MA1521 CALCULUS FOR COMPUTING

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What will we learn in MA1521?

- Chapter 0: Pre-calculus
- Chapter 1: Limits and Continuity
- Chapter 2: Derivatives with Applications
- Chapter 3: Sequences and Series
- Chapter 4: Partial Derivatives
- Chapter 5: Optimization
- Chapter 6: Integrals with Applications
- Chapter 7: Ordinary Differential Equations

Workload and Assessment

- Workload:
  - Lecture: 1.5×2 hours per week (week 1 to 13);
    (Chinese New Year: 11 February);
    Monday, Thursday 12:00 – 1:35pm, LT33.
  - Tutorial: 1 hour per week (week 3 to 13);
- Notes and References:
  - Lecture materials: Available in IVLE,
  - Textbook: `Thomas' Calculus 12th ed.`
- Assessment:
  - Homework Assignments: $5\% \times 3 = 15\%$
  - Tutorial Participation: 5%
  - Mid-Term Test: 20%
  - (11 March, 12:00 – 1:30pm, LT33)
  - Final Exam: 60% (02 May, afternoon)

Sets

- A **set** is a collection of **objects**.
- A set is usually denoted by capital letters $A, B, C, \ldots$.
  - The objects $a, b, c, \ldots$ contained in set A are called the **elements** of A. We write

$$A = \{a, b, c, \ldots\}.$$

  - For example, $\{-1, 1\}$, $\{2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$.
- We can also write a set using **description**:

$$A = \{x \mid \text{properties of } x\}.$$

  - For example, $\{x \mid x^2 = 1\}$, $\{x \mid x \text{ is a prime number}\}$.

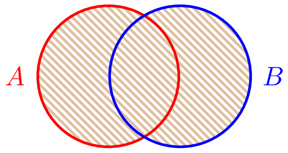
Sets

- If a **is** an element of A, we write $a \in A$;
  If a **is not** an element of A, we write $a \notin A$.
  - Example: $1 \in \{1, 2\}$, $0 \notin \{1, 2\}$.
- If **every** element of set A is also an element of set B, we say A is a **subset** of B, denoted by $A \subseteq B$.

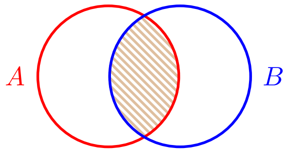
  If A **is not** a subset of B, we write $A \not\subseteq B$.
  - Example: $\{1, 2\} \subseteq \{1, 2, 3\}$, $\{0, 1\} \not\subseteq \{1, 2, 3\}$.
- Two sets are **equal** if they have the same collection of elements, regardless of order.
  In other words, "$A = B$" $\Leftrightarrow$ "$A \subseteq B$ & $B \subseteq A$".
  - Examples:
    - $\{1, 2, 3\} = \{3, 2, 1\}$.
    - $\{x \mid x^2 = 1\} = \{1, -1\}$.

Operations on Sets

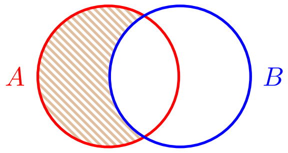
- **Union**: $A \cup B = \{x \mid x \in A$ or $x \in B\}$;



- **Intersection**: $A \cap B = \{x \mid x \in A$ and $x \in B\}$;

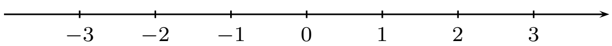


- **Difference**: $A \setminus B = \{x \mid x \in A$ and $x \notin B\}$.



Number System

- $\mathbb{N} = \{1, 2, 3, \ldots\}$: the set of **natural numbers**.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$: the set of **integers**.
  - $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$: **positive integers**;
  - $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$: **negative integers**.
- $\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z}, n \neq 0\}$: **rational numbers**.
- $\mathbb{R}$: the set of **real numbers**.
  - There is a one-to-one correspondence between $\mathbb{R}$ and the points on the number line.



  - $a < b \Leftrightarrow a$ lies to the left of b on the number line.
- $\varnothing$: the **empty set**, the set containing no element.
- Similarly as $\mathbb{Z}^+$ and $\mathbb{Z}^-$, we use $\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{R}^+$, $\mathbb{R}^-$.

Intervals

- Certain subsets of $\mathbb{R}$ can be expressed as **intervals**.
  - **Finite intervals**: (Suppose $a < b$.)
    - $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.

    - $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

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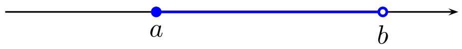
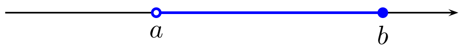


    - $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.

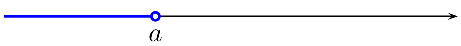


Intervals

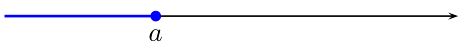
- Certain subsets of $\mathbb{R}$ can be expressed as **intervals**.
  - **Infinite intervals**:
    - $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$.

    - $[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$.

    - $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$.

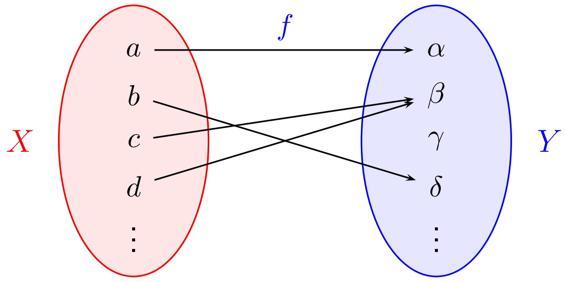


    - $(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$.



Functions

- Let X and Y be two sets.



  - A **function** $f : X \to Y$ is a **rule** which assigns **each** element in X to a **unique** element in Y.
  - If the function f assigns $x \in X$ to $y \in Y$, we say y is the **image** of x under f, denoted by $y = f(x)$.

Functions

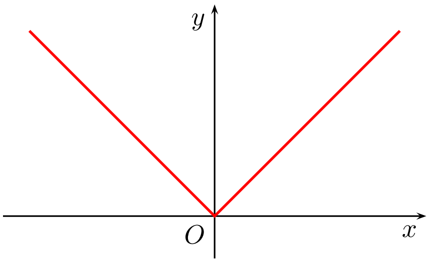
- Let $f : X \to Y$ be a function.
  - X is the **domain** of f;
  - Y is the **codomain** of f.

