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Progress on positivity bounds in SMEFT

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Abstract: In this report, we rederive some current analysis on SMEFT positivity bound. We first rederive the main result in [1], which investigate the vector boson scattering process (VBS) by considering diagrams involving quartic gauge boson couplings (WGC) governed by SMEFT Dim-8 operators.

Contents

1 Positivity bounds in VBS

In this section, we re-interpret some basic concepts, theorems and re-derive (in details) some of the main results and calculations in [1].

1.1 Crossing symmetry

Consider a $2 \to 2$ process, any of the particles can be replaced by its antiparticle on the other side of the interaction. Hence, for
Spin = 0: $M(s,t) = M(u,t)$.
Spin > 0: With the linear polarizations vector $(\epsilon_1^\mu)^* = \epsilon_3^\mu, (\epsilon_2^\mu)^* = \epsilon_4^\mu$, and with restriction to forward limit, we have $M(s,0) = M(u,0)$ (or $\mathcal{A}(s) = \mathcal{A}(u)$).

1.2 Optical theorem

The Optical theorem yield the relation between forward scattering amplitude and cross-section (refer to the Appendix).

$$\text{Im } \mathcal{A}(k_1k_2 \to k_1k_2) = 2E_1E_2|v_1 - v_2|\sigma_t. \tag{1.1}$$

Going into the CM-system, we have $p_1 + p_2 = 0, E_{\text{CM}} = E_1 + E_2, \mathbf{p}_{\text{CM}} = \mathbf{p}_1 = -\mathbf{p}_2$, (with $v = \frac{p}{E}$) we get the optical theorem in the standard form:

$$\text{Im } \mathcal{A}(k_1k_2 \to k_1k_2) = 2E_{\text{CM}}\mathbf{p}_{\text{CM}}\sigma_t. \tag{1.2}$$

when 2 incoming particles are the same, $m = m_1 = m_2$, $E_1 = E_2 = E$, we have $s = 4E^2 = E_{\text{CM}}^2, \mathbf{p}_{\text{CM}} = \sqrt{s^2/4 - m^2}$,)

$$\begin{aligned} \text{Im } \mathcal{A}(k_1k_2 \to k_1k_2) =& 2\sqrt{s}\sqrt{\frac{s}{4} - m^2}\sigma_t & (1.3)\\ =& \sqrt{s(s-4m^2)}\sigma_t. & (1.4) \end{aligned}$$

In general, with 2 different incoming particles, defining $M_+ =$ we have

$$\begin{aligned} & 2(E_1+E_2)\mathbf{p}_{\text{CM}} \\ =& 2\sqrt{(E_1+E_2)^2\mathbf{p}_{\text{CM}}^2} & (1.5)\\ =& 2\sqrt{\mathbf{p}_{\text{CM}}^4 + 2\mathbf{p}_{\text{CM}}^2E_1E_2 + E_1^2E_2^2 - \mathbf{p}_{\text{CM}}^4 + (E_1^2+E_2^2)\mathbf{p}_{\text{CM}}^2 - E_1^2E_2^2} & (1.6)\\ =& \sqrt{(2\mathbf{p}_{\text{CM}}^2 + 2E_1E_2)^2 - 4m_1^2m_2^2} & (1.7)\\ =& \sqrt{\left((E_1+E_2)^2 - 2(m_1^2+m_2^2)\right)^2 - 4m_1^2m_2^2} & (1.8)\\ =& \sqrt{(E_1+E_2)^4 - 2(E_1+E_2)^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2} & (1.9)\\ =& \sqrt{s^2 - s(M_+^2+M_-^2) + M_+^2M_-^2} & (1.10)\\ =& \sqrt{s(s-M_+^2) - M_-^2(s-M_+^2)} & (1.11)\\ =& \sqrt{(s-M_-^2)(s-M_+^2)}. & (1.12) \end{aligned}$$

Hence, Eq. 1.1 yields

$$\begin{aligned}\text{Im } A(k_1k_2 \to k_1k_2) =& 2(E_1\mathbf{p}_2 - E_2\mathbf{p}_1)\sigma_t & (1.13)\\ =& 2(E_1 + E_2)\mathbf{p}_{\text{CM}}\sigma_t & (1.14)\\ =& \sqrt{(s - M_-^2)(s - M_+^2)}\sigma_t. & (1.15)\end{aligned}$$

It yields back the result of Eq. 1.4 when $M_- = 0$.

