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Progress Report on S, T-Dualities and sufficient conditions for the Weak Gravity Conjecture

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Abstract: This note mostly reviewed Weak Gravity Conjecture (WGC) lecture series of Prof. Matt Reece at "Spring School on Superstring Theory and Related Topics", ICTP, 2019; presentations of Prof. Timo Weigand at "KITP's Blackboard Talk", 2020; Ass. Prof. Chethan Krishnan at "Bangalore Area Strings Meeting", 2017; Prof. Gary Shiu at Instituto de Física Teórica workshop, Madrid, 2018; and the papers [1], [2], as well as other relevant literatures. Then we learnt about Superstring theories with Dualites.

Contents

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1 Swampland Program

There are infinitely many families of consistent Quantum Field Theories (QFTs), and we expect the same thing in Quantum Gravity Theories (QGTs). However, the Swampland Program is to restrict much of those possibilities of QGTs that make sense.

Consider the space of consistent Effective Field Theories (EFTs) that pertubatively couple to gravity, not all of these can be completed into full QFT of gravity. The QGTs we know are rather special (various families of continuous moduli space of QGTs, isolated AdS vacua), which are classified as the Landscape of QG. The idea of Swampland Program is to identify the boundary of the Landscape and the theories that can not be UV completed into consistent theories of QG. [3]

For last few years, there has been attempts to go back and forth [4], [5] between **2 approaches:**

- (Top down) To start with sets of general principles (holography, BH physics), we argue that some certain types of EFTs are inconsistent with these principles that could not be completed into QGTs.
- (Bottom up) To study explicit examples having the same features and generalize them - This approach is at risk of leading to fallacy that the theory is can be still on the landscape, just we're not clever enough to include those properties.

1.1 Global symmetries in QG

In QFT, global symmetries occurs frequently and very useful, such as:

- Continuos (or discrete) symmetries acting on local fields, such as: $SU(N)_L \times SU(N)_R$ symmetry in QCD.
- Continuous (or discrete) p-form symmetries acting on extended objects, such as: free U(1) gauge field has an U(1) 1-form symmetry with 2 conserved currents $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, which act on Wilson - 't Hoof loops. (Ref. [6])

Claim: There is none of global symmetries in theory of QG.
Arguments:

- (For perturbative string theories) If we have a perturbative string theory and a global symmetry in space-time, there would be a current operator in the world-sheet of the string, which can be dressed with the factor $e^{ikX}\partial X^\mu$ to make vertex operator that creates a massless spin-1 particle in this space (Ref. [7]). Hence, that global symmetry is gauged.
- (For AdS/CFT) If we have a gauge theory with conserved current j^μ coupled to a gauge field A_μ, then in dual description, there is a massless gauge field A_μ in the bulk. So again, there is a sort of flavor given a conserved current, the gauge field just explicitly has to appear through the holographic dictionary.
- (BHs Physics) If we have a global symmetric theory, then everything in the theory, even BHs respect the symmetry. If we throw things into the BHs that has global symmetric charge, the BHs states have to "know" what the charge is and it has to be conserved as the BHs evaporate (Hawking radiate). But for the global symmetric charge, there is nothing outside the BHs that tell us what the charge is (nothing can be measured, no electric fields) . Ref. [8] [MORE]

All of the arguments so far are only for continuous symmetries. For discrete symmetries, in practice, we can usually prove that it is gauged whenever we can find one.

- (General Arguments) against global symmetries in assymptotically AdS QG, using holography Ref. [9]. To really define a global symmetry (since there are certain pathological cases that we have to exclude) in a local QFT as $U(g, \mathcal{R})$ that involve some operators that denpends on the element of gauge group g and spacial region $\mathcal{R}$. We should be able to construct the operators that only act with that symmetry within some bounded region. In case that we have a conserved current, this just follows from the existence of having a local current but we want to claim that this should be sort of an axiom of what we mean by global symmetry. It's a reasonable claim since if we don't make this kind of assumption there are sort of pathological examples of symmetries that we wouldn't really want to think of as symmetries.[MORE]

1.2 Completeness Conjecture

Claim: Given a QG theory that has some gauge fields, there are lattices of allowed charges under those gauge fields (could be electric charges, magnetic charges, but there's some set of allowed consistent charges), then for each possible charge, there is some state of the theory exists that carries that charge, e.g: Magnetic monopoles exist, things charged under Ramond–Ramond gauge fields exist - D-brane [10].

Remarks: We should be careful about boundary conditions. This is something that seems to be true an asymptotically flat space theories in QG, but in the AdS context, we could have boundary conditions that allow a state of electric charges but not magnetic charges; there could also be magnetically charged objects but we would either need them to come in a pair with something with an anti-charge or we need to change our boundary conditions. So there are some subtleties about boundary conditions, but it's still kind of morally true in the AdS theories that the statement holds. Examples of what it implies is some sort of magnetic monopoles should exist or at least some sort of dyons that if we put them together in a multi particle state with things with electric charge we can make things that have surely a magnetic charges.

One reason why Polchinski was a good person to suggest this was, if we had this idea before D-branes were discovered, we would have known that these were Ramon Ramon gauge fields in string theory and concluded that something charged under Ramond–Ramond gauge fields exist must exist, which is D-branes that's a case is proved to be true it wasn't kind of initially obvious that a charged object exists but they turned out to exist.

This statement is pretty well-established but is super useful for the real world because it doesn't tell us how heavy these objects are. We could say with with confidence that we believe that magnetic monopole should exist in the real world but if we can't tell an experimentalist what its masses that's not a super useful claim to make.

This completeness conjecture, is recently believed to follows from the statement of no global symmetries. The reason is if we don't have objects of all allowed charges we'll have these kind of higher form symmetries that act on the Wilson loops corresponding to the missing charges. E.g: A U(1) gauge theory with charges in multiple of 3 has a Z/3Z 1-form that act on the Wilson loops, which is a global symmetry. If we rule out all kinds of global symmetries, including discrete types and types that act on extended object, then the Completeness Conjecture follows as a consequence. Ref. [9]

1.3 BHs extremality bounds

If we have a BH with the global symmetric charge, there's no way to measure that global symmetry from outside the BH (that's not proof for gauge charges at least for continuous ones). If we have a BH charge under continuous gauge symmetry, it has some external electric field that we can just measure from outside the BH.

BHs with given continuous gauge charge have electric fields associated with a lower bound on BH mass, $M_{\rm BH} > cQ_{BH}$. It's just following from the fact that there's an electric field outside the BHs, the electric field stores energy and that energy contributes to the mass of the black hole. Parametrically we can get the right form of the bound:

$$M_{\rm BH} >\sim \int {\rm d}^3 x E^2 \sim \int r^2 {\rm d}r \left(\frac{Q_{\rm BH} e}{r}\right)^2 \sim \frac{Q_{\rm BH}^2 e^2}{R_{\rm BH}^2}, \tag{1.1}$$

with $R_{\rm BH} \sim M_{\rm BH}/M_{\rm Pl}^2$, hence $M_{\rm BH} >\sim Q_{\rm BH} e M_{\rm Pl}$ (up to a normalization factor that depends on each theory).

In **Reissner - Nordstrom BH** (Maxwell - Einstein action), we have the bounds with a normalization factor of $\sqrt{2}$: $M_{\rm BH} \geq \sqrt{2} Q_{\rm BH} e M_{\rm Pl}$, with $M_{\rm Pl}$: reduced Planck mass $\frac{1}{\sqrt{8\pi G_N}}$, $Q_{\rm BH}$ is a quantized charge, the gauged field kinetic term $-\frac{1}{4e^2}F_{\mu\nu}^2$ when a particles of charge Q couples to the gauge field by integrating A over the gauge field worldline and multiplying by q: $q\int_\gamma A$.

For theories where we also have scalar coupled to the gauge fields, we will get different solutions and the normalziation factor actually changes.

In short, **Exemality bound** means for BHs, gravity dominates over the electric force. **Extremal BHs** are ones that saturate the bounds and the claim is that two extremal BHs of the same charge have no net long-range force between them. So the gravitational attration balances the EM repulsion. Hence, any two BHs that are not both extremal (saturating the bound) will be attracted to each other as gravity will overcome the EM repulsion.

Consider BH extremality generalized to a larger class of other theories. For example, considering Einstein-Maxwell-dilation gravity, with the dilaton means we have an additional massless scalar ϕ with the kinetic term of $\frac{1}{2\kappa}\left(-\frac{1}{2}(\partial\phi)^2\right)$ and the way that scaler interacts but the theory is going to be not only by coupling to gravity but by coupling to the gauge field with the coupling $-\frac{1}{4e^2}e^{-\alpha\phi}F_{\mu\nu}^2$, with some constant α.

Remarks This is familiar for instance from the 10D dilaton in string theory, and it also happens with lots of other things, like Kaluza–Klein (KK) theory for instances, we get similar couplings where ϕ is the radion field that tells us how big the circle that we compactify on, so these couplings show up pretty often.

Claim: In this theory, we can find some classical BHs solutions that carry charge, but now in the solution, the scalar field is turned on, and we get a different extremality bound:

$$M_{\rm BH} \geq \sqrt{\frac{2}{1+\alpha^2}} E Q_{\rm BH} M_{\rm Pl}. \tag{1.2}$$

Remark: Now in these solutions the scalar field is turned on, we can't avoid it because we have an electric field so there's a non zero value of $F_{\mu\nu}^2$, so that sources ϕ in the classical solution and so we get a different extremality bound and that bound turns out to be of the same form as the previous bound with a different constant in front it differs by a factor of the $\sqrt{1+\alpha^2}$. This is quite general there are lots of different variations we can study on how scalar fields couple to gauge fields but we will typically still find an extremality bound in any of those cases. The α is the constant that appear in this coupling of the scalar to the gauge field.

Assumption of Quantized Charge: We have been assuming that the charge is quantinzed, which can be thought as another Swampland conjecture. In QG if we have an abelian gauge field, the gauge group is always U(1), it's always compact and charge is always quantized. In fact that claim can be argued with very similar arguments to the one about no global symmetries. So if we had a theory where we had two particles whose charges were irrational multiples of each other one in the square root of two, say, we could throw those charged particles into BHs and make BHs with any combination of those charges that we want, and let them evaporate. We would end up again with infinitely many states in the given range of charges. That argument also appears in Ref. [8], and so if we found the black hole argument for no global symmetries convincing we should also find it convincing the charge is always going to be quantized. We've really been assuming that in when we said there is a charged lattice and magnetic monopoles are part of it. All of that relied crucially on the assumption of quantized charge.

2 Gauge charges & BH evaboration

We have discussed how BHs is characterized by their masses and charges if we had BHs with global symmetry charges they would Hawking evaporate down to some minimum size, maintaining most of their charge. We would get infinitely many states with given charges. For BH that carries charges under a continuous gauge symmetry, BH exist only below the extremality bounds.

If we have BHs and a theory with gauge charges and they Hawking evaporate, if in additon, there are no light particles with charge that they can emit, then their Hawking evaporation will cause them to lose mass, but they will stop when they hit the extremality bound when they can't evaporate anymore. If there are light charges, then the BHs will tend to preferentially reduce the charges they evaporate and follow some curve like Fig. 1.

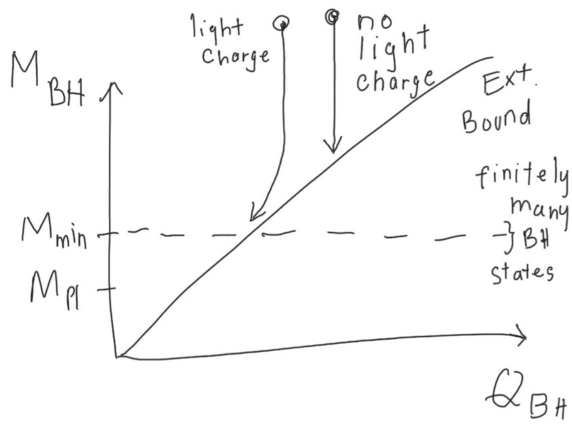


Figure 1: BH evaporation with (non)light particles

But the important point is that, if we have gauge charges (instead of global symmetric charges), we have finitely many BHs state in a given range of mass (for which we trust the EFT).

