Preprint typeset in JHEP style - HYPER VERSION

Progress on positivity bounds in SMEFT

TRAN Quang Loc

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WA, UK

E-mail: loctranq@gmail.com, tq216@cam.ac.uk

Abstract: In this report, we rederive some current analysis on SMEFT positivity bound. We first rederive the main result in [1], which investigate the vector boson scattering process (VBS) by considering diagrams involving quartic gauge boson couplings (WGC) governed by SMEFT Dim-8 operators.

Contents

1 Positivity bounds in VBS

In this section, we re-interpret some basic concepts, theorems and re-derive (in details) some of the main results and calculations in [1].

1.1 Crossing symmetry

Consider a $2 \to 2$ process, any of the particles can be replaced by its antiparticle on the other side of the interaction. Hence, for
Spin $= 0$: $M(s,t) = M(u,t)$.
Spin > 0: With the linear polarizations vector $(\epsilon_1^\mu)^* = \epsilon_3^\mu, (\epsilon_2^\mu)^* = \epsilon_4^\mu$, and with restriction to forward limit, we have $M(s,0) = M(u,0)$ (or $\mathcal{A}(s) = \mathcal{A}(u)$).

1.2 Optical theorem

The Optical theorem yield the relation between forward scattering amplitude and cross-section [REDERIVE THE THEOREM LATER IN THE APPENDIX].

$$\text{Im } \mathcal{A}(k_1k_2 \to k_1k_2) = 2E_1E_2|v_1 - v_2|\sigma_t. \tag{1.1}$$

Going into the CM-system, we have $p_1 + p_2 = 0, E_{\text{CM}} = E_1 + E_2, \mathbf{p}_{\text{CM}} = \mathbf{p}_1 = -\mathbf{p}_2$, (with $v = \frac{p}{E}$) we get the optical theorem in the standard form:

$$\text{Im } \mathcal{A}(k_1 k_2 \to k_1 k_2) = 2E_{\text{CM}}\mathbf{p}_{\text{CM}}\sigma_t. \tag{1.2}$$

when 2 incoming particles are the same, $m = m_1 = m_2$, $E_1 = E_2 = E$, we have $s = 4E^2 = E_{\text{CM}}^2, \mathbf{p}_{\text{CM}} = \sqrt{s^2/4 - m^2}$,)

$$\begin{aligned} \text{Im } \mathcal{A}(k_1 k_2 \to k_1 k_2) =& 2\sqrt{s}\sqrt{\frac{s}{4} - m^2}\sigma_t & (1.3) \\ =& \sqrt{s(s-4m^2)}\sigma_t. & (1.4) \end{aligned}$$

In general, with 2 different incoming particles, defining $M_+ =$ we have

$$\begin{aligned} & 2(E_1+E_2)\mathbf{p}_{\text{CM}} \\ =& 2\sqrt{(E_1+E_2)^2\mathbf{p}_{\text{CM}}^2} & (1.5) \\ =& 2\sqrt{\mathbf{p}_{\text{CM}}^4 + 2\mathbf{p}_{\text{CM}}^2 E_1 E_2 + E_1^2 E_2^2 - \mathbf{p}_{\text{CM}}^4 + (E_1^2+E_2^2)\mathbf{p}_{\text{CM}}^2 - E_1^2 E_2^2} & (1.6) \\ =& \sqrt{(2\mathbf{p}_{\text{CM}}^2 + 2E_1E_2)^2 - 4m_1^2 m_2^2} & (1.7) \\ =& \sqrt{\left((E_1+E_2)^2 - 2(m_1^2+m_2^2)\right)^2 - 4m_1^2 m_2^2} & (1.8) \\ =& \sqrt{(E_1+E_2)^4 - 2(E_1+E_2)^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2} & (1.9) \\ =& \sqrt{s^2 - s(M_+^2 + M_-^2) + M_+^2 M_-^2} & (1.10) \\ =& \sqrt{s(s-M_+^2) - M_-^2(s-M_+^2)} & (1.11) \\ =& \sqrt{(s-M_-^2)(s-M_+^2)}. & (1.12) \end{aligned}$$

Hence, Eq. 1.1 yields

$$\begin{aligned} \text{Im } A(k_1 k_2 \to k_1 k_2) =& 2(E_1\mathbf{p}_2 - E_2\mathbf{p}_1)\sigma_t & (1.13) \\ =& 2(E_1+E_2)\mathbf{p}_{\text{CM}}\sigma_t & (1.14) \\ =& \sqrt{(s-M_-^2)(s-M_+^2)}\sigma_t. & (1.15) \end{aligned}$$