$$\text{Im}A_{ab}^{q_1q_2}(s') = \sqrt{(s' - M_+^2)(s' - M_-^2)}\sigma_{ab}^{q_1q_2}(s') > 0, \quad s' > (\epsilon\Lambda)^2, \tag{1.16}$$

1.3 Scattering amplitude in the forward limits

[ELABORATE MORE FROM [2]]

1.4 Froissart unitary bounds and dispersion relation

Froissart bound: Unitarity forces the high-energy amplitude in the forward limit is bounded by

$$\mathcal{A}(s) < \mathcal{O}(s \ln^2 s) \tag{1.17}$$

It is a necessary condition for the vanishing boundary contribution when we deform the contour integrals from IR to UV regime [PROVE THE BOUND AND ELABORATE MORE ON THE DISPERSION RELATION].

1.5 Positivity bounds (original version)

Physics in the IR regime can be deform to UV (contour C to C'). The boundary constribution varnishes because of the Froissart bound [ELABORATE MORE].

$$f \equiv \frac{1}{2\pi i}\oint_C \mathrm{d}s \frac{\mathcal{A}(s)}{(s-\mu^2)^3} = \frac{1}{2\pi i}\left(\int_{-\infty}^{0} + \int_{4m^2}^{\infty}\right)\mathrm{d}s \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3}, \tag{1.18}$$

with Disc $\mathcal{A}(s) \equiv \mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon)$. From here, we see that the dim-6 and dim-8 operators in low-energy EFT can be constrained by the positivity bound (Disc $M(s,0) \geq 0$) in the UV regime [ADD FIGURE].

In the forward limit ($t \to 0$), we have $s = 4m^2 - u$. Changing the variable with according bounds in the first term, f can be rewrite as:

$$f = \frac{1}{2\pi i}\left(\int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(4m^2 - u)}{(4m^2 - u - \mu^2)^3} + \int_{4m^2}^{\infty} \mathrm{d}u \frac{\text{Disc } \mathcal{A}(s)}{(s-\mu^2)^3}\right). \tag{1.19}$$

The crossing symmmetry reads $\mathcal{A}(4m^2-u)=\mathcal{A}(s)=\mathcal{A}(u)$. Hence,

$$\begin{aligned}\text{Disc }\mathcal{A}(4m^2-u)&=\mathcal{A}(4m^2-u+i\epsilon)-\mathcal{A}(4m^2-u-i\epsilon) &(1.20)\\ &=\mathcal{A}(u-i\epsilon)-\mathcal{A}(u+i\epsilon) &(1.21)\\ &=-\text{Disc }\mathcal{A}(u). &(1.22)\end{aligned}$$

Applying this relation to Eq. 1.19, and replace the variable u by s, we have:

$$f=\frac{1}{2\pi i}\left(\int_{4m^2}^{\infty}\mathrm{d}u\frac{\text{Disc }\mathcal{A}(s)}{(-4m^2+s+\mu^2)^3}+\int_{4m^2}^{\infty}\mathrm{d}u\frac{\text{Disc }\mathcal{A}(s)}{(s-\mu^2)^3}\right). \tag{1.23}$$

Here, taking the Schwarz reflection ($\mathcal{A}(s^*)=\mathcal{A}^*(s)$), we also have

$$\begin{aligned}\text{Disc }\mathcal{A}(u)&=\mathcal{A}(s+i\epsilon)-\mathcal{A}(s-i\epsilon) &(1.24)\\ &=\mathcal{A}(s+i\epsilon)-\mathcal{A}^*(s+i\epsilon) &(1.25)\\ &=2i\text{Im }\mathcal{A}(s), &(1.26)\end{aligned}$$

hence, f becomes:

$$f=\frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}s\left[\frac{1}{(-4m^2+s+\mu^2)^3}+\frac{1}{(s-\mu^2)^3}\right]\text{Im }\mathcal{A}(s). \tag{1.27}$$

From the Optical theorem of $\text{Im }\mathcal{A}(s)=\sqrt{s(s-4m^2)}\sigma_t(s)$, we derive

$$f=\frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}s\left[\frac{1}{(-4m^2+s+\mu^2)^3}+\frac{1}{(s-\mu^2)^3}\right]\sqrt{s(s-4m^2)}\sigma_t(s). \tag{1.28}$$

1.6 Positivity bound for SMEFT

In SMEFT, the multi-particle production of massless particles give rise to the branch cut that covers the whole real axis. [ADD FIGURE]

$$f=\frac{1}{2\pi i}\oint\mathrm{d}s\frac{\mathcal{A}(s)}{(s-\mu^2)^3}. \tag{1.29}$$