If we have very weak coupled gauge theory, it looks like a global symmetry (when gauge couplings reaches 0, there tends to be no different between gauge and global symmetry). However, since the global symmetry should be forbidden and that led to the **Weak Grabity Conjecture (WGC)**: If we have a consistent theory of QG that contain a U(1) gauge field, then the theory should contain at least one charged particle (or object) with $m/q \leq m/q$ of extremal BHs [11].

Remark: For BHs there's a minimum mass/charge ratio and the claim is there should be some something in the theory that violates that bound, whose mass is smaller than it would be allowed to be if that object were a BH. At this point it's important to qualify one point about the previous argument so we said that BH's had a minimal mass because they have an external electric field and we can integrate that electric field down to the BH radius and that stores some energy which contributes to its mass. The distinction between light particles and BHs is that when we make this kind of argument that the mass should be at least as much as the energy stored in the electric field for a light particle, we would only make this argument down to the Compton radius rather than the Schwarzschild radius. The BHs are heavy in Planck units, which means if we're integrating x from infinity we hit their Schwarzschild radius before we get to their Compton radius. Whereas for a light particle like the electron if we're integrating from infinity we hit its Compton radius long before we reach its Schwarzschild radius.

And for that reason, these are different calculations:

$$M > \int_R^\infty \mathrm{d}^3 x E^2(x). \tag{2.1}$$

For the light particles, R accounts for Compton radius, while for BHs, it accounts for Schwarchild radius.
The WGC can be thought of a kinematic condition that allows extremal BHs ($M = Q$) to discharge to objects of ($M > Q$) and ($M < Q$). A WGC-satisfying particle have a stronger electromagnetic (EM) repulsion ($\sim e^2 q^2$) than the gravitational attraction ($\sim m^2/M_{Pl}^2$) i.e. there must exist some particles for which gravity is a weaker force than EM's. The form of the extremality bounds really depends on additional content of the theory, such as dilaton/axion that couples to the gauge field. Two possible statements could be made:

1. Repulsive Force Conjecture (RFC): There is a type of particles that at long distance, they have repulsive force of each other.

2. WGC: There exists a superextremal particle, which violates the extremality bound.

The two statements are equivalent in Maxwell-Einstein theory (only photon & graviton). But if we have a theory with scalar fields, for instance, the WGC only makes reference to how those scalar fields couple to the gauge fields. When we find a solution in theory where our gauge field kinetic term depends on some scalars, we'll get some extremality bounds that depend on that scalar coupling. Whereas this conjecture, depends on how the particle that we're talking about, couples to the scalars because the scalars will mediate some attractive force between two copies of that particle. And depending on whether that particle couples more or less strongly to the scalars than the BHs do, this could be a different statement. So, when we're reading about cases that have scalar fields as many supersymmetric theories through the massless moduli fields, it's important to keep track of which one of these two things is meant by WG conjunction.

One general comment to make about these conjectures is that they all have the form "there exists a particle". It's not a statement that "every particle in the theory has to obey this boundary", and the fact that it has that form sort of signals that these these kinds of statements are not going to be extremely easy to prove. So there can't just be any general argument about taking two of these particles and doing a scattering experiment and applying general principles like unitarity that are going to immediately tell us this result because this is a result of some particles and not to other ones. That in some ways makes it a bit harder to imagine how we would prove this as a general statement.

2.1 BHs & WGC

Another aspect of this of this definition that's a little bit fuzzy is what do we mean by particle? We said "particle" or "object", because in fact if we don't specify that this particle has to be light, it turns out that the WGC could actually be satisfied by BHs themselves. That sounds at first glance like a contradiction because we know that BHs always obey the extremality bound and this inequality is the opposite of the extremality bound.

The first thing to say is in some theories, there are BHs for which BPS bounds, which are consequences of SUSY and extremality bounds coincide (There cases where these are the same bounds). In those cases, BPS BHs saturate the bound, and so we wrote a less-than or equal-to sign over here for a reason, which is that we know theories where there's nothing that's strictly less than because anything strictly less than is just forbidden by SUSY. And so in those cases for the WGC to have any hope of being true at all, we have to allow for the case for these are actually equal, and in that case BPS BHs can be the things that obey the bound.

But if we look at a more general theory where the BPS condition is not the same as the extremality condition, then SUSY doesn't protect the form of the extremality bound. And as a result, the extremality bound gets corrections.

So we know that the extremality bound looks like the mass being bigger than some multiples of the charges, and that's true if we find classical BH solutions in the simplest theory with only a two derivative action. But more generally, we expect that any theory is going to contain higher dimension operators suppressed by some UV scale like the string scale or the Planck scale. And if we find BH solutions in those theories, they will differ from the BH solutions in theory with just a two derivative action.

So, if we could write down various 4-derivative operators in the Lagrangian like $(F_{\mu\nu}^2)^2$, $R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, etc. These operators modify the BH solutions, but because they're higher dimension operators, they don't modify big BHs very much, they have a bigger effect on smaller BHs. If we compute the BH solutions in the presence of these higher dimension operators, we find that there's still an extremality bound, and it has the form that the mass is bigger than the charge times some constant as we saw before but now added some corrections:

$$M_{\rm BH} \geq c_o Q_{\rm BH}\left(1+\frac{c_1}{Q_{\rm BH}^2}+...\right). \tag{2.2}$$

where this coefficient c_1 is some linear combination of coefficients of the 4-derivative operators. So the extremality bound that we're used to is really a statement about Einstein-Maxwell theory with just the Einstein-Hilbert action and the minimal kinetic term for the photons. As soon as we start adding these more complicated terms, we get different results and the calculation of the bound in the presence of these operators was done on a paper [2].

So depending on the sign of this linear combination of operators c_1, we're either going to find that BH states are allowed/not allowed to be quite on the extremal line but have to be a bit heavier in the case that c_1 is positive. Or in the case that c_1 is negative, they're now allowed to lie a little bit below the extremality line as in Fig. 2.

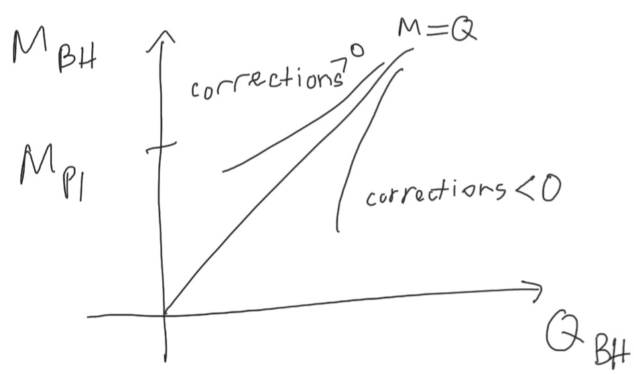


Figure 2: The classical mass-charge relation for extremal BH [2].

There's some evidence that in consistent theories this linear combination is always negative ($c_1 < 0$). And so, in fact BHs themselves can have masses a little bit below what the two derivative extremality bound would have allowed. This hasn't been rigorously proven in general but there's a suggestive argument in a paper [12] also, there's an argument that we think is specific to the case of 4D of Nima Arkani-Hamed & collaborators that have not yet published (but have given some talks on) proves that given certain assumptions, the number c_1 is negative. The [12] paper tries to make an argument based on BH entropy and their arguments make a lot of assumptions there are some reasons to think it really only applies to kind of tree level UV completions, while the Nima's argument is about looking at the renormalization group of running of these operators and arguing that eventually $(F_{\mu\nu}^2)^2$ has the biggest effect and that that one is known to have a positive coefficient for other reasons related to unitarity. So it is supposed that the status of this claim is still not quite rigorous but at least there's enough different suggestive arguments that we might believe that it's just true the BH's inconsistent theories of QG actually can exist slightly below the traditional extremality bound.

What we know before that WGC claims that there should be particles whose mass/charge ratio is smaller than that of an extremal BH, but if extremal BHs don't actually have a definite mass/charge ratio, if we get the curve that bends because of higher derivative operators, it's not quite clear what we mean by this. And so what we should mean by those at least a statement that seems to be an interesting for which there's some evidence is that the mass/charge ratio should be below the asymptotic extremality bound, meaning we should read off the linear part of this bound at large Q and define that to be what we mean by the WGC. And if that's what we mean by the WGC, then in the case that this coefficient is negative, in fact even BHs themselves could be the things that fulfill WGC because they have a mass/charge ratio smaller than the asymptotic slope of that line.

So we would say that this minimal version of the WGC, again, is not a very useful statement if it's not telling us about the existence of light particles is just telling us the big BHs got small corrections. So, the minimal WGC is not quite sharp enough yet to get a lot of phenomenological mileage out of because it's too vague. It doesn't tell us these objects have to be light nor have to be small charge. It just tells us something has to exist that obeys this bound and that thing could just be a BH that gets a small correction.

Remarks: In the modified theories, how do we define the charge? Both the mass and the charge are just read off from the asymptotics of the solution so the charge is just how quickly the electric field is falling off in infinity and the mass is the ADM mass which we can read off from how the metric is falling off.

Remarks: Is there evidence of the same behavior when there is a dilaton coupling? Those calculation has not yet been done but is interesting to do. It get a little bit more complicated because if we start writing down these 4 derivative operators I get a couple dilatons to them in many different ways, so it's not quite clear what choices we would want to make. But certainly it seems to be a well-defined exercise to make some sort of answers here trying to compute how it changes the BH solutions, to see what linear combination of things it depends on, and then see if we have reasons to think that linear combination is positive or negative. IT'S REALLY WORTH DOING!

We are looking to some variations on the WGC that actually do have more power that are more useful statements getting closer to being relevant to phenomenology in the real world. But first, we detour a little bit and just see some examples of settings in the QG context where we can calculate the spectra of particles and see that there are particles that obey this bound. So we quickly review of Kaluza-Klein (KK) theory and also investigate charged particles in the heterotic string spectrum.

2.2 KK Theory

We'll move ahead by talking about some examples where we can calculate spectra. The reason that we should do this is because when we start looking at concrete theories of QG, we rapidly see that not only do they have a particle that obeys the WGC, they tend to have infinitely many particles that obeys. In fact, they have a lot of other common features, that will be part of the story when we talk about generalizing the WGC, and also about the Swampland Distance Conjecture.

The first thing we talk about is KK theory. We want to start with a theory that contains Einstein gravity in D-dimensions and compactify it on a circle so we're going to study it in a space of one lower dimension times a circle $\mathbb{R}^{1,D-2} \times S^1$ from the lower dimensional $d = D - 1$ viewpoint. This has a U(1) gauge symmetry, where the gauge field is just the graviton with one leg pointing along the circle direction.

We're just reviewing to sort of emphasize the features that are going to show up when we talk about some of these other conjectures. We're starting with Einstein gravity and compactifying so what We're doing is just a statement about classical field theory but it's worth saying that we know that KKtheory appears as a sort of a sub-sector of consistent theories and QG like string theory and so all of the statements are going to make can actually be found and real explicit UV complete examples.

