APPENDIX A
PSEUDO-LIPSCHITZ FUNCTION

***Definition* 1.** *For a $k > 1$, a function $f : \mathbb{R}^{n\times 1} \to \mathbb{R}$ is said pseudo-Lipschitz of order k if there exists a constant $L > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq L(1 + \|\mathbf{x}\|^{k-1} + \|\mathbf{y}\|^{k-1})\|\mathbf{x} - \mathbf{y}\|$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n\times 1}$; the first order derivative of f is bounded by a polynomial of order $(k-1)$, i.e., polynomial smoothness.*

APPENDIX B
PROOF OF PROPOSITION 1

This follows from $\sum_{m=1}^{n} \mathbb{E}\big[X_{n,m}^2; |X_{n,m}| > \epsilon\big] \leq \sum_{m=1}^{n} \mathbb{E}\Big[\frac{X_{n,m}^{2+2\alpha}}{\epsilon^{2\alpha}}\Big]$, where the right-hand side converges to zero because $\frac{n}{\epsilon^{2\alpha}} o(n^{-1}) \to 0$ as $n \to \infty$.

APPENDIX C
PROOF OF PROPOSITION 3

Recalling the following subspace decomposition $\mathbf{A} = \mathbf{P}_{\mathbf{M}}^{\perp}\mathbf{A}\mathbf{P}_{\mathbf{Q}}^{\perp} + \mathbf{A}\mathbf{P}_{\mathbf{Q}} + \mathbf{P}_{\mathbf{M}}\mathbf{A} - \mathbf{P}_{\mathbf{M}}\mathbf{A}\mathbf{P}_{\mathbf{Q}}$, the orthogonal projection of $\mathbf{A}$ onto $\mathcal{P}_{\mathbf{X},\mathbf{Y}}$ is given by $\mathbf{A}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}} = \mathbf{P}_{\mathbf{M}}^{\perp}\mathbf{A}\mathbf{P}_{\mathbf{Q}}^{\perp} + \mathbf{B}$. For any integrable function ψ, $\mathbb{E}\left[\psi(\mathbf{A}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}})\right] = \mathbb{E}\left[\psi(\mathbf{P}_{\mathbf{M}}^{\perp}\mathbf{A}\mathbf{P}_{\mathbf{Q}}^{\perp} + \mathbf{B})\right] = \mathbb{E}\left[\psi(\mathbf{P}_{\mathbf{M}}^{\perp}\widetilde{\mathbf{A}}\mathbf{P}_{\mathbf{Q}}^{\perp} + \mathbf{B})\right] = \mathbb{E}\left[\psi(\widetilde{\mathbf{A}}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}})\right]$, where the second equality follows from the fact that $\mathbf{A} \overset{d}{=} \widetilde{\mathbf{A}}$. Hence, $\mathbf{A}|_{\mathcal{P}_{\mathbf{X},\mathbf{Y}}} \overset{d}{=} \mathbf{P}_{\mathbf{M}}^{\perp}\widetilde{\mathbf{A}}\mathbf{P}_{\mathbf{Q}}^{\perp} + \mathbf{B}$, which completes the proof.

APPENDIX D
PROOF OF PROPOSITION 4

Denoting $B = \sqrt{n}A(n) \sim \mu$ yields $\mathbb{E}\big[A^{2+2\alpha}(n)\big] = \mathbb{E}\big[B^{2(1+\alpha)}\big]n^{-(1+\alpha)} = o(n^{-2})$, where the last step uses the facts that $\mathbb{E}\big[B^{2(1+\alpha)}\big]$ is independent of n and $\alpha > 1$.

APPENDIX E
PROOF OF PROPOSITION 5

Denoting $X_{N,ij} = A_{ij}(N)v_j(N)$, then $\{X_{N,ij}\colon 1 \leq j \leq N\}$ is an independent zero-mean triangular array, for $i = 1, 2, \ldots, n$. We claim that $\{X_{N,ij}\colon 1 \leq j \leq N\}$ satisfies two conditions in Proposition 2, $\forall i$. First, we note that $\sum_{j=1}^{N} \mathbb{E}[X_{N,ij}^2] = \sum_{j=1}^{N} \mathbb{E}[A_{ij}^2(N)]v_j^2(N) = \frac{1}{n}\sum_{j=1}^{N} v_j^2(N) = \frac{1}{\rho}\langle \mathbf{v}^2(N)\rangle \to \frac{s_0^2}{\rho}$ as $n \to \infty$. Second, applying Proposition 4 to $A_{ij}(N)$ gives $\mathbb{E}[A_{ij}^{2+2\alpha}(N)] = o(n^{-2})$, leading to $\mathbb{E}[X_{N,ij}^{2+2\alpha}] = \mathbb{E}[A_{ij}^{2+2\alpha}(N)]|v_j(N)|^{2+2\alpha} \leq o(n^{-2})\frac{n}{\rho}\frac{1}{N}\|\mathbf{v}(N)\|_{2+2\alpha}^{2+2\alpha} = o(n^{-1})$, where the last equality holds because $\limsup_{N\to\infty} \frac{1}{N}\|\mathbf{v}(N)\|_{2+2\alpha}^{2+2\alpha} < \infty$ and ρ is a constant. Applying Proposition 2 to $\{X_{N,ij} : 1 \leq j \leq N\}$ leads to $[\mathbf{A}(N)\mathbf{v}(N)]_i = \sum_{j=1}^{N} X_{N,ij} \overset{\mathrm{d}}{\Rightarrow} \mathcal{N}\left(0, \frac{s_0^2}{\rho}\right)$ as $N \to \infty$, $\forall i$. Hence, $\widehat{\mathbf{A}(N)\mathbf{v}}(N) \overset{d}{\Rightarrow} \mathcal{N}\left(0, \frac{s_0^2}{\rho}\right)$ as $N \to \infty$.

APPENDIX F
WELL-KNOWN LEMMAS

***Lemma* 2.** *(Stein's Lemma [39]) For jointly zero-mean Gaussian random variables Z_1 and Z_2, and any function $\phi : \mathbb{R} \to \mathbb{R}$, where $\mathbb{E}[\phi'(Z_2)]$ and $\mathbb{E}[Z_1\phi(Z_2)]$ exist, the following holds $\mathbb{E}[Z_1\phi(Z_2)] = \mathrm{Cov}(Z_1, Z_2)\mathbb{E}[\phi'(Z_2)]$, where $\mathrm{Cov}(Z_1, Z_2)$ is the covariance between Z_1 and Z_2.*

***Lemma* 3.** *(Strong Law of Large Number [40]) Let $\{X_{n,m} : 1 \leq m \leq n\}$ be a triangular array of random variables with $(X_{n,1}, X_{n,2}, \ldots, X_{n,n})$ mutually independent with zero-mean for each n and $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}[|X_{n,m}|^{2+\kappa}] \leq cn^{\kappa/2}$ for some $0 < \kappa < 1$ and $c < \infty$. Then $\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} X_{n,m} \overset{a.s.}{=} 0$.*