For improving the positivity, we introduce the modified amplitude. with the

$$\begin{aligned}B_{\epsilon\Lambda}(s)&\equiv\mathcal{A}(s)-\frac{1}{2\pi i}\int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2}\mathrm{d}s'\frac{\text{Disc}\mathcal{A}(s')}{s'-s} &(1.30)\\ &=\frac{1}{2\pi i}\oint_{\mathrm{C}}\mathrm{d}s'\frac{\mathcal{A}(s')}{s'-s}-\frac{1}{2\pi i}\int_{-(\epsilon\Lambda)^2}^{+(\epsilon\Lambda)^2}\mathrm{d}s'\frac{\text{Disc}\mathcal{A}(s')}{s'-s} &(1.31)\\ &=\frac{1}{2\pi i}\int_{\mathrm{C}'_{\epsilon\Lambda}}\mathrm{d}s'\frac{\mathcal{A}(s')}{s'-s}=\frac{1}{2\pi i}\oint_{\mathrm{C}_{\epsilon\Lambda}}\mathrm{d}s'\frac{B_{,\epsilon\Lambda}(s')}{s'-s}, &(1.32)\end{aligned}$$

the modified amplitude has the same behavior at $s \to \infty$ and satisfies the Froissart bound.

Next, we define:

$$
\begin{aligned}
f_{\epsilon\Lambda}(s) \equiv& \frac{1}{2}\frac{\mathrm{d}^2 B_{\epsilon\Lambda}(s)}{\mathrm{d}s^2} && (1.33)\\
=&\frac{1}{2\pi i}\left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty}\right)\mathrm{d}s'\frac{\text{Disc } B_{\epsilon\Lambda}(s')}{(s'-s)^3} && (1.34)\\
=&\frac{1}{2\pi i}\left(\int_{-\infty}^{-(\epsilon\Lambda)^2} + \int_{+(\epsilon\Lambda)^2}^{\infty}\right)\mathrm{d}s'\frac{\text{Disc } \mathcal{A}(s')}{(s'-s)^3} && (1.35)\\
=&\frac{1}{\pi}\left(\int_{(\epsilon\Lambda)^2+M^2}^{\infty}\mathrm{d}s'\frac{1}{(s'+s-M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty}\mathrm{d}s'\frac{1}{(s'-s)^3}\right)\text{Im } \mathcal{A}(s) && (1.36)\\
=&\frac{1}{\pi}\left(\int_{(\epsilon\Lambda)^2+M^2}^{\infty}\mathrm{d}s'\frac{1}{(s'+s-M^2)^3} + \int_{(\epsilon\Lambda)^2}^{\infty}\mathrm{d}s'\frac{1}{(s'-s)^3}\right)\sqrt{(s-M_-^2)(s-M_+^2)}\sigma_t. && (1.37)
\end{aligned}
$$

Here we follow the same procedure with the original version of positivity bound, applying Froissart bound for the deformation and changing the variable $s' \to M^2 - s'$, where $M^2 = 2m_1^2 + 2m_2^2$. [ADD MORE PHYSICAL INTERPRETATIONS + POLARIZATIONS]

For the positivity bounds on QGC couplings, dim-8 operators are independent of the presence of dim-6 ones, indeed,

$$\sum_i c_i^{(8)} x_i + \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j} > 0, \tag{1.38}$$

or,

$$\sum_i c_i^{(8)} x_i > \sum_{i,j} c_i^{(6)} c_j^{(6)} y_{i,j}. \tag{1.39}$$

While by explicit calculations, we yields that the R.H.S. is already positive definite. Hence, we can imnpose the bound.

$$\sum_i c_i^{(8)} x_i > 0. \tag{1.40}$$

[TO BE CONT.]

1.7 $ZZ \to ZZ$ process

External polarization: Let $p^\mu = (E, 0, 0, p_z)$, thus $p^2 = E^2 - p_z^2 = m^2$. Take the canonical basis that satisfying $p^\mu \epsilon_\mu = 0$ and $\epsilon_\mu^2 = -1$.

$$\epsilon_1^\mu = (0, 1, 0, 0)(\text{traverse}), \tag{1.41}$$

$$\epsilon_2^\mu = (0, 0, 1, 0)(\text{traverse}), \tag{1.42}$$

$$\epsilon_3^\mu = \frac{1}{m}\left(p_z, 0, 0, E\right)(\text{longitudinal}). \tag{1.43}$$