  Unless otherwise stated, X and Y are always taken to be subsets of the set of real numbers $\mathbb{R}$.
- We make the following convention:
  - If X is not stated, the domain of f is taken to be the **largest** possible set $(\subseteq \mathbb{R})$ on which f is defined.
  - If Y is not stated, take $Y = \mathbb{R}$.
- The **range** is the set of images:
  - range of "$f : X \to Y$" $= \{f(x) \mid x \in X\}$.

  By definition, the range is a subset of the codomain.

Absolute Value Function

- The **absolute value function**:
  - $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$



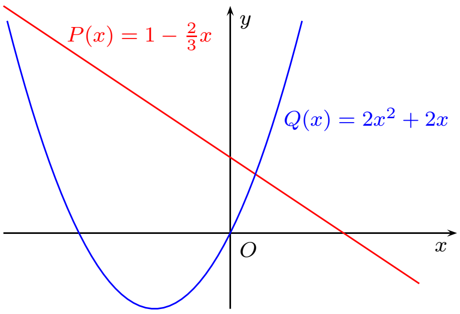
  - Domain: $\mathbb{R}$; Range: $\{x \in \mathbb{R} \mid x \geq 0\}$.
  - $|x|$ represents the **distance** between x and O.
    - $|x| \leq c \Leftrightarrow -c \leq x \leq c$;
    - $|x| < c \Leftrightarrow -c < x < c$.

Polynomials

- A **polynomial** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

  - $a_0, a_1, \ldots, a_n$ are the **coefficients** of $P(x)$;
  - If $a_n \neq 0$, then $n = \deg P(x)$ is the **degree** of $P(x)$.
    - A polynomial of **degree** 1 is a **linear function**.
    - A polynomial of **degree** 2 is a **quadratic function**.

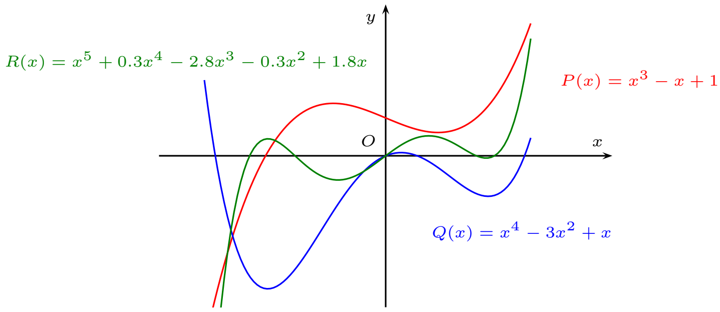


Polynomials

- A **polynomial** is a function of the form

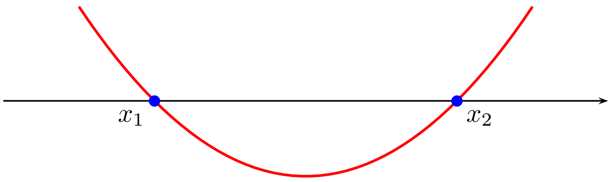
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

  - A polynomial of **degree** 3 is a **cubic function**.
  - A polynomial of **degree** 4 is a **quartic function**.
  - A polynomial of **degree** 5 is a **quintic function**.

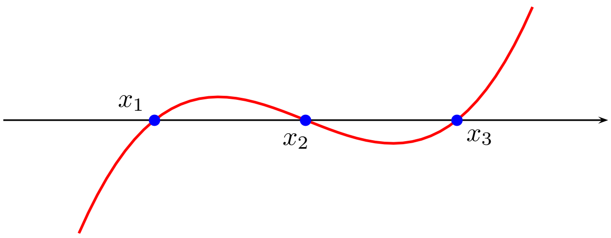


Graphs of Polynomials

- Let $f(x) = (x - x_1)(x - x_2)$, where $x_1 < x_2$.

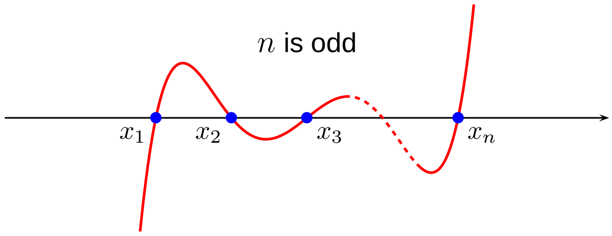


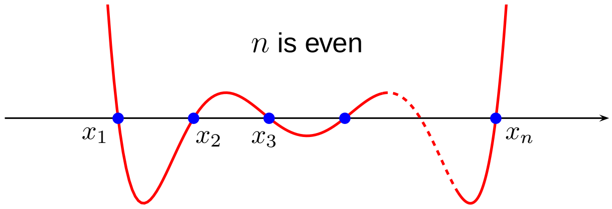
- Let $f(x) = (x - x_1)(x - x_2)(x - x_3)$, $x_1 < x_2 < x_3$.



Graphs of Polynomials

- $f(x) = (x - x_1) \cdots (x - x_n)$, where $x_1 < \cdots < x_n$.



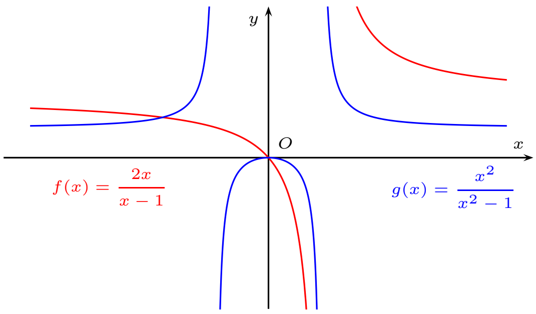


Rational Functions

- A **rational function** $R(x)$ is a function of the form

$$R(x) = \frac{P(x)}{Q(x)},$$

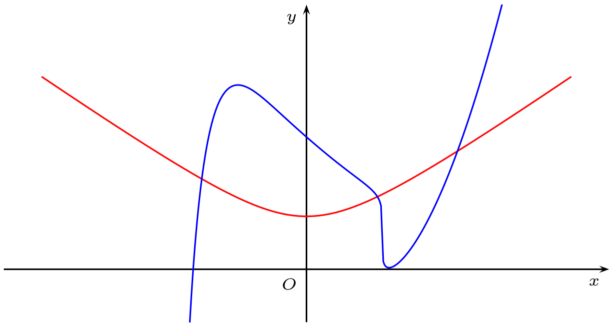
where P, Q are polynomials, $Q(x)$ is not identically zero.



- Every polynomial is a rational function by letting $Q(x) = 1$.

