Contents

1 Introduction

Quantum field theories (QFT) has become one of the pillars of modern physics. Most of them are perturbative theories in that interactions are considered as a tiny perturbation of free theory. From those theories. quantum observables are expressed as a power series of Plank's and coupling constants. Despite their insightfulness, and remarkable successes in explaining physics, non-perturbative QFT is still required for non-perturbative effects which can not be captured in perturbation theories (such as confinements, or mass gap of QCD).

Non-perturbative theories, such as topological quantum field theories or low dimensions (mostly 2D) Conformal Field Theories (CFTs) has been constructed as simple toy models. The 2D CFTs are, however, the building blocks for 2 important areas of physics: condensed matter physics, where they describe critical phenomena on the surface, and string theory where the 2d space-time plays the role of string world-sheet parametrizing the string evolution.

More specifically, we are concerned with Rational 2D-CFTs. In 1984, Belavin, Polyakov, and Zamolodchikov [1] used infinite-dimensional symmetry to reduce an infinite-dimensional field theory problems effectively to finite ones. This chiral symmetry structure is the hallmark of Rational 2D-CFT, which realised the symmetry algebra of Virasoro algebra to solve minimal models.

Then, larger classes of Rational 2D-CFT with similar structures are investigated using the extended infinite-dimensional algebras, which contains Virasoro algebra as their subalgebra. Such algebras include Affine Kac-Moody algebras (or their enveloping algebras) as the simplest infinite-dimensional extensions of ordinary semi-simple Lie algebras.

This essay investigates a particular model of 2D-CFT which realises affine Kac-Moody algebras, namely Wess-Zumino-(Novikov)-Witten (WZW) model.

The layout is as follows, Section 2 discusses the construction of WZW action based on the nonlinear sigma model with an additional WZ term that results in separately conserved (anti)holomorphic currents which localises and enhances (to infinite) symmetries of the WZW action. In Section 3, first, the infinite-dimensional affine Kac-Moody algebra is derived as the algebra realised by the conserved current. Then, the Sugawara construction is derived, verifying the conformal properties of the theory by proving that the energy-momentum tensors satisfies Virasoro algebra. Next, we construct a free fields' theory to illustrate an example of *non-abelian bosonization.* After that, we impose constraints on the representation theory by null vectors. Finally, we derive the Knizhnik-Zamolodchikov equation, which is a tool for performing 4-point correlation function calculation. In Section 4, we introduce the Chern-Simons theory, quantise it, and then derive its correspondence to the WZW model in that the conformal blocks of WZW theory correspond to the states in Hilbert space of quantised Chern-Simons theory.

2 Classical Theory

2.1 Nonlinear sigma model

We first consider Quantum field theory on Riemann Sphere S^2 with the Euclidean metric. The action of the kinetic term reads:

$$S_0 = \frac{1}{4\lambda^2}\int_{S^2} \mathrm{d}^2x\,\mathrm{tr}\big(g^{-1}\partial_\mu g g^{-1}\partial^\mu g\big), \tag{1}$$

where $g \equiv g(z,\bar{z}) : S^2 \to G$ is an element of semi-simple Lie Group. The associated Lie Algebra is denoted as $\mathfrak{g}$, with the inner product defined as $\langle X, Y\rangle = \mathrm{tr}(XY)$. Note that the one form $g^{-1}dg = g^{-1}\partial_\mu g \mathrm{d}x^\mu$ in Eq. (1) is the pullback of Maurer-Cartan form to S^2 under the map g, so it is an element of Lie Algebra. In terms of generators in fundamental representation t^a, the elements of Lie Algebra can be expressed as $X = X^a t^a$. We introduce the normalised trace $\mathrm{tr}\big(t^a t^b\big) = \frac{1}{2}\delta^{a,b}$. Due to the cyclicity property of trace, this action has a global symmetry:

$$g(z,\bar{z}) \mapsto g_L g(z,\bar{z}) g_R^{-1}, \tag{2}$$

where g_L, g_R are constant group elements of G.

$$\begin{aligned}\delta S_0 &= \frac{1}{4\lambda^2}\int_{S^2} \mathrm{d}^2x 2\,\mathrm{tr}\left((g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\delta\partial_\mu g)g^{-1}\partial^\mu g\right) &(3)\\ &= \frac{1}{2\lambda^2}\int_{S^2} \mathrm{d}^2x\,\mathrm{tr}\left(\partial_\mu(g^{-1}\delta g)g^{-1}\partial^\mu g\right) &(4)\\ &= \frac{-1}{2\lambda^2}\int_{S^2} \mathrm{d}^2x\,\mathrm{tr}\left(g^{-1}\delta g\partial_\mu(g^{-1}\partial^\mu g)\right). &(5)\end{aligned}$$

We derive the classical equation of motion:

$$\partial_\mu(g^{-1}\partial^\mu g) = 0. \tag{6}$$

This results in conservation of the current $J^\mu = g^{-1}\partial^\mu g$. It is indeed the current for right multiplicative symmetry $g \mapsto g g_R^{-1}$. Since

$$\partial_\mu(J^\mu) = 0 \iff g\partial_\mu(J^\mu)g^{-1} = \partial_\mu(\partial^\mu g g^{-1}) = 0, \tag{7}$$

it is equivalent to the conserved current $J'^\mu = \partial^\mu g g^{-1}$ for left-multiplication symmetry $g \mapsto g_L g$. Rewriting the current conservation in complex coordinate, with $x_0 = \frac{1}{2}(z+\bar{z}), x_1 = \frac{1}{2i}(z-\bar{z}), \partial_{x_0} = \partial_z + \partial_{\bar{z}}, \partial_{x_1} = i(\partial_z - \partial_{\bar{z}})$, we have:

$$\partial_z \tilde{J}_{\bar{z}} + \partial_{\bar{z}} \tilde{J}_z = 0. \tag{8}$$

Comparing this with the conservation of the energy-momentum tensor in CFT, where $\partial_\mu T^{\mu\nu} = 0$, we have the similar conservation:

$$\begin{cases} \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0, \\ \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0. \end{cases} \tag{9}$$

Since the energy-momentum tensor in CFT is traceless, $T_{z\bar{z}} = T_{\bar{z}z} = 0$, the Eq. (9) tell us that T_{zz} is anti-holomorphic and $T_{\bar{z}\bar{z}}$ is holomorphic. We want to mirror this to our Eq. (8), but we will prove it is not the case for the nonlinear sigma model. J is defined to be pure gauge, hence its field strength varnishes:

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0. \tag{10}$$

Then,

$$\partial_\mu(\epsilon^{\mu\nu} J_\nu) = \frac{1}{2}\epsilon^{\mu\nu}(\partial_\mu J_\nu - \partial_\nu J_\mu) = -\frac{1}{2}\epsilon^{\mu\nu}[J_\mu, J_\nu] = -\epsilon^{\mu\nu} J_\mu J_\nu. \tag{11}$$

This expression is not expected to vanish in general non-Abelian Lie algebra. Hence the current in Eq. (8) is not separately conserved and the theory does not have (anti)holomorphic currents.

2.2 WZ term

The above theory is conformally invariant in classical theories and Abelian (free bosons) quantum theories. However, it yields the issue of non-holomorphicity in the non-Abelian case. We deal with this by introducing an extra term to the action:

$$\Gamma[\tilde{g}] = -\frac{i}{12\pi}\int_{B^3} \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma} \operatorname{tr}\left(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}\right). \tag{12}$$

Alternatively writing in the language of form, with $\mathrm{d}\tilde{g} = \partial_i \tilde{g} \mathrm{d}y^i$, the action is the pull-back of the left-right invariant differential 3-form w on G by $\tilde{g}$:

$$\Gamma[\tilde{g}] = i\int_{B^3} \tilde{g}^*\omega = -\frac{i}{12\pi}\int_{B^3} \operatorname{tr}\big(\tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g} \wedge \tilde{g}^{-1}\mathrm{d}\tilde{g}\big), \tag{13}$$

with

$$w = \frac{1}{12\pi}\,\mathrm{tr}\big(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\big). \tag{14}$$

Note that the term $\Gamma[\tilde{g}]$ still maintains $G \times G$ symmetry.

Here we take the integral over the ball B^3, which is an extension of the Riemann sphere S^2 into its interior ($\partial B = S^2$). The field $g : S^2 \to G$ is then extended to a map $\tilde{g} : B^3 \to G$.

There are many ways to extend the field. Consider 2 extensions, as 2 compact oriented smooth 3-balls B^3, B'^3 with boundary $\partial B^3, \partial B'^3 = S^2$ and $\tilde{g} : B^3 \to G$, $\tilde{g}' : B'^3 \to G$ is respectively an extension over B^3 and B'^3 of g. Gluing the two 3-ball with relatively reversed orientation along their common boundary, we get a new manifold $\mathcal{M} = (B^3 \sqcup B'^3)/S^2$, which is homeomorphic to S^3. Hence, we define the map $\tilde{f} : S^3 \to G$, whose restriction on B and B' are $\tilde{g}$ and $\tilde{g'}$ respectively.

Here we recall that the set of homotopy classes of based maps $\phi : S^n \mapsto G$, denoted by $\pi_n(G)$, are disjoint classes all of whose maps are continuously deformable to each other. The basic result in algebraic topology for homotopy group is $\pi_3(G) = \mathbb{Z}$ with every simple compact Lie Group G. In addition, Bott's theorem states that any continuous mapping of S^3 into a general simple Lie Group can be continuously deformed into an $SU(2)$ subgroup of G. Since the Lie group $SU(2)$ is geometrically a S_3 sphere, the map $\tilde{g}$ is equivalent to S^3 wrapping around itself $\mathbb{Z}$ times.

