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Progress Report on Positivity bounds in low-energy Effective Quantum Gravity

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Abstract: In this report, we rederive some current analysis on SMEFT positivity bound. We first rederive the main result in [1], which investigate the vector boson scattering process (VBS) by considering diagrams involving quartic gauge boson couplings (WGC) governed by SMEFT Dim-8 operators. Then, Scalar photon QED with a spectator field in [2]

Contents

1 Positivity bounds in VBS

In this section, we re-interpret some basic concepts, theorems and re-derive (in details) some of the main results and calculations in [1].

1.1 Crossing symmetry

Consider a $2 \to 2$ process, any of the particles can be replaced by its antiparticle on the other side of the interaction. Hence, for
Spin $= 0$: $M(s,t) = M(u,t)$.
Spin > 0: With the linear polarizations vector $(\epsilon_1^\mu)^* = \epsilon_3^\mu, (\epsilon_2^\mu)^* = \epsilon_4^\mu$, and with restriction to forward limit, we have $M(s,0) = M(u,0)$ (or $\mathcal{A}(s) = \mathcal{A}(u)$).

1.2 Optical theorem

The Optical theorem yield the relation between forward scattering amplitude and cross-section (refer to the Appendix).

$$\text{Im } \mathcal{A}(k_1 k_2 \to k_1 k_2) = 2E_1 E_2 |v_1 - v_2| \sigma_t. \tag{1.1}$$

Going into the CM-system, we have $p_1 + p_2 = 0, E_{\text{CM}} = E_1 + E_2, \mathbf{p}_{\text{CM}} = \mathbf{p}_1 = -\mathbf{p}_2$, (with $v = \frac{p}{E}$) we get the optical theorem in the standard form:

$$\text{Im } \mathcal{A}(k_1 k_2 \to k_1 k_2) = 2E_{\text{CM}} \mathbf{p}_{\text{CM}} \sigma_t. \tag{1.2}$$

when 2 incoming particles are the same, $m = m_1 = m_2$, $E_1 = E_2 = E$, we have $s = 4E^2 = E_{\text{CM}}^2, \mathbf{p}_{\text{CM}} = \sqrt{s^2/4 - m^2}$,)

$$\begin{aligned} \text{Im } \mathcal{A}(k_1 k_2 \to k_1 k_2) =& 2\sqrt{s}\sqrt{\frac{s}{4} - m^2}\sigma_t & (1.3) \\ =& \sqrt{s(s - 4m^2)}\sigma_t. & (1.4) \end{aligned}$$

In general, with 2 different incoming particles, defining $M_+ =$ we have

$$
\begin{aligned}
&2(E_1+E_2)\mathbf{p}_{\mathrm{CM}} \\
=&2\sqrt{(E_1+E_2)^2\mathbf{p}_{\mathrm{CM}}^2} && (1.5)\\
=&2\sqrt{\mathbf{p}_{\mathrm{CM}}^4+2\mathbf{p}_{\mathrm{CM}}^2E_1E_2+E_1^2E_2^2-\mathbf{p}_{\mathrm{CM}}^4+(E_1^2+E_2^2)\mathbf{p}_{\mathrm{CM}}^2-E_1^2E_2^2} && (1.6)\\
=&\sqrt{(2\mathbf{p}_{\mathrm{CM}}^2+2E_1E_2)^2-4m_1^2m_2^2} && (1.7)\\
=&\sqrt{\left((E_1+E_2)^2-2(m_1^2+m_2^2)\right)^2-4m_1^2m_2^2} && (1.8)\\
=&\sqrt{(E_1+E_2)^4-2(E_1+E_2)^2(m_1^2+m_2^2)+(m_1^2-m_2^2)^2} && (1.9)\\
=&\sqrt{s^2-s(M_+^2+M_-^2)+M_+^2M_-^2} && (1.10)\\
=&\sqrt{s(s-M_+^2)-M_-^2(s-M_+^2)} && (1.11)\\
=&\sqrt{(s-M_-^2)(s-M_+^2)}. && (1.12)
\end{aligned}
$$

Hence, Eq. J.29 yields

$$
\begin{aligned}
\text{Im } A(k_1k_2\to k_1k_2) =&2(E_1\mathbf{p}_2-E_2\mathbf{p}_1)\sigma_t && (1.13)\\
=&2(E_1+E_2)\mathbf{p}_{\mathrm{CM}}\sigma_t && (1.14)\\
=&\sqrt{(s-M_-^2)(s-M_+^2)}\sigma_t. && (1.15)
\end{aligned}
$$

It yields back the result of Eq. A.26 when $M_- = 0$.

$$\mathrm{Im}A_{ab}^{q_1q_2}(s') = \sqrt{(s'-M_+^2)(s'-M_-^2)}\sigma_{ab}^{q_1q_2}(s') > 0, \quad s' > (\epsilon\Lambda)^2, \tag{1.16}$$

1.3 Scattering amplitude in the forward limits

[ELABORATE MORE FROM [3]]

1.4 Froissart unitary bounds and dispersion relation

Froissart bound: Unitarity forces the high-energy amplitude in the forward limit is bounded by

$$\mathcal{A}(s) < \mathcal{O}(s\ln^2 s) \tag{1.17}$$

It is a necessary condition for the vanishing boundary contribution when we deform the contour integrals from IR to UV regime [PROVE THE BOUND AND ELABORATE MORE ON THE DISPERSION RELATION].

1.5 Positivity bounds (original version)

Physics in the IR regime can be deform to UV (contour C to C'). The boundary constribution varnishes because of the Froissart bound [ELABORATE MORE].

$$f \equiv \frac{1}{2\pi i} \oint_C \mathrm{d}s \frac{\mathcal{A}(s)}{(s-\mu^2)^3} = \frac{1}{2\pi i} \left(\int_{-\infty}^{0} + \int_{4m^2}^{\infty} \right) \mathrm{d}s \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3}, \tag{1.18}$$

with Disc $\mathcal{A}(s) \equiv \mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon)$. From here, we see that the dim-6 and dim-8 operators in low-energy EFT can be constrained by the positivity bound (Disc $M(s,0) \geq 0$) in the UV regime [ADD FIGURE].

In the forward limit ($t \to 0$), we have $s = 4m^2 - u$. Changing the variable with according bounds in the first term, f can be rewrite as:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(4m^2 - u)}{(4m^2 - u - \mu^2)^3} + \int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3} \right). \tag{1.19}$$

The crossing symmmetry reads $\mathcal{A}(4m^2 - u) = \mathcal{A}(s) = \mathcal{A}(u)$. Hence,

$$\begin{aligned} \text{Disc } \mathcal{A}(4m^2 - u) &= \mathcal{A}(4m^2 - u + i\epsilon) - \mathcal{A}(4m^2 - u - i\epsilon) && (1.20) \\ &= \mathcal{A}(u - i\epsilon) - \mathcal{A}(u + i\epsilon) && (1.21) \\ &= -\text{Disc } \mathcal{A}(u). && (1.22) \end{aligned}$$

Applying this relation to Eq. A.18, and replace the variable u by s, we have:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(s)}{(-4m^2 + s + \mu^2)^3} + \int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3} \right). \tag{1.23}$$

Here, taking the Schwarz reflection ($\mathcal{A}(s^*) = \mathcal{A}^*(s)$), we also have

$$\begin{aligned} \text{Disc } \mathcal{A}(u) &= \mathcal{A}(s + i\epsilon) - \mathcal{A}(s - i\epsilon) && (1.24) \\ &= \mathcal{A}(s + i\epsilon) - \mathcal{A}^*(s + i\epsilon) && (1.25) \\ &= 2i\text{Im } \mathcal{A}(s), && (1.26) \end{aligned}$$

hence, f becomes:

$$f = \frac{1}{\pi} \int_{4m^2}^{\infty} \mathrm{d}s \left[\frac{1}{(-4m^2 + s + \mu^2)^3} + \frac{1}{(s-\mu^2)^3} \right] \text{Im } \mathcal{A}(s). \tag{1.27}$$

From the Optical theorem of Im $\mathcal{A}(s) = \sqrt{s(s-4m^2)}\sigma_t(s)$, we derive

$$f = \frac{1}{\pi} \int_{4m^2}^{\infty} \mathrm{d}s \left[\frac{1}{(-4m^2 + s + \mu^2)^3} + \frac{1}{(s-\mu^2)^3} \right] \sqrt{s(s-4m^2)}\sigma_t(s). \tag{1.28}$$

1.6 Positivity bound for SMEFT

In SMEFT, the multi-particle production of massless particles give rise to the branch cut that covers the whole real axis. [ADD FIGURE]

$$f = \frac{1}{2\pi i} \oint \mathrm{d}s \frac{\mathcal{A}(s)}{(s-\mu^2)^3}. \tag{1.29}$$

For improving the positivity, we introduce the modified amplitude. with the

$$B_{\epsilon\Lambda}(s) \equiv \mathcal{A}(s) - \frac{1}{2\pi i} \int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} \mathrm{d}s' \frac{\mathrm{Disc}\mathcal{A}(s')}{s'-s} \tag{1.30}$$

$$= \frac{1}{2\pi i} \oint_{\mathrm{C}} \mathrm{d}s' \frac{\mathcal{A}(s')}{s'-s} - \frac{1}{2\pi i} \int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} \mathrm{d}s' \frac{\mathrm{Disc}\mathcal{A}(s')}{s'-s} \tag{1.31}$$

$$= \frac{1}{2\pi i} \int_{\mathrm{C}'_{\epsilon\Lambda}} \mathrm{d}s' \frac{\mathcal{A}(s')}{s'-s} = \frac{1}{2\pi i} \oint_{\mathrm{C}_{\epsilon\Lambda}} \mathrm{d}s' \frac{B_{,\epsilon\Lambda}(s')}{s'-s}, \tag{1.32}$$

the modified amplitude has the same behavior at $s \to \infty$ and satisfies the Froissart bound.

Next, we define:

$$f_{\epsilon\Lambda}(s) \equiv \frac{1}{2} \frac{\mathrm{d}^2 B_{\epsilon\Lambda}(s)}{\mathrm{d}s^2} \tag{1.33}$$

$$= \frac{1}{2\pi i} \left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty} \right) \mathrm{d}s' \frac{\mathrm{Disc}\ B_{\epsilon\Lambda}(s')}{(s'-s)^3} \tag{1.34}$$

$$= \frac{1}{2\pi i} \left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty} \right) \mathrm{d}s' \frac{\mathrm{Disc}\ \mathcal{A}(s')}{(s'-s)^3} \tag{1.35}$$

$$= \frac{1}{\pi} \left(\int_{(\epsilon\Lambda)^2+M^2}^{\infty} \mathrm{d}s' \frac{1}{(s'+s-M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} \mathrm{d}s' \frac{1}{(s'-s)^3} \right) \mathrm{Im}\ \mathcal{A}(s) \tag{1.36}$$

$$= \frac{1}{\pi} \left(\int_{(\epsilon\Lambda)^2+M^2}^{\infty} \mathrm{d}s' \frac{1}{(s'+s-M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} \mathrm{d}s' \frac{1}{(s'-s)^3} \right) \sqrt{(s-M_-^2)(s-M_+^2)}\sigma_t. \tag{1.37}$$

Here we follow the same procedure with the original version of positivity bound, applying Froissart bound for the deformation and changing the variable $s' \to M^2 - s'$, where $M^2 = 2m_1^2 + 2m_2^2$. [ADD MORE PHYSICAL INTERPRETATIONS + POLARIZATIONS]

For the positivity bounds on QGC couplings, dim-8 operators are independent of the presence of dim-6 ones, indeed,

$$\sum_i c_i^{(8)} x_i + \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j} > 0, \tag{1.38}$$

or,

$$\sum_i c_i^{(8)} x_i > \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j}. \tag{1.39}$$

While by explicit calculations, we yields that the R.H.S. is already positive definite. Hence, we can imnpose the bound.

$$\sum_i c_i^{(8)} x_i > 0. \tag{1.40}$$

[TO BE CONT.]

1.7 $ZZ \to ZZ$ process

External polarization: Let $p^\mu = (E, 0, 0, p_z)$, thus $p^2 = E^2 - p_z^2 = m^2$. Take the canonical basis that satisfying $p^\mu \epsilon_\mu = 0$ and $\epsilon_\mu^2 = -1$.

$$\epsilon_1^\mu = (0,1,0,0)(\text{traverse}), \tag{1.41}$$

$$\epsilon_2^\mu = (0,0,1,0)(\text{traverse}), \tag{1.42}$$

$$\epsilon_3^\mu = \frac{1}{m}\left(p_z, 0, 0, E\right)(\text{longitudinal}). \tag{1.43}$$

We can parameterize the polarization vectors of 2 imcoming Z bosons as:

$$\epsilon^\mu(V_1) = \sum_{i=1}^{3} a_i \epsilon_i^\mu = (a_3 \frac{p_1}{m_1}, a_1, a_2, a_3 \frac{E_1}{m_1}), \tag{1.44}$$

$$\epsilon^\mu(V_2) = \sum_{i=1}^{3} a_i \epsilon_i^\mu = (b_3 \frac{p_2}{m_2}, b_1, b_2, b_3 \frac{E_2}{m_2}). \tag{1.45}$$

1.8 Dim-8 operators included in QGC

Operators involved in quartic gauge boson couplings (QGC) has been studied in [4], [5], [6] and are listed into 3 categories as followed:

1.8.1 Operators containing just $D_\mu\Phi$

The two independent operators in this class are

$$\mathcal{L}_{S,0} = \left[\left(D_\mu\Phi\right)^\dagger D_\nu\Phi\right] \times \left[\left(D^\mu\Phi\right)^\dagger D^\nu\Phi\right] \tag{1.46}$$

$$\mathcal{L}_{S,1} = \left[\left(D_\mu\Phi\right)^\dagger D^\mu\Phi\right] \times \left[\left(D_\nu\Phi\right)^\dagger D^\nu\Phi\right] \tag{1.47}$$

$$\mathcal{L}_{S,2} = \left[\left(D_\mu\Phi\right)^\dagger D_\nu\Phi\right] \times \left[\left(D^\mu\Phi\right)^\dagger D^\nu\Phi\right] \tag{1.48}$$

1.8.2 Operators containing $D_\mu\Phi$ and field strength

The operators in this class are:

$$\mathcal{L}_{M,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{1.49}$$

$$\mathcal{L}_{M,1} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{1.50}$$

$$\mathcal{L}_{M,2} = \left[B_{\mu\nu}B^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{1.51}$$

$$\mathcal{L}_{M,3} = \left[B_{\mu\nu}B^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{1.52}$$

$$\mathcal{L}_{M,4} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}D^\mu\Phi\right] \times B^{\beta\nu} \tag{1.53}$$

$$\mathcal{L}_{M,5} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}D^\nu\Phi\right] \times B^{\beta\mu} \tag{1.54}$$

$$\mathcal{L}_{M,6} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\nu}D^\mu\Phi\right] \tag{1.55}$$

$$\mathcal{L}_{M,7} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\mu}D^\nu\Phi\right] \tag{1.56}$$

1.8.3 Operators containing just the field strength tensor

The following operators containing just the field strength tensor also lead to quartic anomalous couplings:

$$\mathcal{L}_{T,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \mathrm{Tr}\left[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}\right] \tag{1.57}$$

$$\mathcal{L}_{T,1} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}\right] \tag{1.58}$$

$$\mathcal{L}_{T,2} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}\right] \tag{1.59}$$

$$\mathcal{L}_{T,3} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\hat{W}^{\nu\alpha}\right] \times B_{\beta\nu} \tag{1.60}$$

$$\mathcal{L}_{T,4} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\alpha\mu}\hat{W}^{\beta\nu}\right] \times B_{\beta\nu} \tag{1.61}$$

$$\mathcal{L}_{T,5} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times B_{\alpha\beta}B^{\alpha\beta} \tag{1.62}$$

$$\mathcal{L}_{T,6} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times B_{\mu\beta}B^{\alpha\nu} \tag{1.63}$$

$$\mathcal{L}_{T,7} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times B_{\beta\nu}B^{\nu\alpha} \tag{1.64}$$

$$\mathcal{L}_{T,8} = B_{\mu\nu}B^{\mu\nu}B_{\alpha\beta}B^{\alpha\beta} \tag{1.65}$$

$$\mathcal{L}_{T,9} = B_{\alpha\mu}B^{\mu\beta}B_{\beta\nu}B^{\nu\alpha} \tag{1.66}$$

2 Scalar photon QED with a spectator field

In this session, we review [2] **My comments:**

- I spotted a difference from my vertices definition E.8 versus Eq. (B.1) in [2] at a factor $\frac{1}{2}$. Indeed, with Eq. (B.1), the Eq. (B.3) should yield $\frac{i}{4M_{\rm Pl}^2 t}su$. I am not sure whether they have added a factor of 4 (for symmetry?) implicitly there, or I have made some errors in the calculation. Please have a cross-check!

- In their Eq. (B.3), I suppose the $V_m^{\mu\nu}(k_1,k_3)$ should be changed to $V_0^{\mu\nu}(k_1,k_3)$ as we have assumed the scalar to be massless. Otherwise, there will be an additional term in the amplitude.

2.1 Field Redefinition

Under field redefinition, as in Eq. (3.5) of [2]

$$g_{\mu\nu} = g_{\mu\nu} + 2C\frac{\alpha^2}{M^2 M_{\rm Pl}^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right), \tag{2.1}$$

the terms (inside the bracket) of Eq. (4.2) reads,

$$\begin{aligned}
\frac{M_{\rm Pl}^2}{2}R =& \frac{M_{\rm Pl}^2}{2}R^{\mu\nu}g_{\mu\nu} && (2.2)\\
\to& \frac{M_{\rm Pl}^2}{2}R^{\mu\nu}\left[g_{\mu\nu} + 2C\frac{\alpha^2}{M^2 M_{\rm Pl}^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right)\right] && (2.3)\\
=& \frac{M_{\rm Pl}^2}{2}R + C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}C\frac{\alpha^2}{M^2}R(\partial\phi)^2, && (2.4)
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2}(\partial\phi)^2 =& -\frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi && (2.5)\\
\to& -\frac{1}{2}\left[g_{\mu\nu} + 2C\frac{\alpha^2}{M^2 M_{\rm Pl}^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right)\right]\partial^\mu\phi\partial^\nu\phi && (2.6)\\
=& -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}C\frac{\alpha^2}{M^2 M_{\rm Pl}^2}(\partial\phi)^4, && (2.7)
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2}(\partial\chi)^2 =& -\frac{1}{2}g_{\mu\nu}\partial^\mu\chi\partial^\nu\chi && (2.8)\\
\to& -\frac{1}{2}\left[g_{\mu\nu} + 2C\frac{\alpha^2}{M^2 M_{\rm Pl}^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right)\right]\partial^\mu\chi\partial^\nu\chi && (2.9)\\
=& -\frac{1}{2}(\partial\chi)^2 - C\frac{\alpha^2}{M^2 M_{\rm Pl}^2}(\partial\phi\partial\chi)^2 + \frac{1}{2}C\frac{\alpha^2}{M^2 M_{\rm Pl}^2}(\partial\phi)^2(\partial\chi)^2. && (2.10)
\end{aligned}$$

All other terms are added with negligible higher-order (H. O.) terms.
The metric determinant transforms as,

$$
\begin{aligned}
\sqrt{-g} =& \sqrt{-\det g_{\mu\nu}} && (2.11)\\
\to& \sqrt{-\det\left[g_{\mu\nu} + 2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)g_{\mu\nu}\right)\right]} && (2.12)\\
\to& \sqrt{-\det\left[g_{\mu\alpha}\left(\delta^\alpha{}_\nu + 2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)\right]} && (2.13)\\
\sim& \sqrt{-\det\left[g_{\mu\alpha}\exp\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)\right]} && (2.14)\\
=& \sqrt{-\det(g_{\mu\alpha})}\sqrt{\det\exp\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)} && (2.15)\\
=& \sqrt{-g}\sqrt{\exp\mathrm{tr}\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial^\alpha\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)\delta^\alpha{}_\nu\right)\right)} && (2.16)\\
=& \sqrt{-g}\sqrt{\exp\left(2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(-(\partial\phi)^2\right)\right)} && (2.17)\\
\sim& \sqrt{-g}\sqrt{1 - 2C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial\phi\right)^2} && (2.18)\\
\sim& \sqrt{-g}\left(1 - C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial\phi\right)^2\right). && (2.19)
\end{aligned}
$$

Hence, subtitute $\sqrt{-g}$ by $\sqrt{-g}\left(1 + C\frac{\alpha^2}{M^2M_{\rm Pl}^2}\left(\partial\phi\right)^2\right)$, the Lagrangian terms transform as,

$$
\begin{aligned}
\sqrt{-g}\frac{M_{\rm Pl}^2}{2}R \to& \sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R + C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}C\frac{\alpha^2}{M^2}R(\partial\phi)^2\right.\\
&\left. + \frac{1}{2}C\frac{\alpha^2}{M^2}R(\partial\phi)^2 + \text{H. O.}\right). && (2.20)\\
=& \sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R + C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \text{H. O.}\right), && (2.21)
\end{aligned}
$$

$$\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2\right) \tag{2.22}$$

$$\to\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4-\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4+\text{H. O.}\right) \tag{2.23}$$

$$=\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4+\text{H. O.}\right), \tag{2.24}$$

$$\sqrt{-g}\left(-\frac{1}{2}(\partial\chi)^2\right) \tag{2.25}$$

$$\begin{aligned}\to\sqrt{-g}\Big(&-\frac{1}{2}(\partial\chi)^2-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi\partial\chi)^2+\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^2(\partial\chi)^2\\ &-\frac{1}{2}C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^2(\partial\chi)^2+\text{H. O.}\Big)\end{aligned} \tag{2.26}$$

$$=\sqrt{-g}\Big(-\frac{1}{2}(\partial\chi)^2-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi\partial\chi)^2+\text{H. O.}\Big), \tag{2.27}$$

Given the above transformation, we deduce the extra terms,

$$\sqrt{-g}\left(C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4-C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi\partial\chi)^2\right). \tag{2.28}$$

Hence, the IR action (4.2) in [2]

$$\begin{aligned}\mathcal{L}_{\rm IR}^{(J)}=\sqrt{-g}\Bigg[&\frac{M_{\rm Pl}^2}{2}R-\frac{1}{2}(\partial\chi)^2-\frac{1}{2}(\partial\phi)^2-\frac{\alpha^3M}{(2\pi)^2}\frac{\phi^3}{3!}+\frac{\alpha^4}{2\pi^2}\frac{\phi^4}{4!}\\ &+C\frac{\alpha^2}{M^2}R^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\tilde{C}\frac{\alpha^4}{M^4}(\partial\phi)^4\Bigg],\end{aligned} \tag{2.29}$$

reduces to (4.3) of [2],

$$\begin{aligned}\mathcal{L}_{\rm IR}=\sqrt{-g}\Bigg[&\frac{M_{\rm Pl}^2}{2}R-\frac{1}{2}(\partial\chi)^2-\frac{1}{2}(\partial\phi)^2-\frac{\alpha^3M}{(2\pi)^2}\frac{\phi^3}{3!}+\frac{\alpha^4}{2\pi^2}\frac{\phi^4}{4!}\\ &+C'\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial\phi)^4+C\frac{\alpha^2}{M^2M_{\rm Pl}^2}(\partial_\mu\phi\,\partial^\mu\chi)^2+\dots\Bigg].\end{aligned} \tag{2.30}$$