We can parameterize the polarization vectors of 2 imcoming Z bosons as:

$$\epsilon^\mu(V_1) = \sum_{i=1}^{3} a_i \epsilon_i^\mu = (a_3 \frac{p_1}{m_1}, a_1, a_2, a_3 \frac{E_1}{m_1}), \tag{1.44}$$

$$\epsilon^\mu(V_2) = \sum_{i=1}^{3} a_i \epsilon_i^\mu = (b_3 \frac{p_2}{m_2}, b_1, b_2, b_3 \frac{E_2}{m_2}). \tag{1.45}$$

1.8 Dim-8 operators included in QGC

Operators involved in quartic gauge boson couplings (QGC) has been studied in [3], [4], [5] and are listed into 3 categories as followed:

1.8.1 Operators containing just $D_\mu \Phi$

The two independent operators in this class are

$$\mathcal{L}_{S,0} = \left[(D_\mu \Phi)^\dagger D_\nu \Phi\right] \times \left[(D^\mu \Phi)^\dagger D^\nu \Phi\right] \tag{1.46}$$

$$\mathcal{L}_{S,1} = \left[(D_\mu \Phi)^\dagger D^\mu \Phi\right] \times \left[(D_\nu \Phi)^\dagger D^\nu \Phi\right] \tag{1.47}$$

$$\mathcal{L}_{S,2} = \left[(D_\mu \Phi)^\dagger D_\nu \Phi\right] \times \left[(D^\mu \Phi)^\dagger D^\nu \Phi\right] \tag{1.48}$$

1.8.2 Operators containing $D_\mu\Phi$ and field strength

The operators in this class are:

$$\mathcal{L}_{M,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{1.49}$$

$$\mathcal{L}_{M,1} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{1.50}$$

$$\mathcal{L}_{M,2} = \left[B_{\mu\nu}B^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{1.51}$$

$$\mathcal{L}_{M,3} = \left[B_{\mu\nu}B^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{1.52}$$

$$\mathcal{L}_{M,4} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} D^\mu\Phi\right] \times B^{\beta\nu} \tag{1.53}$$

$$\mathcal{L}_{M,5} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu} D^\nu\Phi\right] \times B^{\beta\mu} \tag{1.54}$$

$$\mathcal{L}_{M,6} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\nu} D^\mu\Phi\right] \tag{1.55}$$

$$\mathcal{L}_{M,7} = \left[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\mu} D^\nu\Phi\right] \tag{1.56}$$

1.8.3 Operators containing just the field strength tensor

The following operators containing just the field strength tensor also lead to quartic anomalous couplings:

$$\mathcal{L}_{T,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \mathrm{Tr}\left[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}\right] \tag{1.57}$$

$$\mathcal{L}_{T,1} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}\right] \tag{1.58}$$

$$\mathcal{L}_{T,2} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}\right] \tag{1.59}$$

$$\mathcal{L}_{T,3} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\hat{W}^{\nu\alpha}\right] \times B_{\beta\nu} \tag{1.60}$$

$$\mathcal{L}_{T,4} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\alpha\mu}\hat{W}^{\beta\nu}\right] \times B_{\beta\nu} \tag{1.61}$$

$$\mathcal{L}_{T,5} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times B_{\alpha\beta}B^{\alpha\beta} \tag{1.62}$$

$$\mathcal{L}_{T,6} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times B_{\mu\beta}B^{\alpha\nu} \tag{1.63}$$

$$\mathcal{L}_{T,7} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times B_{\beta\nu}B^{\nu\alpha} \tag{1.64}$$

$$\mathcal{L}_{T,8} = B_{\mu\nu}B^{\mu\nu}B_{\alpha\beta}B^{\alpha\beta} \tag{1.65}$$

$$\mathcal{L}_{T,9} = B_{\alpha\mu}B^{\mu\beta}B_{\beta\nu}B^{\nu\alpha} \tag{1.66}$$

1.9 Discussions and Questions

Question 1: Why lightest heavy state?

