

Progress on positivity bounds in SMEFT

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ABSTRACT: In this report, we rederive some current analysis on SMEFT positivity bound. We first rederive the main result in [1], which investigate the vector boson scattering process (VBS) by considering diagrams involving quartic gauge boson couplings (WGC) governed by SMEFT Dim-8 operators.

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1 Positivity bounds in VBS

In this section, we re-interpret some basic concepts, theorems and re-derive (in details) some of the main results and calculations in [1].

1.1 Crossing symmetry

Consider a $2 \rightarrow 2$ process, any of the particles can be replaced by its antiparticle on the other side of the interaction. Hence, for

Spin = 0: $M(s, t) = M(u, t)$.

Spin > 0: With the linear polarizations vector $(\epsilon_1^\mu)^* = \epsilon_3^\mu$, $(\epsilon_2^\mu)^* = \epsilon_4^\mu$, and with restriction to forward limit, we have $M(s, 0) = M(u, 0)$ (or $\mathcal{A}(s) = \mathcal{A}(u)$).

1.2 Optical theorem

The Optical theorem yield the relation between forward scattering amplitude and cross-section [REDERIVE THE THEOREM LATER IN THE APPENDIX].

$$\text{Im } \mathcal{A}(k_1 k_2 \rightarrow k_1 k_2) = 2E_1 E_2 |v_1 - v_2| \sigma_t. \quad (1.1)$$

Going into the CM-system, we have $p_1 + p_2 = 0$, $E_{\text{CM}} = E_1 + E_2$, $\mathbf{p}_{\text{CM}} = \mathbf{p}_1 = -\mathbf{p}_2$, (with $v = \frac{p}{E}$) we get the optical theorem in the standard form:

$$\text{Im } \mathcal{A}(k_1 k_2 \rightarrow k_1 k_2) = 2E_{\text{CM}} \mathbf{p}_{\text{CM}} \sigma_t. \quad (1.2)$$

when 2 incoming particles are the same, $m = m_1 = m_2$, $E_1 = E_2 = E$, we have $s = 4E^2 = E_{\text{CM}}^2$, $\mathbf{p}_{\text{CM}} = \sqrt{s^2/4 - m^2}$,)

$$\text{Im } \mathcal{A}(k_1 k_2 \rightarrow k_1 k_2) = 2\sqrt{s} \sqrt{\frac{s}{4} - m^2} \sigma_t \quad (1.3)$$

$$= \sqrt{s(s - 4m^2)} \sigma_t. \quad (1.4)$$

In general, with 2 different incoming particles, defining $M_+ =$ we have

$$\begin{aligned} & 2(E_1 + E_2) \mathbf{p}_{\text{CM}} \\ &= 2\sqrt{(E_1 + E_2)^2 \mathbf{p}_{\text{CM}}^2} \end{aligned} \quad (1.5)$$

$$= 2\sqrt{\mathbf{p}_{\text{CM}}^4 + 2\mathbf{p}_{\text{CM}}^2 E_1 E_2 + E_1^2 E_2^2 - \mathbf{p}_{\text{CM}}^4 + (E_1^2 + E_2^2) \mathbf{p}_{\text{CM}}^2 - E_1^2 E_2^2} \quad (1.6)$$

$$= \sqrt{(2\mathbf{p}_{\text{CM}}^2 + 2E_1 E_2)^2 - 4m_1^2 m_2^2} \quad (1.7)$$

$$= \sqrt{((E_1 + E_2)^2 - 2(m_1^2 + m_2^2))^2 - 4m_1^2 m_2^2} \quad (1.8)$$

$$= \sqrt{(E_1 + E_2)^4 - 2(E_1 + E_2)^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2} \quad (1.9)$$

$$= \sqrt{s^2 - s(M_+^2 + M_-^2) + M_+^2 M_-^2} \quad (1.10)$$

$$= \sqrt{s(s - M_+^2) - M_-^2(s - M_+^2)} \quad (1.11)$$

$$= \sqrt{(s - M_-^2)(s - M_+^2)}. \quad (1.12)$$

Hence, Eq. 1.1 yields

$$\text{Im } A(k_1 k_2 \rightarrow k_1 k_2) = 2(E_1 \mathbf{p}_2 - E_2 \mathbf{p}_1) \sigma_t \quad (1.13)$$

$$= 2(E_1 + E_2) \mathbf{p}_{\text{CM}} \sigma_t \quad (1.14)$$

$$= \sqrt{(s - M_-^2)(s - M_+^2)} \sigma_t. \quad (1.15)$$

It yields back the result of Eq. 1.4 when $M_- = 0$.

