1 Solution to problem 4.8,page 138

Problem:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \tag{1}$$

1. First step:
   - Multiple A^T with A

$$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \tag{2}$$

2. Second step:Finding eigenvalues from the determinant:

$$A^T A - \Lambda I = \begin{bmatrix} 13-\Lambda & 12 & 2 \\ 12 & 13-\Lambda & -2 \\ 2 & -2 & 8-\Lambda \end{bmatrix} \tag{3}$$

$$\begin{aligned} Det(A^T A - \Lambda I) = (13-\Lambda).(13-\Lambda).(8-\Lambda) + 12.(-2).2 + 2.12.(-2) \\ - (-2).(13-\Lambda).2 - (-2).(-2).(13-\Lambda I) - (8-\Lambda).12.12 \\ = (8-\Lambda)^3 + 10.(8-\Lambda)^2 - 127.(8-\Lambda) - 136 \end{aligned} \tag{4}$$

$$(8-\Lambda)^3 + 10.(8-\Lambda)^2 - 127.(8-\Lambda) - 136 = 0 \tag{5}$$

Solving the equation above,we get:

$$\Lambda \in \{25, 0, 9\}$$

as eigenvalues of matrix A

3. Third step:Finding eigenvectors from eigenvalues
   - Pluging in 25,0,9 for equation (3) and find reduced row echelon form

$$\begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

- Finding solution for the following equations

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

$$\rightarrow v_1 = 1, v_2 = 1, v_3 = 0$$

$$\rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- Unit vector:

$$\rightarrow \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$\rightarrow v_1 = -2, v_2 = -2, v_3 = 1$$

$$\rightarrow \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

- Unit vector:

$$\rightarrow \begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\rightarrow v_1 = 1, v_2 = -1, v_3 = 4$$

$$\rightarrow \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

- Unit vector:

$$\rightarrow \begin{bmatrix} \frac{\sqrt{2}}{6} \\ \frac{-\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix}$$

- From 3 eigenvectors(that has been converted),we can have matrix of eigenvectors and matrix of eigenvalues

$$\mathbf{V} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & \frac{-2}{3} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{6}}{2} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix} \tag{12}$$

$$\mathbf{\Sigma} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \tag{13}$$

4. Fourth Step:Finding U matrix
   - U can be found by finding matrix of eigenvectors of AA^T or by using the following formula:

$$\begin{aligned} \mathbf{U} &= \mathbf{AV\Sigma^{\boldsymbol{T}}} \\ &= \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & \frac{-2}{3} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned} \tag{14}$$

Unit vector:

$$\mathbf{U} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix} \tag{15}$$