***Lemma* 4.** *(Holder's inequality [41]) For random variables X and Y, $\mathbb{E}[|X + Y|^r] \leq c_r(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r])$, where c_r = 1 if $0 < r \leq 1$ and $c_r = 2^{r-1}$ otherwise. In particular, the inequality becomes $\mathbb{E}[|X + y|^r] \leq c_r(\mathbb{E}[|X|^r] + |y|^r])$ when $Y = y$ being a constant.*

***Lemma* 5.** *(Lyapunov's inequality [41]) Suppose a random variable X and a constant κ with $0 < \kappa < 1$, then $|\mathbb{E}[X]|^{2+\kappa} \leq \mathbb{E}[|X|^{2+\kappa}]$.*

***Proposition* 7.** *Suppose the $\mathbf{P}_{\mathbf{M}(n)} = \left(\frac{1}{\sqrt{n}}\mathbf{V}(n)\right)\left(\frac{1}{\sqrt{n}}\mathbf{V}(n)\right)^*$, where $\mathbf{M}(n) \in \mathbb{R}^{n\times t}$ $(t \leq n)$, t is a fixed constant, and $\mathbf{V}(n)=[\mathbf{v}_1(n), \mathbf{v}_2(n), \ldots, \mathbf{v}_t(n)] \in \mathbb{R}^{n\times t}$ is an orthogonal basis of $\mathbf{M}(n)$ such that $\mathbf{V}^*(n)\mathbf{V}(n) = n\mathbf{I}$. If we let $\mathbf{a}(n) \in \mathbb{R}^{n\times 1}$ be a random vector with independent entries, which have zero mean and finite variance σ_a^2, then $\lim_{n\to\infty} \mathbf{P}_{\mathbf{M}}(n)\mathbf{a}(n) \overset{a.s.}{=} \mathbf{0}_n$, where $\mathbf{0}_n$ is the $n \times 1$ all-zero vector.*

Proof. Denoting $\widetilde{\mathbf{a}}(n) = \frac{\mathbf{a}(n)}{\|\mathbf{a}(n)\|_2}$ yields $\mathbf{P}_{\mathbf{M}(n)}\mathbf{a}(n) = \mathbf{V}(n)\frac{\|\mathbf{a}(n)\|_2}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\mathbf{V}^*(n)\right)\widetilde{\mathbf{a}}(n)$. The proposition follows from the fact that $\frac{\|\mathbf{a}(n)\|_2}{\sqrt{n}} \overset{a.s.}{=} \sigma_a$ and $\left(\frac{1}{\sqrt{n}}\mathbf{V}^*(n)\right)\widetilde{\mathbf{a}}(n) \overset{a.s.}{=} \mathbf{0}_n$ as $n \to \infty$. □

APPENDIX G
PROOF OF SE IN (7) THEOREM 1

Substituting $\phi_b(\mathbf{u}_i^t) = (b_i^t)^2$ into (16) gives $\lim_{n\to\infty}\langle \mathbf{b}^t, \mathbf{b}^t\rangle \overset{a.s}{=} \mathbb{E}\left[\sigma_t^2\widetilde{Z}_t^2\right] = \sigma_t^2$. Using (14) with $t_1 = t_2 = t$ yields $\lim_{N\to\infty} \frac{1}{\rho}\langle \mathbf{q}^t, \mathbf{q}^t\rangle = \sigma_t^2$. Then substituting $\phi_h(\mathbf{v}_i^{t-1}) = f_t^2(h_i^t, x_{0i}) = (q_i^t)^2$ into (17) leads to $\lim_{N\to\infty}\langle \mathbf{q}^t, \mathbf{q}^t\rangle \overset{a.s}{=} \mathbb{E}\left[f_t^2(\tau_{t-1}Z_{t-1}, X_0)\right]$, resulting in $\sigma_t^2 = \frac{1}{\rho}\mathbb{E}\left[f_t^2(\tau_{t-1}Z_{t-1}, X_0)\right]$. Showing the rest half $\tau_t^2 = \mathbb{E}\left[g_t^2(\sigma_t Z, W)\right]$ of the SE in (7) follows from the exactly same procedure as the above. Setting $\phi_h(\mathbf{v}_i^t) = (h_i^{t+1})^2$ in (17) gives $\lim_{N\to\infty}\langle \mathbf{h}^{t+1}, \mathbf{h}^{t+1}\rangle = \tau_t^2$. Using (15) with $t_1 = t_2 = t$ yields $\lim_{N\to\infty}\langle \mathbf{m}^t, \mathbf{m}^t\rangle = \tau_t^2$. Applying (16) to $\phi_b(\mathbf{u}_i^t) = g_t^2(b_i^t, w_i)$ yields $\lim_{N\to\infty}\langle \mathbf{m}^t, \mathbf{m}^t\rangle = \mathbb{E}\left[g_t^2(\sigma_t Z, W)\right]$. Therefore, $\tau_t^2 = \mathbb{E}\left[g_t^2(\sigma_t Z, W)\right]$, concluding the proof.

APPENDIX H
PROOF OF THEOREM 1: STEPS 3 AND 4

A. Step 3: We show a), b), c), and d) of Theorem 1 conditioning on $\mathcal{F}_{t,t} = \{\mathbf{b}^0, \ldots, \mathbf{b}^{t-1}, \mathbf{m}^0, \ldots, \mathbf{m}^{t-1}, \mathbf{h}^1, \ldots, \mathbf{h}^t, \mathbf{q}^0, \ldots, \mathbf{q}^t, \mathbf{x}_0, \mathbf{w}\}$.

a) Note that conditioning on $\mathcal{F}_{t,t}$ is equivalent to conditioning on $\mathcal{P}_{\mathbf{X}_t,\mathbf{Y}_t} = \{\mathbf{A}|\mathbf{A}^*\mathbf{M}_t = \mathbf{X}_t, \mathbf{A}\mathbf{Q}_t = \mathbf{Y}_t\}$. Applying Proposition 3 to obtain the conditional distribution $\mathbf{A}|_{\mathcal{F}_{t,t}}$

and following the same procedure as in [27, Lemma 1a], the conditional distribution of $\mathbf{b}^t$ on $\mathcal{F}_{t,t}$ is expressed as

$$\mathbf{b}^t|_{\mathcal{F}_{t,t}} \overset{d}{=} \sum_{j=0}^{t-1} \beta_j \mathbf{b}^j + \widetilde{\mathbf{A}}\mathbf{q}_\perp^t - \mathbf{P}_{\mathbf{M}_t}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t + \mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1). \quad (29)$$

