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Weak Gravity EFT

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Abstract: In this report, we review [1]

Contents

1 Main calculations

My comments:

- I spotted a difference from my vertices definition 1.38 versus Eq. (B.1) in [1] at a factor $\frac{1}{2}$. Indeed, with Eq. (B.1), the Eq. (B.3) should yield $\frac{i}{4M_{\rm Pl}^2 t} su$. I am not sure whether they have added a factor of 4 (for symmetry?) implicitly there, or I have made some errors in the calculation. Please have a cross-check!

- In their Eq. (B.3), I suppose the $V_m^{\mu\nu}(k_1, k_3)$ should be changed to $V_0^{\mu\nu}(k_1, k_3)$ as we have assumed the scalar to be massless. Otherwise, there will be an additional term in the amplitude.

1.1 Field Redefinition

Under field redefinition, as in Eq. (3.5) of [1]

$$g_{\mu\nu} = g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\rm Pl}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right), \tag{1.1}$$

the terms (inside the bracket) of Eq. (4.2) reads,

$$\begin{aligned} \frac{M_{\rm Pl}^2}{2} R =& \frac{M_{\rm Pl}^2}{2} R^{\mu\nu} g_{\mu\nu} && (1.2) \\ \to& \frac{M_{\rm Pl}^2}{2} R^{\mu\nu} \left[g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\rm Pl}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) \right] && (1.3) \\ =& \frac{M_{\rm Pl}^2}{2} R + C \frac{\alpha^2}{M^2} R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} C \frac{\alpha^2}{M^2} R (\partial\phi)^2, && (1.4) \end{aligned}$$

$$\begin{aligned} -\frac{1}{2} (\partial\phi)^2 =& -\frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi && (1.5) \\ \to& -\frac{1}{2} \left[g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\rm Pl}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) \right] \partial^\mu \phi \partial^\nu \phi && (1.6) \\ =& -\frac{1}{2} (\partial\phi)^2 - \frac{1}{2} C \frac{\alpha^2}{M^2 M_{\rm Pl}^2} (\partial\phi)^4, && (1.7) \end{aligned}$$

$$\begin{aligned} -\frac{1}{2} (\partial\chi)^2 =& -\frac{1}{2} g_{\mu\nu} \partial^\mu \chi \partial^\nu \chi && (1.8) \\ \to& -\frac{1}{2} \left[g_{\mu\nu} + 2C \frac{\alpha^2}{M^2 M_{\rm Pl}^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) \right] \partial^\mu \chi \partial^\nu \chi && (1.9) \\ =& -\frac{1}{2} (\partial\chi)^2 - C \frac{\alpha^2}{M^2 M_{\rm Pl}^2} (\partial\phi \partial\chi)^2 + \frac{1}{2} C \frac{\alpha^2}{M^2 M_{\rm Pl}^2} (\partial\phi)^2 (\partial\chi)^2. && (1.10) \end{aligned}$$

All other terms are added with negligible higher-order (H. O.) terms.
The metric determinant transforms as,

$$
\begin{aligned}
\sqrt{-g} =& \sqrt{-\det g_{\mu\nu}} && (1.11)\\
\to& \sqrt{-\det\left[g_{\mu\nu}+2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}(\partial\phi)g_{\mu\nu}\right)\right]} && (1.12)\\
\to& \sqrt{-\det\left[g_{\mu\alpha}\left(\delta^\alpha{}_\nu+2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\phi\partial_\nu\phi-\frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)\right]} && (1.13)\\
\sim& \sqrt{-\det\left[g_{\mu\alpha}\exp\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\partial_\nu\phi-\frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)\right]} && (1.14)\\
=& \sqrt{-\det(g_{\mu\alpha})}\sqrt{\det\exp\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\phi\partial_\nu\phi-\frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)} && (1.15)\\
=& \sqrt{-g}\sqrt{\exp\mathrm{tr}\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\phi\partial_\nu\phi-\frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)} && (1.16)\\
=& \sqrt{-g}\sqrt{\exp\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(-(\partial\phi)^2\right)\right)} && (1.17)\\
\sim& \sqrt{-g}\sqrt{1-2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial\phi\right)^2} && (1.18)\\
\sim& \sqrt{-g}\left(1-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial\phi\right)^2\right). && (1.19)
\end{aligned}
$$