A Calculation

A.1 S-type operators

We have,

$$D_\mu\Phi = \left(\partial_\mu + i\frac{g}{2}W_\mu^j\sigma^j + i\frac{g'}{2}B_\mu\right)\Phi. \tag{A.1}$$

$$= \begin{pmatrix} \partial_\mu + i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu & i\frac{g}{2}W_\mu^1 + \frac{g'}{2}W_\mu^2 \\ i\frac{g}{2}W_\mu^1 - \frac{g'}{2}W_\mu^2 & \partial_\mu - i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{A.2}$$

$$= \frac{v}{\sqrt{2}} \begin{pmatrix} i\frac{g}{2}W_\mu^1 + \frac{g'}{2}W_\mu^2 \\ -i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu \end{pmatrix}. \tag{A.3}$$

Hence,

$$(D_\mu\Phi)^\dagger D_\nu\Phi = \frac{v^2}{8}\Bigg[g^2W_\mu^1W_\nu^1 + g^2W_\mu^2W_\nu^2 + igg'(W_\mu^1W_\nu^2 - W_\nu^1W_\mu^2) \tag{A.4}$$

$$+ (g'B_\mu - gW_\mu^3)(g'B_\nu - gW_\nu^3)\Bigg], \tag{A.5}$$

with the gauge boson diagonalization reads:

$$\begin{cases} B_\mu = \cos\theta A_\mu - \sin\theta Z_\mu \\ W_\mu^3 = \sin\theta A_\mu + \cos\theta Z_\mu \end{cases}, \tag{A.6}$$

with $\tan\theta = \frac{g'}{g}$. Hence,

$$(D_\mu\Phi)^\dagger D_\nu\Phi \supset \frac{v^2}{8}(g'B_\mu - gW_\mu^3)(g'B_\nu - gW_\nu^3) \tag{A.7}$$

$$\supset \frac{v^2}{8}(g'\sin\theta + g\cos\theta)^2 Z_\mu Z_\nu \tag{A.8}$$

$$= \frac{v^2g^2}{8}\left(\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right)^2 Z_\mu Z_\nu \tag{A.9}$$

$$= \frac{v^2g^2}{8\cos^2\theta}Z_\mu Z_\nu \tag{A.10}$$

$$= \frac{m^2}{2}Z_\mu Z_\nu. \tag{A.11}$$

Here, we have used the relation $g'\cos\theta = g\sin\theta (= e)$, and $vg = 2m_Z\cos\theta$. So we have,

$$\mathcal{L}_{S,0} = \left[(D_\mu\Phi)^\dagger D_\nu\Phi\right]\left[(D^\mu\Phi)^\dagger D^\nu\Phi\right] \supset \frac{m^4}{4}Z_\mu Z^\nu Z^\mu Z_\nu. \tag{A.12}$$

We also derive the exact same result for $\mathcal{L}_{S,1}, \mathcal{L}_{S,2}$ for Z^4 vertices. However, for W^4, $(WZ)^2$ (and W^2Z^2) vertices the indices for those terms are not the same due to different polarization of $W^{\pm}$.

Since S-type operators only admit longitudinal polarization, the polarization vectors for $1, 2 \to 1, 2$ process (with $p_1 = -p_2 = p, E_1 = E_2 = E$) read:

$$\epsilon_1^\mu =(a_3 \frac{p}{m}, 0, 0, a_3 \frac{E}{m}), \tag{A.13}$$

$$\epsilon_2^\mu =(-b_3 \frac{p}{m}, 0, 0, b_3 \frac{E}{m}). \tag{A.14}$$

Hence, for later convenience, we note here some relations:

$$\epsilon_1^{(*)} \epsilon_2^{(*)} = - a_3^{(*)} b_3^{(*)} \left(\frac{E^2 + p^2}{m^2} \right), \tag{A.15}$$

$$\epsilon_1^* \epsilon_1 = - |a_3|^2, \tag{A.16}$$

$$\epsilon_2^* \epsilon_2 = - |b_3|^2. \tag{A.17}$$

The Z field read:

$$Z^\mu(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \sqrt{2E_k}} \sum_{i=0}^{3} \left[a_{k,j} \epsilon_j^\mu e^{-ikx} + a_{k,j}^\dagger \epsilon_j^{*\mu} e^{ikx} \right]. \tag{A.18}$$

in which the creation/annihilation operators obey the commutation relation $[a_{k_1}, a_{k_2}^\dagger] = (2\pi)^3 \delta^3(\mathbf{k}_1 - \mathbf{k}_2)$, $[a_{k_1}, a_{k_2}] = [a_{k_1}^\dagger, a_{k_2}^\dagger] = 0$. The incoming and outgoing states read:

$$\langle f| = \langle p_1, p_2| =2E a_{p_1} a_{p_2}, \tag{A.19}$$

$$|i\rangle = |p_1, p_2\rangle =2E a_{p_1}^\dagger a_{p_2}^\dagger. \tag{A.20}$$

Now consider the interacting vertices of $\langle f| : Z_\mu Z^\nu Z^\mu Z_\nu : |i\rangle$, we extract all the posible combination of creation/annihilation operators $(a, a^\dagger)$ for the $ZZ \to ZZ$ chanel.