2.2 Scattering Matrix

2.2.1 Matter Langrangian & Scalars-Graviton vertices

Scalar matter fields interact with the gravitational field as described by the action,

$$S_m=\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2\right]. \tag{2.31}$$

The quantum fluctuation of gravitational fields can be expanded about a smooth background metric $\eta_{\mu\nu}$, with the flutuations suppressed by the Planck scale $M_{\rm Pl} = \frac{1}{\sqrt{8\pi G}}$ as,

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\rm Pl}} h_{\mu\nu}. \tag{2.32}$$

Einstein action are expanded as,

$$S_{\rm g} = \int \mathrm{d}^4x \sqrt{-g} \left[\mathcal{L}_g^{(0)} + \mathcal{L}_g^{(1)} + \mathcal{L}_g^{(2)} + ...\right] \tag{2.33}$$

with the expanding terms,

$$\mathcal{L}_g^{(0)} = \frac{M_{\rm Pl}^2}{2} R \tag{2.34}$$

$$\mathcal{L}_g^{(1)} = \frac{2}{M_{\rm Pl}} h_{\mu\nu}[\eta^{\mu\nu} R - 2R^{\mu\nu}] \tag{2.35}$$

Here, we skip the discussion about gauge fixing and ghost Lagrangian. A similar expansion for matter Lagrangian yields,

$$S_m = \int \mathrm{d}^4x \sqrt{-g} \left(\mathcal{L}_m^{(0)} + \mathcal{L}_m^{(1)} + \mathcal{L}_m^{(2)} + ...\right) \tag{2.36}$$

Here, $T^{\mu\nu}$ is derived from variation of J.29 as,

$$T^{\mu\nu} = \frac{2}{\sqrt{-\eta}} \frac{\delta S_m}{\delta \eta^{\mu\nu}} = \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \left(\partial_\tau \phi \partial^\tau \phi - m^2 \phi^2\right)\right) \tag{2.37}$$

Feynman rules for $\mathcal{L}_m^{(1)}$ in momentum space reads,

$$V_{\mu\nu}^{\phi\phi h}(p_1, p_2, m) = \frac{i}{M_{\rm Pl}} \left[(p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) - \eta_{\mu\nu}(p_1 p_2 - m^2)\right]. \tag{2.38}$$

Hence, the massless scalar field ϕ and χ reads

$$V_{\mu\nu}^{\phi\phi h}(p_1, p_3, 0) = \frac{i}{M_{\rm Pl}} \left[(p_{1\mu} p_{3\nu} + p_{1\mu} p_{3\nu}) - \eta_{\mu\nu}(p_1 p_3)\right]. \tag{2.39}$$

$$V_{\alpha\beta}^{\chi\chi h}(p_2, p_4, 0) = \frac{i}{M_{\rm Pl}} \left[(p_{2\alpha} p_{4\beta} + p_{2\alpha} p_{4\beta}) - \eta_{\alpha\beta}(p_2 p_4)\right]. \tag{2.40}$$

2.2.2 Graviton Propagator

Consider the Lagrangian

$$\sqrt{-g}\mathcal{L} = \sqrt{-g} \left(\frac{M_{\rm Pl}^2}{2} R + \mathcal{L}_m + \mathcal{L}_{\rm GF}\right), \tag{2.41}$$

to the second order in $h_{\mu\nu}$,

$$\frac{M_{\rm Pl}^2}{2}R = \frac{M_{\rm Pl}^2}{2}(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h) + \frac{1}{2}\left[\partial_\tau h_{\mu\nu}\partial^\tau \bar{h}^{\mu\nu} - 2\partial^\tau \bar{h}_{\mu\tau}\partial_\sigma \bar{h}^{\mu\sigma}\right], \tag{2.42}$$

with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.43}$$

as mentioned in the Appendix. The harmonic gauge means to choose $\xi = 1$ of the gauge fixing term,

$$\mathcal{L}_{\rm GF} = \xi\partial_\mu \bar{h}^{\mu\nu}\partial^\tau \bar{h}_{\tau\nu} \tag{2.44}$$

The Lagrangian become

$$\sqrt{-g}\mathcal{L} = \frac{1}{2}\partial_\tau h_{\mu\nu}\partial^\tau h^{\mu\nu} - \frac{1}{4}\partial_\tau h\partial^\tau h - \frac{1}{M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}. \tag{2.45}$$

Taking integration by part, we have

$$\mathcal{L} = \frac{1}{2}h_{\mu\nu}\Box\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta} + \frac{1}{M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}., \tag{2.46}$$

with the "identity" tensor $I^{\mu\nu\alpha\beta}$ defined as in A.18. The E.O.M. follows,

$$\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta}\Box D_{\alpha\beta\gamma\delta} = I^{\mu\nu}{}_{\gamma\delta}. \tag{2.47}$$

This equation admitted the solution of A.26, assuming that the initial condition correspond to Feynman propagator $D^{\alpha\beta\gamma\sigma}(x-y)$,

$$D^{\alpha\beta\gamma\delta}(x-y) = \begin{cases} G^{\alpha\beta\gamma\delta}(x-y) & \text{if } x^0 > y^0, \\ G^{\alpha\beta\gamma\delta}(y-x) & \text{if } x^0 < y^0, \end{cases} \tag{2.48}$$

we obtain

$$iD^{\alpha\beta\gamma\sigma}(x) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{i}{q^2+i\epsilon}e^{-iqx}P^{\alpha\beta\gamma\delta}, \tag{2.49}$$

with

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}\right). \tag{2.50}$$

Hence, the propagator reads,

$$\frac{iP^{\mu\alpha\nu\beta}}{k^2} = \frac{i}{2}\frac{\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}}{(p_1+p_3)^2}. \tag{2.51}$$

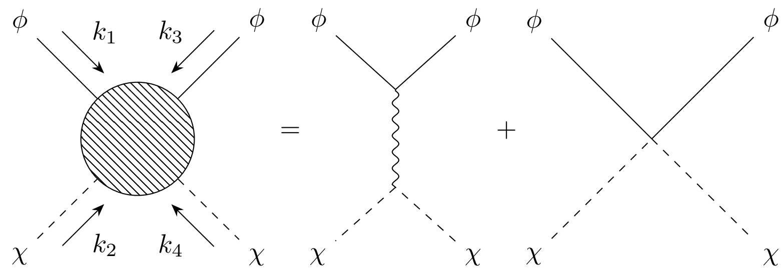

Figure 1: Feynman diagrams of the process, taken from [2]

2.2.3 Scattering matrix of effective 4-scalar vetices

We now have all the ingredients to derive the scattering matrix.
The t-channel reads,

$$i\mathcal{M}_1 = -V^{\phi\phi h}_{\mu\nu}(p_1,p_3,0)\frac{iP^{\mu\alpha\nu\beta}}{(p_1+p_3)^2}V^{\chi\chi h}_{\alpha\beta}(p_2,p_4,0) \tag{2.52}$$

$$\begin{aligned}=&-\frac{i^3}{2M_{\rm Pl}^2(p_1+p_3)^2}\big[4p_{1\mu}p_{3\nu}\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)p_{2\alpha}p_{4\beta}\\&-2\eta_{\mu\nu}(p_1p_3)\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)p_{2\alpha}p_{4\beta}\\&-2p_{1\mu}p_{3\nu}\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)\eta_{\alpha\beta}(p_2p_4)\\&+\eta_{\mu\nu}(p_1p_3)\left(\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\alpha\nu}-\eta^{\mu\alpha}\eta^{\nu\beta}\right)\eta_{\alpha\beta}(p_2p_4)\big]\end{aligned} \tag{2.53}$$

$$\begin{aligned}=&\frac{i}{4M_{\rm Pl}^2(p_1p_3)}\big[4[(p_1p_2)(p_3p_4)+(p_1p_4)(p_2p_3)-(p_1p_3)(p_2p_4)]\\&-2(p_1p_3)(1+1-4)(p_2p_4)-2(p_1p_3)(1+1-4)(p_2p_4)\\&+(p_1p_3)(4+4-16)(p_2p_4)\big]\end{aligned} \tag{2.54}$$

$$=\frac{i}{M_{\rm Pl}^2(p_1p_3)}[(p_1p_2)(p_3p_4)+(p_1p_4)(p_2p_3)-(p_1p_3)(p_2p_4)]. \tag{2.55}$$

Here, we have used the fact that $\eta^{\mu\nu}\eta_{\mu\alpha}=\delta^{\nu}{}_{\alpha}$, and $\eta^{\mu\nu}\eta_{\mu\nu}=4$. In addition, with $p_i^2=m_i^2=0$ for all $i=1,2,3,4$, the Madelstam variables read:

$$s\equiv-(p_1+p_2)^2=-(p_3+p_4)^2=-2p_1p_2=-2p_3p_4, \tag{2.56}$$

$$t\equiv-(p_1+p_3)^2=-(p_2+p_4)^2=-2p_1p_3=-2p_2p_4, \tag{2.57}$$

$$-s-t=u\equiv-(p_1+p_4)^2=-(p_2+p_3)^2=-2p_1p_4=-2p_2p_3. \tag{2.58}$$

Therefore, the term is rewriten to

$$i\mathcal{M}_1=-i\frac{s(s+t)}{M_{\rm Pl}^2t}. \tag{2.59}$$

Now, consider the 4-vertice diagram,

$$
\begin{aligned}
i\mathcal{M}_2 =& 2iC\frac{\alpha^2}{M^2 M_{\mathrm{Pl}}^2}\left[p_{1\mu}p_{3\nu}\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}\right)p_{2\alpha}p_{4\beta}\right] && (2.60)\\
=& 2iC\frac{\alpha^2}{M^2 M_{\mathrm{Pl}}^2}\left[(p_1p_2)(p_3p_4)+(p_1p_4)(p_2p_3)\right] && (2.61)\\
=& 2iC\frac{\alpha^2}{M^2 M_{\mathrm{Pl}}^2}\left[s(s+t)+\frac{t^2}{2}\right]. && (2.62)
\end{aligned}
$$

The effective 4-scalar scattering matrix yields,

$$
\begin{aligned}
\mathcal{M} =& \mathcal{M}_1+\mathcal{M}_2 && (2.63)\\
=& -\frac{s(s+t)}{M_{\mathrm{Pl}}^2 t}+2C\frac{\alpha^2}{M^2 M_{\mathrm{Pl}}^2}\left[s(s+t)+\frac{t^2}{2}\right]. && (2.64)
\end{aligned}
$$

3 Photon wavefunction Renormalization

with $l \equiv p - xk$, the relation $l^\sigma l^\rho = \frac{\eta^{\sigma\rho}}{d} l^2$, and using Feynman's trick in Appendix J.29

$$\pi^{\rho\sigma}(k^2) \tag{3.1}$$

$$=e^2\,\mu^\epsilon \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(2p-k)^\rho(2p-k)^\sigma}{(p^2+m^2)((p-k)^2+m^2)} - 2e^2\eta^{\rho\sigma}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2+m^2} \tag{3.2}$$

$$=e^2\,\mu^\epsilon \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(2p-k)^\rho(2p-k)^\sigma - 2\eta^{\rho\sigma}((p-k)^2+m^2)}{(p^2+m^2)((p-k)^2+m^2)} \tag{3.3}$$

$$=e^2\,\mu^\epsilon \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{(2p-k)^\rho(2p-k)^\sigma - 2\eta^{\rho\sigma}((p-k)^2+m^2)}{\left((p-xk)^2+m^2+xk^2-x^2k^2\right)^2} \tag{3.4}$$

$$\begin{aligned}=&e^2\,\mu^\epsilon \int \frac{\mathrm{d}^d l}{(2\pi)^d} \int_0^1 \mathrm{d}x \left[\frac{(4l^\rho l^\sigma - 2\eta^{\rho\sigma}l^2}{(l^2+m^2+xk^2(1-x))^2}\right.\\ &\left.+\frac{k^\rho k^\sigma(2x-1)^2 - 2\eta^{\rho\sigma}(x^2k^2-2xk^2+k^2+m^2)+2(2x-1)(l^\rho k^\sigma + l^\sigma k^\rho) - 4\eta^{\rho\sigma}l(x-1)k}{(l^2+m^2+xk^2(1-x))^2}\right]\end{aligned} \tag{3.5}$$

$$\begin{aligned}=&e^2\,\mu^\epsilon \int \frac{\mathrm{d}^d l}{(2\pi)^d} \int_0^1 \mathrm{d}x \left[\frac{\left(\frac{4}{d}-2\right)\eta^{\rho\sigma}l^2}{(l^2+m^2+xk^2(1-x))^2}+\right.\\ &\left.\frac{-2\eta^{\rho\sigma}(k^2+m^2+k^2x^2-2k^2x)+k^\rho k^\sigma(1-4x+4x^2)}{(l^2+m^2+xk^2(1-x))^2}\right]\end{aligned} \tag{3.6}$$

$$\begin{aligned}=&e^2\,\mu^\epsilon \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l}{(2\pi)^d}\left[-\left(\frac{4}{d}-2\right)\eta^{\mu\nu}\frac{i(-1)^1}{(-4\pi)^{\frac{d}{2}}}\frac{d}{2}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)}(m^2-xk^2(x-1))^{\frac{d}{2}-1}\right.\\ &\left.+\frac{i(-1)^2}{(-4\pi)^{\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}(m^2-xk^2(x-1))^{\frac{d}{2}-2}(-2\eta^{\mu\nu}(k^2+m^2+k^2x(x-2)+k^\mu k^\nu(2x-1)^2))\right],\end{aligned} \tag{3.7}$$

We have,

$$
\begin{aligned}
&\left(\frac{4}{d}-2\right)\frac{i\eta^{\mu\nu}}{(-4\pi)^{d/2}}\frac{d}{2}\Gamma(1-\frac{d}{2})(m^2-k^2x(x-1))^{\frac{d}{2}-1}\mu^{4-d} \\
=&(2-d)\frac{i\eta^{\mu\nu}}{(-4\pi)^{d/2}}\frac{d}{2}\Gamma(1-\frac{d}{2})\left(\frac{m^2-k^2x(x-1)}{\mu^2}\right)^{\frac{d}{2}-2}(m^2-k^2x(x-1)) \qquad (3.8)\\
=&(-2-d+4)i\eta^{\mu\nu}\Bigg(\frac{1}{(4\pi)^2}+\frac{\ln(-4\pi)}{(4\pi)^2}(2-\frac{d}{2}))(\frac{-2}{4-d}+\delta_E-1) \\
&(1+(\frac{d}{2}-2)\ln\left(\frac{m^2-kx(x-1)}{\mu^2}\right))(m^2-k^2(x-1)x)\Bigg) \qquad (3.9)\\
=&i\eta^{\mu\nu}\left[\frac{2}{(4\pi)^2}(\frac{2}{4-d}-\delta_E+1)-\frac{2}{(4\pi)^2}\ln\left(\frac{m^2-xk^2(x-1)}{\mu^2}\right)+\frac{2}{(4\pi)^2}\ln(-4\pi)-\frac{2}{(4\pi)^2}\right] \\
&(m^2-xk^2(x-1)) \qquad (3.10)\\
=&\frac{i2\eta^{\mu\nu}}{4\pi^2}\left[\Delta+\ln\left(\frac{\mu^2}{m^2-xk^2(x-1)}\right)\right](m^2-xk^2(x-1)). \qquad (3.11)
\end{aligned}
$$

with $\Delta=\frac{2}{4-d}-\gamma_E+\ln(-4\pi)$. In the similar fashion, for the other term, we also have

$$
\begin{aligned}
&\frac{i\Gamma(2-\frac{d}{2})}{(-4\pi)^{\frac{d}{2}}}(m^2-k^2x(x-1))^{\frac{d}{2}-2}\mu^{4-d} \qquad (3.12)\\
=&i\left(\frac{1}{(4\pi)^2}+\frac{\ln(-4\pi)}{(4\pi)^2}\left(2-\frac{d}{2}\right)\right)\left(\frac{1}{2-\frac{d}{2}}-\gamma_E\right)\left(1+\left(\frac{d}{2}-2\right)\ln\left(\frac{m^2-xk^2(x-1}{\mu^2}\right)\right) \\
& \qquad (3.13)\\
=&\frac{i}{4\pi^2}\left(\Delta+\ln\left(\frac{\mu^2}{m^2-xk^2(x+1)}\right)\right) \qquad (3.14)
\end{aligned}
$$

Subtitute them back, we have

$$
\begin{aligned}
&\frac{-ie^2}{4\pi^2}\int_0^1 \mathrm{d}x \ln\left(\frac{\mu^2}{m^2-xk^2(x+1)}\right)\Bigg(-k^\mu k^\nu(2x-1)^2 \\
&+2\eta^{\mu\nu}(k^2+m^2+k^2x(x-2)+xk^2(x-1)-m^2)\Bigg) \\
=&\frac{-ie^2}{4\pi^2}\int_0^1 \mathrm{d}x \ln\left(\frac{\mu^2}{m^2-xk^2(x+1)}\right)\left(2\eta^{\mu\nu}k^2(x-1)(2x-1)-k^\mu k^\nu(2x-1)^2\right)
\end{aligned}
\tag{3.15}
$$

$$
=\frac{-ie^2}{4\pi^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}y \ln\left(\frac{\mu^2}{m^2-k^2(y^2-\frac{1}{4})}\right)\left(4\eta^{\mu\nu}k^2y^2-y^\mu k^\nu y^2-2\eta^{\mu\nu}k^2y\right) \tag{3.16}
$$

$$
=\frac{-ie^2}{4\pi^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}y y^2 \ln\left(\frac{\mu^2}{m^2-k^2(y^2-\frac{1}{4})}\right)\left(k^2\eta^{\mu\nu}-k^\mu k^\nu\right) \tag{3.17}
$$

A Weak-Field Gravity

A.1 Linearised theory

Expanding metric arround flat-space Minkowski background, with $M_{\mathrm{Pl}}=\frac{1}{\sqrt{8\pi G}}$,

$$
g_{\mu\nu}=\eta_{\mu\nu}+\sqrt{32\pi G}h_{\mu\nu}=\eta_{\mu\nu}+\frac{2}{M_{\mathrm{Pl}}}h_{\mu\nu}. \tag{A.1}
$$

To leading order, the inverse metric reads,

$$
g^{\mu\nu}=\eta^{\mu\nu}-\frac{2}{M_{\mathrm{Pl}}}h^{\mu\nu}. \tag{A.2}
$$

Christoffel symbols are then,

$$
\Gamma^\sigma_{\nu\rho}=\frac{1}{M_{\mathrm{Pl}}}\eta^{\sigma\lambda}\left(\partial_\nu h_\lambda\rho+\partial_\rho h_{\nu\lambda}-\partial_\lambda h_{\nu\rho}\right). \tag{A.3}
$$

The Rieman tensor reads,

$$
R^\sigma{}_{\rho\mu\nu}=\partial_\mu\Gamma^\sigma_{\nu\rho}-\partial_\nu\Gamma^\sigma_{\mu\rho}+\Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda}-\Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda}. \tag{A.4}
$$

The $\Gamma\Gamma$ term is at $\mathcal{O}(h^2)$, hence, to the linear order,

$$
R^\sigma{}_{\rho\mu\nu}=\partial_\mu\Gamma^\sigma_{\nu\rho}-\partial_\nu\Gamma^\sigma_{\mu\rho}+\mathcal{O}(h^2) \tag{A.5}
$$

$$
=\frac{1}{M_{\mathrm{Pl}}}\eta^{\sigma\lambda}(\partial_\mu\partial_\rho h_{\nu\rho}-\partial_\mu\partial_\lambda h_{\nu\rho}-\partial_\nu\partial_\rho h_{\mu\lambda}+\partial_\nu\partial_\lambda h_{\mu\rho})+\mathcal{O}(h^2). \tag{A.6}
$$

The Ricci tensors read

$$R_{\mu\nu} = \frac{1}{M_{\rm Pl}} \left(\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h \right). \tag{A.7}$$

with $h \equiv h^\mu{}_\mu$, $\Box \equiv \partial^\mu \partial_\mu$. The Ricci scalar follows,

$$R = \frac{1}{M_{\rm Pl}} \left(\partial^\mu \partial^\nu h_{\mu\nu} - \Box h \right). \tag{A.8}$$

Hence, we deduce the Einstein equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{A.9}$$

with

$$G_{\mu\nu} = \frac{1}{M_{\rm Pl}} \left[\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - (\partial^\rho \partial^\sigma h_{\rho\sigma} - \Box h)\eta_{\mu\nu} \right]. \tag{A.10}$$

A.2 Green function

We have derived Ricci tensor and scalar,

$$R_{\mu\nu} = \frac{1}{M_{\rm Pl}} \left(\partial_\mu \partial_\gamma h^\gamma{}_\nu + \partial_\nu \partial_\gamma h^\gamma{}_\mu - \partial_\mu \partial_\nu h^\gamma{}_\gamma - \Box h_{\mu\nu} \right) + \mathcal{O}(h^2), \tag{A.11}$$

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{1}{M_{\rm Pl}} \partial_\mu \partial_\gamma h^{\mu\gamma} + \mathcal{O}(h^2). \tag{A.12}$$

The Einstein equation reads,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{A.13}$$

$$\simeq \frac{1}{M_{\rm Pl}} \left[\left(\delta_{(\mu}{}^\alpha \delta_{\nu)}{}^\beta - \eta_{\mu\nu} \eta^{\alpha\beta} \right) \Box - 2\delta_{(\mu}{}^{(\alpha} \partial_{\nu)} \partial^{\beta)} + \eta^{\alpha\beta} \partial_\mu \partial_\nu + \eta_{\mu\nu} \partial^\alpha \partial^\beta \right] h_{\alpha\beta} \tag{A.14}$$

$$\equiv \frac{1}{M_{\rm Pl}} O_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \tag{A.15}$$

$$= \frac{1}{M_{\rm Pl}^2} T_{\mu\nu}. \tag{A.16}$$

The Green function follows,

$$O_{\mu\nu}{}^{\alpha\beta} G_{\alpha\beta\gamma\delta}(x-y) = \frac{1}{2} I_{\mu\nu\gamma\delta} \delta_D^{(4)}(x-y), \tag{A.17}$$

with "identity" tensor defined as

$$I_{\mu\nu\gamma\delta} \equiv \frac{1}{2} \left(\eta_{\mu\gamma} \eta_{\nu\delta} + \eta_{\mu\delta} \eta_{\nu\gamma} \right), \tag{A.18}$$

We then concern gauge fixing as the operator $O_{\mu\nu}{}^{\alpha\beta}$ cannot be inverted.

A.3 Gauge transformation

Under the coordinate transformation,

$$x^\mu \to \tilde{x}^\mu = x^\mu + \kappa\xi^\mu(x), \tag{A.19}$$

the metric reads,

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \tag{A.20}$$

We choose the de Donder gauge, which reads,

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{A.21}$$

It is useful to define the field $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, from which we can rewrite the A.16 as,

$$\Box\bar{h}_{\mu\nu} = -\frac{\kappa}{2}T_{\mu\nu} \tag{A.22}$$

The Green function A.17 can be re-expressed as,

$$\left(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)\Box G_{\alpha\beta\gamma\delta} = I^{\mu\nu}{}_{\gamma\delta}, \tag{A.23}$$

from which the gravitational field $h_{\mu\nu}$ can be extracted, using the ansatz $G_{\alpha\beta\gamma\delta} = aI_{\alpha\beta\gamma\delta} + b\eta_{\alpha\beta}\eta_{\gamma\delta}$ which yields,

$$G_{\alpha\beta\gamma\delta} = I_{\alpha\beta\gamma\delta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}. \tag{A.24}$$

In the position representation, the Green function reads,

$$G_{\mu\nu\alpha\beta} = \frac{1}{2\Box}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} + \eta_{\mu\nu}\eta_{\alpha\beta})\delta_D^{(4)}(x-y) \tag{A.25}$$

$$= -\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} + \eta_{\mu\nu}\eta_{\alpha\beta})\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2}. \tag{A.26}$$

This expression should come with the initial condition that defined by the choice of the contour on complex k_0-plane.