We're studying the D-dimensional theory where the D-dimensional metric is given by

$$\mathrm{d}s_D^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + R^2\phi(x)^2(\mathrm{d}\theta + A_\mu \mathrm{d}x^\mu)^2. \tag{2.3}$$

where $\theta \simeq \theta + 2\pi$ is a periodic variable that is a coordinate on the circle that we compactify on. The size of the internal dimension is R times the vev of the field ϕ: $R\langle\phi\rangle 2\pi$. and so this field ϕ is the "radion" field that controls the size of the circle is sort of our prototype of a "volume modulus", and we're going to be talking a lot more about moduli fields and how they're related to the WGC as we go along. So this is the simplest setting in which such a field appears. There's an exercise which is to take the metric 2.3 and plug it into the Einstein-Hilbert action and see what the action looks like in lower dimensions. And the action looks like:

$$S = -\frac{1}{16\pi G_D}\int \mathrm{d}^d x\sqrt{-g}2\pi R\phi(x)\mathcal{R}_d + \text{terms involving } A_\mu. \tag{2.4}$$

It inherits a prefactor that depends on the higher-dimensional Newton's constant, then we get the size of the extra dimension $2\pi R\phi(x)$, then we've got curlier R which is the Ricci scalar of the D-dimensional theory.

We will find that there is no kinetic term for ϕ (term involving derivative of ϕ^2), but $\phi(x)$ has the kinetic mixing with the graviton through its coupling to the Ricci scalar.

We can define the lower d-dimensional Newton's constant via the higher D-dimensional Newton's constant:

$$G_d = \frac{G_D}{2\pi R\langle\phi\rangle}. \tag{2.5}$$

If we doesn't have a potential so the vev could really be anything. But here we're just imagining that we look at solutions where ϕ is asymptotically approaching some constant value and that's what we're calling the vev.

That's the action that we got at the beginning for KK theory. Now we should be free to work with that action if we want, but it's often convenient to do what's called "Einstein frame" which means eliminating the kinetic mixing between the scalar and the metric with a Weyl rescaling (we can do a field redefinition where we replace our metric with this power of the scalar multiplying the metric).

$$g_{\mu\nu} \mapsto \left(\frac{\phi(x)}{\langle\phi\rangle}\right)^{-\frac{2}{d-2}} g_{\mu\nu}. \tag{2.6}$$

Notice that this is a field redefinition, it is not a diffeomorphism nor a change in coordinates. There's a general formula for what happens to the Ricci scalar when we do such a Weyl transformation. That if we rescale a field as: $g_{\mu\nu} = \Omega(x)\tilde{g}_{\mu\nu}$, then the Ricci scalar changes in this way (This is also a good exercise to do!):

$$\mathcal{R} = \frac{1}{\Omega}\left[\tilde{R} - (d-1)\nabla^2(\ln\Omega) - \frac{(d-2)(d-1)}{4}\partial_\mu(\ln\Omega)\partial^\mu(\ln\Omega)\right]. \tag{2.7}$$

In particular, the fact that it depends on the logarithm of the thing we've rescaled by will be important for our discussion. Using this fact in doing this field redefinition, we find that our action becomes different action

$$\sqrt{-g}\phi(x)\mathcal{R} \to \langle\phi\rangle\sqrt{-g}\left[\mathcal{R} - \frac{d-1}{d-2}\left[\partial_\mu(\ln\phi)\right]^2\right] + \text{bound.terms}. \tag{2.8}$$

Right now, we no longer have a scalar coupling to the Ricci scalar, we just have its vev and we now have a kinetic term which depends on the log of our radion field, and there was also a boundary term that we can dropp.

Now we could also gotten to this just by directly starting with the different ansat,

$$\mathrm{d}s_D^2 = \left(\frac{\phi(x)}{\langle\phi\rangle}\right)^{-\frac{2}{d-2}} g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathcal{R}^2\phi^2(x)(\mathrm{d}\theta + A_\mu\mathrm{d}x^\mu)^2. \tag{2.9}$$

But this way would not have been an obvious guess probably, if we didn't start with the above and then try to figure out how to eliminate the kinetic mixing.

Now once we've done the field redefinition we find that the gauge field couples to the scalar it's kinetic term becomes:

$$-\frac{R^2\phi^2(x)}{8k_d^2}\left(\frac{\phi(x)}{\langle\phi\rangle}\right)^{\frac{2}{d-2}} F_{\mu\nu}F^{\mu\nu}. \tag{2.10}$$

So our gauge field has a prefactor that's a power of ϕ but the kinetic term depends on the $\ln(\phi)$. So in terms of the canonically normalized field, this is an exponential coupling similar to the dilaton coupling that we wrote earlier.

In addition to the massless fields that we've been working with, we have KK modes, i.e. the excitations around the circle, and a mode of KK charge n, i.e. n units of momentum around the circle, as a mass given by

$$m_n^2 = \frac{n^2}{R^2\langle\phi\rangle^2}. \tag{2.11}$$

We have infinitely many of these modes from all the different momentum.

We did we go through all these details because we want to emphasize some features that turned out to be a pretty universal and that formed the foundation for several of the swampland conjectures. Some observations are the matching of the Newton's constants depend on the radius. If we assume that radius was big in fundamental Planck units of the higher D-dimensional theory, then the lower d-dimensional theory has a higher Planck scale than the higher D-dimensional theory, which means that if we measure Newton's constant on the lower dimensional theory and then we do scattering experiments and see when gravity becomes strongly interacting, it happens much earlier than we would have estimated from the low-energy measurement. So theories that have these large internal dimensions have very low UV cut-offs relative to what we might naively expect. QG cut off should be so much lower than the d-dimensional Planck scale.

The next observation is that the limit where we make the $U(1)$ gauge coupling e_{KK} small by making the prefactor of the term big is the limit when the charge particles are light. So in fact, the masses of these charge particles when written in units of the d-dimensional block scale are just

$$m_n^2 = \frac{1}{2}n^2 e_{\mathrm{KK}}^2 M_d^{d-2}. \tag{2.12}$$

These masses saturate the WGC if we were to work out what the external BHs look like we would find this coefficient of 1/2 is exactly what we want for extremal BHs. And that's not an accident, it's because if this theory were super symmetric, then this is also a BPS condition because our U(1) gauge symmetry is just part of the higher dimensional space-time symmetries and so SUSY will protect the corresponding extremality bounds.

The next observation is the kinetic term is logarithmic scalar field ϕ (that also means if we were to write it as a function of the gauge coupling, it would be logarithmic as a function of that too), so the canonically normalized field has the form

$$\hat{\phi} = \sqrt{\frac{d-1}{d-2}}\frac{1}{k_d}\ln\left(\frac{\phi}{\langle\phi\rangle}\right). \tag{2.13}$$

And if the coupling of ϕ with the gauge field in terms of the canonically normalized field it's exponential as $e^{-c\hat{\phi}}F_{\mu\nu}^2$. Final related fact to point out is because of that, the number of KK modes below the cut-off scale M_D is <u>exponentially large</u> as a function of this canonical a normalized field $\hat{\phi}$. So we have a modulus field as it goes to large values, we find exponentially many modes appearing carrying different charges under the gauge group, all of which obey the WGC. So this's one example where we find that not only is the WGC obeyed, that's obeyed in a way that's much stronger than what it had to be.

2.3 Heteroic String Theory

Let us briefly mention one other example, it's 10d heterotic string. For concreteness, $SO(32)$ string. So we haven't talked yet about what the WGC means for non-abelian gauge groups, but there's a $U(1)^{16} \subset SO(32)$. So let's just look at the $U(1)$ charges (the charges under the Cartan generators of the Lie algebra). And with this gauge group we can find particles whose charges are labeled by these 16 numbers $\mathbf{q} = (q_1, ..., q_{16})$ with the q_i either all integers or all half integers and the sum of them is an even integer $\sum q_i \in 2\mathbb{Z}$. That's the charge lattice of $SO(32)$. We can work out the spectrum of particles that carry these charges in the perturbative string spectrum and let just jump to the answer that the <u>lightest</u> charged particle for $\mathbf{q} \neq 0$ turns out to have a mass given by

$$m^2 = \frac{2}{\alpha'}\left(|\mathbf{q}|^2 - 2\right). \tag{2.14}$$

So this spectrum is asymptotically linear but if we were to plot mass/charge it stays below the diagonal line and in fact, asymptotically this matches on to the BH extremality bound.

So our gauge coupling and 10-dimensional Plank scale reads

$$e^2 = g_S^2(2\pi)^7\alpha'^3 \tag{2.15}$$

$$M_{10}^{-8}, = \frac{1}{2}g_s^2(2\pi)^7\alpha'^4. \tag{2.16}$$

where string coupling $g_S = e^{\langle\Phi\rangle}$, with Φ has a canonical kinetic term of $(\partial\Phi)^2$ (not logarithm of Φ).

Again, as in the previous case we find a coupling which is exponential as a function of some modulus. We find a set of states of different charges whose masses are bounded in some way by the charge. In fact, we can work out the BH extremality bound in this case:

$$M_{\rm BH}^2 \geq e^2 |\mathbf{Q}_{\rm BH}|^2 M_{10}^8 = \frac{2|\mathbf{Q}_{\rm BH}|^2}{\alpha'}. \tag{2.17}$$

In terms of α', this linear relationship is just the extremality bound. So the heterotic string spectrum has infinitely many charge particles all of which obey the WGC and what are asymptotically approaching the BH spectrum. So we have our two examples which's KK theory and the heteroic string and we see that in both cases not only did a particle exist as the WGC told us, but infinitely many particles exists. And next, we will see how we are led by those to some more powerful conjectures that extend the weak gravity condition.

3 U(1)xU(1)

Ref. [13]. WGC for gauge group U(1)xU(1), we have two charged particles, each of which obeys each U(1) bound. But it turns out that's not really sufficient. The reason is the form of the extremality bound in this case (simplest case where these gauge field are not mixing with each other) :

$$M_{\rm BH}^2 \geq 2(\sum_i e_i^2 Q_{\rm BH,\ i}^2) M_{\rm Pl}^2. \tag{3.1}$$

So if we have a BH that carries charges under both gauge groups, there is an extremality bound as above. This is because there are electric fields for each of these gauge groups outside the BHs of each of which are storing some energy. But now the claim is that if we just satisfy the WGC for each U(1) individually, we don't necessarily allow all the BHs that saturate this bound to discharge.

Look at a toy example, suppose 2 charged particles charge (1,0) $m_1 = \sqrt{2} e_1 M_{\rm Pl}$ and (0,1) $m_2 = \sqrt{2} e_2 M_{\rm Pl}$. If we look at each U(1) individually, this would be enough. But the claim is that it is not enough if we have a BHs that carry both charges. So suppose we have an extremal BH with charge $(Q_{\rm BH,1}, Q_{\rm BH,2})$, we can just ask what would happen to the BH if it emitted one of 2 particles? How would its masses and charges change?

So suppose we can just emit particle 1: $M'_{\rm BH} = (M_{\rm BH} - m_1)^2$ (minimum mass loss, particle without momentum), $Q'_{\rm BH} = (Q_{\rm BH,1} - 1, Q_{\rm BH,2})$, under the first gauge group, its charge will decrease by 1.

$$M'^2_{\rm BH} - 2e_1^2(Q_{\rm BH,1} - 1)^2 M_{\rm Pl}^2 - 2e_2^2 Q^2_{\rm BH,2} M^2_{\rm Pl}. \tag{3.2}$$

If we just evaluate this quantity is it + or not, if we expand out the square, and subtract everything, what we will find is something look like

$$4e_1^2 Q_{\rm BH,1} M^2_{\rm Pl}\left(1 - \sqrt{1 + \frac{e_2^2 Q^2_{\rm BH,2}}{e_1^2 Q^2_{\rm BH,1}}}\right) < 0. \tag{3.3}$$

This decay is not kinematically possible, since the BH by emitting this particle, we turn into another BH that just isn't a good classical solution. It does not obey the extremality bound. And so adding these 2 particles is not sufficient. (These particles do not allow general extremal BHs to discharge.)

So this tell us whatever it is, the right condition is a theory with more than 1 gauge group, it is going to be much more complicated than just saying that we have to individually obey the bound for each gauge group independently.