Algebraic Functions

- An **algebraic function** is a function constructed from **polynomials** using **algebraic operations**:
  - addition, subtraction, multiplication, division, taking roots, composite
  - $f(x) = \sqrt{x^2+1}, \quad g(x) = \dfrac{x^3+1}{x+2} + (x-2)\sqrt[5]{x^3-1}$

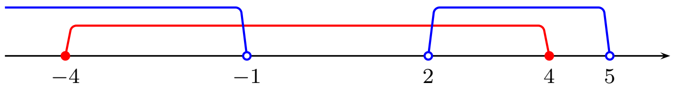


Examples

- Find the domain of $f(x) = \dfrac{1}{x^4-16} + \dfrac{2}{x^4+16}$.
- For fraction, $\dfrac{P(x)}{Q(x)}$ is defined $\Leftrightarrow Q(x) \neq 0$.
  - $\dfrac{1}{x^4-16}$:
    - $x^4 - 16 = (x^2+4)(x-2)(x+2)$.
    - $x^4 - 16 \neq 0 \Leftrightarrow x \neq \pm 2$.
  - $\dfrac{1}{x^4+16}$:
    - $x^4 + 16 \geq 0 + 16 = 16 > 0$.
    - $x^4 + 16 \neq 0$ for all $x \in \mathbb{R}$.
  - Therefore, the domain of f is $\{x \in \mathbb{R} \mid x \neq \pm 2\}$.

Examples

- $f(x) = \sqrt{16 - x^2} + \dfrac{6}{\sqrt{(x+1)(x-2)(5-x)}}$.
- For square root function, $\sqrt{g(x)}$ is defined $\Leftrightarrow g(x) \geq 0$.
    - $\sqrt{16 - x^2}$:
        - $16 - x^2 = (4 - x)(4 + x) = -(x - 4)(x + 4)$.
        - $16 - x^2 \geq 0 \Leftrightarrow (x - 4)(x + 4) \leq 0 \Leftrightarrow -4 \leq x \leq 4$.
    - $\dfrac{6}{\sqrt{(x+1)(x-2)(5-x)}}$:
        - $(x+1)(x-2)(5-x) \geq 0$ and $\neq 0$ (i.e., > 0)
        - $x < -1$ or $2 < x < 5$.



    - Domain: $\{x \in \mathbb{R} \mid -4 \leq x < -1 \text{ or } 2 < x \leq 4\}$.

Examples

- Solve the inequalities $-1 > -x^3 > -8$.
    - $-1 > -x^3 \Leftrightarrow 1 < x^3 \Leftrightarrow \sqrt[3]{1} < \sqrt[3]{x^3} \Leftrightarrow 1 < x$.
    - $-x^3 > -8 \Leftrightarrow x^3 < 8 \Leftrightarrow \sqrt[3]{x^3} < \sqrt[3]{8} \Leftrightarrow x < 2$.

  In general, we need to use polynomial factorization:
    - $-1 > -x^3 \Leftrightarrow x^3 - 1 > 0$.
        - $x^3 - 1 = (x - 1)(x^2 + x + 1) > 0$.
          $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$.

      $\therefore \quad x^3 - 1 > 0 \Leftrightarrow x > 1$.
    - $-x^3 > -8 \Leftrightarrow x^3 - 8 < 0$.
        - $x^3 - 8 = (x - 2)(x^2 + 2x + 4) < 0$.
          $x^2 + 2x + 4 = (x + 1)^2 + 3 > 0$.

      $\therefore \quad x^3 - 8 < 0 \Leftrightarrow x < 2$.

  Therefore, $1 < x < 2$.

Examples

- Solve $|x+2| \geq |2x-3|$.

$$\begin{aligned} |a| \geq |b| &\Leftrightarrow a^2 \geq b^2 \Leftrightarrow a^2 - b^2 \geq 0 \\ &\Leftrightarrow (a-b)(a+b) \geq 0 \end{aligned}$$

$$\begin{aligned} & |x+2| \geq |2x-3| \\ \Leftrightarrow\ & [(x+2)-(2x-3)][(x+2)+(2x-3)] \geq 0 \\ \Leftrightarrow\ & (-x+5)(3x-1) \geq 0 \\ \Leftrightarrow\ & (x-5)(3x-1) \leq 0 \\ \Leftrightarrow\ & \tfrac{1}{3} \leq x \leq 5. \end{aligned}$$

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Examples

- Solve the inequality $x \leq \dfrac{3x}{x+2}$.

$$\begin{aligned} x \leq \frac{3x}{x+2} &\Leftrightarrow x - \frac{3x}{x+2} \leq 0 \\ &\Leftrightarrow \frac{x(x+2)-3x}{x+2} \leq 0 \\ &\Leftrightarrow \frac{x(x-1)}{x+2} \leq 0 \\ &\Leftrightarrow x(x-1)(x+2) \leq 0 \text{ and } x+2 \neq 0. \end{aligned}$$

  - $x(x-1)(x+2) \leq 0 \Leftrightarrow x \leq -2$ or $0 \leq x \leq 1$.

  Hence, the answer is "$x < -2$ or $0 \leq x \leq 1$".

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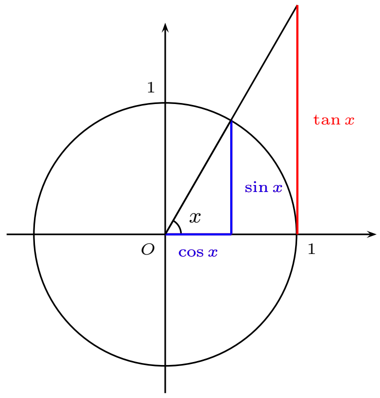
Trigonometric Functions

- The **trigonometric functions**
  - $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$.

  are the ratios of the sides of a right angle triangle.



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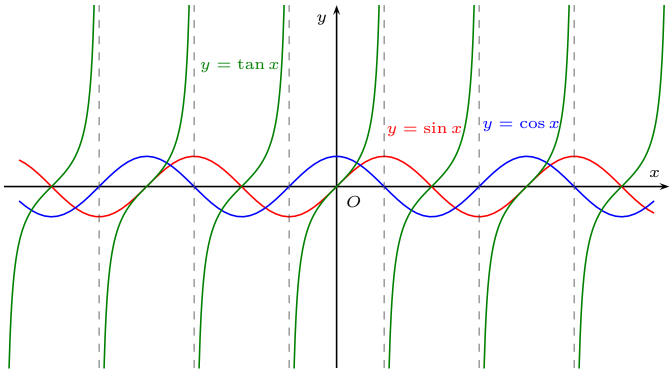
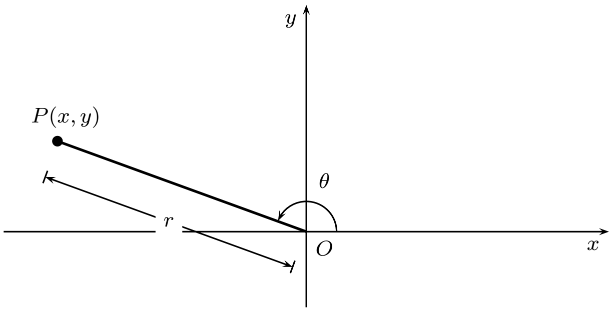


Trigonometric Functions

- Let $P(x, y)$ be a point and let $|OP| = r = \sqrt{x^2 + y^2}$.