Next, we consider Γ under the change of homotopy class. Take a point y on S^3 and parameterised it by y^μ satisfying $(y^0)^2 + (y^i)^2 = 1$ with Pauli matrices σ_i $(i = 1, 2, 3)$. Consider the case S^3 wrapping once around itself. We define a map $\tilde{f} : S^3 \to G$, which is close to identity:

$$\tilde{f}(y) = y^0 \mathbb{1}_2 - i y^k \sigma_k. \tag{15}$$

The difference between 2 extensions reads:

$$\begin{aligned}
\Delta\Gamma = &- \frac{i}{12\pi}\bigg(\int_{B^3} \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}\big(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}\big) \\
&- \int_{B'^3} \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}\big(\tilde{g}'^{-1}\partial^\alpha \tilde{g}'\tilde{g}'^{-1}\partial^\beta \tilde{g}'\tilde{g}'^{-1}\partial^\gamma \tilde{g}'\big)\bigg) && (16) \\
= &- i \int_{\mathcal{M}\cong S^3} \tilde{f}^* \omega && (17) \\
= &- \frac{i}{12\pi}\int_{S^3} \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}\Big(\tilde{f}^{-1}\partial^\alpha \tilde{f}\tilde{f}^{-1}\partial^\beta \tilde{f}\tilde{f}^{-1}\partial^\gamma \tilde{f}\Big). && (18)
\end{aligned}$$

Here, we substitute $\tilde{f}^{-1}\partial^k \tilde{f} = -i\sigma^k$, with the volume of unit 3-sphere $\int_{S^3} d^3 y = 2\pi^2$, since the integrand does not depend on y, we have:

$$\begin{aligned}\Delta\Gamma =& -\frac{i(-i)^3}{12\pi}2\pi^2\epsilon_{ijk}\operatorname{tr}\big(\sigma^i\sigma^j\sigma^k\big) && (19)\\ =&\frac{\pi}{12}\epsilon_{ijk}\operatorname{tr}\big([\sigma^i,\sigma^j]\sigma^k\big) && (20)\\ =&\frac{\pi}{12}2i\epsilon_{ijk}\operatorname{tr}\big(\epsilon^{ijl}\sigma_l\sigma^k\big) && (21)\\ =&\frac{i\pi}{6}2\operatorname{tr}\big(\sigma_k\sigma^k\big) && (22)\\ =&2\pi i. && (23)\end{aligned}$$

From now on, for convenience, we will denote the extension of g as also "g". We define WZW action:

$$S[g] \equiv S_0[g] + k\Gamma[g], \quad k\in\mathbb{R}. \tag{24}$$

Consider the Feynman path-integral:

$$Z = \int \mathcal{D}g e^{-S[g]}, \tag{25}$$

from the quantum theory point of view, we have to make sure that the $S[g]$ takes a unique value at different topological extensions of the same g. We see that the difference of $2\pi i\mathbb{Z}$ in the exponential does not change the result. Since $\Delta\Gamma = 2\pi i$, we conclude that the quantum theory can only be well-defined for $k\in\mathbb{Z}$. For compact Lie groups, the k is called the *"level"* of the model. Next, we shall derive the equation of motion for the action. Consider Γ under the perturbation of extension g:

$$\begin{aligned}\delta\Gamma =&\frac{-i}{12\pi}\int_{B^3}3\operatorname{tr}\left((-g^{-1}\delta g g^{-1}\,\mathrm{d}g + g^{-1}\mathrm{d}\delta g)\wedge g^{-1}\,\mathrm{d}g\wedge g^{-1}\,\mathrm{d}g\right) && (26)\\ =&\frac{-i}{4\pi}\int_{B^3}\mathrm{d}\operatorname{tr}\left(g^{-1}\delta g g^{-1}\,\mathrm{d}g\wedge g^{-1}\,\mathrm{d}g\right) && (27)\\ =&\frac{-i}{4\pi}\int_{S^2}\operatorname{tr}\left(g^{-1}\delta g g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\right) && (28)\\ =&\frac{-i}{4\pi}\int_{S^2}\operatorname{tr}\left(g^{-1}\delta g\,\mathrm{d}\big(g^{-1}\mathrm{d}g\big)\right). && (29)\end{aligned}$$

Or, equivalently,

$$\delta\Gamma = \frac{-i}{4\pi}\int_{S^2}\mathrm{d}x^2\epsilon_{\mu\nu}\operatorname{tr}\left(g^{-1}\delta g\partial^\mu(g^{-1}\partial^\nu g)\right). \tag{30}$$

Here we also see Γ is invariant if variation of g varnishes on S^2. Translating the term into complex coordinate, denoting $\partial\equiv\partial_z, \bar\partial\equiv\partial_{\bar z}$, we get:

$$\begin{aligned}\partial\Gamma =&\frac{-i}{4\pi}\int_{S^2}\operatorname{tr}\left(g^{-1}\delta g\mathrm{d}\left(g^{-1}\partial g\mathrm{d}z + g^{-1}\bar\partial g\mathrm{d}\bar z\right)\right) && (31)\\ =&\frac{-i}{4\pi}\int_{S^2}\operatorname{tr}\left(g^{-1}\delta g\left(\bar\partial(g^{-1}\partial g)\mathrm{d}\bar z\wedge\mathrm{d}z + \partial(g^{-1}\bar\partial g)dz\wedge d\bar z\right)\right) && (32)\\ =&\frac{-i}{4\pi}\int_{S^2}\mathrm{d}z\mathrm{d}\bar z\operatorname{tr}\left(g^{-1}\delta g\left(-\bar\partial(g^{-1}\partial g) + \partial(g^{-1}\bar\partial g)\right)\right) && (33)\end{aligned}$$

From Eq. (5), the variation of S_0 reads:

$$\delta S_0 = \frac{-i}{4\lambda^2}\int_{S^2} \mathrm{d}z\mathrm{d}\bar{z}\,\mathrm{tr}\left(g^{-1}\delta g(\partial(g^{-1}\bar{\partial}g) + \bar{\partial}(g^{-1}\partial g))\right) \tag{34}$$

Hence, we derive the equation of motion for $S[g]$:

$$\partial(g^{-1}\bar{\partial}g)\left(1+\frac{\lambda^2 k}{\pi}\right) + \bar{\partial}(g^{-1}\partial g)\left(1-\frac{\lambda^2 k}{\pi}\right) = 0 \tag{35}$$

Here we see the equation of motion still obey the $G \times G$ symmetry. For $\lambda^2 = \pi/k$, with $k > 0$, we deduce the conservation:

$$\partial(g^{-1}\bar{\partial}g) = 0. \tag{36}$$

as well as:

$$g\partial(g^{-1}\bar{\partial}g)g^{-1} = \bar{\partial}(\partial g g^{-1}) = 0. \tag{37}$$

Hence, we get the holomorphic and anti-holomorphic conserved current $\bar{J} = -k\partial g g^{-1}$ and $J = kg^{-1}\bar{\partial}g$ respectively. It enhances the symmetry of the theory. $G \times G$ global is now localised by holomorphic maps

$$g(z,\bar{z}) \mapsto g_L(z)g(z,\bar{z})g_R^{-1}(z), \tag{38}$$

where $g_L(z)$, $g_R\bar{z}$ is any holomorphic, anti-holomorphic maps $S^2 \to G$ respectively . This exhibits the local $G(z) \times G(\bar{z})$ symmetry. This property plays a pivotal role since we can now get infinitely many conserved currents by multiply one with arbitrary (anti)holomorphic functions, and hence, yields infinitely many symmetries in our theory. As we shall see in the next section, this fact is also followed by the infinite-dimensional affine Kac-Moody algebra that the currents realise when we quantise the theory.

3 Quantum theory

3.1 Current algebra

Now we examine the conformal property of the WZW model on the quantum level. Our starting point is to describe the algebraic structure of OPE $J^a(z)J^b(w)$. Instead of canonically quantising the theory, we take the shortcut that constrains the possible OPE as much as we can and we shall see it is already enough to derive all of its structure. Consider the OPE $J^a(z)J^b(w)$ having conformal dimension 2 in the form:

$$J^a(z)J^b(w) \sim \sum_p \frac{X_p(w)}{(z-w)^p}. \tag{39}$$

By dimensional analysis, the holomorphic field X_p has the conformal dimension $2-p$. The unitarity of CFT imposes the constrain on the dimension of operators that the lowest energy of CFT is the vacuum, which has dimension 0. Hence, we require the dimensions to be non-negative, which means we have two possible terms of $p = 2$ and $p = 1$. The OPE can be defined as:

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^a(w)}{z-w}. \tag{40}$$

Now we constrain the constants κ^{ab} and $f^{ab}{}_c$. First, since OPEs hold inside the correlation functions, we have $J^a(z)J^b(w) \sim J^b(w)J^a(z)$, which yields:

$$\kappa^{ab} = \kappa^{ba}, \; f^{ab}{}_c = -f^{ba}{}_c. \tag{41}$$

Second, the associative properties of OPE $[J^a(z)J^b(w)]J^c(t) = J^a(z)[J^b(w)J^c(t)]$ yields further constraints:

$$\kappa^{cd} f^{ab}{}_d = \kappa^{bd} f^{ca}{}_d = \kappa^{ad} f^{bc}{}_d, \tag{42}$$

$$f^{ab}{}_c f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0. \tag{43}$$

We see that for simple Lie Algebra, $f^{ab}{}_c$ plays the role of structure constants since it satisfies the anti-symmetric and Jacobian properties. In addition, the above constraints tell us that κ^{ab} is the invariant form of the Lie algebra. The only one invariant form we have of simple algebra is the Killing form, which is proportional to $tr(t^a t^b)$ and hence, is proportional to δ^{ab}. Therefore, we can express it as $\kappa^{ab} = k\delta^{ab}$ for some constant k. We then derive the OPE for holomorphic and anti-holomorphic currents:

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{ab}_c J^c(w)}{z-w}. \tag{44}$$

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{if^{ab}_c \bar{J}^c(\bar{w})}{\bar{z}-\bar{w}}. \tag{45}$$

This relation is called the *"current algebra"* and is the central reference point for calculations in the WZW model. Here, there is only 1 free constant k in the construction. As we shall see later, this constant is indeed exactly the same as the quantised *level* k of the WZW action we defined above.

3.2 Affine Kac-Moody algebra

Consider the current algebra, as we would have expected in any CFT, if we just had the left and right global symmetry, the currents would just realise some the finite dimensional algebras. In WZW model, however, the localized symmetry imposes that there are infinitely many conserved current. Therefore, the algebra realised by currents should be enhanced to infinite dimensions. In this subsection, we shall derive such algebra, namely *"affine Kac-Moody algebra"*, as a central extension of the loop algebra.

