle + \lim_{N\to\infty} \frac{\mathbf{m}_\perp^{t^*} \widetilde{\mathbf{A}} \mathbf{h}^{t_1+1}}{N}, \tag{48}$$

where (48) is by the induction hypothesis (15). Note that $\frac{\mathbf{m}_\perp^{t^*}}{\|\mathbf{m}_\perp^t\|_2} \widetilde{\mathbf{A}} \frac{\mathbf{h}^{t_1+1}}{\|\mathbf{h}^{t_1+1}\|_2} \stackrel{d}{=} \frac{Z}{\sqrt{n}}$ due to Proposition 6. The second term in (48) is represented as

$$\begin{aligned} \lim_{N\to\infty} \frac{\mathbf{m}_\perp^{t^*} \widetilde{\mathbf{A}} \mathbf{h}^{t_1+1}}{N} &\stackrel{d}{=} \lim_{N\to\infty} \frac{\|\mathbf{m}_\perp^t\|_2}{\sqrt{n}} \frac{\|\mathbf{h}^{t_1+1}\|_2}{\sqrt{N}} \frac{\sqrt{n}}{\sqrt{N}} \frac{Z}{\sqrt{n}}, \\ &= \sqrt{\rho} \lim_{N\to\infty} \sqrt{\langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t\rangle \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t_1+1}\rangle} \frac{Z}{\sqrt{n}} \stackrel{a.s.}{=} 0, \end{aligned} \tag{49}$$

Substituting (49) into (48) yields

$$\begin{aligned} \lim_{N\to\infty} \langle \mathbf{h}^{t_1+1}, \mathbf{h}^{t+1}\rangle &= \lim_{n\to\infty} \langle \mathbf{m}^{t_1}, \sum_{j=0}^{t-1} \alpha_j \mathbf{m}^j\rangle = \lim_{n\to\infty} \langle \mathbf{m}^{t_1}, \mathbf{m}_{||}^t\rangle, \\ &= \lim_{n\to\infty} \langle \mathbf{m}^{t_1}, \mathbf{m}_{||}^t\rangle + \lim_{n\to\infty} \langle \mathbf{m}^{t_1}, \mathbf{m}_\perp^t\rangle = \lim_{n\to\infty} \langle \mathbf{m}^{t_1}, \mathbf{m}^t\rangle, \end{aligned}$$

concluding (15) when $t_1 < t$ and $t_2 = t$.

For $t_1 = t_2 = t$, by (47),

$$\begin{aligned} \lim_{N\to\infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1}\rangle &\stackrel{d}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{N\to\infty} \langle \mathbf{h}^{i+1}, \mathbf{h}^{j+1}\rangle \\ &+ 2\sum_{i=0}^{t-1} \alpha_i \lim_{N\to\infty} \langle \mathbf{h}^{i+1}, \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t\rangle + \lim_{N\to\infty} \langle \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t\rangle. \end{aligned} \tag{50}$$

Then, by (49), the following holds

$$\begin{aligned} \lim_{N\to\infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1}\rangle &\stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{N\to\infty} \langle \mathbf{h}^{i+1}, \mathbf{h}^{j+1}\rangle + \\ &\lim_{N\to\infty} \langle \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t\rangle. \end{aligned} \tag{51}$$

By Proposition 5, the empirical distribution of $\widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t$ converges to $\widehat{\widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t} \stackrel{d}{\Rightarrow} \mathcal{N}(0, \lim_{n\to\infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t\rangle)$. Hence, the second moment of $\widehat{\widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t}$ converges to

$$\lim_{N\to\infty} \langle \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t, \widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t\rangle \stackrel{a.s.}{=} \lim_{n\to\infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t\rangle. \tag{52}$$

Substituting (52) into (51) leads to $\lim_{N\to\infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1}\rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{N\to\infty} \langle \mathbf{h}^{i+1}, \mathbf{h}^{j+1}\rangle + \lim_{n\to\infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t\rangle$, implying

$$\begin{aligned} \lim_{N\to\infty} \langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1}\rangle &\stackrel{a.s.}{=} \sum_{i,j=0}^{t-1} \alpha_i \alpha_j \lim_{n\to\infty} \langle \mathbf{m}^i, \mathbf{m}^j\rangle + \lim_{n\to\infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t\rangle, \\ &= \lim_{n\to\infty} \langle \mathbf{m}_{||}^t, \mathbf{m}_{||}^t\rangle + \lim_{n\to\infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t\rangle = \lim_{n\to\infty} \langle \mathbf{m}^t, \mathbf{m}^t\rangle. \end{aligned}$$

Therefore, (15) also holds for $t_1 = t_2 = t$, which completes the proof.