Hence, subtitute $\sqrt{-g}$ by $\sqrt{-g}\left(1+C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial\phi\right)^2\right)$, the Lagrangian terms transform as,

$$
\begin{aligned}
\sqrt{-g}\frac{M_{\rm Pl}^2}{2}R \to& \sqrt{-g}\bigg(\frac{M_{\rm Pl}^2}{2}R+C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}C\frac{\alpha^2}{M^2}R(\partial\phi)^2\\
&+\frac{1}{2}C\frac{\alpha^2}{M^2}R(\partial\phi)^2+\text{H. O.}\bigg). && (1.20)\\
=& \sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R+C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\text{H. O.}\right), && (1.21)
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2\right) && (1.22)\\
\to&\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4-\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4+\text{H. O.}\right) && (1.23)\\
=&\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4+\text{H. O.}\right), && (1.24)
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{-g}\left(-\frac{1}{2}(\partial\chi)^2\right) && (1.25)\\
\to&\sqrt{-g}\bigg(-\frac{1}{2}(\partial\chi)^2-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi\partial\chi)^2+\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^2(\partial\chi)^2\\
&-\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^2(\partial\chi)^2+\text{H. O.}\bigg) && (1.26)\\
=&\sqrt{-g}\bigg(-\frac{1}{2}(\partial\chi)^2-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi\partial\chi)^2+\text{H. O.}\bigg), && (1.27)
\end{aligned}
$$

Given the above transformation, we deduce the extra terms,

$$
\sqrt{-g}\left(C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi\partial\chi)^2\right). \tag{1.28}
$$

Hence, the IR action (4.2) in [1]

$$
\begin{aligned}
\mathcal{L}_{\rm IR}^{(J)}=\sqrt{-g}\bigg[&\frac{M_{\rm Pl}^2}{2}R-\frac{1}{2}(\partial\chi)^2-\frac{1}{2}(\partial\phi)^2-\frac{\alpha^3M}{(2\pi)^2}\frac{\phi^3}{3!}+\frac{\alpha^4}{2\pi^2}\frac{\phi^4}{4!}\\
&+C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\tilde{C}\frac{\alpha^4}{M^4}(\partial\phi)^4\bigg],
\end{aligned} \tag{1.29}
$$

reduces to (4.3) of [1],

$$
\begin{aligned}
\mathcal{L}_{\rm IR}=\sqrt{-g}\bigg[&\frac{M_{\rm Pl}^2}{2}R-\frac{1}{2}(\partial\chi)^2-\frac{1}{2}(\partial\phi)^2-\frac{\alpha^3M}{(2\pi)^2}\frac{\phi^3}{3!}+\frac{\alpha^4}{2\pi^2}\frac{\phi^4}{4!}\\
&+C'\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4+C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial_\mu\phi\,\partial^\mu\chi)^2+\dots\bigg].
\end{aligned} \tag{1.30}
$$

1.2 Scattering Matrix

1.2.1 Matter Langrangian & Scalars-Graviton vertices

Scalar matter fields interact with the gravitational field as described by the action,

$$
S_m=\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2\right]. \tag{1.31}
$$

The quantum fluctuation of gravitational fields can be expanded about a smooth background metric $\eta_{\mu\nu}$, with the flutuations suppressed by the Planck scale $M_{\rm Pl} = \frac{1}{\sqrt{8\pi G}}$ as,

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\rm Pl}} h_{\mu\nu}. \tag{1.32}$$

Einstein action are expanded as,

$$S_{\rm g} = \int {\rm d}^4 x \sqrt{-g} \left[\mathcal{L}_g^{(0)} + \mathcal{L}_g^{(1)} + \mathcal{L}_g^{(2)} + ... \right] \tag{1.33}$$

with the expanding terms,

$$\mathcal{L}_g^{(0)} = \frac{M_{\rm Pl}^2}{2} R \tag{1.34}$$

$$\mathcal{L}_g^{(1)} = \frac{2}{M_{\rm Pl}} h_{\mu\nu} [\eta^{\mu\nu} R - 2R^{\mu\nu}] \tag{1.35}$$

Here, we skip the discussion about gauge fixing and ghost Lagrangian. A similar expansion for matter Lagrangian yields,

$$S_m = \int {\rm d}^4 x \sqrt{-g} \left(\mathcal{L}_m^{(0)} + \mathcal{L}_m^{(1)} + \mathcal{L}_m^{(2)} + ... \right) \tag{1.36}$$