It turns out that there is a simple geometric criterion to understand what is the correct condition to allow these BHs to discharge. Consider the charge/mass ratio variables, we have a set of variable:

$$Z_{i,j} \equiv \frac{\sqrt{2} e_i q_{i,j} M_{\rm Pl}}{m_j}. \tag{3.4}$$

index i: charge, index j: particle.

For a given particle j, think of the collection for that j: $(Z_i)_j = \mathbf{Z}_j$ as a vector in sojme space. BH extremality bound says that if we such vector for a BH, then $\mathbf{Z}_{\rm BH}\mathbf{Z}_{\rm BH} \leq 1$. Its charge cannot be too big for a given mass. So in the space of this charge/mass ratio, there some region in which BHs can live. (this space is many dimensions if we have many gauge group). There is some sort of balls inside this region that contains all the BHs (BH regions). Extremal BHs would live in the boundary of this region if we compute using the two derivatives action (as mentions in last lecture, if we have higher dimension operators, the BHs could move slightly outside/inside this region, unless there is some other physics like BPS bound followed by SUSY that could force them to live exactly on the boundary of the region).

But what Ref. [13] pointed out is that a condition that allows for all of the BHs to be able to shed their charge is to have some set of charged particles in the space, such that they are convex hull contains the BHs region.

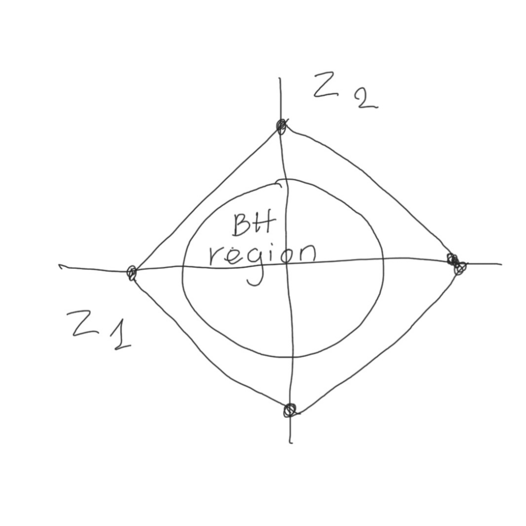


Figure 3: Convex hull in charge/mass representation [13].

The convex hulls meaning, that we connect up each of these particles and anti-particles in such a way to buid the convex space. And this convex hull should contain the BHs region.

So the example we gave before, we just have 2 particles and we saw not all BHs get discharged as an example where we have 2 U(1), so the BHs region is the interior of some circle in a plane, and we was choosing to have a particle at (1,0) & (0,1), there would be also anti-particles at (-1,0) & (0,-1). What we were seeing is that if we go in a general direction that does not point along one of the axes, the convex hull does not fully contain the BH region in that direction. So there is region that are not contained, and that means the WGC fails as Fig. 4(a). On the other hand, if we had chosen the particles to have smaller masses, sufficiently far away from the WG boundary in a single direction, then the convex hull fully contain the BH region and everything would be able to discharge as Fig. 4(b). Or we could have a more complicated spectrum, we could have particles carrying both charges as Fig. 4(c).

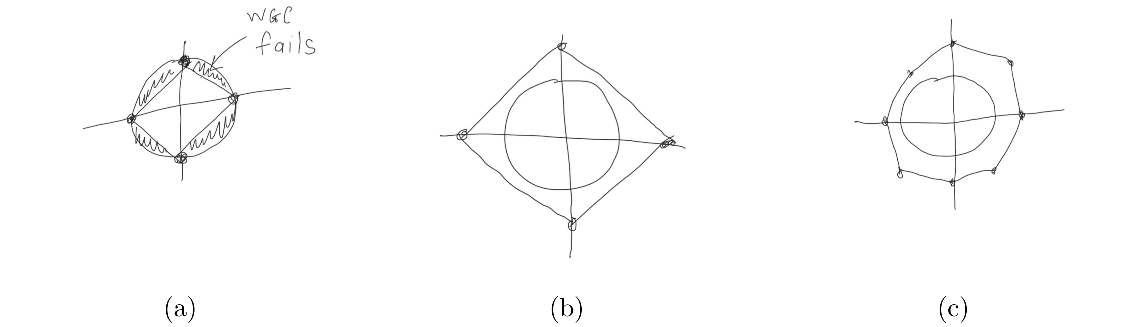


Figure 4: Convex hull for different charged species (Matt Reece's lectures on WGC).

So the minimal version of the WGC does not tell us exactly wehat charges the particles have to have. It just says whatever those charges are, the masses should be light enough the convex hull contain the BHs region. So as we saw before, there are examples, like KK theory, that would be very much stronger properties. If we look at KK theory, we would just find charged particles living on the boundary of this region at every direction as in Fig. 5.

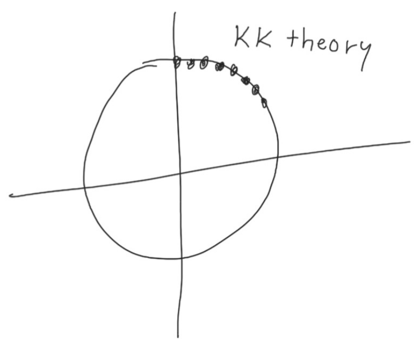


Figure 5: Charged particles living on the boundary in KK theory (Matt Reece's lectures on WGC).

And the reason for that is because in this case, the extremality bound coincides with the BPS bound. If we study theory of scalars, we wiould find that BH extremality bound can sometimes take a different form. So for one example, if we take a string theory, and compactify it, we will have both U(1) that correspond to KK charge or momentum around the circle, but also U(1) correspond to winding, how many times the string winds arround the circle.

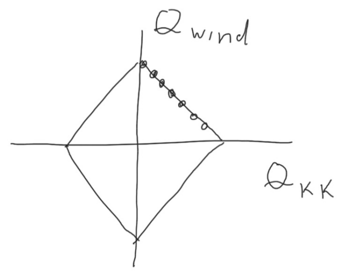


Figure 6: BHs' linear region boundary in compactifying String theory (Matt Reece's lectures on WGC).

And if we work out what BHs look like in that theory. we will find that the formula we had before (with the squareroot of some squares) is not the right one, that only work for the Reissner Nordstrom case, but in this case we get a more interesting story where we have the linear region, so we can get kind of a diamon shape as in Fig. 6. But again, the convex hull condition is the right criterion to impose. And in this case again because these are BPS charges, we find that we just have actual states carrying all of these charges in all directions along the boundary.

So everything up to now is the definition of the WGC for a general gauge group is going to be this condition that we have particles carrying charges under the gauge group such that their convex hull contains all posible BHs & allows extremal BHs to discharge

There's another way to formulate the WGC in theories of scalars that contrast with us Ref. [14]defined something called the "WGC with scalar fields"

$$e^2Q^2M_{\rm Pl}^2 \geq m^2 + G^{ij}(\phi)\frac{\partial m}{\partial \phi^i}\frac{\partial m}{\partial \phi^j}M_{\rm Pl}^2 \tag{3.5}$$

which is that the particles should exist that obeys the inequality, the EM force should be bigger than the gravitational force coming from a mass plus scalar forces that depend on how the mass changes if we vary the value of the scalar field, and this is the version of the so-called Repulsive Force Conjecture. The statement is that the repulsion of electromagnetism should overcome the combined attraction of gravity and scalars. And this is well-defined for U(1), but again, it is not so obvious how to define this if we have more than 1 gauge group (so there's no clear definition of this in the case of more than one gauge group, but there're works in progress). It seems like this alternative conjecture and the above conjecture might both be true. But if the particle masses depend on the scalar in a different way than the masses of BHs depend on the scalar, these are 2 different statements.

Another question we could ask is what happens to the non abelian gauge groups? We sort of mention this before when we worked on the example of charged state in the heterotic string theory. We can take the $\mathbf{q}$ to be charges under the Cartan generator of the Lie algebra (the maximal torus, the U(1) contained inside the gauge group), and then we can still apply this condition. But because the gluons themselves are charged in massless, they're infinitely far out in the space, and so this become sort of a trivial condition for non-abelian gauge theory to satisfy

(we would apply Hull to any gauge group, but what we mean by charge in the non-abelian case is not necessarily obvious, so what we mean by it is take the U(1) that live inside the non-abelian gauge group and make those charge under those. One way to see that there should be something related non Abelian gauge group is as if we have a non Abelian gauge theory, we can compactify that theory on a circle and turn on the Wilson lines for the gauge group around the circle, and break it down to the U(1) that lives inside the non Abelian gauge group and so, if the WGC applies on that lower dimensional theory, it should tell us that something had to apply in the higher-dimensional theory) These are so far kinds of minimal WGC. They may be not as useful as we would like because they could be satisfied just by saying that BHs get small corrections, it pushed them out of the BH region defined by the extremality bound on asymptotically big BHs.

So what we want to talk about are some extensions of this idea that are much more powerful. We talk about KK, heteroic string and we saw that in both cases, we had not just a single charged particle obeyed the WGC but an infinite family of charged particles from different charges, all of which obey the conjecture. This is closely related to another idea SDC.

3.1 Swampland Distance Conjecture

Ref. [15] They had some set of conjectures, all of which are related to the fact that QG has a moduli space $\mathcal{M}$. Known theories of QG have moduli spaces of scalar fields with flat potential that are protected by SUSY. But in fact, similar statements (for light fields) apply even away from the SUSY context. Although it is harder to make these statements sharp, we saw in KK theory that we have a redian mode that control the size of the circle even in a non-supersymmetric theory that fields can be very light compared to UV cut-off. But for now, let's talk as if the moduli space exists, and the scalars are massless because it is easier to formulate everything in that context.

So there first conjecture about moduli spaces is that If we were given a moduli space, there exist points at the infinitely far away (given a point p_0 in the moduli space $\mathcal{M}$ and some positive number $T > 0$, we can find some other point in the moduli space) whose distance is bigger than that number ($p \in \mathcal{M}$) with $d(p, p_0) > T$. So moduli spaces allow us to go infinitely far away. And the distance is defined by the scalar field kinetic terms.

So there is some metric $g_{ij}(\phi)\partial\phi^i\partial\phi^j$ (it's the same metric that appear in Palty's statement of his WGC with scalar fields, when we compute the attractive force on charged objects to the scalars, we need to know the metric on that space).

The next statements where things become more interesting and make a connection to the WGC: Theory at p with $d(p, p_0) > T$. They say that if we go far away some distance T from where we started, we will find an infinite tower of light particles with mass is exponentially small (of order $e^{-\alpha T}$, $\alpha < 0$, $\alpha \sim \mathcal{O}(1)$ in Planck unit) [16].

This means in QG, first, there are big moduli spaces. The scalar field let us vary the parameters of the theory by arbitrarily large amounts, but whenever we try to vary them too much, we find our EFT starts to break down because lots of particles are starting to become light and so if we don't incorporate those particles in our theory, we don't have a good understanding of the physics anymore. (EFT breaks down for super-Plankian field values).

e.g.: KK theory, where we have a scalar with the kinetic term that loosely looks like $M_{\text{Pl}}^2\partial_\mu(\ln\phi)\partial^\mu(\ln\phi)$ where the masses of the KK mode: $m_n \sim \frac{n}{\mathcal{R}\phi}$. So the kinetic terms tells us that if we vary this number ϕ, by an order 1 amount that we're going an order n Plank distance in the field space and if we write this mass in terms of the canonically normalized field or the field that's actually measuring that distance we see that the mass is decreasing exponentially. So that's an example, similarly for the heteroic string, we saw that our masses roughly went like $m_n \sim \frac{n}{\sqrt{\alpha'}}$ (n times the string mass scale) but that's exponentially small in the distance traversed by the dilaton.

