

Progress Report on Positivity bounds in low-energy Effective Quantum Gravity

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ABSTRACT: In this report, we rederive some current analysis on SMEFT positivity bound. We first rederive the main result in [1], which investigate the vector boson scattering process (VBS) by considering diagrams involving quartic gauge boson couplings (WGC) governed by SMEFT Dim-8 operators.

Contents

1	Positivity bounds in VBS	1
1.1	Crossing symmetry	1
1.2	Optical theorem	1
1.3	Scattering amplitude in the forward limits	2
1.4	Froissart unitary bounds and dispersion relation	2
1.5	Positivity bounds (original version)	3
1.6	Positivity bound for SMEFT	4
1.7	$ZZ \rightarrow ZZ$ process	5
1.8	Dim-8 operators included in QGC	5
1.8.1	Operators containing just $D_\mu \Phi$	5
1.8.2	Operators containing $D_\mu \Phi$ and field strength	6
1.8.3	Operators containing just the field strength tensor	6
2	Calculations	7
2.1	Field Redefinition	7
2.2	Scattering Matrix	9
2.2.1	Matter Lagrangian & Scalars-Graviton vertices	9
2.2.2	Graviton Propagator	10
2.2.3	Scattering matrix of effective 4-scalar vertices	12
A	Weak-Field Gravity	13
A.1	Linearised theory	13
A.2	Green function	14
A.3	Gauge transformation	15
B	Field Redefinition	15
C	Effective Field Theory in Gravity	16
D	Derivation of the positivity bound for $ZZ \rightarrow ZZ$ process	17
D.1	S-type operators	17
D.2	T-type operators	20
D.2.1	$\mathcal{L}_{T,0}, \mathcal{L}_{T,1}, \mathcal{L}_{T,5}, \mathcal{L}_{T,6}, \mathcal{L}_{T,8}$	21
D.2.2	$\mathcal{L}_{T,2}, \mathcal{L}_{T,7}, \mathcal{L}_{T,9}$	23
D.3	M-type operators	25

D.3.1	$\mathcal{L}_{M,0}, \mathcal{L}_{M,2}, \mathcal{L}_{M,4}$	26
D.3.2	$\mathcal{L}_{M,1}, \mathcal{L}_{M,3}, \mathcal{L}_{M,5}, \mathcal{L}_{M,7}$	28
D.4	Results	30
E	Proof of Optical theorem	31
E.1	A	33
F	Hahn-Banach separation theorem	33

1 Positivity bounds in VBS

In this section, we re-interpret some basic concepts, theorems and re-derive (in details) some of the main results and calculations in [1].

1.1 Crossing symmetry

Consider a $2 \rightarrow 2$ process, any of the particles can be replaced by its antiparticle on the other side of the interaction. Hence, for

Spin = 0: $M(s, t) = M(u, t)$.

Spin > 0: With the linear polarizations vector $(\epsilon_1^\mu)^* = \epsilon_3^\mu, (\epsilon_2^\mu)^* = \epsilon_4^\mu$, and with restriction to forward limit, we have $M(s, 0) = M(u, 0)$ (or $\mathcal{A}(s) = \mathcal{A}(u)$).

1.2 Optical theorem

The Optical theorem yield the relation between forward scattering amplitude and cross-section (refer to the Appendix).

$$\text{Im } \mathcal{A}(k_1 k_2 \rightarrow k_1 k_2) = 2E_1 E_2 |v_1 - v_2| \sigma_t. \quad (1.1)$$

Going into the CM-system, we have $p_1 + p_2 = 0, E_{\text{CM}} = E_1 + E_2, \mathbf{p}_{\text{CM}} = \mathbf{p}_1 = -\mathbf{p}_2$, (with $v = \frac{p}{E}$) we get the optical theorem in the standard form:

$$\text{Im } \mathcal{A}(k_1 k_2 \rightarrow k_1 k_2) = 2E_{\text{CM}} \mathbf{p}_{\text{CM}} \sigma_t. \quad (1.2)$$

when 2 incoming particles are the same, $m = m_1 = m_2, E_1 = E_2 = E$, we have $s = 4E^2 = E_{\text{CM}}^2, \mathbf{p}_{\text{CM}} = \sqrt{s^2/4 - m^2},$)

$$\text{Im } \mathcal{A}(k_1 k_2 \rightarrow k_1 k_2) = 2\sqrt{s} \sqrt{\frac{s}{4} - m^2} \sigma_t \quad (1.3)$$

$$= \sqrt{s(s - 4m^2)} \sigma_t. \quad (1.4)$$

In general, with 2 different incoming particles, defining $M_+ =$ we have

$$\begin{aligned} & 2(E_1 + E_2)\mathbf{p}_{\text{CM}} \\ &= 2\sqrt{(E_1 + E_2)^2\mathbf{p}_{\text{CM}}^2} \end{aligned} \quad (1.5)$$

$$= 2\sqrt{\mathbf{p}_{\text{CM}}^4 + 2\mathbf{p}_{\text{CM}}^2 E_1 E_2 + E_1^2 E_2^2 - \mathbf{p}_{\text{CM}}^4 + (E_1^2 + E_2^2)\mathbf{p}_{\text{CM}}^2 - E_1^2 E_2^2} \quad (1.6)$$

$$= \sqrt{(2\mathbf{p}_{\text{CM}}^2 + 2E_1 E_2)^2 - 4m_1^2 m_2^2} \quad (1.7)$$

$$= \sqrt{((E_1 + E_2)^2 - 2(m_1^2 + m_2^2))^2 - 4m_1^2 m_2^2} \quad (1.8)$$

$$= \sqrt{(E_1 + E_2)^4 - 2(E_1 + E_2)^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2} \quad (1.9)$$

$$= \sqrt{s^2 - s(M_+^2 + M_-^2) + M_+^2 M_-^2} \quad (1.10)$$

$$= \sqrt{s(s - M_+^2) - M_-^2(s - M_+^2)} \quad (1.11)$$

$$= \sqrt{(s - M_-^2)(s - M_+^2)}. \quad (1.12)$$

Hence, Eq. 2.31 yields

$$\text{Im } A(k_1 k_2 \rightarrow k_1 k_2) = 2(E_1 \mathbf{p}_2 - E_2 \mathbf{p}_1) \sigma_t \quad (1.13)$$

$$= 2(E_1 + E_2) \mathbf{p}_{\text{CM}} \sigma_t \quad (1.14)$$

$$= \sqrt{(s - M_-^2)(s - M_+^2)} \sigma_t. \quad (1.15)$$

It yields back the result of Eq. A.26 when $M_- = 0$.

$$\text{Im } A_{ab}^{q_1 q_2}(s') = \sqrt{(s' - M_+^2)(s' - M_-^2)} \sigma_{ab}^{q_1 q_2}(s') > 0, \quad s' > (\epsilon \Lambda)^2, \quad (1.16)$$

1.3 Scattering amplitude in the forward limits

[ELABORATE MORE FROM [2]]

1.4 Froissart unitary bounds and dispersion relation

Froissart bound: Unitarity forces the high-energy amplitude in the forward limit is bounded by

$$\mathcal{A}(s) < \mathcal{O}(s \ln^2 s) \quad (1.17)$$

It is a necessary condition for the vanishing boundary contribution when we deform the contour integrals from IR to UV regime [PROVE THE BOUND AND ELABORATE MORE ON THE DISPERSION RELATION].

1.5 Positivity bounds (original version)

Physics in the IR regime can be deformed to UV (contour C to C'). The boundary contribution vanishes because of the Froissart bound [ELABORATE MORE].

$$f \equiv \frac{1}{2\pi i} \oint_C ds \frac{\mathcal{A}(s)}{(s - \mu^2)^3} = \frac{1}{2\pi i} \left(\int_{-\infty}^0 + \int_{4m^2}^{\infty} \right) ds \frac{\text{Disc } \mathcal{A}(s)}{(s - \mu^2)^3}, \quad (1.18)$$

with $\text{Disc } \mathcal{A}(s) \equiv \mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon)$. From here, we see that the dim-6 and dim-8 operators in low-energy EFT can be constrained by the positivity bound ($\text{Disc } M(s, 0) \geq 0$) in the UV regime [ADD FIGURE].

In the forward limit ($t \rightarrow 0$), we have $s = 4m^2 - u$. Changing the variable with according bounds in the first term, f can be rewritten as:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(4m^2 - u)}{(4m^2 - u - \mu^2)^3} + \int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(s)}{(s - \mu^2)^3} \right). \quad (1.19)$$

The crossing symmetry reads $\mathcal{A}(4m^2 - u) = \mathcal{A}(s) = \mathcal{A}(u)$. Hence,

$$\text{Disc } \mathcal{A}(4m^2 - u) = \mathcal{A}(4m^2 - u + i\epsilon) - \mathcal{A}(4m^2 - u - i\epsilon) \quad (1.20)$$

$$= \mathcal{A}(u - i\epsilon) - \mathcal{A}(u + i\epsilon) \quad (1.21)$$

$$= -\text{Disc } \mathcal{A}(u). \quad (1.22)$$

Applying this relation to Eq. A.18, and replace the variable u by s , we have:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(s)}{(-4m^2 + s + \mu^2)^3} + \int_{4m^2}^{\infty} du \frac{\text{Disc } \mathcal{A}(s)}{(s - \mu^2)^3} \right). \quad (1.23)$$

Here, taking the Schwarz reflection ($\mathcal{A}(s^*) = \mathcal{A}^*(s)$), we also have

$$\text{Disc } \mathcal{A}(u) = \mathcal{A}(s + i\epsilon) - \mathcal{A}(s - i\epsilon) \quad (1.24)$$

$$= \mathcal{A}(s + i\epsilon) - \mathcal{A}^*(s + i\epsilon) \quad (1.25)$$

$$= 2i \text{Im } \mathcal{A}(s), \quad (1.26)$$

hence, f becomes:

$$f = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \left[\frac{1}{(-4m^2 + s + \mu^2)^3} + \frac{1}{(s - \mu^2)^3} \right] \text{Im } \mathcal{A}(s). \quad (1.27)$$

From the Optical theorem of $\text{Im } \mathcal{A}(s) = \sqrt{s(s - 4m^2)} \sigma_t(s)$, we derive

$$f = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \left[\frac{1}{(-4m^2 + s + \mu^2)^3} + \frac{1}{(s - \mu^2)^3} \right] \sqrt{s(s - 4m^2)} \sigma_t(s). \quad (1.28)$$

1.6 Positivity bound for SMEFT

In SMEFT, the multi-particle production of massless particles give rise to the branch cut that covers the whole real axis. [ADD FIGURE]

$$f = \frac{1}{2\pi i} \oint ds \frac{\mathcal{A}(s)}{(s - \mu^2)^3}. \quad (1.29)$$

For improving the positivity, we introduce the modified amplitude. with the

$$B_{\epsilon\Lambda}(s) \equiv \mathcal{A}(s) - \frac{1}{2\pi i} \int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} ds' \frac{\text{Disc}\mathcal{A}(s')}{s' - s} \quad (1.30)$$

$$= \frac{1}{2\pi i} \oint_C ds' \frac{\mathcal{A}(s')}{s' - s} - \frac{1}{2\pi i} \int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} ds' \frac{\text{Disc}\mathcal{A}(s')}{s' - s} \quad (1.31)$$

$$= \frac{1}{2\pi i} \int_{C'_{\epsilon\Lambda}} ds' \frac{\mathcal{A}(s')}{s' - s} = \frac{1}{2\pi i} \oint_{C_{\epsilon\Lambda}} ds' \frac{B_{\epsilon\Lambda}(s')}{s' - s}, \quad (1.32)$$

the modified amplitude has the same behavior at $s \rightarrow \infty$ and satisfies the Froissart bound.