  - $\sin\theta = \dfrac{y}{r}$; $\quad \cos\theta = \dfrac{x}{r}$; $\quad \tan\theta = \dfrac{y}{x}$;
  - $\csc\theta = \dfrac{r}{y}$; $\quad \sec\theta = \dfrac{r}{x}$; $\quad \cot\theta = \dfrac{x}{y}$.

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Trigonometric Identities

- Identities on trigonometric functions:
  - $\sin^2\theta + \cos^2\theta = 1$; $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$;
  - $\sec^2\theta - \tan^2\theta = 1$; $\quad \csc^2\theta - \cot^2\theta = 1$.
- Trigonometric functions of compounded angles:
  - $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$;
  - $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.
- Double-angled formulas:
  - $\sin 2\alpha = 2\sin\alpha\cos\alpha$;
  - $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.
- Periodicity:
  - $\sin(\alpha + 2\pi) = \sin\alpha$; $\quad \cos(\alpha + 2\pi) = \cos\alpha$;
  - $\tan(\alpha + \pi) = \tan\alpha$.

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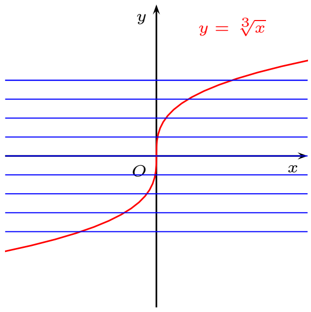
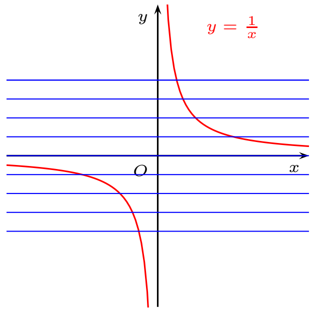
One to One Functions

- Consider the two functions $f(x) = \sqrt[3]{x}$ and $g(x) = 1/x$.
  - Do they have any common property?





  - Every horizontal line cuts each graph **at most once**.
  - In other words, f and g *never take on the same value twice* (or more).
  - This lends to the definition of **one to one function**.

One to One Functions

- **Definition**. Let f be a function with domain D.
  - f is said to be **one to one** if
    - for any $a, b \in D$, $a \neq b \Rightarrow f(a) \neq f(b)$.

    Or equivalently, ("$P \Rightarrow Q$" $\Leftrightarrow$ "not $Q \Rightarrow$ not P"),
    - for any $a, b \in D$, $f(a) = f(b) \Rightarrow a = b$.

    In short, **one to one** means **not many to one**.

- **Examples.** $f(x) = \sqrt[3]{x}$ and $g(x) = 1/x$.
  - Suppose $f(a) = f(b)$, i.e., $\sqrt[3]{a} = \sqrt[3]{b}$.
    - Then $(\sqrt[3]{a})^3 = (\sqrt[3]{b})^3$. That is, $a = b$.
  - Suppose $g(a) = g(b)$, i.e., $1/a = 1/b$.
    - Then $(1/a)^{-1} = (1/b)^{-1}$. That is, $a = b$.

Inverse Functions

- $f(x) = \sqrt[3]{x}$ and $h(x) = x^3$ are the **inverse operations** of each other.

$$\boxed{x} \xrightarrow{\quad f \quad} \boxed{\sqrt[3]{x}} \xrightarrow{\quad h \quad} \boxed{(\sqrt[3]{x})^3 = x}$$

$$\boxed{x} \xrightarrow{\quad h \quad} \boxed{x^3} \xrightarrow{\quad f \quad} \boxed{\sqrt[3]{x^3} = x}$$

Definition. Let f be a **one to one function** with

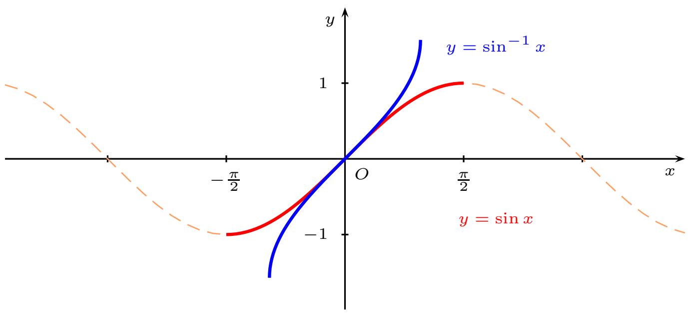
- domain A and range B.

Its **inverse function** f^{-1} is the function with

- domain B and range A, and
- $f^{-1}(y) = x \Leftrightarrow y = f(x)$ for any $x \in A, y \in B$.

Inverse Sine Function

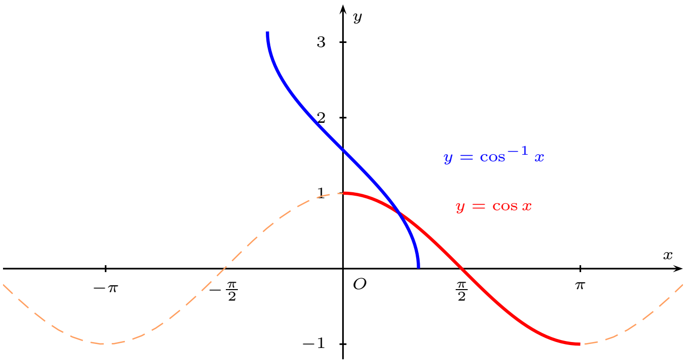
- Let $y = \sin x$. It is not one to one on $\mathbb{R}$.



- But we can restrict the domain on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
  Then $\sin x$ is one to one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, range $= [-1, 1]$.
- The **inverse sine function** is
  - $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
  - $\sin^{-1} x = y \Leftrightarrow x = \sin y$.