These examples that we have seen are illustrations that kind of capture all of the general features of moduli spaces and theories of quantum gravity

There are many more examples in string theory, so for example let's say we build a 3+1 D gauge theory in type IIB strings with D7 branes wrapped on 4-cycles. Something that might be phenomenologically relevant if we're trying to construct something like the Standard Model within String Theory, then our gauge coupling $1/g^2 \sim \text{Vol}(\Sigma_4)$ (volume of these 4D cycles) and those volumes can be parameterized by some scalar fields (scalar moduli) and again if we go a long distance in field space (small coupling g), we're making this volume big and we're bringing down some KK modes, not as simple these previous KK modes that live on a circle but still some set of modes are becoming light as we make the volume big.

SDC is already a very interesting conjecture from the viewpoint of thinking about things like theories of inflation where we might want fields to travel long distances over time as inflation is happening. One distinction in that context is the field space distance that appears here is the geodesic distance in field space that depends only on the kinetic terms.

So we have some space of scalar fields we know a metric on that space We are given two points and asked to measure the distance. We do it in the usual geometric way just using the metric. But if we have fields that are actually evolving in time and we're not in this supersymmetric limit where we have a perfectly flat moduli space then the fields might not actually be following geodesics and field space because there could be a potential and the potential could change how the fields are evolving (field space distance does not depend on $V(\phi)$). This might be an important distinction if we're trying to apply this to realistic theories of the real world that have SUSY braking where the potential can be very important for figuring out what the fields are doing. Two papers that discuss the swampland distance conjecture in this context and how this distinction might be important [17], [18].

Remarks: There's been various other recent works along these lines so it's not completely straightforward to say that just because if we go along geodesic distance in the field space, we'll bring down an tower of modes, but inflationary theories with long field ranges are excluded, we have to be careful in trying to make these kinds of arguments because these statements are really statements about the limit of exact moduli spaces, which is not the limit that's useful for the real world.

Now we circle back the WGC, this conjecture made no reference to gauge fields, we were just talking about theories of QG that contain scalar fields acting as moduli. But in string theory or KK theory as we've seen in examples, the values of gauge couplings are determined by the values of moduli. (They're not just fixed parameters that we can't change but depend on the values of some scalar fields). This starts to bring us to a link between these ideas and the WGC. The WGC grew out of the statement of no global symmetries which is a statement that QG doesn't want us to have exact global symmetries, and therefore if we have a gauge coupling and we try to make it very small, the theory should somehow break down.

3.2 Tower WGC

In a lot of contexts what happens when we try to make a gauge coupling very small is precisely that we're taking a long distance limited moduli space, the small gauge coupling corresponds to a very big value of some scalar field just as happened here in

KK theory, the gauge coupling is set by the value of ϕ. So if we want to make a small gauge coupling we have to go a long distance in field space and then what we have is that some tower mode should start to become light, and we should worry that we're going to lose control of our field theory because we don't know about all of these modes that are coming down from the UV cutoff and entering the energy scales that we would like to calculate. Given these ideas together with the WGC, it's tempting to conjecture something that's a little sharper than the WGC, which is that in the case where we try to send the gauge coupling to zero, a tower of modes is going to become light and in some cases, that tower of modes as we've seen is exactly the same tower of modes that will become light.

Putting these pieces together, it's tempting to guess but if we have a complete theory of gravity coupled to a gauge field with a very small gauge coupling ($g << 1$), there must exist an infinite tower of modes of different charge q, each one of which obeys something like the WGC, the mass is less than its charge $M <\sim eqM_{\rm Pl}$. So the statement follows from other statements that we've made, we're just saying that given the examples that we have and pick up our statements, it's tempting to guess that. So the statement in the sort of loose form that we've have written is something we might call the **Tower WGC** [19].

3.3 Sublattice WGC

We could get something even sharper than this and the statement that's true and all examples that we're aware of is what we call a **Sublattice WGC**: In any consistenr QG theory, there's sublattice of charges under the gauge group that is allowed and there's a sub lattice of the full charge lattice (of the same dimension). (It's not like we could have a five dimensional charge lattice and we're picking a lattice that only goes in one of the directions). It's got to be the full dimensionality for which every site in the sub lattice contains at least one superextremal particle.

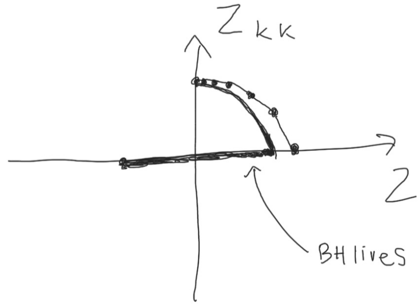
That's along the lines of the tower conjecture but it's a little bit sharper, saying that these can't just have any random set of charges, there should at least be some sub lattice so maybe it's all multiples of charge 2, maybe it's all multiples of charge 3, but there's some sublattice for which we can find an example of a particle obeying the WGC for every site in the sublattice. [20], [21]

We were led to guess this not by the logic started from thinking about the SDC and examples but rather just by thinking about the WGC itself and trying to sort of test it and see if it was consistent.

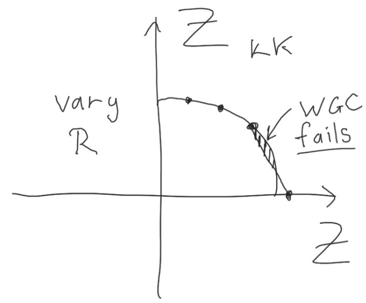
Claim:[20]: The Convex Hull Condition is not preserved by dimensional reduction. (There's sort of a flaw in the convex hull condition, the minimal version of the WGC which is that we could give a theory that satisfies it and we could compactify that theory on a circle and get a new theory that doesn't satisfy it anymore.)

The reason is, the gauge group gets bigger $G \times U(1)_{\mathrm{KK}}$, so our compactified theory now have the KK charged along the circle and as we saw the convex hull condition tells us that when we have a bigger gauge group it's harder to satisfy the conjecture than when the smaller one. We need to pull the charges farther away in order for the convex hull to enclose the BH region. And so, if we just had some particle with a sufficiently big charge to mass ratio to satisfy the conjecture in the one dimensional case for instance, where our BHs live on the line segment as in Fig. 7(a).

when we compactified the theory we're going to get a bunch of KK modes of this particle so there's going to be a mode with charge (1,1), charge (1,2), there's going to be a bunch of other KK modes that sort of accumulate close to the axis of pure KK charge and so they will have some convex hull and then there's some BH region. This example looks fine the way we find it is as we vary the radius of the compactification, as in Fig. 7(b). There will always be some radii for which our KK modes fail to fully enclosed the BH region and the WGC can fail.



(a)



(b)

Figure 7: Convex hull in KK theory.

So the claim this that the original version of minimal WGC generalized the multiple gauge groups in the natural way as [13] is not a completely satisfying conjecture because it's somehow not sufficient if we give we a theory that obeys it, we can change that theory only in the infrared by compactifying, and get a theory that doesn't obey it anymore. We noticed this and then we went looking for examples in string theory so we could just understand how is the WGC satisfied in actual theories of QG and what we found was that in all the examples that we could check, there were these towers of particles and the towers of particles rescued the statement because now we have many different particles of different charges and so they have KK modes in different locations and when we start putting them all together we always find, that after we compactify, the theory still obeys the conjecture.

Claim: these tower WGCs are more robust under compactification than the original WGCs so that's one reason to think that these might be plausible statement. Another reason to think they're possible again is they connect nicely to these SDC. The final reason, is in the perturbative string theoy, we can prove the sublattice WGC from modular invariance [20], [21]. This was really the main technical development in those papers is that in perturbative string theory we can actually just prove that the statement is true and in fact it follows from modular invariance. So it's possible to show just by studying a modular invariant partition function that has a chemical potential turned on, the charge states exist within a sublattice and they obey a bound that looks like the WGC bound.

At least, in a perturbative string theory this is a well-established fact. Of course, perturbative string theory is not all of QG and we can ask what about particles that are charged under gauge fields that live on D-brane for instance and we can't really say anything about that although there have been some recent interesting papers showing that a class of F-Theory compactification[22] also contain states that obey the slWGC. So we have a lot of examples where this is just a known fact about about actual theories of QG that we can construct

Remark: The reason why it has to be a sublattice instead of a full lattice is we can just construct counter examples where they're not the full lattice and the flavor of these counter examples is that we start with something that has multiple charges(like compactify on a torus) and then take an orbifold where one of the directions gets the gauge field projected out. so we no longer have a gauge field corresponding to some of the charges, but states are still labeled by some some kind of momentum in that direction that's not really conserved but a state that had charge in that direction before we orbifold it will still have a bigger mass because of that extra momentum. and so what we can do is to cook up a theory in which for certain sites in our charged lattice, the lightest state that is not projected out by the orbifold is a state that had momentum in that extra direction. And that adds to its mass but doesn't add to its charge because we removed the charge associated with that direction. We can find the details and the papers but the interesting thing is that in the examples that we know, where we can check what states obey the WGC, it's never a very sparse sublattice so it maybe at most half the sites in the charged lattice fail to obey the conjecture. It's never like every tenth site or every seventeenth site. And whether that's just an artifact of looking at examples that we know how to construct or whether it's a deeper statement We're not really sure. But to the extent that it seems to be true that the supplies to sub lattices that are not very sparse. This avoids the problem with the original WGC that we could obey it just put BHs. So the original conjecture we could have a small correction to an extremal BH, and that obeyed the conjecture but that extremal BH has an enormous charge in order to be in the regime in which we trust semi classical gravity. Whereas if we require this to be true for sublattices whose sparseness is only sort of an order one number, then we're going to find states of small charge that obey the conjecture and those states are going to be lighter than the Planck scale. So this is now telling us about actual particles not just about BHs, so again we don't have a proof that there are no examples where this is satisfied with a sparse sublattice even in the perturbative string context but we've looked at lots of different examples of heterotic string orbifolds that we understand and in those examples it never was.

4 Natural Inflation

Suppose Inflation is an axion Ref. [23], so it's aperiodic field $\phi \cong \phi + 2\pi F$, and the potential has to be built of periodic things

$$V(\phi) \approx c_1 \Lambda_1^4 \cos\frac{\phi}{F} + c_2 \Lambda_2^4 \cos\frac{2\phi}{F} + ... \tag{4.1}$$

As long as the coefficient decreases fast enough, this can give us a nice, smooth, periodic potential that can be used to have a model of inflation ($F > M_{\rm Pl}$).

Looking at these model in string theory, consider UV completions, there is a nice way to get axion-liked field that have the periodic structure out of higher dimesional gauge theories, by integrating a p-form gauge field over a p-cycle to get a periodic scalar $\int_\Sigma A_p$.

Explain: if we have a circle as an extradimension and we integrate along the coordinate of that circle, the corresponding component of our gauge field

$$\theta(x) = \int_0^R \mathrm{d}y A_y(x, y), \tag{4.2}$$

Then, if we do a gauge transformation

$$A_y(x, y) \to A_y(x, y) + \partial_y \alpha \tag{4.3}$$

where the gauge function α winds arround the circle, then the gauge transformation will just shift the scalar field by 2π: $\theta \simeq \theta + 2\pi$. The similar thing happens for higher dimensional cycles, and so there is a natural way to get periodic scalar variables from higher dimensional gauge theories. Axion in strong CP problem in early days of superstring theory Ref. [24], Ref. [25].

Extranatural inflation Ref. [26]: higher dimensional gauge theory gives us the periodic scalars that are potential candidates for inflation. These model are particularly useful in bulding and working with EFTs. However, when people find such models in string theory, since it is difficult to get the periods to be big in Planck units ($F > M_{\rm Pl}$) Ref. [27].

We can ask what give rise to the mass of that axion? The asnwer is the mass comes from charged particles in Kaluza Klein (KK) mode from the higher dimensional theory's that run in loops, and correct the mass of lower dimensional axion fields.