B Field Redefinition

Equivalent theorem [7] states that under reparameterization of field operators, the Scattering matrix remains unchanged, which means under redefinition of the field $\phi \to \tilde{\phi} = \phi + a_i\phi^i$, the generating functional,

$$\mathcal{Z} = \int D[\phi]\exp\left(i\int d^4x\mathcal{L}(\phi,\partial_\mu\phi)\right) \tag{B.1}$$

does not change as long as the Jacobian is essentially one [7]. Hence, we can use this technique to simplify the Lagrangian, for example,

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{M_{\text{Pl}}}[a_1 h_{\mu\gamma} h_\nu{}^\gamma + a_2 h_{\mu\nu} h_\gamma{}^\gamma]. \tag{B.2}$$

Substituting this into the triple graviton vertex gives,

$$h_{\mu\nu}\partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta \to h_{\mu\nu}\partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta + a_1 \frac{2}{M_{\text{Pl}}} h_{\mu\gamma} h_\nu{}^\gamma \partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta \tag{B.3}$$

$$+ a_2 \frac{2}{M_{\text{Pl}}} h_{\mu\nu} h_\gamma{}^\gamma \partial^\mu h^{\nu\alpha}\partial_\alpha h_\beta{}^\beta. \tag{B.4}$$

Hence, the field definition generates two addition quadruple graviton vertex with two parameter a_1, a_2. With a proper choice of these parameters, we can cancel some of the quadruple graviton vertex contributions in the original Lagrangian.

C Effective Field Theory in Gravity

In gravity, the quantities which can be used to construct higher operators of the effective Lagrangian are Rienman tensor $R_{\mu\nu\alpha\beta}$, Ricci tensor $R_{\mu\nu}$, and Ricci scalar R. Those quantities contain 2 partial derivative ($\sim \partial\partial h$), hence the Lagrangian has only even dimention terms.

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + ... \tag{C.1}$$

where

- $\mathcal{L}^{(0)}$: Constants (such as Cosmology constant Λ - which usually be neglected)
- $\mathcal{L}^{(2)}$: Only one term R
- $\mathcal{L}^{(4)}$: 3 possible terms $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, R_{\mu\nu}R^{\mu\nu}, R^2$
- $\mathcal{L}^{(6)}$: 4 possible terms $R_{\mu\nu\alpha\beta}R^{\mu\nu}R^{\alpha\beta}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}R, R_{\mu\nu}R^{\mu\nu}R, R^3$.

which are Lorentz invariant and General Coordinate Transformations invariant.

When we neglect the higher operators, the effective theory is non-local at high energy while restored locality at low energy. We can utilize this properties to reduce loops of the full theory to effective vertices at low-energy.

D Derivation of the positivity bound for $ZZ \to ZZ$ process

D.1 S-type operators

We have,

$$D_\mu \Phi = \left(\partial_\mu + i\frac{g}{2}W_\mu^j \sigma^j + i\frac{g'}{2}B_\mu\right)\Phi. \tag{D.1}$$

$$= \begin{pmatrix} \partial_\mu + i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu & i\frac{g}{2}W_\mu^1 + \frac{g'}{2}W_\mu^2 \\ i\frac{g}{2}W_\mu^1 - \frac{g'}{2}W_\mu^2 & \partial_\mu - i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{D.2}$$

$$= \frac{v}{\sqrt{2}} \begin{pmatrix} i\frac{g}{2}W_\mu^1 + \frac{g'}{2}W_\mu^2 \\ -i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu \end{pmatrix}. \tag{D.3}$$

Hence,

$$(D_\mu\Phi)^\dagger D_\nu \Phi = \frac{v^2}{8}\Bigg[g^2 W_\mu^1 W_\nu^1 + g^2 W_\mu^2 W_\nu^2 + igg'(W_\mu^1 W_\nu^2 - W_\nu^1 W_\mu^2) \tag{D.4}$$

$$+ (g'B_\mu - gW_\mu^3)(g'B_\nu - gW_\nu^3)\Bigg], \tag{D.5}$$

with the gauge boson diagonalization reads:

$$\begin{cases} B_\mu = \cos\theta A_\mu - \sin\theta Z_\mu \\ W_\mu^3 = \sin\theta A_\mu + \cos\theta Z_\mu \end{cases}, \tag{D.6}$$

with $\tan\theta = \frac{g'}{g}$. Hence,

$$(D_\mu\Phi)^\dagger D_\nu\Phi \supset \frac{v^2}{8}(g'B_\mu - gW_\mu^3)(g'B_\nu - gW_\nu^3) \tag{D.7}$$

$$\supset \frac{v^2}{8}(g'\sin\theta + g\cos\theta)^2 Z_\mu Z_\nu \tag{D.8}$$

$$= \frac{v^2 g^2}{8}\left(\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right)^2 Z_\mu Z_\nu \tag{D.9}$$

$$= \frac{v^2 g^2}{8\cos^2\theta} Z_\mu Z_\nu \tag{D.10}$$

$$= \frac{m^2}{2} Z_\mu Z_\nu. \tag{D.11}$$

Here, we have used the relation $g'\cos\theta = g\sin\theta (= e)$, and $vg = 2m_Z\cos\theta$. So we have,

$$\mathcal{L}_{S,0} = \left[(D_\mu\Phi)^\dagger D_\nu\Phi\right]\left[(D^\mu\Phi)^\dagger D^\nu\Phi\right] \supset \frac{m^4}{4} Z_\mu Z^\nu Z^\mu Z_\nu. \tag{D.12}$$

We also derive the exact same result for $\mathcal{L}_{S,1}, \mathcal{L}_{S,2}$ for Z^4 vertices. However, for W^4, $(WZ)^2$ (and W^2Z^2) vertices the indices for those terms are not the same due to different polarization of $W^{\pm}$.

Since S-type operators only admit longitudinal polarization, the polarization vectors for $1,2 \to 1,2$ process (with $p_1 = -p_2 = p, E_1 = E_2 = E$) read:

$$\epsilon_1^\mu =(a_3 \frac{p}{m}, 0, 0, a_3 \frac{E}{m}), \tag{D.13}$$

$$\epsilon_2^\mu =(-b_3 \frac{p}{m}, 0, 0, b_3 \frac{E}{m}). \tag{D.14}$$

Hence, for later convenience, we note here some relations:

$$\epsilon_1^{(*)}\epsilon_2^{(*)} = - a_3^{(*)} b_3^{(*)} \left(\frac{E^2 + p^2}{m^2} \right), \tag{D.15}$$

$$\epsilon_1^* \epsilon_1 = - |a_3|^2, \tag{D.16}$$

$$\epsilon_2^* \epsilon_2 = - |b_3|^2. \tag{D.17}$$

The Z field read:

$$Z^\mu(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \sqrt{2E_k}} \sum_{i=0}^{3} \left[a_{k,j} \epsilon_j^\mu e^{-ikx} + a_{k,j}^\dagger \epsilon_j^{*\mu} e^{ikx} \right]. \tag{D.18}$$

in which the creation/annihilation operators obey the commutation relation $[a_{k_1}, a_{k_2}^\dagger] = (2\pi)^3 \delta^3(\mathbf{k}_1 - \mathbf{k}_2)$, $[a_{k_1}, a_{k_2}] = [a_{k_1}^\dagger, a_{k_2}^\dagger] = 0$. The incoming and outgoing states read:

$$\langle f| = \langle p_1, p_2| =2E a_{p_1} a_{p_2}, \tag{D.19}$$

$$|i\rangle = |p_1, p_2\rangle =2E a_{p_1}^\dagger a_{p_2}^\dagger. \tag{D.20}$$

Now consider the interacting vertices of $\langle f| : Z_\mu Z^\nu Z^\mu Z_\nu : |i\rangle$, we extract all the posible combination of creation/annihilation operators $(a, a^\dagger)$ for the $ZZ \to ZZ$ chanel.

First, we see that all the terms have to contain $2a$ and $2a^\dagger$. Then, of those $4Z$, there are six ways to exact such terms. We first consider the 2 cases when $2a$ is extracted from ZZ pair with the same Lorentz indices ($Z_\mu Z^\mu$ or $Z_\nu Z^\nu$), and then other 4 cases when $2a$ is extracted from ZZ pair with the different indices ($Z_\mu Z^\nu$ or $Z_\mu Z_\nu$ or $Z^\mu Z^\nu$ or $Z^\mu Z_\nu$). We'll investigate each cases in the 2 group and see that the rest obey the same rules.

Let's consider the case when exacting $2a$ from $Z_\mu Z^\mu$ (and $2a^\dagger$ from $Z_\nu Z^\nu$). For $1,2 \to 1,2$ process, using the result from commutation relation that $\langle p_1, p_2| a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} |p_1, p_2\rangle =$

$4E^2 \langle 0| [a_{p_1}, a^\dagger_{k_1}][a_{p_2}, a^\dagger_{k_2}] + [a_{p_1}, a^\dagger_{k_2}][a_{p_2}, a^\dagger_{k_1}] + [a_{k_3}, a^\dagger_{p_1}][a_{k_4}, a^\dagger_{p_2}] + [a_{k_4}, a^\dagger_{p_1}][a_{k_3}, a^\dagger_{p_2}] |0\rangle$, we get 4 terms from the extraction as follow:

Z_μ	Z^ν	Z^μ	Z_ν
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
1	2	2	1
2	1	1	2
2	2	1	1

From D.15, the first term read:

$$\epsilon_{1\mu}\epsilon_1^{*\nu}\epsilon_2^{\mu}\epsilon_{2\nu}^* = (-1)^2(a_3 b_3 a_3^* b_3^*)\left(\frac{p^2+E^2}{m^2}\right)^2 \tag{D.21}$$

$$= |a_3|^2|b_3|^2\left(\frac{p^2+E^2}{m^2}\right)^2 \tag{D.22}$$

All the rest 3 terms yield the same result since the Lorentz indices are contracted between 1 and 2 (x4). Moreover, the case when $2a$ is extracted from $Z_\nu Z^\nu$ also yields the same results (x2).

For the rest 4 cases, we investigate the first one,

Z_μ	Z^ν	Z^μ	Z_ν
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
2	1	1	2
1	2	2	1

Using D.15, D.16, the first term read:

$$\epsilon_{1\mu}\epsilon_2^{*\nu}\epsilon_1^{\mu}\epsilon_{2\nu}^* = |a_3|^2|b_3|^2. \tag{D.23}$$

The second term has Lorentz indices contracted in the same fashion and yields the same result (x2). The third term read:

$$\epsilon_{2\mu}\epsilon_1^{*\nu}\epsilon_1^{\mu}\epsilon_{2\nu}^* = |a_3|^2|b_3|^2\left(\frac{p^2+E^2}{m^2}\right)^2, \tag{D.24}$$

which is also the result for the fourth term (x2). Moreover, we see that all the rest 3 cases yield exactly the same results (x4)

To sum up, and then to drop the m^4 component, we have same the result for $\mathcal{L}_{S,0}, \mathcal{L}_{S,1}, \mathcal{L}_{S,2}$:

$$\begin{aligned}
&\frac{m^4}{4} Z_\mu Z^\nu Z^\mu Z_\nu \left[(4.2+2.4)|a_3|^2|b_3|^2 \left(\frac{p^2+E^2}{m^2}\right)^2 + 2.4|a_3|^2|b_3|^2 \right] \\
\supset& 16|a_3|^2|b_3|^2 E^2 p^2 \qquad \text{(D.25)} \\
\equiv& 16 A_6 E^2 p^2. \qquad \text{(D.26)}
\end{aligned}$$

D.2 T-type operators

First, we expand,

$$\begin{aligned}
\hat{W}^{\mu\nu} \equiv& ig\frac{\sigma^j}{2} W^{j,\mu\nu} \qquad \text{(D.27)} \\
=& i\frac{g}{2} \begin{pmatrix} W^{3\mu\nu} & W^{1\mu\nu} - iW^{2\mu\nu} \\ W^{1\mu\nu} - iW^{2\mu\nu} & -W^{3\mu\nu} \end{pmatrix} \qquad \text{(D.28)} \\
=& i\frac{g}{2} \begin{pmatrix} \cos\theta Z^{\mu\nu} + \sin\theta A^{\mu\nu} & \sqrt{2} W^{+\mu\nu} \\ \sqrt{2} W^{-\mu\nu} & -\cos\theta Z^{\mu\nu} - \sin\theta A^{\mu\nu} \end{pmatrix}. \qquad \text{(D.29)}
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\alpha\beta}\right] \supset& -\frac{g^2}{2}\cos^2\theta Z_{\mu\nu} Z^{\alpha\beta} \qquad \text{(D.30)} \\
=& -\frac{e^2}{2}\cot^2\theta Z_{\mu\nu} Z^{\alpha\beta}. \qquad \text{(D.31)}
\end{aligned}$$

Also,

$$\hat{B}_{\mu\nu} \equiv i\frac{g'}{2} B_{\mu\nu} \supset -i\frac{g'}{2}\sin\theta Z_{\mu\nu} = -i\frac{e}{2}\tan\theta Z_{\mu\nu}, \qquad \text{(D.32)}$$

and,

$$\hat{B}_{\mu\nu}\hat{B}^{\alpha\beta} \supset -\frac{e^2}{4}\tan^2\theta Z_{\mu\nu} Z^{\alpha\beta}. \qquad \text{(D.33)}$$

We can see that replacing $\mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\alpha\beta}\right]$ by $\hat{B}_{\mu\nu}\hat{B}^{\alpha\beta}$ yields a factor of $\frac{\tan^4\theta}{2}$. Hence, we only need to calculate $\mathcal{L}_{T,0}, \mathcal{L}_{T,1}, \mathcal{L}_{T,2}$ and all other T-type operators can be derive by adding that factors. The derivative of Z field reads:

$$\partial^\nu Z^\mu(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2E_k}} \sum_{i=0}^{3} \left[-ik^\nu a_{k,j}\epsilon_j^\mu e^{-ikx} + ik^\nu a_{k,j}^\dagger \epsilon_j^{*\mu} e^{ikx} \right]. \qquad \text{(D.34)}$$

The $\mathcal{L}_{T,i}$ vertices only admit tranverse component of polarization, hence, they read:

$$\epsilon_1^\mu = (0, a_1, a_2, 0), \tag{D.35}$$
$$\epsilon_2^\mu = (0, b_1, b_2, 0). \tag{D.36}$$

Some useful relations:

$$\epsilon_1^{(*)}\epsilon_2^{(*)} = -a_1^{(*)}b_1^{(*)} - a_2^{(*)}b_2^{(*)}, \tag{D.37}$$
$$\epsilon_1^*\epsilon_1 = -|a_1|^2 - |a_2|^2, \tag{D.38}$$
$$\epsilon_2^*\epsilon_2 = -|b_1|^2 - |b_2|^2. \tag{D.39}$$

D.2.1 $\mathcal{L}_{T,0}$, $\mathcal{L}_{T,1}$, $\mathcal{L}_{T,5}$, $\mathcal{L}_{T,6}$, $\mathcal{L}_{T,8}$

First consider $\mathcal{L}_{T,0}$,

$$\begin{aligned}
&\mathrm{Tr}\Big[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\Big] \times \mathrm{Tr}\Big[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}\Big] \\
\supset&\frac{e^4}{4}\cot\theta^4 Z_{\mu\nu}Z^{\mu\nu}Z_{\alpha\beta}Z^{\alpha\beta} &\text{(D.40)}\\
\supset&\frac{e^4}{4}\cot\theta^4(\partial_\mu Z_\nu\partial^\mu Z^\nu\partial_\alpha Z_\beta\partial^\alpha Z^\beta + \partial_\mu Z_\nu\partial^\mu Z^\nu\partial_\beta Z_\alpha\partial^\beta Z_\alpha \\
&+ \partial_\nu Z_\mu\partial^\nu Z^\mu\partial_\alpha Z_\beta\partial^\alpha Z^\beta + \partial_\nu Z_\mu\partial^\nu Z^\mu\partial_\beta Z_\alpha\partial^\beta Z_\alpha). &\text{(D.41)}
\end{aligned}$$

We derived D.41 by droping all 12 terms contain $Z_\gamma\partial^\gamma$ since contracting them yield $\epsilon_\gamma p^\gamma = 0$ (no matter they belong to the "1" or "2" states as we only have tranverse polarizations for T-type operators).

We see that those 4 term are contracted in the identical way (symmetric in $\mu \leftrightarrow \nu, \alpha \leftrightarrow \beta$), hence we only need to consider 1 term and then multiply the results by 4 (x4). Let's pick the $\partial_\nu Z_\mu\partial^\nu Z^\mu\partial_\alpha Z_\beta\partial^\alpha Z^\beta$ to investigate. There are also 6 cases of extracting $: a^\dagger a^\dagger a a :$. Consider the first case, we have 4 terms:

$\partial_\nu Z_\mu$	$\partial^\nu Z^\mu$	$\partial_\alpha Z_\beta$	$\partial^\alpha Z^\beta$
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
2	1	1	2
1	2	2	1

The first term read:

$$p_{1\nu}\epsilon_{1\mu}p_2^\nu\epsilon_2^\mu p_{1\alpha}\epsilon_{1\beta}^* p_2^\alpha\epsilon_2^{\beta *} = (E^2 + p^2)^2(a_1^*b_1^* + a_2^*b_2^*)(a_1b_1 + a_2b_2). \tag{D.42}$$

The rest 3 terms yield the same result (x4) as all the contractions are between "1" and "2" state. The second case where we swap all $a \leftrightarrow a^\dagger$ follows the same principle and ends up in the same results (x2). Consider one of the rest 4 cases, there are also 4 terms in it:

$\partial_\nu Z_\mu$	$\partial^\nu Z^\mu$	$\partial_\alpha Z_\beta$	$\partial^\alpha Z^\beta$
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
2	2	1	1
1	2	2	1
2	1	1	2

The first one read:

$$p_{1\nu}\epsilon_{1\mu}p_1^\nu \epsilon_1^{\mu *} p_{\alpha 2}\epsilon_{\beta 2}p_2^\alpha \epsilon_2^{\beta *} = m^4(|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2). \tag{D.43}$$

The second term yields the same results as it only exchange "$1 \leftrightarrow 2$" (x2). The third term read:

$$p_{1\nu}\epsilon_{1\mu}p_2^\nu \epsilon_2^{\nu *} p_{\alpha 2}\epsilon_{\beta 2}p_1^\alpha \epsilon_1^{\beta *} = (E^2 + p^2)^2(a_1 b_1^* + a_2 b_2^*)(a_1^* b_1 + a_2^* b_2) \tag{D.44}$$

This is also the result for the final term (x2). All the rest 3 cases yields the same results (x4) as they only differ by the exchange of $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$. To sum up and then to drop m^4 term, we have

$$\begin{aligned}
&\frac{e^4}{4}\cot\theta^4 \Big[(E^2+p^2)^2 4\left[4.2(a_1^* b_1^* + a_2^* b_2^*)(a_1 b_1 + a_2 b_2) + 2.4(a_1 b_1^* + a_2 b_2^*)(a_1^* b_1 + a_2^* b_2)\right] \\
&+4.2.4m^4(|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)\Big] \\
&\supset e^4 \cot\theta^4 \big[64\left(|a_1|^2|a_2|^2 + |a_2|^2|b_2|^2\right) + 32\left(a_1 a_2^* b_1^* b_2 + c.c.\right) + 32(a_1 a_2^* b_1 b_2^* + c.c)\big] E^2 p^2
\end{aligned} \tag{D.45}$$

$$\equiv e^4 \cot\theta^4 \left[64 A_1 + 32 A_4 + 32 A_4'\right] E^2 p^2. \tag{D.46}$$

$\mathcal{L}_{T,1}$ yields the same result as they only exchange $\mu \leftrightarrow \alpha, \beta \leftrightarrow \nu$ in the term D.40. $\mathcal{L}_{T,5}$ is also equal $\mathcal{L}_{T,6}$ and can be calculated as,

$$\begin{aligned}
&\frac{\tan^4\theta}{2} e^4 \cot\theta^4 \left[64 A_1 + 32 A_4 + 32 A_4'\right] E^2 p^2 \\
=& e^4 \left[32 A_1 + 16 A_4 + 16 A_4'\right] E^2 p^2.
\end{aligned} \tag{D.47}$$

$\mathcal{L}_{T,8}$ is derived as,

$$\begin{aligned}
&\frac{\tan^4\theta}{2} e^4 \left[32 A_1 + 16 A_4 + 16 A_4'\right] E^2 p^2 \\
=& e^4 \tan^4\theta \left[16 A_1 + 8 A_4 + 8 A_4'\right] E^2 p^2.
\end{aligned} \tag{D.48}$$

D.2.2 $\mathcal{L}_{T,2}, \mathcal{L}_{T,7}, \mathcal{L}_{T,9}$

Next, consider $\mathcal{L}_{T,2}$,

$$
\begin{aligned}
&\mathrm{Tr}\Big[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\Big] \times \mathrm{Tr}\Big[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}\Big] \\
\supset &\frac{g^4\cos^4\theta}{4} Z_{\alpha\mu}Z^{\mu\beta}Z_{\beta\nu}Z^{\nu\alpha} \qquad \text{(D.49)}\\
\supset &\frac{g^4\cos^4\theta}{4}(\partial_\alpha Z_\mu \partial^\beta Z^\mu \partial_\beta Z_\nu \partial^\alpha Z^\nu + \partial_\mu Z_\alpha \partial^\mu Z^\beta \partial_\nu Z_\beta \partial^\nu Z^\alpha). \qquad \text{(D.50)}
\end{aligned}
$$

In D.50, we also dropped all 14 terms contain $Z_\gamma\partial^\gamma$. The two above term differ only by swapping $\alpha \leftrightarrow \mu, \nu \leftrightarrow \beta$, and hence yield the same result (x2). Let's investigate the first one. There are also 6 cases of extracting $: a^\dagger a^\dagger a a :$. First, consider:

$\partial_\alpha Z_\mu$	$\partial^\beta Z^\mu$	$\partial_\beta Z_\nu$	$\partial^\alpha Z^\nu$
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
1	2	2	1
2	1	1	2

The first term read:

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^\beta \epsilon_2^\mu p_{\beta 1}\epsilon^*_{\nu 1}p_2^\alpha \epsilon_2^{\nu *} = (E^2+p^2)^2(a_1b_1+a_2b_2)(a_1^*b_1^*+a_2^*b_2^*). \qquad \text{(D.51)}$$

The second term yield the same result(x2). Let's consider the third one,

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^\beta \epsilon_2^\mu p_{\beta 2}\epsilon^*_{\nu 2}p_1^\alpha \epsilon_1^{\nu *} = m^4(a_1b_1+a_2b_2)(a_1^*b_1^*+a_2^*b_2^*). \qquad \text{(D.52)}$$

This is also the result for the final term (x2).