d) Defining $\lim_{n\to\infty} \langle \mathbf{m}_\perp^t, \mathbf{m}_\perp^t\rangle \stackrel{a.s.}{=} \Gamma_t^2$, we can write

$$\widehat{\widetilde{\mathbf{A}}^* \mathbf{m}_\perp^t} \stackrel{d}{\Rightarrow} \mathcal{N}(0, \Gamma_t^2). \tag{53}$$

Using (53) and (47), the following convergence holds

$$h_i^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{\Rightarrow} \sum_{j=0}^{t-1} \alpha_j h_i^{j+1} + \Gamma_t Z, \tag{54}$$

where $Z \sim \mathcal{N}(0,1)$. Similar to Step 3d), we can write, using (54), $\mathbf{v}_i^t \stackrel{d}{\Rightarrow} \widetilde{\mathbf{v}}_i^t$, where $\mathbf{v}_i^t = (h_i^1, .., h_i^{t+1}, x_{0i})$ and $\widetilde{\mathbf{v}}_i^t = (h_i^1, \ldots, h_i^t, \sum_{j=0}^{t-1} \alpha_j h_i^{j+1} + \Gamma_t Z, x_{0i})$. Hence, to prove (17) we first claim that $\left| \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \phi_h(\widetilde{\mathbf{v}}_i^t) - \mathbb{E}\left[\phi_h(\tau_0 Z_0, \ldots, \tau_t Z_t, X_0)\right]\right| \stackrel{a.s.}{=} 0$. Similar to Step 2d), using the triangular inequality, we verify that $\lim_{N\to\infty} Y_1^t \stackrel{a.s.}{=} 0$ and $\lim_{N\to\infty} Y_2^t \stackrel{a.s.}{=} 0$, where $Y_1^t = \left|\frac{1}{N} \sum_{i=1}^{N} \left(\phi_h(\widetilde{\mathbf{v}}_i^t) - \widetilde{\phi}_h(\mathbf{v}_i^{t-1})]\right)\right|$ and $Y_2^t = \left|\frac{1}{N} \sum_{i=1}^{N} \widetilde{\phi}_h(\mathbf{v}_i^{t-1}) - \mathbb{E}\left[\phi_h(\tau_0 Z_0, \ldots, \tau_t Z_t, X_0)\right]\right|$, and $\widetilde{\phi}_h(\mathbf{v}_i^{t-1}) = \mathbb{E}_Z[\phi_h(\widetilde{\mathbf{v}}_i^t)]$, $\forall i$.

First showing $\lim_{N\to\infty} Y_1^t$ is of interest. By (5), $|\phi_h(\widetilde{\mathbf{v}}_i^t)| \leq d_1^t \exp\left[d_2^t \left(\sum_{j=0}^{t-1} |\alpha_j h_i^{j+1}|^\lambda + \Gamma_t^\lambda |Z_i|^\lambda + |x_{0i}|^\lambda\right)\right]$, where $d_1^t > 0$, $d_2^t > 0$, and $1 \leq \lambda < 2$ are constant. Hence,

$$\mathbb{E}_Z\left[|\phi_b(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}\right] \leq d_5^t \exp\left[d_4^t \Big(\sum_{j=0}^{t-1} |\alpha_j h_i^{j+1}|^\lambda + |x_{0i}|^\lambda\Big)\right],$$

where $0 < \kappa < 1$, $d_4^t = d_2^t(2+\kappa)$, and $d_5^t = (d_1^t)^{2+\kappa} \mathbb{E}_Z\left[\exp(d_4^t \Gamma_t^\lambda |Z_i|^\lambda)\right]$ are constants. Define $Y_{N,i}^t = \phi_h(\widetilde{\mathbf{v}}_i^t) - \mathbb{E}_Z[\phi_h(\widetilde{\mathbf{v}}_i^t)]$, $\forall i$. To prove the convergence $\lim_{n\to\infty} Y_1^t \stackrel{a.s.}{=} 0$, we will show that $\{Y_{N,i}^t\}_{i=1}^N$ satisfy the condition in Lemma 3 in Appendix F. Indeed, the $\mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}]$ is upper bounded as follows.

$$\begin{aligned}
\mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}] &\le 2^{1+\kappa}\Big[\mathbb{E}_Z[|\phi_h(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}] + |\mathbb{E}_Z[\phi_h(\widetilde{\mathbf{v}}_i^t)]|^{2+\kappa}\Big], \\
&\le 2^{1+\kappa}\Big[\mathbb{E}_Z[|\phi_h(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}] + \mathbb{E}_Z[|\phi_h(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}]\Big], \\
&\le 2^{2+\kappa}\Big[\mathbb{E}_Z[|\phi_h(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}]\Big] \\
&\le d_6^t \exp\Big[d_4^t\Big(\sum_{j=0}^{t-1}|\alpha_j h_i^{j+1}|^{\lambda} + |x_{0i}|^{\lambda}\Big)\Big], \quad \text{(55a)}
\end{aligned}$$