We will now derive affine algebra structure through the Polyakov-Wiegmann identity. First, under group multiplication, the principle chiral term transform as:

$$S_0[gh] = S_0[g] + S_0[h]. \tag{46}$$

Next, we consider the WZ term. With the transformation:

$$(gh)^{-1}d(gh) = h^{-1}(g^{-1}\mathrm{d}g)h^{-1} + h^{-1}\mathrm{d}h = h^{-1}(g^{-1}\mathrm{d}g + \mathrm{d}hh^{-1})h, \tag{47}$$

the WZ term yields:

$$\begin{aligned}
\Gamma[gh] =& -\frac{i}{12\pi}\int_{B^3} \mathrm{tr}\big((gh)^{-1}\,\mathrm{d}(gh)\wedge(gh)^{-1}\,\mathrm{d}(gh)\wedge(gh)^{-1}\,\mathrm{d}(gh)\big) && (48)\\
=&\Gamma[g]+\Gamma[h] - \frac{3i}{12\pi}\int_{B^3}\mathrm{tr}\left(g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g\wedge \mathrm{d}hh^{-1} + g^{-1}\mathrm{d}g\wedge\mathrm{d}hh^{-1}\wedge\mathrm{d}hh^{-1}\right) && (49)\\
=&\Gamma[g]+\Gamma[h] - \frac{i}{4\pi}\int_{B^3}\mathrm{tr}\left(-d(g^{-1}\mathrm{d}g)\wedge\mathrm{d}hh^{-1} - g^{-1}\mathrm{d}g\wedge\mathrm{d}\big(\mathrm{d}hh^{-1}\big)\right) && (50)\\
=&\Gamma[g]+\Gamma[h] - \frac{i}{4\pi}\int_{B^3}\mathrm{d}\,\mathrm{tr}\left(g^{-1}\mathrm{d}g\wedge\mathrm{d}hh^{-1}\right) && (51)\\
=&\Gamma[g]+\Gamma[h] - \frac{i}{4\pi}\int_{S^2}\mathrm{tr}\left(g^{-1}\mathrm{d}g\wedge\mathrm{d}hh^{-1}\right). && (52)
\end{aligned}$$

Hence, the full WZW action reads, in complex coordinate:

$$\begin{aligned}
S[gh] &= S[g]+S[h]+\frac{k}{8\pi}\int_{S^2}\mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{tr}\left(g^{-1}\bar{\partial}g\wedge\partial hh^{-1}\right). && (53)\\
&= S[g]+S[h]+W_k[g,h], && (54)
\end{aligned}$$

with our convention of $W_k[g,h] \equiv kW_1[g,h] \equiv \frac{k}{8\pi}\int_{S^2}\mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{tr}\left(g^{-1}\bar{\partial}g\wedge\partial hh^{-1}\right)$. This relation is called the *"Polyakov-Wiegmann identity"*, and we will derive the affine algebra from it. First we see that the term W[g,h] transform as a 2-cocycle:

$$\begin{aligned}
&S[(gh)k] = S[g(hk)]\\
\iff\ & W_k[gh,k]+W_k[g,h] = W_k[g,hk]+W_k[h,k]. && (55)
\end{aligned}$$

This illustrates the current form a projective representation of the loop group LG of G, or non-projectively, the representation of central extension $\hat{G}$ (of loop group LG). Note that this extension's Lie algebra $\hat{\mathfrak{g}}$ is the affine Kac-Moody algebra, which is also an extension of the loop algebra $\mathcal{L}\mathfrak{g}$ of the loop group LG. This construction is similar to the quantization of conventional CFTs, where the Virasoro algebra is the central extension of Witt algebra (the Appendix A show how this algebra is constructed from the purely mathematical point of view). Next, we will show that the commutation relation indeed follows this affine algebra. Consider holomorphic field $J^a(z)$ with conformal weight 1, we have it's Laurent expansion around the origin:

$$J^a(z) \equiv \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}, \tag{56}$$

where the generalised modes are expressed as:

$$J^a_n = \oint dz J^a(z) z^n. \tag{57}$$

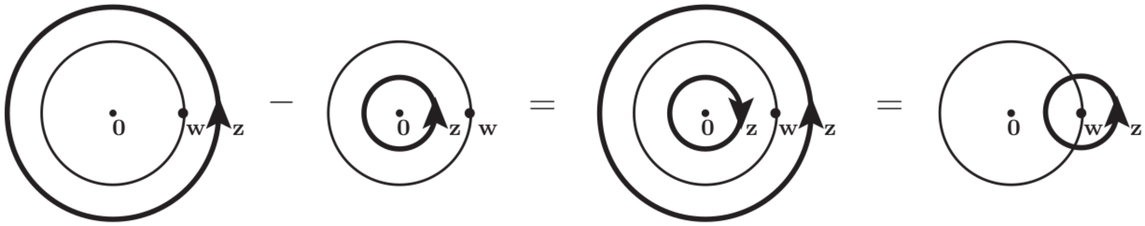

Figure 1: Subtraction rule of the contour integral

Its conserved charge reads:

$$J_0^a = \oint dz J^a(z). \tag{58}$$

We compute the commutation relation is calculated following the rule for the subtraction of contour integral as in the Fig. 1:

$$\begin{aligned}
\left[J_n^a, J_m^b\right] &= \frac{1}{(2\pi i)^2}\left(\oint dz \oint_{|z|>|w|} dw - \oint dz \oint_{|w|>|z|} dw\right) z^n w^m J^a(z) J^b(w) && (59)\\
&= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz z^n w^m \left[\frac{i f^{ab}{}_c J^c(z)}{z-w} + \frac{k\delta^{ab}}{(z-w)^2}\right] && (60)\\
&= \frac{1}{2\pi i}\oint_0 dw \left[i f^{ab}{}_c J^c(w) w^{n+m} + kn\delta^{ab} w^{n+m-1}\right] && (61)\\
&= i f^{ab}{}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}. && (62)
\end{aligned}$$

This commutation relation reflects the affine Kac-Moody Algebra $\hat{\mathfrak{g}}$, with the central term of $kn\delta^{ab}\delta_{n+m,0}$. We see that it arises naturally as a central extension of loop algebra $\mathcal{L}\mathfrak{g}$, generated by $J^a \otimes s_n$, with $n \in \mathbb{Z}$ and s is unit circle on complex plane, as they are g-valued functions on S^1:

$$\left[J^a \otimes s_n, J^b \otimes s_m\right] = \left[J^a, J^b\right] \otimes s_{n+m} + kn\delta^{ab}\delta_{n+m,0}. \tag{63}$$

It obeys the the infinite-dimensional affine Kac-Moody algebra $\hat{\mathfrak{g}}_k$ at level k. The algebra yields back the finite-dimensional Lie Algebra $\mathfrak{g}$ when $n = m = 0$.

$$\left[J_0^a, J_0^b\right] = i f^{ab}{}_c J_0^c. \tag{64}$$

The theory yields back the standard Lie Algebra structure with the conserved charges and they realise the symmetry algebra. However, the affine Kac-Moody is not the symmetry algebra of the theory since all higher modes of the currents do not commute with the Hamiltonian. Instead, this affine algebra is named the *"spectrum-generating algebra"* of the theory.

3.3 Sugawara construction

We have done many works with currents, now we have not shown anything in the context of CFT, which realises the Virasoro algebra. Our next task is to construct the energy-momentum tensor of the theory, and show that it is indeed conformal.
In classical theory, the energy-momentum tensor is constructed in the form:

$$T(z) = \gamma' J^a(z) J^a(z), \tag{65}$$

with the parameter γ' is simply given by $\gamma' = \frac{1}{2k}$. In the quantum version, we need to redefine it using normal ordering of the currents, and the renormalisation of parameter γ due to the quantum effects:

$$T(z) = \gamma (J^a J^a)(z) \tag{66}$$

The normal-ordering is defined as the contour integral over regular part of the OPE:

$$(J^a J^a)(z) \equiv \frac{1}{2\pi i} \oint_z \frac{\mathrm{d}x}{x-z} J^a(x) J^a(z) \tag{67}$$

We wish to prove that the energy-momentum tensor satisfies the Virasoro algebra, in terms of OPE:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{68}$$

First, we compute the OPE between $J^a(z)$ and $T(w)$, considering the singular part, denoted by contraction:

$$\begin{aligned}
&J^a(z)T(w) \\
=&\frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} J^a(z)\left(J^b(x) J^b(w)\right) && (69)\\
=&\frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \left(J^a(z) J^b(x) J^b(w) + J^b(x) J^a(z) J^b(w)\right) && (70)\\
=&\frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \left(\left[\frac{k\delta^{ab}}{(z-x)^2} + \frac{if^{ab}{}_c J^c(x)}{z-x}\right] J^b(w) + J^b(x)\left[\frac{k\delta^{ab}}{(z-x)^2} + \frac{if^{ab}{}_c J^c(x)}{z-x}\right]\right) && (71)\\
=&\frac{\gamma}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \left(\frac{k\delta^{ab} J^b(w)}{(z-x)^2} + \frac{if^{ab}{}_c}{z-x}\left[\frac{k\delta^{cb}}{(x-w)^2} + \frac{if^{cb}{}_d J^d(w)}{x-w} + (J^c J^b)(w) + (\partial J^c J^b)(w) + \ldots\right]\right. \\
&\left. + \frac{k\delta^{ab} J^b(x)}{(z-w)^2} + \frac{if^{ab}{}_c}{z-w}\left[\frac{k\delta^{bc}}{(x-w)^2} + \frac{if^{bc}{}_d J^d(w)}{x-w} + (J^b J^c)(w) + (\partial J^b J^c)(w) + \ldots\right]\right). && (72)
\end{aligned}$$

Here, both regular and singular term appears in the JJ's OPE. Since δ^{bc} is symmetric and $f^{ab}{}_c$ is totally anti-symmetric, the second order pole is canceled out. Furthermore, the sum of normal-order terms (pairwise) $(J^c J^b)(w) + (J^c J^b)(w)$, $(\partial J^c J^b)(w) + (\partial J^b J^c)(w)$, etc. are symmetric, they all varnish by the anti-symmetry in b and c of the structure constant. The term $\frac{\gamma}{2\pi i}\oint_w \frac{\mathrm{d}x}{x-w}\frac{if^{ab}{}_c}{z-w}\frac{if^{bc}_d J^d(w)}{x-w}$ is a pure second order pole and has no residue, so it varnishes by complex analysis. We are only left with:

$$\overbrace{J^a(z)T}(w) = \gamma\left(\frac{2k\delta^{ab}J^b(w)}{(w-z)^2} - \frac{f^{ab}{}_c f^{cb}{}_d J^d(w)}{(w-z)^2}\right).$$

Recall the definition of the quadratic Casimir of the adjoint representation, the Coxeter number $h^\vee$:

$$f^{ab}{}_c f^{bc}{}_d = 2h^\vee\delta^{ad}. \tag{73}$$

The result reads, up to the singular terms:

$$\begin{aligned}\overbrace{T(z)J^a}(w) = \overbrace{J^a(w)T}(z) &\sim 2\gamma(k+h^\vee)\frac{J^a(z)}{(w-z)^2} &(74)\\ &\sim 2\gamma(k+h^\vee)\left(\frac{J^a(w)}{(z-w)^2}+\frac{\partial J^a(w)}{z-w}\right). &(75)\end{aligned}$$