Now, since swapping $a \leftrightarrow a^\dagger$ is equivalent to swapping the indices pairwise, we have a second case that yield exactly the same results as the first one (x2).

Next, consider the third case of

$\partial_\alpha Z_\mu$	$\partial^\beta Z^\mu$	$\partial_\beta Z_\nu$	$\partial^\alpha Z^\nu$
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
2	2	1	1
1	2	2	1
2	1	1	2

The first term (and also the second) (x2) yields,

$$p_{\alpha 1}\epsilon_{\mu 1}p_1^{\beta}\epsilon_1^{\mu *}p_{\beta 2}\epsilon_{\nu 2}p_2^{\alpha}\epsilon_2^{\nu *} = (E^2+p^2)^2(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2). \tag{D.53}$$

The third (and also fourth) (x2) yields,

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^{\beta}\epsilon_2^{\mu *}p_{\beta 2}\epsilon_{\nu 2}p_1^{\alpha}\epsilon_1^{\nu *} = m^4(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2). \tag{D.54}$$

Swaping $a \leftrightarrow a^\dagger$ does not change the result, hence, we have the fourth case that also yield the same result (x2). Now for the last 2 cases. First, consider,

$\partial_\alpha Z_\mu$	$\partial^\beta Z^\mu$	$\partial_\beta Z_\nu$	$\partial^\alpha Z^\nu$
a	$a^\dagger$	$a^\dagger$	a
1	1	2	2
2	2	1	1
1	2	1	2
2	1	2	1

The first (and also second) term (x2) read,

$$p_{\alpha 1}\epsilon_{\mu 1}p_1^{\beta}\epsilon_1^{\mu *}p_{\beta 2}\epsilon_{\nu 2}^{*}p_2^{\alpha}\epsilon_2^{\nu} = (E^2+p^2)^2(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2). \tag{D.55}$$

The third (and fourth) (x2) read,

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^{\beta}\epsilon_2^{\mu *}p_{\beta_1}\epsilon_{\nu 1}^{*}p_2^{\alpha}\epsilon_2^{\nu} = (E^2+p^2)^2(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2). \tag{D.56}$$

Swaping $a \leftrightarrow a^\dagger$ does not change the result, hence, we have the sixth case that also yield the same result (x2).

We have completed calculating all the posibilities for this vetices. Summing up and then subtracting the m^4 term we have the result for $\mathcal{L}_{T,2}$:

$$\begin{aligned}
&\frac{e^4}{4}\cot\theta^4\Big[(E^2+p^2)\big[2.2.2(a_1b_1+a_2b_2)(a_1^*b_1^*+a_2^*b_2^*)\\
&+2.2.2.2(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2)+2.2.2(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2)\big]\\
&+2.2.2m^4(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2)\Big]\\
&\supset e^4\cot\theta^4\Big[32\left(|a_1|^2|b_1|^2+|a_2|^2|b_2|^2\right)+32\left(|a_1|^2|b_2|^2+|a_2|^2|b_1|^2\right) && \text{(D.57)}\\
&\quad+8\left(a_1a_2^*b_1^*b_2+c.c.\right)+8(a_1a_2^*b_1b_2^*+c.c)\Big]E^2p^2 && \text{(D.58)}\\
&\equiv e^4\cot^4\theta\left[32A_1+16A_2+8A_4+8A_4'\right]E^2p^2. && \text{(D.59)}
\end{aligned}$$

From there, the $\mathcal{L}_{T,7}$ can be derived,

$$\frac{\tan^4\theta}{2}e^4\cot^4\theta\left[32A_1+16A_2+8A_4+8A_4'\right]E^2p^2 \tag{D.60}$$
$$=e^4\left[16A_1+8A_2+4A_4+4A_4'\right]E^2p^2. \tag{D.61}$$

And also, the $\mathcal{L}_{T,9}$

$$\frac{\tan^4\theta}{2}e^4\left[32A_1+16A_2+8A_4+8A_4'\right]E^2p^2 \tag{D.62}$$
$$=e^4\tan^4\theta\left[8A_1+4A_2+2A_4+2A_4'\right]E^2p^2. \tag{D.63}$$

D.3 M-type operators

We have,

$$D_\mu\Phi \supset \frac{v}{\sqrt{2}}\begin{pmatrix}0\\ -i\frac{g}{2}W_\mu^3+i\frac{g'}{2}B_\mu\end{pmatrix} \tag{D.64}$$
$$\supset -\frac{im_Z}{\sqrt{2}}\begin{pmatrix}0\\ Z_\mu\end{pmatrix}. \tag{D.65}$$

Also,

$$\hat{W}_{\mu\nu}\supset\frac{ie\cot\theta}{2}\begin{pmatrix}Z^{\mu\nu} & 0\\ 0 & -Z^{\mu\nu}\end{pmatrix}. \tag{D.66}$$

Hence,

$$[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\mu\Phi]\times\hat{B}^{\beta\nu}\supset\frac{im}{\sqrt{2}}\frac{-ie\cot\theta}{2}\frac{-im}{\sqrt{2}}\frac{-ie\tan\theta}{2}Z_\mu Z_{\beta\nu}Z^\mu Z^{\beta\nu} \tag{D.67}$$
$$=-\frac{m^2e^2}{8}Z_\mu Z_{\beta\nu}Z^\mu Z^{\beta\nu}. \tag{D.68}$$

Similarly,

$$[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\nu\Phi]\times\hat{B}^{\beta\mu}\supset-\frac{m^2e^2}{8}Z_\mu Z_{\beta\nu}Z^\nu Z^{\beta\mu}. \tag{D.69}$$

With this result, we will see later that $\mathcal{L}_{M,4}$ (and $\mathcal{L}_{M,5}$) only differs from $\mathcal{L}_{M,0}$ (and $\mathcal{L}_{M,1}$) by a factor of $-\frac{\tan^2\theta}{2}$. We also have,

$$\hat{W}_{\beta\nu}\hat{W}^{\beta\mu}\supset-\frac{e^2\cot^2\theta}{4}\begin{pmatrix}Z_{\beta\nu}Z^{\beta\mu} & 0\\ 0 & Z_{\beta\nu}Z^{\beta\mu}\end{pmatrix} \tag{D.70}$$

Hence,

$$[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\mu}D^\nu\Phi] \supset \frac{im}{\sqrt{2}}\frac{-e^2\cot^2\theta}{4}\frac{-im}{\sqrt{2}}Z_\mu Z_{\beta\nu}Z^{\beta\mu}Z^\nu \tag{D.71}$$

$$= -\frac{e^2}{8}\cot^2\theta m^2 Z_\mu Z_{\beta\nu}Z^{\beta\mu}Z^\nu, \tag{D.72}$$

from which we see that $\mathcal{L}_{M,7}$ differs from $\mathcal{L}_{M,1}$ by a factor of $-\frac{1}{2}$.

D.3.1 $\mathcal{L}_{M,0}, \mathcal{L}_{M,2}, \mathcal{L}_{M,4}$

First, consider $\mathcal{L}_{M,0}$,

$$\mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right]\left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{D.73}$$

$$\supset \frac{e^2\cot^2\theta}{2}Z_{\mu\nu}Z^{\mu\nu}\frac{m^2}{2}Z_\beta Z^\beta \tag{D.74}$$

$$\supset \frac{e^2}{4}\cot^2\theta m^2\left(\partial_\mu Z_\nu\partial^\mu Z^\nu + \partial_\nu Z_\mu\partial^\nu Z^\mu\right)Z_\beta Z^\beta. \tag{D.75}$$

Those 2 terms also yield same results (x2). Consider one of them ($\partial_\mu Z_\nu \partial^\mu Z^\nu Z_\beta Z^\beta$), there are 6 ways of extracting :$a^\dagger a^\dagger a a$:. Consider the first way,

$\partial_\mu Z_\nu$	$\partial^\mu Z^\nu$	Z_β	Z^β
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
1	2	2	1
2	1	1	2
2	1	2	1

First term reads,

$$p_{1\mu}\epsilon_{1\nu}p_2^\mu\epsilon_2^\nu\epsilon_{1\beta}^*\epsilon_2^{\beta *} = \frac{(E^2+p^2)^2}{m^2}(a_1b_1+a_2b_2)a_3^*b_3^*. \tag{D.76}$$

All the rest 3 terms read the same result (x4) since swaping "1" $\leftrightarrow$ "2" with a Lorenzt contracted term does not change the result.

Consider the next case of swaping all $a \leftrightarrow a^\dagger$, we have the result is the complex conjugate of the first case.

Moving on to the next case, we consider,

$\partial_\mu Z_\nu$	$\partial^\mu Z^\nu$	Z_β	Z^β
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
1	2	2	1
2	2	1	1
2	1	1	2

First term read,

$$p_{1\mu}\epsilon_{1\nu}p_1^{\mu}\epsilon_1^{\nu *}\epsilon_{2\beta}\epsilon_2^{\beta *} = m^2(|a_1|^2 + |a_2|^2)|b_3|^2. \tag{D.77}$$

Second term read,

$$p_{1\mu}\epsilon_{1\nu}p_2^{\mu}\epsilon_2^{\nu *}\epsilon_{2\beta}\epsilon_1^{\beta *} = -\frac{(E^2+p^2)^2}{m^2}(a_1b_1^* + a_2b_2^*)a_3^*b_3. \tag{D.78}$$

Third term read,

$$p_{2\mu}\epsilon_{2\nu}p_2^{\mu}\epsilon_2^{\nu *}\epsilon_{1\beta}\epsilon_1^{\beta *} = m^2(|b_1|^2 + |b_2|^2)|a_3|^2. \tag{D.79}$$

Last term read,

$$p_{2\mu}\epsilon_{2\nu}p_1^{\mu}\epsilon_1^{\nu *}\epsilon_{1\beta}\epsilon_2^{\beta *} = -\frac{(E^2+p^2)^2}{m^2}(a_1^*b_1 + a_2^*b_2)a_3b_3^*. \tag{D.80}$$

All the rest 3 cases yield exactly the same results since they only swap $a \leftrightarrow a^\dagger$ within Lorent invariant terms (x4).

To sum up, then to drop the m^4 term, we have the result for $\mathcal{L}_{M,0}$:

$$\begin{aligned}
&\frac{e^2}{4}\cot^2\theta m^2\Bigg[\frac{(E^2+p^2)^2}{m^2}\Big[4(a_1b_1+a_2b_2)a_3^*b_3^* + c.c. \\
&-4(a_1b_1^*+a_2b_2^*)a_3^*b_3 - c.c.\Big] \\
&+4m^2\Big[(|a_1|^2+|a_2|^2)|b_3|^2 + (|b_1|^2+|b_2|^2)|a_3|^2\Big]\Bigg] \\
\supset& e^2\cot^2\theta\Big[4(a_1b_1+a_2b_2)a_3^*b_3^* + c.c. - 4(a_1b_1^*+a_2b_2^*)a_3^*b_3 - c.c.\Big]E^2p^2 && \text{(D.81)} \\
\equiv& e^2\cot^2\theta\,[4A_5 + 4A_5']\,E^2p^2. && \text{(D.82)}
\end{aligned}$$

We can also derive the result for $\mathcal{L}_{M,2}$ which is

$$\frac{\tan^4\theta}{2}e^2\cot^2\theta\,[4A_5+4A_5']\,E^2p^2 = e^2\tan^2\theta\,[2A_5+2A_5']\,E^2p^2. \tag{D.83}$$

And the result for $\mathcal{L}_{M,4}$,

$$-\frac{\tan^2\theta}{2}e^2\cot^2\theta\,[4A_5+4A_5']\,E^2p^2 = -e^2\,[2A_5+2A_5']\,E^2p^2. \tag{D.84}$$

D.3.2 $\mathcal{L}_{M,1}, \mathcal{L}_{M,3}, \mathcal{L}_{M,5}, \mathcal{L}_{M,7}$

First, consider $\mathcal{L}_{M,1}$

$$\mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\nu\beta}\right]\left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{D.85}$$

$$\supset\frac{e^2\cot^2\theta}{2}\cos^2\theta Z_{\mu\nu}Z^{\nu\beta}\frac{m^2}{2}\cos^2\theta Z_\beta Z^\mu \tag{D.86}$$

$$\supset\frac{e^2}{4}\cot^2\theta m^2\left(-\partial_\mu Z_\nu\partial^\beta Z^\nu - \partial_\nu Z_\mu\partial^\nu Z^\beta\right)Z_\beta Z^\mu. \tag{D.87}$$

Note that the contraction in the first term $(Z_\beta Z_\mu\partial^\beta\partial^\mu)$ doesn't varnish. However, the second term varnish since it contracts the pure longitudinal with transversal polarised Z-boson. Hence, we only consider the first term. There are some relation for contracting the longitudinal polarization vectors $\epsilon_{1\mu} = (a_3\frac{p}{m}, 0, 0, -a_3\frac{p}{m}), \epsilon_{2\mu} = (-b_3\frac{p}{m}, 0, 0, -b_3\frac{p}{m})$ with the 4-momentum $p_1^\mu = (E, 0, 0, p), p_2^\mu = (E, 0, 0, -p)$ as,

$$p_1\epsilon_1^{(*)} = p_2\epsilon_2^{(*)} = 0, \tag{D.88}$$

$$p_2\epsilon_1^{(*)} = 2a_3^{(*)}\frac{Ep}{m}, \tag{D.89}$$

$$p_1\epsilon_2^{(*)} = -2b_3^{(*)}\frac{Ep}{m}. \tag{D.90}$$

For the first term, there are also 6 cases, we calculate the first case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
$a^\dagger$	a	a	$a^\dagger$
1	1	2	2
2	2	1	1
2	1	2	1
1	2	1	2

The first term read,

$$p_{1\mu}\epsilon_{1\nu}^* p_1^\beta\epsilon_1^\nu\epsilon_{2\beta}\epsilon_2^{\mu*} = -4\frac{E^2p^2}{m^2}|b_3|^2(|a_1|^2 + |a_2|^2). \tag{D.91}$$

The second term read,

$$p_{2\mu}\epsilon_{2\nu}^* p_2^\beta\epsilon_2^\nu\epsilon_{1\beta}\epsilon_1^{\mu*} = -4\frac{E^2p^2}{m^2}|a_3|^2(|b_1|^2 + |b_2|^2). \tag{D.92}$$

The third term read:

$$p_{2\mu}\epsilon_{2\nu}^* p_1^\beta\epsilon_1^\nu\epsilon_{2\beta}\epsilon_1^{\mu*} = 4\frac{E^2p^2}{m^2}a_3^*b_3(a_1b_1^* + a_2b_2^*). \tag{D.93}$$

The fourth term read:

$$p_{1\mu}\epsilon^*_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon_{1\beta}\epsilon^{\mu *}_2 = 4\frac{E^2p^2}{m^2}a_3b^*_3(a^*_1b_1 + a^*_2b_2). \tag{D.94}$$

Swaping $a \leftrightarrow a^\dagger$ yields the same result, we got the second case (x2). Consider the third case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
$a^\dagger$	a	$a^\dagger$	a
1	1	2	2
2	2	1	1
1	2	2	1
2	1	1	2

The first term read,

$$p_{1\mu}\epsilon^*_{1\nu}p^\beta_1\epsilon^\nu_1\epsilon^*_{2\beta}\epsilon^\mu_2 = -4\frac{E^2p^2}{m^2}|a_3|^2(|b_1|^2 + |b_2|^2). \tag{D.95}$$

The second term read,

$$p_{2\mu}\epsilon^*_{2\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{1\beta}\epsilon^\mu_1 = -4\frac{E^2p^2}{m^2}|b_3|^2(|a_1|^2 + |a_2|^2). \tag{D.96}$$

For the third term, since $p_1\epsilon_1 = p_2\epsilon_2 = 0$, we have,

$$p_{1\mu}\epsilon^*_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{2\beta}\epsilon^\mu_1 = 0 \tag{D.97}$$

Since swaping $a \leftrightarrow a^\dagger$ yields the same result, we derive the same thing for the fourth term as well (x2). Consider the fifth case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
1	2	2	1
2	1	1	2

The first term reads,

$$p_{1\mu}\epsilon_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{1\beta}\epsilon^{\mu *}_2 = 4\frac{E^2p^2}{m^2}a^*_3b^*_3(a_1b_1 + a_2b_2). \tag{D.98}$$

The second term reads the same result (x2). The third term reads,

$$p_{1\mu}\epsilon_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{2\beta}\epsilon^{\mu *}_1 = 0. \tag{D.99}$$

This result also hold for the fourth term (x2). Finally, we get the sixth case by swaping $a \leftrightarrow a^\dagger$. It yields the complex conjugate of the fifth case (+c.c.). To sum up and then to subtract the m^4 term, we have,

$$\begin{aligned}
&-\frac{e^2}{4}\cot^2\theta m^2 4\frac{E^2p^2}{m^2}\Big[2.2|a_3|^2(|b_1|^2+|b_2|^2)+2.2|b_3|^2(|a_1|^2+|a_2|^2) \\
&-2a_3b_3^*(a_1^*b_1+a_2^*b_2)+c.c.+2a_3^*b_3^*(a_1b_1+a_2b_2)+c.c.\Big] \qquad \text{(D.100)} \\
=&-e^2\cot^2\theta\Big[4|a_3|^2(|b_1|^2+|b_2|^2)+4|b_3|^2(|a_1|^2+|a_2|^2) \\
&-2a_3b_3^*(a_1^*b_1+a_2^*b_2)+c.c.+2a_3^*b_3^*(a_1b_1+a_2b_2)+c.c.\Big]E^2p^2. \qquad \text{(D.101)} \\
\equiv&-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2. \qquad \text{(D.102)}
\end{aligned}$$

We can also derive the $\mathcal{L}_{M,3}$

$$\begin{aligned}
&\frac{\tan^4\theta}{2}\Big[-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2\Big] \\
=&-e^2\tan^2\theta(2A_3+2A_3'+A_5'+A_5)E^2p^2. \qquad \text{(D.103)}
\end{aligned}$$

and $\mathcal{L}_{M,5}$,

$$\begin{aligned}
&-\frac{\tan^2\theta}{2}\Big[-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2\Big] \qquad \text{(D.104)} \\
=&e^2(2A_3+2A_3'+A_5'+A_5)E^2p^2. \qquad \text{(D.105)}
\end{aligned}$$

and $\mathcal{L}_{M,7}$,

$$\begin{aligned}
&-\frac{1}{2}\Big[-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2\Big] \qquad \text{(D.106)} \\
=&+e^2\cot^2\theta(2A_3+2A_3'+A_5'+A_5)E^2p^2. \qquad \text{(D.107)}
\end{aligned}$$

D.4 Results

Summing up all the operators, with the coefficients are redefined as,

$$F_{S,i}\equiv f_{S,i};\, F_{M,i}\equiv e^2f_{M,i};\, F_{T,i}\equiv e^4f_{T,i}, \qquad \text{(D.108)}$$

we get,

ZZ:

$$\begin{aligned}
&16A_6(F_{S,0}+F_{S,1}+F_{S,2}) \\
&+[16A_1+8(A_4+A_4')]\left[4\cot^4\theta(F_{T,0}+F_{T,1})+2(F_{T,5}+F_{T,6})+\tan^4\theta F_{T,8}\right] \\
&+(8A_1+4A_2+2A_4+2A_4')(4\cot^4\theta F_{T,2}+2F_{T,7}+\tan^4\theta F_{T,9}) \\
&+(4A_5+4A_5')(2\cot^2\theta F_{M,0}+\tan^2\theta F_{M,2}-F_{M,4}) \\
&+(2A_3+2A_3'+A_5'+A_5)(-2\cot^2\theta F_{M,1}-\tan^2\theta F_{M,3}+F_{M,5}+\cot^2\theta F_{M,7})\geq 0. \qquad \text{(D.109)}
\end{aligned}$$

With the convention of,

$$\begin{aligned}
&A_1 \equiv |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2, && A_4 \equiv a_1 a_2^* b_1 b_2^* + c.c., \\
&A_2 \equiv |a_1|^2|b_2|^2 + |a_2|^2|b_1|^2, && A_4' \equiv a_1 a_2^* b_1^* b_2 + c.c., \\
&A_3 \equiv (|b_1|^2 + |b_2|^2)|a_3|^2, && A_5 \equiv (a_1 b_1 + a_2 b_2) a_3^* b_3^* + c.c., \\
&A_3' \equiv (|a_1|^2 + |a_2|^2)|b_3|^2, && A_5' \equiv -(a_1 b_1^* + a_2 b_2^*) a_3^* b_3 + c.c. \\
&A_3'' \equiv |b_1|^2|a_3|^2 && A_6 \equiv |a_3|^2|b_3|^2,
\end{aligned} \tag{D.110}$$

When we consider only real polarizations, $(A_5 + A_5')$ varnishes. Multiplying $\frac{\tan^4\theta}{2}$, we get,

$$\begin{aligned}
&ZZ: \\
&8At_W^4\left(F_{S,0} + F_{S,1} + F_{S,2}\right) + Dt_W^2\left(-t_W^4 F_{M,3} + t_W^2 F_{M,5} - 2F_{M,1} + F_{M,7}\right) \\
&+ (B + C)\left(2t_W^8 F_{T,9} + 4t_W^4 F_{T,7} + 8F_{T,2}\right) + 8B\left[t_W^4\left(t_W^4 F_{T,8} + 2F_{T,5} + 2F_{T,6}\right) + 4F_{T,0} + 4F_{T,1}\right].
\end{aligned} \tag{D.111}$$

with the convention of

$$\begin{aligned}
&A \equiv a_3^2 b_3^2, && E \equiv a_3 b_3\left(a_1 b_1 + a_2 b_2\right), \\
&B \equiv \left(a_1 b_1 + a_2 b_2\right)^2, && F \equiv \left(a_1 b_3 - a_3 b_1\right)^2 + \left(a_2 b_3 - a_3 b_2\right)^2, \\
&C \equiv \left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right), && G \equiv \left(a_3 b_1 + a_1 b_3\right)^2 + \left(a_3 b_2 + a_2 b_3\right)^2, \\
&D \equiv a_3^2\left(b_1^2 + b_2^2\right) + \left(a_1^2 + a_2^2\right) b_3^2, && H \equiv a_3^2\left(b_1^2 + b_2^2\right).
\end{aligned} \tag{D.112}$$

E Proof of Optical theorem

Consider a process of $p_1 + p_2 \to \{k_f\}$, with $f = 1, .., n$, the total cross-section reads:

$$\sigma_t = \sum_n \mathrm{d}\Pi_n (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_f k_f) \frac{M^2(p_1 p_2 \to \{k_f\})}{2E_1 2E_2 |v_1 - v_2|}, \tag{E.1}$$

with

$$\mathrm{d}\Pi_n \equiv \prod_{f=1}^{n} \int \frac{\mathrm{d}k_f}{(2\pi)^3} \frac{1}{2E_f}. \tag{E.2}$$

The interacting T-matrix is defined from S-matrix as:

$$S = \mathbb{1} + iT. \tag{E.3}$$

Unitarity of S-matrix implies the following property of T:

$$S^\dagger S \Rightarrow (1 - iT^\dagger)(1 + iT^\dagger) = 1 \tag{E.4}$$
$$\Rightarrow T^\dagger T = i(T^\dagger - T). \tag{E.5}$$

The interaction reads:

$$i \langle \{k_f\}| T |p_1, p_2\rangle = iM(p_1 p_2 \to k_f)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_f k_f). \tag{E.6}$$