Here, $T^{\mu\nu}$ is derived from variation of 1.31 as,

$$T^{\mu\nu} = \frac{2}{\sqrt{-\eta}} \frac{\delta S_m}{\delta \eta^{\mu\nu}} = \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \left(\partial_\tau \phi \partial^\tau \phi - m^2 \phi^2 \right) \right) \tag{1.37}$$

Feynman rules for $\mathcal{L}_m^{(1)}$ in momentum space reads,

$$V_{\mu\nu}^{\phi\phi h}(p_1, p_2, m) = \frac{i}{M_{\rm Pl}} \left[(p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) - \eta_{\mu\nu} (p_1 p_2 - m^2) \right]. \tag{1.38}$$

Hence, the massless scalar field ϕ and χ reads

$$V_{\mu\nu}^{\phi\phi h}(p_1, p_3, 0) = \frac{i}{M_{\rm Pl}} \left[(p_{1\mu} p_{3\nu} + p_{1\mu} p_{3\nu}) - \eta_{\mu\nu} (p_1 p_3) \right]. \tag{1.39}$$

$$V_{\alpha\beta}^{\chi\chi h}(p_2, p_4, 0) = \frac{i}{M_{\rm Pl}} \left[(p_{2\alpha} p_{4\beta} + p_{2\alpha} p_{4\beta}) - \eta_{\alpha\beta} (p_2 p_4) \right]. \tag{1.40}$$

1.2.2 Graviton Propagator

Consider the Lagrangian

$$\sqrt{-g} \mathcal{L} = \sqrt{-g} \left(\frac{M_{\rm Pl}^2}{2} R + \mathcal{L}_m + \mathcal{L}_{\rm GF} \right), \tag{1.41}$$

to the second order in $h_{\mu\nu}$,

$$\frac{M_{\rm Pl}^2}{2}R = \frac{M_{\rm Pl}^2}{2}(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) + \frac{1}{2}\left[\partial_\tau h_{\mu\nu}\partial^\tau \bar{h}^{\mu\nu} - 2\partial^\tau \bar{h}_{\mu\tau}\partial_\sigma \bar{h}^{\mu\sigma}\right], \tag{1.42}$$

with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{1.43}$$

as mentioned in the Appendix. The harmonic gauge means to choose $\xi = 1$ of the gauge fixing term,

$$\mathcal{L}_{\rm GF} = \xi\partial_\mu \bar{h}^{\mu\nu}\partial^\tau \bar{h}_{\tau\nu} \tag{1.44}$$

The Lagrangian become

$$\sqrt{-g}\mathcal{L} = \frac{1}{2}\partial_\tau h_{\mu\nu}\partial^\tau h^{\mu\nu} - \frac{1}{4}\partial_\tau h\partial^\tau h - \frac{1}{M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}. \tag{1.45}$$

Taking integration by part, we have

$$\mathcal{L} = \frac{1}{2}h_{\mu\nu}\Box\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta} + \frac{1}{M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}., \tag{1.46}$$

with the "identity" tensor $I^{\mu\nu\alpha\beta}$ defined as in A.18. The E.O.M. follows,

$$\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta}\Box D_{\alpha\beta\gamma\delta} = I^{\mu\nu}{}_{\gamma\delta}. \tag{1.47}$$

This equation admitted the solution of A.26, assuming that the initial condition correspond to Feynman propagator $D^{\alpha\beta\gamma\sigma}(x-y)$,

$$D^{\alpha\beta\gamma\delta}(x-y) = \begin{cases} G^{\alpha\beta\gamma\delta}(x-y) & \text{if } x^0 > y^0, \\ G^{\alpha\beta\gamma\delta}(y-x) & \text{if } x^0 < y^0, \end{cases} \tag{1.48}$$

we obtain

$$iD^{\alpha\beta\gamma\sigma}(x) = \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2+i\epsilon}e^{-iqx}P^{\alpha\beta\gamma\delta}, \tag{1.49}$$

with

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}\right). \tag{1.50}$$

Hence, the propagator reads,

$$\frac{iP^{\mu\alpha\nu\beta}}{k^2} = \frac{i}{2}\frac{\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}}{(p_1+p_3)^2}. \tag{1.51}$$

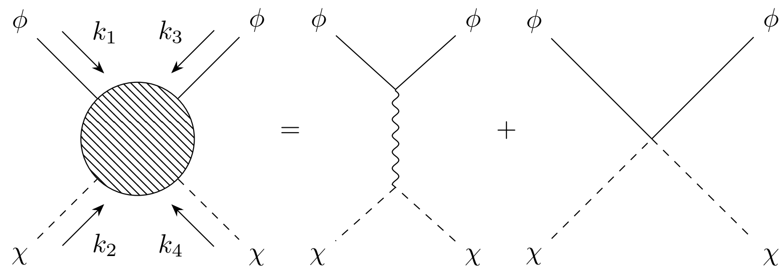

Figure 1: Feynman diagrams of the process, taken from [1]