Inverse Cosine Function

- Let $y = \cos x$. It is not one to one on $\mathbb{R}$.

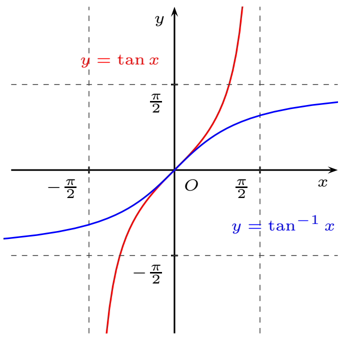


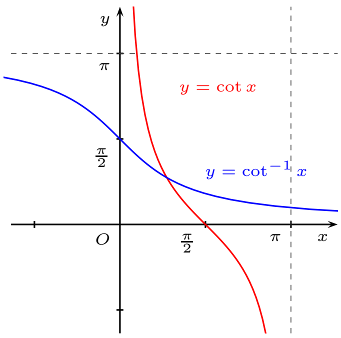
But $\cos x$ is one to one on $[0, \pi]$, range $= [-1, 1]$.

- The **inverse cosine function** is
  - $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$.
  - $\cos^{-1} x = y \Leftrightarrow x = \cos y$.

Inverse Trigonometric Functions

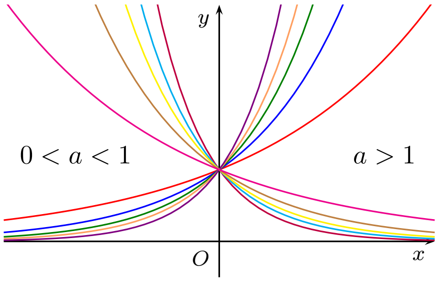
- We have the following **inverse trigonometric functions** from the given domain to its range:
  - $\tan^{-1} x : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.
  - $\cot^{-1} x : \mathbb{R} \longrightarrow (0, \pi)$.





Exponential Functions

- Let $a > 0$ and $a \neq 1$. Consider the **exponential function** $f(x) = a^x$, $x \in \mathbb{R}$.



  - a^x is **one to one** on $\mathbb{R}$, and its range is $\mathbb{R}^+$.
  - It admits an inverse function $\log_a : \mathbb{R}^+ \to \mathbb{R}$:
    - $y = \log_a x \Leftrightarrow x = a^y$.

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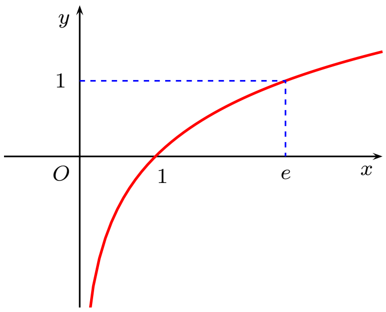
Logarithmic Functions

- Let e denote the **Euler number**:
  - $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = 2.71828\cdots$

  We call $\log_e x = \ln x$ the **natural logarithm function**.



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Properties

- Exponential Functions: Let $a > 0$.
  - $a^0 = 1$; $a^{-x} = 1/a^x$;
  - $a^x a^y = a^{x+y}$; $(a^x)^y = a^{xy}$.
- Logarithmic Functions: Let $a, x, y > 0$ with $a \neq 1$.
  - $\log_a x + \log_a y = \log_a xy$;
  - $\log_a x - \log_a y = \log_a (x/y)$;
  - $\log_a (x^b) = b \log_a x$;
  - $\log_b x = \dfrac{\log_a x}{\log_a b}$, where $b > 0$ and $b \neq 1$.
- Relations:
  - $a^{\log_a x} = x$ for all $x > 0$ and $a > 0$, $a \neq 1$.
  - $\log_a (a^x) = x$ for all $a > 0$, $a \neq 1$.

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Examples

- Prove that $a^{\ln b} = b^{\ln a}$ for all $a > 0$ and $b > 0$.
  - Let $X = a^{\ln b}$. Then
    - $\ln X = \ln(a^{\ln b}) = \ln b \cdot \ln a$.
  - Let $Y = b^{\ln a}$. Then
    - $\ln Y = \ln(b^{\ln a}) = \ln a \cdot \ln b$.

    So $\ln X = \ln Y$. It follows that
    - $X = e^{\ln X} = e^{\ln Y} = Y$.
  - That is, $a^{\ln b} = b^{\ln a}$.

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Examples

- Find the domain of $f(x) = \cos\ln(5-x) + \tan\sqrt{x-3}$.
  - $\ln x$ is defined on positive numbers.
    - $\ln(5-x)$: $\quad 5-x>0 \Leftrightarrow x<5$.
  - $\sqrt{x}$ is defined on nonnegative numbers.
    - $\sqrt{x-3}$: $\quad x-3\geq 0 \Leftrightarrow x\geq 3$.
  - $\tan x = \dfrac{\sin x}{\cos x}$ is defined when $\cos x \neq 0$.
    - So $\sqrt{x-3} \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$.

    $\sqrt{x-3} < \sqrt{5-3} = \sqrt{2} \approx 1.414 < 1.57 \approx \frac{\pi}{2} < \frac{3\pi}{2} < \cdots$

  Hence, the domain is $[3,5)$.

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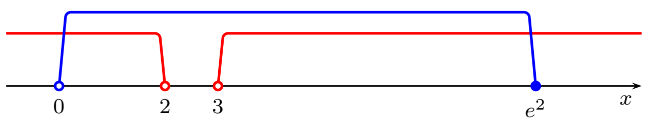
Examples

- Find the domain of $f(x) = \dfrac{x^2+x+2}{\sqrt{x^2-5x+6}} + \sqrt{2-\ln x}$.
  - $\dfrac{x^2+x+2}{\sqrt{x^2-5x+6}}$:
    - $x^2-5x+6 \geq 0$ and $\neq 0$ (i.e., >0).
    - $x^2-5x+6 = (x-2)(x-3) > 0 \Leftrightarrow x<2$ or $x>3$.
  - $\sqrt{2-\ln x}$:
    - $\ln x$ is defined: $\quad x>0$.
    - $2-\ln x \geq 0 \Leftrightarrow \ln x \leq 2 \Leftrightarrow x \leq e^2$.



  Hence, the domain is $(0,2)\cup(3,e^2]$.