The Completeness relation in Hilbert space reads:

$$\mathbb{1} = \sum_n \mathrm{d}\Pi_n |\{k_f\}\rangle \langle \{k_f\}| \,. \tag{E.7}$$

Using the Completeness relation, we have,

$$\langle q_1, q_2| T^\dagger T |p_1, p_2\rangle = \sum_n \mathrm{d}\Pi_n \langle q_1, q_2| T^\dagger |\{k_f\}\rangle \langle \{k_f\}| T |p_1, p_2\rangle \tag{E.8}$$

Using Eq. E.5, the L.H.S of Eq. E.8 can be rewrite as

$$\begin{aligned}
&\langle q_1, q_2| T^\dagger T |p_1, p_2\rangle \\
=& i \left(\langle q_1, q_2| T^\dagger |p_1, p_2\rangle - \langle q_1, q_2| T |p_1, p_2\rangle\right) && \text{(E.9)} \\
=& i(2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \left[M^*(q_1 q_2 \to p_1 p_2) - M(p_1 p_2 \to q_1 q_2)\right]. && \text{(E.10)}
\end{aligned}$$

Using Eq. E.6, the R.H.S of Eq. E.5 reads:

$$\begin{aligned}
&\sum_n \mathrm{d}\Pi_n \langle q_1, q_2| T^\dagger |\{k_f\}\rangle \langle \{k_f\}| T |p_1, p_2\rangle \\
=& \sum_n \mathrm{d}\Pi_n (2\pi)^8 \delta^{(4)}(q_1 + q_2 - k_f) \delta^{(4)}(p_1 + p_2 - k_f) M^*(q_1 q_2 \to k_f) M(p_1 p_2 \to k_f).
\end{aligned} \tag{E.11}$$

Consolidate Eq. E.10 and Eq. E.11, we get:

$$M(p_1 p_2 \to q_1 q_2) - M^*(q_1 q_2 \to p_1 p_2) = \sum_n \mathrm{d}\Pi_n \,(2\pi)^4 i \delta^{(4)}(p_1 + p_2 - \sum_f k_f) |M(p_1 p_2 \to k_f)|^2 \tag{E.12}$$

Or,

$$2\mathrm{Im}\ M(p_1 p_2 \to p_1 p_2) = 2E_1 2E_2 |v_1 - v_2| \sigma_t. \tag{E.13}$$

When we have the same incoming particles, it becomes the standard form of

$$\mathrm{Im}\ M(p_1 p_2 \to p_1 p_2) = 2E_{\mathrm{CM}} |\mathbf{p}_{\mathrm{CM}}| \sigma_t. \tag{E.14}$$

E.1 A

$$iM = \frac{(i\lambda)^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{(\frac{p}{2}-k)^2-m^2+i\epsilon}\frac{i}{(\frac{p}{2}+k)^2-m^2+i\epsilon} \tag{E.15}$$

$$= \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(\frac{p_0}{2}-k_0)^2-E_k^2+i\epsilon}\frac{1}{(\frac{p_0}{2}+k_0)^2-E_k^2+i\epsilon}. \tag{E.16}$$

In the C.M. frame, $p=(p_0,\mathbf{0})$, $k=(k_0,\mathbf{k})$, $E_k^2=|\mathbf{k}|^2+m^2$.
The inegration has poles at:

$$k_0 = \pm(E_k - i\epsilon) - \frac{p_0}{2} \tag{E.17}$$

$$k_0 = \pm(E_k - i\epsilon) + \frac{p_0}{2}. \tag{E.18}$$

$$\frac{1}{\left(\frac{p_0}{2}+k\right)^2-m^2+i\epsilon} \to -2\pi i\delta\left(\left(\frac{p_0}{2}+k\right)^2-m^2\right) \tag{E.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}k_0}\left(\left(\frac{p_0}{2}-k_0\right)^2+E_k^2\right)\Bigg|_{k_0=E_k-\frac{p_0}{2}} = 2E_k. \tag{E.20}$$

$$\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4} \tag{E.21}$$

F Hahn-Banach separation theorem

Lemma (Zorn's Lemma). If P is a nonempty partially ordered set such that every chain has an upper bound, then P contains a maximal element. Let V: a real vector space, $U\subset V$, p a subaddictive, positive homogenerous functional on V. If f is a linear functional on U such that $f(v)\le p(v)$ for all $v\in U$, then there is a linear functional F on V such that $F(v)=f(v)$ for all $v\in U$ and $F(v)\le p(v)$ for all $v\in V$.
This theorem allows linear functional defined in subspace to be extended into the whole vector space.
Proof: Let Z be the subspace of V that contains U, and g is a linear functional on Z that is equal to f on U and is less than or equal to p on Z, and P be the collection of all ordered pairs (Z,g). Since P contains (U,f), it is nonempty. We establish a partial ordering on P by $(Z_1,g_1)\prec(Z_2,g_2)$ if $Z_1\subset Z_2$ and $g_1=g_2$ for g_2 in Z_1. If $\{(Z_\alpha,g_\alpha|\alpha\in A)\}$ is a chain (where A is an arbitrary indexing set), then it would have an upper bound (Z,g), where $Z=\bigcup_{\alpha\in A}Z_\alpha$ and g is a linear functional on Z defined

by $g(v) = g_\alpha(v)$ for all $v \in X_\alpha$.
Here, Z and g are well-defined and satisfied the necessary condition. Z is clearly a subspace of V containing U. To see that g is well-defined, suppose that $v \in Z_\alpha$ and $v \in Z_\beta$. Since Z_α and Z_β are in the chain, either $(Z_\alpha, g_\alpha) \prec (Z_\beta, g_\beta)$ or $(Z_\beta, g_\beta) \prec (Z_\alpha, g_\alpha)$. Without loss of generality, we assume the former, then $Z_\alpha \subset Z_\beta$ and $g_\beta = g_\alpha$ for g_β in Z_α, which means $g_\beta(v) = g_\alpha(v)$ for $v \in Z_\alpha$. This implies that g is well-defined. Additionally, g is linear since there exists some α such that $u, v \in Z_\alpha$, so then for $\gamma, \eta \in \mathbb{R}$, we have $g(\gamma u + \eta v) = g_\alpha(\gamma u + \eta v) = \gamma g_a(u) + \eta g_a(v) = \gamma g(u) + \eta g(v)$. Hence, we conclude that $(Z, g) \in P$ is an upper bound of the chain $\{(Z_\alpha, g_\alpha) | \alpha \in A\}$ i.e. $(Z_\alpha, g_\alpha) \prec (Z, g)$ for all α.
Now, we have P satisfied the assumption of Zorn's Lemma, so P has a matrix element, says (Y, F). We now show that Y is in fact equal to V, in which case F is our desired extension of f into V. Assume by contradiction that $Y \neq V$, then there exists some $v_0 \in V \setminus Y$. Let $Y' = Y + \text{span}\{v_0\} = \{v + \lambda v_0 | v \in Y, \lambda \in R\}$. Now, we have to find some $(Y', F') \in P$ such that $(Y, F) \prec (Y', F')$, which contradicts the maximality of (Y, F).
We already have that $Y \subset Y'$ and that Y' is a subspace of V that contains U. Now, we need to show that $F' = F$ for F' in Y, and that F' fulfills the conditions for $(Y', F")$ be in the collection P (i.e.) we need to show that F' is a linear functional that coincides with f on U and satisfies $F' \leq p$ on Y'). For some $\alpha \in \mathbb{R}$ fixed, we define $F'(v + \lambda v_0) = F(v) + \lambda\alpha$ for all $v \in Y, \lambda \in \mathbb{R}$, with $F'(v_0) = \alpha$. Then it follows immediately that F' is linear and that $F' = F$ for F' in Y (which implies that F' coincides with f on U, since (Y, F) is in P and is maximal, meaning $(U, f) \preceq (Y, F)$). Finally, we need to show an α can be chosen such that $F' \leq p$ on Y'. Particularly, we need an α such that for $\lambda > 0$, we have,

$$F'(v + \lambda v_0) = F(v) + \lambda\alpha \leq p(v + \lambda v_0) \tag{F.1}$$

when $\lambda < 0$, by letting $\lambda = -\mu$, we have,

$$F'(v - \mu v_0) = F(v) - \mu\alpha \leq p(v - \mu v_0) \tag{F.2}$$

for all $v \in Y$. We see that F.1 is equivalent to

$$\alpha \leq p(w + v_0) - F(w) \tag{F.3}$$

for all $w \in Y$ if we divide by λ and let $w = \frac{v}{\lambda}$. We see that F.2 is equivalent to

$$\alpha \geq F(x) - p(x - v_0) \tag{F.4}$$

for all $x \in Y$, if we divide by μ and let $x = \frac{v}{\mu}$. We then combine F.3 and F.4 to obtain

$$p(w + v_0) - F(w) \geq \alpha \geq F(x) - p(x - v_0) \tag{F.5}$$

for all $w, v \in Y$. Hence, to see that α exists and is well-defined, we need to show that

$$\inf_{w \in Y} \{p(w + v_0) - F(w)\} \geq \sup_{x \in Y} \{F(x) - p(x - v_0)\} \tag{F.6}$$

which means showing that

$$p(w + v_0) - F(w) \geq F(x) - p(x - v_0) \tag{F.7}$$

holds for all $w, x \in Y$. This equivalent to,

$$F(w) + F(x) = F(w + x) \leq p(w + v_0) + p(x - v_0). \tag{F.8}$$

Now, we can show that F.8 is true for all $w, x \in Y$, which will establish that we can find a suitable α,

$$\begin{aligned} F(w + x) &\leq p(w + x) && \text{(F.9)} \\ &= p(w + v_0 - v_0 + x) && \text{(F.10)} \\ &\leq p(w + v_0) + p(x - v_0). && \text{(F.11)} \end{aligned}$$

So, we are finally able to verify the existence of a proper α such that $F' \leq p$. Thus, we can conclude that (Y', F') is in the collection P and that $(Y, F) \prec (Y', F')$. This contradicts the maximality of (Y', F') established by Zorn's Lemma, so we can conclude that $Y = V$, with F being the desired extension of f into V. $\square$

G Krein-Milman theorem

Definition: A nonemty set F is a face of A if whenever $ax + (1 - a)y = z \in F$, for some $0 \leq \alpha \leq 1$ and $x, y \in A$ then $x, y \in F$.

Lemma: Take any element $l \in X^*$ (continuous linear functional). We claim that a set $F_l = \{y \in A : l(y) = \max_{x \in A} l(x)\}$ is a face.

Proof: First, since A is compact and l continuous therefore F_l is nonempty. Suppose $\alpha x + (1 - \alpha)y = z \in F_l$. Then $\max_{x \in A} l(x) = l(z) = \alpha l(x) + (1 - \alpha)l(y) \leq \alpha \max_{x \in A} l(x) + \max_{x \in A} l(x) \leq \max_{x \in A} l(x)$. On the other hand, we always have the equality which can be fullfield if and only if $l(y) = \max_{x \in A} l(x)$ and $l(x) = \max_{x \in A} l(x)$.

Thus $x, y \in F_l$.

Krein-Milman theorem: Let X be a locally convex linear toplogical vector space. Let A be a convex compact in X. Then the set of extrem points is not empty and A is a closure of the convex hull of its extrem epoints. Now we return to the proof of the first part of the theorem. By Corns lemma we do the following procedure: if A consists with only one point then we are done. If there are two distinct points say $x \neq y$ both belonging to A, then by Hahn-Banach theorem there exists $l \in X^*$ which strictly separates these two points. In other words $l(x) \geq l(y)$. Now we construct the face F_l surely it does not contain the point y. Then we look at F and make the same procedure. Thus we obtain the sequence of faces $\{F_l\}$. It is linearly ordered (ordered by inclusion) set. They are compact (as a closed (indeed) subset of compact set) so they have an upper bound, for example intersecion of compacts is not empty. We choose the minimal element. Note that a minimal element is a face (easy). If it contains more that 1 point then we make the same procedure which will bring us to the contradiction with minimality. Thus we obtain the extreme point.

Now we are ready to prove the second part of the theorem. Let E be a set of extreme points in A. Let CConv E be a closure of its convex hull.

Firstly note that CConv $E \subseteq A$. We need to prove the convers inclusion. From contrary, let $x \in A\setminus$ CConv E. Then we use the Hahn-Banach theorem to the point x (as a compact set) and closed convex set CConv E. There exists $l \in x^*$ such that we have $\sup_{y \in \mathrm{CConv} E} l(y) < l(y)$. Then we constract the face F_l. Surely it does not intersect the set CConv E and by the first part of the theorem it has an extreme point. So we obtain the contradiction.

H Photon Loops

$$I_n(A) = \int d^D k \frac{1}{(k^2 - A + i\epsilon)^n}, \quad A > 0 \tag{H.1}$$

We have the poles,

$$k_0 = \pm\sqrt{\mathbf{k}^2 + A} \mp i\epsilon' \tag{H.2}$$

Wick rotation,

$$\int_{-\infty}^{\infty} dk_0 \to \int_{-i\infty}^{i\infty} dk_0 = i \int_{-\infty}^{\infty} dq_{E,0}, \tag{H.3}$$

with,

$$k_0 = iq_{E,0}, \tag{H.4}$$

$$\mathbf{k} = \mathbf{q}_E \tag{H.5}$$

$$k^2 = k_0^2 - \mathbf{k}^2 = -q_{E,0}^2 - \mathbf{q}_E^2 \equiv -q_E^2 \tag{H.6}$$

$$I_n(A) = \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} d\mathbf{k}^{D-1}(q^2 - A + i\epsilon)^{-n} \tag{H.7}$$

$$= i\int_{-\infty}^{\infty} dq_{E,0} \int_{-\infty}^{\infty} d^{D-1}\mathbf{q}_E(-q^2 - A + i\epsilon)^{-n} \tag{H.8}$$

$$= i\int_{-\infty}^{\infty} d^D q_E(-1)^n(q_E^2 + A - i\epsilon)^{-n} \tag{H.9}$$

Integration in Polar coordinate, with, $\sqrt{k} \equiv q$, we have,

$$q^{D-1}dq = \frac{1}{2}k^{\frac{D}{2}-1}dk = \frac{1}{2}(q_E^2)^{\frac{D}{2}-1}dq_E^2. \tag{H.10}$$

Hence,

$$\int d^D q_E = \int_0^{\infty} dq_E q_E^{D-1} \int_{\Omega_D} d\Omega_D \tag{H.11}$$

$$= \frac{1}{2}\int_0^{\infty} dq_E^2 (q_E^2)^{\frac{D}{2}-1}\Omega_D. \tag{H.12}$$

I Mathematics for loops calculations

I.1 Logarithms and Powers

The natural logarithm $\log(z)$ is defined as

$$\log(z) = \log(|z|) + i\arg(z), \tag{I.1}$$

with $-\pi \leq \arg(z) \leq \pi$. The logarithm $\log(z)$ has a branch cut along the negative real axis. The general power $w = z^\alpha$ (α is a complex constant) is defined follow the exponential function :

$$z^\alpha = \left(e^{\log(z)}\right)^\alpha = e^{\alpha\log(z)}. \tag{I.2}$$

With the above definitions having the following properties :

$$\log(z_1z_2) = \log(z_1) + \log(z_2) + \eta(z_1, z_2), \tag{I.3}$$

$$\eta(z_1, z_2) = 2\pi i\left[\theta(-\mathrm{Im}z_1)\theta(-\mathrm{Im}z_2)\theta(\mathrm{Im}z_1z_2) - \theta(\mathrm{Im}z_1)\theta(\mathrm{Im}z_2)\theta(-\mathrm{Im}z_1z_2)\right], \tag{I.4}$$

$$(z_1z_2)^\alpha = e^{\alpha\log(z_1z_2)} = e^{\alpha[\log(z_1)+\log(z_2)+\eta(z_1,z_2)]} = e^{\alpha\eta(z_1,z_2)}z_1^\alpha z_2^\alpha, \tag{I.5}$$

so :

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) \quad \text{If } \mathrm{Im}(z_1) \text{ and } \mathrm{Im}(z_2) \text{ have different sign} \tag{I.6}$$

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2) \quad \text{If } \mathrm{Im}(z_1) \text{ and } \mathrm{Im}(z_2) \text{ have the same sign} \tag{I.7}$$

$$(z_1 z_2)^\alpha = z_1^\alpha + z_2^\alpha \quad \text{If } \mathrm{Im}(z_1) \text{ and } \mathrm{Im}(z_2) \text{ have different sign.} \tag{I.8}$$

$$\tag{I.9}$$

For $-z = a - i\rho$ with a is real and $\rho \to 0^+$:

$$\log(-z) = \begin{cases} \log(|a|) & \text{If } a > 0 \\ \log(|a|) - i\pi & \text{If } a < 0 \end{cases} \tag{I.10}$$

$$\arg(-z) = \arg z - \pi \tag{I.11}$$

$$\log(-z) = \log(|z|) + i\,\arg(-z) = \log(|z|) + i\,\arg(z) - i\pi = \log(z) - i\pi \tag{I.12}$$

$$(-z)^\alpha = e^{-i\pi\alpha} e^{\alpha \log(z)} = e^{-i\pi\alpha} z^\alpha. \tag{I.13}$$

If A and B are real then :

$$\log(AB - i\rho) = \log(A - i\rho') + \log(B - i\rho/A), \tag{I.14}$$

where ρ' is infinitesimal and has the same sign as rho. We get :

$$(AB - i\rho)^\alpha = e^{\alpha \log(AB - i\rho)} = e^{\alpha[\log(A - i\rho') + \log(B - i\rho/A)]} = (A - i\rho')^\alpha (B - i\rho/A)^\alpha. \tag{I.15}$$

I.2 Dilogarithms

The dilogarithm or Spence function is defined [8]

$$Sp(z) = -\int_0^1 dt \frac{\log(1 - zt)}{t}, \tag{I.16}$$

where z is complex number. The dilogarithm function has a cut along the positive real axis from 1 to $+\infty$ due to branch cut of logarithm function. When one is in a problematic situation, the following transformation formulate maybe useful :

$$Sp(z) = -Sp(\frac{1}{z}) - \frac{1}{6}\pi^2 - \frac{1}{2}\log^2(-z), \tag{I.17}$$

$$Sp(z) = -Sp(1 - z) + \frac{1}{6}\pi^2 - \log(1 - z)\log(z). \tag{I.18}$$

I.3 Gamma and Beta functions

The Gamma function $\Gamma(z)$ is one commonly used extension of the factorial function to complex numbers. The gamma function is defined for all complex numbers except the non-positive integers. For positive integer n:

$$\Gamma(n+1) = n\Gamma(n)! = n!, \tag{I.19}$$

follow the derivation by Daniel Bernoulli, for complex numbers z with a positive real part $\mathrm{Re}(z)$ the gamma function is defined by :

$$\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}. \tag{I.20}$$

$\Gamma(z)$ is analytical everywhere, except at the points $z = 0, -1, -1, -3, ...$ The following properties of the gamma function are very useful :

$$\Gamma(z+1) = z\Gamma(z) \tag{I.21}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{I.22}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \quad \epsilon \to 0, \tag{I.23}$$

where $\gamma_E = -\Gamma'(1)$ is Euler constant.

The beta function is defined by

$$B(p,q) = \int_0^1 dt t^{p-1}(1-t)^{q-1}, \tag{I.24}$$

where $\mathrm{Re}p > 0$ and $\mathrm{Re}q > 0$; the principal values of the various powers are to be taken. The analytical continuation of $B(p,q)$ onto the left halves of the p and q planes is achieved by using

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{I.25}$$

J D-dimensional Integral

The below formulas are cited form [9].

J.1 D-dimension properties

Transforming 4-dimension to D-dimension ($D = 4 - 2\epsilon$, $\epsilon \to 0$, D can be complex number) with some changed properties :

- Tensor metric $g^{\mu\nu}$ is D-dimension with $\mu, \nu = 0, 1, 2, .., D-1$.
  And $g_{\mu\nu} g^{\mu\nu} = D$.

- Dirac matrix γ^μ:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}$$

$$\mathrm{Tr}\{\mathbf{1}\} = 4.$$

- Remaining Lorentz invariant, Gauge invariant

- Integral :

$$\int \frac{d^4 q}{(2\pi)^4} \to \mu^{4-D} \frac{d^D q}{(2\pi)^D}$$

  with μ is normalized factor (mass scale of dimensional regularization).

J.2 Basic integral

$$I_n(A) = \int d^D k \frac{1}{(k^2 - A + i\epsilon)^n}, \qquad A > 0. \tag{J.1}$$

1. Wick rotation

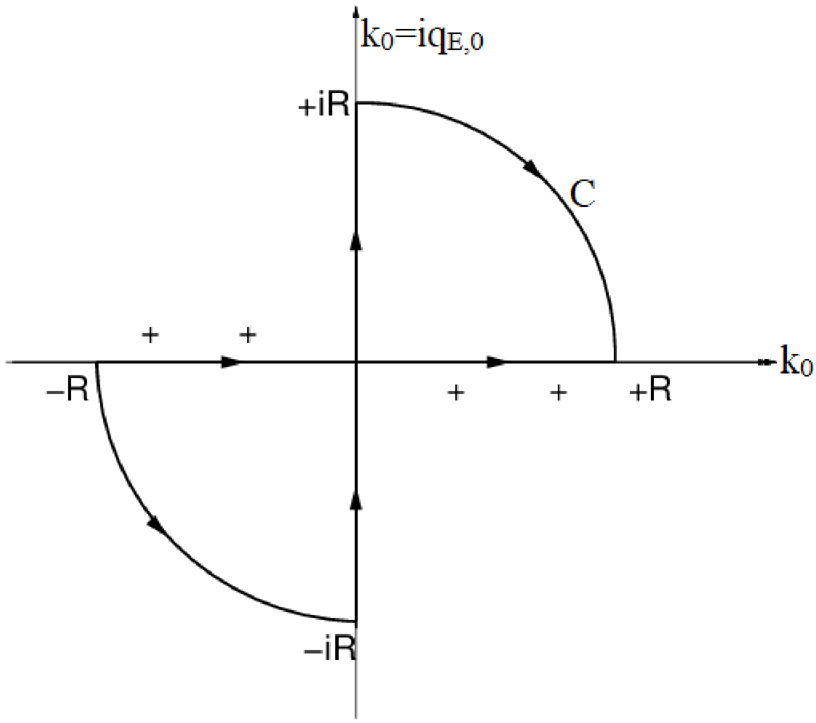

Figure 2: Wick rotation

k_0 has two poles :

$$k_0 = \pm\sqrt{\vec{k}^2 + A} \mp i\epsilon'. \tag{J.2}$$

In Complex Calculus, we get the result :

$$\oint_C dk_0 (k^2 - A + i\epsilon)^{-n} = 0 \tag{J.3}$$

$$\to \int_{-\infty}^{\infty} dk_0 ... = \int_{-i\infty}^{i\infty} dk_0 ... \tag{J.4}$$

Substitution :

$$k_0 = iq_{E,0}, \quad \vec{k} = \vec{q}_E, \tag{J.5}$$

$$k^2 = -q_E^2 \leq 0, \tag{J.6}$$

$$\int_{-i\infty}^{i\infty} dk_0 ... = i \int_{-\infty}^{\infty} dq_{E,0} ... \tag{J.7}$$

$$\rightarrow I_n(A) = \int_{-\infty}^{\infty} dk_0 \int d^{D-1}\vec{k}(q^2 - A + i\epsilon)^{-n} \tag{J.8}$$

$$= i\int_{-\infty}^{\infty} dq_{E,0} \int d^{D-1}\vec{q}_E(-q_{E,0}^2 - \vec{q}_E^{\,2} - A + i\epsilon)^{-n} \tag{J.9}$$

$$= i\int d^D q_E(-1 + i\epsilon')^n(q_E^2 + A - i\epsilon)^{-n} \tag{J.10}$$

$$= i\int d^D q_E(-1)^n(q_E^2 + A - i\epsilon)^{-n}, \tag{J.11}$$

with n is a integer .