By Proposition 7 in Appendix F, the third term on the r.h.s of (29) converges to $\lim_{n\to\infty} \mathbf{P}_{\mathbf{M}_t}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t \overset{a.s.}{=} \mathbf{0}_n$. Similar to Step 2a), we verify the convergence $\lim_{n\to\infty} \mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1) \overset{a.s.}{=} \mathbf{0}_n$ by characterizing the expectation and variance of its empirical distribution $\widehat{\mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1)}$ as $n \to \infty$. Indeed, $\lim_{n\to\infty} |\langle \mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1)\rangle| \le \lim_{n\to\infty} |o(1)| \frac{1}{n} \sum_{i=1}^n \left|\sum_{j=0}^{t-1} m_i^j\right| \le \lim_{n\to\infty} |o(1)| \sum_{j=0}^{t-1} \frac{1}{n} \sum_{i=1}^n |m_i^j| \overset{a.s.}{=} 0$, where the last equality holds because applying $\phi_b(b_i^j, w_i) = g_j(b_i^j, w_i)$ to the induction hypothesis of (16), for $j < t$, leads to $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n |m_i^j| \overset{a.s.}{=} \mathbb{E}[|g_j(\sigma_j \widetilde{Z}_j, W)|] < \infty$. Hence, $\lim_{n\to\infty} \langle \mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1)\rangle \overset{a.s.}{=} 0$. For the variance of $\widehat{\mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1)}$, we get

$$\begin{aligned}
\lim_{n\to\infty} \langle \mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1)\rangle_2 &= \lim_{n\to\infty} \frac{1}{n}[o(1)]^2 \sum_{i=1}^n \left(\sum_{j=0}^{t-1} m_i^j\right)^2, \\
&\le \lim_{n\to\infty} [o(1)]^2 t \frac{1}{n} \sum_{i=1}^n \sum_{j=0}^{t-1} (m_i^j)^2, \quad (30a)\\
&= \lim_{n\to\infty} [o(1)]^2 t \sum_{j=0}^{t-1} \langle \mathbf{m}^j, \mathbf{m}^j\rangle \overset{a.s.}{=} 0, \quad (30b)
\end{aligned}$$

where (30a) follows from the Cauchy-Schwarz inequality and (30b) holds because $\lim_{n\to\infty} \langle \mathbf{m}^j, \mathbf{m}^j\rangle \overset{a.s.}{=} \mathbb{E}[g_j^2(\sigma_j \widetilde{Z}_j, W)] < \infty$, $\forall j$, which is a direct consequence of the induction hypothesis of (16) with $\phi_b(b_i^j, w_i) = g_j^2(b_i^j, w_i)$. Hence, (30b) is equivalent to $\lim_{n\to\infty} \langle \mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1)\rangle_2 \overset{a.s.}{=} 0$. Therefore, $\lim_{n\to\infty} \mathbf{M}_t \overrightarrow{\mathbf{o}}_t(1) \overset{a.s.}{=} \mathbf{0}_n$, implying

$$\mathbf{b}^t|_{\mathcal{F}_{t,t}} \overset{d}{\Longrightarrow} \sum_{j=0}^{t-1} \beta_j \mathbf{b}^j + \widetilde{\mathbf{A}}\mathbf{q}_\perp^t. \quad (31)$$

b) Note that by the induction hypothesis of (17) for $\phi_h(h_i^t, x_{0i}) = |f_t(h_i^t, x_{0i})|^{2+2\alpha}$, we get $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N |q_i^t|^{2+2\alpha} \overset{a.s.}{=} \mathbb{E}[|f_t(\tau_{t-1} Z_{t-1}, X_0)|^{2+2\alpha}] < \infty$. On the other hand, $\sum_{i=1}^N |q_{\perp i}^t|^{2+2\alpha} < \sum_{i=1}^N |q_i^t|^{2+2\alpha}$. Thus, we have $\limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^N |q_{\perp i}^t|^{2+2\alpha} < \infty$, which concludes (12).

c) For $t_1 < t$ and $t_2 = t$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t\rangle &\overset{d}{=} \lim_{n\to\infty} \sum_{j=0}^{t-1} \beta_j \langle \mathbf{b}^{t_1}, \mathbf{b}^j\rangle + \lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle, \quad (32a)\\
&\overset{a.s.}{=} \sum_{j=0}^{t-1} \beta_j \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^j\rangle}{\rho} + \lim_{n\to\infty} \frac{\mathbf{b}^{t_1 *}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}{n}, \quad (32b)
\end{aligned}$$

where (32a) follows from (31) and (32b) results from the induction hypothesis (14) for $t_1 < t$ and $t_2 = j < t$. Now, using Proposition 6, we get $\frac{\mathbf{b}^{t_1 *}}{\|\mathbf{b}^{t_1}\|_2} \widetilde{\mathbf{A}} \frac{\mathbf{q}_\perp^t}{\|\mathbf{q}_\perp^t\|_2} \overset{d}{=} \frac{Z}{\sqrt{n}}$, where $Z \sim \mathcal{N}(0,1)$. Hence, $\frac{\mathbf{b}^{t_1 *}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}{n} \overset{d}{=} \frac{\|\mathbf{b}^{t_1}\|_2}{\sqrt{n}} \frac{\|\mathbf{q}_\perp^t\|_2}{\sqrt{N}} \frac{1}{\sqrt{\rho}} \frac{Z}{\sqrt{n}}$, i.e.,

$$\frac{\mathbf{b}^{t_1 *}\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}{n} \overset{d}{=} \frac{1}{\sqrt{\rho}} \sqrt{\langle \mathbf{b}^{t_1}, \mathbf{b}^{t_1}\rangle \langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle} \frac{Z}{\sqrt{n}}. \quad (33)$$

By the induction hypothesis of (14), we have $\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^{t_1}\rangle = \lim_{n\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^{t_1}\rangle}{\rho} < \infty$. Moreover, the $\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle$ converges to $\lim_{N\to\infty} \langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle < \lim_{N\to\infty} \langle \mathbf{q}^t, \mathbf{q}^t\rangle < \infty$ because using the induction hypothesis (17) we have $\langle \mathbf{q}^t, \mathbf{q}^t\rangle = \frac{1}{N} \sum_{i=1}^N f_t^2(h_i^t, x_{0i}) \overset{a.s.}{=} \mathbb{E}[f_t^2(\tau_{t-1} Z_{t-1}, X_0)] < \infty$. Thus, for $t_1 < t$,

$$\lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \mathbf{b}^{t_1}\rangle \overset{a.s.}{=} 0. \quad (34)$$

Substituting (34) into (32b) gives $\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t\rangle \overset{a.s.}{=} \sum_{j=0}^{t-1} \beta_j \lim_{n\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^j\rangle}{\rho} = \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_{||}^t\rangle}{\rho}$ due to (8), implying

$$\begin{aligned}
\lim_{n\to\infty} \langle \mathbf{b}^{t_1}, \mathbf{b}^t\rangle &\overset{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_{||}^t\rangle}{\rho} + \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}_\perp^t\rangle}{\rho}, \quad (35a)\\
&= \lim_{N\to\infty} \frac{\langle \mathbf{q}^{t_1}, \mathbf{q}^t\rangle}{\rho}, \quad (35b)
\end{aligned}$$

where (35a) follows from the fact that $\mathbf{q}^j$ is orthogonal to $\mathbf{q}_\perp^t$, for $j < t$, and (35b) holds due to (9), concluding (14) when $t_1 < t$ and $t_2 = t$.