The calculation (at least in the limit of heavy particles) is that the potential comes from wraping a worldline of these massive particles around the circle that we compactify on, or in the string theory context, where we have higher dimensional cyclone wraping the Euclidean brane of the appropriate dimension around the cycle (Euclidean worldline instanton). Ref. [11]

Because of that, the mass of the potential for the axion depends on the spectrum of particles that are charged under the gauge field that we started with, but that's where become interesting to connect this to the WGC as WGC tells us something about charge particles in the higher-dimensional gauge theory.

In the case when we compact 5D to 4D, where we have a massive particle in the higher dimensional theory, its contribution is exponentially supressed, and the cosine funtion of axion fields

$$V(\theta) \simeq \frac{1}{R^4} e^{-2\pi mR} \cos(\theta) f(mR) + \text{higher order terms} \sim e^{-nS_{\rm inst}} \tag{4.4}$$

we can think of the action $S_{inst} = 2\pi mR$ for the Euclidean worldline wrapped on the circle. There will be exponentially suppressed if we wrap the particle n-times.

The WGC in 5D tells us there is some particles who masses are bounded:

$$m^2 < e_5^2 q^2 M_5^{(5-2)}. \tag{4.5}$$

But for the period of the axion $F(\theta)$, we can work out just by dimensionally reducing the kinetic term of the gauge field in 5D to the kinetic term for θ in 4D, and we find f^2 depends inversely on the radius of compactifcation and on e_5^2:"

$$f^2 = \frac{1}{2\pi R e_5^2}. \tag{4.6}$$

The 4D Planck scale are $M_{\text{Pl}}^2 = M_5^3 2\pi R$. If we multiply both sides of Eq. 4.5 by R^2, we have:

$$m^2R^2 < \frac{1}{f^2}q^2M_{\text{Pl}}^2 \tag{4.7}$$

If we want $f > M_{\text{Pl}}$, $mR < 1$, and S_{inst} is not large, we cannot trust the leading answer anymore Ref. [11]. What it shows us is that we should at least be careful about building models of inflation that come from higher dimensional gauge fields because they are generically going to be some set of corrections and these theories that are hard to control. That does not mean that it is impossible, and in particular, in the 5D examples, in Ref. [28] pointed out that if we just do the loop calculation, the n-th order terms in the limit of very small masses $m << \frac{1}{R}$, where the exponential is not suppressed at all, goes like $1/n^5$, and that's already enough to give us a smooth potential. So they conclude that we do not actually need to suppressed all these instantons to have a nice smooth potential that we can inflate with, we just need some really light particles .

5 WGC from Unitarity and Causality

This section is based on [1] with Gary Shiu's presentation on "Vistas over the Swampland", Instituto de Física Teórica workshop, Madrid, 19 September 2018.

5.1 A web of Swampland Conjecture (WGC-TWGC-sLWCG)

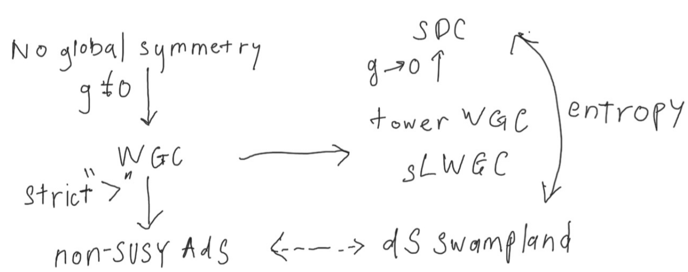


Figure 8: A Web of Swampland Conjectures (Gary Shiu's lecture on WGC).

WGC as an upgrade of the conjecture that there are no global symmetries in QG since global symmetries can be thought of gauge symmetries with varnishing gauge couplings. It's natural to suspect that something go wrong when the coupling is weak. When we refine WGC to a strict inequality, we are led to the AdS instability conjecture.

Consistencies appoints KK reduction suggest stronger version of the WGC, such as tower WGC, sub lacttice WGC that requires infinitely many charge states, and there are some conjectures that seem naively unrelated, like a Swampland Distance Conjecture (SDC), and the dS Swampland conjecture.

SDC: for any consistent theory of QG, there must be a maximum field range beyond which given QFT break down. The reason is there is a towel of light state that become exponentially light and that invalidate the effective field theory we start with, and interestingly, as it turns out, the TWGC & sLWGC can be thought of as special cases of distance conjecture as we send the gauge couplings to zero, which take us to infinite distance field space. The tower of states are required by the WGC indeed become light.

The SDC conjecture can be shown to be related to the Swampland distance conjecture through entropy considerations at weak couplings. So, what QG has given us is if all this conjectures were random speculations, they would not have to fit into this intraceted web, and this also give us confident that perhaps we can understand very well of perhaps prove one of these conjectures, we may be able to find out what is the underlying reasons that unifies the different statements about QG.

5.2 Brief recap on WGC

WGC: For any U(1) gauge theory that admits UV completion with gravity, there must exist a state whose charge to mass ratio is bigger than that of the extremal BHs, this is sometimes know as the mild form of the WGC.

Remarks: This conjecture, which implies that an extremal BHs can decay, is well supported by many examples in string theory. A heuristic argument to motivate this conjecture is to consider theories that violate it, which means that all the particles in the theory have a charge smaller than its mass. If this is the case, than the gravitational attraction between these particles is stronger than the gauge repulsion, so they can form bound state. This bound state cannnot lose their charge, and do not Hawking radiate once they reach the extremal limit.

We therefore end up having an indefinite number of stable states. So what the WGC really does is to postulate that there should not be a large number of exactly stable states that are not protected by symmetry (stated in original paper). To avoid the situation from happening, we can demand the existence of a state whose charge to mass ratio is bigger than 1, so this bound state cannot decay. Unlike the no global symmetry conjecture, we do not run into a similar problem with remnants. Even though there are infinite number of stable states, there is only a finite number of them within a finite mass range, so it's less clear why the WGC should hold, given that it is the strongest statement.

Evidence for the WGC: These are very general argument, but suppose we take into account additional features that come along with string theory, such as modular invariance, one seems to find an evidence that require the conjecture takes even stronger form, not just the existence of such a state, but perhaps, an infinite number of them.

We forcus on the mildest form of the WGC, and how to derive it using **Unitarity** and **Causality**.

Mild form of WGC requires only the existence of some states for an extremal BHs to decay to, so the natural question one can ask is can an extremal BHs be such a state? The point is that the extremal condition Q = M is only a classical result. Any correction like higher derivatives corrections can modify the BH solutions and the extremal conditions. So this effect is typically small as they are suppressed by the mass of the BHs, so for very large BHs, these corrections are unimportant, but not so when the BHs are not so large. Indeed, in the context of heteroic string, this higher derivatives corrections are shown to make some of the 4D extrema BHs lighter than the classical. The paper will show that this behavior follow from unitarity at least in a wide classes of theories that can be naturally realized in string theory.

5.3 To what extend does the proof apply?

If we already found a super extremal particle in the spectrum, then we are done. But even if not, what we find is that an extremal BHs can satisfy the mild form of the WGC in at least 2 classes of theories. The first class of theory are those with **light parity even scalars** or **spin-2 particles** in the spectrum. Light means that they are lighter than the scale which we call Λ_{QFT}, where the ordinary description of QFT will break down. So in string theory, we can imagine this scalar could be the dilation or any other moduli which are stabilized at a mass below the string scale. While this seems to be a rather natural in string theory setup, this is not the most general case. Nonetheless, it is general enough to subsume the class of theory to which the entropic proof of Ref. [12] would apply. [PICs]

But we will see that Unitarity would allows us to say a bit more. We know what kind of particles would be needed to satisfy this mild form of the WGC, we need particles that parity even, and we know about the parity and as well as the spin. Moreover, we will see that the WG conditions that we derive is a strict inequality. Even if there is no such light particles in the spectrum, our proof will still apply to theories that admit tree-level SUSY UV completion, which mean the higher derivative 4-point amplitudes are generated by tree level exchange and they respect super symmetry at the tree level.

5.4 Higher Derivatives Corrections

Let's start by enumerating the higher derivative operators to Einstein-Maxwell theory, and we assume that in the infared, the dynamic of the BH is described by an EFT of photons & gravition. In 4D, the most general effective action up to 4 derivatives operators are given as:

$$\begin{aligned}\delta\mathcal{L} =& c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau} + c_4 R F_{\mu\nu}F^{\mu\nu} + c_5 R_{\mu\nu}H^{\mu\rho}F^{\nu}{}_{\rho} \\ &+ c_6 R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + c_7 F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + c_8 F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\nu}.\end{aligned} \tag{5.1}$$

$$\begin{aligned}S =& S_{\text{Einstein-Maxwell}} + \\ &\int \mathrm{d}^4x\sqrt{-g}\left[\frac{\alpha_1}{4M_{\mathrm{Pl}}(F_{\mu\nu}F^{\mu\nu})^2} + \frac{\alpha_2}{4M_{\mathrm{Pl}}^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \frac{\alpha_3}{2M_{\mathrm{Pl}}^2}F_{\mu\nu}F_{\rho\sigma}W^{\mu\nu\rho\sigma}\right].\end{aligned} \tag{5.2}$$

Using field redefinition, we can recast all higher derivative terms into three separated terms that we call α_1, α_2, α_3, where $W^{\mu\nu\rho\sigma}$ is the Weyl tensor:

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + \frac{1}{2}(g_{\mu[\rho}R_{\sigma]}\nu - g_{\nu[\sigma}R_{\rho]\mu}) - \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}. \tag{5.3}$$

This higher derivative corrections would modify the BHs solution and the extremal conditions, and so the charge to mass ratio of the extremal BH is no longer 1 but 1 up to some corrections.

The formula of Ref. [2] is valid as long as the higher derivative corrections are small, which is the case when the BHs (mass & charge) are large (when we evaluate the Ricci scalar and the energy of the gauge kinetic function, we find that they are all ghost-like, which means inverse with the mass of BHs). So this is going to be a small corrections to the extremal condition.

5.5 WGC from Unitarity

If we want to prove the mild form, the require conjecture is just amount to deriving the inequality $2\alpha_1 - \alpha_3 \geq 0$ for the Wilson coefficients (so large BHs can decay into smaller extremal BHs). This one with a strict inequality follows from Unitarity, which means the large BHs would prefer to decay to smaller ones. So in supersymmetric cases (we will see later), if α_1 is not 0, then it would be a strain in equality.

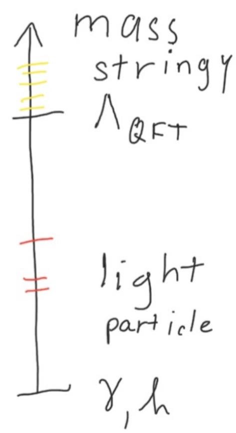


Figure 9: A schematic picture of the particle spectrum [1].

We could enumerate as possible sources of the higher dimensional operators that will contribute to $\alpha_1, \alpha_2, \alpha_3$, and which contributions dominate depends on our particle spectrum (Fig. 9). We would assume, through our discussion that we have a weakly coupled UV completion, and there is some energy scales Λ_{QFT} beyond which the ordinary QFT description breakdown, and in string theory, this is the string scale above which we would have to take into account infinitely many local fields. Now there are 2 possibilities in addition to this heavy states that are above Λ_{QFT}, there could be some light massive particles (as they're lighter than Λ_{QFT}).

5.6 Sources of Higer Dimensional Operators

There are 3 sources of this higher dimensional operators: Neutral bosons, loop effects from charged bosons, and UV effects.

- Neutral particles can generate this higher derivative operators at the tree level, they're neutral bosons (dilaton, axion, moduli).
- Charged particles cannot contribute to this high derivative operators at tree level, but they contribute at loops. The leading contribution is at one loop.
- Contribution from UV physics which are suppressed by Λ_{QFT}. And we will discuss in turn the unitary constraints on this various contribution.