1.2.3 Scattering matrix of effective 4-scalar vetices

We now have all the ingredients to derive the scattering matrix.
The t-channel reads,

$$
\begin{aligned}
i\mathcal{M}_1 =& - V^{\phi\phi h}_{\mu\nu}(p_1, p_3, 0)\frac{iP^{\mu\alpha\nu\beta}}{(p_1+p_3)^2}V^{\chi\chi h}_{\alpha\beta}(p_2, p_4, 0) && (1.52)\\
=& -\frac{i^3}{2M_{\text{Pl}}^2(p_1+p_3)^2}\big[4p_{1\mu}p_{3\nu}\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)p_{2\alpha}p_{4\beta}\\
&-2\eta_{\mu\nu}(p_1p_3)\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)p_{2\alpha}p_{4\beta}\\
&-2p_{1\mu}p_{3\nu}\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)\eta_{\alpha\beta}(p_2p_4)\\
&+\eta_{\mu\nu}(p_1p_3)\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)\eta_{\alpha\beta}(p_2p_4)\big] && (1.53)\\
=&\frac{i}{4M_{\text{Pl}}^2(p_1p_3)}\big[4[(p_1p_2)(p_3p_4)+(p_1p_4)(p_2p_3)-(p_1p_3)(p_2p_4)]\\
&-2(p_1p_3)(1+1-4)(p_2p_4)-2(p_1p_3)(1+1-4)(p_2p_4)\\
&+(p_1p_3)(4+4-16)(p_2p_4)\big] && (1.54)\\
=&\frac{i}{M_{\text{Pl}}^2(p_1p_3)}[(p_1p_2)(p_3p_4)+(p_1p_4)(p_2p_3)-(p_1p_3)(p_2p_4)]. && (1.55)
\end{aligned}
$$

Here, we have used the fact that $\eta^{\mu\nu}\eta_{\mu\alpha} = \delta^{\nu}{}_{\alpha}$, and $\eta^{\mu\nu}\eta_{\mu\nu} = 4$. In addition, with $p_i^2 = m_i^2 = 0$ for all $i = 1, 2, 3, 4$, the Madelstam variables read:

$$
\begin{aligned}
s \equiv& -(p_1+p_2)^2 = -(p_3+p_4)^2 = -2p_1p_2 = -2p_3p_4, && (1.56)\\
t \equiv& -(p_1+p_3)^2 = -(p_2+p_4)^2 = -2p_1p_3 = -2p_2p_4, && (1.57)\\
-s-t = u \equiv& -(p_1+p_4)^2 = -(p_2+p_3)^2 = -2p_1p_4 = -2p_2p_3. && (1.58)
\end{aligned}
$$

Therefore, the term is rewriten to

$$
i\mathcal{M}_1 = -i\frac{s(s+t)}{M_{\text{Pl}}^2 t}. \tag{1.59}
$$

Now, consider the 4-vertice diagram,

$$i\mathcal{M}_2 = 2iC\frac{\alpha^2}{M^2 M_{\text{Pl}}^2}\left[p_{1\mu}p_{3\nu}\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}\right)p_{2\alpha}p_{4\beta}\right] \tag{1.60}$$

$$= 2iC\frac{\alpha^2}{M^2 M_{\text{Pl}}^2}\left[(p_1p_2)(p_3p_4)+(p_1p_4)(p_2p_3)\right] \tag{1.61}$$

$$= 2iC\frac{\alpha^2}{M^2 M_{\text{Pl}}^2}\left[s(s+t)+\frac{t^2}{2}\right]. \tag{1.62}$$

The effective 4-scalar scattering matrix yields,

$$\mathcal{M} = \mathcal{M}_1+\mathcal{M}_2 \tag{1.63}$$

$$= -\frac{s(s+t)}{M_{\text{Pl}}^2 t}+2C\frac{\alpha^2}{M^2 M_{\text{Pl}}^2}\left[s(s+t)+\frac{t^2}{2}\right]. \tag{1.64}$$