Next, we define:

$$f_{\epsilon\Lambda}(s) \equiv \frac{1}{2} \frac{d^2 B_{\epsilon\Lambda}(s)}{ds^2} \quad (1.33)$$

$$= \frac{1}{2\pi i} \left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty} \right) ds' \frac{\text{Disc } B_{\epsilon\Lambda}(s')}{(s' - s)^3} \quad (1.34)$$

$$= \frac{1}{2\pi i} \left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty} \right) ds' \frac{\text{Disc } \mathcal{A}(s')}{(s' - s)^3} \quad (1.35)$$

$$= \frac{1}{\pi} \left(\int_{(\epsilon\Lambda)^2 + M^2}^{\infty} ds' \frac{1}{(s' + s - M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} ds' \frac{1}{(s' - s)^3} \right) \text{Im } \mathcal{A}(s) \quad (1.36)$$

$$= \frac{1}{\pi} \left(\int_{(\epsilon\Lambda)^2 + M^2}^{\infty} ds' \frac{1}{(s' + s - M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} ds' \frac{1}{(s' - s)^3} \right) \sqrt{(s - M_-^2)(s - M_+^2)} \sigma_t. \quad (1.37)$$

Here we follow the same procedure with the original version of positivity bound, applying Froissart bound for the deformation and changing the variable $s' \rightarrow M^2 - s'$, where $M^2 = 2m_1^2 + 2m_2^2$. [ADD MORE PHYSICAL INTERPRETATIONS + POLARIZATIONS]

For the positivity bounds on QGC couplings, dim-8 operators are independent of the presence of dim-6 ones, indeed,

$$\sum_i c_i^{(8)} x_i + \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j} > 0, \quad (1.38)$$

or,

$$\sum_i c_i^{(8)} x_i > \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j}. \quad (1.39)$$

While by explicit calculations, we yields that the R.H.S. is already positive definite. Hence, we can impose the bound.

$$\sum_i c_i^{(8)} x_i > 0. \quad (1.40)$$

[TO BE CONT.]

1.7 $ZZ \rightarrow ZZ$ process

External polarization: Let $p^\mu = (E, 0, 0, p_z)$, thus $p^2 = E^2 - p_z^2 = m^2$. Take the canonical basis that satisfying $p^\mu \epsilon_\mu = 0$ and $\epsilon_\mu^2 = -1$.

$$\epsilon_1^\mu = (0, 1, 0, 0) (\text{traverse}), \quad (1.41)$$

$$\epsilon_2^\mu = (0, 0, 1, 0) (\text{traverse}), \quad (1.42)$$

$$\epsilon_3^\mu = \frac{1}{m} (p_z, 0, 0, E) (\text{longitudinal}). \quad (1.43)$$

We can parameterize the polarization vectors of 2 incoming Z bosons as:

$$\epsilon^\mu(V_1) = \sum_{i=1}^3 a_i \epsilon_i^\mu = (a_3 \frac{p_1}{m_1}, a_1, a_2, a_3 \frac{E_1}{m_1}), \quad (1.44)$$

$$\epsilon^\mu(V_2) = \sum_{i=1}^3 a_i \epsilon_i^\mu = (b_3 \frac{p_2}{m_2}, b_1, b_2, b_3 \frac{E_2}{m_2}). \quad (1.45)$$

1.8 Dim-8 operators included in QGC

Operators involved in quartic gauge boson couplings (QGC) has been studied in [3], [4], [5] and are listed into 3 categories as followed:

1.8.1 Operators containing just $D_\mu \Phi$

The two independent operators in this class are

$$\mathcal{L}_{S,0} = \left[(D_\mu \Phi)^\dagger D_\nu \Phi \right] \times \left[(D^\mu \Phi)^\dagger D^\nu \Phi \right] \quad (1.46)$$

$$\mathcal{L}_{S,1} = \left[(D_\mu \Phi)^\dagger D^\mu \Phi \right] \times \left[(D_\nu \Phi)^\dagger D^\nu \Phi \right] \quad (1.47)$$

$$\mathcal{L}_{S,2} = \left[(D_\mu \Phi)^\dagger D_\nu \Phi \right] \times \left[(D^\mu \Phi)^\dagger D^\nu \Phi \right] \quad (1.48)$$

1.8.2 Operators containing $D_\mu\Phi$ and field strength

The operators in this class are:

$$\mathcal{L}_{M,0} = \text{Tr} [\hat{W}_{\mu\nu} \hat{W}^{\mu\nu}] \times [(D_\beta\Phi)^\dagger D^\beta\Phi] \quad (1.49)$$

$$\mathcal{L}_{M,1} = \text{Tr} [\hat{W}_{\mu\nu} \hat{W}^{\nu\beta}] \times [(D_\beta\Phi)^\dagger D^\mu\Phi] \quad (1.50)$$

$$\mathcal{L}_{M,2} = [B_{\mu\nu} B^{\mu\nu}] \times [(D_\beta\Phi)^\dagger D^\beta\Phi] \quad (1.51)$$

$$\mathcal{L}_{M,3} = [B_{\mu\nu} B^{\nu\beta}] \times [(D_\beta\Phi)^\dagger D^\mu\Phi] \quad (1.52)$$

$$\mathcal{L}_{M,4} = [(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} D^\mu\Phi] \times B^{\beta\nu} \quad (1.53)$$

$$\mathcal{L}_{M,5} = [(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} D^\nu\Phi] \times B^{\beta\mu} \quad (1.54)$$

$$\mathcal{L}_{M,6} = [(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} \hat{W}^{\beta\nu} D^\mu\Phi] \quad (1.55)$$

$$\mathcal{L}_{M,7} = [(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} \hat{W}^{\beta\mu} D^\nu\Phi] \quad (1.56)$$

1.8.3 Operators containing just the field strength tensor

The following operators containing just the field strength tensor also lead to quartic anomalous couplings:

$$\mathcal{L}_{T,0} = \text{Tr} [\hat{W}_{\mu\nu} \hat{W}^{\mu\nu}] \times \text{Tr} [\hat{W}_{\alpha\beta} \hat{W}^{\alpha\beta}] \quad (1.57)$$

$$\mathcal{L}_{T,1} = \text{Tr} [\hat{W}_{\alpha\nu} \hat{W}^{\mu\beta}] \times \text{Tr} [\hat{W}_{\mu\beta} \hat{W}^{\alpha\nu}] \quad (1.58)$$

$$\mathcal{L}_{T,2} = \text{Tr} [\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta}] \times \text{Tr} [\hat{W}_{\beta\nu} \hat{W}^{\nu\alpha}] \quad (1.59)$$

$$\mathcal{L}_{T,3} = \text{Tr} [\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta} \hat{W}^{\nu\alpha}] \times B_{\beta\nu} \quad (1.60)$$

$$\mathcal{L}_{T,4} = \text{Tr} [\hat{W}_{\alpha\mu} \hat{W}^{\alpha\mu} \hat{W}^{\beta\nu}] \times B_{\beta\nu} \quad (1.61)$$

$$\mathcal{L}_{T,5} = \text{Tr} [\hat{W}_{\mu\nu} \hat{W}^{\mu\nu}] \times B_{\alpha\beta} B^{\alpha\beta} \quad (1.62)$$

$$\mathcal{L}_{T,6} = \text{Tr} [\hat{W}_{\alpha\nu} \hat{W}^{\mu\beta}] \times B_{\mu\beta} B^{\alpha\nu} \quad (1.63)$$

$$\mathcal{L}_{T,7} = \text{Tr} [\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta}] \times B_{\beta\nu} B^{\nu\alpha} \quad (1.64)$$

$$\mathcal{L}_{T,8} = B_{\mu\nu} B^{\mu\nu} B_{\alpha\beta} B^{\alpha\beta} \quad (1.65)$$

$$\mathcal{L}_{T,9} = B_{\alpha\mu} B^{\mu\beta} B_{\beta\nu} B^{\nu\alpha} \quad (1.66)$$

2 Scalar photon QED with a spectator field

In this session, we review [6] **My comments:**

- I spotted a difference from my vertices definition E.8 versus Eq. (B.1) in [6] at a factor $\frac{1}{2}$. Indeed, with Eq. (B.1), the Eq. (B.3) should yield $\frac{i}{4M_{\text{Pl}}^2 t} su$. I am not sure whether they have added a factor of 4 (for symmetry?) implicitly there, or I have made some errors in the calculation. Please have a cross-check!
- In their Eq. (B.3), I suppose the $V_m^{\mu\nu}(k_1, k_3)$ should be changed to $V_0^{\mu\nu}(k_1, k_3)$ as we have assumed the scalar to be massless. Otherwise, there will be an additional term in the amplitude.

2.1 Field Redefinition

Under field redefinition, as in Eq. (3.5) of [6]

$$g_{\mu\nu} = g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right), \quad (2.1)$$

the terms (inside the bracket) of Eq. (4.2) reads,

$$\frac{M_{\text{Pl}}^2}{2} R = \frac{M_{\text{Pl}}^2}{2} R^{\mu\nu} g_{\mu\nu} \quad (2.2)$$

$$\rightarrow \frac{M_{\text{Pl}}^2}{2} R^{\mu\nu} \left[g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) \right] \quad (2.3)$$

$$= \frac{M_{\text{Pl}}^2}{2} R + C \frac{\alpha^2}{M^2} R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} C \frac{\alpha^2}{M^2} R (\partial\phi)^2, \quad (2.4)$$

$$-\frac{1}{2} (\partial\phi)^2 = -\frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \quad (2.5)$$

$$\rightarrow -\frac{1}{2} \left[g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) \right] \partial^\mu \phi \partial^\nu \phi \quad (2.6)$$

$$= -\frac{1}{2} (\partial\phi)^2 - \frac{1}{2} C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^4, \quad (2.7)$$

$$-\frac{1}{2} (\partial\chi)^2 = -\frac{1}{2} g_{\mu\nu} \partial^\mu \chi \partial^\nu \chi \quad (2.8)$$

$$\rightarrow -\frac{1}{2} \left[g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) \right] \partial^\mu \chi \partial^\nu \chi \quad (2.9)$$

$$= -\frac{1}{2} (\partial\chi)^2 - C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi \partial\chi)^2 + \frac{1}{2} C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^2 (\partial\chi)^2. \quad (2.10)$$

All other terms are added with negligible higher-order (H. O.) terms.
The metric determinant transforms as,

$$\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} \quad (2.11)$$

$$\rightarrow \sqrt{-\det \left[g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi) g_{\mu\nu} \right) \right]} \quad (2.12)$$

$$\rightarrow \sqrt{-\det \left[g_{\mu\alpha} \left(\delta^\alpha_\nu + 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial^\alpha \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi) \delta^\alpha_\nu \right) \right) \right]} \quad (2.13)$$

$$\sim \sqrt{-\det \left[g_{\mu\alpha} \exp \left(2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial^\alpha \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi) \delta^\alpha_\nu \right) \right) \right]} \quad (2.14)$$

$$= \sqrt{-\det(g_{\mu\alpha})} \sqrt{\det \exp \left(2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial^\alpha \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi) \delta^\alpha_\nu \right) \right)} \quad (2.15)$$

$$= \sqrt{-g} \sqrt{\exp \text{tr} \left(2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left(\partial^\alpha \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi) \delta^\alpha_\nu \right) \right)} \quad (2.16)$$

$$= \sqrt{-g} \sqrt{\exp \left(2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (-(\partial\phi)^2) \right)} \quad (2.17)$$

$$\sim \sqrt{-g} \sqrt{1 - 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^2} \quad (2.18)$$

$$\sim \sqrt{-g} \left(1 - C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^2 \right). \quad (2.19)$$

Hence, substitute $\sqrt{-g}$ by $\sqrt{-g} \left(1 + C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^2 \right)$, the Lagrangian terms transform as,

$$\begin{aligned} \sqrt{-g} \frac{M_{\text{Pl}}^2}{2} R &\rightarrow \sqrt{-g} \left(\frac{M_{\text{Pl}}^2}{2} R + C \frac{\alpha^2}{M^2} R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} C \frac{\alpha^2}{M^2} R (\partial\phi)^2 \right. \\ &\quad \left. + \frac{1}{2} C \frac{\alpha^2}{M^2} R (\partial\phi)^2 + \text{H. O.} \right). \end{aligned} \quad (2.20)$$

$$= \sqrt{-g} \left(\frac{M_{\text{Pl}}^2}{2} R + C \frac{\alpha^2}{M^2} R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \text{H. O.} \right), \quad (2.21)$$

$$\sqrt{-g} \left(-\frac{1}{2}(\partial\phi)^2 \right) \quad (2.22)$$

$$\rightarrow \sqrt{-g} \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^4 - \frac{1}{2}C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^4 + \text{H. O.} \right) \quad (2.23)$$

$$= \sqrt{-g} \left(-\frac{1}{2}(\partial\phi)^2 - C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^4 + \text{H. O.} \right), \quad (2.24)$$

$$\sqrt{-g} \left(-\frac{1}{2}(\partial\chi)^2 \right) \quad (2.25)$$

$$\rightarrow \sqrt{-g} \left(-\frac{1}{2}(\partial\chi)^2 - C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi\partial\chi)^2 + \frac{1}{2}C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^2 (\partial\chi)^2 \right. \\ \left. - \frac{1}{2}C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^2 (\partial\chi)^2 + \text{H. O.} \right) \quad (2.26)$$

$$= \sqrt{-g} \left(-\frac{1}{2}(\partial\chi)^2 - C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi\partial\chi)^2 + \text{H. O.} \right), \quad (2.27)$$