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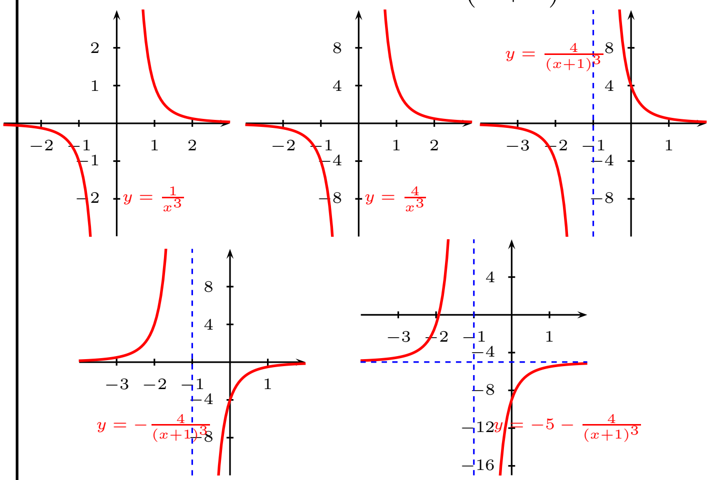
Graph Sketching

- Given a function $f(x)$.
  - Basic Transformation of Graphs:
    - $f(x-k)$: k units to the right;
    - $f(x+k)$: k units to the left;
    - $f(x)+k$: k units up;
    - $f(x)-k$: k units down;
    - $f(-x)$: reflection about y-axis;
    - $-f(x)$: reflection about x-axis;
    - $kf(x)$: scale along y-axis by k;
    - $f(kx)$: scale along x-axis by $1/k$.

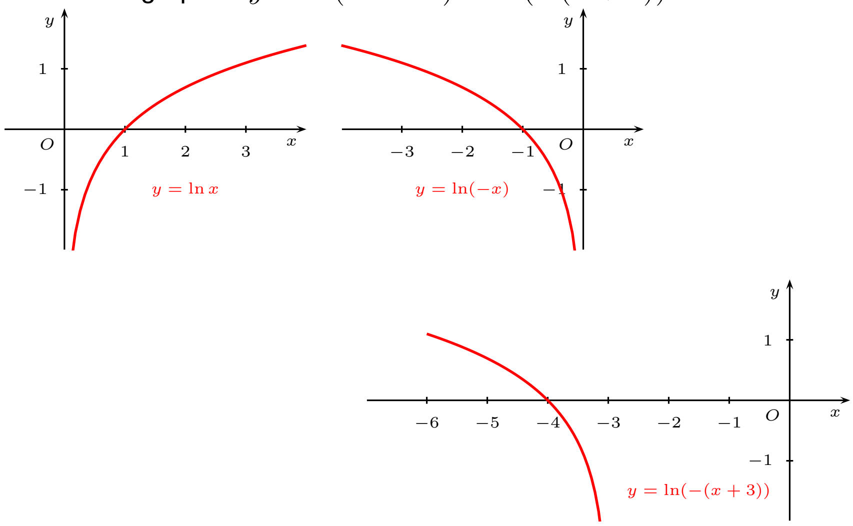
Examples

- Sketch the graph of $y = -5 - \dfrac{4}{(x+1)^3}$.



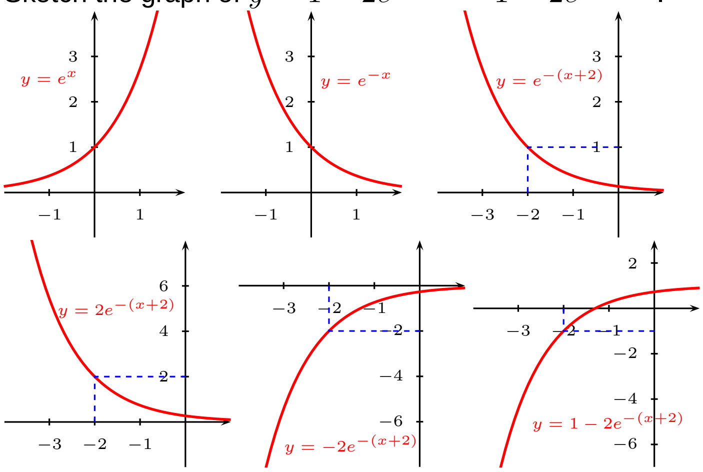
Examples

- Sketch the graph of $y = \ln(-x-3) = \ln(-(x+3))$.



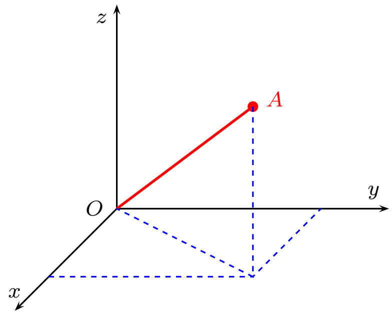
Examples

- Sketch the graph of $y = 1 - 2e^{-x-2} = 1 - 2e^{-(x+2)}$.



Three-Dimensional Space

- The **three-dimensional space** is the set of points:
  - $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.
- Let $A(x_1, y_1, z_1)$ be a point and $O(0, 0, 0)$ be the origin.
  - The vector $\overrightarrow{OA}$ is called the **position vector** of A.
  - Its **length** is denoted by $|\overrightarrow{OA}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$.



Three-Dimensional Space

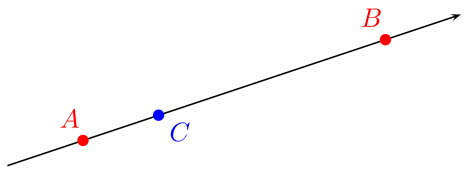
- If $\mathbf{v} \neq \mathbf{0} (= (0, 0, 0))$, then $\mathbf{v} = |\mathbf{v}| \left(\dfrac{\mathbf{v}}{|\mathbf{v}|} \right)$.
  - $\left| \dfrac{\mathbf{v}}{|\mathbf{v}|} \right| = \dfrac{1}{|\mathbf{v}|} |\mathbf{v}| = 1 \Rightarrow \dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a **unit vector**.
- Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if and only if
  - $\mathbf{u} = \lambda \mathbf{v}$ for some $\lambda \in \mathbb{R}$, denoted by $\mathbf{u} \parallel \mathbf{v}$.
- Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. Then
  - $\overrightarrow{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

  Without ambiguity, sometimes we may also write
  - $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Lines in Three-Dimensional Space

- Consider the straight line L passing through points A, B.
  - $\mathbf{u} = \overrightarrow{AB}$ is a **direction vector**.