A Weak-Field Gravity

A.1 Linearised theory

Expanding metric arround flat-space Minkowski background, with $M_{\text{Pl}} = \frac{1}{\sqrt{8\pi G}}$,

$$g_{\mu\nu} = \eta_{\mu\nu}+\sqrt{32\pi G}h_{\mu\nu} = \eta_{\mu\nu}+\frac{2}{M_{\text{Pl}}}h_{\mu\nu}. \tag{A.1}$$

To leading order, the inverse metric reads,

$$g^{\mu\nu} = \eta^{\mu\nu}-\frac{2}{M_{\text{Pl}}}h^{\mu\nu}. \tag{A.2}$$

Christoffel symbols are then,

$$\Gamma^{\sigma}_{\nu\rho} = \frac{1}{M_{\text{Pl}}}\eta^{\sigma\lambda}\left(\partial_\nu h_\lambda\rho+\partial_\rho h_{\nu\lambda}-\partial_\lambda h_{\nu\rho}\right). \tag{A.3}$$

The Rieman tensor reads,

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu\Gamma^{\sigma}_{\nu\rho}-\partial_\nu\Gamma^{\sigma}_{\mu\rho}+\Gamma^{\lambda}_{\nu\rho}\Gamma^{\sigma}_{\mu\lambda}-\Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\nu\lambda}. \tag{A.4}$$

The $\Gamma\Gamma$ term is at $\mathcal{O}(h^2)$, hence, to the linear order,

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu\Gamma^{\sigma}_{\nu\rho}-\partial_\nu\Gamma^{\sigma}_{\mu\rho}+\mathcal{O}(h^2) \tag{A.5}$$

$$= \frac{1}{M_{\text{Pl}}}\eta^{\sigma\lambda}(\partial_\mu\partial_\rho h_{\nu\rho}-\partial_\mu\partial_\lambda h_{\nu\rho}-\partial_\nu\partial_\rho h_{\mu\lambda}+\partial_\nu\partial_\lambda h_{\mu\rho})+\mathcal{O}(h^2). \tag{A.6}$$

The Ricci tensors read

$$R_{\mu\nu} = \frac{1}{M_{\rm Pl}}\left(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right). \tag{A.7}$$

with $h \equiv h^\mu{}_\mu$, $\Box \equiv \partial^\mu\partial_\mu$. The Ricci scalar follows,

$$R = \frac{1}{M_{\rm Pl}}\left(\partial^\mu\partial^\nu h_{\mu\nu} - \Box h\right). \tag{A.8}$$

Hence, we deduce the Einstein equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{A.9}$$

with

$$G_{\mu\nu} = \frac{1}{M_{\rm Pl}}\left[\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - (\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h)\eta_{\mu\nu}\right]. \tag{A.10}$$

A.2 Green function

We have derived Ricci tensor and scalar,

$$R_{\mu\nu} = \frac{1}{M_{\rm Pl}}\left(\partial_\mu\partial_\gamma h^\gamma{}_\nu + \partial_\nu\partial_\gamma h^\gamma{}_\mu - \partial_\mu\partial_\nu h^\gamma{}_\gamma - \Box h_{\mu\nu}\right) + \mathcal{O}(h^2), \tag{A.11}$$

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{1}{M_{\rm Pl}}\partial_\mu\partial_\gamma h^{\mu\gamma} + \mathcal{O}(h^2). \tag{A.12}$$

The Einstein equation reads,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{A.13}$$

$$\simeq \frac{1}{M_{\rm Pl}}\left[\left(\delta_{(\mu}{}^\alpha\delta_{\nu)}{}^\beta - \eta_{\mu\nu}\eta^{\alpha\beta}\right)\Box - 2\delta_{(\mu}{}^{(\alpha}\partial_{\nu)}\partial^{\beta)} + \eta^{\alpha\beta}\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial^\alpha\partial^\beta\right] h_{\alpha\beta} \tag{A.14}$$

$$\equiv \frac{1}{M_{\rm Pl}} O_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \tag{A.15}$$

$$= \frac{1}{M_{\rm Pl}^2} T_{\mu\nu}. \tag{A.16}$$

The Green function follows,

$$O_{\mu\nu}{}^{\alpha\beta} G_{\alpha\beta\gamma\delta}(x-y) = \frac{1}{2} I_{\mu\nu\gamma\delta}\delta_D^{(4)}(x-y), \tag{A.17}$$

with "identity" tensor defined as

$$I_{\mu\nu\gamma\delta} \equiv \frac{1}{2}\left(\eta_{\mu\gamma}\eta_{\nu\delta} + \eta_{\mu\delta}\eta_{\nu\gamma}\right), \tag{A.18}$$

We then concern gauge fixing as the operator $O_{\mu\nu}{}^{\alpha\beta}$ cannot be inverted.