It yields back the result of Eq. 1.4 when $M_- = 0$.

$$\text{Im}A_{ab}^{q_1 q_2}(s') = \sqrt{(s'-M_+^2)(s'-M_-^2)}\sigma_{ab}^{q_1 q_2}(s') > 0, \quad s' > (\epsilon\Lambda)^2, \tag{1.16}$$

1.3 Scattering amplitude in the forward limits

[ELABORATE MORE FROM [2]]

1.4 Froissart unitary bounds and dispersion relation

Froissart bound: Unitarity forces the high-energy amplitude in the forward limit is bounded by

$$\mathcal{A}(s) < \mathcal{O}(s \ln^2 s) \tag{1.17}$$

It is a necessary condition for the vanishing boundary contribution when we deform the contour integrals from IR to UV regime [PROVE THE BOUND AND ELABORATE MORE ON THE DISPERSION RELATION].

1.5 Positivity bounds (original version)

Physics in the IR regime can be deform to UV (contour C to C'). The boundary constribution varnishes because of the Froissart bound [ELABORATE MORE].

$$f \equiv \frac{1}{2\pi i} \oint_C \mathrm{d}s \frac{\mathcal{A}(s)}{(s-\mu^2)^3} = \frac{1}{2\pi i} \left(\int_{-\infty}^{0} + \int_{4m^2}^{\infty} \right) \mathrm{d}s \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3}, \tag{1.18}$$

with Disc $\mathcal{A}(s) \equiv \mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon)$. From here, we see that the dim-6 and dim-8 operators in low-energy EFT can be constrained by the positivity bound (Disc $M(s,0) \geq 0$) in the UV regime [ADD FIGURE].

In the forward limit ($t \to 0$), we have $s = 4m^2 - u$. Changing the variable with according bounds in the first term, f can be rewrite as:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(4m^2-u)}{(4m^2-u-\mu^2)^3} + \int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3} \right). \tag{1.19}$$

The crossing symmmetry reads $\mathcal{A}(4m^2 - u) = \mathcal{A}(s) = \mathcal{A}(u)$. Hence,

$$\begin{aligned} \text{Disc } \mathcal{A}(4m^2-u) &= \mathcal{A}(4m^2-u+i\epsilon) - \mathcal{A}(4m^2-u-i\epsilon) && (1.20)\\ &= \mathcal{A}(u-i\epsilon) - \mathcal{A}(u+i\epsilon) && (1.21)\\ &= -\text{Disc } \mathcal{A}(u). && (1.22) \end{aligned}$$

Applying this relation to Eq. 1.19, and replace the variable u by s, we have:

$$f = \frac{1}{2\pi i} \left(\int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(s)}{(-4m^2+s+\mu^2)^3} + \int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3} \right). \tag{1.23}$$

Here, taking the Schwarz reflection ($\mathcal{A}(s^*) = \mathcal{A}^*(s)$), we also have

$$\begin{aligned} \text{Disc } \mathcal{A}(u) &= \mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon) && (1.24)\\ &= \mathcal{A}(s+i\epsilon) - \mathcal{A}^*(s+i\epsilon) && (1.25)\\ &= 2i\text{Im } \mathcal{A}(s), && (1.26) \end{aligned}$$

hence, f becomes:

$$f = \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}s \left[\frac{1}{(-4m^2+s+\mu^2)^3} + \frac{1}{(s-\mu^2)^3}\right] \mathrm{Im}\ \mathcal{A}(s). \tag{1.27}$$

From the Optical theorem of $\mathrm{Im}\ \mathcal{A}(s) = \sqrt{s(s-4m^2)}\sigma_t(s)$, we derive

$$f = \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}s \left[\frac{1}{(-4m^2+s+\mu^2)^3} + \frac{1}{(s-\mu^2)^3}\right] \sqrt{s(s-4m^2)}\sigma_t(s). \tag{1.28}$$

1.6 Positivity bound for SMEFT

In SMEFT, the multi-particle production of massless particles give rise to the branch cut that covers the whole real axis. [ADD FIGURE]

$$f = \frac{1}{2\pi i}\oint \mathrm{d}s \frac{\mathcal{A}(s)}{(s-\mu^2)^3}. \tag{1.29}$$

