# 1 Latent GP and its Time Derivative as a Model of Stellar Activity

Picture this: there exists a latent GP (or set of latent GPs) that describes stellar activity. We have noisy observations (photometry and radial velocities) that are a linear combination of this process, $f(t)$, and its time derivative, $\dot{f}(t)$.

The relevant covariances for this process and its derivative are:

$$\begin{aligned}
\mathrm{Cov}\Big[f(t_n), f(t_m)\Big] &= k(t_n, t_m), \\
\mathrm{Cov}\Big[f(t_n), \dot{f}(t_m)\Big] &= \frac{dk(t_n, t_m)}{dt_m}, \\
\mathrm{Cov}\Big[\dot{f}(t_n), f(t_m)\Big] &= \frac{dk(t_n, t_m)}{dt_n}, \\
\mathrm{Cov}\Big[\dot{f}(t_n), \dot{f}(t_m)\Big] &= \frac{d^2k(t_n, t_m)}{dt_n dt_m}.
\end{aligned} \tag{1}$$

where the middle two definitions are derived in Section 1.2.

Let us now define the noisy observations, photometric and radial velocity measurements at some time $t$, as:

$$\begin{aligned}
\mathrm{F}(t) &= \mathrm{A}f(t) + \mathrm{B}\dot{f}(t), \\
\mathrm{RV}(t) &= \mathrm{C}f(t) + \mathrm{D}\dot{f}(t).
\end{aligned} \tag{2}$$

We want to find the individual covariance functions for each of these observables in terms of the covariances of the latent GP and its derivative (Equation 1). Using these functions, we form the covariance block matrix, $\mathcal{K}$. This block matrix will have dimensions $2 \times 2$ and be positive semi-definite (and symmetric). The diagonals of this matrix will be the variances of F and RV, $K_{\mathrm{F,F}}$ and $K_{\mathrm{RV,RV}}$, and the off-diagonal elements are the relevant covariances, $K_{\mathrm{F,RV}}$ and $K_{\mathrm{RV,F}}$.