For the case of $t_1 = t_2 = t$, it is similarly given by $\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle \overset{d}{=} \sum_{i,j=0}^{t-1} \beta_i\beta_j \lim_{n\to\infty} \langle \mathbf{b}^i, \mathbf{b}^j\rangle + 2\sum_{i=0}^{t-1} \beta_i \lim_{n\to\infty} \langle \mathbf{b}^i, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle + \lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle$ due to (31). Then, by (34), the following holds

$$\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle \overset{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i\beta_j \lim_{n\to\infty} \langle \mathbf{b}^i, \mathbf{b}^j\rangle + \lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle. \quad (36)$$

Using Proposition 5, $\widehat{\widetilde{\mathbf{A}}\mathbf{q}_\perp^t} \overset{d}{\Longrightarrow} \mathcal{N}\left(0, \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}\right)$. Thus, the second moment of $\widehat{\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}$ is

$$\lim_{n\to\infty} \langle \widetilde{\mathbf{A}}\mathbf{q}_\perp^t, \widetilde{\mathbf{A}}\mathbf{q}_\perp^t\rangle \overset{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}. \quad (37)$$

Now, incorporating (37) in (36), $\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle \overset{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i\beta_j \lim_{n\to\infty} \langle \mathbf{b}^i, \mathbf{b}^j\rangle + \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}$, resulting in

$$\begin{aligned}
\lim_{n\to\infty} \langle \mathbf{b}^t, \mathbf{b}^t\rangle &\overset{a.s.}{=} \sum_{i,j=0}^{t-1} \beta_i\beta_j \lim_{N\to\infty} \frac{\langle \mathbf{q}^i, \mathbf{q}^j\rangle}{\rho} + \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}, \quad (38a)\\
&\overset{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}_{||}^t, \mathbf{q}_{||}^t\rangle}{\rho} + \lim_{N\to\infty} \frac{\langle \mathbf{q}_\perp^t, \mathbf{q}_\perp^t\rangle}{\rho}, \\
&\overset{a.s.}{=} \lim_{N\to\infty} \frac{\langle \mathbf{q}^t, \mathbf{q}^t\rangle}{\rho},
\end{aligned}$$

where (38a) is due to the induction hypothesis (14) for $0 \le t_1 = i, t_2 = j \le t-1$. This completes the proof of (14) at the tth iteration.

d) Defining $\lim_{N\to\infty}\frac{\langle \mathbf{q}_\perp^t,\mathbf{q}_\perp^t\rangle}{\rho}\overset{a.s.}{=}\gamma_t^2$, we can write by (37) that $\widehat{\widetilde{\mathbf{A}}\mathbf{q}_\perp^t}\overset{d}{\Longrightarrow}\mathcal{N}(0,\gamma_t^2)$. Using (31) in conjunction with the latter, we get

$$b_i^t|_{\mathcal{F}_{t,t}}\overset{d}{\Longrightarrow}\sum_{j=0}^{t-1}\beta_j b_i^j+\gamma_t Z,\ \text{for } i=1,2,\dots,n, \tag{39}$$

where $Z\sim\mathcal{N}(0,1)$. Similar to Step 1d), using (39) $\mathbf{u}_i^t\overset{d}{\Longrightarrow}\widetilde{\mathbf{u}}_i^t$, where $\mathbf{u}_i^t=(b_i^0\dots,b_i^t,w_i)$ and $\widetilde{\mathbf{u}}_i^t=(b_i^0,\dots,b_i^{t-1},\sum_{j=0}^{t-1}\beta_j b_i^j+\gamma_t Z,w_i)$, $\forall i$. To prove (16), we first claim that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\phi_b(\widetilde{\mathbf{u}}_i^t)-\mathbb{E}\Big[\phi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_t\widetilde{Z}_t,W)\Big]\overset{a.s.}{=}0$. By the triangular inequality, $\Big|\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\phi_b(\widetilde{\mathbf{u}}_i^t)-\mathbb{E}\Big[\phi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_t\widetilde{Z}_t,W)\Big]\Big|\le X_1^t+X_2^t$, where $X_1^t=\Big|\frac{1}{n}\sum_{i=1}^n\Big(\phi_b(\widetilde{\mathbf{u}}_i^t)-\widetilde{\phi}_b(\mathbf{u}_i^{t-1})\Big)\Big|$, $X_2^t=\Big|\frac{1}{n}\sum_{i=1}^n\widetilde{\phi}_b(\mathbf{u}_i^{t-1})-\mathbb{E}[\phi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_t\widetilde{Z}_t,W)]\Big|$, and $\widetilde{\phi}_b(\mathbf{u}_i^{t-1})=\mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]$. Similar to Step 1d), we verify $\lim_{n\to\infty}X_1^t\overset{a.s.}{=}0$ and $\lim_{n\to\infty}X_2^t\overset{a.s.}{=}0$.

First showing $\lim_{n\to\infty}X_1^t\overset{a.s.}{=}0$ is of interest. By (5), $|\phi_b(\widetilde{\mathbf{u}}_i^t)|\le c_1^t\exp\Big(c_2^t\Big(\sum_{j=0}^{t-1}|b_i^t|^\lambda+\Big|\sum_{j=0}^{t-1}\beta_j b_i^j+\gamma_t Z\Big|^\lambda+|w_i|^\lambda\Big)\Big)$, where $c_1^t>0$, $c_2^t>0$, and $1\le\lambda<2$ are constants. Using the inequality $\|\mathbf{x}\|_1^\lambda\le(t+1)^{\lambda-1}\|\mathbf{x}\|_\lambda^\lambda$ for $\mathbf{x}\in\mathbb{R}^{(t+1)\times 1}$, we get $|\phi_b(\widetilde{\mathbf{u}}_i^t)|\le c_1^t\exp\Big(c_2^t\Big(\sum_{j=0}^{t-1}(1+(t+1)^{\lambda-1}|\beta_j|^\lambda)|b_i^t|^\lambda+(t+1)^{\lambda-1}|\gamma_t|^\lambda|Z|^\lambda+|w_i|^\lambda\Big)\Big)$. Hence, $\mathbb{E}_Z[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}]\le c_3^t\exp\Big(c_4^t\Big(\sum_{j=0}^{t-1}|b_i^j|^\lambda+|w_i|^\lambda\Big)\Big)\mathbb{E}_Z\left[\exp(c_4^t|Z|^\lambda)\right]$, where $0<\kappa<1$, $c_3^t=(c_1^t)^{2+\kappa}$, and $c_4^t=(2+\kappa)c_2^t\max\Big\{1+(t+1)^{\lambda-1}|\beta_0|^\lambda,\dots,1+(t+1)^{\lambda-1}|\beta_{t-1}|^\lambda,(t+1)^{\lambda-1}|\gamma_t|^\lambda\Big\}$, resulting in