Given the above transformation, we deduce the extra terms,

$$\sqrt{-g} \left(C \frac{\alpha^2}{M^2} R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^4 - C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi\partial\chi)^2 \right). \quad (2.28)$$

Hence, the IR action (4.2) in [6]

$$\mathcal{L}_{\text{IR}}^{(J)} = \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{\alpha^3 M}{(2\pi)^2} \frac{\phi^3}{3!} + \frac{\alpha^4}{2\pi^2} \frac{\phi^4}{4!} \right. \\ \left. + C \frac{\alpha^2}{M^2} R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \tilde{C} \frac{\alpha^4}{M^4} (\partial\phi)^4 \right], \quad (2.29)$$

reduces to (4.3) of [6],

$$\mathcal{L}_{\text{IR}} = \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{\alpha^3 M}{(2\pi)^2} \frac{\phi^3}{3!} + \frac{\alpha^4}{2\pi^2} \frac{\phi^4}{4!} \right. \\ \left. + C' \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial\phi)^4 + C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} (\partial_\mu \phi \partial^\mu \chi)^2 + \dots \right]. \quad (2.30)$$

2.2 Scattering Matrix

2.2.1 Matter Lagrangian & Scalars-Graviton vertices

Scalar matter fields interact with the gravitational field as described by the action,

$$S_m = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right]. \quad (2.31)$$

The quantum fluctuation of gravitational fields can be expanded about a smooth background metric $\eta_{\mu\nu}$, with the fluctuations suppressed by the Planck scale $M_{\text{Pl}} = \frac{1}{\sqrt{8\pi G}}$ as,

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\text{Pl}}} h_{\mu\nu}. \quad (2.32)$$

Einstein action are expanded as,

$$S_g = \int d^4x \sqrt{-g} [\mathcal{L}_g^{(0)} + \mathcal{L}_g^{(1)} + \mathcal{L}_g^{(2)} + \dots] \quad (2.33)$$

with the expanding terms,

$$\mathcal{L}_g^{(0)} = \frac{M_{\text{Pl}}^2}{2} R \quad (2.34)$$

$$\mathcal{L}_g^{(1)} = \frac{2}{M_{\text{Pl}}} h_{\mu\nu} [\eta^{\mu\nu} R - 2R^{\mu\nu}] \quad (2.35)$$

Here, we skip the discussion about gauge fixing and ghost Lagrangian. A similar expansion for matter Lagrangian yields,

$$S_m = \int d^4x \sqrt{-g} (\mathcal{L}_m^{(0)} + \mathcal{L}_m^{(1)} + \mathcal{L}_m^{(2)} + \dots) \quad (2.36)$$

Here, $T^{\mu\nu}$ is derived from variation of 2.31 as,

$$T^{\mu\nu} = \frac{2}{\sqrt{-\eta}} \frac{\delta S_m}{\delta \eta^{\mu\nu}} = \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial_\tau \phi \partial^\tau \phi - m^2 \phi^2) \right) \quad (2.37)$$

Feynman rules for $\mathcal{L}_m^{(1)}$ in momentum space reads,

$$V_{\mu\nu}^{\phi\phi h}(p_1, p_2, m) = \frac{i}{M_{\text{Pl}}} [(p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) - \eta_{\mu\nu} (p_1 p_2 - m^2)]. \quad (2.38)$$

Hence, the massless scalar field ϕ and χ reads

$$V_{\mu\nu}^{\phi\phi h}(p_1, p_3, 0) = \frac{i}{M_{\text{Pl}}} [(p_{1\mu} p_{3\nu} + p_{1\nu} p_{3\mu}) - \eta_{\mu\nu} (p_1 p_3)]. \quad (2.39)$$

$$V_{\alpha\beta}^{\chi\chi h}(p_2, p_4, 0) = \frac{i}{M_{\text{Pl}}} [(p_{2\alpha} p_{4\beta} + p_{2\beta} p_{4\alpha}) - \eta_{\alpha\beta} (p_2 p_4)]. \quad (2.40)$$

2.2.2 Graviton Propagator

Consider the Lagrangian

$$\sqrt{-g} \mathcal{L} = \sqrt{-g} \left(\frac{M_{\text{Pl}}^2}{2} R + \mathcal{L}_m + \mathcal{L}_{\text{GF}} \right), \quad (2.41)$$

to the second order in $h_{\mu\nu}$,

$$\frac{M_{\text{Pl}}^2}{2}R = \frac{M_{\text{Pl}}^2}{2}(\partial_\mu\partial_\nu h^{\mu\nu} - \square h) + \frac{1}{2}[\partial_\tau h_{\mu\nu}\partial^\tau \bar{h}^{\mu\nu} - 2\partial^\tau \bar{h}_{\mu\tau}\partial_\sigma \bar{h}^{\mu\sigma}], \quad (2.42)$$

with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad (2.43)$$

as mentioned in the Appendix. The harmonic gauge means to choose $\xi = 1$ of the gauge fixing term,

$$\mathcal{L}_{\text{GF}} = \xi\partial_\mu \bar{h}^{\mu\nu}\partial^\tau \bar{h}_{\tau\nu} \quad (2.44)$$

The Lagrangian become

$$\sqrt{-g}\mathcal{L} = \frac{1}{2}\partial_\tau h_{\mu\nu}\partial^\tau h^{\mu\nu} - \frac{1}{4}\partial_\tau h\partial^\tau h - \frac{1}{M_{\text{Pl}}}h^{\mu\nu}T_{\mu\nu}. \quad (2.45)$$

Taking integration by part, we have

$$\mathcal{L} = \frac{1}{2}h_{\mu\nu}\square\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta} + \frac{1}{M_{\text{Pl}}}h^{\mu\nu}T_{\mu\nu}, \quad (2.46)$$

with the “identity” tensor $I^{\mu\nu\alpha\beta}$ defined as in A.18. The E.O.M. follows,

$$\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta}\square D_{\alpha\beta\gamma\delta} = I^{\mu\nu}{}_{\gamma\delta}. \quad (2.47)$$

This equation admitted the solution of A.26, assuming that the initial condition correspond to Feynman propagator $D^{\alpha\beta\gamma\sigma}(x-y)$,

$$D^{\alpha\beta\gamma\delta}(x-y) = \begin{cases} G^{\alpha\beta\gamma\delta}(x-y) & \text{if } x^0 > y^0, \\ G^{\alpha\beta\gamma\delta}(y-x) & \text{if } x^0 < y^0, \end{cases} \quad (2.48)$$

we obtain

$$iD^{\alpha\beta\gamma\sigma}(x) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 + i\epsilon} e^{-iqx} P^{\alpha\beta\gamma\delta}, \quad (2.49)$$

with

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2}(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}). \quad (2.50)$$

Hence, the propagator reads,

$$\frac{iP^{\mu\alpha\nu\beta}}{k^2} = \frac{i}{2} \frac{\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}}{(p_1 + p_3)^2}. \quad (2.51)$$

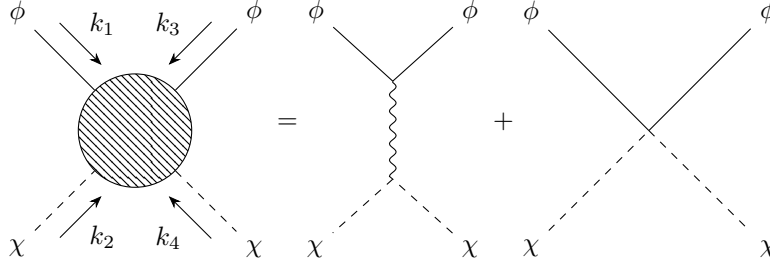


Figure 1: Feynman diagrams of the process, taken from [6]

2.2.3 Scattering matrix of effective 4-scalar vetices

We now have all the ingredients to derive the scattering matrix.

The t-channel reads,

$$i\mathcal{M}_1 = -V_{\mu\nu}^{\phi\phi h}(p_1, p_3, 0) \frac{iP^{\mu\alpha\nu\beta}}{(p_1 + p_3)^2} V_{\alpha\beta}^{\chi\chi h}(p_2, p_4, 0) \quad (2.52)$$

$$\begin{aligned} &= -\frac{i^3}{2M_{\text{Pl}}^2(p_1 + p_3)^2} [4p_{1\mu}p_{3\nu} (\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}) p_{2\alpha}p_{4\beta} \\ &\quad - 2\eta_{\mu\nu}(p_1p_3) (\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}) p_{2\alpha}p_{4\beta} \\ &\quad - 2p_{1\mu}p_{3\nu} (\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}) \eta_{\alpha\beta}(p_2p_4) \\ &\quad + \eta_{\mu\nu}(p_1p_3) (\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}) \eta_{\alpha\beta}(p_2p_4)] \end{aligned} \quad (2.53)$$

$$\begin{aligned} &= \frac{i}{4M_{\text{Pl}}^2(p_1p_3)} [4[(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3) - (p_1p_3)(p_2p_4)] \\ &\quad - 2(p_1p_3)(1 + 1 - 4)(p_2p_4) - 2(p_1p_3)(1 + 1 - 4)(p_2p_4) \\ &\quad + (p_1p_3)(4 + 4 - 16)(p_2p_4)] \end{aligned} \quad (2.54)$$

$$= \frac{i}{M_{\text{Pl}}^2(p_1p_3)} [(p_1p_2)(p_3p_4) + (p_1p_4)(p_2p_3) - (p_1p_3)(p_2p_4)]. \quad (2.55)$$

Here, we have used the fact that $\eta^{\mu\nu}\eta_{\mu\alpha} = \delta^\nu_\alpha$, and $\eta^{\mu\nu}\eta_{\mu\nu} = 4$. In addition, with $p_i^2 = m_i^2 = 0$ for all $i = 1, 2, 3, 4$, the Madelstam variables read:

$$s \equiv -(p_1 + p_2)^2 = -(p_3 + p_4)^2 = -2p_1p_2 = -2p_3p_4, \quad (2.56)$$

$$t \equiv -(p_1 + p_3)^2 = -(p_2 + p_4)^2 = -2p_1p_3 = -2p_2p_4, \quad (2.57)$$

$$-s - t = u \equiv -(p_1 + p_4)^2 = -(p_2 + p_3)^2 = -2p_1p_4 = -2p_2p_3. \quad (2.58)$$

Therefore, the term is rewritten to

$$i\mathcal{M}_1 = -i \frac{s(s+t)}{M_{\text{Pl}}^2 t}. \quad (2.59)$$

Now, consider the 4-vertice diagram,

$$i\mathcal{M}_2 = 2iC \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} [p_{1\mu} p_{3\nu} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha}) p_{2\alpha} p_{4\beta}] \quad (2.60)$$

$$= 2iC \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} [(p_1 p_2)(p_3 p_4) + (p_1 p_4)(p_2 p_3)] \quad (2.61)$$

$$= 2iC \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left[s(s+t) + \frac{t^2}{2} \right]. \quad (2.62)$$

The effective 4-scalar scattering matrix yields,

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \quad (2.63)$$

$$= -\frac{s(s+t)}{M_{\text{Pl}}^2 t} + 2C \frac{\alpha^2}{M^2 M_{\text{Pl}}^2} \left[s(s+t) + \frac{t^2}{2} \right]. \quad (2.64)$$

A Weak-Field Gravity

A.1 Linearised theory

Expanding metric around flat-space Minkowski background, with $M_{\text{Pl}} = \frac{1}{\sqrt{8\pi G}}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\text{Pl}}} h_{\mu\nu}. \quad (\text{A.1})$$