A.3 Gauge transformation

Under the coordinate transformation,

$$x^\mu \to \tilde{x}^\mu = x^\mu + \kappa\xi^\mu(x), \tag{A.19}$$

the metric reads,

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \tag{A.20}$$

We choose the de Donder gauge, which reads,

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{A.21}$$

It is useful to define the field $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, from which we can rewrite the A.16 as,

$$\Box\bar{h}_{\mu\nu} = -\frac{\kappa}{2}T_{\mu\nu} \tag{A.22}$$

The Green function A.17 can be re-expressed as,

$$\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)\Box G_{\alpha\beta\gamma\delta} = I^{\mu\nu}{}_{\gamma\delta}, \tag{A.23}$$

from which the gravitational field $h_{\mu\nu}$ can be extracted, using the ansatz $G_{\alpha\beta\gamma\delta} = aI_{\alpha\beta\gamma\delta} + b\eta_{\alpha\beta}\eta_{\gamma\delta}$ which yields,

$$G_{\alpha\beta\gamma\delta} = I_{\alpha\beta\gamma\delta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}. \tag{A.24}$$

In the position representation, the Green function reads,

$$\begin{aligned} G_{\mu\nu\alpha\beta} &= \frac{1}{2\Box}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} + \eta_{\mu\nu}\eta_{\alpha\beta})\delta_D^{(4)}(x-y) && \text{(A.25)} \\ &= -\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} + \eta_{\mu\nu}\eta_{\alpha\beta})\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2}. && \text{(A.26)} \end{aligned}$$

This expression should come with the initial condition that defined by the choice of the contour on complex k_0-plane.

B Field Redefinition

Equivalent theorem [2] states that under reparameterization of field operators, the Scattering matrix remains unchanged, which means under redefinition of the field $\phi \to \tilde{\phi} = \phi + a_i\phi^i$, the generating functional,

$$\mathcal{Z} = \int D[\phi]\exp\left(i\int d^4x\mathcal{L}(\phi, \partial_\mu\phi)\right) \tag{B.1}$$

does not change as long as the Jacobian is essentially one [2]. Hence, we can use this technique to simplify the Lagrangian, for example,

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{M_{\rm Pl}}[a_1 h_{\mu\gamma} h_\nu{}^\gamma + a_2 h_{\mu\nu} h_\gamma{}^\gamma]. \tag{B.2}$$

Substituting this into the triple graviton vertex gives,

$$h_{\mu\nu}\partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta \to h_{\mu\nu}\partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta + a_1 \frac{2}{M_{\rm Pl}} h_{\mu\gamma} h_\nu{}^\gamma \partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta \tag{B.3}$$

$$+ a_2 \frac{2}{M_{\rm Pl}} h_{\mu\nu} h_\gamma{}^\gamma \partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta. \tag{B.4}$$

Hence, the field definition generates two addition quadruple graviton vertex with two parameter a_1, a_2. With a proper choice of these parameters, we can cancel some of the quadruple graviton vertex contributions in the original Lagrangian.

C Effective Field Theory in Gravity

In gravity, the quantities which can be used to construct higher operators of the effective Lagrangian are Rienman tensor $R_{\mu\nu\alpha\beta}$, Ricci tensor $R_{\mu\nu}$, and Ricci scalar R. Those quantities contain 2 partial derivative ($\sim \partial\partial h$), hence the Lagrangian has only even dimention terms.

$$\mathcal{L}_{\rm eff} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + ... \tag{C.1}$$

where

- $\mathcal{L}^{(0)}$: Constants (such as Cosmology constant Λ - which usually be neglected)
- $\mathcal{L}^{(2)}$: Only one term R
- $\mathcal{L}^{(4)}$: 3 possible terms $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, R_{\mu\nu}R^{\mu\nu}, R^2$
- $\mathcal{L}^{(6)}$: 4 possible terms $R_{\mu\nu\alpha\beta}R^{\mu\nu}R^{\alpha\beta}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}R, R_{\mu\nu}R^{\mu\nu}R, R^3$.

which are Lorentz invariant and General Coordinate Transformations invariant.

When we neglect the higher operators, the effective theory is non-local at high energy while restored locality at low energy. We can utilize this properties to reduce loops of the full theory to effective vertices at low-energy.

References

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