  Let $\mathbf{a}, \mathbf{b}$ be the position vectors of A, B, respectively.
  Let $\mathbf{r}$ be the position vector of $C \in L$.



  - $(\mathbf{r} - \mathbf{a}) \parallel (\mathbf{b} - \mathbf{a}) \Rightarrow (\mathbf{r} - \mathbf{a}) = \lambda(\mathbf{b} - \mathbf{a})$, $\lambda \in \mathbb{R}$.

  Therefore, L can be represented by

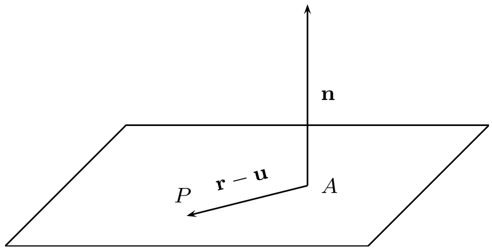
  - $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, $\lambda \in \mathbb{R}$.

Scalar Products

- Let $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$.
  - Their **scalar product** is defined by
    - $\mathbf{u} \bullet \mathbf{v} = x_1x_2 + y_1y_2 + z_1z_2$.
  - Geometric meaning:
    - $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$, θ is the angle between $\mathbf{u}, \mathbf{v}$.
- Properties:
  - $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$;
  - $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$;
  - $\lambda(\mathbf{u} \bullet \mathbf{v}) = (\lambda\mathbf{u}) \bullet \mathbf{v} = (\mathbf{u}) \bullet (\lambda\mathbf{v})$;
  - $\mathbf{u} \bullet \mathbf{u} = |\mathbf{u}|^2$;
  - $\mathbf{u} \bullet \mathbf{v} = 0 \Leftrightarrow \mathbf{u} \perp \mathbf{v}$ ($\mathbf{u}$ and $\mathbf{v}$ are perpendicular).

Planes in Three-Dimensional Space

- Let Π be a plane containing point A with position vector $\mathbf{u}$, and let $\mathbf{n}$ be a **normal vector** of Π (i.e., $\mathbf{n} \perp \Pi$).
  - Let $\mathbf{r}$ be the position vector of any point P.



$$\begin{aligned} P \in \Pi &\Leftrightarrow \overrightarrow{AP} \perp \mathbf{n} \\ &\Leftrightarrow (\mathbf{r} - \mathbf{u}) \bullet \mathbf{n} = 0 \\ &\Leftrightarrow \mathbf{r} \bullet \mathbf{n} = \mathbf{u} \bullet \mathbf{n}. \end{aligned}$$

  - Therefore, the equation of Π is given by $\mathbf{r} \bullet \mathbf{n} = \mathbf{u} \bullet \mathbf{n}$.

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Vector Products

- Let $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$.
  - Their **vector product** is defined by

    $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$; equivalently, it equals
    $(y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$.
  - Geometric meaning:
    - $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, and its direction is given by the **right-hand rule**.
    - $|\mathbf{u} \times \mathbf{v}|$ represents the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$:
      - $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,\sin\theta$, θ: angle between $\mathbf{u}, \mathbf{v}$.

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Vector Products

- Properties of vector products:
  - $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$;
  - $\mathbf{i} \times \mathbf{j} = \mathbf{k}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$;
  - $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$;
  - $\lambda(\mathbf{u} \times \mathbf{v}) = (\lambda\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\lambda\mathbf{v})$;
  - $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{u} \parallel \mathbf{v}$;
  - $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \bullet \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \bullet \mathbf{v}$.
- Applications of vector products:
  - Suppose vectors $\mathbf{u}$ and $\mathbf{v}$ are non-parallel vectors which are parallel to a plane Π.
    - A normal vector of Π is given by $\mathbf{u} \times \mathbf{v}$.

Examples

- Find the equation of the line L passing through points $(-2, -1, 0)$ and $(3, 2, -3)$.
  - A direction vector of L is given by
    - $(3, 2, -3) - (-2, -1, 0) = (5, 3, -3)$.
  - Any point on L can be written as
    - $(-2, -1, 0) + \lambda(5, 3, -3), \quad \lambda \in \mathbb{R}$.
  - Therefore, the equation of L is
    - $\mathbf{r} = (-2 + 5\lambda)\mathbf{i} + (-1 + 3\lambda)\mathbf{j} - 3\lambda\mathbf{k}$.
- **Remark**. Note that the representation is not unique.
  - $(3, 2, 3) + \lambda(5, 3, -3)$, $\lambda \in \mathbb{R}$, is also a solution.

Examples

- Find the equation of the plane Π, if
  - Π contains $(2, 2, 2)$, and
  - Π is perpendicular to $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
- **Solution**.
  - A normal vector: $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$;
    A point in Π: $(2, 2, 2)$.
  - Therefore, the equation of Π is:
    - $(x, y, z) \bullet (3, 1, -2) = (2, 2, 2) \bullet (3, 1, -2)$.

    That is,
    - $3x + y - 2z = 4$.

Examples

- Find the equation of the plane Π, if
  - Π contains $(2, 2, 2)$ and
    the line $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$, $\lambda \in \mathbb{R}$.
- **Solution**.
  - In order to get a normal vector to Π, we need two vectors parallel to Π:
    - One is given by $(3, 1, -2) \parallel \Pi$;
    - Let $\lambda = 0$ in the line. $(1, -2, 3) \in \Pi$.
      - $(2, 2, 2) - (1, -2, 3) = (1, 4, -1) \parallel \Pi$.
  - Normal vector: $(3, 1, -2) \times (1, 4, -1) = (7, 1, 11)$.
  - Equation of Π:
    - $(x, y, z) \bullet (7, 1, 11) = (2, 2, 2) \bullet (7, 1, 11)$;
    - That is, $7x + y + 11z = 38$.

Examples

- Find the equation of the plane Π, if Π contains lines
  - $L_1 : \mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda_1(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$, $\lambda_1 \in \mathbb{R}$, and
  - $L_2 : \mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda_2(\mathbf{i} + 3\mathbf{k})$, $\lambda_2 \in \mathbb{R}$.
- **Solution**.
  - $(3, 1, -2)$ and $(1, 0, 3)$ are vectors parallel to Π.
  - Normal vector to Π:
    - $(3, 1, -2) \times (1, 0, 3) = (3, -11, -1)$.
  - Let $\lambda_1 = 0$ in L_1 (or $\lambda_2 = 0$ in L_2).
    - $(1, -2, 3)$ is a point in Π.
  - Equation of Π is given by
    - $(x, y, z) \bullet (3, -11, -1) = (1, -2, 3) \bullet (3, -11, -1)$
    - That is, $3x - 11y - z = 22$.