To leading order, the inverse metric reads,

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{2}{M_{\text{Pl}}} h^{\mu\nu}. \quad (\text{A.2})$$

Christoffel symbols are then,

$$\Gamma_{\nu\rho}^\sigma = \frac{1}{M_{\text{Pl}}} \eta^{\sigma\lambda} (\partial_\nu h_{\lambda\rho} + \partial_\rho h_{\nu\lambda} - \partial_\lambda h_{\nu\rho}). \quad (\text{A.3})$$

The Rieman tensor reads,

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma_{\nu\rho}^\sigma - \partial_\nu \Gamma_{\mu\rho}^\sigma + \Gamma_{\nu\rho}^\lambda \Gamma_{\mu\lambda}^\sigma - \Gamma_{\mu\rho}^\lambda \Gamma_{\nu\lambda}^\sigma. \quad (\text{A.4})$$

The $\Gamma\Gamma$ term is at $\mathcal{O}(h^2)$, hence, to the linear order,

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma_{\nu\rho}^\sigma - \partial_\nu \Gamma_{\mu\rho}^\sigma + \mathcal{O}(h^2) \quad (\text{A.5})$$

$$= \frac{1}{M_{\text{Pl}}} \eta^{\sigma\lambda} (\partial_\mu \partial_\rho h_{\nu\lambda} - \partial_\mu \partial_\lambda h_{\nu\rho} - \partial_\nu \partial_\rho h_{\mu\lambda} + \partial_\nu \partial_\lambda h_{\mu\rho}) + \mathcal{O}(h^2). \quad (\text{A.6})$$

The Ricci tensors read

$$R_{\mu\nu} = \frac{1}{M_{\text{Pl}}} (\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \square h_{\mu\nu} - \partial_\mu \partial_\nu h). \quad (\text{A.7})$$

with $h \equiv h^\mu{}_\mu$, $\square \equiv \partial^\mu \partial_\mu$. The Ricci scalar follows,

$$R = \frac{1}{M_{\text{Pl}}} (\partial^\mu \partial^\nu h_{\mu\nu} - \square h). \quad (\text{A.8})$$

Hence, we deduce the Einstein equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (\text{A.9})$$

with

$$G_{\mu\nu} = \frac{1}{M_{\text{Pl}}} [\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \square h_{\mu\nu} - \partial_\mu \partial_\nu h - (\partial^\rho \partial^\sigma h_{\rho\sigma} - \square h) \eta_{\mu\nu}]. \quad (\text{A.10})$$

A.2 Green function

We have derived Ricci tensor and scalar,

$$R_{\mu\nu} = \frac{1}{M_{\text{Pl}}} (\partial_\mu \partial_\gamma h^\gamma{}_\nu + \partial_\nu \partial_\gamma h^\gamma{}_\mu - \partial_\mu \partial_\nu h^\gamma{}_\gamma - \square h_{\mu\nu}) + \mathcal{O}(h^2), \quad (\text{A.11})$$

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{1}{M_{\text{Pl}}} \partial_\mu \partial_\gamma h^{\mu\gamma} + \mathcal{O}(h^2). \quad (\text{A.12})$$

The Einstein equation reads,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (\text{A.13})$$

$$\simeq \frac{1}{M_{\text{Pl}}} \left[(\delta_{(\mu}{}^\alpha \delta_{\nu)}{}^\beta - \eta_{\mu\nu} \eta^{\alpha\beta}) \square - 2\delta_{(\mu}{}^{(\alpha} \partial_{\nu)} \partial^{\beta)} + \eta^{\alpha\beta} \partial_\mu \partial_\nu + \eta_{\mu\nu} \partial^\alpha \partial^\beta \right] h_{\alpha\beta} \quad (\text{A.14})$$

$$\equiv \frac{1}{M_{\text{Pl}}} O_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \quad (\text{A.15})$$

$$= \frac{1}{M_{\text{Pl}}^2} T_{\mu\nu}. \quad (\text{A.16})$$

The Green function follows,

$$O_{\mu\nu}{}^{\alpha\beta} G_{\alpha\beta\gamma\delta}(x-y) = \frac{1}{2} I_{\mu\nu\gamma\delta} \delta_D^{(4)}(x-y), \quad (\text{A.17})$$

with “identity” tensor defined as

$$I_{\mu\nu\gamma\delta} \equiv \frac{1}{2} (\eta_{\mu\gamma} \eta_{\nu\delta} + \eta_{\mu\delta} \eta_{\nu\gamma}), \quad (\text{A.18})$$

We then concern gauge fixing as the operator $O_{\mu\nu}{}^{\alpha\beta}$ cannot be inverted.

A.3 Gauge transformation

Under the coordinate transformation,

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \kappa \xi^\mu(x), \quad (\text{A.19})$$

the metric reads,

$$h_{\mu\nu} \rightarrow \tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \quad (\text{A.20})$$

We choose the de Donder gauge, which reads,

$$\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h = 0. \quad (\text{A.21})$$

It is useful to define the field $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$, from which we can rewrite the [A.16](#) as,

$$\square \bar{h}_{\mu\nu} = -\frac{\kappa}{2} T_{\mu\nu} \quad (\text{A.22})$$

The Green function [A.17](#) can be re-expressed as,

$$\left(I^{\mu\nu\alpha\beta} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta} \right) \square G_{\alpha\beta\gamma\delta} = I^{\mu\nu}{}_{\gamma\delta}, \quad (\text{A.23})$$

from which the gravitational field $h_{\mu\nu}$ can be extracted, using the ansatz $G_{\alpha\beta\gamma\delta} = a I_{\alpha\beta\gamma\delta} + b \eta_{\alpha\beta} \eta_{\gamma\delta}$ which yields,

$$G_{\alpha\beta\gamma\delta} = I_{\alpha\beta\gamma\delta} - \frac{1}{2} \eta_{\alpha\beta} \eta_{\gamma\delta}. \quad (\text{A.24})$$

In the position representation, the Green function reads,

$$G_{\mu\nu\alpha\beta} = \frac{1}{2\square} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} + \eta_{\mu\nu} \eta_{\alpha\beta}) \delta_D^{(4)}(x-y) \quad (\text{A.25})$$

$$= -\frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} + \eta_{\mu\nu} \eta_{\alpha\beta}) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2}. \quad (\text{A.26})$$

This expression should come with the initial condition that defined by the choice of the contour on complex k_0 -plane.

B Field Redefinition

Equivalent theorem [\[7\]](#) states that under reparameterization of field operators, the Scattering matrix remains unchanged, which means under redefinition of the field $\phi \rightarrow \tilde{\phi} = \phi + a_i \phi^i$, the generating functional,

$$\mathcal{Z} = \int D[\phi] \exp \left(i \int d^4 x \mathcal{L}(\phi, \partial_\mu \phi) \right) \quad (\text{B.1})$$

does not change as long as the Jacobian is essentially one [7]. Hence, we can use this technique to simplify the Lagrangian, for example,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{2}{M_{\text{Pl}}} [a_1 h_{\mu\gamma} h_{\nu}{}^{\gamma} + a_2 h_{\mu\nu} h_{\gamma}{}^{\gamma}]. \quad (\text{B.2})$$

Substituting this into the triple graviton vertex gives,

$$h_{\mu\nu} \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta} \rightarrow h_{\mu\nu} \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta} + a_1 \frac{2}{M_{\text{Pl}}} h_{\mu\gamma} h_{\nu}{}^{\gamma} \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta} \quad (\text{B.3})$$

$$+ a_2 \frac{2}{M_{\text{Pl}}} h_{\mu\nu} h_{\gamma}{}^{\gamma} \partial^{\mu} h^{\nu\alpha} \partial_{\alpha} h_{\beta}{}^{\beta}. \quad (\text{B.4})$$

Hence, the field definition generates two addition quadruple graviton vertex with two parameter a_1, a_2 . With a proper choice of these parameters, we can cancel some of the quadruple graviton vertex contributions in the original Lagrangian.

C Effective Field Theory in Gravity

In gravity, the quantities which can be used to construct higher operators of the effective Lagrangian are Riemann tensor $R_{\mu\nu\alpha\beta}$, Ricci tensor $R_{\mu\nu}$, and Ricci scalar R . Those quantities contain 2 partial derivative ($\sim \partial\partial h$), hence the Lagrangian has only even dimension terms.

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots \quad (\text{C.1})$$

where

- $\mathcal{L}^{(0)}$: Constants (such as Cosmology constant Λ - which usually be neglected)
- $\mathcal{L}^{(2)}$: Only one term R
- $\mathcal{L}^{(4)}$: 3 possible terms $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}, R_{\mu\nu} R^{\mu\nu}, R^2$
- $\mathcal{L}^{(6)}$: 4 possible terms $R_{\mu\nu\alpha\beta} R^{\mu\nu} R^{\alpha\beta}, R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} R, R_{\mu\nu} R^{\mu\nu} R, R^3$.

which are Lorentz invariant and General Coordinate Transformations invariant.

When we neglect the higher operators, the effective theory is non-local at high energy while restored locality at low energy. We can utilize this properties to reduce loops of the full theory to effective vertices at low-energy.

D Derivation of the positivity bound for $ZZ \rightarrow ZZ$ process

D.1 S-type operators

We have,

$$D_\mu \Phi = \left(\partial_\mu + i \frac{g}{2} W_\mu^j \sigma^j + i \frac{g'}{2} B_\mu \right) \Phi. \quad (\text{D.1})$$

$$= \begin{pmatrix} \partial_\mu + i \frac{g}{2} W_\mu^3 + i \frac{g'}{2} B_\mu & i \frac{g}{2} W_\mu^1 + \frac{g'}{2} W_\mu^2 \\ i \frac{g}{2} W_\mu^1 - \frac{g'}{2} W_\mu^2 & \partial_\mu - i \frac{g}{2} W_\mu^3 + i \frac{g'}{2} B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \quad (\text{D.2})$$

$$= \frac{v}{\sqrt{2}} \begin{pmatrix} i \frac{g}{2} W_\mu^1 + \frac{g'}{2} W_\mu^2 \\ -i \frac{g}{2} W_\mu^3 + i \frac{g'}{2} B_\mu \end{pmatrix}. \quad (\text{D.3})$$

Hence,

$$(D_\mu \Phi)^\dagger D_\nu \Phi = \frac{v^2}{8} \left[g^2 W_\mu^1 W_\nu^1 + g^2 W_\mu^2 W_\nu^2 + i g g' (W_\mu^1 W_\nu^2 - W_\nu^1 W_\mu^2) \right. \quad (\text{D.4})$$

$$\left. + (g' B_\mu - g W_\mu^3)(g' B_\nu - g W_\nu^3) \right], \quad (\text{D.5})$$

with the gauge boson diagonalization reads:

$$\begin{cases} B_\mu = \cos \theta A_\mu - \sin \theta Z_\mu \\ W_\mu^3 = \sin \theta A_\mu + \cos \theta Z_\mu \end{cases}, \quad (\text{D.6})$$

with $\tan \theta = \frac{g'}{g}$. Hence,

$$(D_\mu \Phi)^\dagger D_\nu \Phi \supset \frac{v^2}{8} (g' B_\mu - g W_\mu^3)(g' B_\nu - g W_\nu^3) \quad (\text{D.7})$$

$$\supset \frac{v^2}{8} (g' \sin \theta + g \cos \theta)^2 Z_\mu Z_\nu \quad (\text{D.8})$$

$$= \frac{v^2 g^2}{8} \left(\frac{\sin^2 \theta}{\cos \theta} + \cos \theta \right)^2 Z_\mu Z_\nu \quad (\text{D.9})$$

$$= \frac{v^2 g^2}{8 \cos^2 \theta} Z_\mu Z_\nu \quad (\text{D.10})$$

$$= \frac{m^2}{2} Z_\mu Z_\nu. \quad (\text{D.11})$$

Here, we have used the relation $g' \cos \theta = g \sin \theta (= e)$, and $vg = 2m_Z \cos \theta$. So we have,

$$\mathcal{L}_{S,0} = [(D_\mu \Phi)^\dagger D_\nu \Phi] [(D^\mu \Phi)^\dagger D^\nu \Phi] \supset \frac{m^4}{4} Z_\mu Z^\nu Z^\mu Z_\nu. \quad (\text{D.12})$$

We also derive the exact same result for $\mathcal{L}_{S,1}, \mathcal{L}_{S,2}$ for Z^4 vertices. However, for W^4 , $(WZ)^2$ (and W^2Z^2) vertices the indices for those terms are not the same due to different polarization of W^\pm .