First, we see that all the terms have to contain $2a$ and $2a^\dagger$. Then, of those $4Z$, there are six ways to exact such terms. We first consider the 2 cases when $2a$ is extracted from ZZ pair with the same Lorentz indices ($Z_\mu Z^\mu$ or $Z_\nu Z^\nu$), and then other 4 cases when $2a$ is extracted from ZZ pair with the different indices ($Z_\mu Z^\nu$ or $Z_\mu Z_\nu$ or $Z^\mu Z^\nu$ or $Z^\mu Z_\nu$). We'll investigate each cases in the 2 group and see that the rest obey the same rules.

Let's consider the case when exacting $2a$ from $Z_\mu Z^\mu$ (and $2a^\dagger$ from $Z_\nu Z^\nu$). For $1, 2 \to 1, 2$ process, using the result from commutation relation that $\langle p_1, p_2| a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} |p_1, p_2\rangle =$

$4E^2 \langle 0| [a_{p_1}, a^\dagger_{k_1}][a_{p_2}, a^\dagger_{k_2}] + [a_{p_1}, a^\dagger_{k_2}][a_{p_2}, a^\dagger_{k_1}] + [a_{k_3}, a^\dagger_{p_1}][a_{k_4}, a^\dagger_{p_2}] + [a_{k_4}, a^\dagger_{p_1}][a_{k_3}, a^\dagger_{p_2}] |0\rangle$, we get 4 terms from the extraction as follow:

Z_μ	Z^ν	Z^μ	Z_ν
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
1	2	2	1
2	1	1	2
2	2	1	1

From A.15, the first term read:

$$\epsilon_{1\mu}\epsilon_1^{*\nu}\epsilon_2^{\mu}\epsilon_{2\nu}^{*} = (-1)^2(a_3 b_3 a_3^* b_3^*)\left(\frac{p^2+E^2}{m^2}\right)^2 \tag{A.21}$$

$$= |a_3|^2|b_3|^2\left(\frac{p^2+E^2}{m^2}\right)^2 \tag{A.22}$$

All the rest 3 terms yield the same result since the Lorentz indices are contracted between 1 and 2 (x4). Moreover, the case when $2a$ is extracted from $Z_\nu Z^\nu$ also yields the same results (x2).

For the rest 4 cases, we investigate the first one,

Z_μ	Z^ν	Z^μ	Z_ν
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
2	1	1	2
1	2	2	1

Using A.15, A.16, the first term read:

$$\epsilon_{1\mu}\epsilon_2^{*\nu}\epsilon_1^{\mu}\epsilon_{2\nu}^{*} = |a_3|^2|b_3|^2. \tag{A.23}$$

The second term has Lorentz indices contracted in the same fashion and yields the same result (x2). The third term read:

$$\epsilon_{2\mu}\epsilon_1^{*\nu}\epsilon_1^{\mu}\epsilon_{2\nu}^{*} = |a_3|^2|b_3|^2\left(\frac{p^2+E^2}{m^2}\right)^2, \tag{A.24}$$

which is also the result for the fourth term (x2). Moreover, we see that all the rest 3 cases yield exactly the same results (x4)

To sum up, and then to drop the m^4 component, we have same the result for $\mathcal{L}_{S,0}, \mathcal{L}_{S,1}, \mathcal{L}_{S,2}$:

$$
\begin{aligned}
&\frac{m^4}{4} Z_\mu Z^\nu Z^\mu Z_\nu \left[(4.2+2.4)|a_3|^2 |b_3|^2 \left(\frac{p^2+E^2}{m^2} \right)^2 + 2.4|a_3|^2|b_3|^2 \right] \\
\supset& 16|a_3|^2|b_3|^2 E^2 p^2 && \text{(A.25)} \\
\equiv& 16 A_6 E^2 p^2. && \text{(A.26)}
\end{aligned}
$$