5.6.1 Light neural bosons

Consider a theory with non-light scalar (dilaton) or pseudoscalar (axion). Integrate them out lead to tree-level effective couplings. This tree-level effective couplings are suppressed by the mass of the particles, and if they are sufficiently light compared with the UV scale, their contributions dominate over the UV effects. Unitarity is what fixes the positivity of these rules and coefficients. More generally, unitarity implies that α_1 is positive when the photon is coupled to a parity even scalar or spin-2 particle. Whereas, unitarity would imply that α_2 is positive if the photon is coupled to a parity odd scalar field or spin-2 particle. Notice that the spin-2 particles can carry any parity in either case.

5.6.2 Charged particles

Charged particles contribute to leading order at one loop, and we can consider, for example, a minimal coupled charged particles, and the 1 loop effective couplings generated by a minimally coupled charged particles can be estimated in term of the charge to mass ratio (z). So α_1, α_2 goes like order z to the 4th, whereas α_3 only go at most to the z^2.

So, if we find a particle with a large charge/mass ratio, then α_3 is sub dominant. There is a hierarchy that is enjoyed by this Wilson coefficient. Morevover, in this limit, gravity is negligible, and unitarity for QFT would implies that both α_1 & α_2 are positive definite. Ref. [29], Ref. [30]. This is the case that when we already have a super extremal particle in our spectrum, so, not only can we satisfy the WGC with a super extremal particle, extremal BHs themselves can also satisfy the WGC. But, not only in this case, we are interested in the posibility that charge to mass ratio of the particles are not large, in which case, the effective couplings are of order 1, and in this limit, no vigrous unitarity bound is known on this Wilson coefficients, but this is also the case where other effect that come in a tree level will dominate.

5.6.3 UV effects

This third type of effects come from integrating out UV physics above Λ_{QFT}, and they are suppressed by Λ_{QFT}. And if we are adressing these questions in the context of string theory, these are the α' effects. In general, it is difficult to fix the sign of this Wilson coefficient from unitarity. There is no general statement one can make about the sign of all this α_1 that are generated by integrating of stringy effect. However, if the higher derivative operators describing the four-point amplitudes are generated by some intermediate states, then we can say something concrete. So, in a weakly coupled UV completion of gravity, this effects would be dominated by the tree level exchange.

in such a case, we could put a constraints on the Wilson coefficients α_i from unitarity just as the tree level effects from the neutral boson case.

5.7 Unitarity constrain for various cases

We see that unitarity put a bound on some of the Wilson coefficients, not all. It does not look like we have enough to prove that the WGC. One easy observation one can make is that when the loops effect dominate, then one can clearly say that the high derivative corrections to the extremal conditions of the BHs would make an extremal BHs lighter than the classical one. When this effect domintae, this is the case when we already have a super extremal particle satisfying the WGC. So, if we are interest in whether extremal BHs can play the role of the WG state when no such a super extremal particles exist, we are in the situation where the other effects dominates. It may appear that we don't have enough to prove the positivity of these corrections because unitarity only constrain two of the Wilson coefficients but not the third. As it turned out, the third is significantly constrained by other principle like **Causality** & **Super symmetry**.

5.8 SUSY in WGC

The point is that the effective operators α_3 would generate for us some new photon-photon-graviton helicity amplitude that are not present in the Einstein-Maxwell theory. In particular, there are helicities and amplitudes in the all incoming notation. Some helicities are incompatible with the SUSY Ward-Takahashi identity, essentially because the helicity symmetry broken too much. So in SUSY theory, $\alpha_3 = 0$. So the mild form of the WGC follows from Unitarity because we know that Unitarity has $\alpha_1 > 0$. Hence, $2\alpha_1 - \alpha_3 > 0$.

5.9 Tree-Level SUSY

Notice that we have only used SUSY at the **tree-level** amplitudes to set $\alpha_3 = 0$, so our argument is applicable even to the non SUSY theories as long as the tree level scattering of photons and gravitons is compatible with SUSY. So this is the case, for example, for the $O(16) \times O(16)$ Ref. [31], where space-time SUSY is broken by an unconventional projection but the tree level vertices of the bosonic factor is the same as super heterotic string. Now there may be other principles that allows us to set $\alpha_3 = 0$, for example, if we make a stronger condition of not demanding only the quantic interactions to be generated by tree level exchange, but also the cubic interations are also generated by tree level exchange of some massive particles that $\alpha_3 = 0$ follows as long as all the photon & graviton does not kinnetically mix with the heavy particles.

5.10 Causality

Now, what if we don't have tree level SUSY? Another point one can make is that the helicity amplitudes written down before also leads to causality violation at an energy scale of the order of $M_{\rm Pl}/\sqrt{\alpha_3}$, and so, this scale must be above our Λ_{QFT}. Moreover, it was argued in the situation like this, an infinite tower of massive higher spin states is required to UV completed theory. So whatever that UV completion is, we know that this effect will be classified into what we call UV effect. So in other words, causality implies that effect A (tree level effect coming from neutral bosons) can never give a dominant contributions to α_3. So if the effect coming from neutral boson exchange dominates, causality would imply this relation, which means that the mild form of the WGC can be satisfied and the requirement is for the low energy spectrum to contain a parity-even scalar neutral or spin-2 neutral particles with a mass smaller than UV scale.

5.11 Summary of the Constraints

We can summarize the coverage of our argument. Depending on the situation, we can prove the WGC using combinations of causality or tree-level SUSY, and we can see that if the dominant contribution to the 4-derivative operator come from a new boson exchange, we are in the situation where the mild form of the WGC follows from causality and unitarity. This was the case covered by the entropic argument of Ref. [12], but, our arguments can be applied to other situations as well, when the loops effects or the UV effects dominate. There is also a nice interpretation of our finding in terms of open-closed duality. So in the string theory, charged particles are typically associate with open strings, so if we find a super extremal particle in our spectrum, then we are already done. If not, that means that the charge to mass ratio is small, and that also means that the corresponding open string must be long in order to make the charge/mass ratio small. And this is the regime where we should probably interpret the open string loop more appropriately in term of closed string exchange. And so we can think of the extremal BH that satisfy the WGC as the closed string channel version of the super extremal particles in the open string case.

5.12 Stronger forms of the WGC

One may ask the stronger version of WGC such as the **convex hull condition** for theory with multiple U(1) Ref. [13]. Or the **tower WGC** [19] or **sLWGC** [20], [21] also follow from unitarity? There we find that we can prove these stronger forms but only if we make some addition assumptions on the UV. This is not surprising if we do not priori assume that gravity is negligible, the unitarity bound would depend also on the UV sensitive high derivative terms of gravity. This, however does not mean that using the unitarity arguments to constrain WG is an empty statement because we vary mild assumptions on the UV, several strong conditions came out. In particular, by considering the scattering amplitude and demanding unitarity, we finds that when we consider scattering of particles different photons, we not only get the convex hull condition, but also the necessity of having a bifundamental field,

5.13 Implications for KK Reduction

This has implications for KK reduction because when we reduce our theory to lower dimensions, we get in addition to the original U(1) a KK U(1), and if we look at the spectrum for some value of the radius, the spectrum does not contain bifundamental field, which violate the positivity bound, and one way to save the day is to introduce a tower of state that are charged under the original U(1).

This lattice is what we call the tower WGC and it's interesting to see how this condition comes out through an independent argument. So several stronger versions of the WGC that require infinitely many charged states have been proposed on different grounds, either from moduli invariance of the worldsheet theory, or from other consistency requirements. But here, from the unitarity, we also see the requirement of having even infinitely many charged states.

5.14 Summarise

To summarise, we show that the mild form of the WGC can be satisfied by extremal BHs for a wide class of theory, including those that can be naturally realized in string theory. For example, we finds in low energy spectrum some dilaton-liked fields or moduli fields which coupled to the photon, then they can play the role of satisfying the WGC.

One find that stronger versions of WGC can also follow from unitarity if we make some additional assumptions on the UV. And what we find quite reemarkable is not only that the various swampland conjecture are related, their proof are somewhat related as well. It will be interesting to find out what's their connections.

6 Symmetries

6.1 more to added

KK theory, Brans-Dicke, Dilaton

6.2 S-duality -SL(2,$\mathcal{Z}$)

S-duality (strong-weak duality) relates 2 different physical theories (like QFT and String theory) $SL(2,\mathcal{Z})$ transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{6.1}$$

with $(a, b, c, d \in \mathcal{Z} : ad - bc = 1)$.
The role of $SL(2,\mathcal{Z})$ in $SL(2,\mathcal{R})$ is somewhat like $\mathcal{Z}$ in $\mathcal{R}$. with $S\&T$ matrix defined as:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{6.2}$$

We shall prove that they generate the $SL(2,\mathcal{Z})$ by 2 ways. The algebraic method, consider the action on a matrix,

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \tag{6.3}$$

$$T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix} \tag{6.4}$$

Suppose that $c \neq 0$, we have $a = cq_0 + r_0$, with $0 \leq r_0 < |c|$. Hence, with $r_0 = a - q_0 c$, we have,

$$T^{-q_0} A = \begin{pmatrix} r_0 & b - q_0 d \\ c & d \end{pmatrix} \tag{6.5}$$

7 Recap on GR

p-form

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \tag{7.1}$$

Hodge dual of p-form ω reads

$$(*\omega)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \sqrt{|g|} \epsilon_{\mu_1 \dots \mu_{n-p} \nu_1 \dots \nu_p} \omega^{\nu_1 \dots \nu_p}. \tag{7.2}$$

Some properties:

$$*(*\omega) = \pm (-1)^{p(n-p)} \omega, \tag{7.3}$$

$$\langle \eta, \omega \rangle = \int_{\mathcal{M}} \eta \wedge *\omega, \tag{7.4}$$

$$\langle d\alpha, \omega \rangle = \langle \alpha, d^\dagger \omega \rangle, \tag{7.5}$$

with

$$d^\dagger = \pm(-1)^{np+p-1} * d * . \tag{7.6}$$

Exterior derivative is a map $d : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})$

$$d\omega = \frac{1}{p!}\frac{\partial \omega_{\mu_1\ldots\mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{7.7}$$

Properties

$$\begin{aligned} d^2\omega =& 0. & (7.8)\\ d(\omega \wedge \eta) =& d\omega \wedge \eta + (-1)^p \omega \wedge d\eta & (7.9)\\ d(\psi^*\omega) =& \psi^* d\omega. & (7.10) \end{aligned}$$

Wedge product

$$(\omega \wedge \eta)_{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} = \frac{(p+q)!}{p!q!}\omega_{[\mu_1\ldots\mu_p}\eta_{\nu_1\ldots\nu_p]}. \tag{7.11}$$

Properties:

$$\begin{aligned} \omega \wedge \eta &= (-1)^{pq}\eta \wedge \omega, & (7.12)\\ \omega \wedge \omega &= 0 \text{ if } \omega \text{ odd}, & (7.13)\\ \omega \wedge (\eta \wedge \gamma) &= (\omega \wedge \eta) \wedge \gamma. & (7.14) \end{aligned}$$

Covariance derivative is the map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ such that,

$$\begin{aligned} \nabla_X(Y+Z) =& \nabla_X Y + \nabla_X Z, & (7.15)\\ \nabla_{(fX+gY)} Z =& f\nabla_X Z + g\nabla_X Y, & (7.16)\\ \nabla_X(fY) =& (\nabla_X f)Y + f\nabla_X Y & (7.17)\\ =& X(f)Y + f\nabla_X Y. & (7.18) \end{aligned}$$

$$\nabla_{e_\rho} e_\nu = \Gamma^\mu{}_{\rho\nu} e_\mu \tag{7.19}$$

$$\begin{aligned} \nabla_X Y =& \nabla_X(Y^\nu e_\nu) = X(Y^\nu)e_\nu + Y^\nu \nabla_X e_\nu & (7.20)\\ =& X^\mu(e_\mu Y^\nu)e_\nu + Y^\nu X^\mu \Gamma^\sigma{}_{\mu\nu} e_\sigma & (7.21)\\ =& X^\mu(e_\mu Y^\nu + Y^\sigma \Gamma^\nu{}_{\mu\sigma})e_\nu & (7.22)\\ \Rightarrow \nabla_\nu Y =& (e_\mu Y^\nu + Y^\sigma \Gamma^\nu{}_{\mu\sigma})e_\nu. & (7.23) \end{aligned}$$