Since S-type operators only admit longitudinal polarization, the polarization vectors for $1, 2 \rightarrow 1, 2$ process (with $p_1 = -p_2 = p, E_1 = E_2 = E$) read:

$$\epsilon_1^\mu = (a_3 \frac{p}{m}, 0, 0, a_3 \frac{E}{m}), \quad (\text{D.13})$$

$$\epsilon_2^\mu = (-b_3 \frac{p}{m}, 0, 0, b_3 \frac{E}{m}). \quad (\text{D.14})$$

Hence, for later convenience, we note here some relations:

$$\epsilon_1^{(*)} \epsilon_2^{(*)} = -a_3^{(*)} b_3^{(*)} \left(\frac{E^2 + p^2}{m^2} \right), \quad (\text{D.15})$$

$$\epsilon_1^* \epsilon_1 = -|a_3|^2, \quad (\text{D.16})$$

$$\epsilon_2^* \epsilon_2 = -|b_3|^2. \quad (\text{D.17})$$

The Z field read:

$$Z^\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 \sqrt{2E_k}} \sum_{i=0}^3 \left[a_{k,j} \epsilon_j^\mu e^{-ikx} + a_{k,j}^\dagger \epsilon_j^{*\mu} e^{ikx} \right]. \quad (\text{D.18})$$

in which the creation/annihilation operators obey the commutation relation $[a_{k_1}, a_{k_2}^\dagger] = (2\pi)^3 \delta^3(\mathbf{k}_1 - \mathbf{k}_2)$, $[a_{k_1}, a_{k_2}] = [a_{k_1}^\dagger, a_{k_2}^\dagger] = 0$. The incoming and outgoing states read:

$$\langle f | = \langle p_1, p_2 | = 2E a_{p_1} a_{p_2}, \quad (\text{D.19})$$

$$|i\rangle = |p_1, p_2\rangle = 2E a_{p_1}^\dagger a_{p_2}^\dagger. \quad (\text{D.20})$$

Now consider the interacting vertices of $\langle f | : Z_\mu Z^\nu Z^\mu Z_\nu : |i\rangle$, we extract all the possible combination of creation/annihilation operators (a, a^\dagger) for the $ZZ \rightarrow ZZ$ channel.

First, we see that all the terms have to contain $2a$ and $2a^\dagger$. Then, of those $4Z$, there are six ways to exact such terms. We first consider the 2 cases when $2a$ is extracted from ZZ pair with the same Lorentz indices ($Z_\mu Z^\mu$ or $Z_\nu Z^\nu$), and then other 4 cases when $2a$ is extracted from ZZ pair with the different indices ($Z_\mu Z^\nu$ or $Z_\mu Z_\nu$ or $Z^\mu Z^\nu$ or $Z^\mu Z_\nu$). We'll investigate each cases in the 2 group and see that the rest obey the same rules.

Let's consider the case when exacting $2a$ from $Z_\mu Z^\mu$ (and $2a^\dagger$ from $Z_\nu Z^\nu$). For $1, 2 \rightarrow 1, 2$ process, using the result from commutation relation that $\langle p_1, p_2 | a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} | p_1, p_2 \rangle =$

$4E^2 \langle 0 | [a_{p_1}, a_{k_1}^\dagger][a_{p_2}, a_{k_2}^\dagger] + [a_{p_1}, a_{k_2}^\dagger][a_{p_2}, a_{k_1}^\dagger] + [a_{k_3}, a_{p_1}^\dagger][a_{k_4}, a_{p_2}^\dagger] + [a_{k_4}, a_{p_1}^\dagger][a_{k_3}, a_{p_2}^\dagger] | 0 \rangle$, we get 4 terms from the extraction as follow:

$$\begin{array}{cccc} Z_\mu & Z^\nu & Z^\mu & Z_\nu \\ a & a^\dagger & a & a^\dagger \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{array}$$

From D.15, the first term read:

$$\epsilon_{1\mu}\epsilon_1^{*\nu}\epsilon_2^\mu\epsilon_{2\nu}^* = (-1)^2 (a_3 b_3 a_3^* b_3^*) \left(\frac{p^2 + E^2}{m^2} \right)^2 \quad (\text{D.21})$$

$$= |a_3|^2 |b_3|^2 \left(\frac{p^2 + E^2}{m^2} \right)^2 \quad (\text{D.22})$$

All the rest 3 terms yield the same result since the Lorentz indices are contracted between 1 and 2 (x4). Moreover, the case when $2a$ is extracted from $Z_\nu Z^\nu$ also yields the same results (x2).

For the rest 4 cases, we investigate the first one,

$$\begin{array}{cccc} Z_\mu & Z^\nu & Z^\mu & Z_\nu \\ a & a & a^\dagger & a^\dagger \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{array}$$

Using D.15, D.16, the first term read:

$$\epsilon_{1\mu}\epsilon_2^{*\nu}\epsilon_1^\mu\epsilon_{2\nu}^* = |a_3|^2 |b_3|^2. \quad (\text{D.23})$$

The second term has Lorentz indices contracted in the same fashion and yields the same result (x2). The third term read:

$$\epsilon_{2\mu}\epsilon_1^{*\nu}\epsilon_1^\mu\epsilon_{2\nu}^* = |a_3|^2 |b_3|^2 \left(\frac{p^2 + E^2}{m^2} \right)^2, \quad (\text{D.24})$$

which is also the result for the fourth term (x2). Moreover, we see that all the rest 3 cases yield exactly the same results (x4)

To sum up, and then to drop the m^4 component, we have same the result for $\mathcal{L}_{S,0}, \mathcal{L}_{S,1}, \mathcal{L}_{S,2}$:

$$\begin{aligned} & \frac{m^4}{4} Z_\mu Z^\nu Z^\mu Z_\nu \left[(4.2 + 2.4) |a_3|^2 |b_3|^2 \left(\frac{p^2 + E^2}{m^2} \right)^2 + 2.4 |a_3|^2 |b_3|^2 \right] \\ & \supset 16 |a_3|^2 |b_3|^2 E^2 p^2 \end{aligned} \quad (\text{D.25})$$

$$\equiv 16 A_6 E^2 p^2. \quad (\text{D.26})$$

D.2 T-type operators

First, we expand,

$$\hat{W}^{\mu\nu} \equiv ig \frac{\sigma^j}{2} W^{j,\mu\nu} \quad (\text{D.27})$$

$$= i \frac{g}{2} \begin{pmatrix} W^{3\mu\nu} & W^{1\mu\nu} - iW^{2\mu\nu} \\ W^{1\mu\nu} - iW^{2\mu\nu} & -W^{3\mu\nu} \end{pmatrix} \quad (\text{D.28})$$

$$= i \frac{g}{2} \begin{pmatrix} \cos \theta Z^{\mu\nu} + \sin \theta A^{\mu\nu} & \sqrt{2} W^{+\mu\nu} \\ \sqrt{2} W^{-\mu\nu} & -\cos \theta Z^{\mu\nu} - \sin \theta A^{\mu\nu} \end{pmatrix}. \quad (\text{D.29})$$

Hence,

$$\text{Tr} [\hat{W}_{\mu\nu} \hat{W}^{\alpha\beta}] \supset -\frac{g^2}{2} \cos^2 \theta Z_{\mu\nu} Z^{\alpha\beta} \quad (\text{D.30})$$

$$= -\frac{e^2}{2} \cot^2 \theta Z_{\mu\nu} Z^{\alpha\beta}. \quad (\text{D.31})$$

Also,

$$\hat{B}_{\mu\nu} \equiv i \frac{g'}{2} B_{\mu\nu} \supset -i \frac{g'}{2} \sin \theta Z_{\mu\nu} = -i \frac{e}{2} \tan \theta Z_{\mu\nu}, \quad (\text{D.32})$$

and,

$$\hat{B}_{\mu\nu} \hat{B}^{\alpha\beta} \supset -\frac{e^2}{4} \tan^2 \theta Z_{\mu\nu} Z^{\alpha\beta}. \quad (\text{D.33})$$

We can see that replacing $\text{Tr} [\hat{W}_{\mu\nu} \hat{W}^{\alpha\beta}]$ by $\hat{B}_{\mu\nu} \hat{B}^{\alpha\beta}$ yields a factor of $\frac{\tan^4 \theta}{2}$. Hence, we only need to calculate $\mathcal{L}_{T,0}, \mathcal{L}_{T,1}, \mathcal{L}_{T,2}$ and all other T-type operators can be derive by adding that factors. The derivative of Z field reads:

$$\partial^\nu Z^\mu(x) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3 \sqrt{2E_k}} \sum_{i=0}^3 \left[-ik^\nu a_{k,j} \epsilon_j^\mu e^{-ikx} + ik^\nu a_{k,j}^\dagger \epsilon_j^{*\mu} e^{ikx} \right]. \quad (\text{D.34})$$

The $\mathcal{L}_{T,i}$ vertices only admit tranverse component of polarization, hence, they read:

$$\epsilon_1^\mu = (0, a_1, a_2, 0), \quad (\text{D.35})$$

$$\epsilon_2^\mu = (0, b_1, b_2, 0). \quad (\text{D.36})$$

Some useful relations:

$$\epsilon_1^{(*)} \epsilon_2^{(*)} = -a_1^{(*)} b_1^{(*)} - a_2^{(*)} b_2^{(*)}, \quad (\text{D.37})$$

$$\epsilon_1^* \epsilon_1 = -|a_1|^2 - |a_2|^2, \quad (\text{D.38})$$

$$\epsilon_2^* \epsilon_2 = -|b_1|^2 - |b_2|^2. \quad (\text{D.39})$$

D.2.1 $\mathcal{L}_{T,0}, \mathcal{L}_{T,1}, \mathcal{L}_{T,5}, \mathcal{L}_{T,6}, \mathcal{L}_{T,8}$

First consider $\mathcal{L}_{T,0}$,

$$\begin{aligned} & \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\mu\nu} \right] \times \text{Tr} \left[\hat{W}_{\alpha\beta} \hat{W}^{\alpha\beta} \right] \\ & \supset \frac{e^4}{4} \cot \theta^4 Z_{\mu\nu} Z^{\mu\nu} Z_{\alpha\beta} Z^{\alpha\beta} \end{aligned} \quad (\text{D.40})$$

$$\begin{aligned} & \supset \frac{e^4}{4} \cot \theta^4 (\partial_\mu Z_\nu \partial^\mu Z^\nu \partial_\alpha Z_\beta \partial^\alpha Z^\beta + \partial_\mu Z_\nu \partial^\mu Z^\nu \partial_\beta Z_\alpha \partial^\beta Z^\alpha \\ & + \partial_\nu Z_\mu \partial^\nu Z^\mu \partial_\alpha Z_\beta \partial^\alpha Z^\beta + \partial_\nu Z_\mu \partial^\nu Z^\mu \partial_\beta Z_\alpha \partial^\beta Z^\alpha). \end{aligned} \quad (\text{D.41})$$

We derived D.41 by dropping all 12 terms contain $Z_\gamma \partial^\gamma$ since contracting them yield $\epsilon_\gamma p^\gamma = 0$ (no matter they belong to the “1” or “2” states as we only have tranverse polarizations for T-type operators).