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Examples

- Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Find the unit vector $\mathbf{v}$ such that
  - $\mathbf{u} \bullet \mathbf{v}$ has the largest possible value;
  - $\mathbf{u} \bullet \mathbf{v}$ has the smallest possible value.
- **Solution.**
  - Recall: $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, θ: angle between $\mathbf{u}, \mathbf{v}$.
  - $|\mathbf{u}|$ is given $(= 3)$; and $|\mathbf{v}| = 1$.
  - $\mathbf{u} \bullet \mathbf{v}$ is the largest $\Leftrightarrow \cos\theta = 1 \Leftrightarrow \theta = 0$.
    - $\mathbf{v}$ is parallel to $\mathbf{u}$ of the same direction.
    - $\mathbf{v} = \mathbf{u}/|\mathbf{u}| = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.
  - $\mathbf{u} \bullet \mathbf{v}$ is the smallest $\Leftrightarrow \cos\theta = -1 \Leftrightarrow \theta = \pi$.
    - $\mathbf{v}$ is parallel to $\mathbf{u}$ of the opposite direction.
    - $\mathbf{v} = -\mathbf{u}/|\mathbf{u}| = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$.

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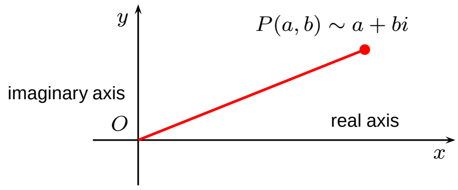
Complex Numbers

- Let $i = \sqrt{-1}$. The set of **complex numbers** is
  - $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$.
- Let $z = a + bi$ be a complex number, $a, b \in \mathbb{R}$.
  - a is the **real part** of z, denoted by $\mathrm{Re}\, z$;
  - b is the **imaginary part** of z, denoted by $\mathrm{Im}\, z$.
  - $a - bi$ is the **conjugate** of z, denoted by z^*.
  - $\sqrt{a^2 + b^2}$ is the **modulus** of z, denoted by $|z|$.
- A complex number $z = a + bi$, $a, b \in \mathbb{R}$, can be identified as a point $(a, b) \in \mathbb{R}^2$.



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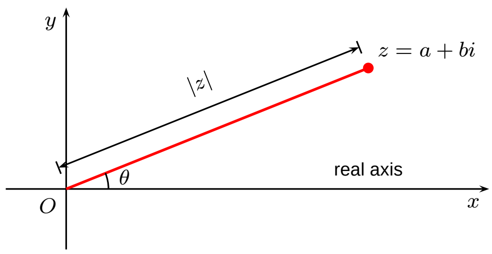
Complex Numbers

- Let $z = a + bi$, $a, b \in \mathbb{R}$.
  - The angle $\theta \in (-\pi, \pi]$ between z and the real axis is called the argument of z, denoted by $\arg z$.



  - Then $a = |z| \cos\theta$ and $b = |z| \sin\theta$.
  - $z = |z|(\cos\theta + i \sin\theta)$ is the **polar form** of z.
  - **Euler's formula**:
    - $e^{i\theta} = \cos\theta + i \sin\theta$.

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Arithmetic Operations on $\mathbb{C}$

- Addition and Subtraction:
  - $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$.
- Multiplication:
  - $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$.
  - Polar form: $z_1 = |z_1|e^{i\alpha}$ and $z_2 = |z_2|e^{i\beta}$;
    - $z_1 z_2 = |z_1|\,|z_2|\,e^{i(\alpha+\beta)}$.
  - In particular, let $z = a+bi$, $a, b \in \mathbb{R}$.
    - $zz^* = (a+bi)(a-bi) = a^2 + b^2 = |z|^2$.
  - **De Moivre's theorem**:
    - $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n \in \mathbb{Z}$.

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Arithmetic Operations on $\mathbb{C}$

- Division:
  - $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di}$
    - Multiply numerator and denominator by $(c+di)^*$.
    - $\dfrac{a+bi}{c+di} = \dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}$.
  - Polar form: $z_1 = |z_1|e^{i\alpha}$ and $z_2 = |z_2|e^{i\beta}$;
    - $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}\,e^{i(\alpha-\beta)}$.
- **Example**. Evaluate $\dfrac{1+2i}{3+4i}$.
  - $\dfrac{1+2i}{3+4i} = \dfrac{1+2i}{3+4i}\,\dfrac{3-4i}{3-4i} = \dfrac{11+2i}{25}$.

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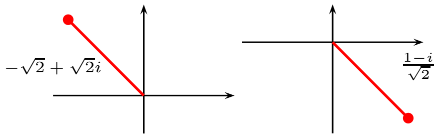
Examples

- Evaluate $\left(\dfrac{1}{-\sqrt{2}+\sqrt{2}\,i}\right)^{2011}$.
  - Let $z=-\sqrt{2}+\sqrt{2}\,i$. Then $|z|=\sqrt{2+2}=2$.



  - $\arg z=\frac{3\pi}{4}$. $\quad z=2e^{i\frac{3\pi}{4}}$.
  - LHS $=\left(\frac{1}{z}\right)^{2011}=2^{-2011}e^{-2011\times\frac{3\pi}{4}i}$

$$=2^{-2011}e^{-(1508\pi i+\frac{\pi i}{4})}=2^{-2011}e^{-\frac{\pi i}{4}}$$
$$=2^{-2011}\cdot\frac{1-i}{\sqrt{2}}$$
$$=\frac{1}{2^{2011}\sqrt{2}}-\frac{1}{2^{2011}\sqrt{2}}i.$$

Examples

- Prove that for any θ with $\theta\neq\pm\pi,\pm3\pi,\pm5\pi,\ldots$
  - $\dfrac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}=\cos\theta+i\sin\theta.$
- **Proof.**
  - $(1+\cos\theta-i\sin\theta)(\cos\theta+i\sin\theta)$
    $=\cos\theta+\cos^2\theta-i\sin\theta\cos\theta+i\sin\theta+i\cos\theta\sin\theta+\sin^2\theta$
    $=1+\cos\theta+i\sin\theta.$

  Alternative Proof.

  - Let $z=\cos\theta+i\sin\theta$.
    - The identity becomes $\dfrac{1+z}{1+z^*}=z$, where $|z|=1$.
    - Apply the multiplication:
      - $(1+z^*)z=z+z^*z=z+|z|^2=z+1.$