This is the OPE of energy-momentum tensor T with a current of a primary field $J^a(z)$, which has conformal weight one. We fixed γ to be:

$$\gamma \equiv \frac{1}{2(k+h^\vee)}. \tag{76}$$

Note that in the classical case, $\gamma = \frac{1}{2k}$. Here, the quantum version yields an extra correction given by the Coexter number (we can also think of putting an $\hbar$ next to $h^\vee$ for this quantum correction).
Now we have all the tools to compute TT's OPE:

$$\begin{aligned}\overbrace{T(z)T}(w) =& \frac{1}{4\pi i(k+h^\vee)}\oint\frac{\mathrm{d}x}{x-w}\left(\overbrace{T(z)J^a}(x)J^a(w) + \overbrace{T(z)J^a(x)J^a}(w)\right) &(77)\\ =& \frac{1}{4\pi i(k+h^\vee)}\oint\frac{\mathrm{d}x}{x-w}\Bigg(\left[\frac{J^a(x)}{(z-x)^2}+\frac{\partial J^a(x)}{z-x}\right]J^a(w)\\ &+ J^a(x)\left[\frac{J^a(w)}{(z-w)^2}+\frac{\partial J^a(w)}{z-w}\right]\Bigg) &(78)\\ =& \frac{1}{4\pi i(k+h^\vee)}\oint\frac{\mathrm{d}x}{x-w}\Bigg(k\dim\mathfrak{g}\bigg(\frac{1}{(z-x)^2(x-w)^2}\\ &- \frac{2}{(z-x)(x-w)^3}+\frac{1}{(x-w)^2(z-w)^2}+\frac{2}{(z-x)(x-w)^3}\bigg)\\ &+ \frac{(J^aJ^a)(w)}{(z-x)^2}+\frac{(\partial J^aJ^a)(w)}{z-x}+\frac{(J^aJ^a)(w)}{(z-w)^2}+\frac{(J^a\partial J^a)(w)}{z-w}\Bigg) &(79)\\ =& \frac{k\dim\mathfrak{g}}{2(k+h^\vee)(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}. &(80)\end{aligned}$$

We see that $T(z)T(w)$ obeys Virasoro algebra, with central charge:

$$c \equiv \frac{k \dim \mathfrak{g}}{k + h^\vee}. \tag{81}$$

We have derived the Virasoro algebra in the language of OPE. To see it in more explicit form (of Eq. (210) in the Appendix A), we rewrite the normal ordering in terms of mode. We denote:

$$L_n \equiv \gamma \left(\sum_{m \leq -1} J^a_m J^a_{n-m} + \sum_{m \geq 0} J^a_{n-m} J^a_m \right) = \gamma \sum_{m \in \mathbb{Z}} : J^a_m J^a_{n-m} : . \tag{82}$$

Analogous to the OPE calculation process, we first compute the commutation relation between L_n and J^a_m:

$$\begin{aligned}
&[L_n, J^a_m] \\
=&\gamma \left[\sum_{l \leq -1} J^b_l J^b_{n-l} + \sum_{l \geq 0} J^b_{n-l} J^b_l, J^a_m \right] && (83) \\
=&\gamma \Bigg(\sum_{l \leq -1} \left(J^b_l \left(k(n-l) \delta^{ab} \delta_{n+m-l,0} + i f^{ba}{}_c J^c_{n+m-l} \right) + \left(k l \delta^{ab} \delta_{l+m,0} + i f^{ab}{}_c J^c_{l+m} \right) J^b_{n-l} \right) \\
&+ \sum \left(J^b_{n-l} (k l \delta^a b \delta_{l+m,0} + i f^{ab}{}_c J^c_{l+m}) + \left(k(n-l) \delta^{ab} \delta_{n+m-l,0} + i f^{ab}{}_c J^c_{n+m-l} \right) J^b_l \right) \Bigg). && (84)
\end{aligned}$$

The central term reads:

$$\begin{aligned}
[L_n, J^a_m]_{\text{central}} &= \gamma \left(\sum_{l \in \mathbb{Z}} k(n-l) \delta^{ab} J^b_l \delta_{n+m-l,0} + \sum_{l \in \mathbb{Z}} k l \delta^{ab} J^b_{n-l} \delta_{l+m,0} \right) && (85) \\
&= -2 \gamma k m J^a_{n+m}. && (86)
\end{aligned}$$

The non-central term reads:

$$\begin{aligned}
[L_n, J^a_m]_{\text{non-central}} =& \gamma i \Bigg(\sum_{l \leq -1} f^{ba}{}_c J^b_l J^c_{n+m-l} + \sum_{l \geq 0} f^{ba}{}_c J^c_{n+m-l} J^b_l + \sum_{l \leq m-1} f^{ba}{}_c J^c_l J^b_{n+m-l} \\
&+ \sum_{l \geq m} f^{ba}{}_c J^b_{n+m-l} J^c_l \Bigg) && (87) \\
=& i f^{ba}{}_c (J^b J^c)_{n+m} + i f^{ba}{}_c (J^c J^b)_{n+m} + \sum_{l=0}^{m-1} i f^{ba}{}_c \left[J^c_l, J^b_{n+m-l} \right] && (88) \\
=& \gamma \sum_{l=0}^{m-1} i f^{ba}{}_c \left(k l \delta^{bc} \delta_{n+m,0} + i f^{cb}{}_d J^d_{n+m} \right) && (89) \\
=& \gamma n f^{ab}_c f^{cb}{}_d J^d_{n+m} && (90) \\
=& -2 \gamma h^\vee n J^a_{n+m}. && (91)
\end{aligned}$$

Taking the sum of both terms yields:

$$[L_n, J^a_m] = -2\gamma(k + h^\vee)mJ^a_{n+m} = -mJ^a_{n+m}. \tag{92}$$

We derive the algebra:

$$\begin{aligned}
&[L_n, L_m] \\
=&\gamma \left[L_n, \sum_{l\le -1} J^a_l J^a_{n-l} + \sum_{l\ge 0} J^a_{n-l} J^a_l\right] && (93)\\
=&\gamma \left(\sum_{l\le -1}\left(-lJ^a_{n+l}J^a_{m-l} - (m-l)J^a_l J^a_{n+m-l}\right) + \sum_{l\ge 0}\left(-lJ^a_{n+l}J^a_{m-l} - (m-l)J^a_l J^a_{n+m-l}\right)\right) && (94)\\
=&\gamma\Bigg(\sum_{l\le n-1}(n-l)J^a_l J^a_{n+m-l} - \sum_{l\le -1}(m-l)J^a_l J^a_{n+m-l} + \sum_{l\ge n}(n-l)J^a_{n+m-l}J^a_l && (95)\\
&-\sum_{l\ge 0}(m-l)J^a_{n+m-l}J^a_l)\Bigg) && (96)\\
=&\gamma(n-m)\left(\sum_{l\le -1} J^a_l J^a_{n+m-l} + \sum_{l\ge 0} J^a_{n+m-l}J^a_l\right) + \gamma\sum_{l=0}^{n-1}(n-l)\left[J^a_l, J^a_{n+m-l}\right] && (97)\\
=&(n-m)L_{n+m} + \gamma\sum_{l=0}^{n-1}(n-l)lk\delta^{aa}\delta_{n+m,0} && (98)\\
=&(n-m)L_{n+m} + \frac{k}{6}\gamma\dim(\mathfrak{g})n(n^2-1)\sigma_{n+m,0}. && (99)
\end{aligned}$$

With the central charge defined in Eq. (81) as:

$$c = \frac{k\dim\mathfrak{g}}{k+h^\vee} = 2k\gamma\dim(\mathfrak{g}), \tag{100}$$

we see that the commutation relation satisfies the Virasoro algebra (denoted by Vir).

$$[L_n, L_m] =(n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{101}$$

Similar to the case of current algebra, the energy momentum tensors satisfying infinite-dimensional Virasoro algebra goes along with the fact that we have infinitely many conserved energy-momentum tensors (as well as their associated conserved charges), generated by multiplying one with arbitrary (anti)holomorphic functions. Together with the relations we have derived:

$$[J^a_n, J^b_m] = if^{ab}{}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}, \tag{102}$$
$$[L_n, J^a_m] = -mJ^a_{n+m}, \tag{103}$$

we see that the Virasoro algebra naturally arises as subalgebra contained in the universal enveloping algebra of the affine Kac-Moody algebra, which is the semi-direct product of the two Vir $\ltimes$ $\hat{\mathfrak{g}}_k$. Furthermore, the Eq. (103) illustrates that affine Kac-Moody algebra is a Lie ideal of this universal enveloping algebra. We see that this rich mathematical structures provide infinite symmetries (along with the infinite numbers of conserved currents/energy-momentum tensors) lead to the solvability of WZW models.

3.4 Free field construction

In this subsection, we consider the representation of n-free fermions and prove that it is just the same theory with $\mathfrak{so}(n)_1$ of the WZW model. It illustrates the fact that different CFTs can become equivalent at the quantum level, and is an example of the occurrence called *"non-abelian bosonization"*.

First, we consider the action of n massless (due to conformal invariance condition) free fermion:

$$S = \int_{S^2} \mathrm{d}^2 z \bar{\psi}^i \gamma^\mu \partial_\mu \psi^i, \quad (i = 1, ..., n). \tag{104}$$

This action has $SO(n)$ symmetry. Hence, there will be conserved currents associated with holomorphic and antiholomorphic fields $\psi, \bar{\psi}$ respectively:

$$J^a(z) = \frac{1}{2} T^a_{ij} (\psi^i \psi^j)(z), \tag{105}$$

$$\bar{J}^a(\bar{z}) = \frac{1}{2} T^a_{ij} (\bar{\psi}^i \bar{\psi}^j)(\bar{z}), \tag{106}$$

where T^a_{ij} are generators of fundamental representation of $\mathfrak{so}(n)$. We shall prove that the currents satisfy Kac-Moody algebra as in Eq. (62). First, we have the OPE:

$$\psi^i(z)\psi^j(w) \sim \frac{\delta^{ij}}{z-w}. \tag{107}$$

The normal ordered products of fields expressed in terms of OPE as:

$$J^a(z) = \frac{1}{2} T^a_{ij} \oint_z \frac{\mathrm{d}x}{2\pi i(x-z)} \psi^i(x)\psi^j(z). \tag{108}$$