2. Integrals in D-Dimensional Euclid space

$$\int d^D q_E = \int_{\Omega_D} d\Omega_D \int_0^{\infty} dq_E q_E^{D-1} = \int_{\Omega_D} d\Omega_D \int_0^{\infty} dq_E^2 \frac{1}{2}(q_E^2)^{D-1}, \tag{J.12}$$

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} = \text{D-dimensional solid angle.} \tag{J.13}$$

Proof :

$$\sqrt{\pi}^D = (\int_0^{\infty} dx.e^{-x^2})^D = \int dx_1...dx_D.e^{-\sum_i x_i^2} = \int d^D x.e^{-x^2} \tag{J.14}$$

$$= \int d\Omega_D \int_0^{\infty} dx x^{D-1} e^{-x^2} = \int d\Omega_D \frac{1}{2}\int_0^{\infty} dx^2 (x^2)^{(D-2)/2} e^{-x^2} \tag{J.15}$$

$$= \int d\Omega_D \frac{1}{2}\Gamma(D/2) \rightarrow \Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{J.16}$$

$\rightarrow I_n(A)$

$$= i(-1)^n \Omega_D \int_0^{\infty} dq_E^2 \frac{1}{2}(q_E^2)^{D/2-1}(q_E^2 + A - i\epsilon)^{-n} \tag{J.17}$$

$$= i(-1)^n \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^{\infty} dx \frac{1}{2} x^{D/2-1}(x + A - i\epsilon)^{-n} \tag{J.18}$$

$$= i(-1)^n \frac{\pi^{D/2}}{\Gamma(D/2)}(A - i\epsilon)^{D/2-n} \int_0^{\infty} dy(y+1)^{-n} y^{D/2-1} \qquad \text{set : } y = \frac{x}{A - i\epsilon} \tag{J.19}$$

$$= i(-1)^n \frac{\pi^{D/2}}{\Gamma(D/2)}(A - i\epsilon)^{D/2-n} B\left(\frac{D}{2}, n - \frac{D}{2}\right) \qquad \text{Beta function : } B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{J.20}$$

$$= i(-1)^n \pi^{D/2} \frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)}(A - i\epsilon)^{D/2-n}. \tag{J.21}$$

Similarly, another D-dimensional integrals [10] :

$$\int \frac{d^Dq}{(2\pi)^D} \frac{q^2}{(q^2-A)^n} = \frac{i(-1)^{n-1}}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)} (A)^{1+\frac{D}{2}-n}, \tag{J.22}$$

$$\int \frac{d^Dq}{(2\pi)^D} \frac{q^\mu q^\nu}{(q^2-A)^n} = \frac{i(-1)^{n-1}}{(4\pi)^{D/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)} (A)^{1+\frac{D}{2}-n}, \tag{J.23}$$

$$\int \frac{d^Dq}{(2\pi)^D} \frac{(q^2)^2}{(q^2-A)^n} = \frac{i(-1)^{n}}{(4\pi)^{D/2}} \frac{D(D+2)}{4} \frac{\Gamma(n-\frac{D}{2}-2)}{\Gamma(n)} (A)^{2+\frac{D}{2}-n}, \tag{J.24}$$

$$\int \frac{d^Dq}{(2\pi)^D} \frac{q^\mu q^\nu q^\rho q^\eta}{(q^2-A)^n} = \frac{i(-1)^{n}}{(4\pi)^{D/2}} \frac{\Gamma(n-\frac{D}{2}-2)}{\Gamma(n)} (A)^{2+\frac{D}{2}-n} \frac{1}{4} (g^{\mu\nu}g^{\rho\eta} + g^{\mu\rho}g^{\nu\eta} + g^{\mu\eta}g^{\rho\nu}). \tag{J.25}$$

In some cases, these integrals will be UV-divergent behaviour when $D \to 4$, we should use the following expanded identities :

$$A^x = 1 + x\log(A) + O(x^2) \tag{J.26}$$

$$\Gamma(x) = \frac{1}{x} - \gamma_E + O(x), \tag{J.27}$$

with $x \to 0$.

3. Feynman parametrization :
   Using Feynman's trick to transform complicated integrals to basic integral :

$$\frac{1}{A_1A_2} = \int_0^1 dx \frac{1}{(A_1x + (1-x)A_2)^2}, \tag{J.28}$$

general formula :

$$\frac{1}{A_1...A_n} = (n-1)! \int_0^1 dx_1 ... \int_0^1 dx_n \frac{\delta(1-\sum_{k=1}^n x_k)}{(\sum_{k=1}^n x_kA_k)^n}. \tag{J.29}$$

K N-point integrals

K.1 One-point function

$$A_0(m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq(q^2 - m^2 + i\epsilon)^{-1} \tag{K.1}$$

$$= \frac{(2\pi\mu)^{4-D}}{i\pi^2} I_1(m^2) \tag{K.2}$$

$$= -m^2\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}}\Gamma(\frac{2-D}{2}), \tag{K.3}$$

when $D \to 4$:

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} = 1 + \frac{D-4}{2}\log\left(\frac{m^2}{4\pi\mu^2}\right) + O((D-4)^2), \tag{K.4}$$

$$\Gamma\left(\frac{2-D}{2}\right) = -\left(\frac{2}{4-D} - \gamma_E + 1\right) + O(D-4), \tag{K.5}$$

$$\Rightarrow A_0(m) = m^2\left[\underbrace{\frac{2}{4-D} - \gamma_E + \log(4\pi)}_{=\triangle} - \log\left(\frac{m^2}{\mu^2}\right) + 1\right] + O(D-4) \tag{K.6}$$

$$= m^2\left[\triangle - \log\left(\frac{m^2}{\mu^2}\right) + 1\right] + O(D-4). \tag{K.7}$$

Note : $(D-4)A_0(m) = -2m^2 + O(D-4)$.

K.2 Two-point functions

K.2.1 Scalar

$$B_0(p, m_0, m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{1}{(q^2 - m_0^2 + i\epsilon)[(q+p)^2 - m^2 + i\epsilon]}, \tag{K.8}$$

using Feynman parametrization :

$$
\begin{aligned}
&\to B_0(p, m_0, m) \\
&= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \int_0^1 dx \frac{1}{\left[(q+xp)^2 - x^2p^2 + x(p^2 - m^2 + m_0^2) - m_0^2 + i\epsilon\right]^2} && \text{(K.9)} \\
&= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int_0^1 \int \underbrace{d^D q' \frac{1}{\left(q'^2 - A + i\epsilon\right)^2}}_{=I_2(A)} \qquad \text{set : } q' = q + xp && \text{(K.10)} \\
&= (4\pi\mu^2)^{\frac{4-D}{2}} \Gamma\left(\frac{4-D}{2}\right) \int_0^1 dx \left[x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon\right]^{\frac{D-4}{2}} && \text{(K.11)} \\
&= \triangle - \int_0^1 dx \log\left[\frac{x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\epsilon}{\mu^2}\right] + O(D-4). && \text{(K.12)}
\end{aligned}
$$

Alternative notation $B_0(p, m_0, m) \to B_0(p^2, m_0, m^2)$ and $(D-4)B_0 = -2 + O(D-4)$. Some special cases of Two-point fucntions :

$$
\begin{aligned}
B_0(p^2, 0, m) &= \triangle - \log\left(\frac{m^2}{\mu^2}\right) + 2 + \frac{m^2 - p^2}{p^2} \log\left(\frac{m^2 - p^2 - i\epsilon}{m^2}\right) && \text{(K.13)} \\
B_0(0, 0, m) &= \triangle - \log\left(\frac{m^2}{\mu^2}\right) + 1 && \text{(K.14)} \\
B_0(m^2, 0, m) &= \triangle - \log\left(\frac{m^2}{\mu^2}\right) + 2 && \text{(K.15)} \\
B_0(p^2, 0, 0) &= \triangle - \log\left(\frac{-p^2 - i\epsilon}{\mu^2}\right) + 2 && \text{(K.16)} \\
B_0(0, m, m) &= \triangle - \log\left(\frac{m^2}{\mu^2}\right) && \text{(K.17)} \\
A_0(m) &= m^2 B_0(0, 0, m) = m^2(B_0(0, m, m) + 1) && \text{(K.18)} \\
B_0(m^2, 0, m) &= B_0(0, m, m) + 2. && \text{(K.19)}
\end{aligned}
$$

K.2.2 Tensor

In all Tensor integral cases, we should use the Passarino-Veltman reduction method [9].

1.

$$
B_\mu(p_1, m_0, m_1) = \langle | \frac{q_\mu}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} | \rangle_q = p_{1\mu} B_1(p_1, m_0, m_1) \qquad \text{(K.20)}
$$

Multiply by p_1^μ :

$$
\begin{aligned}
&p_1^2 B_1(p_1^2, m_0, m_1) \\
&= \langle| \frac{q p_1}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_q && \text{(K.21)} \\
&= \langle| \frac{\frac{1}{2}\left[(q+p_1)^2 - m_1^2\right] - \frac{1}{2}(q^2 - m_0^2) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_q && \text{(K.22)} \\
&= \frac{1}{2}A(m_0) - \frac{1}{2}A(m_1) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)B_0(p_1^2, m_0, m_1) && \text{(K.23)}
\end{aligned}
$$

$$
\Rightarrow B_1(p_1^2, m_0, m_1) = \frac{1}{2p^2}\left[A(m_0) - A(m_1) - (p_1^2 - m_1^2 + m_0^2)B_0(p_1^2, m_0, m_1)\right]. \tag{K.24}
$$

When $D \to 4$: $(D-4)B_1(p_1^2, m_0, m_1) = 1 + 0(D-4)$.

2.

$$
\begin{aligned}
&B_{\mu\nu}(p_1, m_0, m_1) \\
=&\langle| \frac{q_\mu q_\nu}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_q && \text{(K.25)} \\
=&g_{\mu\nu} B_{00}(p_1^2, m_0, m_1) + p_{1\mu} p_{1\nu} B_{11}(p_1^2, m_0, m_1) && \text{(K.26)}
\end{aligned}
$$

- Multiply by $g^{\mu\nu}$:

$$
\begin{aligned}
&D B_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1) \\
&= \langle| \frac{(q^2 - m_0^2) + m_0^2}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_q && \text{(K.27)} \\
&= A_0(m_1) + m_0^2 B_1(p_1^2, m_0, m_1) && \text{(K.28)}
\end{aligned}
$$

- Multiply by p_1^μ

$$
\begin{aligned}
&p_{1\nu}\left[B_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1)\right] \\
&= \langle| \frac{q_\nu\left\{\frac{1}{2}\left[(q+p_1)^2 - m_1^2\right] - \frac{1}{2}(q^2 - m_0^2) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)\right\}}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_1 && \text{(K.29)} \\
&= p_{1\nu}\left[\frac{1}{2}A_0(m_1) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1)\right],
\end{aligned}
$$

we get set of equations :

$$
\begin{cases}
D B_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1) = A_0(m_1) + m_0^2 B_1(p_1^2, m_0, m_1) \\
B_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1) = \frac{1}{2}A_0(m_1) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1)
\end{cases}, \tag{K.30}
$$

$$\Rightarrow \begin{cases} B_{00} = \frac{1}{2(D-1)}\left[A_0(m_1) + 2m_0^2 B_0(p_1^2, m_0, m_1) + (p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1)\right] \\ B_{11} = \frac{1}{2(D-1)p_1^2}\left[(D-2)A_0(m_1) - 2m_0^2 B_0(p_1^2, m_0, m_1) - D(p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1)\right] \end{cases} . \tag{K.31}$$

When $D \to 4$:

$$\begin{cases} B_{00} = \frac{1}{6}\left[A_0(m_1) + 2m_0^2 B_0 + (p_1^2 - m_1^2 + m_0^2)B_1 + m_0^2 + m_1^2 - \frac{p_1^2}{3}\right] \\ B_{11} = \frac{1}{6p_1^2}\left[2A_0(m_1) - 2m_0^2 B_0 - 4(p_1^2 - m_1^2 + m_0^2)B_1 - m_0^2 - m_1^2 + \frac{p_1^2}{3}\right] \end{cases} , \tag{K.32}$$

and :

$$\begin{cases} (D-4)B_{00}(p_1^2, m_0, m_1) = \frac{-1}{6}(p_1^2 - 3m_0^2 - 3m_1^2) + O(D-4) \\ (D-4)B_{11}(p_1^2, m_0, m_1) = \frac{2}{3} \qquad\qquad\qquad\qquad + O(D-4) \end{cases} . \tag{K.33}$$

K.2.3 The derivative of two-point function

1.

$$B_0(p^2, \lambda, m) = \langle |\, (q^2 - \lambda^2)\left[(q+p)^2 - m^2\right] |\rangle_q \tag{K.34}$$

$$B_0'(p^2, \lambda, m)\big|_{p^2=m^2} = \left.\frac{\partial B_0(p^2, \lambda, m)}{\partial p^2}\right|_{p^2=m} \tag{K.35}$$

$$= \frac{\partial}{\partial p^2}\left\{-\int_0^1 dx \quad \log\left[\frac{x^2p^2 - x(p^2 - m^2 + \lambda^2) + \lambda^2 + i\epsilon}{\mu^2}\right]\right\}\Bigg|_{p^2=m^2} \tag{K.36}$$

$$= \frac{\partial}{\partial p^2}\left\{-\int_0^1 dx \quad \log\left[\frac{x^2p^2}{m^2} - x\frac{p^2 - m^2 + \lambda^2}{m^2} + \frac{\lambda^2}{m^2} - i\epsilon\right]\right\}\Bigg|_{p^2=m^2} \tag{K.37}$$

$$= -\int_0^1 dx \frac{1}{m^2}\frac{x^2 - x}{x^2 - \frac{\lambda^2}{m^2}x + \frac{\lambda^2}{m^2} - i\epsilon} \approx -\int_0^1 dx \frac{1}{m^2}\frac{x^2 - x}{x^2 + \frac{\lambda^2}{m^2} - i\epsilon} \tag{K.38}$$

Because $\left.\frac{\partial B_0(p^2,\lambda,m)}{\partial p^2}\right|_{p^2=m^2}$ is real,

proof:

$$\left.\frac{\partial B_0(p^2, \lambda, m)}{\partial p^2}\right|_{p^2=m^2} - \left.\frac{\partial B_0^*(p^2, \lambda, m)}{\partial p^2}\right|_{p^2=m^2}$$

$$= -\int_0^1 dx \frac{1}{m^2}\left[\frac{x^2 - x}{x^2 - \frac{\lambda^2}{m^2}x + \frac{\lambda^2}{m^2} - i\epsilon} - \frac{x^2 - x}{x^2 - \frac{\lambda^2}{m^2}x + \frac{\lambda^2}{m^2} + i\epsilon}\right] \tag{K.39}$$

$$= -2i\epsilon\int_0^1 dx \frac{1}{m^2}\frac{x^2 - x}{\left|x^2 - \frac{\lambda^2}{m^2}x + \frac{\lambda^2}{m^2} - i\epsilon\right|^2} \quad \xrightarrow{\epsilon \to 0} \quad 0. \tag{K.40}$$

So equation Eq.(K.38) become :

$$Re\left[-\int_0^1 dx\frac{1}{m^2}\frac{x^2-x}{\left(x-\frac{i\lambda}{m}\right)\left(x+\frac{i\lambda}{m}\right)}\right] \tag{K.41}$$

$$= Re\left[-\frac{1}{2m^2}\int_0^1 dx\left(\frac{x-1}{x-\frac{i\lambda}{m}}+\frac{x-1}{x+\frac{i\lambda}{m}}\right)\right] \tag{K.42}$$

$$\begin{aligned} &= -\frac{1}{2m^2}Re\left\{1+\left(\frac{i\lambda}{m}-1\right)\left[\log\left(1-\frac{i\lambda}{m}\right)-\log\left(\frac{-i\lambda}{m}\right)\right]\right. \\ &\qquad\qquad \left. +1+\left(\frac{-i\lambda}{m}-1\right)\left[\log\left(1+\frac{i\lambda}{m}\right)-\log\left(\frac{i\lambda}{m}\right)\right]\right\} \end{aligned} \tag{K.43}$$

$$= \frac{-1}{2m^2}\left[2+2\log\left(\frac{\lambda}{m}\right)\right] = \frac{-1}{m^2}\left[\log\left(\frac{\lambda}{m}\right)+1\right]. \tag{K.44}$$

2.

$$B_0'(0,m,m) = \frac{\partial}{\partial p^2}B_0(p^2,m,m)\bigg|_{p^2=0} = \frac{1}{6m^2}. \tag{K.45}$$

K.3 Three-point functions

K.3.1 Scalar

1. We will use the mass-regularization to compute it :

$$\begin{aligned} &C_0(p_1,p_2,\lambda,m,m) \\ &= \frac{1}{i\pi^2}\int d^4q\left\{(q^2-\lambda^2)[(q+p_1)^2-m^2][(q+p_2)^2-m^2]\right\}^{-1} \end{aligned} \tag{K.46}$$

$$= \langle|\left\{(q^2-\lambda^2)[(q+p_1)^2-m^2][(q+p_2)^2-m^2]\right\}^{-1}|\rangle_q \tag{K.47}$$

Using the Feynman parametrization [11]:

$$\left\{(q^2-\lambda^2)[(q+p_1)^2-m^2][(q+p_2)^2-m^2]\right\}^{-1} \tag{K.48}$$

$$= 2\int_0^1 dx\int_0^{1-x} dy\left\{(q^2-\lambda^2)x+[(q+p_1)^2-m^2](1-x-y)+[(q+p_2)^2-m^2]y\right\}^{-3} \tag{K.49}$$

$$\begin{aligned} &= 2\int_0^1 dx\int_0^{1-x} dy\left\{[q+p_1(1-x-y)+yp_2]^2-p_1^2(1-x-y)^2\right. \\ &\qquad\qquad \left.-y^2p_2^2-2p_1p_2y(1-x-y)-\lambda^2x+i\epsilon\right\}^{-3} \end{aligned} \tag{K.50}$$

with $p_1^2 = p_2^2 = m^2$.

$$\Rightarrow C_0(p_1, p_2, \lambda, m, m) \tag{K.51}$$

$$= 2\int_0^1 dx \int_0^{1-x} dy \, \langle| \left\{ [q + p_1(1-x-y) + yp_2]^2 - p_1^2(1-x-y)^2 \right. \tag{K.52}$$

$$\left. -y^2p_2^2 - 2p_1p_2y(1-x-y) - \lambda^2 x + i\epsilon \right\}^{-3} |\rangle_q$$

$$= -\int_0^1 dx \int_0^{1-x} dy \left\{ [p_1(1-x-y) + yp_2]^2 + \lambda^2 x - i\epsilon \right\}^{-1} \tag{K.53}$$

$$= -\int_0^1 dx \int_0^{x} dy \left\{ [p_1(x-y) + yp_2]^2 + (1-x)\lambda^2 - i\epsilon \right\}^{-1} \tag{K.54}$$

$$= -\int_0^1 dx \int_0^{1} dy \quad x \left\{ [p_1x(1-y) + xyp_2]^2 + (1-x)\lambda^2 - i\epsilon \right\}^{-1} \qquad (\text{set } y_n = \frac{y_o}{x}) \tag{K.55}$$

$$= -\int_0^1 dx \int_0^1 dy \frac{y}{y^2 [p_1(1-x) + xp_2]^2 + (1-y)\lambda^2 - i\epsilon} \qquad (x \leftrightarrow y). \tag{K.56}$$

Set $P_x^2 = [p_1(1-x) + xp_2]^2$ and $\bar{P}_x^2 = [p_1(1-x) + xp_2]^2 - i\epsilon$:

$$\Rightarrow C_0(p_1, p_2, \lambda, m, m)$$

$$= -\int_0^1 dx \int_0^1 dy \frac{y}{y^2\bar{P}_x^2 + (1-y)\lambda^2} \tag{K.57}$$

$$= -\int_0^1 dx \int_0^1 dy \frac{1}{2\bar{P}_x^2} \left[\frac{2y\bar{P}_x^2 - \lambda^2}{y^2\bar{P}_x^2 + (1-y)\lambda^2} + \frac{\lambda^2}{y^2\bar{P}_x^2 + (1-y)\lambda^2} \right], \tag{K.58}$$

take the limit $\lambda \to 0$, the second term be vanished :

$$C_0(p_1, p_2, \lambda, m, m$$

$$= -\int_0^1 dx \frac{1}{2\bar{P}_x^2} \left(\log \bar{P}_x^2 - \log \lambda^2 \right) \tag{K.59}$$

$$= \log\left(\frac{\lambda^2}{m^2}\right) \int_0^1 dx \frac{1}{2\bar{P}_x^2} - \int_0^1 dx \frac{1}{2\bar{P}_x^2} \log \frac{\bar{P}_x^2}{m^2}. \tag{K.60}$$

Calculating the divergent term $\log\left(\frac{\lambda^2}{m^2}\right) \int_0^1 dx \frac{1}{2\bar{P}_x^2}$:

$$\int_0^1 \frac{dx}{\bar{P}_x^2} = \int_0^1 \frac{dx}{[p_1(1-x) + xp_2]^2 - i\epsilon} \tag{K.61}$$

$$= \int_0^1 \frac{dx}{\bar{t}x^2 - \bar{t}x + m^2} \tag{K.62}$$

$$= \int_0^1 \frac{dx}{\bar{t}\,(x - x_1)\,(x - x_2)}, \tag{K.63}$$

with x_1, x_2 is solution of $\bar{P}_x^2 = m^2 - \bar{t}x + \bar{t}x^2 = 0$ and $\bar{t} = t + i\epsilon = (p_1 - p_2)^2 + i\epsilon$.

$$\Rightarrow \int_0^1 \frac{dx}{\bar{P}_x^2} = \int_0^1 \frac{dx}{\bar{t}\,(x_1 - x_2)} \left[\frac{1}{x - x_1} - \frac{1}{x - x_2}\right] \tag{K.64}$$

$$= \frac{1}{\bar{t}\,(x_1 - x_2)} \left[\log\left(\frac{x_1 - 1}{x_1}\right) - \log\left(\frac{x_2 - 1}{x_2}\right)\right]. \tag{K.65}$$

With:

$$x_t = \frac{\sqrt{1 - 4m^2/\bar{t}} - 1}{\sqrt{1 - 4m^2/\bar{t}} + 1} = \frac{x_1 - 1}{x_1} = \frac{x_2}{x_2 - 1} \tag{K.66}$$

$$\text{and:} \qquad \frac{x_t}{m^2(1 - x_t^2)} = \frac{1}{\bar{t}\,(x_2 - x_1)}. \tag{K.67}$$

$$\Rightarrow \int_0^1 \frac{dx}{\bar{P}_x^2} = \frac{-x_t}{m^2\,(1 - x_t^2)} 2\log(x_t), \tag{K.68}$$

we get final result of divergent term containing the singularity of λ :

$$\log\left(\frac{\lambda^2}{m^2}\right) \int_0^1 dx \frac{1}{2\bar{P}_x^2} = \frac{-x_t}{m^2\,(1 - x_t^2)} \log(x_t) \log\left(\frac{\lambda^2}{m^2}\right). \tag{K.69}$$