$$\mathbb{E}_Z[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}]\le c_5^t\exp\Big(c_4^t\Big(\sum_{j=0}^{t-1}|\beta_j b_i^j|^\lambda+|w_i|^\lambda\Big)\Big), \tag{40}$$

and $c_5^t=c_3^t\mathbb{E}_Z\left[\exp(c_4^t\gamma_t^\lambda|Z|^\lambda)\right]$ is constant. We define $X_{n,i}^t=\phi_b(\widetilde{\mathbf{u}}_i^t)-\widetilde{\phi}_b(\mathbf{u}_i^{t-1})=\phi_b(\widetilde{\mathbf{u}}_i^t)-\mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]$ such that $X_1^t=|\frac{1}{n}\sum_{i=1}^n X_{n,i}^t|$. To prove $\lim_{n\to\infty}X_1^t\overset{a.s.}{=}0$, we show that $\{X_{n,i}^t\}_{i=1}^n$ satisfy Lemma 3 in Appendix F. Indeed, $\mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}]$ is upper bouned as follows,

$$\begin{aligned}\mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}]&\le 2^{1+\kappa}\Big(\mathbb{E}_Z\left[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}\right]+\big|\mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]\big|^{2+\kappa}\Big), &(41a)\\ &\le 2^{2+\kappa}\mathbb{E}_Z\left[|\phi_b(\widetilde{\mathbf{u}}_i^t)|^{2+\kappa}\right], &(41b)\\ &\le c_6^t\exp\Big(c_4^t\Big(\sum_{j=0}^{t-1}|\beta_j b_i^j|^\lambda+|w_i|^\lambda\Big)\Big), &(41c)\end{aligned}$$

where (41a) follows from Lemma 4 (Holder's inequality) in Appendix F, (41b) follows from Lemma 5 (Lyapunov's inequality) in Appendix F, and (41c) follows from (40) with $c_6^t=2^{2+\kappa}c_5^t$. We denote the last term of (41c) as $\psi_b(\mathbf{u}_i^{t-1})=c_6^t\exp\Big(c_4^t\Big(\sum_{j=0}^{t-1}|\beta_j b_i^j|^\lambda+|w_i|^\lambda\Big)\Big)$. Then, $\psi_b(\mathbf{u}_i^{t-1})$ is a controlled function. From (41c), we get, for n is sufficiently large,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n\mathbb{E}_Z[|X_{n,i}^t|^{2+\kappa}]\ &\le\ \frac{1}{n}\sum_{i=1}^n\psi_b(\mathbf{u}_i^{t-1}),\\ &\overset{a.s.}{=}\mathbb{E}[\psi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_{t-1}\widetilde{Z}_{t-1},W)], &(42a)\\ &<\ cn^{\kappa/2}, &(42b)\end{aligned}$$

where c is a positive constant, (42a) is due to the induction hypothesis (16), and (42b) holds because $\mathbb{E}[\psi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_{t-1}\widetilde{Z}_{t-1},W)]=c_7^t<\infty$ and there exists n_t, a positive constant, such that $c_7^t<cn^{\kappa/2}$ for $n>n_t$. By Lemma 3 in Appendix F, we get $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n X_{n,i}^t\overset{a.s.}{=}0$, implying

$$\frac{1}{n}\sum_{i=1}^n\Big(\phi_b(\widetilde{\mathbf{u}}_i^t)-\widetilde{\phi}_b(\mathbf{u}_i^{t-1})\Big)\overset{a.s.}{=}0, \tag{43}$$

which proving $\lim_{n\to\infty}X_1^t\overset{a.s.}{=}0$.

Now, showing $\lim_{n\to\infty}X_2^t\overset{a.s.}{=}0$ is of interest. By the induction hypothesis (16), $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\widetilde{\phi}_b(\mathbf{u}_i^{t-1})\overset{a.s.}{=}\mathbb{E}[\widetilde{\phi}_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_{t-1}\widetilde{Z}_{t-1},W)]$, resulting in

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\widetilde{\phi}_b(\mathbf{u}_i^{t-1})\\ &=\mathbb{E}\left[\mathbb{E}_Z\left[\phi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_{t-1}\widetilde{Z}_{t-1},\sum_{j=0}^{t-1}\beta_j\sigma_j\widetilde{Z}_j+\gamma_t Z,W)\right]\right],\\ &=\mathbb{E}\left[\phi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_{t-1}\widetilde{Z}_{t-1},\sum_{j=0}^{t-1}\beta_j\sigma_j\widetilde{Z}_j+\gamma_t Z,W)\right],\end{aligned} \tag{44}$$

where (44) follows from the substitution $\widetilde{\phi}_b(\mathbf{u}_i^{t-1})=\mathbb{E}_Z[\phi_b(\widetilde{\mathbf{u}}_i^t)]$. Therefore, showing $\lim_{n\to\infty}X_2^t\overset{a.s.}{=}0$ is equivalent to proving $\sum_{j=0}^{t-1}\beta_j\sigma_j\widetilde{Z}_j+\gamma_t Z=\sigma_t\widetilde{Z}_t$, where $\widetilde{Z}_t\sim\mathcal{N}(0,1)$ and σ_t is defined in (7).

In particular, for $\phi_b(\mathbf{u}_i^t)=(b_i^t)^2$, we get $\phi_b(\widetilde{\mathbf{u}}_i^t)=\Big(\sum_{j=0}^{t-1}\beta_j b_i^j+\gamma_t Z\Big)^2$ because $\widetilde{\mathbf{u}}_i^t=(b_i^0,\dots,b_i^{t-1},\sum_{j=0}^{t-1}\beta_j b_i^j+\gamma_t Z,w_i)$. Combining (43) and (44),

$$\begin{aligned}\lim_{n\to\infty}\langle\mathbf{b}^t,\mathbf{b}^t\rangle&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\phi_b(\mathbf{u}_i^t)\\ &\overset{d}{=}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\phi_b(\widetilde{\mathbf{u}}_i^t)\overset{a.s.}{=}\mathbb{E}\Big[\Big(\sum_{j=0}^{t-1}\beta_j\sigma_j\widetilde{Z}_j+\gamma_t Z\Big)^2\Big].\end{aligned} \tag{45}$$

Using (14), $\lim_{n\to\infty}\langle\mathbf{b}^t,\mathbf{b}^t\rangle\overset{a.s.}{=}\lim_{N\to\infty}\frac{\langle\mathbf{q}^t,\mathbf{q}^t\rangle}{\rho}=\sigma_t^2$, where the last equality holds because by the induction hypothesis (17) for $\phi_h(\mathbf{v}_i^{t-1})=f_t^2(h_i^t,x_{0i})$ in (17), $\frac{1}{\rho}\lim_{N\to\infty}\langle\mathbf{q}^t,\mathbf{q}^t\rangle\overset{a.s.}{=}\frac{1}{\rho}\mathbb{E}[f_t^2(\tau_{t-1}Z,X_0)]=\sigma_t^2$. Hence, $\mathbb{E}\left[\Big(\sum_{j=0}^{t-1}\beta_j\sigma_j\widetilde{Z}_j+\gamma_t Z\Big)^2\right]\overset{a.s.}{=}\sigma_t^2$, implying $\sum_{j=0}^{t-1}\beta_j\sigma_j\widetilde{Z}_j+\gamma_t Z=\sigma_t\widetilde{Z}_t$ due to (45), verifying that $\lim_{n\to\infty}X_2^t\overset{a.s.}{=}0$, which completes the proof of (16).

