MATHEMATICAL REPRESENT THE GAME PLAY OF NIM

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I. INTRODUCTION

The game nim has been well known in game theory, with the strategy using nimber. This article will give you a mathematic explanation.

Nim is an impartial game that given couples of piles of sticks, and two players alternatively take turn to remove some number of sticks from some pile. The player who take the last stick will be the winner.

The nimber addition was invented in 19th century. Today, we know it's equivalent to the XOR operator, denoted as $\bigoplus$. I'll also use the XOR (Exclusive OR) in this paper.

II. BACKGROUND

1. Binary representation:

Let's get started with properties of binary numbers and XOR:

Binary representation of a nonnegative decimal number x can be represent as a finite sequence $\overline{x_n x_{n-1}\, x_{n-2} \dots x_0}_2$ of $(n+1)$ digits 0 or 1, such that:

$$x = \sum_{i=0}^{n}\left(x_i \times 2^i\right), \text{ where } x_i \in \{0,1\} \text{ for all } 0 \le i \le n.$$

The number of binary digits is $n+1$.

Also, it's followed that each binary digit x_i can be determined by:

$$x_i = \left\lfloor \frac{x}{2^i} \right\rfloor \bmod 2$$

For example,

$$0 = 0_2$$

$$1 = 1 \times 2^0 = 1_2$$

$$2 = 1 \times 2^1 + (0 \times 2^0) = 10_2$$

$$3 = 2^1 + 2^0 = 11_2$$

$$10 = 2^3 + 2^1 = 1010_2$$

2. XOR operation $\bigoplus$
   a. Definition

We can rewrite the definition of XOR operation as:

Let $a = \sum_{i=0}^{n}\left(a_i \times 2^i\right)$ and $b = \sum_{i=0}^{n}\left(b_i \times 2^i\right)$ be any non-negative integers a and b, where $a_i, b_i \in \{0,1\}$ for all $0 \le i \le n$.

Then, $c = a \bigoplus b$ is defined as $c = \sum_{i=0}^{n}\left(c_i \times 2^i\right)$ such that $c_i = |a_i - b_i|$

The definition is also equivalent to: $c = a \oplus b = \sum_{i=0}^{n} \left(\left(\left(\left\lfloor \frac{a}{2^i} \right\rfloor + \left\lfloor \frac{b}{2^i} \right\rfloor \right) \ mod\ 2 \right) \times 2^i \right)$

For example:

$$1 \oplus 2 = 01_2 \oplus 10_2 = 11_2 = 3$$

$$4 \oplus 5 = 100_2 \oplus 101_2 = 001_2 = 1$$

$$10 \oplus 5 = 1010_2 \oplus 0101_2 = 1111_2 = 15$$

$$15 \oplus 5 = 1111_2 \oplus 0101_2 = 1010_2 = 10$$

$$15 \oplus 10 = 1111_2 \oplus 1010_2 = 0101_2 = 5$$

$$8 \oplus 7 = 10s00_2 \oplus 0111_2 = 1111_2 = 15$$

b. Properties of XOR operation:

For all non-negative integers a, b, c:

- $\mathrm{a} \oplus 0 = 0 \oplus \mathrm{a} = \mathrm{a}$ (0 is the identity)
- $\mathrm{a} \oplus \mathrm{b} = \mathrm{b} \oplus \mathrm{a}$ (symmetricity)
- $\mathrm{a} \oplus \mathrm{b} \oplus \mathrm{c} = (\mathrm{a} \oplus \mathrm{b}) \oplus \mathrm{c} = \mathrm{a} \oplus (\mathrm{b} \oplus \mathrm{c}) = (\mathrm{a} \oplus \mathrm{c}) \oplus \mathrm{b}$ (associativity)
- $\mathrm{a} \oplus \mathrm{b} = 0 \Leftrightarrow \mathrm{a} = \mathrm{b}$
- $\mathrm{a} \oplus \mathrm{a} = 0$
- $\underbrace{a \oplus a \oplus a \oplus \ldots \oplus a}_{n\ times\ occurences\ of\ a} = \begin{cases} a & if\ n\ is\ odd \\ 0 & otherwise \end{cases}$

III. WINNING STRATEGY FOR NIM

We know the strategy is to force a 0 sum XOR to opponent, but let's see how we can be able to do it for all next move.