The first and second Cartan structure relation reads,

$$\begin{aligned} d\hat{\theta}^a =& -\omega^a{}_b \wedge \hat{\theta}^b, & (7.24)\\ R^a{}_b =& d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. & (7.25) \end{aligned}$$

Cartan-subalgebra
Maximality

8 O(d,d)

9 KK Theory

Extracts 4D gravity limit of a higher dimesions theory

$$\hat{\theta}^a = e^a_\mu dx^\mu \equiv dx^a \tag{9.1}$$
$$\hat{\theta}^\psi = e^{\sigma(x)}(d\psi + A_\mu dx^\mu). \tag{9.2}$$

Hence,

$$d\hat{\theta}^a = -\,\omega^a{}_b \wedge \hat{\theta}^b - \omega^a{}_\psi \wedge \hat{\theta}^\psi \equiv -\omega^a_{0\,b} \wedge \hat{\theta}^b. \tag{9.3}$$
$$d\hat{\theta}^\psi = \sigma_{,a} e^{\sigma(x)} dx^a \wedge d\psi + \sigma_{,a} e^{\sigma(x)} A_\mu dx^a \wedge dx^\mu + e^{\sigma(x)} \partial_a A_\mu dx^a \wedge dx^\mu \tag{9.4}$$
$$= \sigma_{,a} \hat{\theta}^a \wedge \hat{\theta}^\psi + e^{\sigma(x)} \frac{F}{2} \quad (\text{with } F = 2\partial_{[a} A_{\mu]} dx^a \wedge dx^\mu) \tag{9.5}$$
$$= -\,\omega^\psi{}_a \wedge \hat{\theta}^a. \tag{9.6}$$

From 9.6, we deduce that

$$\omega^\psi{}_a = \sigma_{,a} \hat{\theta}^\psi + e^{\sigma(x)} \frac{1}{2} F_{ab} \hat{\theta}^b \tag{9.7}$$

Hence,

$$\omega^a{}_\psi \wedge \hat{\theta}^\psi = -\frac{1}{2} e^{\sigma(x)} F^a{}_b \hat{\theta}^b \wedge \hat{\theta}^\psi \tag{9.8}$$

Therefore, from 9.3, we have

$$\omega^a{}_b = \omega^a_{0\,b} - \omega^a{}_\psi = \omega^a_{0\,b} + \frac{1}{2} e^\sigma(x) F^a{}_b \hat{\theta}^\psi \tag{9.9}$$

Hence,

$$R^a{}_b$$
$$= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b + \omega^a{}_\psi \wedge \omega^\psi{}_b \tag{9.10}$$
$$= d\omega^a_{0\,b} + \frac{1}{2} d\left(e^{\sigma(x)} F^a{}_b \hat{\theta}^\psi\right) + \left(\omega^a_{0\,c} + \frac{1}{2} e^{\sigma(x)} F^a{}_c \hat{\theta}^\psi\right) \wedge \left(\omega^c_{0b} + \frac{1}{2} e^{\sigma(x)} F^c{}_b \hat{\theta}^\psi\right)$$
$$+ \left(\sigma^{,a} \hat{\theta}_\psi + \frac{1}{2} e^{\sigma(x)} F^a{}_c \hat{\theta}^c\right) \wedge \left(\sigma_{,b} \hat{\theta}^\psi + \frac{1}{2} e^{\sigma(x)} F_{bd} \hat{\theta}^d\right) \tag{9.11}$$
$$= d\omega^a_{0\,b} + \frac{1}{2}\Big[\sigma_{,c} e^{\sigma(x)} F^a{}_b \hat{\theta}^c \wedge \hat{\theta}^\psi + e^{\sigma(x)} F^a{}_{b,c} \hat{\theta}^c \wedge \hat{\theta^\psi} + e^{\sigma(x)} F^a{}_b \Big(\sigma_{,c} \hat{\theta}^c \wedge \hat{\theta^\psi}$$
$$+ e^{\sigma(x)} F\Big)\Big] + \omega^a_{0\,c} \wedge \omega^c_{0b} + \frac{1}{2} e^{\sigma(x)} \left(\omega^a_{0\,c} \wedge F^c{}_b \hat{\theta}^\psi + F^a{}_c \hat{\theta}^c \wedge \sigma_{,a} \hat{\theta^\psi}\right) \tag{9.12}$$
$$+ \frac{1}{4} e^{2\sigma(x)} F^a{}_c \hat{\theta}^c \wedge F_{bd} \hat{\theta}^d. \tag{9.13}$$

$$R^{\psi}{}_{a}$$

$$=d\omega^{\psi}{}_{a} + \omega^{\psi}{}_{b} \wedge \omega^{b}{}_{a} \tag{9.14}$$

$$=d\left(\sigma_{,a}\hat{\theta}^{\psi} + \frac{1}{2}e^{\sigma(x)}F_{ab}\hat{\theta}^{b}\right) + \left(\sigma_{,b}\hat{\theta}^{\psi} + \frac{1}{2}e^{\sigma(x)}F_{bd}\hat{\theta}^{d}\right) \wedge \left(\omega^{b}_{0a} + \frac{1}{2}e^{\sigma(x)}F^{b}_{a}\hat{\theta}^{\psi}\right) \tag{9.15}$$

$$=\sigma_{,ab}\hat{\theta}^{b} \wedge \hat{\theta^{\psi}} + \frac{1}{2}e^{\sigma(x)}\left(\sigma_{,c}F_{ab}\hat{\theta}^{c} \wedge \hat{\theta}^{b} + F_{ab,c}\hat{\theta}^{c} \wedge \hat{\theta}^{b} - F_{ab}\omega^{b}_{0c} \wedge \hat{\theta}^{c}\right)$$

$$+\sigma_{,a}\left(\sigma_{,b}\hat{\theta}^{b} \wedge \hat{\theta}^{\psi} + e^{\sigma(x)}F\right) + \sigma_{,b}\hat{\theta^{\psi}} \wedge \omega^{b}_{0a} + \frac{1}{2}e^{\sigma(x)}F_{bd}\hat{\theta}^{d} \wedge \omega^{b}_{0a} + \frac{1}{4}e^{2\sigma}F_{bd}F^{b}a\hat{\theta}^{d} \wedge \hat{\theta^{\psi}}. \tag{9.16}$$

Extract,

$$R^{\psi}{}_{a\psi b} = -\sigma_{,ab} - \sigma_{,a}\sigma_{,b} + \sigma_{,c}\Gamma^{c}{}_{ab} - \frac{1}{4}e^{2\sigma(x)}F_{ac}F_{b}{}^{c} \tag{9.17}$$

Hence,

$$R^{\psi}{}_{\psi} = R^{\psi}{}_{a\psi a} = -\Box\sigma(\nabla\sigma)^{2} - \frac{1}{4}e^{2\sigma}F^{2}, \tag{9.18}$$

$$R_{ab} = R_{adbd} = R_{0ab} - \nabla_{a}\nabla_{b}\sigma - \nabla_{a}\sigma\nabla_{b}\sigma - \frac{1}{4}e^{2\sigma(x)}F_{ac}F_{b}{}^{c} + \frac{1}{2}e^{2\sigma}F_{ac}F_{b}{}^{c} \tag{9.19}$$

$$= R_{0ab} - \nabla_{a}\nabla_{b}\sigma - \nabla_{a}\sigma\nabla_{b}\sigma + \frac{1}{4}e^{2\sigma(x)}F_{ac}F_{b}{}^{c}. \tag{9.20}$$

Extract

$$R = R_{0} + \frac{1}{4}e^{2\sigma}F^{2} - 2\Box\sigma - 2(\nabla\sigma)^{2} \tag{9.21}$$

$$= R_{0} + \frac{1}{4}e^{2\sigma}F^{2} - 2e^{-\sigma}\Box e^{\sigma}. \tag{9.22}$$

Since $\Box\sigma + \nabla\sigma\nabla\sigma = e^{-\sigma}\left(\Box\sigma + \nabla\sigma\nabla\sigma\right)e^{\sigma} = e^{-\sigma}\nabla\left[\left(\nabla\sigma\right)e^{\sigma}\right] = e^{-\sigma}\Box e^{\sigma}$.

$$S_{5} = -\frac{1}{16\pi G_{5}}\int d^{5}x\sqrt{g_{5}}R_{5} \tag{9.23}$$

$$= -\frac{\int d\psi}{16\pi G_{5}}\int d^{4}x\sqrt{\tilde{g}}e^{\sigma(x)}\left(\tilde{R} + \frac{e^{2\sigma}}{4}F^{2}\right) \tag{9.24}$$

Scalar-tensor gravity can change our "metric" to make action look like S_{EM}. Conformal transformation:

$$\tilde{g}_{ab} = \Omega^{2}(x)g_{ab}. \tag{9.25}$$

Original frame to Einstein frame

$$\tilde{R} = \Omega^{-2}(R - 6\Omega^{-1}\Box\Omega). \tag{9.26}$$

Set $\Omega = e^{-sigma/2}$,

$$\begin{aligned} \sqrt{\tilde{g}}\tilde{R} =& \Omega^4 \sqrt{g}\Omega^{-2}(R - 6\Omega^{-1}\Box\Omega) && (9.27) \\ =& e^{-\sigma}\sqrt{g}\left(R + 3\Box\sigma - \frac{3}{2}(\nabla\sigma)^2\right) && (9.28) \end{aligned}$$

Finally, we write $\phi = \sigma/\sqrt{3}$ to obtain,

$$S = \frac{L}{16\pi G_5}\int dx^4 \left(-R + \frac{1}{2}(\nabla\sigma)^2 - \frac{e^{\sqrt{3}\phi}}{4}F^2\right)\sqrt{g}. \tag{9.29}$$

Note that $L = \int d\psi$, $ds^2 = e^{-\sqrt{3}\phi}g_{\mu\nu}dx^\mu dx^\nu - e^{2\sqrt{3}\phi}(d\psi + A_\mu dx^\mu)^2$, and $G_4 = \frac{G_5}{L}$

10 KK Black holes

Note: Compactification induces a "rest frame",

$$\begin{aligned} \psi \sim& \psi + L, && (10.1) \\ t \sim& t. && (10.2) \end{aligned}$$

$$\begin{aligned} t' =& \gamma(t - v\psi), && (10.3) \\ \psi' =& \gamma(\psi - vt). && (10.4) \end{aligned}$$

look at black string

$$\begin{aligned} ds^2 =& \left(1 - \frac{2GM}{r}\right)\frac{(dt + vd\psi)^2}{1 - v^2} - \frac{(d\psi + vdt)^2}{1 - v^2} - \frac{dr^2}{1 - \frac{2GM}{r}} - r^2 d\Omega^2_\sigma. && (10.5) \\ =& -\left(1 + \frac{2GMv^2}{r(1 - v^2)}\right)\left(d\psi + \frac{2GMvdt}{(1 - v^2)r + 2GMv^2}\right)^2 + \left(1 - \frac{2GM}{(1 - v^2)r + 2GMv^2}\right)dt^2 - dr^2 ... && (10.6) \end{aligned}$$

$$\hat{r} = r + \frac{2GM}{1 - v^2} \tag{10.7}$$

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