We derive the OPE between $J^a(z)$ and $\psi^j(w)$ as:

$$\begin{aligned}
J^a(z)\psi^j(w) &\sim \frac{T^a_{ij}}{2} \oint_z \frac{\mathrm{d}z}{2\pi i(x-z)} \psi^i(x)\psi^j(z)\psi^l(w) && (109)\\
&= \frac{T^a_{ij}}{2} \oint_z \frac{\mathrm{d}z}{2\pi i(x-z)} \left(-\frac{\delta^{il}\psi^j(z)}{z-w} + \frac{\delta^{jl}\psi^i(x)}{z-w} \right) && (110)\\
&= \frac{-T^a_{lj}\psi^j(z) + T^a_{il}\psi^i(w)}{2(z-w)} && (111)\\
&= -\frac{T^a_{lj}\psi^j(w)}{z-w}. && (112)
\end{aligned}$$

Hence, OPE between currents reads:

$$\begin{aligned}
J^a(z)J^b(w) &\sim \frac{T^b_{ij}}{2}\oint_w \frac{\mathrm{d}x}{2\pi i(x-z)} J^a(z)\psi^i(x)\psi^j(w) && (113)\\
&= \frac{T^b_{ij}}{2}\oint_w \frac{\mathrm{d}x}{2\pi i(x-z)} \left(-\frac{T^a_{il}\psi^l(x)\psi^j(w)}{z-x} - \frac{T^a_{jl}\psi^i(x)\psi^l(w)}{z-w}\right) && (114)\\
&= \frac{T^b_{ij}}{2}\oint_w \frac{\mathrm{d}x}{2\pi i(x-z)} \Bigg(-\frac{T^a_{il}\delta^l j}{(z-x)(x-w)} - \frac{T^a_{jl}\delta^{il}}{(z-w)(x-w)} \\
&\quad - \frac{T^a_{il}(\psi^l\psi^j)(w)}{z-x} - \frac{T^a_{jl}(\psi^i\psi^l)(w)}{z-w}\Bigg) && (115)\\
&\sim -\frac{T^b_{ij}T^a_{ij}}{2(z-w)^2} + \frac{T^b_{ij}T^a_{il}(\psi^l\psi^j)(w)}{2(z-w)} - \frac{T^b_{ij}T^a_{jl}(\psi^i\psi^j)(w)}{2(z-w)} && (116)\\
&= \frac{\operatorname{tr}\left(T^aT^b\right)}{2(z-w)^2} + \frac{\left[T^a,T^b\right]_{jl}(\psi^j\psi^l)(w)}{2(z-w)} && (117)\\
&= \frac{\operatorname{tr}\left(T^aT^b\right)}{2(z-w)^2} + \frac{if^{ab}{}_cT^c_{jl}(\psi^j\psi^l)(w)}{2(z-w)}. && (118)
\end{aligned}$$

We have the quadratic casimir $(T^aT^a)_{ij} = \mathcal{C}(\lambda)\delta_{ij}$. Taking the trace, we deduce that $\operatorname{tr}(T^aT^a) = \mathcal{C}(\lambda)\dim(\lambda)$. Hence, $\operatorname{tr}\left(T^aT^b\right)$ should be proportional to δ^{ab}, expressed as:

$$\operatorname{tr}\left(T^aT^b\right) = \frac{\mathcal{C}(\lambda)\dim(\lambda)}{\dim(\mathfrak{g})}\delta^{ab}. \tag{119}$$

Substituting into Eq. (118), we have:

$$J^a(z)J^b(w) \sim \frac{\mathcal{C}(\lambda)\dim(\lambda)}{2\dim(\mathfrak{g})}\frac{\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_cJ^c(w)}{z-w}, \tag{120}$$

Here, we can draw a more general conclusion that the current J^a satisfies the current algebra at level $k = \frac{\mathcal{C}(\lambda)\dim(\lambda)}{2\dim(\mathfrak{g})}$. Particularly, for fundamental representation of $\mathfrak{so}(n)$, we have $k = \frac{(n-1)n}{n(n-1)} = 1$. We see that the fermionic action of (104) describes the same physics with bosonic WZW action. This is an example of non-abelian bosonization, a procedure that uses (non-local) transformations to transform local fermionic fields to bosonic fields, which obtains all symmetries of the original theory [2].

3.5 Representation

Let $|\lambda\rangle$ be the Primary field of current algebra. We want all positive current mode annihilate the state:

$$J^a_n|\lambda\rangle = 0, \qquad \forall n > 0. \tag{121}$$

We have several zero-modes which satisfy the Lie Algebra in Eq. (64), and hence, form a representation on $|\lambda\rangle$. We hence, impose the condition that $|\lambda\rangle$ is the highest weight state

of the highest weight representation with respect to the zero-mode algebra.
We consider a particular example of $\mathfrak{su}(2)_k$ with three oscilator J_n^3, J_n^+, J_n^- satisfying:

$$\left[J_m^+, J_n^+\right] = \left[J_m^-, J_n^-\right] = 0, \tag{122}$$
$$\left[J_n^3, J_n^\pm\right] = \pm J_{n+n}^\pm, \tag{123}$$
$$\left[J_m^3, J_n^3\right] = \frac{k}{2} n \delta_{m+n,0}, \tag{124}$$
$$\left[J_m^+, J_n^-\right] = km\delta_{m+n,0} + 2J_{m+n}^3. \tag{125}$$

Rewriting the condition (121) for $\mathfrak{su}(2)$, with the representation labeled by the spins l (more generally, λ can be thought of the Dynkin labels):

$$J_n^{3,\pm} |l\rangle = 0, \qquad \forall n > 0. \tag{126}$$

Since $|l\rangle$ is the highest weight state, we also have $J_0^+ |l\rangle = 0$, and $J_0^3 |l\rangle = l |l\rangle$. and $J_0^- |l\rangle$ is another state in the $\mathfrak{su}(2)$ representation, which we call *decendants.* With those constraints, we can act all the possible oscillators to $|l\rangle$ to generate all possible descendants:

$$\left(\prod_{i=1}^{\infty} (J_{-i}^-)^{n_i^-} (J_{-i}^3)^{n_i^3} (J_{-i}^+)^{n_i^+}\right) (J_0^-)^{n_0^-} |l\rangle \,, \tag{127}$$

with all $n_i \geq 0$. This construction forms the *"Verma module"* of the representation. This Verma module has *"null vectors"* - which are also primary fields. An trivial example of them is when we run down the global $\mathfrak{su}(2)$ representation and hit the lowest weight state of $(J_0^-)^{2l} |l\rangle$, with $J_0^- \left((J_0^-)^{2l} |l\rangle\right) = 0$, with the condition that l is a positive half-integer. Here we note an intriguing point that if l is not a half-integer, then we would run down with the lowering operators to $-\infty$, which causes the representation to be non-unitary. However, in this essay, we only consider unitary WZW models. In the context of unitary CFT, the null vectors are important since they restrict our fusion rules, and are a good sign for the theory that being a *Rational CFT*, as we shall see later in our theory.
For later convenience, we introduce the hermitian properties:

$$(J_n^+)^\dagger = J_{-n}^-, \qquad\qquad (J_n^3)^\dagger = J_{-n}^3. \tag{128}$$

Now, we consider a non-trivial null vector $\mathcal{N} = (J_{-1}^+)^{k+1-2l} |l\rangle$. To show that $|\mathcal{N}\rangle$ is a primary state, we will show that it is annihilated by all positive modes. However, we indeed only need to show that it is annihilated by mode 1 since other modes can be expressed as commutation relation of mode 1, for example: $J_2^3 = \frac{1}{2}[J_1^+, J_1^-]$. Since J_0^+, J_1^+ commutes with J_{-1}^+, we first have the trivial result:

$$J_0^+ |\mathcal{N}\rangle = J_1^+ |\mathcal{N}\rangle = 0. \tag{129}$$

Next, we check:

$$
\begin{aligned}
J_1^3 |\mathcal{N}\rangle =& [J_1^3, (J_{-1}^+)^{k+1-2l}] |l\rangle & (130)\\
=& \sum_{n=0}^{k-2l} (J_{-1}^+)^n [J_1^3, J_{-1}^+] (J_{-1}^+)^{k+2l-n} |l\rangle & (131)\\
=& \sum_{n=0}^{k-2l} (J_{-1}^+)^n J_0^+ (J_{-1}^+)^{k+2l-n} |l\rangle & (132)\\
=& \sum_{m=0}^{k-2l} (J_{-1}^+)^n (J_{-1}^+)^{k+2l-n} J_0^+ |l\rangle = 0. & (133)
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
J_1^- |\mathcal{N}\rangle &= \sum_{n=0}^{k-2l} (J_{-1}^+)^n [J_1^-, J_{-1}^+] (J_{-1}^+)^{k-2l-n} |l\rangle & (134)\\
&= \sum_{n=0}^{k-2l} (J_{-1}^+)^n (k - 2J_0^3) (J_{-1}^+)^{k-2l-n} |l\rangle & (135)\\
&= \sum_{n=0}^{k-2l} (J_{-1}^+)^n (k - 2(l + k - 2l - n)) (J_{-1}^+)^{k-2l-n} |l\rangle & (136)\\
&= \sum_{n=0}^{k-2l} (J_{-1}^+)^n (2l - k + 2n) (J_{-1}^+)^{k-2l-n} |l\rangle & (137)\\
&= \left[(2l-k)(k-2l+1) + (k-2l)(k-2l+1)\right] (J_{-1}^+)^{k-2l} |l\rangle = 0. & (138)
\end{aligned}
$$

We have found the null vector $|\mathcal{N}\rangle$, now we concern about the unitarity property of the theory. Considering the norm $\left|(J_{-1}^+)^N |l\rangle\right|^2$, with normalised $|l\rangle$, we shall prove by induction that $\left|(J_{-1}^+)^N |l\rangle\right|^2 = \prod_{n=1}^{N} n(k+1-n-2l)$.. First, it is trivial that the equation is true for

$N = 0$. Suppose it is also true for $N - 1$, we have:

$$\begin{aligned}
\left|(J_{-1}^{+})^{N}\left|l\right>\right|^{2} =& \left<l\right|(J_{1}^{-})^{N}(J_{-1}^{+})^{N}\left|l\right> & (139)\\
=& \left<l\right|(J_{1}^{-})^{N-1}[J_{1}^{-},(J_{-1}^{+})^{N}]\left|l\right> & (140)\\
=& \sum_{n=0}^{N-1}\left<l\right|(J_{1}^{-})^{N-1}(J_{-1}^{+})^{n}(k-2J_{0}^{3})(J_{-1}^{+})^{N-1-n}\left|l\right> & (141)\\
=& \sum_{n=0}^{N-1}\left<l\right|(J_{1}^{-})^{N-1}(J_{-1}^{+})^{n}\left(k-2(l+N-1-n)\right)(J_{-1}^{+})^{N-1-n}\left|l\right> & (142)\\
=& \sum_{n=0}^{N-1}\left(k-2(l+N-1-n)\right)\left<l\right|(J_{1}^{-})^{N-1}(J_{-1}^{+})^{n}(J_{-1}^{+})^{N-1-n}\left|l\right> & (143)\\
=& N(k+1-N-2l)\prod_{n=1}^{N-1}n(k+1-n-2l) & (144)\\
=& \prod_{n=1}^{N}n(k+1-n-2l). & (145)
\end{aligned}$$

If $N = k + 1 - 2l$, the norm is zero, which confirms the result for our null vector $|\mathcal{N}\rangle$ having the zero norm. If we go on putting more descendants with higher N, the norm still stays zero. However, if there was no null vector where the norm varnishes, the norm would become negative at some point keeping increase N. Hence if we want a unitary representation of the algebra, the only chance is to constraint the $|\mathcal{N}\rangle$ to be the null vector, the first constrain we can think of is $k + 1 - 2l \in \mathbb{Z}$, which implies $k \in \mathbb{Z}$ (since l is a half-integer). Note that we have drawn this constraint when constructing the action. Here, we re-derive exactly the same conclusion from the perspective of representation theory.