B. Step 4: We show a), b), c), and d) of Theorem 1 conditioning on $\mathcal{F}_{t+1,t} = \{\mathbf{b}^0,\ldots,\mathbf{b}^t,\mathbf{m}^0,\ldots,\mathbf{m}^t,\mathbf{h}^1,\ldots,\mathbf{h}^t,\mathbf{q}^0,\ldots,\mathbf{q}^t,\mathbf{x}_0,\mathbf{w}\}$.

The proof of Step 4 is similar to the proof of Step 3. Thus, we only present the features that are unique in Step 4.

a) Similar to Step 3a), using Proposition 3 to characterize $\mathbf{A}|_{\mathcal{F}_{t+1,t}}$ and following the same procedure as in [27, Lemma 1a], the $\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}}$ is

$$\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{=} \sum_{j=0}^{t-1}\zeta_j\mathbf{h}^{j+1} + \widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t - \mathbf{P}_{\mathbf{Q}_{t+1}}\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t + \mathbf{Q}_t\overrightarrow{\mathbf{o}}_t(1). \tag{46}$$

By Proposition 7 in Appendix F, $\lim_{N\to\infty}\mathbf{P}_{\mathbf{Q}_{t+1}}\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t \stackrel{a.s.}{=} \mathbf{0}_N$. Similar to Step 3a), we verify that $\lim_{N\to\infty}\mathbf{Q}_t\overrightarrow{\mathbf{o}}_t(1) \stackrel{a.s.}{=} \mathbf{0}_N$ by characterizing (i) the expectation of the empirical distribution $\widehat{\mathbf{Q}_t\overrightarrow{\mathbf{o}}_t(1)}$ is bounded as $\lim_{N\to\infty}|\langle\mathbf{Q}_t\overrightarrow{\mathbf{o}}_t(1)\rangle| \le \lim_{N\to\infty}|o(1)|\sum_{j=0}^{t-1}\frac{1}{N}\sum_{i=1}^N|q_i^j| \stackrel{a.s.}{=} 0$ and (ii) the empirical variance of $\widehat{\mathbf{Q}_t\overrightarrow{\mathbf{o}}_t(1)}$ is bounded an converges to $\lim_{N\to\infty}\langle\mathbf{Q}_t\overrightarrow{\mathbf{o}}_t(1)\rangle_2 \le \lim_{N\to\infty}[o(1)]^2 t\sum_{j=0}^{t-1}\langle\mathbf{q}^j,\mathbf{q}^j\rangle \stackrel{a.s.}{=} 0$. Therefore, using $\lim_{N\to\infty}\mathbf{Q}_t\overrightarrow{\mathbf{o}}_t(1) \stackrel{a.s.}{=} \mathbf{0}_N$, we get

$$\mathbf{h}^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{\Longrightarrow} \sum_{j=0}^{t-1}\zeta_j\mathbf{h}^{j+1} + \widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t, \tag{47}$$

which completes the proof of (11).

b) Using (16) for $\phi_b(b_i^t,w_i) = |g_t(b_i^t,w_i)|^{2+2\alpha}$, we get $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n|m_i^t|^{2+2\alpha} \stackrel{a.s.}{=} \mathbb{E}[|g_t(\sigma_t Z_t,W)|^{2+2\alpha}] < \infty$. Because $\sum_{i=1}^n|m_{\perp i}^t|^{2+2\alpha} < \sum_{i=1}^n|m_i^t|^{2+2\alpha}$, the following holds $\limsup_{n\to\infty}\sum_{i=1}^n|m_{\perp i}^t|^{2+2\alpha} < \infty$, which concludes (13).

c) For $t_1 < t$ and $t_2 = t$, we have $\lim_{N\to\infty}\langle\mathbf{h}^{t_1+1},\mathbf{h}^{t+1}\rangle \stackrel{d}{=} \lim_{N\to\infty}\sum_{j=0}^{t-1}\zeta_j\langle\mathbf{h}^{t_1+1},\mathbf{h}^j\rangle + \lim_{N\to\infty}\langle\mathbf{h}^{t_1+1},\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t\rangle$ due to (47), resulting in

$$\lim_{N\to\infty}\langle\mathbf{h}^{t_1+1},\mathbf{h}^{t+1}\rangle \stackrel{a.s.}{=} \sum_{j=0}^{t-1}\zeta_j\lim_{n\to\infty}\langle\mathbf{m}^{t_1},\mathbf{m}^j\rangle + \lim_{N\to\infty}\frac{\mathbf{m}_\perp^{t^*}\widetilde{\mathbf{A}}\mathbf{h}^{t_1+1}}{N}, \tag{48}$$

where (48) is by the induction hypothesis (15). Note that $\frac{\mathbf{m}_\perp^{t^*}}{\|\mathbf{m}_\perp^t\|_2}\widetilde{\mathbf{A}}\frac{\mathbf{h}^{t_1+1}}{\|\mathbf{h}^{t_1+1}\|_2} \stackrel{d}{=} \frac{Z}{\sqrt{n}}$ due to Proposition 6. The second term in (48) is represented as

$$\begin{aligned}\lim_{N\to\infty}\frac{\mathbf{m}_\perp^{t^*}\widetilde{\mathbf{A}}\mathbf{h}^{t_1+1}}{N} &\stackrel{d}{=} \lim_{N\to\infty}\frac{\|\mathbf{m}_\perp^t\|_2}{\sqrt{n}}\frac{\|\mathbf{h}^{t_1+1}\|_2}{\sqrt{N}}\frac{\sqrt{n}}{\sqrt{N}}\frac{Z}{\sqrt{n}},\\ &= \sqrt{\rho}\lim_{N\to\infty}\sqrt{\langle\mathbf{m}_\perp^t,\mathbf{m}_\perp^t\rangle\langle\mathbf{h}^{t_1+1},\mathbf{h}^{t_1+1}\rangle}\frac{Z}{\sqrt{n}} \stackrel{a.s.}{=} 0,\end{aligned} \tag{49}$$

Substituting (49) into (48) yields

$$\begin{aligned}\lim_{N\to\infty}\langle\mathbf{h}^{t_1+1},\mathbf{h}^{t+1}\rangle &= \lim_{n\to\infty}\langle\mathbf{m}^{t_1},\sum_{j=0}^{t-1}\zeta_j\mathbf{m}^j\rangle = \lim_{n\to\infty}\langle\mathbf{m}^{t_1},\mathbf{m}_{||}^t\rangle,\\ &= \lim_{n\to\infty}\langle\mathbf{m}^{t_1},\mathbf{m}_{||}^t\rangle + \lim_{n\to\infty}\langle\mathbf{m}^{t_1},\mathbf{m}_\perp^t\rangle = \lim_{n\to\infty}\langle\mathbf{m}^{t_1},\mathbf{m}^t\rangle,\end{aligned}$$

concluding (15) when $t_1 < t$ and $t_2 = t$.