Calcualting the finite term $-\int_0^1 dx \frac{1}{2\bar{P}_x^2} \log \frac{\bar{P}_x^2}{m^2}$:

$$
\begin{aligned}
&-\int_0^1 dx \frac{1}{2\bar{P}_x^2} \log \frac{\bar{P}_x^2}{m^2} \\
&= -\int_0^1 \frac{dx}{2} \frac{1}{\bar{t}\,(x_1 - x_2)} \left[\frac{1}{x - x_1} - \frac{1}{x - x_2} \right] \left[\log(x - x_1) + \log(x - x_2) + \log \frac{\bar{t}}{m^2} \right]
\end{aligned} \tag{K.70}
$$

$$
\begin{aligned}
&= \frac{x_t}{m^2(1 - x_t^2)} \int_0^1 \frac{dx}{2} \left\{ \left[\frac{1}{x - x_1} - \frac{1}{x - x_2} \right] [\log(x - x_1) + \log(x - x_2)] \right. \\
&\left. + \left[\frac{1}{x - x_1} - \frac{1}{x - x_2} \right] \log \frac{\bar{t}}{m^2} \right\}
\end{aligned} \tag{K.71}
$$

$$
\begin{aligned}
&= \frac{x_t}{2m^2(1 - x_t^2)} \left\{ \left[\log \left(\frac{x_1 - 1}{x_1} \right) - \log \left(\frac{x_2 - 1}{x_2} \right) \right] \log \frac{\bar{t}}{m^2} \right\} \\
&+ \frac{x_t}{2m^2(1 - x_t^2)} \int_0^1 dx \left[\frac{1}{x - x_1} - \frac{1}{x - x_2} \right] \times \left\{ \left[\log(x - x_1) + \log(x - x_2) \right. \right. \\
&\left. - \log(x_1 - x_2) - \log(x_2 - x_1) \right] + [\log(x_2 - x_1) + \log(x_1 - x_2)] \}
\end{aligned} \tag{K.72}
$$

$$
\begin{aligned}
&= \frac{x_t}{2m^2(1 - x_t^2)} \left\{ \left[\log \frac{\bar{t}}{m^2} + \log(x_1 - x_2) + \log(x_2 + x_1) \right] 2 \log(x_t) \right\} \\
&+ \frac{x_t}{2m^2(1 - x_t^2)} \int_0^1 dx \left[\frac{1}{x - x_1} - \frac{1}{x - x_2} \right] \left[\log(x - x_1) + \log(x - x_2) \right. \\
&\left. - \log(x_1 - x_2) - \log(x_2 - x_1) \right].
\end{aligned} \tag{K.73}
$$

We have :

$$
\frac{\partial}{\partial z} \mathrm{Li}_2 \left(\frac{z - a}{z - b} \right) = -\left(\frac{1}{z - a} - \frac{1}{z - b} \right) \log \left(\frac{a - b}{z - b} \right) \tag{K.74}
$$

$$
\Rightarrow -\mathrm{Li}_2 \left(\frac{A - a}{A - b} \right) + \mathrm{Li}_2 \left(\frac{a}{b} \right) = \int_0^A dz \left(\frac{1}{z - a} - \frac{1}{z - b} \right) [\log(a - b) - \log(z - b)] . \tag{K.75}
$$

$$
\begin{aligned}
&\Rightarrow -\int_0^1 dx \frac{1}{2\bar{P}_x^2}\log\frac{\bar{P}_x^2}{m^2} \\
&=\frac{x_t}{2m^2(1-x_t^2)}\left\{\left[\log\frac{\bar{t}}{m^2}+\log(x_1-x_2)+\log(x_2+x_1)\right]2\log(x_t)\right. && \text{(K.76)}\\
&\quad+\left.\left[\mathrm{Li}_2\frac{x_2}{x_1}-\mathrm{Li}_2\frac{1-x_2}{1-x_1}+\mathrm{Li}_2\frac{1-x_1}{1-x_2}-\mathrm{Li}_2\frac{x_1}{x_2}\right]\right\} && \text{(K.77)}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{x_t}{2m^2(1-x_t^2)}\left\{\log\left[\frac{-\bar{t}}{m^2}(x_1-x_2)^2\right]2\log(x_t)+\left[\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)\right.\right. \\
&\left.\left.+\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)\right]\right\} && \text{(K.78)}\\
&=\frac{x_t}{2m^2(1-x_t^2)}\left[2\log(x_t)\log\frac{(1+x_t)^2}{x_t}+4\mathrm{Li}_2(-x_t)+\frac{2\pi^2}{6}+\log^2(x_t)\right] && \text{(K.79)}\\
&=\frac{x_t}{m^2(1-x_t^2)}\left\{\log(x_t)\left[2\log(1+x_t)-\frac{1}{2}\log^2(x_t)\right]+2\mathrm{Li}_2(-x_t)+\frac{\pi^2}{6}\right\}. && \text{(K.80)}
\end{aligned}
$$

We used some below correlation :

$$
\log\frac{\bar{t}}{m^2}+\log(x_1-x_2)+\log(x_2-x_1)=\log\left[\frac{-\bar{t}}{m^2}(x_1-x_2)^2\right] \tag{K.81}
$$

$$
\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)+\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)=4\mathrm{Li}_2(-x_t)+\frac{2\pi^2}{6}+\log^2 x_t \tag{K.82}
$$

$$
\log\left[\frac{-\bar{t}}{m^2}(x_1-x_2)^2\right]=\log(1+x_t)^2-\log(x_t) \tag{K.83}
$$

$$
-\mathrm{Li}_2\frac{1}{z}-\frac{\pi^2}{6}-\frac{1}{2}\log^2(-z)=\mathrm{Li}_2(z). \tag{K.84}
$$

2. Convergent three-point function

$$
C_0(q,p,0,0,m)=\langle|\frac{1}{n^2(n+q)^2\left[(n+p)^2-m^2\right]}|\rangle_n \qquad \text{with : } p^2=m^2 \tag{K.85}
$$

$$= -\int_0^1 dx \int_0^1 dyx \left\{ [px(1-y)+yxq]^2 - q^2yx \right\}^{-1} \tag{K.86}$$

$$= -\int_0^1 dx \int_0^1 dy \left\{ x\,[p(1-y)+yq]^2 - q^2y \right\}^{-1} \tag{K.87}$$

$$= -\int_0^1 dy \frac{\log\left[\frac{(p(1-y)+yq)^2-q^2y}{-q^2y}\right]}{(p(1-y)+yq)^2} \qquad \text{apply : } pq = \frac{q^2}{2} \tag{K.88}$$

$$= -\int_0^1 dy \frac{\log\left[\frac{p^2(1-y)^2}{-q^2y}\right]}{p^2(1-y)^2+q^2y} = -\int_0^1 dy \frac{\log\left[\frac{p^2y^2}{-q^2(1-y)}\right]}{p^2y^2+q^2(1-y)}. \tag{K.89}$$

Consider :

$$\int_0^1 \frac{\log A(x) - \log B(x)}{A(x)-B(x)} dx = \int_0^1 \frac{\log A(x) - \log B(x)}{a(x-x_1)(x-x_2)} dx, \tag{K.90}$$

with x_1,x_2 are the solutions of equation $A(x) - B(x) = 0$.

$$\int_0^1 \frac{\log A(x) - \log B(x)}{A(x)-B(x)} dx = \int_0^1 [\log A(x) - \log B(x)] \left(\frac{1}{x-x_1} - \frac{1}{x-x_2} \right) \frac{1}{a(x_1-x_2)}, \tag{K.91}$$

with the x_1 terms :

$$\int_0^1 \frac{\log A(x) - \log B(x)}{x-x_1} dx = \int_0^1 \frac{\log A(x) - \log A(x_1)}{x-x_1} dx + \int_0^1 \frac{\log B(x_1) - \log B(x)}{x-x_1} dx, \tag{K.92}$$

similar for x_2.

From Eq. (K.89), we get $0 < y_1 < 1$, $y_2 < 0$ are the solutions of equation $y^2 + \frac{q^2}{p^2}(1-y) = 0$. We split into two parts corresponding to y_1, y_2 :

- $$\int_0^1 dy \frac{\log\left[\frac{p^2y^2}{-q^2(1-y)}\right]}{y-y_1} = \int_0^1 dy \frac{\log(p^2y^2) - \log\left[-q^2(1-y)\right]}{y-y_1} \tag{K.93}$$

+

$$\begin{aligned}
&\int_0^1 \frac{2\log y - 2\log y_1}{y-y_1}dy \\
&= \int_{-y_1}^{1-y_1} \frac{2\log\left(1+\frac{y}{y_1}\right)}{y}dy && \text{(K.94)}\\
&= \left(\int_0^{1-y_1} - \int_0^{-y_1}\right)\frac{2\log\left(1+\frac{y}{y_1}\right)}{y}dy && \text{(K.95)}\\
&= \int_0^1 \frac{dy}{y} 2\log\left(1+y\frac{1-y_1}{y_1}\right) - \int_0^1 \frac{dy}{y} 2\log\left(1-y\right) && \text{(K.96)}\\
&= -2\mathrm{Li}_2\left(\frac{y_1-1}{y_1}\right) + 2\mathrm{Li}_2(1) && \text{(K.97)}
\end{aligned}$$

+

$$\int_0^1 \frac{\log[-q^2(1-y_1)] - \log[-q^2(1-y)]}{y-y_1}dy = -\int_0^1 \frac{\log y - \log(1-y_1)}{1-y_1-y}dy \tag{K.98}$$

$$= \int_{y_1-1}^{y_1} \frac{\log\left(1+\frac{y}{1-y_1}\right)}{y}dy = \left(\int_0^{y_1} - \int_0^{y_1-1}\right)\frac{\log\left(1+\frac{y}{1-y_1}\right)}{y}dy \tag{K.99}$$

$$= \int_0^1 \frac{dy}{y}\log\left(1+y\frac{y_1}{1-y_1}\right) - \int_0^1 \frac{dy}{y}\log(1-y) = -\mathrm{Li}_2\left(\frac{y_1}{y_1-1}\right) + \mathrm{Li}_2(1). \tag{K.100}$$

•

$$\int_0^1 dy \frac{\log\left[\frac{p^2y^2}{-q^2(1-y)}\right]}{y-y_2} = \int_0^1 dy \frac{\log\left(\frac{p^2}{-q^2}\right) + \log\left(\frac{y^2}{1-y}\right)}{y-y_2} \tag{K.101}$$

+

$$\int_0^1 dy \frac{\log\left(\frac{p^2}{-q^2}\right)}{y-y_2} = \log\left(\frac{p^2}{-q^2}\right)\log\left(\frac{y_2-1}{y_2}\right) \tag{K.102}$$

$+$

$$\int_0^1 dy \frac{\log\left(\frac{y^2}{1-y}\right)}{y-y_2} = \int_0^1 \frac{2\log y}{y-y_2}dy + \int_0^1 \frac{\log y}{y-(1-y_2)}dy \tag{K.103}$$

$$= \int_0^1 dy \int_1^y 2\frac{dt}{t}\frac{1}{y-y_2} + \int_0^1 dy \int_1^y \frac{dt}{t}\frac{1}{y-(1-y_2)} \tag{K.104}$$

$$= -2\int_0^1 \frac{dt}{t}\int_0^t dy \frac{1}{y-y_2} - \int_0^1 \frac{dt}{t}\int_0^t dy \frac{1}{y-(1-y_2)} \tag{K.105}$$

$$= -2\int_0^1 \frac{dt}{t}\log\left(1-\frac{t}{y_2}\right) - \int_0^1 \frac{dt}{t}\log\left(1-\frac{t}{1-y_2}\right) \tag{K.106}$$

$$= 2\mathrm{Li}_2\left(\frac{1}{y_2}\right) + \mathrm{Li}_2\left(\frac{1}{1-y_2}\right). \tag{K.107}$$

$$\Rightarrow C_0(q,p,0,0,m) = \langle| \frac{1}{n^2(n+q)^2\left[(n+p)^2-m^2\right]} |\rangle_n \tag{K.108}$$

$$= \frac{1}{p^2(y_2-y_1)}\Bigg[-2\mathrm{Li}_2\left(\frac{y_1-1}{y_1}\right) + 3\mathrm{Li}_2(1) - \mathrm{Li}_2\left(\frac{y_1}{y_1-1}\right) \tag{K.109}$$

$$-\log\left(\frac{p^2}{-q^2}\right)\log\left(\frac{y_2-1}{y_2}\right) - 2\mathrm{Li}_2\left(\frac{1}{y_2}\right) - \mathrm{Li}_2\left(\frac{1}{1-y_2}\right)\Bigg]. \tag{K.110}$$

K.3.2 Tensor

1.

$$\begin{aligned} C_\mu(p_1,p_2,m_0,m_1,m_2) &= \langle| \frac{q_\mu}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]} |\rangle_q \\ &= p_{1\mu}C_1 + p_{2\mu}C2, \end{aligned} \tag{K.111}$$

with $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$.

- Multiply by p_1^μ :

$$p_1^2 C_1 + p_1 p_2 C_2 = \langle| \frac{\frac{1}{2}\left[(q+p_1)^2-m_1^2\right] - \frac{1}{2}(q^2-m_0^2) - \frac{1}{2}m_0^2}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]} |\rangle_q \tag{K.112}$$

$$= \frac{1}{2}B_0(p_1^2,m_0,m_1) - \frac{1}{2}B_0((p_1-p_2)^2,m_2,m_1) - \frac{1}{2}m_0^2 C_0. \tag{K.113}$$

- Multiply by p_2^μ :

$$p_1p_2C_1 + p_2^2C_2 = \langle| \frac{\frac{1}{2}\left[(q+p_2)^2 - m_2^2\right] - \frac{1}{2}(q^2 - m_0^2) - \frac{1}{2}m_0^2}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]\left[(q+p_2)^2 - m_2^2\right]} |\rangle_q \quad \text{(K.114)}$$

$$= \frac{1}{2}B_0(p_2^2, m_0, m_2) - \frac{1}{2}B_0((p_1 - p_2)^2, m_1, m_2) - \frac{1}{2}m_0^2C_0. \quad \text{(K.115)}$$

The solutions $C_1(p_1, p_2, m_0, m_1, m_2), C_2(p_1, p_2, m_0, m_1, m_2)$:

$$\begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}B_0(p_1^2, m_0, m_1) - \frac{1}{2}B_0((p_1 - p_2)^2, m_2, m_1) - \frac{1}{2}m_0^2C_0 \\ \frac{1}{2}B_0(p_2^2, m_0, m_2) - \frac{1}{2}B_0((p_1 - p_2)^2, m_1, m_2) - \frac{1}{2}m_0^2C_0 \end{pmatrix}. \quad \text{(K.116)}$$

When $D \to 4$: $(D-4)C_\mu = O(D-4) \Rightarrow$ UV-convergent $C\mu(p_1, p_2, m_0, m_1, m_2)$.
Special case : $m_0 = 0 \to$ IR-convergent $C_\mu(p_1, p_2, 0, m_1, m_2)$.

2. Convergent 1st-order tensor three-point function

$$C_\beta(-q, p, 0, 0, m) = \langle| \frac{n_\beta}{n^2(n-q)^2\left[(n+p)^2 - m^3\right]} |\rangle_n \qquad \text{with : } p^2 = m^2$$

$$= -q_\beta C_1(-q, p, 0, 0, m) + p_\beta C_2(-q, p, 0, 0, m). \quad \text{(K.117)}$$

- Multiply by q^β :

$$\begin{aligned} -q^2C_1 + qpC_2 &= \langle| \frac{qn}{n^2(n-q)^2\left[(n+p)^2 - m^3\right]} |\rangle_n \\ &= \frac{1}{2}\langle| \frac{n^2 + q^2 - (n-q)^2}{n^2(n-q)^2\left[(n+p)^2 - m^3\right]} |\rangle_n \\ &= \frac{1}{2}\left[B_0((q+p)^2, 0, m) - B_0(p^2, 0, m) + q^2C_0(-q, p, 0, 0, m)\right] \\ &= \frac{1}{2}q^2C_0(-q, p, 0, 0, m). \end{aligned} \quad \text{(K.118)}$$

- Multiply by p^β :

$$\begin{aligned} -pqC_1 + m^2C_2 &= \langle| \frac{pn}{n^2(n-q)^2\left[(n+p)^2 - m^3\right]} |\rangle_n \\ &= \frac{1}{2}\langle| \frac{(n+p)^2 - n^2 - m^2}{n^2(n-q)^2\left[(n+p)^2 - m^3\right]} |\rangle_n \\ &= \frac{1}{2}\left[B_0(q^2, 0, 0) - B_0((q+p)^2, 0, m)\right]. \end{aligned} \quad \text{(K.119)}$$

The solutions of $C_1(-q,p,0,0,m),\ C_2(-q,p,0,0,m)$:

$$\begin{pmatrix} -q^2 & qp \\ -pq & p^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} q^2 C_0(-q,p,0,0,m) \\ \frac{1}{2}\left[B_0(q^2,0,0) - B_0((q+p)^2,0,m)\right] \end{pmatrix}. \tag{K.120}$$

$\Rightarrow C_\beta(-q,p,0,0,m) \to$ convergent.

3.

$$\begin{aligned} C_{\mu\nu}(p_1,p_2,0,m,m) &= \langle | \frac{q_\mu q_\nu}{q^2\left[(q+p_1)^2 - m^2\right]\left[(q+p_2)^2 - m^2\right]} | \rangle_q \\ &= g_{\mu\nu} C_{00} + p_{1\mu} p_{1\nu} C_{11} + p_{2\mu} p_{2\nu} C_{22} + (p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) C_{12}, \end{aligned} \tag{K.121}$$

with $p_1^2 = p_2^2 = m^2$.

- Multiply by $g^{\mu\nu}$:

$$DC_{00} + p_1^2 C_{11} + p_2^2 C_{22} + 2p_1p_2C_{12} = B_0((p_1-p_2)^2,m,m) \tag{K.122}$$

- Multiply by p_1^μ :

$$\begin{cases} C_{00} + p_1^2 C_{11} + p_1p_2C_{12} = \frac{1}{2}\left[B_1((p_1-p_2)^2,m,m) + B_0((p_1-p_2)^2,m,m)\right] \\ p_1^2 C_{12} + p_1p_2C_{22} = \frac{1}{2}\left[B_1(p_2^2,0,m) - B_1((p_1-p_2)^2,m,m)\right] \end{cases} \tag{K.123}$$

- Multiply by p_2^μ :

$$\begin{cases} C_{00} + p_2^2 C_{22} + p_1p_2C_{12} = \frac{1}{2}\left[B_1((p_1-p_2)^2,m,m) + B_0((p_1-p_2)^2,m,m)\right] \\ p_2^2 C_{12} + p_1p_2C_{11} = \frac{1}{2}\left[B_1(p_1^2,0,m) - B_1((p_1-p_2)^2,m,m)\right] \end{cases} \tag{K.124}$$

$$\Rightarrow C_{00} = \frac{1}{2-D}\left[B_1((p_1-p_2)^2,m,m)\right] \tag{K.125}$$

$$= \frac{1}{2}\left[-B_1((p_1-p_2)^2,m,m) + \frac{1}{2}\right] + O(D-4). \tag{K.126}$$

$$= \frac{\triangle}{4} + \text{finite term} \tag{K.127}$$

When $D \to 4$: $(D-4)C_{00} = -\frac{1}{2}$. The C_{11}, C_{22}, C_{12} solutions:

$$\begin{pmatrix} p_2^2 & p_1p_2 \\ p_1p_2 & p_1^2 \end{pmatrix} \begin{pmatrix} C_{22} \\ C_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left[B_0((p_1-p_2)^2,m,m) - B_1((p_1-p_2)^2,m,m)\right] + (1-D)C_{00} \\ \frac{1}{2}\left[B_1(p_2^2,0,m) - B_1((p_1-p_2)^2,m,m)\right] \end{pmatrix}, \tag{K.128}$$

and :

$$\begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left[B_0((p_1-p_2)^2, m, m) - B_1((p_1-p_2)^2, m, m)\right] + (1-D)C_{00} \\ \frac{1}{2}\left[B_1(p_1^2, 0, m) - B_1((p_1-p_2)^2, m, m)\right] \end{pmatrix}. \tag{K.129}$$

$\Rightarrow C_{11}, C_{22}, C_{12}$: convergent.

K.4 Four-point functions

K.4.1 Scalar

Scalar four-point function in the first Box diagram Fig. (**??**) :

$$\begin{aligned} D_0 &= \langle| \frac{1}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \\ &= \langle| \frac{1}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} |\rangle_n . \end{aligned} \tag{K.130}$$

Because of $C_0(-k', p', \lambda, m_e, m_\mu) = C_0(-k, p, \lambda, m_e, m_\mu) = C_0(m_e^2, s, m_\mu^2, \lambda, m_e, m_\mu)$.

$$q^2 D_0 = \langle| \frac{-2nq}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} |\rangle_n \tag{K.131}$$

$$\Rightarrow \frac{-q^2}{2} D_0(q, -k, p, \lambda, \lambda, m_e, m_\mu) =$$
$$\langle| \frac{\left(n+\frac{1}{2}q\right)^2 - \frac{1}{4}q^2}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} |\rangle_n - C_0(k'^2, s, p'^2, \lambda, m_e, m_\mu) \tag{K.132}$$

We split into two parts, the finite term $\langle| \frac{\left(n+\frac{1}{2}q\right)^2 - \frac{1}{4}q^2}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} |\rangle_n$ and the
IR-divergent term $C_0(k'^2, s, p'^2, \lambda, m_e, m_\mu)$.