We see that those 4 term are contracted in the identical way (symmetric in $\mu \leftrightarrow \nu, \alpha \leftrightarrow \beta$), hence we only need to consider 1 term and then multiply the results by 4 (x4). Let's pick the $\partial_\nu Z_\mu \partial^\nu Z^\mu \partial_\alpha Z_\beta \partial^\alpha Z^\beta$ to investigate. There are also 6 cases of extracting : $a^\dagger a^\dagger a a$:. Consider the first case, we have 4 terms:

$\partial_\nu Z_\mu$	$\partial^\nu Z^\mu$	$\partial_\alpha Z_\beta$	$\partial^\alpha Z^\beta$
a	a	a^\dagger	a^\dagger
1	2	1	2
2	1	2	1
2	1	1	2
1	2	2	1

The first term read:

$$p_{1\nu} \epsilon_{1\mu} p_2^\nu \epsilon_2^\mu p_{1\alpha} \epsilon_{1\beta}^* p_2^\alpha \epsilon_2^{\beta*} = (E^2 + p^2)^2 (a_1^* b_1^* + a_2^* b_2^*) (a_1 b_1 + a_2 b_2). \quad (\text{D.42})$$

The rest 3 terms yield the same result (x4) as all the contractions are between “1” and “2” state. The second case where we swap all $a \leftrightarrow a^\dagger$ follows the same principle and ends up in the same results (x2). Consider one of the rest 4 cases, there are also 4 terms in it:

$$\begin{array}{cccc}
\partial_\nu Z_\mu & \partial^\nu Z^\mu & \partial_\alpha Z_\beta & \partial^\alpha Z^\beta \\
a & a^\dagger & a & a^\dagger \\
1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 \\
2 & 1 & 1 & 2
\end{array}$$

The first one read:

$$p_{1\nu}\epsilon_{1\mu}p_1^\nu\epsilon_1^{\mu*}p_{\alpha 2}\epsilon_{\beta 2}p_2^\alpha\epsilon_2^{\beta*}=m^4(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2). \quad (\text{D.43})$$

The second term yields the same results as it only exchange “1 \leftrightarrow 2” (x2). The third term read:

$$p_{1\nu}\epsilon_{1\mu}p_2^\nu\epsilon_2^{\nu*}p_{\alpha 2}\epsilon_{\beta 2}p_1^\alpha\epsilon_1^{\beta*}=(E^2+p^2)^2(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2) \quad (\text{D.44})$$

This is also the result for the final term (x2). All the rest 3 cases yields the same results (x4) as they only differ by the exchange of $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$. To sum up and then to drop m^4 term, we have

$$\begin{aligned}
& \frac{e^4}{4} \cot \theta^4 \left[(E^2 + p^2)^2 4 [4.2(a_1^*b_1^* + a_2^*b_2^*)(a_1b_1 + a_2b_2) + 2.4(a_1b_1^* + a_2b_2^*)(a_1^*b_1 + a_2^*b_2)] \right. \\
& \left. + 4.2.4m^4(|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2) \right] \\
& \supset e^4 \cot \theta^4 [64(|a_1|^2|a_2|^2 + |a_2|^2|b_2|^2) + 32(a_1a_2^*b_1^*b_2 + c.c.) + 32(a_1a_2^*b_1b_2^* + c.c.)] E^2 p^2 \\
& \hspace{15em} (\text{D.45})
\end{aligned}$$

$$\equiv e^4 \cot \theta^4 [64A_1 + 32A_4 + 32A'_4] E^2 p^2. \quad (\text{D.46})$$

$\mathcal{L}_{T,1}$ yields the same result as they only exchange $\mu \leftrightarrow \alpha, \beta \leftrightarrow \nu$ in the term [D.40](#).

$\mathcal{L}_{T,5}$ is also equal $\mathcal{L}_{T,6}$ and can be calculated as,

$$\begin{aligned}
& \frac{\tan^4 \theta}{2} e^4 \cot \theta^4 [64A_1 + 32A_4 + 32A'_4] E^2 p^2 \\
& = e^4 [32A_1 + 16A_4 + 16A'_4] E^2 p^2.
\end{aligned} \quad (\text{D.47})$$

$\mathcal{L}_{T,8}$ is derived as,

$$\begin{aligned}
& \frac{\tan^4 \theta}{2} e^4 [32A_1 + 16A_4 + 16A'_4] E^2 p^2 \\
& = e^4 \tan^4 \theta [16A_1 + 8A_4 + 8A'_4] E^2 p^2.
\end{aligned} \quad (\text{D.48})$$

D.2.2 $\mathcal{L}_{T,2}, \mathcal{L}_{T,7}, \mathcal{L}_{T,9}$

Next, consider $\mathcal{L}_{T,2}$,

$$\begin{aligned} & \text{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}] \times \text{Tr}[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}] \\ & \supset \frac{g^4 \cos^4 \theta}{4} Z_{\alpha\mu} Z^{\mu\beta} Z_{\beta\nu} Z^{\nu\alpha} \end{aligned} \quad (\text{D.49})$$

$$\supset \frac{g^4 \cos^4 \theta}{4} (\partial_\alpha Z_\mu \partial^\beta Z^\mu \partial_\beta Z_\nu \partial^\alpha Z^\nu + \partial_\mu Z_\alpha \partial^\mu Z^\beta \partial_\nu Z_\beta \partial^\nu Z^\alpha). \quad (\text{D.50})$$

In D.50, we also dropped all 14 terms contain $Z_\gamma \partial^\gamma$. The two above term differ only by swapping $\alpha \leftrightarrow \mu, \nu \leftrightarrow \beta$, and hence yield the same result (x2). Let's investigate the first one. There are also 6 cases of extracting : $a^\dagger a^\dagger a a$:. First, consider:

$$\begin{array}{cccc} \partial_\alpha Z_\mu & \partial^\beta Z^\mu & \partial_\beta Z_\nu & \partial^\alpha Z^\nu \\ a & a & a^\dagger & a^\dagger \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{array}$$

The first term read:

$$p_{\alpha 1} \epsilon_{\mu 1} p_2^\beta \epsilon_2^\mu p_{\beta 1} \epsilon_{\nu 1}^* p_2^\alpha \epsilon_2^{\nu*} = (E^2 + p^2)^2 (a_1 b_1 + a_2 b_2) (a_1^* b_1^* + a_2^* b_2^*). \quad (\text{D.51})$$

The second term yield the same result(x2). Let's consider the third one,

$$p_{\alpha 1} \epsilon_{\mu 1} p_2^\beta \epsilon_2^\mu p_{\beta 2} \epsilon_{\nu 2}^* p_1^\alpha \epsilon_1^{\nu*} = m^4 (a_1 b_1 + a_2 b_2) (a_1^* b_1^* + a_2^* b_2^*). \quad (\text{D.52})$$

This is also the result for the final term (x2).

Now, since swapping $a \leftrightarrow a^\dagger$ is equivalent to swapping the indices pairwise, we have a second case that yield exactly the same results as the first one (x2).

Next, consider the third case of

$$\begin{array}{cccc} \partial_\alpha Z_\mu & \partial^\beta Z^\mu & \partial_\beta Z_\nu & \partial^\alpha Z^\nu \\ a & a^\dagger & a & a^\dagger \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{array}$$

The first term (and also the second) (x2) yields,

$$p_{\alpha 1} \epsilon_{\mu 1} p_1^\beta \epsilon_1^{\mu*} p_{\beta 2} \epsilon_{\nu 2} p_2^\alpha \epsilon_2^{\nu*} = (E^2 + p^2)^2 (|a_1|^2 + |a_2|^2) (|b_1|^2 + |b_2|^2). \quad (\text{D.53})$$

The third (and also fourth) (x2) yields,

$$p_{\alpha 1} \epsilon_{\mu 1} p_2^\beta \epsilon_2^{\mu*} p_{\beta 2} \epsilon_{\nu 2} p_1^\alpha \epsilon_1^{\nu*} = m^4 (a_1 b_1^* + a_2 b_2^*) (a_1^* b_1 + a_2^* b_2). \quad (\text{D.54})$$

Swaping $a \leftrightarrow a^\dagger$ does not change the result, hence, we have the fourth case that also yield the same result (x2). Now for the last 2 cases. First, consider,

$\partial_\alpha Z_\mu$	$\partial^\beta Z^\mu$	$\partial_\beta Z_\nu$	$\partial^\alpha Z^\nu$
a	a^\dagger	a^\dagger	a
1	1	2	2
2	2	1	1
1	2	1	2
2	1	2	1

The first (and also second) term (x2) read,

$$p_{\alpha 1} \epsilon_{\mu 1} p_1^\beta \epsilon_1^{\mu*} p_{\beta 2} \epsilon_{\nu 2}^* p_2^\alpha \epsilon_2^\nu = (E^2 + p^2)^2 (|a_1|^2 + |a_2|^2) (|b_1|^2 + |b_2|^2). \quad (\text{D.55})$$

The third (and fourth) (x2) read,

$$p_{\alpha 1} \epsilon_{\mu 1} p_2^\beta \epsilon_2^{\mu*} p_{\beta 1} \epsilon_{\nu 1}^* p_2^\alpha \epsilon_2^\nu = (E^2 + p^2)^2 (a_1 b_1^* + a_2 b_2^*) (a_1^* b_1 + a_2^* b_2). \quad (\text{D.56})$$

Swaping $a \leftrightarrow a^\dagger$ does not change the result, hence, we have the sixth case that also yield the same result (x2).

We have completed calculating all the possibilities for this vetices. Summing up and then subtracting the m^4 term we have the result for $\mathcal{L}_{T,2}$:

$$\begin{aligned} & \frac{e^4}{4} \cot \theta^4 \left[(E^2 + p^2) [2.2.2(a_1 b_1 + a_2 b_2)(a_1^* b_1^* + a_2^* b_2^*) \right. \\ & + 2.2.2.2(|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2) + 2.2.2(a_1 b_1^* + a_2 b_2^*)(a_1^* b_1 + a_2^* b_2)] \\ & \left. + 2.2.2m^4(a_1 b_1^* + a_2 b_2^*)(a_1^* b_1 + a_2^* b_2) \right] \\ & \supset e^4 \cot \theta^4 \left[32(|a_1|^2 |b_1|^2 + |a_2|^2 |b_2|^2) + 32(|a_1|^2 |b_2|^2 + |a_2|^2 |b_1|^2) \right. \\ & \left. + 8(a_1 a_2^* b_1^* b_2 + c.c.) + 8(a_1 a_2^* b_1 b_2^* + c.c.) \right] E^2 p^2 \end{aligned} \quad (\text{D.57})$$

$$+ 8(a_1 a_2^* b_1^* b_2 + c.c.) + 8(a_1 a_2^* b_1 b_2^* + c.c.) \Big] E^2 p^2 \quad (\text{D.58})$$

$$\equiv e^4 \cot^4 \theta [32A_1 + 16A_2 + 8A_4 + 8A_4'] E^2 p^2. \quad (\text{D.59})$$

From there, the $\mathcal{L}_{T,7}$ can be derived,

$$\frac{\tan^4 \theta}{2} e^4 \cot^4 \theta [32A_1 + 16A_2 + 8A_4 + 8A'_4] E^2 p^2 \quad (\text{D.60})$$

$$= e^4 [16A_1 + 8A_2 + 4A_4 + 4A'_4] E^2 p^2. \quad (\text{D.61})$$

And also, the $\mathcal{L}_{T,9}$

$$\frac{\tan^4 \theta}{2} e^4 [32A_1 + 16A_2 + 8A_4 + 8A'_4] E^2 p^2 \quad (\text{D.62})$$

$$= e^4 \tan^4 \theta [8A_1 + 4A_2 + 2A_4 + 2A'_4] E^2 p^2. \quad (\text{D.63})$$

D.3 M-type operators

We have,

$$D_\mu \Phi \supset \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ -i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu \end{pmatrix} \quad (\text{D.64})$$

$$\supset -\frac{im_Z}{\sqrt{2}} \begin{pmatrix} 0 \\ Z_\mu \end{pmatrix}. \quad (\text{D.65})$$

Also,

$$\hat{W}_{\mu\nu} \supset \frac{ie \cot \theta}{2} \begin{pmatrix} Z^{\mu\nu} & 0 \\ 0 & -Z^{\mu\nu} \end{pmatrix}. \quad (\text{D.66})$$

Hence,

$$[(D_\mu \Phi)^\dagger \hat{W}_{\beta\nu} D^\mu \Phi] \times \hat{B}^{\beta\nu} \supset \frac{im}{\sqrt{2}} \frac{-ie \cot \theta}{2} \frac{-im}{\sqrt{2}} \frac{-ie \tan \theta}{2} Z_\mu Z_{\beta\nu} Z^\mu Z^{\beta\nu} \quad (\text{D.67})$$

$$= -\frac{m^2 e^2}{8} Z_\mu Z_{\beta\nu} Z^\mu Z^{\beta\nu}. \quad (\text{D.68})$$

Similarly,

$$[(D_\mu \Phi)^\dagger \hat{W}_{\beta\nu} D^\nu \Phi] \times \hat{B}^{\beta\mu} \supset -\frac{m^2 e^2}{8} Z_\mu Z_{\beta\nu} Z^\nu Z^{\beta\mu}. \quad (\text{D.69})$$

With this result, we will see later that $\mathcal{L}_{M,4}$ (and $\mathcal{L}_{M,5}$) only differs from $\mathcal{L}_{M,0}$ (and $\mathcal{L}_{M,1}$) by a factor of $-\frac{\tan^2 \theta}{2}$. We also have,

$$\hat{W}_{\beta\nu} \hat{W}^{\beta\mu} \supset -\frac{e^2 \cot^2 \theta}{4} \begin{pmatrix} Z_{\beta\nu} Z^{\beta\mu} & 0 \\ 0 & Z_{\beta\nu} Z^{\beta\mu} \end{pmatrix} \quad (\text{D.70})$$

Hence,

$$[(D_\mu \Phi)^\dagger \hat{W}_{\beta\nu} \hat{W}^{\beta\mu} D^\nu \Phi] \supset \frac{im}{\sqrt{2}} \frac{-e^2 \cot^2 \theta}{4} \frac{-im}{\sqrt{2}} Z_\mu Z_{\beta\nu} Z^{\beta\mu} Z^\nu \quad (\text{D.71})$$

$$= -\frac{e^2}{8} \cot^2 \theta m^2 Z_\mu Z_{\beta\nu} Z^{\beta\mu} Z^\nu, \quad (\text{D.72})$$

from which we see that $\mathcal{L}_{M,7}$ differs from $\mathcal{L}_{M,1}$ by a factor of $-\frac{1}{2}$.