$$\text{Im } A_{ab}^{q_1 q_2}(s') = \sqrt{(s' - M_+^2)(s' - M_-^2)} \sigma_{ab}^{q_1 q_2}(s') > 0, \quad s' > (\epsilon \Lambda)^2, \quad (1.16)$$

1.3 Scattering amplitude in the forward limits

[ELABORATE MORE FROM [2]]

1.4 Froissart unitary bounds and dispersion relation

Froissart bound: Unitarity forces the high-energy amplitude in the forward limit is bounded by

$$\mathcal{A}(s) < \mathcal{O}(s \ln^2 s) \quad (1.17)$$

It is a necessary condition for the vanishing boundary contribution when we deform the contour integrals from IR to UV regime [PROVE THE BOUND AND ELABORATE MORE ON THE DISPERSION RELATION].

1.5 Positivity bounds (original version)

Physics in the IR regime can be deform to UV (contour C to C'). The boundary contribution vanishes because of the Froissart bound [ELABORATE MORE].

$$f \equiv \frac{1}{2\pi i} \oint_C ds \frac{\mathcal{A}(s)}{(s - \mu^2)^3} = \frac{1}{2\pi i} \left(\int_{-\infty}^0 + \int_{4m^2}^{\infty} \right) ds \frac{\text{Disc } \mathcal{A}(s)}{(s - \mu^2)^3}, \quad (1.18)$$

with $\text{Disc } \mathcal{A}(s) \equiv \mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon)$. From here, we see that the dim-6 and dim-8 operators in low-energy EFT can be constrained by the positivity bound ($\text{Disc } M(s, 0) \geq 0$) in the UV regime [ADD FIGURE].

In the forward limit ($t \rightarrow 0$), we have $s = 4m^2 - u$. Changing the variable with according bounds in the first term, f can be rewrite as:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(4m^2 - u)}{(4m^2 - u - \mu^2)^3} + \int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(s)}{(s - \mu^2)^3} \right). \quad (1.19)$$

The crossing symmetry reads $\mathcal{A}(4m^2 - u) = \mathcal{A}(s) = \mathcal{A}(u)$. Hence,

$$\text{Disc } \mathcal{A}(4m^2 - u) = \mathcal{A}(4m^2 - u + i\epsilon) - \mathcal{A}(4m^2 - u - i\epsilon) \quad (1.20)$$

$$= \mathcal{A}(u - i\epsilon) - \mathcal{A}(u + i\epsilon) \quad (1.21)$$

$$= -\text{Disc } \mathcal{A}(u). \quad (1.22)$$

Applying this relation to Eq. 1.19, and replace the variable u by s , we have:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(s)}{(-4m^2 + s + \mu^2)^3} + \int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(s)}{(s - \mu^2)^3} \right). \quad (1.23)$$

Here, taking the Schwarz reflection ($\mathcal{A}(s^*) = \mathcal{A}^*(s)$), we also have

$$\text{Disc } \mathcal{A}(u) = \mathcal{A}(s + i\epsilon) - \mathcal{A}(s - i\epsilon) \quad (1.24)$$

$$= \mathcal{A}(s + i\epsilon) - \mathcal{A}^*(s + i\epsilon) \quad (1.25)$$

$$= 2i \text{Im } \mathcal{A}(s), \quad (1.26)$$

hence, f becomes:

$$f = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \left[\frac{1}{(-4m^2 + s + \mu^2)^3} + \frac{1}{(s - \mu^2)^3} \right] \text{Im } \mathcal{A}(s). \quad (1.27)$$

From the Optical theorem of $\text{Im } \mathcal{A}(s) = \sqrt{s(s - 4m^2)} \sigma_t(s)$, we derive

$$f = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \left[\frac{1}{(-4m^2 + s + \mu^2)^3} + \frac{1}{(s - \mu^2)^3} \right] \sqrt{s(s - 4m^2)} \sigma_t(s). \quad (1.28)$$

1.6 Positivity bound for SMEFT

In SMEFT, the multi-particle production of massless particles give rise to the branch cut that covers the whole real axis. [ADD FIGURE]

$$f = \frac{1}{2\pi i} \oint ds \frac{\mathcal{A}(s)}{(s - \mu^2)^3}. \quad (1.29)$$