## 1.1 Elements of the Block Matrix

$$
\begin{aligned}
K_{\mathrm{F,F}} &= \mathrm{Cov}\Big[\mathrm{F}(t_n), \mathrm{F}(t_m)\Big], \\
&= \mathrm{Cov}\Big[\mathrm{A}f(t_n) + \mathrm{B}\dot{f}(t_n), \mathrm{A}f(t_m) + \mathrm{B}\dot{f}(t_m)\Big], \\
&= \mathrm{E}\Big\{\Big[\mathrm{A}f(t_n) + \mathrm{B}\dot{f}(t_n)\Big]\Big[\mathrm{A}f(t_m) + \mathrm{B}\dot{f}(t_m)\Big]\Big\}, \\
&= \mathrm{E}\Big[\mathrm{A}^2 f(t_n)f(t_m) + \mathrm{AB}f(t_n)\dot{f}(t_m) \\
&+ \mathrm{BA}\dot{f}(t_n)f(t_m) + \mathrm{B}^2\dot{f}(t_n)\dot{f}(t_m)\Big], \\
&= \mathrm{E}\Big[\mathrm{A}^2 f(t_n)f(t_m)\Big] + \mathrm{E}\Big[\mathrm{AB}f(t_n)\dot{f}(t_m)\Big] \\
&+ \mathrm{E}\Big[\mathrm{BA}\dot{f}(t_n)f(t_m)\Big] + \mathrm{E}\Big[\mathrm{B}^2\dot{f}(t_n)\dot{f}(t_m)\Big], \\
&= \mathrm{A}^2\mathrm{Cov}\Big[f(t_n), f(t_m)\Big] + \mathrm{ABCov}\Big[f(t_n), \dot{f}(t_m)\Big] \\
&+ \mathrm{BACov}\Big[\dot{f}(t_n), f(t_m)\Big] + \mathrm{B}^2\mathrm{Cov}\Big[\dot{f}(t_n)\dot{f}(t_m)\Big], \\
&= \mathrm{A}^2 k(t_n, t_m) + \mathrm{AB}\frac{dk(t_n, t_m)}{dt_m} \\
&+ \mathrm{BA}\frac{dk(t_n, t_m)}{dt_n} + \mathrm{B}^2\frac{d^2k(t_n, t_m)}{dt_n dt_m}.
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
K_{\mathrm{F,RV}} &= \mathrm{Cov}\Big[\mathrm{F}(t_n), \mathrm{RV}(t_m)\Big], \\
&= \mathrm{Cov}\Big[\mathrm{A}f(t_n) + \mathrm{B}\dot{f}(t_n), \mathrm{C}f(t_m) + \mathrm{D}\dot{f}(t_m)\Big], \\
&= \mathrm{ACCov}\Big[f(t_n)f(t_m)\Big] + \mathrm{ADCov}\Big[f(t_n)\dot{f}(t_m)\Big] \\
&+ \mathrm{BCCov}\Big[\dot{f}(t_n)f(t_m)\Big] + \mathrm{BDCov}\Big[\dot{f}(t_n)\dot{f}(t_m)\Big], \\
&= \mathrm{AC}k(t_n, t_m) + \mathrm{AD}\frac{dk(t_n, t_m)}{dt_m} \\
&+ \mathrm{BC}\frac{dk(t_n, t_m)}{dt_n} + \mathrm{BD}\frac{d^2k(t_n, t_m)}{dt_n dt_m}.
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
K_{\mathrm{RV,f}} &= \mathrm{Cov}\Big[\mathrm{RV}(t_n), \mathrm{F}(t_m)\Big], \\
&= \mathrm{Cov}\Big[\mathrm{C}f(t_n) + \mathrm{D}\dot{f}(t_n), \mathrm{A}f(t_m) + \mathrm{B}\dot{f}(t_m)\Big], \\
&= \mathrm{CA}k(t_n, t_m) + \mathrm{CB}\frac{dk(t_n, t_m)}{dt_m} \\
&+ \mathrm{DA}\frac{dk(t_n, t_m)}{dt_n} + \mathrm{DB}\frac{d^2k(t_n, t_m)}{dt_n dt_m}.
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
K_{\mathrm{RV,RV}} &= \mathrm{Cov}\Big[\mathrm{RV}(t_n), \mathrm{RV}(t_m)\Big], \\
&= \mathrm{Cov}\Big[\mathrm{C}f(t_n) + \mathrm{D}\dot{f}(t_n), \mathrm{C}f(t_m) + \mathrm{D}\dot{f}(t_m)\Big], \\
&= \mathrm{C}^2k(t_n, t_m) + \mathrm{CD}\frac{dk(t_n, t_m)}{dt_m} \\
&+ \mathrm{DC}\frac{dk(t_n, t_m)}{dt_n} + \mathrm{D}^2\frac{d^2k(t_n, t_m)}{dt_n dt_m}.
\end{aligned} \tag{6}
$$

Then, the final covariance block matrix between the photometry, $F(t)$, and the radial velocities, $\mathrm{RV}(t)$, is:

$$
\mathcal{K} = \begin{bmatrix} K_{\mathrm{F,F}} & K_{\mathrm{F,RV}} \\ K_{\mathrm{RV,F}} & K_{\mathrm{RV,RV}} \end{bmatrix} \tag{7}
$$

Note that the covariance block matrix is symmetric, as required (i.e., $K_{\mathrm{RV,F}}$ is the transpose of $K_{\mathrm{F,RV}}$, or $K_{\mathrm{F,RV}} = K_{\mathrm{RV,F}}^{\mathrm{T}}$).

Some properties about the block matrix **(???)**:

1. Each section within the block matrix has contributions from all four covariance functions: $k$, $dk/dt_n$, $dk/dt_m$, and $d^2k/dt_n dt_m$. Does this require that $t_n$ and $t_m$ be the same dimensions?

## 1.2 Derivative of Covariance

$$
\begin{aligned}
\frac{d}{dt_n}\Big[k(t_n, t_m)\Big] &= \frac{d}{dt_n}\Big\{\mathrm{Cov}\Big[f(t_n), f(t_m)\Big]\Big\}, \\
&= \frac{d}{dt_n}\Big\{\mathrm{E}\Big[f(t_n) \cdot f(t_m)\Big]\Big\}, \\
&= \mathrm{E}\left[\frac{df(t_n)}{dt_n} \cdot f(t_m) + \frac{df(t_m)}{dt_n} \cdot f(t_n)\right], \\
&= \mathrm{E}\left[\frac{df(t_n)}{dt_n} \cdot f(t_m)\right], \\
&= \mathrm{E}\Big[\dot{f}(t_n) \cdot f(t_m)\Big], \\
&= \mathrm{Cov}\Big[\dot{f}(t_n), f(t_m)\Big].
\end{aligned} \tag{8}
$$