The IR-divergent term

We apply the same calculating process $C_0(p_1, p_2, \lambda, m, m)$ Eq. (K.47) :

$$C_0(k'^2, s, p'^2, \lambda, m_e, m_\mu) = -\int_0^1 \frac{dx}{2\bar{P}_x^2}\left(\log \bar{P}_x - \log \lambda^2\right) \tag{K.133}$$
$$= \log\left(\frac{\lambda^2}{-\bar{q}^2}\right)\int_0^1 \frac{dx}{2\bar{P}_x^2} - \int_0^1 \frac{dx}{2\bar{P}_x^2}\log\left(\frac{\bar{P}_x^2}{-\bar{q}^2}\right) \tag{K.134}$$
$$= \log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{2\bar{P}_x^2} - \int_0^1 \frac{dx}{2\bar{P}_x^2}\log\left(\frac{\bar{P}_x^2}{-\bar{t}}\right). \tag{K.135}$$

The second finite term : $-\int_0^1 \frac{dx}{2\bar{P}_x^2}\log\left(\frac{\bar{P}_x^2}{-\bar{t}}\right)$ should be cancelled by the initial finite term : $\langle | \frac{\left(n+\frac{1}{2}q\right)^2-\frac{1}{4}q^2}{(n^2-\lambda^2)[(n+q)^2-\lambda^2][(n-k)^2-m_e^2]\left[(n+p)^2-m_\mu^2\right]} | \rangle_n$, but I have not been able to prove this. We will only consider the first IR-divergent term :

$$\log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{2\bar{P}_x^2} = \frac{1}{2}\log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{\bar{s}x^2+(m_e^2-m_\mu^2-\bar{s})x+m_\mu^2} \tag{K.136}$$

$$= \frac{1}{2}\log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{\bar{s}(x-x_1)(x-x_2)} \tag{K.137}$$

$$= \frac{1}{2}\log\left(\frac{\lambda^2}{-\bar{t}}\right)\frac{1}{\bar{s}(x_1-x_2)}\left[\log\left(\frac{x_1-1}{x_1}\right)-\log\left(\frac{x_2-1}{x_2}\right)\right], \tag{K.138}$$

with x_1,x_2 are the solutions of :

$$\bar{s}x^2+(m_e^2-m_\mu^2-\bar{s})x+m_\mu^2 = 0, \tag{K.139}$$

$$\triangle = (\bar{s}-m_e^2-m_\mu^2)^2-4m_\mu^2m_e^2 \tag{K.140}$$

$$= \left[\bar{s}-(m_e-m_\mu)^2-4m_\mu m_e\right]\left[\bar{s}-(m_e-m_\mu)^2\right] \tag{K.141}$$

$$= \left[\bar{s}-(m_e-m_\mu)^2\right]^2\left[1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}\right], \tag{K.142}$$

$$\Rightarrow x_1, x_2 = \frac{\bar{s}+m_\mu^2-m_e^2\pm\sqrt{\triangle}}{2\bar{s}}. \tag{K.143}$$

And :

$$x_s = \frac{\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}-1}{\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}+1}.$$

We have :

(+)

$$1-x_s^2 = \frac{4\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}+1\right)^2} \tag{K.144}$$

$$\Rightarrow \frac{(1-x_s^2)}{-x_s} = \frac{\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}+1}{1-\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}}\frac{4\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}+1\right)^2} \tag{K.145}$$

$$= \frac{4\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1-\left(\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}\right)^2} = \frac{\left[\bar{s}-(m_e-m_\mu)^2\right]\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{m_em_\mu}, \tag{K.146}$$

$$\Rightarrow \frac{(1-x_s^2)m_e m_\mu}{-x_s} = \left[\bar{s}-(m_e-m_\mu)^2\right]\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} = \sqrt{\triangle} = (x_1-x_2)\bar{s}. \tag{K.147}$$

(+)

$$\frac{x_1-1}{x_1}.\frac{x_2}{x_2-1} = \frac{x_1x_2-x_2}{x_1x_2-x_1} = \frac{\bar{s}-m_\mu^2-m_e^2-\sqrt{\triangle}}{\bar{s}-m_\mu^2-m_e^2+\sqrt{\triangle}} \tag{K.148}$$

$$\Rightarrow 1-\frac{\bar{s}-m_\mu^2-m_e^2-\sqrt{\triangle}}{\bar{s}-m_\mu^2-m_e^2+\sqrt{\triangle}} = \frac{2\sqrt{\triangle}}{\bar{s}-m_\mu^2-m_e^2+\sqrt{\triangle}} \tag{K.149}$$

$$= \frac{2\left[\bar{s}-(m_\mu-m_e)^2\right]\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\bar{s}-m_\mu^2-m_e^2+\sqrt{\triangle}} \tag{K.150}$$

$$= \frac{4\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\frac{2(\bar{s}-m_\mu^2-m_e^2)+2\sqrt{\triangle}}{\bar{s}-(m_\mu-m_e)^2}} = \frac{4\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\frac{2[\bar{s}-(m_\mu-m_e)^2-2m_\mu m_e]}{\bar{s}-(m_\mu-m_e)^2}+2\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} \tag{K.151}$$

$$= \frac{4\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1+1-\frac{4m_e m_\mu}{\bar{s}-(m_\mu-m_e)^2}+2\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} \tag{K.152}$$

$$= \frac{4\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1+\left(\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}\right)^2+2\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} = \frac{4\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}+1\right)^2} \tag{K.153}$$

$$= 1-x_s^2 \tag{K.154}$$

$$\Rightarrow \frac{\bar{s}-m_\mu^2-m_e^2-\sqrt{\triangle}}{\bar{s}-m_\mu^2-m_e^2+\sqrt{\triangle}} = \frac{x_1-1}{x_1}.\frac{x_2}{x_2-1} = x_s^2 \tag{K.155}$$

Using Eq. (K.155) and Eq. (K.147), we get the final result of Eq .(K.138) :

$$\log\left(\frac{\lambda^2}{-\bar{t}}\right)\frac{1}{2\bar{s}(x_1-x_2)}\left[\log\left(\frac{x_1-1}{x_1}\right)-\log\left(\frac{x_2-1}{x_2}\right)\right] = \frac{-x_s}{(1-x_s^2)m_e m_\mu}\log(x_s)\log\left(\frac{\lambda^2}{-\bar{t}}\right), \tag{K.156}$$

or :

$$D_0(q,-k,p,\lambda,\lambda,m_e,m_\mu) = \frac{-2x_s}{(1-x_s^2)q^2 m_e m_\mu}\log(x_s)\log\left(\frac{\lambda^2}{-\bar{t}}\right). \tag{K.157}$$

The same result for $D_0(-q,-k',-p,\lambda,\lambda,m_e,m_\mu)$ in the second Box diagram with substitution of $x_s \to x_u$.

K.4.2 Tensor

1. Fist rank tensor in the Box diagrams Fig. (**??**) :

$$
\begin{aligned}
&D_\mu(-q,-k',p',\lambda,\lambda,m_e,m_\mu)\\
&=\langle|\frac{n_\mu}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n\\
&=-q_\mu D_1-k'_\mu D_2+p'_\mu D_3.
\end{aligned}
$$

We are using below identities :

$$
k'q=\frac{q^2}{2}\qquad kq=\frac{-q^2}{2}, \tag{K.158}
$$

$$
pq=\frac{q^2}{2}\qquad p'q=\frac{-q^2}{2}. \tag{K.159}
$$

- Multilpy by q^μ :

$$
-q^2D_1-\frac{q^2}{2}D_2-\frac{q^2}{2}D_3 \tag{K.160}
$$

$$
=\langle|\frac{nq}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n \tag{K.161}
$$

$$
=\frac{1}{2}\langle|\frac{-(n-q)^2+n^2+q^2}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n \tag{K.162}
$$

$$
\begin{aligned}
=&\frac{1}{2}\Bigg[\langle|\frac{-1}{(n^2-\lambda^2)\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n\\
&+\langle|\frac{1}{\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n\\
&+q^2D_0(-q,-k',p',\lambda,\lambda,m_e,m_\mu)\Bigg]
\end{aligned}
$$

$$
=\frac{1}{2}q^2D_0(-q,-k',p',\lambda,\lambda,m_e,m_\mu), \tag{K.163}
$$

- Multilpy by k'^{μ} :

$$
\begin{aligned}
&-\frac{q^2}{2}D_1 - m_e^2 D_2 + k'p' D_3 \\
=&\langle| \frac{nk'}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.164)}\\
=&\frac{1}{2}\langle| \frac{n^2-\left[(n-k')^2-m_e^2\right]}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.165)}\\
=&\frac{1}{2}\left[C_0(-k,p,\lambda,m_e,m_\mu) - \langle| \frac{1}{n^2(n-q)^2\left[(n+p')^2-m_\mu^2\right]} |\rangle_n\right]. \quad \text{(K.166)}
\end{aligned}
$$

- Multilpy by p'^{μ} :

$$
\begin{aligned}
&\frac{q^2}{2}D_1 - p'k' D_2 + m_\mu^2 D_3 \quad \text{(K.167)}\\
=&\langle| \frac{np'}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.168)}\\
=&\frac{1}{2}\langle| \frac{-n^2+\left[(n-p')^2-m_\mu^2\right]}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.169)}\\
=&\frac{1}{2}\left[-C_0(-k,p,\lambda,m_e,m_\mu) + \langle| \frac{1}{n^2(n-q)^2\left[(n-k')^2-m_e^2\right]} |\rangle_n\right]. \quad \text{(K.170)}
\end{aligned}
$$

The solutions $D_1(-q,-k',p',\lambda,\lambda,m_e,m_\mu)$, $D_2(-q,-k',p',\lambda,\lambda,m_e,m_\mu)$, $D_3(-q,-k',p',\lambda,\lambda,m_e,m_\mu)$:

$$
\begin{pmatrix} -q^2 & -\frac{q^2}{2} & -\frac{q^2}{2} \\ -\frac{q^2}{2} & -m_e^2 & k'p' \\ \frac{q^2}{2} & -p'k' & m_\mu^2 \end{pmatrix}
\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} =
\begin{pmatrix} \frac{1}{2}q^2 D_0(-q,-k',p',\lambda,\lambda,m_e,m_\mu) \\ \frac{1}{2}\left[C_0(-k,p,\lambda,m_e,m_\mu) - \langle| \frac{1}{n^2(n-q)^2\left[(n+p')^2-m_\mu^2\right]} |\rangle_n\right] \\ \frac{1}{2}\left[-C_0(-k,p,\lambda,m_e,m_\mu) + \langle| \frac{1}{n^2(n-q)^2\left[(n-k')^2-m_e^2\right]} |\rangle_n\right] \end{pmatrix}. \quad \text{(K.171)}
$$

$$
\begin{aligned}
&\Rightarrow D_1(-q,-k',p',\lambda,\lambda,m_e,m_\mu) \sim \frac{-C_0(-k,p,\lambda,m_e,m_\mu)}{q^2} \\
&\to D_\mu(-q,-k',p',\lambda,\lambda,m_e,m_\mu) \sim q_\mu \frac{C_0(-k,p,\lambda,m_e,m_\mu)}{q^2}.
\end{aligned} \quad \text{(K.172)}
$$

The same for $D_\mu(-q,-k',-p,\lambda,\lambda,m_e,m_\mu)$.

2. Second rank tensor in the Box diagrams Fig. (**??**) :

$$
\begin{aligned}
& D_{\alpha\beta}(-q,-k',p',\lambda,\lambda,m_e,m_\mu) && \text{(K.173)}\\
=& \langle| \frac{n_\alpha n_\beta}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n && \text{(K.174)}\\
=& g_{\alpha\beta}D_{00}+q_\alpha q_\beta D_{11}+k'_\alpha k'_\beta D_{22}+p'_\alpha p'_\beta D_{33}+(q_\alpha k'_\beta+k'_\alpha q_\beta)D_{12} && \text{(K.175)}\\
& -(q_\alpha p'_\beta+p'_\alpha q_\beta)D_{13}-(k'_\alpha p'_\beta+p'_\alpha k'_\beta)D_{23}. && \text{(K.176)}
\end{aligned}
$$

- Multiply by $g^{\alpha\beta}$:

$$
\begin{aligned}
& 4D_{00}+q^2D_{11}+m_e^2D_{22}+m_\mu^2D_{33}+q^2D_{12}+q^2D_{13}-2k'p'D_{23} && \text{(K.177)}\\
=& \langle| \frac{n^2}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n && \text{(K.178)}\\
=& C_0(-k,p,\lambda,m_e,m_\mu). && \text{(K.179)}
\end{aligned}
$$

- Multiply by q^α :

$$
\begin{aligned}
& q_\beta D_{00}+q^2q_\beta D_{11}+\frac{q^2}{2}k'_\beta D_{22}-\frac{q^2}{2}p'_\beta D_{33}+(q^2k'_\beta+\frac{q^2}{2}q_\beta)D_{12} && \text{(K.180)}\\
& -(q^2p'_\beta-\frac{q^2}{2}q_\beta)D_{13}-(\frac{q^2}{2}p'_\beta-\frac{q^2}{2}k'_\beta)D_{23} && \text{(K.181)}\\
=& \langle| \frac{n_\beta nq}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n && \text{(K.182)}\\
=& \frac{1}{2}\langle| \frac{n_\beta\left[n^2+q^2-(n-q)^2\right]}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n && \text{(K.183)}\\
=& \frac{1}{2}\Bigg[C_\beta(-k,p,\lambda,m_e,m_\mu)+q_\beta C_0(-k,p,\lambda,m_e,m_\mu) && \\
& -C_\beta(-k',p',\lambda,m_e,m_\mu)+q^2D_\beta(-q,-k',p',\lambda,\lambda,m_e,m_\mu)\Bigg]. && \text{(K.184)}
\end{aligned}
$$

- Multiply by k'^{α} :

$$
\begin{aligned}
&k'_\beta D_{00} + \frac{q^2}{2} q_\beta D_{11} + m_e^2 k'_\beta D_{22} + k'p' p'_\beta D_{33} + (\frac{q^2}{2} k'_\beta + m_e^2 q_\beta) D_{12} \\
&- (\frac{q^2}{2} p'_\beta + k'p' q_\beta) D_{13} - (m_e^2 p'_\beta + k'p' k'_\beta) D_{23} \\
=&\langle | \frac{n_\beta n k'}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.185)} \\
=&\frac{1}{2} \langle | \frac{n_\beta \left(n^2 - \left[(n-k')^2-m_e^2\right]\right)}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.186)} \\
=&\frac{1}{2} \left[C_\beta(-k,p,\lambda,m_e,m_\mu) + q_\beta C_0(-k,p,\lambda,m_e,m_\mu) - C_\beta(-q,p',\lambda,\lambda,m_\mu) \right]. \quad \text{(K.187)}
\end{aligned}
$$

- Multiply by p'^{α} :

$$
\begin{aligned}
&p'_\beta D_{00} - \frac{q^2}{2} q_\beta D_{11} + p'k' k'_\beta D_{22} + m_\mu^2 p'_\beta D_{33} + (-\frac{q^2}{2} k'_\beta + k'p' q_\beta) D_{12} \\
&- (-\frac{q^2}{2} p'_\beta + m_\mu^2 q_\beta) D_{13} - (k'p' p'_\beta + m_\mu^2 k'_\beta) D_{23} \quad \text{(K.188)} \\
=&\langle | \frac{n_\beta n p'}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.189)} \\
=&\frac{1}{2} \langle | \frac{n_\beta \left(\left[(n+p')^2-m_\mu^2\right] - n^2\right)}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \quad \text{(K.190)} \\
=&\frac{1}{2} \left[C_\beta(-q,-k',\lambda,\lambda,m_e) - q_\beta C_0(-k,p,\lambda,m_e,m_\mu) - C_\beta(-k,p,\lambda,m_e,m_\mu) \right]. \quad \text{(K.191)}
\end{aligned}
$$

So we get set of equations :

-

$$
\begin{aligned}
&4D_{00} + q^2 D_{11} + m_e^2 D_{22} + m_\mu^2 D_{33} + q^2 D_{12} + q^2 D_{13} - 2k'p' D_{23} \\
=&C_0(-k,p,\lambda,m_e,m_\mu) \quad \text{(K.192)}
\end{aligned}
$$

- $$\begin{aligned}&D_{00}+q^2D_{11}+\frac{q^2}{2}D_{12}+\frac{q^2}{2}D_{13}\\=&\frac{1}{2}\left[C_0(-k,p,\lambda,m_e,m_\mu)+C_1(-k,p,\lambda,m_e,m_\mu)\right.\\&\left.+C_2(-k,p,\lambda,m_e,m_\mu)-q^2D_1(-q,-k',p',\lambda,\lambda,m_e,m_\mu)\right]\end{aligned}\tag{K.193}$$

- $$\frac{q^2}{2}D_{22}+q^2D_{12}+\frac{q^2}{2}D_{23}=\frac{1}{2}\left[C_1(-k',p',\lambda,m_e,m_\mu)-C_1(-k,p,\lambda,m_e,m_\mu)\right]\tag{K.194}$$

- $$\frac{-q^2}{2}D_{33}-q^2D_{13}-\frac{q^2}{2}D_{23}=\frac{1}{2}\left[-C_2(-k',p',\lambda,m_e,m_\mu)+C_2(-k,p,\lambda,m_e,m_\mu)\right]\tag{K.195}$$

- $$\begin{aligned}\frac{q^2}{2}D_{11}+m_e^2D_{12}-k'p'D_{13}=\frac{1}{2}&\left[C_0(-k,p,\lambda,m_e,m_\mu)+C_1(-k,p,\lambda,m_e,m_\mu)\right.\\&\left.+C_2(-k,p,\lambda,m_e,m_\mu)+C_1(-q,p',\lambda,\lambda,m_\mu)\right]\end{aligned}\tag{K.196}$$

- $$D_{00}+m_e^2D_{22}+\frac{q^2}{2}D_12-k'p'D_{23}=-\frac{1}{2}C_1(-k,p,\lambda,m_e,m_\mu)\tag{K.197}$$

- $$k'p'D_{33}-\frac{q^2}{2}D_{13}-m_e^2D_{23}=\frac{1}{2}\left[C_2(-k,p,\lambda,m_e,m_\mu)-C_2(-q,p',\lambda,\lambda,m_\mu)\right]\tag{K.198}$$

- $$\frac{-q^2}{2}D_{11}+k'p'D_{12}-m_\mu^2D_{23}\tag{K.199}$$
$$\begin{aligned}=&-\frac{1}{2}\left[C_1(-q,-k',\lambda,\lambda,m_e)+C_1(-k,p,\lambda,m_e,m_\mu)\right.\\&\left.+C_0(-k,p,\lambda,m_e,m_\mu)\right]\end{aligned}\tag{K.200}$$

-

$$p'k'D_{22} - \frac{q^2}{2}D_{12} - m_\mu^2 D_{23} = \frac{1}{2}\left[C_1(-k,p,\lambda,m_e,m_\mu) - C_2(-q,-k',\lambda,\lambda,m_e)\right] \tag{K.201}$$

-

$$D_{00} + m_\mu^2 D_{33} + \frac{q^2}{2}D_{13} - k'p'D_{23} = -\frac{1}{2}C_2(-k,p,\lambda,m_e,m_\mu). \tag{K.202}$$

From the above set of equations, we can easy deduce out which parts being proportional to IR-divergent value :

$$\begin{aligned}\Rightarrow D_{11}(-q,-k',p',\lambda,\lambda,m_e,m_\mu) &\sim \frac{C_0(-k,p,\lambda,m_e,m_\mu)}{q^2}\\ \to D_{\mu\nu}(-q,-k',p',\lambda,\lambda,m_e,m_\mu) &\sim q_\mu q_\nu \frac{C_0(-k,p,\lambda,m_e,m_\mu)}{q^2}.\end{aligned} \tag{K.203}$$

Similar for $D_{\mu\nu}(-q,-k',-p,\lambda,\lambda,m_e,m_\mu)$.

K.5 Proving the relations

- Eq.(K.74) :

$$\frac{\partial}{\partial z}\mathrm{Li}_2\frac{z-a}{z-b} = -\frac{\partial}{\partial z}\left\{\int_0^1 \frac{dt}{t}\log\left(1-\frac{z-a}{z-b}t\right)\right\} \tag{K.204}$$

$$= \int_0^1 \frac{dt}{t}\frac{\left(\frac{z-a}{z-b}\right)' t}{1-\frac{z-a}{z-b}t} = -\int_0^1 dt\left(\frac{b-a}{z-b}\right)\frac{1}{z-b-(z-a)t}, \tag{K.205}$$

$$\left(\frac{b-a}{z-b}\right)\frac{\log\frac{z-b-(z-a)}{z-b}}{z-a} = \frac{b-a}{(z-b)(z-a)}\log\frac{a-b}{z-b} \tag{K.206}$$

$$= \left[\frac{1}{z-b} - \frac{1}{z-a}\right]\log\frac{a-b}{z-b}. \tag{K.207}$$

- Eq.(K.81), we have :

$$x_1 - x_2 = \sqrt{1-4m^2/\bar{t}} \tag{K.208}$$

$$\Rightarrow \log\frac{\bar{t}}{m^2} + \log(x_1-x_2) + \log(x_2-x_1) = \log\frac{\bar{t}}{m^2} + \log\left[-(x_1-x_2)^2\right] \tag{K.209}$$

$$= \log\frac{\bar{t}}{m^2} + \log\left(4m^2/\bar{t}-1\right) = \log\left[\frac{\bar{t}}{m^2}\left(4m^2/\bar{t}-1\right)\right] = \log\left[-\frac{\bar{t}}{m^2}(x_2-x_1)^2\right]. \tag{K.210}$$

- Eq.(K.82) :

$$\mathrm{Li}_2(-x_t) - \mathrm{Li}_2\frac{-1}{x_t} + \mathrm{Li}_2(-x_t) - \mathrm{Li}_2\frac{-1}{x_t} = 2\mathrm{Li}_2(-x_t) - 2\mathrm{Li}_2\left(\frac{-1}{x_t}\right) \tag{K.211}$$

$$= 2\mathrm{Li}_2(-x_t) - 2\left[\log(-x_t) + \frac{\pi^2}{6} + \frac{1}{2}\log^2\frac{1}{x_t}\right] = 4\mathrm{Li}_2(-x_t) + \frac{\pi^2}{3} + \log^2 x_t. \tag{K.212}$$

- Eq.(K.83), considering :

$$1 + x_t = \frac{x_1 - x_2}{1 - x_2} \Rightarrow \frac{(x_1 - x_2)^2}{x_t} = \frac{-x_1}{x_2}\frac{(x_1 - x_2)^2}{(1 - x2)^2} = \frac{(x_1 - x_2)^2}{-x_1 x_2}, \tag{K.213}$$

because :

$$x_t = \frac{x_1 - 1}{1 - x_2} = \frac{-x_2}{x_1} \qquad x_1 + x_2 = 1; \tag{K.214}$$

and

$$x_2.x_1 = \frac{m^2}{\bar{t}}. \tag{K.215}$$

$$\Rightarrow \frac{(1 + x_t)^2}{x_t} = \frac{(x_1 - x_2)^2}{\frac{-m^2}{\bar{t}}} = \frac{-\bar{t}}{m^2}(x_1 - x_2)^2, \tag{K.216}$$

$$\Rightarrow \log\left[\frac{-\bar{t}}{m^2}(x_1 - x_2)^2\right] = \log\frac{(1 + x_t)^2}{x_t}. \tag{K.217}$$

We have :

$$\frac{(1 + x_t)^2}{x_t} = \frac{-\bar{t}}{m^2}(x_1 - x_2)^2 = 4 - \frac{\bar{t}}{m^2} \tag{K.218}$$

$$\Rightarrow \mathrm{Im}\left[\frac{(1 + x_t)^2}{x_t}\right] < 0; \tag{K.219}$$

$$x_t = \frac{\sqrt{1 - 4m^2/\bar{t}} - 1}{\sqrt{1 - 4m^2/\bar{t}} + 1} \Rightarrow x_t - 1 = \frac{-1}{x_1} \tag{K.220}$$

$$\mathrm{Im}\,(x_1) = \mathrm{Im}\left(\frac{\sqrt{1 - 4m^2/\bar{t}}}{2}\right) > 0 \tag{K.221}$$

$$\Rightarrow \mathrm{Im}\,(x_t) = \mathrm{Im}\,(x_t - 1) = \mathrm{Im}\left(\frac{-1}{x_1}\right) > 0 \tag{K.222}$$

$$\Rightarrow \log\left[\frac{-\bar{t}}{m^2}(x_1 - x_2)^2\right] = \log\frac{(1 + x_t)^2}{x_t} = \log(1 + x_t)^2 - \log x_t, \tag{K.223}$$

because $\mathrm{Im}(\frac{(1+x_t)^2}{x_t})$ and $\mathrm{Im}(\frac{1}{x_t})$ are always same sign.

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