A.2 T-type operators

First, we expand,

$$
\begin{aligned}
\hat{W}^{\mu\nu} \equiv& ig\frac{\sigma^j}{2} W^{j,\mu\nu} && \text{(A.27)} \\
=& i\frac{g}{2} \begin{pmatrix} W^{3\mu\nu} & W^{1\mu\nu} - iW^{2\mu\nu} \\ W^{1\mu\nu} - iW^{2\mu\nu} & -W^{3\mu\nu} \end{pmatrix} && \text{(A.28)} \\
=& i\frac{g}{2} \begin{pmatrix} \cos\theta Z^{\mu\nu} + \sin\theta A^{\mu\nu} & \sqrt{2} W^{+\mu\nu} \\ \sqrt{2} W^{-\mu\nu} & -\cos\theta Z^{\mu\nu} - \sin\theta A^{\mu\nu} \end{pmatrix}. && \text{(A.29)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{Tr}\left[\hat{W}_{\mu\nu} \hat{W}^{\alpha\beta} \right] \supset& -\frac{g^2}{2} \cos^2\theta Z_{\mu\nu} Z^{\alpha\beta} && \text{(A.30)} \\
=& -\frac{e^2}{2} \cot^2\theta Z_{\mu\nu} Z^{\alpha\beta}. && \text{(A.31)}
\end{aligned}
$$

Also,

$$
\hat{B}_{\mu\nu} \equiv i\frac{g'}{2} B_{\mu\nu} \supset -i\frac{g'}{2} \sin\theta Z_{\mu\nu} = -i\frac{e}{2} \tan\theta Z_{\mu\nu}, \tag{A.32}
$$

and,

$$
\hat{B}_{\mu\nu} \hat{B}^{\alpha\beta} \supset -\frac{e^2}{4} \tan^2\theta Z_{\mu\nu} Z^{\alpha\beta}. \tag{A.33}
$$

We can see that replacing $\mathrm{Tr}\left[\hat{W}_{\mu\nu} \hat{W}^{\alpha\beta} \right]$ by $\hat{B}_{\mu\nu} \hat{B}^{\alpha\beta}$ yields a factor of $\frac{\tan^4\theta}{2}$. Hence, we only need to calculate $\mathcal{L}_{T,0}, \mathcal{L}_{T,1}, \mathcal{L}_{T,2}$ and all other T-type operators can be derive by adding that factors. The derivative of Z field reads:

$$
\partial^\nu Z^\mu(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \sqrt{2E_k}} \sum_{i=0}^{3} \left[-ik^\nu a_{k,j} \epsilon_j^\mu e^{-ikx} + ik^\nu a_{k,j}^\dagger \epsilon_j^{*\mu} e^{ikx} \right]. \tag{A.34}
$$

The $\mathcal{L}_{T,i}$ vertices only admit tranverse component of polarization, hence, they read:

$$\epsilon_1^\mu = (0, a_1, a_2, 0), \tag{A.35}$$
$$\epsilon_2^\mu = (0, b_1, b_2, 0). \tag{A.36}$$

Some useful relations:

$$\epsilon_1^{(*)}\epsilon_2^{(*)} = -a_1^{(*)}b_1^{(*)} - a_2^{(*)}b_2^{(*)}, \tag{A.37}$$
$$\epsilon_1^*\epsilon_1 = -|a_1|^2 - |a_2|^2, \tag{A.38}$$
$$\epsilon_2^*\epsilon_2 = -|b_1|^2 - |b_2|^2. \tag{A.39}$$

A.2.1 $\mathcal{L}_{T,0}$, $\mathcal{L}_{T,1}$, $\mathcal{L}_{T,5}$, $\mathcal{L}_{T,6}$, $\mathcal{L}_{T,8}$

First consider $\mathcal{L}_{T,0}$,

$$\begin{aligned}
&\mathrm{Tr}\Big[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\Big] \times \mathrm{Tr}\Big[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}\Big] \\
\supset&\frac{e^4}{4}\cot\theta^4 Z_{\mu\nu}Z^{\mu\nu}Z_{\alpha\beta}Z^{\alpha\beta} &\text{(A.40)}\\
\supset&\frac{e^4}{4}\cot\theta^4(\partial_\mu Z_\nu\partial^\mu Z^\nu\partial_\alpha Z_\beta\partial^\alpha Z^\beta + \partial_\mu Z_\nu\partial^\mu Z^\nu\partial_\beta Z_\alpha\partial^\beta Z_\alpha \\
&+ \partial_\nu Z_\mu\partial^\nu Z^\mu\partial_\alpha Z_\beta\partial^\alpha Z^\beta + \partial_\nu Z_\mu\partial^\nu Z^\mu\partial_\beta Z_\alpha\partial^\beta Z_\alpha). &\text{(A.41)}
\end{aligned}$$

We derived A.41 by droping all 12 terms contain $Z_\gamma\partial^\gamma$ since contracting them yield $\epsilon_\gamma p^\gamma = 0$ (no matter they belong to the "1" or "2" states as we only have tranverse polarizations for T-type operators).