However, that is not the only conclusion. We further require that the null vector appears at the positive level, which means imposing the condition $k + 1 - 2l > 0 \iff l \leq k/2$, otherwise, it would be already 0 at the first excitation. Therefore, in $\mathfrak{su}2$ WZW model, the spin can not be arbitrarily high but is bounded in ground state representation. We hence draw the important conclusion that unitarity implies that there are finitely many representations, which defines the *"Rational CFT"*. The term "Rational" means that the Central charges and conformal weights in the CFT rational numbers.

We can generalise this result. Consider the highest weight states $|\lambda\rangle$ in affine Kac-Moody algebra. We want to show its connection to the Virasoro representation theory. First, consider the affine Kac-Moody descendants:

$$L_n\left|\lambda\right> = \gamma(J^aJ^a)_n\left|\lambda\right> = \left(\sum_{m\leq -1}J_m^aJ_{n-m}^a + \sum_{m\geq 0}J_{n-m}^aJ_m^a\right)\left|\lambda\right>. \tag{146}$$

With $n > 0$, the positive J operator is always on the right and eliminate the $|\lambda\rangle$. Hence, $L_n|\lambda\rangle = 0, \forall n > 0$. Which means that affine Kac-Moody descendant λ is also the Virasoro primary. Therefore, there are many Virasoro primaries in the Verma module. We then determine the conformal weight for the lowest weight states λ:

$$L_0 |\lambda\rangle = \gamma(J^a J^a)_0 |\lambda\rangle = \frac{1}{2(k+h^\vee)} J_0^a J_0^a |\lambda\rangle = \frac{1}{2(k+h^\vee)} \mathrm{Cas}(\lambda) |\lambda\rangle , \tag{147}$$

where $\mathrm{Cas}(\lambda)$ is the Casimir of representation. In particular, the Casimir for $\mathfrak{su}(2)$ representation is $\mathrm{Cas}(l) = 2l(l+1)$. The conformal weight in $\mathfrak{su}(2)$ is then $\Delta_l = \frac{l(l+1)}{k+2}$ as $h^\vee = 2$. When we know the conformal weight of the highest weight state, we can derive the conformal weigh of any decendant since the oscillators change the conformal weight by an integer. To be explicit:

$$\Delta(J_{-n_1}^{a_1} ... J_{-n_m}^{a_m} |\lambda\rangle) = \frac{\mathrm{Cas}(\lambda)}{2(k+h^\vee)} + \sum_{i=1}^{m} n_i. \tag{148}$$

3.6 Knizhnik–Zamolodchikov equations

One of the important results of the Sugawara construction is Knizhnik–Zamolodchikov equations. Consider the field ϕ_λ is the field corresponds to primary state $|\lambda\rangle$. Since λ is primary, only the field that associates to the state $J_{p-1}^a |\lambda\rangle$ with $p = 1$ survive. Hence we have the OPE:

$$J^a(z)\phi_\lambda(w) \sim \frac{t^a \phi_\lambda}{(z-w)} \tag{149}$$

where t^a are matrices of the representation of the zero-modes. Considering the OPE between $(J^a J^a)_{-1}$ and ϕ^{λ_i}, we compute the first order pole:

$$(J^a J^a)_{-1} \phi^{\lambda_i}(z_i) = \oint_{z_i} \frac{\mathrm{d}z}{2\pi i} (J^a J^a)(z) \phi^{\lambda_i}(z_i) \tag{150}$$

$$= \oint_{z_i} \frac{\mathrm{d}z}{2\pi i} \oint_{z} \frac{\mathrm{d}x}{2\pi i(x-z)} J^a(x) J^a(z) \phi_i^\lambda(z_i). \tag{151}$$

Encircling the complement of z, the singularity from z_i reads:

$$\langle \phi^{\lambda_1}(z_1) ... (J^a J^a)_{-1} \phi^{\lambda_i}(z_i) ... \phi^{\lambda_n}(z_n) \rangle$$

$$= -\sum_{j=1}^{n} \oint_{z_i} \frac{dz}{2\pi i} \oint_{z_j} \frac{dx}{2\pi i(x-z)} \langle J^a(x) J^a(z) \phi^{\lambda_1}(z_1) ... \phi^{\lambda_n}(z_n) \rangle \tag{152}$$

$$= -\sum_{j=1}^{n} \oint_{z_i} \frac{dz}{2\pi i} \oint_{z_j} \frac{dx}{2\pi i(x-z)} \frac{t_j^a}{x-z_j} \langle J^a(z) \phi^{\lambda_1}(z_1) ... \phi^{\lambda_n}(z_n) \rangle \tag{153}$$

$$= -\sum_{j=1}^{n} \oint_{z_i} \frac{dz}{2\pi i} \frac{t_i^a \otimes t_j^a}{(z_i - z)(z_j - z_i)} \langle \phi^{\lambda_1}(z_1) ... \phi^{\lambda_n}(z_n) \rangle \tag{154}$$

$$= \sum_{j=1}^{n} \frac{t_i^a \otimes t_j^a}{z_i - z_j} \langle \phi^{\lambda_1}(z_1) ... \phi^{\lambda_n}(z_n) \rangle. \tag{155}$$

We use the Sugawara construction to derive the analog of BPZ-equation for WZW models. The identity between mode and OPE expression read $L_{-1} = \frac{(J^aJ^a)_{-1}}{k+h^\vee}$:

$$0 = \left\langle \phi^{\lambda_1}(z_1) \ldots \left(L_{-1} - \frac{(J^aJ^a)_{-1}}{k+h^\vee} \right) \phi^{\lambda_i}(z_i) \ldots \phi^{\lambda_n}(z_n) \right\rangle . \tag{156}$$

L_{-1} can be identified with the infinitesimal translation operator ∂_{z_1}. We deduced that the correlator of primary fields follows Knizhnik-Zamolodchikov (KZ) equation:

$$\left(\partial_{z_i} - \frac{1}{k+h^\vee} \sum_{j=1}^{n} \frac{t_i^a \otimes t_j^a}{z_i - z_j} \right) \langle \phi^{\lambda_1}(z_1) \ldots \phi^{\lambda_n}(z_n) \rangle = 0. \tag{157}$$

The KZ equation shows the constraints of n-point functions. Solving it is a challenging task, but it is a great way to calculate 4-point correlation functions [3].

4 Correspondence to Chern-Simons theory

One incarnation of the holographic principle is the correspondence between 2D WZW model, (as the boundary field theory) and 2+1 dimension Chern-Simons (CS) theory (as the bulk field theory). In this section, we will illustrate how states in the Hilbert space of the canonical quantization Chern-Simons theory corresponds to correlation functions of the WZW model. First, we introduce the gauge CS action:

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} \operatorname{tr} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right), \tag{158}$$

with A is the connection of the principal G-bundle on $\mathcal{M}$. This theory is an example of topological field theory since the action is written in differential form and hence, is independent of the metric. Here, similar to WZ term in WZW model, since $\pi(G) = \mathbb{Z}$, the gauge invariance of compact and simple group G implies k is an integer. Varying the action yields:

$$\begin{aligned} \delta S_{CS} &= \frac{k}{4\pi} \int_{\mathcal{M}} \operatorname{tr} \left(A \wedge \mathrm{d}\delta A + \mathrm{d}A \wedge \delta A + 2\delta A \wedge \mathrm{d}A + 2\delta A \wedge A \wedge A \right) && (159) \\ &= \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{d}(A \wedge \delta A) + \frac{k}{2\pi} \int_{\mathcal{M}} \delta A \wedge (\mathrm{d}A + A \wedge A) && (160) \\ &= \frac{k}{4\pi} \int_{\partial\mathcal{M}} A \wedge \delta A + \frac{k}{2\pi} \int_{\mathcal{M}} \delta A \wedge F. && (161) \end{aligned}$$

Now we want to (canonically) quantise our theory. We take the manifold as a direct product $\mathcal{M} = \mathbb{R} \times \Sigma$, where Σ is 2D Riemann surface, and $\mathbb{R}$ represents time. We make a temporal decomposition $d = \partial_0 \mathrm{d}x^0 + \tilde{\mathrm{d}}$ with $\tilde{\mathrm{d}} = \partial_i \mathrm{d}x^i$, and $A = A_0 \mathrm{d}t + \tilde{A}$, where $\tilde{A} = A_i \mathrm{d}x^i$ (with $i = 1, 2$).

$$S_{CS} = -\frac{k}{4\pi} \int_{\mathcal{M}} \operatorname{tr}\left(\tilde{A} \wedge \partial_t \tilde{A} \right) \wedge \mathrm{d}t + \frac{k}{2\pi} \int_{\mathcal{M}} \operatorname{tr}\left(A_0 \mathrm{d}t \wedge \tilde{F} \right), \tag{162}$$

with $\tilde{F} \equiv \tilde{\mathrm{d}}\tilde{A} + \tilde{A} \wedge \tilde{A} = 0$ is the curvature of vector bundle over Σ. Or in another form:

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \mathrm{tr}\big(\epsilon^{ij} \left(A_i \partial_t A_j + A_0 F_{ij}\right)\big). \tag{163}$$

with $F_{ij} \equiv \partial_i A_j - \partial_j A_i + [A_i, A_j]$. In this action, A_0 serves as the Lagrange multiplier, we fix the axial gauge as $A_0 = 0$ on the whole manifold to enforce the constraint $F_{ij} = 0$. We get the action:

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \mathrm{tr}\big(\epsilon^{ij} A_i \partial_t A_j\big). \tag{164}$$

