CSE 417: Algorithms and Computational Complexity

6: Dynamic Programming, III Longest Increasing Subseq.

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Instructor: W. L. Ruzzo

Three Steps to Dynamic Programming

- Formulate the answer as a recurrence relation or recursive algorithm
- Show that number of different parameters in the recursive algorithm is "small" (e.g., bounded by a low-degree polynomial)
- Specify an order of evaluation for the recurrence so that already have the partial results ready when you need them.

Longest Increasing Subsequence

- Given a sequence of integers $s_1,\ldots,s_n$ find a subsequence $s_{i_1} < s_{i_2} < \ldots < s_{i_k}$ with $i_1 < \ldots < i_k$ so that k is as large as possible.
- e.g. Given 9,5,2,8,7,3,1,6,4 as input,
  - possible increasing subsequence is 5,7
  - better is 2,3,6 or 2,3,4 (either or which would be a correct output to our problem)

Find recursive algorithm

- Solve sub-problem on $s_1,\ldots,s_{n-1}$ and then try to extend using s_n
- Two cases:
  - s_n is not used
    - answer is the same answer as on $s_1,\ldots,s_{n-1}$
  - s_n is used
    - answer is s_n preceded by the longest increasing subsequence in $s_1,\ldots,s_{n-1}$ that ends in a number smaller than s_n

Refined recursive idea (stronger notion of subproblem)

- Suppose that we knew for each $i<n$ the longest increasing subsequence in $s_1,\ldots,s_n$ that ends in s_i.
  - $i=n-1$ is just the $n-1$ size sub-problem we tried before.
- Now to compute value for $i=n$ find
  - s_n preceded by the maximum over all $i<n$ such that $s_i<s_n$ of the longest increasing subsequence ending in s_i
- First find the best **length** rather than trying to actually compute the sequence itself.

Recurrence

- Let $L[i]$=length of longest increasing subsequence in $s_1,\ldots,s_n$ that ends in s_i.
- $L[j]=1+\max\{L[i] : i<j \text{ and } s_i<s_j\}$
  (where max of an empty set is 0)
- Length of longest increasing subsequence:
  - $\max\{L[i]: 1\le i \le n\}$

Computing the actual sequence

- For each j, we computed
  L[j]=1+max{L[i] **:** i<j and s_i<s_j}
  (where max of an empty set is 0)
- Also maintain P[j] the value of the i that achieved that max
  - this will be the index of the predecessor of s_j in a longest increasing subsequence that ends in s_j
  - by following the P[j] values we can reconstruct the whole sequence in linear time.

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Longest Increasing Subsequence Algorithm

- **for** j=1 **to** n **do**
  L[j]←1
  P[j]←0
  for i=1 **to** j-1 **do**
  if (s_i<s_j & L[i]+1>L[j]) **then**
  P[j] ←i
  L[j] ←L[i]+1
  endfor
  endfor
- Now find **j** such that **L[j]** is largest and walk backwards through **P[j]** pointers to find the sequence

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Example

i	1	2	3	4	5	6	7	8	9
s_i									
l_i									
p_i									

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Example

i	1	2	3	4	5	6	7	8	9
s_i	90	50	20	80	70	30	10	60	40
l_i	1	1	1	2	2	2	1	3	3
p_i	0	0	0	2	2	3	0	6	6

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