For improving the positivity, we introduce the modified amplitude. with the

$$\begin{aligned} B_{\epsilon\Lambda}(s) &\equiv \mathcal{A}(s) - \frac{1}{2\pi i}\int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} \mathrm{d}s' \frac{\mathrm{Disc}\mathcal{A}(s')}{s'-s} && (1.30)\\ &= \frac{1}{2\pi i}\oint_{\mathrm{C}} \mathrm{d}s' \frac{\mathcal{A}(s')}{s'-s} - \frac{1}{2\pi i}\int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2} \mathrm{d}s' \frac{\mathrm{Disc}\mathcal{A}(s')}{s'-s} && (1.31)\\ &= \frac{1}{2\pi i}\int_{\mathrm{C}'_{\epsilon\Lambda}} \mathrm{d}s' \frac{\mathcal{A}(s')}{s'-s} = \frac{1}{2\pi i}\oint_{\mathrm{C}_{\epsilon\Lambda}} \mathrm{d}s' \frac{B_{,\epsilon\Lambda}(s')}{s'-s}, && (1.32) \end{aligned}$$

the modified amplitude has the same behavior at $s \to \infty$ and satisfies the Froissart bound.

Next, we define:

$$\begin{aligned} f_{\epsilon\Lambda}(s) &\equiv \frac{1}{2}\frac{\mathrm{d}^2 B_{\epsilon\Lambda}(s)}{\mathrm{d}s^2} && (1.33)\\ =&\frac{1}{2\pi i}\left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty}\right) \mathrm{d}s' \frac{\mathrm{Disc}\ B_{\epsilon\Lambda}(s')}{(s'-s)^3} && (1.34)\\ =&\frac{1}{2\pi i}\left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty}\right) \mathrm{d}s' \frac{\mathrm{Disc}\ \mathcal{A}(s')}{(s'-s)^3} && (1.35)\\ =&\frac{1}{\pi}\left(\int_{(\epsilon\Lambda)^2+M^2}^{\infty} \mathrm{d}s' \frac{1}{(s'+s-M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} \mathrm{d}s' \frac{1}{(s'-s)^3}\right) \mathrm{Im}\ \mathcal{A}(s) && (1.36)\\ =&\frac{1}{\pi}\left(\int_{(\epsilon\Lambda)^2+M^2}^{\infty} \mathrm{d}s' \frac{1}{(s'+s-M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty} \mathrm{d}s' \frac{1}{(s'-s)^3}\right) \sqrt{(s-M_-^2)(s-M_+^2)}\sigma_t. && (1.37) \end{aligned}$$

Here we follow the same procedure with the original version of positivity bound, applying Froissart bound for the deformation and changing the variable $s' \to M^2 - s'$, where $M^2 = 2m_1^2 + 2m_2^2$. [ADD MORE PHYSICAL INTERPRETATIONS + POLARIZATIONS]

1.7 $ZZ \to ZZ$ process

External polarization: Let $p^\mu = (E, 0, 0, p_z)$, thus $p^2 = E^2 - p_z^2 = m^2$. Take the canonical basis that satisfying $p^\mu \epsilon_\mu = 0$ and $\epsilon_\mu^2 = -1$.

$$\epsilon_1^\mu = (0,1,0,0)(\text{traverse}), \tag{1.38}$$
$$\epsilon_2^\mu = (0,0,1,0)(\text{traverse}), \tag{1.39}$$
$$\epsilon_3^\mu = \frac{1}{m}\,(p_z, 0, 0, E)\,(\text{longitudinal}). \tag{1.40}$$

We can parameterize the polarization vectors of 2 imcoming Z bosons as:

$$\epsilon^\mu(V_1) = \sum_{i=1}^{3} a_i \epsilon_i^\mu = (a_3 \frac{p_1}{m_1}, a_1, a_2, a_3 \frac{E_1}{m_1}), \tag{1.41}$$
$$\epsilon^\mu(V_2) = \sum_{i=1}^{3} a_i \epsilon_i^\mu = (b_3 \frac{p_2}{m_2}, b_1, b_2, b_3 \frac{E_2}{m_2}). \tag{1.42}$$

1.8 Dim-8 operators included in QGC

Operators involved in quartic gauge boson couplings (QGC) has been studied in [3], [4], [5] and are listed into 3 categories as followed:

1.8.1 Operators containing just $D_\mu \Phi$

The two independent operators in this class are

$$\mathcal{L}_{S,0} = \left[(D_\mu\Phi)^\dagger D_\nu\Phi\right] \times \left[(D^\mu\Phi)^\dagger D^\nu\Phi\right] \tag{1.43}$$
$$\mathcal{L}_{S,1} = \left[(D_\mu\Phi)^\dagger D^\mu\Phi\right] \times \left[(D_\nu\Phi)^\dagger D^\nu\Phi\right] \tag{1.44}$$
$$\mathcal{L}_{S,2} = \left[(D_\mu\Phi)^\dagger D_\nu\Phi\right] \times \left[(D^\mu\Phi)^\dagger D^\nu\Phi\right] \tag{1.45}$$

1.8.2 Operators containing $D_\mu\Phi$ and field strength

The operators in this class are:

$$\mathcal{L}_{M,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{1.46}$$

$$\mathcal{L}_{M,1} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{1.47}$$

$$\mathcal{L}_{M,2} = [B_{\mu\nu}B^{\mu\nu}] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{1.48}$$

$$\mathcal{L}_{M,3} = \left[B_{\mu\nu}B^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{1.49}$$

$$\mathcal{L}_{M,4} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}D^\mu\Phi\right] \times B^{\beta\nu} \tag{1.50}$$

$$\mathcal{L}_{M,5} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}D^\nu\Phi\right] \times B^{\beta\mu} \tag{1.51}$$

$$\mathcal{L}_{M,6} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\nu}D^\mu\Phi\right] \tag{1.52}$$

$$\mathcal{L}_{M,7} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\mu}D^\nu\Phi\right] \tag{1.53}$$

1.8.3 Operators containing just the field strength tensor

The following operators containing just the field strength tensor also lead to quartic anomalous couplings:

$$\mathcal{L}_{T,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \mathrm{Tr}\left[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}\right] \tag{1.54}$$

$$\mathcal{L}_{T,1} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}\right] \tag{1.55}$$

$$\mathcal{L}_{T,2} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}\right] \tag{1.56}$$

$$\mathcal{L}_{T,3} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\hat{W}^{\nu\alpha}\right] \times B_{\beta\nu} \tag{1.57}$$

$$\mathcal{L}_{T,4} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\alpha\mu}\hat{W}^{\beta\nu}\right] \times B_{\beta\nu} \tag{1.58}$$

$$\mathcal{L}_{T,5} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times B_{\alpha\beta}B^{\alpha\beta} \tag{1.59}$$

$$\mathcal{L}_{T,6} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times B_{\mu\beta}B^{\alpha\nu} \tag{1.60}$$

$$\mathcal{L}_{T,7} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times B_{\beta\nu}B^{\nu\alpha} \tag{1.61}$$

$$\mathcal{L}_{T,8} = B_{\mu\nu}B^{\mu\nu}B_{\alpha\beta}B^{\alpha\beta} \tag{1.62}$$

$$\mathcal{L}_{T,9} = B_{\alpha\mu}B^{\mu\beta}B_{\beta\nu}B^{\nu\alpha} \tag{1.63}$$

1.9 Discussions and Questions

Question 1: Why lightest heavy state?

References

[1] C. Zhang and S.-Y. Zhou, *Positivity bounds on vector boson scattering at the LHC*, *Phys. Rev. D* **100** (2019) 095003 [1808.00010].

[2] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, *Causality, analyticity and an IR obstruction to UV completion*, *JHEP* **10** (2006) 014 [hep-th/0602178].

[3] O. Eboli, M. Gonzalez-Garcia and J. Mizukoshi, *p p —¿ j j e+- mu+- nu nu and j j e+- mu-+ nu nu at O(alpha(em)**6) and O(alpha(em)**4 alpha(s)**2) for the study of the quartic electroweak gauge boson vertex at CERN LHC*, *Phys. Rev. D* **74** (2006) 073005 [hep-ph/0606118].

[4] C. Degrande, O. Eboli, B. Feigl, B. Jäger, W. Kilian, O. Mattelaer et al., *Monte Carlo tools for studies of non-standard electroweak gauge boson interactions in multi-boson processes: A Snowmass White Paper*, in *Community Summer Study 2013: Snowmass on the Mississippi*, 9, 2013, 1309.7890.

[5] O. J. P. Éboli and M. C. Gonzalez-Garcia, *Classifying the bosonic quartic couplings*, *Phys. Rev. D* **93** (2016) 093013 [1604.03555].