In a zero sum XOR, we know that every term 2^i has an even number of occurrences, for all $0 \leq i \leq n$. Keep in mind, that we break every number into term of 2^i.

The opponent must take some sticks out, and give us back a position with non-zero sum XOR. We can simplify this move as:

- Let $x_b \geq 1$ (b stands for "before") be the number of sticks in his selected pile, before he makes move.
- Let $x_a < x_b$ (a stands for "after") be the number of sticks in his selected pile.
  - This means he took out $(x_b - x_a)$ sticks from the pile.

- Then, the new sum XOR gives back to us is:

$$s = \overbrace{\underbrace{0 \oplus x_b}_{exclude\ old\ pile\ from\ old\ sum} \oplus x_a}^{add\ a\ new\ pile\ of\ x_a\ sticks} = x_b \oplus x_a > 0$$

Notice that, the 0 stands for the old sum XOR.
If you get confused by the formula, let's take a look at the lemma:

$$(\mathrm{a} \oplus \mathrm{b} \oplus \mathrm{c}) \oplus a = (a \oplus a) \oplus b \oplus c = b \oplus c$$

So, excluding a term from a sum XOR is equivalent to XOR it with the sum one more time.

So, we have a new non-zero sum XOR $s > 0$ back. Let's find a way to him a new sum XOR.

- We know $s \oplus s = 0$. However, a pile of exactly s sticks is not always existed in the position.
- But if we expressed into term 2^i:

$$s = \sum_{i=0}^{n} \left(s_i \times 2^i\right)$$

Whenever $s_i = 1$, we know that term 2^i occurred some odd times in the position. Notice that, if we have a pile of 2^i we can turn the pile to any number less than 2^i sticks with our move.

Then, consider $k = \max(\{i \mid s_i = 1\})$, we can make a move by removing a term 2^k, and turn its to $\left(s - 2^k\right)$, which is our recommended move. Notice that: $\left(s - 2^k\right) = s \oplus 2^k = a$ should have form as $a = \sum_{i=0}^{n}\left(a_i \times 2^i\right)$ such that $a_i = \begin{cases} 0 & if\ i = k \\ s_i & otherwise \end{cases}$.

There're also other equivalent definitions for this k:

$$k = \lfloor \log_2 s \rfloor = \left\lfloor \frac{\ln s}{\ln 2} \right\rfloor \text{ or } k = \max\left(\left\{i \middle| \left\lfloor \frac{s}{2^i} \right\rfloor mod\ 2 = 1\right\}\right)$$

Or, "k is the minimum integer i such that $s < 2^{i+1}$".

The new sum XOR giving to opponent is

$$s \oplus 2^k \oplus \left(s - 2^k\right) = 0$$

So, the idea is not to subtract the whole s to gain 0 sum XOR, but convert 2^k any less number that match s.

Summary, our steps to force as new 0 sum XOR to opponent are

1. Calculate new sum XOR $s > 0$ given to us. Express s as binary form:

$$s = \sum_{i=0}^{n} \left(s_i \times 2^i\right)$$

2. Find $k = \lfloor \log_2 s \rfloor$
3. Select a pile of x_b sticks such that $\left\lfloor \frac{x_b}{2^k} \right\rfloor mod\ 2 = 1$
4. We'll leave the pile with exactly x_a sticks with our turn. Then, x_a need to satisfy:

$$s \oplus x_b \oplus x_a = 0$$
$$\Rightarrow x_a \oplus 0 = x_a \oplus (s \oplus x_b \oplus x_a)$$
$$\Leftrightarrow x_a = s \oplus x_b$$

So, we'll take away $x_b - x_a$ sticks from the pile and complete our turn. We've successfully forced a new zero sum XOR into opponent.

IV. References

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