For $t_1 = t_2 = t$, by (47),

$$\lim_{N\to\infty}\langle\mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle \stackrel{d}{=} \sum_{i,j=0}^{t-1}\zeta_i\zeta_j\lim_{N\to\infty}\langle\mathbf{h}^{i+1},\mathbf{h}^{j+1}\rangle$$
$$+ 2\sum_{i=0}^{t-1}\zeta_i\lim_{N\to\infty}\langle\mathbf{h}^{i+1},\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t\rangle + \lim_{N\to\infty}\langle\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t,\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t\rangle. \tag{50}$$

Then, by (49), the following holds

$$\lim_{N\to\infty}\langle\mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1}\zeta_i\zeta_j\lim_{N\to\infty}\langle\mathbf{h}^{i+1},\mathbf{h}^{j+1}\rangle + \lim_{N\to\infty}\langle\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t,\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t\rangle. \tag{51}$$

By Proposition 5, the empirical distribution of $\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t$ converges to $\widehat{\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t} \stackrel{d}{\Longrightarrow} \mathcal{N}(0,\lim_{n\to\infty}\langle\mathbf{m}_\perp^t,\mathbf{m}_\perp^t\rangle)$. Hence, the second moment of $\widehat{\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t}$ converges to

$$\lim_{N\to\infty}\langle\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t,\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t\rangle \stackrel{a.s.}{=} \lim_{n\to\infty}\langle\mathbf{m}_\perp^t,\mathbf{m}_\perp^t\rangle. \tag{52}$$

Substituting (52) into (51) leads to $\lim_{N\to\infty}\langle\mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle \stackrel{a.s.}{=} \sum_{i,j=0}^{t-1}\zeta_i\zeta_j\lim_{N\to\infty}\langle\mathbf{h}^{i+1},\mathbf{h}^{j+1}\rangle + \lim_{n\to\infty}\langle\mathbf{m}_\perp^t,\mathbf{m}_\perp^t\rangle$, implying

$$\begin{aligned}\lim_{N\to\infty}\langle\mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle &\stackrel{a.s.}{=} \sum_{i,j=0}^{t-1}\zeta_i\zeta_j\lim_{n\to\infty}\langle\mathbf{m}^i,\mathbf{m}^j\rangle + \lim_{n\to\infty}\langle\mathbf{m}_\perp^t,\mathbf{m}_\perp^t\rangle,\\ &= \lim_{n\to\infty}\langle\mathbf{m}_{||}^t,\mathbf{m}_{||}^t\rangle + \lim_{n\to\infty}\langle\mathbf{m}_\perp^t,\mathbf{m}_\perp^t\rangle = \lim_{n\to\infty}\langle\mathbf{m}^t,\mathbf{m}^t\rangle.\end{aligned}$$

Therefore, (15) also holds for $t_1 = t_2 = t$, which completes the proof.

d) Defining $\lim_{n\to\infty}\langle\mathbf{m}_\perp^t,\mathbf{m}_\perp^t\rangle \stackrel{a.s.}{=} \Gamma_t^2$, we can write

$$\widehat{\widetilde{\mathbf{A}}^*\mathbf{m}_\perp^t} \stackrel{d}{\Longrightarrow} \mathcal{N}(0,\Gamma_t^2). \tag{53}$$

Using (53) and (47), the following convergence holds

$$h_i^{t+1}|_{\mathcal{F}_{t+1,t}} \stackrel{d}{\Longrightarrow} \sum_{j=0}^{t-1}\zeta_j h_i^{j+1} + \Gamma_t Z, \tag{54}$$

where $Z \sim \mathcal{N}(0,1)$. Similar to Step 3d), we can write, using (54), $\mathbf{v}_i^t \stackrel{d}{\Longrightarrow} \widetilde{\mathbf{v}}_i^t$, where $\mathbf{v}_i^t = (h_i^1,..,h_i^{t+1},x_{0i})$ and $\widetilde{\mathbf{v}}_i^t = (h_i^1,\ldots,h_i^t,\sum_{j=0}^{t-1}\zeta_j h_i^{j+1} + \Gamma_t Z, x_{0i})$. Hence, to prove (17) we first claim that $\Big|\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\phi_h(\widetilde{\mathbf{v}}_i^t) - \mathbb{E}\Big[\phi_h(\tau_0 Z_0,\ldots,\tau_t Z_t,X_0)\Big]\Big| \stackrel{a.s.}{=} 0$. Similar to Step 2d), using the triangular inequality, we verify that $\lim_{N\to\infty}Y_1^t \stackrel{a.s.}{=} 0$ and $\lim_{N\to\infty}Y_2^t \stackrel{a.s.}{=} 0$, where $Y_1^t = \Big|\frac{1}{N}\sum_{i=1}^N\Big(\phi_h(\widetilde{\mathbf{v}}_i^t) - \widetilde{\phi}_h(\mathbf{v}_i^{t-1})]\Big)\Big|$ and $Y_2^t = \Big|\frac{1}{N}\sum_{i=1}^N\widetilde{\phi}_h(\mathbf{v}_i^{t-1}) - \mathbb{E}\Big[\phi_h(\tau_0 Z_0,\ldots,\tau_t Z_t,X_0)\Big]\Big|$, and $\widetilde{\phi}_h(\mathbf{v}_i^{t-1}) = \mathbb{E}_Z[\phi_h(\widetilde{\mathbf{v}}_i^t)]$, $\forall i$.

First, showing $\lim_{N\to\infty}Y_1^t \stackrel{a.s.}{=} 0$ is of interest. By (5), $|\phi_h(\widetilde{\mathbf{v}}_i^t)| \le d_1^t\exp\Big(d_2^t\Big(\sum_{j=0}^{t-1}|h_i^{j+1}|^\lambda + \Big|\sum_{j=0}^{t-1}\zeta_j h_i^{j+1} + \Gamma_t Z_i\Big|^\lambda + |x_{0i}|^\lambda\Big)\Big)$, where $d_1^t > 0$, $d_2^t > 0$, and $1 \le \lambda < 2$ are constants. Using the inequality $\|\mathbf{x}\|_1^\lambda \le (t+1)^{\lambda-1}\|\mathbf{x}\|_\lambda^\lambda$ for