D.3.1 $\mathcal{L}_{M,0}, \mathcal{L}_{M,2}, \mathcal{L}_{M,4}$

First, consider $\mathcal{L}_{M,0}$,

$$\text{Tr} [\hat{W}_{\mu\nu} \hat{W}^{\mu\nu}] [(D_\beta \Phi)^\dagger D^\beta \Phi] \quad (\text{D.73})$$

$$\supset \frac{e^2 \cot^2 \theta}{2} Z_{\mu\nu} Z^{\mu\nu} \frac{m^2}{2} Z_\beta Z^\beta \quad (\text{D.74})$$

$$\supset \frac{e^2}{4} \cot^2 \theta m^2 (\partial_\mu Z_\nu \partial^\mu Z^\nu + \partial_\nu Z_\mu \partial^\nu Z^\mu) Z_\beta Z^\beta. \quad (\text{D.75})$$

Those 2 terms also yield same results (x2). Consider one of them $(\partial_\mu Z_\nu \partial^\mu Z^\nu Z_\beta Z^\beta)$, there are 6 ways of extracting $:a^\dagger a^\dagger a a:$. Consider the first way,

$\partial_\mu Z_\nu$	$\partial^\mu Z^\nu$	Z_β	Z^β
a	a	a^\dagger	a^\dagger
1	2	1	2
1	2	2	1
2	1	1	2
2	1	2	1

First term reads,

$$p_{1\mu} \epsilon_{1\nu} p_2^\mu \epsilon_2^\nu \epsilon_{1\beta}^* \epsilon_2^{\beta*} = \frac{(E^2 + p^2)^2}{m^2} (a_1 b_1 + a_2 b_2) a_3^* b_3^*. \quad (\text{D.76})$$

All the rest 3 terms read the same result (x4) since swaping “1” \leftrightarrow “2” with a Lorentz contracted term does not change the result.

Consider the next case of swaping all $a \leftrightarrow a^\dagger$, we have the result is the complex conjugate of the first case.

Moving on to the next case, we consider,

$\partial_\mu Z_\nu$	$\partial^\mu Z^\nu$	Z_β	Z^β
a	a^\dagger	a	a^\dagger
1	1	2	2
1	2	2	1
2	2	1	1
2	1	1	2

First term read,

$$p_{1\mu}\epsilon_{1\nu}p_1^\mu\epsilon_1^{\nu*}\epsilon_{2\beta}\epsilon_2^{\beta*} = m^2(|a_1|^2 + |a_2|^2)|b_3|^2. \quad (\text{D.77})$$

Second term read,

$$p_{1\mu}\epsilon_{1\nu}p_2^\mu\epsilon_2^{\nu*}\epsilon_{2\beta}\epsilon_1^{\beta*} = -\frac{(E^2 + p^2)^2}{m^2}(a_1b_1^* + a_2b_2^*)a_3^*b_3. \quad (\text{D.78})$$

Third term read,

$$p_{2\mu}\epsilon_{2\nu}p_2^\mu\epsilon_2^{\nu*}\epsilon_{1\beta}\epsilon_1^{\beta*} = m^2(|b_1|^2 + |b_2|^2)|a_3|^2. \quad (\text{D.79})$$

Last term read,

$$p_{2\mu}\epsilon_{2\nu}p_1^\mu\epsilon_1^{\nu*}\epsilon_{1\beta}\epsilon_2^{\beta*} = -\frac{(E^2 + p^2)^2}{m^2}(a_1^*b_1 + a_2^*b_2)a_3b_3^*. \quad (\text{D.80})$$

All the rest 3 cases yield exactly the same results since they only swap $a \leftrightarrow a^\dagger$ within Lorent invariant terms (x4).

To sum up, then to drop the m^4 term, we have the result for $\mathcal{L}_{M,0}$:

$$\begin{aligned} & \frac{e^2}{4} \cot^2 \theta m^2 \left[\frac{(E^2 + p^2)^2}{m^2} \left[4(a_1b_1 + a_2b_2)a_3^*b_3^* + c.c. \right. \right. \\ & \quad \left. \left. - 4(a_1b_1^* + a_2b_2^*)a_3^*b_3 - c.c. \right] \right. \\ & \quad \left. + 4m^2 \left[(|a_1|^2 + |a_2|^2)|b_3|^2 + (|b_1|^2 + |b_2|^2)|a_3|^2 \right] \right] \end{aligned}$$

$$\supset e^2 \cot^2 \theta \left[4(a_1b_1 + a_2b_2)a_3^*b_3^* + c.c. - 4(a_1b_1^* + a_2b_2^*)a_3^*b_3 - c.c. \right] E^2 p^2 \quad (\text{D.81})$$

$$\equiv e^2 \cot^2 \theta [4A_5 + 4A'_5] E^2 p^2. \quad (\text{D.82})$$

We can also derive the result for $\mathcal{L}_{M,2}$ which is

$$\frac{\tan^4 \theta}{2} e^2 \cot^2 \theta [4A_5 + 4A'_5] E^2 p^2 = e^2 \tan^2 \theta [2A_5 + 2A'_5] E^2 p^2. \quad (\text{D.83})$$

And the result for $\mathcal{L}_{M,4}$,

$$-\frac{\tan^2 \theta}{2} e^2 \cot^2 \theta [4A_5 + 4A'_5] E^2 p^2 = -e^2 [2A_5 + 2A'_5] E^2 p^2. \quad (\text{D.84})$$

D.3.2 $\mathcal{L}_{M,1}, \mathcal{L}_{M,3}, \mathcal{L}_{M,5}, \mathcal{L}_{M,7}$

First, consider $\mathcal{L}_{M,1}$

$$\text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\nu\beta} \right] \left[(D_\beta \Phi)^\dagger D^\mu \Phi \right] \quad (\text{D.85})$$

$$\supset \frac{e^2 \cot^2 \theta}{2} \cos^2 \theta Z_{\mu\nu} Z^{\nu\beta} \frac{m^2}{2} \cos^2 \theta Z_\beta Z^\mu \quad (\text{D.86})$$

$$\supset \frac{e^2}{4} \cot^2 \theta m^2 \left(-\partial_\mu Z_\nu \partial^\beta Z^\nu - \partial_\nu Z_\mu \partial^\nu Z^\beta \right) Z_\beta Z^\mu. \quad (\text{D.87})$$

Note that the contraction in the first term $(Z_\beta Z_\mu \partial^\beta \partial^\mu)$ doesn't vanish. However, the second term vanishes since it contracts the pure longitudinal with transversal polarised Z-boson. Hence, we only consider the first term. There are some relations for contracting the longitudinal polarization vectors $\epsilon_{1\mu} = (a_3 \frac{p}{m}, 0, 0, -a_3 \frac{p}{m})$, $\epsilon_{2\mu} = (-b_3 \frac{p}{m}, 0, 0, -b_3 \frac{p}{m})$ with the 4-momentum $p_1^\mu = (E, 0, 0, p)$, $p_2^\mu = (E, 0, 0, -p)$ as,

$$p_1 \epsilon_1^{(*)} = p_2 \epsilon_2^{(*)} = 0, \quad (\text{D.88})$$

$$p_2 \epsilon_1^{(*)} = 2a_3^{(*)} \frac{Ep}{m}, \quad (\text{D.89})$$

$$p_1 \epsilon_2^{(*)} = -2b_3^{(*)} \frac{Ep}{m}. \quad (\text{D.90})$$

For the first term, there are also 6 cases, we calculate the first case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
a^\dagger	a	a	a^\dagger
1	1	2	2
2	2	1	1
2	1	2	1
1	2	1	2

The first term reads,

$$p_{1\mu} \epsilon_{1\nu}^* p_1^\beta \epsilon_1^\nu \epsilon_{2\beta} \epsilon_2^{\mu*} = -4 \frac{E^2 p^2}{m^2} |b_3|^2 (|a_1|^2 + |a_2|^2). \quad (\text{D.91})$$

The second term reads,

$$p_{2\mu} \epsilon_{2\nu}^* p_2^\beta \epsilon_2^\nu \epsilon_{1\beta} \epsilon_1^{\mu*} = -4 \frac{E^2 p^2}{m^2} |a_3|^2 (|b_1|^2 + |b_2|^2). \quad (\text{D.92})$$

The third term reads:

$$p_{2\mu} \epsilon_{2\nu}^* p_1^\beta \epsilon_1^\nu \epsilon_{2\beta} \epsilon_1^{\mu*} = 4 \frac{E^2 p^2}{m^2} a_3^* b_3 (a_1 b_1^* + a_2 b_2^*). \quad (\text{D.93})$$

The fourth term read:

$$p_{1\mu}\epsilon_{1\nu}^*p_2^\beta\epsilon_2^\nu\epsilon_{1\beta}\epsilon_2^{\mu*} = 4\frac{E^2p^2}{m^2}a_3b_3^*(a_1^*b_1 + a_2^*b_2). \quad (\text{D.94})$$

Swaping $a \leftrightarrow a^\dagger$ yields the same result, we got the second case (x2). Consider the third case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
a^\dagger	a	a^\dagger	a
1	1	2	2
2	2	1	1
1	2	2	1
2	1	1	2

The first term read,

$$p_{1\mu}\epsilon_{1\nu}^*p_1^\beta\epsilon_1^\nu\epsilon_{2\beta}^*\epsilon_2^\mu = -4\frac{E^2p^2}{m^2}|a_3|^2(|b_1|^2 + |b_2|^2). \quad (\text{D.95})$$

The second term read,

$$p_{2\mu}\epsilon_{2\nu}^*p_2^\beta\epsilon_2^\nu\epsilon_{1\beta}^*\epsilon_1^\mu = -4\frac{E^2p^2}{m^2}|b_3|^2(|a_1|^2 + |a_2|^2). \quad (\text{D.96})$$

For the third term, since $p_1\epsilon_1 = p_2\epsilon_2 = 0$, we have,

$$p_{1\mu}\epsilon_{1\nu}^*p_2^\beta\epsilon_2^\nu\epsilon_{2\beta}^*\epsilon_1^\mu = 0 \quad (\text{D.97})$$

Since swaping $a \leftrightarrow a^\dagger$ yields the same result, we derive the same thing for the fourth term as well (x2). Consider the fifth case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
a	a	a^\dagger	a^\dagger
1	2	1	2
2	1	2	1
1	2	2	1
2	1	1	2

The first term reads,

$$p_{1\mu}\epsilon_{1\nu}p_2^\beta\epsilon_2^\nu\epsilon_{1\beta}^*\epsilon_2^{\mu*} = 4\frac{E^2p^2}{m^2}a_3^*b_3^*(a_1b_1 + a_2b_2). \quad (\text{D.98})$$