Denoting the last term of (55a) as $\psi_h(\mathbf{v}_i^{t-1}) = d_6^t \exp\Big[d_4^t\Big(\sum_{j=0}^{t-1}|\alpha_j h_i^{j+1}|^{\lambda} + |x_{0i}|^{\lambda}\Big)\Big]$, then $\psi_h(\mathbf{v}_i^{t-1})$ is a controlled function. From (55a), we get, for N is sufficiently large,

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{n}\mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}] &\le \frac{1}{N}\sum_{i=1}^{n}\psi_h(\mathbf{v}_i^{t-1}), \\
&\overset{a.s.}{=} \mathbb{E}[\psi_h(\tau_0 Z_0,\ldots,\tau_{t-1}Z_{t-1},X_0)], \\
&< cN^{\kappa/2}, \quad \text{(56a)}
\end{aligned}$$

where c is a positive constant and (56a) holds because $\mathbb{E}[\psi_b(\sigma_0\widetilde{Z}_0,\ldots,\sigma_{t-1}\widetilde{Z}_{t-1},W)] = d_7^t < \infty$ and there exists N_t, a positive constant, such that $d_7^t < cN^{\kappa/2}$ for $N > N_t$. Using Lemma 3 in Appendix F, $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}Y_{N,i}^t \overset{a.s.}{=} 0$, implying $\lim_{N\to\infty}Y_1^t \overset{a.s.}{=} 0$.

We are now ready to verify the convergence $\lim_{N\to\infty}Y_2^t \overset{a.s.}{=} 0$. Applying the induction hypothesis (17) for $\widetilde{\phi}_b(\mathbf{v}_i^{t-1})$ gives

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\widetilde{\phi}_h(\mathbf{v}_i^{t-1}) \overset{a.s.}{=} \mathbb{E}[\widetilde{\phi}_h(\tau_0 Z_0,\ldots,\tau_{t-1}Z_{t-1},X_0)], \\
&= \mathbb{E}\Big[\mathbb{E}_Z[\phi_h(\tau_0 Z_0,\ldots,\tau_{t-1}Z_{t-1},\sum_{j=0}^{t-1}\alpha_j\tau_j Z_j + \Gamma_t Z, X_0)]\Big], \\
&= \mathbb{E}\Big[\phi_h(\tau_0 Z_0,\ldots,\tau_{t-1}Z_{t-1},\sum_{j=0}^{t-1}\alpha_j\tau_j Z_j + \Gamma_t Z, X_0)\Big].
\end{aligned}$$

Therefore, showing $\lim_{N\to\infty}Y_2^t \overset{a.s.}{=} 0$ is equivalent to proving $\sum_{j=0}^{t-1}\alpha_j\tau_j Z_j + \Gamma_t Z = \tau_t Z_t$, where $Z_t \sim \mathcal{N}(0,1)$ and τ_t is defined in (7). Similar to the proof of $\lim_{n\to\infty}X_2^t \overset{a.s.}{=} 0$ in Step 3d), setting $\phi_h(\mathbf{v}_i^t) = (h_i^t)^2$, i.e., $\phi_h(\widetilde{\mathbf{v}}_i^t) = \Big(\sum_{j=0}^{t-1}\alpha_j h_i^{j+1} + \Gamma_t Z\Big)^2$, we get

$$\begin{aligned}
&\lim_{N\to\infty}\langle\mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\phi_h(\mathbf{v}_i^t) \\
&\overset{a.s.}{=} \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\phi_h(\widetilde{\mathbf{v}}_i^t) \overset{a.s.}{=} \mathbb{E}\Big[\Big(\sum_{j=0}^{t-1}\alpha_j\tau_j Z_j + \Gamma_t Z\Big)^2\Big].
\end{aligned}$$

Using (15), we get $\lim_{N\to\infty}\langle\mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle \overset{a.s.}{=} \lim_{n\to\infty}\langle\mathbf{m}^t,\mathbf{m}^t\rangle = \tau_t^2$, where the last equality holds by the induction hypothesis (16) for $\phi_b(\mathbf{u}_i^t) = g_t^2(b_i^t, w_i)$, resulting in $\lim_{n\to\infty}\langle\mathbf{m}^t,\mathbf{m}^t\rangle \overset{a.s.}{=} \mathbb{E}[g_t^2(\sigma_t\widetilde{Z}_t, W)] = \tau_t^2$. Hence, $\mathbb{E}\Big[\Big(\sum_{j=0}^{t-1}\alpha_j\tau_j Z_j + \Gamma_t Z\Big)^2\Big] \overset{a.s.}{=} \tau_t^2$, which implies $\sum_{j=0}^{t-1}\alpha_j\tau_j Z_j + \Gamma_t Z = \tau_t Z_t$. Thus, it is verified that $\lim_{N\to\infty}Y_2^t \overset{a.s.}{=} 0$, which completes the proof of (17).