$\mathbf{x} \in \mathbb{R}^{(t+1)\times 1}$, we get $|\phi_h(\widetilde{\mathbf{v}}_i^t)| \le d_1^t \exp\Big(d_2^t\Big(\sum_{j=0}^{t-1}(1+(t+1)^{\lambda-1}|\zeta_j|^\lambda)|h_i^{j+1}|^\lambda + (t+1)^{\lambda-1}|\Gamma_t|^\lambda|Z_i|^\lambda + |x_{0i}|^\lambda\Big)\Big)$. Hence, $\mathbb{E}_Z\Big[|\phi_b(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}\Big] \le d_5^t \exp\Big[d_4^t\Big(\sum_{j=0}^{t-1}|h_i^{j+1}|^\lambda + |x_{0i}|^\lambda\Big)\Big]$, where $0 < \kappa < 1$, $d_4^t = d_2^t(2+\kappa)\max\Big\{1+(t+1)^{\lambda-1}|\alpha_0|^\lambda,\dots,1+(t+1)^{\lambda-1}|\alpha_{t-1}|^\lambda,(t+1)^{\lambda-1}|\Gamma_t|^\lambda\Big\}$, and $d_5^t = (d_1^t)^{2+\kappa}\mathbb{E}_Z\Big[\exp(d_4^t|Z_i|^\lambda)\Big]$ are constants. Define $Y_{N,i}^t = \phi_h(\widetilde{\mathbf{v}}_i^t) - \mathbb{E}_Z[\phi_h(\widetilde{\mathbf{v}}_i^t)]$, $\forall i$. To prove the convergence $\lim_{n\to\infty} Y_1^t \overset{a.s.}{=} 0$, we will show that $\{Y_{N,i}^t\}_{i=1}^N$ satisfy the condition in Lemma 3 in Appendix F. Indeed, the $\mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}]$ is upper bounded as follows.

$$
\begin{aligned}
\mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}] &\le 2^{1+\kappa}\Big(\mathbb{E}_Z[|\phi_h(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}] + |\mathbb{E}_Z[\phi_h(\widetilde{\mathbf{v}}_i^t)]|^{2+\kappa}\Big),\\
&\le 2^{2+\kappa}\mathbb{E}_Z[|\phi_h(\widetilde{\mathbf{v}}_i^t)|^{2+\kappa}]\\
&\le d_6^t \exp\left(d_4^t\left(\sum_{j=0}^{t-1}|\zeta_j h_i^{j+1}|^\lambda + |x_{0i}|^\lambda\right)\right),\\
&\triangleq \psi_h(\mathbf{v}_i^{t-1}). \qquad (55a)
\end{aligned}
$$

Then, $\psi_h(\mathbf{v}_i^{t-1})$ is a controlled function. From (55a), we get, for N is sufficiently large,

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^n \mathbb{E}_Z[|Y_{N,i}^t|^{2+\kappa}] \quad &\le \quad \frac{1}{N}\sum_{i=1}^n \psi_h(\mathbf{v}_i^{t-1}),\\
&\overset{a.s.}{=} \quad \mathbb{E}[\psi_h(\tau_0 Z_0,\dots,\tau_{t-1}Z_{t-1},X_0)],\\
&< \quad cN^{\kappa/2}, \qquad (56a)
\end{aligned}
$$

where c is a positive constant and (56a) holds because $\mathbb{E}[\psi_b(\sigma_0\widetilde{Z}_0,\dots,\sigma_{t-1}\widetilde{Z}_{t-1},W)] = d_7^t < \infty$ and there exists N_t, a positive constant, such that $d_7^t < cN^{\kappa/2}$ for $N > N_t$. Using Lemma 3 in Appendix F, $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N Y_{N,i}^t \overset{a.s.}{=} 0$, implying $\lim_{N\to\infty} Y_1^t \overset{a.s.}{=} 0$.

We are now ready to verify the convergence $\lim_{N\to\infty} Y_2^t \overset{a.s.}{=} 0$. Applying the induction hypothesis (17) for $\widetilde{\phi}_b(\mathbf{v}_i^{t-1})$ gives

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \widetilde{\phi}_h(\mathbf{v}_i^{t-1}) \overset{a.s.}{=} \mathbb{E}[\widetilde{\phi}_h(\tau_0 Z_0,\dots,\tau_{t-1}Z_{t-1},X_0)],\\
&= \mathbb{E}\left[\mathbb{E}_Z[\phi_h(\tau_0 Z_0,\dots,\tau_{t-1}Z_{t-1},\sum_{j=0}^{t-1}\zeta_j\tau_j Z_j + \Gamma_t Z, X_0)]\right],\\
&= \mathbb{E}\left[\phi_h(\tau_0 Z_0,\dots,\tau_{t-1}Z_{t-1},\sum_{j=0}^{t-1}\zeta_j\tau_j Z_j + \Gamma_t Z, X_0)\right].
\end{aligned}
$$

Therefore, showing $\lim_{N\to\infty} Y_2^t \overset{a.s.}{=} 0$ is equivalent to proving $\sum_{j=0}^{t-1}\zeta_j\tau_j Z_j + \Gamma_t Z = \tau_t Z_t$, where $Z_t \sim \mathcal{N}(0,1)$ and τ_t is defined in (7). Similar to the proof of $\lim_{n\to\infty} X_2^t \overset{a.s.}{=} 0$ in Step 3d), setting $\phi_h(\mathbf{v}_i^t) = (h_i^t)^2$, i.e., $\phi_h(\widetilde{\mathbf{v}}_i^t) = \Big(\sum_{j=0}^{t-1}\zeta_j h_i^{j+1} + \Gamma_t Z\Big)^2$, we get

$$
\begin{aligned}
&\lim_{N\to\infty}\langle \mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \phi_h(\mathbf{v}_i^t)\\
&\overset{d}{=} \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \phi_h(\widetilde{\mathbf{v}}_i^t) \overset{a.s.}{=} \mathbb{E}\left[\left(\sum_{j=0}^{t-1}\zeta_j\tau_j Z_j + \Gamma_t Z\right)^2\right].
\end{aligned}
$$

Using (15), we get $\lim_{N\to\infty}\langle \mathbf{h}^{t+1},\mathbf{h}^{t+1}\rangle \overset{a.s.}{=} \lim_{n\to\infty}\langle \mathbf{m}^t,\mathbf{m}^t\rangle = \tau_t^2$, where the last equality holds by the induction hypothesis (16) for $\phi_b(\mathbf{u}_i^t) = g_t^2(b_i^t, w_i)$, resulting in $\lim_{n\to\infty}\langle \mathbf{m}^t,\mathbf{m}^t\rangle \overset{a.s.}{=} \mathbb{E}[g_t^2(\sigma_t\widetilde{Z}_t, W)] = \tau_t^2$. Hence, $\mathbb{E}\Big[\Big(\sum_{j=0}^{t-1}\zeta_j\tau_j Z_j + \Gamma_t Z\Big)^2\Big] \overset{a.s.}{=} \tau_t^2$, which implies $\sum_{j=0}^{t-1}\zeta_j\tau_j Z_j + \Gamma_t Z = \tau_t Z_t$. Thus, it is verified that $\lim_{N\to\infty} Y_2^t \overset{a.s.}{=} 0$, which completes the proof of (17).