The second term reads the same result (x2). The third term reads,

$$p_{1\mu}\epsilon_{1\nu}p_2^\beta\epsilon_2^\nu\epsilon_{2\beta}^*\epsilon_1^{\mu*} = 0. \quad (\text{D.99})$$

This result also hold for the fourth term (x2). Finally, we get the sixth case by swaping $a \leftrightarrow a^\dagger$. It yields the complex conjugate of the fifth case (+c.c.). To sum up and then to subtract the m^4 term, we have,

$$- \frac{e^2}{4} \cot^2 \theta m^2 4 \frac{E^2 p^2}{m^2} \left[2.2 |a_3|^2 (|b_1|^2 + |b_2|^2) + 2.2 |b_3|^2 (|a_1|^2 + |a_2|^2) \right. \\ \left. - 2a_3 b_3^* (a_1^* b_1 + a_2^* b_2) + c.c. + 2a_3^* b_3^* (a_1 b_1 + a_2 b_2) + c.c. \right] \quad (\text{D.100})$$

$$= -e^2 \cot^2 \theta \left[4 |a_3|^2 (|b_1|^2 + |b_2|^2) + 4 |b_3|^2 (|a_1|^2 + |a_2|^2) \right. \\ \left. - 2a_3 b_3^* (a_1^* b_1 + a_2^* b_2) + c.c. + 2a_3^* b_3^* (a_1 b_1 + a_2 b_2) + c.c. \right] E^2 p^2. \quad (\text{D.101})$$

$$\equiv -e^2 \cot^2 \theta (4A_3 + 4A'_3 + 2A'_5 + 2A_5) E^2 p^2. \quad (\text{D.102})$$

We can also derive the $\mathcal{L}_{M,3}$

$$\frac{\tan^4 \theta}{2} \left[-e^2 \cot^2 \theta (4A_3 + 4A'_3 + 2A'_5 + 2A_5) E^2 p^2 \right] \\ = -e^2 \tan^2 \theta (2A_3 + 2A'_3 + A'_5 + A_5) E^2 p^2. \quad (\text{D.103})$$

and $\mathcal{L}_{M,5}$,

$$- \frac{\tan^2 \theta}{2} \left[-e^2 \cot^2 \theta (4A_3 + 4A'_3 + 2A'_5 + 2A_5) E^2 p^2 \right] \quad (\text{D.104})$$

$$= e^2 (2A_3 + 2A'_3 + A'_5 + A_5) E^2 p^2. \quad (\text{D.105})$$

and $\mathcal{L}_{M,7}$,

$$- \frac{1}{2} \left[-e^2 \cot^2 \theta (4A_3 + 4A'_3 + 2A'_5 + 2A_5) E^2 p^2 \right] \quad (\text{D.106})$$

$$= +e^2 \cot^2 \theta (2A_3 + 2A'_3 + A'_5 + A_5) E^2 p^2. \quad (\text{D.107})$$

D.4 Results

Summing up all the operators, with the coefficients are redefined as,

$$F_{S,i} \equiv f_{S,i}; F_{M,i} \equiv e^2 f_{M,i}; F_{T,i} \equiv e^4 f_{T,i}, \quad (\text{D.108})$$

we get,

ZZ:

$$16A_6 (F_{S,0} + F_{S,1} + F_{S,2}) \\ + [16A_1 + 8(A_4 + A'_4)] [4 \cot^4 \theta (F_{T,0} + F_{T,1}) + 2(F_{T,5} + F_{T,6}) + \tan^4 \theta F_{T,8}] \\ + (8A_1 + 4A_2 + 2A_4 + 2A'_4) (4 \cot^4 \theta F_{T,2} + 2F_{T,7} + \tan^4 \theta F_{T,9}) \\ + (4A_5 + 4A'_5) (2 \cot^2 \theta F_{M,0} + \tan^2 \theta F_{M,2} - F_{M,4}) \\ + (2A_3 + 2A'_3 + A'_5 + A_5) (-2 \cot^2 \theta F_{M,1} - \tan^2 \theta F_{M,3} + F_{M,5} + \cot^2 \theta F_{M,7}) \geq 0. \quad (\text{D.109})$$

With the convention of,

$$\begin{aligned}
A_1 &\equiv |a_1|^2 |b_1|^2 + |a_2|^2 |b_2|^2, & A_4 &\equiv a_1 a_2^* b_1 b_2^* + c.c., \\
A_2 &\equiv |a_1|^2 |b_2|^2 + |a_2|^2 |b_1|^2, & A_4' &\equiv a_1 a_2^* b_1^* b_2 + c.c., \\
A_3 &\equiv (|b_1|^2 + |b_2|^2) |a_3|^2, & A_5 &\equiv (a_1 b_1 + a_2 b_2) a_3^* b_3^* + c.c., \\
A_3' &\equiv (|a_1|^2 + |a_2|^2) |b_3|^2, & A_5' &\equiv -(a_1 b_1^* + a_2 b_2^*) a_3^* b_3 + c.c. \\
A_3'' &\equiv |b_1|^2 |a_3|^2 & A_6 &\equiv |a_3|^2 |b_3|^2,
\end{aligned} \tag{D.110}$$

When we consider only real polarizations, $(A_5 + A_5')$ vanishes. Multiplying $\frac{\tan^4 \theta}{2}$, we get,

ZZ :

$$\begin{aligned}
&8At_W^4 (F_{S,0} + F_{S,1} + F_{S,2}) + Dt_W^2 (-t_W^4 F_{M,3} + t_W^2 F_{M,5} - 2F_{M,1} + F_{M,7}) \\
&+ (B + C) (2t_W^8 F_{T,9} + 4t_W^4 F_{T,7} + 8F_{T,2}) + 8B [t_W^4 (t_W^4 F_{T,8} + 2F_{T,5} + 2F_{T,6}) + 4F_{T,0} + 4F_{T,1}] .
\end{aligned} \tag{D.111}$$

with the convention of

$$\begin{aligned}
A &\equiv a_3^2 b_3^2, & E &\equiv a_3 b_3 (a_1 b_1 + a_2 b_2), \\
B &\equiv (a_1 b_1 + a_2 b_2)^2, & F &\equiv (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2, \\
C &\equiv (a_1^2 + a_2^2) (b_1^2 + b_2^2), & G &\equiv (a_3 b_1 + a_1 b_3)^2 + (a_3 b_2 + a_2 b_3)^2, \\
D &\equiv a_3^2 (b_1^2 + b_2^2) + (a_1^2 + a_2^2) b_3^2, & H &\equiv a_3^2 (b_1^2 + b_2^2).
\end{aligned} \tag{D.112}$$

E Proof of Optical theorem

Consider a process of $p_1 + p_2 \rightarrow \{k_f\}$, with $f = 1, \dots, n$, the total cross-section reads:

$$\sigma_t = \sum_n d\Pi_n (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_f k_f) \frac{M^2(p_1 p_2 \rightarrow \{k_f\})}{2E_1 2E_2 |v_1 - v_2|}, \tag{E.1}$$

with

$$d\Pi_n \equiv \prod_{f=1}^n \int \frac{dk_f}{(2\pi)^3} \frac{1}{2E_f}. \tag{E.2}$$

The interacting T-matrix is defined from S-matrix as:

$$S = \mathbb{1} + iT. \tag{E.3}$$

Unitarity of S-matrix implies the following property of T:

$$S^\dagger S \Rightarrow (1 - iT^\dagger)(1 + iT^\dagger) = 1 \quad (\text{E.4})$$

$$\Rightarrow T^\dagger T = i(T^\dagger - T). \quad (\text{E.5})$$

The interaction reads:

$$i \langle \{k_f\} | T | p_1, p_2 \rangle = iM(p_1 p_2 \rightarrow k_f) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_f k_f). \quad (\text{E.6})$$

The Completeness relation in Hilbert space reads:

$$\mathbb{1} = \sum_n d\Pi_n |\{k_f\}\rangle \langle \{k_f\}|. \quad (\text{E.7})$$

Using the Completeness relation, we have,

$$\langle q_1, q_2 | T^\dagger T | p_1, p_2 \rangle = \sum_n d\Pi_n \langle q_1, q_2 | T^\dagger | \{k_f\} \rangle \langle \{k_f\} | T | p_1, p_2 \rangle \quad (\text{E.8})$$

Using Eq. E.5, the L.H.S of Eq. E.8 can be rewrite as

$$\begin{aligned} & \langle q_1, q_2 | T^\dagger T | p_1, p_2 \rangle \\ &= i (\langle q_1, q_2 | T^\dagger | p_1, p_2 \rangle - \langle q_1, q_2 | T | p_1, p_2 \rangle) \end{aligned} \quad (\text{E.9})$$

$$= i(2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) [M^*(q_1 q_2 \rightarrow p_1 p_2) - M(p_1 p_2 \rightarrow q_1 q_2)]. \quad (\text{E.10})$$

Using Eq. E.6, the R.H.S of Eq. E.5 reads:

$$\begin{aligned} & \sum_n d\Pi_n \langle q_1, q_2 | T^\dagger | \{k_f\} \rangle \langle \{k_f\} | T | p_1, p_2 \rangle \\ &= \sum_n d\Pi_n (2\pi)^8 \delta^{(4)}(q_1 + q_2 - k_f) \delta^{(4)}(p_1 + p_2 - k_f) M^*(q_1 q_2 \rightarrow k_f) M(p_1 p_2 \rightarrow k_f). \end{aligned} \quad (\text{E.11})$$

Consolidate Eq. E.10 and Eq. E.11, we get:

$$M(p_1 p_2 \rightarrow q_1 q_2) - M^*(q_1 q_2 \rightarrow p_1 p_2) = \sum_n d\Pi_n (2\pi)^4 i \delta^{(4)}(p_1 + p_2 - \sum_f k_f) |M(p_1 p_2 \rightarrow k_f)|^2 \quad (\text{E.12})$$

Or,

$$2\text{Im } M(p_1 p_2 \rightarrow p_1 p_2) = 2E_1 2E_2 |v_1 - v_2| \sigma_t. \quad (\text{E.13})$$

When we have the same incoming particles, it becomes the standard form of

$$\text{Im } M(p_1 p_2 \rightarrow p_1 p_2) = 2E_{\text{CM}} |\mathbf{p}_{\text{CM}}| \sigma_t. \quad (\text{E.14})$$

E.1 A

$$iM = \frac{(i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(\frac{p}{2} - k)^2 - m^2 + i\epsilon} \frac{i}{(\frac{p}{2} + k)^2 - m^2 + i\epsilon} \quad (\text{E.15})$$

$$= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(\frac{p_0}{2} - k_0)^2 - E_k^2 + i\epsilon} \frac{1}{(\frac{p_0}{2} + k_0)^2 - E_k^2 + i\epsilon}. \quad (\text{E.16})$$

In the C.M. frame, $p = (p_0, \mathbf{0})$, $k = (k_0, \mathbf{k})$, $E_k^2 = |\mathbf{k}|^2 + m^2$.

The integration has poles at:

$$k_0 = \pm(E_k - i\epsilon) - \frac{p_0}{2} \quad (\text{E.17})$$

$$k_0 = \pm(E_k - i\epsilon) + \frac{p_0}{2}. \quad (\text{E.18})$$

$$\frac{1}{(\frac{p_0}{2} + k)^2 - m^2 + i\epsilon} \rightarrow -2\pi i \delta\left(\left(\frac{p_0}{2} + k\right)^2 - m^2\right) \quad (\text{E.19})$$

$$\left. \frac{d}{dk_0} \left(\left(\frac{p_0}{2} - k_0 \right)^2 + E_k^2 \right) \right|_{k_0 = E_k - \frac{p_0}{2}} = 2E_k. \quad (\text{E.20})$$

$$\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \quad (\text{E.21})$$

F Hahn-Banach separation theorem

Let V : a real vector space, $U \subset V$, p a subaddictive, positive homogenous functional on V . If f is a linear functional on U such that $f(v) \leq p(v)$ for all $v \in U$, then there is a linear functional F on V such that $F(v) = f(v)$ for all $v \in U$ and $F(v) \leq p(v)$ for all $v \in V$.

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