For improving the positivity, we introduce the modified amplitude. with the

$$B_{\epsilon\Lambda}(s) \equiv \mathcal{A}(s) - \frac{1}{2\pi i} \int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} ds' \frac{\text{Disc } \mathcal{A}(s')}{s' - s} \quad (1.30)$$

$$= \frac{1}{2\pi i} \oint_C ds' \frac{\mathcal{A}(s')}{s' - s} - \frac{1}{2\pi i} \int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} ds' \frac{\text{Disc } \mathcal{A}(s')}{s' - s} \quad (1.31)$$

$$= \frac{1}{2\pi i} \int_{C'_{\epsilon\Lambda}} ds' \frac{\mathcal{A}(s')}{s' - s} = \frac{1}{2\pi i} \oint_{C_{\epsilon\Lambda}} ds' \frac{B_{\epsilon\Lambda}(s')}{s' - s}, \quad (1.32)$$

the modified amplitude has the same behavior at $s \rightarrow \infty$ and satisfies the Froissart bound.

Next, we define:

$$f_{\epsilon\Lambda}(s) \equiv \frac{1}{2} \frac{d^2 B_{\epsilon\Lambda}(s)}{ds^2} \quad (1.33)$$

$$= \frac{1}{2\pi i} \left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty} \right) ds' \frac{\text{Disc } B_{\epsilon\Lambda}(s')}{(s' - s)^3} \quad (1.34)$$

$$= \frac{1}{2\pi i} \left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty} \right) ds' \frac{\text{Disc } \mathcal{A}(s')}{(s' - s)^3} \quad (1.35)$$

$$= \frac{1}{\pi} \left(\int_{(\epsilon\Lambda)^2 + M^2}^{\infty} ds' \frac{1}{(s' + s - M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} ds' \frac{1}{(s' - s)^3} \right) \text{Im } \mathcal{A}(s) \quad (1.36)$$

$$= \frac{1}{\pi} \left(\int_{(\epsilon\Lambda)^2 + M^2}^{\infty} ds' \frac{1}{(s' + s - M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} ds' \frac{1}{(s' - s)^3} \right) \sqrt{(s - M_-^2)(s - M_+^2)} \sigma_t. \quad (1.37)$$

Here we follow the same procedure with the original version of positivity bound, applying Froissart bound for the deformation and changing the variable $s' \rightarrow M^2 - s'$, where $M^2 = 2m_1^2 + 2m_2^2$. [ADD MORE PHYSICAL INTERPRETATIONS + POLARIZATIONS]

For the positivity bounds on QGC couplings, dim-8 operators are independent of the presence of dim-6 ones, indeed,

$$\sum_i c_i^{(8)} x_i + \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j} > 0, \quad (1.38)$$

or,

$$\sum_i c_i^{(8)} x_i > \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j}. \quad (1.39)$$

While by explicit calculations, we yields that the R.H.S. is already positive definite. Hence, we can impose the bound.

$$\sum_i c_i^{(8)} x_i > 0. \quad (1.40)$$

[TO BE CONT.]

1.7 $ZZ \rightarrow ZZ$ process

External polarization: Let $p^\mu = (E, 0, 0, p_z)$, thus $p^2 = E^2 - p_z^2 = m^2$. Take the canonical basis that satisfying $p^\mu \epsilon_\mu = 0$ and $\epsilon_\mu^2 = -1$.

$$\epsilon_1^\mu = (0, 1, 0, 0) (\text{traverse}), \quad (1.41)$$

$$\epsilon_2^\mu = (0, 0, 1, 0) (\text{traverse}), \quad (1.42)$$

$$\epsilon_3^\mu = \frac{1}{m} (p_z, 0, 0, E) (\text{longitudinal}). \quad (1.43)$$

We can parameterize the polarization vectors of 2 incoming Z bosons as:

$$\epsilon^\mu(V_1) = \sum_{i=1}^3 a_i \epsilon_i^\mu = (a_3 \frac{p_1}{m_1}, a_1, a_2, a_3 \frac{E_1}{m_1}), \quad (1.44)$$

$$\epsilon^\mu(V_2) = \sum_{i=1}^3 a_i \epsilon_i^\mu = (b_3 \frac{p_2}{m_2}, b_1, b_2, b_3 \frac{E_2}{m_2}). \quad (1.45)$$