We see that this action is of the form in phase space $S = \int p_i \partial_t q^i$, hence the two gauge connection A_i and $\frac{k}{\pi}\epsilon^{ij} A_j$ is canonical conjugate to each other. We extract the Poisson bracket:

$$\big\{A_i^a(x), A_j^b(y)\big\} = \frac{4\pi}{k} \delta^{ab} \epsilon_{ij} \delta^2(x-y). \tag{165}$$

We promote the commutation relation:

$$\big[A_i^a(x), A_j^b(y)\big] = \frac{\pi i}{k} \delta^{ab} \epsilon_{ij} \delta^2(x-y). \tag{166}$$

To make use of the complex structure $(z, \bar{z})$ (as a holomorphic quantization process) we transform the commutator to complex coordinate:

$$\big[A_z^a(z_1, \bar{z}_1), A_{\bar{z}}^b(z_2, \bar{z}_2)\big] = \frac{\pi}{k} \delta^{ab} \delta(z_1 - z_2) \delta(\bar{z}_1 - \bar{z}_2). \tag{167}$$

Now we regard the Hilbert space $\mathcal{H}$ as the space of all holomorphic functional $\Psi(A_{\bar{z}})$ and consider A_z as its functional derivatives.

$$A_z^a \Psi(A_{\bar{z}}) = \frac{\pi}{k} \frac{\delta}{\delta A_{\bar{z}}^a} \Psi(A_{\bar{z}}). \tag{168}$$

The action becomes:

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int \mathrm{d}t \int \mathrm{tr}(A_z \partial_t A_{\bar{z}} + A_t F_{z\bar{z}}). \tag{169}$$

And the Lagrange multiplier means $kF_{z\bar{z}} = 0$, we have

$$k\partial_z A_{\bar{z}} - k\partial_{\bar{z}} A_z + k[A_z, A_{\bar{z}}] = 0 \tag{170}$$

$$\Rightarrow k\partial_z A_{\bar{z}} - \pi \partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + \pi \left[\frac{\delta}{\delta A_{\bar{z}}}, A_{\bar{z}}\right] = 0. \tag{171}$$

Hence,

$$\left(\partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}\right]\right) \Psi(A_z) = \frac{k}{\pi} \partial_z A_{\bar{z}} \Psi(A_z). \tag{172}$$

Now we consider WZW theory. Consider the WZW action:

$$Z[A] = \left\langle \exp\left(\frac{1}{\pi} \int A^b_{\bar{w}} J^b_w \right) \right\rangle_{\mathrm{WZW}}, \tag{173}$$

We recall the Eq. (44):

$$J^a(z) J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}_c J^c(w)}{z-w}. \tag{174}$$

With the derivative of pole yields delta function,

$$\partial_{\bar{z}} \frac{1}{z-w} = \pi \delta^2(z-w), \tag{175}$$

Taking its derivatives, we have:

$$\partial_{\bar{z}} J^a(z) J^b(w) = k\pi \delta^{ab} \partial_z \delta^2(z-w) - i\pi f^{ab}{}_c \delta^2(z-w) J^c(w). \tag{176}$$

We also have:

$$\frac{\delta}{\delta A^a_{\bar{z}}} Z[A] = \left\langle \frac{1}{\pi} J^a(z) \exp\left(\frac{1}{\pi} \int A^c_{\bar{w}} J^c(w) \right) \right\rangle \tag{177}$$

$$= \frac{1}{\pi} J^a(z) Z[A]. \tag{178}$$

Hence,

$$\partial_{\bar{z}} \frac{\delta}{\delta A^a_{\bar{z}}} Z[A]$$

$$= \left\langle \frac{1}{\pi} \partial_{\bar{z}} J^a(z) \exp\left(\frac{1}{\pi} \int A^d_{\bar{w}} J^d(w) \right) \right\rangle \tag{179}$$

$$= \left\langle \frac{1}{\pi} \partial_{\bar{z}} J^a(z) \sum_{n \geq 0} \frac{1^n}{\pi^n n!} \sum_{i=1}^{n} \int A^{d_1}_{\bar{w}} \ldots A^{d_i}_{\bar{w}} \ldots A^{d_n}_{\bar{w}} : J^{d_1}(w) \ldots J^{d_i}(w) \ldots J^{d_n}(w) : \right\rangle \tag{180}$$

$$= \left\langle \frac{1}{\pi^2} \int \left[k\pi \delta^{ab} \partial_z \delta^2(z-w) - \pi f^{ab}{}_c \delta^2(z-w) J^c(w) \right] A^b_{\bar{w}} \sum_{n>0} \frac{n}{\pi^{n-1} n!} A^{d_1}_{\bar{w}} \ldots A^{d_{n-1}}_{\bar{w}} \right. \tag{181}$$

$$\left. : J^{b_1}(w) \ldots J^{b_{n-1}}(w) : \right\rangle \quad (\text{with index } b \equiv d_i) \tag{182}$$

$$= \left\langle \left(\frac{k}{\pi} \partial_z A^a_{\bar{z}} - \frac{1}{\pi} f^{ab}{}_c J^c(z) A^b_{\bar{z}} \right) \sum_{m \geq 0} \frac{1}{\pi^m m!} A^{d_1}_{\bar{w}} \ldots A^{d_m}_{\bar{w}} : J^{d_1}(w) \ldots J^{d_m}(w) : \right\rangle \tag{183}$$

(with index $m \equiv n-1$)

$$= \left\langle \left(\frac{k}{\pi} \partial_z A^a_{\bar{z}} - f^{ab}{}_c \frac{\delta}{\delta A^c_{\bar{z}}}(z) A^b_{\bar{z}} \right) \exp\left(\frac{1}{\pi} A^d_{\bar{w}} J^d(w) \right) \right\rangle \tag{184}$$

$$= \left(\frac{k}{\pi} \partial_z A^a_{\bar{z}} - \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}} \right]^a \right) Z[A]. \tag{185}$$

We deduce the relation:

$$\left(\partial\frac{\delta}{\delta A_{\bar{z}}}+\left[A_{\bar{z}},\frac{\delta}{\delta A_{\bar{z}}}\right]\right)Z[A]=\frac{k}{\pi}\partial_z A_{\bar{z}}Z[A]. \tag{186}$$

We see that it yields back the Eq. (172). In context of WZW, we call this Ward-Takahashi identity. Furthermore, turning on A_z componnent, the partition function of WZW can be decompose into finite sum of holomorphic and anti-holomorphic component:

$$Z(A_z,A_{\bar{z}})=\sum_{\alpha}\Psi_\alpha(A_{\bar{z}})\bar{\Psi}_\alpha(A_z). \tag{187}$$

Each $\Psi(A_{\bar{z}})$, $\bar{\Psi}_\alpha(A_z)$ can be thought as wave function for the canonical quantization of WZW model on Σ. They are called *conformal blocks* and satisfy the Ward-Takahashi identity.

Hence we see a connection between the state vectors in Hilbert space of CS theory and the partition function of WZW models as a generating function of a given source (with the source is the connection on the 2D Riemann surface). In another word, there is an explicit correspondence between the wave function of the canonical quantization of the CS theory on $\mathbb{R}\times\Sigma$ corresponds to conformal blocks of the WZW theory on Σ. It is a peculiar connection between two seemingly different theories: The CS is a 2+1 dimensional, topological theory, whose action does not require a metric, while the WZW is a 2D theory whose action depends on complex structures of Riemann surface. This relation can be regarded as a prototype of AdS/CFT correspondence where a CFT in lower dimensions is related to gravity theory in higher dimensions.

Since we have known much about WZW theory, this connection can be exploited to calculate observables in CS theory, for example, the vacuum expectation values of n Wilson loops on S^3. By separating the Wilson loops through their boundary S^2 and then determining the transformation of conformal blocks under monodromy transformations, one can associate the n Wilson loops to n trivial knots and calculate their vacuum expectation values.

5 Conclusion and outlooks

We have studies the construction of WZW theory as a 2D CFT along with its rich underlying mathematical structures, and also, its correspondence to the quantised Chern-Simon theory. Among several results we have derived, the two most crucial points of our study are that:

- The separately conserved (anti)holomorphic currents/energy-momentum tensors in the WZW model give rise to infinite amounts of symmetries. This fact is followed by the underlying mathematical structure of infinite-dimensional affine Kac-Moody algebra and Virasoro algebra, which determines the exact solvability of the WZW models.

- Through the construction of free fermion field theory , we illustrated an example of the "non-abelian bosonization". By investigating the representation of the theory, we constructed the Verma module, finding the Null vectors and showed that the theory is indeed a Rational CFT.

- The connection to CS theory in 2+1 dimensions in which the wave functions in Hilbert space of the canonical quantised CS theory corresponds to conformal blocks of the WZW model, which can be utilized to compute CS theory's observables such as vacuum expectation values of n Wilson loops on S^3.

Let us mention a few topics that we have not had time to fully elaborate on:

- The formal mathematical structure of affine Kac-Moody algebra and the derivation of Virasoro algebra is described in Appendix A. Besides, one can further refer to [4] for the representation theory, character formula, weight system, unitarisability, etc. which is constructed in a purely mathematical viewpoint.

- We have mentioned briefly correlation functions of primary fields are given by solving the KZ equation. Although solving the differential equation is demanding, 4-point correlation functions are derived in this fashion. A nice discussion and detailed calculation for on them can be found in [3], [5].

- We have restricted the WZW models at positive integer levels. However, other levels can also appear in Non-unitary CFT. A particularly intriguing class of them is called *"admissible levels"*, which still has null vectors in the Verma module. A good reference can be found in [3].

Some further applications of WZW models include:

- WZW models whose Lie group are universal cover of $SL(2, R)$ are used to describe bosonic strings on AdS_3 [6], [7], [8].

- Plateau transition in the integer quantum Hall effect [9].

- $SL(2,\mathbb{R})/U(1)$ gauged WZW model as Witten's two dimensional Euclidean black holes [10].

Acknowledgements

It is a pleasure to thank Prof. David Skinner for his dedicated guidance. Thank also to all other lecturers in Part III of the Applied Mathematics and Theoretical Physics for their instructions. I also acknowledge the unconditional supports from my family, as well as the helpful discussions and supports from my friends, especially Mr. Tian Chen and Mr. Thong Nguyen.