1.8 Dim-8 operators included in QGC

Operators involved in quartic gauge boson couplings (QGC) has been studied in [3], [4], [5] and are listed into 3 categories as followed:

1.8.1 Operators containing just $D_\mu\Phi$

The two independent operators in this class are

$$\mathcal{L}_{S,0} = \left[(D_\mu\Phi)^\dagger D_\nu\Phi \right] \times \left[(D^\mu\Phi)^\dagger D^\nu\Phi \right] \quad (1.46)$$

$$\mathcal{L}_{S,1} = \left[(D_\mu\Phi)^\dagger D^\mu\Phi \right] \times \left[(D_\nu\Phi)^\dagger D^\nu\Phi \right] \quad (1.47)$$

$$\mathcal{L}_{S,2} = \left[(D_\mu\Phi)^\dagger D_\nu\Phi \right] \times \left[(D^\mu\Phi)^\dagger D^\nu\Phi \right] \quad (1.48)$$

1.8.2 Operators containing $D_\mu\Phi$ and field strength

The operators in this class are:

$$\mathcal{L}_{M,0} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\mu\nu} \right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi \right] \quad (1.49)$$

$$\mathcal{L}_{M,1} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\nu\beta} \right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi \right] \quad (1.50)$$

$$\mathcal{L}_{M,2} = [B_{\mu\nu} B^{\mu\nu}] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi \right] \quad (1.51)$$

$$\mathcal{L}_{M,3} = [B_{\mu\nu} B^{\nu\beta}] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi \right] \quad (1.52)$$

$$\mathcal{L}_{M,4} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} D^\mu\Phi \right] \times B^{\beta\nu} \quad (1.53)$$

$$\mathcal{L}_{M,5} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} D^\nu\Phi \right] \times B^{\beta\mu} \quad (1.54)$$

$$\mathcal{L}_{M,6} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} \hat{W}^{\beta\nu} D^\mu\Phi \right] \quad (1.55)$$

$$\mathcal{L}_{M,7} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} \hat{W}^{\beta\mu} D^\nu\Phi \right] \quad (1.56)$$

1.8.3 Operators containing just the field strength tensor

The following operators containing just the field strength tensor also lead to quartic anomalous couplings:

$$\mathcal{L}_{T,0} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\mu\nu} \right] \times \text{Tr} \left[\hat{W}_{\alpha\beta} \hat{W}^{\alpha\beta} \right] \quad (1.57)$$

$$\mathcal{L}_{T,1} = \text{Tr} \left[\hat{W}_{\alpha\nu} \hat{W}^{\mu\beta} \right] \times \text{Tr} \left[\hat{W}_{\mu\beta} \hat{W}^{\alpha\nu} \right] \quad (1.58)$$

$$\mathcal{L}_{T,2} = \text{Tr} \left[\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta} \right] \times \text{Tr} \left[\hat{W}_{\beta\nu} \hat{W}^{\nu\alpha} \right] \quad (1.59)$$

$$\mathcal{L}_{T,3} = \text{Tr} \left[\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta} \hat{W}^{\nu\alpha} \right] \times B_{\beta\nu} \quad (1.60)$$

$$\mathcal{L}_{T,4} = \text{Tr} \left[\hat{W}_{\alpha\mu} \hat{W}^{\alpha\mu} \hat{W}^{\beta\nu} \right] \times B_{\beta\nu} \quad (1.61)$$

$$\mathcal{L}_{T,5} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\mu\nu} \right] \times B_{\alpha\beta} B^{\alpha\beta} \quad (1.62)$$

$$\mathcal{L}_{T,6} = \text{Tr} \left[\hat{W}_{\alpha\nu} \hat{W}^{\mu\beta} \right] \times B_{\mu\beta} B^{\alpha\nu} \quad (1.63)$$

$$\mathcal{L}_{T,7} = \text{Tr} \left[\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta} \right] \times B_{\beta\nu} B^{\nu\alpha} \quad (1.64)$$

$$\mathcal{L}_{T,8} = B_{\mu\nu} B^{\mu\nu} B_{\alpha\beta} B^{\alpha\beta} \quad (1.65)$$

$$\mathcal{L}_{T,9} = B_{\alpha\mu} B^{\mu\beta} B_{\beta\nu} B^{\nu\alpha} \quad (1.66)$$

1.9 Discussions and Questions

Question 1: Why lightest heavy state?

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