A Mathematical construction of Affine Kac-Moody algebra

Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials $f = \sum_{k\in\mathbb{Z}} d_k t^k$. f is a linear functional on $\mathcal{L}$ with $\operatorname{Res} f := d_{-1}$. It is defined by the properties:

$$\operatorname{Res}\bigl(t^{-1}\bigr) = 1, \quad \operatorname{Res}\frac{\mathrm{d}f}{\mathrm{d}t} = 0. \tag{188}$$

We construct loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathcal{L}$ with the Lie algebra structure defined by:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg \qquad (X, Y \in \mathfrak{g}). \tag{189}$$

We define the central extension of loop algebra $\mathcal{L}\mathfrak{g}$ as $\hat{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$, with c is center of $\mathcal{L}\mathfrak{g}$, i.e. $[c, \xi] = 0, \forall \xi \in \mathcal{L}\mathfrak{g}$. We define the commutation relation for $\hat{\mathfrak{g}}$ (with $x, y \in \mathcal{L}\mathfrak{g}$, and $\alpha, \beta \in \mathbb{C}$) as:

$$[x + \alpha c, y + \beta c] = [x, y] + \sigma(x, y)c. \tag{190}$$

with the $\mathbb{C}$-valued 2-cocycle on the loop algebra $\mathcal{L}\mathfrak{g}$: $\mathcal{L}\mathfrak{g} \times \mathcal{L}\mathfrak{g} \to \mathbb{C}$ defined as:

$$\sigma(x, y) = \sigma(X \otimes f, Y \otimes g) \tag{191}$$

$$= \kappa(X, Y) \operatorname{Res}_{t=0} \left(\frac{\mathrm{d}f}{\mathrm{d}t} g \right), \tag{192}$$

with $\kappa(X, Y) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is the Cartan-Killing form of Lie algebra $\mathfrak{g}$. We prove that the 2-cocycle satisfy:

$$\begin{cases} \sigma(x, y) = -\sigma(y, x), \\ \sigma([x, y], z) + \sigma([y, z], x) + \sigma([z, x], y) = 0. \end{cases} \tag{193}$$

Proof:

$$\sigma(X \otimes f, Y \otimes g) + \sigma(Y \otimes g, X \otimes f) = (X, Y) \operatorname{Res}\left(\frac{\mathrm{d}f}{\mathrm{d}t} g + f \frac{\mathrm{d}f}{\mathrm{d}t} \right) \tag{194}$$

$$= \operatorname{Res}\left(\frac{\mathrm{d}(fg)}{dt} \right) = 0. \tag{195}$$

$$\sigma([X \otimes f, Y \otimes g], Z \otimes h) + \sigma([Y \otimes g, Z \otimes h], X \otimes f) + \sigma([Z \otimes h, X \otimes f], Y \otimes g)$$

$$= ([X, Y]Z) \operatorname{Res}\left(\frac{\mathrm{d}}{\mathrm{d}t}(fg)h \right) + ([Y, Z], X) \operatorname{Res}\left(\frac{\mathrm{d}}{\mathrm{d}t}(gh)f \right) + ([Y, Z], X) \operatorname{Res}\left(\frac{\mathrm{d}}{\mathrm{d}t}(hf)g \right) \tag{196}$$

$$= ([X, Y], Z) \operatorname{Res}\left(\frac{\mathrm{d}}{\mathrm{d}t}(fgh) \right) = 0. \tag{197}$$

Hence, we can rewrite the Lie bracket for $\hat{\mathfrak{g}}$ as:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \kappa(X, Y) \operatorname{Res}_{t=0} \left(\frac{\mathrm{d}f}{dt} g \right) c. \tag{198}$$

With the 2-cocycle $\sigma : \mathcal{L}\mathfrak{g} \times \mathcal{L}\mathfrak{g} \to \mathbb{C}$, we define the bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$: $\sigma_{m,n}(X, Y) = \sigma(X \otimes t^m, Y \otimes t^n)$. $\sigma_{m,n}$ is symmetric since Lie algebra $\mathfrak{g}$ is simple. Combining with 2 conditions for 2-cocycle in Eq. (193), we derive:

$$\begin{cases} \alpha_{m,n} = -\alpha_{n,m} \\ \alpha_{m+n,p} + \alpha_{n+p,m} + \alpha_{p+m,n} = 0. \end{cases} \tag{199}$$

Substituting $n = p = 0$, we have $\alpha_{m,0} = 0$. Substituting $p = -m-n$, we have $\alpha_{m+n,-m-n} = \alpha_{m,-m} + \alpha_{n,-n}$, hence $\alpha_{m,-m} = m\alpha_{1,-1}$. Putting $p = q-m-n$, we get $\alpha_{q-m-n,m+n} = \alpha_{q-m,m} + \alpha_{q-n,n}$, hence $\alpha_{q-k,k} = k\alpha_{q-1,1}$. For $k = 1$, this lead to $\alpha_{m,n} = 0$ when $m \neq -n$. We derive:

$$\sigma_{m,n} = m\delta_{m+n,0}\sigma_{1,-1}. \tag{200}$$

We define the affine Lie algebra associated with $\mathfrak{g}$ as the central extension $\hat{\mathfrak{g}}$ equipped with such 2-cocycle. Its Lie bracket reads:

$$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{n+m} + \kappa(X, Y)n\delta_{n+m,0}c. \tag{201}$$

By the construction of affine Lie algebra, we have the following exact sequence of the algebra:

$$0 \to \mathbb{R} \to \hat{\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0, \tag{202}$$

corresponding to the exact sequence of group:

$$1 \to U(1) \to \hat{G} \to LG \to 1, \tag{203}$$

where $LG = \{g : S^1 \to G\}$ is the loop group of G and $\hat{G}$ is a central extension of the loop group LG by the circle group $U(1)$. Now we shall also derive Virasoro algebra as the central extension of Witt algebra. Consider the algebra:

$$A = \left\{ f(z)\frac{d}{dz} \;\middle|\; f(z) \in \mathbb{C}\left[z, z^{-1}\right] \right\}. \tag{204}$$

It has the Lie Algebra structure, with the Lie bracket:

$$\left[f(z)\frac{\mathrm{d}}{dz}, g(z)\frac{\mathrm{d}}{dz}\right] = \left(f(z)g'(z) - g(z)f'(z)\right)\frac{\mathrm{d}}{dz}. \tag{205}$$

With basis $L_n = -z^{n+1}\frac{\mathrm{d}}{dz}$ $(n \in \mathbb{Z})$, we derive the Witt algebra:

$$[L_n, L_m] = (n-m)L_{n+m}, \tag{206}$$

Now, we consider its central extension of the form $A \oplus \mathbb{C}c$:

$$\left[f\frac{\mathrm{d}}{dz} + \alpha c, g\frac{\mathrm{d}}{dz} + \beta c\right] = \left[f\frac{\mathrm{d}}{dz}, g\frac{\mathrm{d}}{dz}\right] + \omega\left(f\frac{\mathrm{d}}{dz}, g\frac{\mathrm{d}}{dz}\right)c. \tag{207}$$

Define $\alpha_{p,q} = \alpha(L_p, L_q)$, from the Eq. (200), we derive $q\sigma_{p,-p} = p\sigma_{q,-q}$, combining with the above property of $\alpha_{q-m-n,m+n} = \alpha_{q-m,m} + \alpha_{q-n,n}$, we derive:

$$(p-q)\sigma_{p+q,-p-q} = (p+2q)\sigma_{p,-p} - (2p+q)\sigma_{q,-q} \tag{208}$$

This is the difference equation, which is generally solved by:

$$\sigma_{p,-p} = \lambda p^3 + \mu p \tag{209}$$

Setting $\lambda = \frac{1}{12}, \mu = -\frac{1}{12}$, we derive the Virasoro Algebra, expressed as:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{m^3 - m}{12}c\delta_{m+n,0}. \tag{210}$$

References

[1] A. Belavin, A. M. Polyakov and A. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl. Phys. B* **241** (1984) 333.

[2] E. Witten, *Nonabelian Bosonization in Two-Dimensions*, *Commun. Math. Phys.* **92** (1984) 455.

[3] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997, 10.1007/978-1-4612-2256-9.

[4] V. G. Kac, *Infinite-Dimensional Lie Algebras*. Cambridge University Press, 3 ed., 1990, 10.1017/CBO9780511626234.

[5] V. Knizhnik and A. Zamolodchikov, *Current Algebra and Wess-Zumino Model in Two-Dimensions*, *Nucl. Phys. B* **247** (1984) 83.

[6] J. M. Maldacena and H. Ooguri, *Strings in AdS(3) and SL(2,R) WZW model 1.: The Spectrum*, *J. Math. Phys.* **42** (2001) 2929 [`hep-th/0001053`].

[7] J. Maldacena, H. Ooguri and J. Son, *Strings in ads3 and the sl(2,r) wzw model. ii: Euclidean black hole*, *Journal of Mathematical Physics* **42** (2001) 2961–2977.

[8] J. M. Maldacena and H. Ooguri, *Strings in AdS(3) and the SL(2,R) WZW model. Part 3. Correlation functions*, *Phys. Rev. D* **65** (2002) 106006 [`hep-th/0111180`].

[9] M. R. Zirnbauer, *The integer quantum hall plateau transition is a current algebra after all*, *Nuclear Physics B* **941** (2019) 458–506.

[10] E. Witten, *String theory and black holes*, *Phys. Rev. D* **44** (1991) 314.

[11] E. Witten, *On Holomorphic factorization of WZW and coset models*, *Commun. Math. Phys.* **144** (1992) 189.

[12] S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory*, *Nucl. Phys. B* **326** (1989) 108.

[13] K. Gawedzki, *Conformal field theory: A Case study*, `hep-th/9904145`.

[14] P. Goddard and D. I. Olive, *Kac-Moody and Virasoro Algebras in Relation to Quantum Physics*, *Int. J. Mod. Phys. A* **1** (1986) 303.

[15] R. Blumenhagen and E. Plauschinn, *Introduction to conformal field theory: with applications to String theory*, vol. 779. 2009, 10.1007/978-3-642-00450-6.

[16] T. Kohno and A. M. Society, *Conformal Field Theory and Topology*, Iwanami series in modern mathematics. American Mathematical Society, 2002, 10.1090/mmono/210.

[17] M. Bershadsky and H. Ooguri, *Hidden SL(n) Symmetry in Conformal Field Theories*, *Commun. Math. Phys.* **126** (1989) 49.

[18] E. Witten, *Quantization of Chern-Simons Gauge Theory With Complex Gauge Group*, *Commun. Math. Phys.* **137** (1991) 29.

[19] H. Murayama, *Explicit Quantization of the Chern-Simons Action*, *Z. Phys. C* **48** (1990) 79.

[20] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** (1989) 351.

[21] J. Fuchs, I. Runkel and C. Schweigert, *Twenty-five years of two-dimensional rational conformal field theory*, *J. Math. Phys.* **51** (2010) 015210 [0910.3145].