We see that those 4 term are contracted in the identical way (symmetric in $\mu \leftrightarrow \nu, \alpha \leftrightarrow \beta$), hence we only need to consider 1 term and then multiply the results by 4 (x4). Let's pick the $\partial_\nu Z_\mu\partial^\nu Z^\mu\partial_\alpha Z_\beta\partial^\alpha Z^\beta$ to investigate. There are also 6 cases of extracting : $a^\dagger a^\dagger aa$:. Consider the first case, we have 4 terms:

$\partial_\nu Z_\mu$	$\partial^\nu Z^\mu$	$\partial_\alpha Z_\beta$	$\partial^\alpha Z^\beta$
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
2	1	1	2
1	2	2	1

The first term read:

$$p_{1\nu}\epsilon_{1\mu}p_2^\nu\epsilon_2^\mu p_{1\alpha}\epsilon_{1\beta}^* p_2^\alpha\epsilon_2^{\beta *} = (E^2+p^2)^2(a_1^*b_1^* + a_2^*b_2^*)(a_1b_1 + a_2b_2). \tag{A.42}$$

The rest 3 terms yield the same result (x4) as all the contractions are between "1" and "2" state. The second case where we swap all $a \leftrightarrow a^\dagger$ follows the same principle and ends up in the same results (x2). Consider one of the rest 4 cases, there are also 4 terms in it:

$\partial_\nu Z_\mu$	$\partial^\nu Z^\mu$	$\partial_\alpha Z_\beta$	$\partial^\alpha Z^\beta$
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
2	2	1	1
1	2	2	1
2	1	1	2

The first one read:

$$p_{1\nu}\epsilon_{1\mu}p_1^\nu\epsilon_1^{\mu *}p_{\alpha 2}\epsilon_{\beta 2}p_2^\alpha\epsilon_2^{\beta *} = m^4(|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2). \tag{A.43}$$

The second term yields the same results as it only exchange "$1 \leftrightarrow 2$" (x2). The third term read:

$$p_{1\nu}\epsilon_{1\mu}p_2^\nu\epsilon_2^{\nu *}p_{\alpha 2}\epsilon_{\beta 2}p_1^\alpha\epsilon_1^{\beta *} = (E^2 + p^2)^2(a_1b_1^* + a_2b_2^*)(a_1^*b_1 + a_2^*b_2) \tag{A.44}$$

This is also the result for the final term (x2). All the rest 3 cases yields the same results (x4) as they only differ by the exchange of $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$. To sum up and then to drop m^4 term, we have

$$\begin{aligned}
&\frac{e^4}{4}\cot\theta^4\Big[(E^2+p^2)^2 4\left[4.2(a_1^*b_1^* + a_2^*b_2^*)(a_1b_1 + a_2b_2) + 2.4(a_1b_1^* + a_2b_2^*)(a_1^*b_1 + a_2^*b_2)\right] \\
&+4.2.4m^4(|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)\Big] \\
&\supset e^4\cot\theta^4\Big[64\left(|a_1|^2|a_2|^2 + |a_2|^2|b_2|^2\right) + 32\left(a_1a_2^*b_1^*b_2 + c.c.\right) + 32(a_1a_2^*b_1b_2^* + c.c)\Big]E^2p^2 \qquad \text{(A.45)} \\
&\equiv e^4\cot\theta^4\left[64A_1 + 32A_4 + 32A_4'\right]E^2p^2. \qquad \text{(A.46)}
\end{aligned}$$

$\mathcal{L}_{T,1}$ yields the same result as they only exchange $\mu \leftrightarrow \alpha, \beta \leftrightarrow \nu$ in the term A.40.
$\mathcal{L}_{T,5}$ is also equal $\mathcal{L}_{T,6}$ and can be calculated as,

$$\begin{aligned}
&\frac{\tan^4\theta}{2}e^4\cot\theta^4\left[64A_1 + 32A_4 + 32A_4'\right]E^2p^2 \\
=&e^4\left[32A_1 + 16A_4 + 16A_4'\right]E^2p^2.
\end{aligned} \tag{A.47}$$

$\mathcal{L}_{T,8}$ is derived as,

$$\begin{aligned}
&\frac{\tan^4\theta}{2}e^4\left[32A_1 + 16A_4 + 16A_4'\right]E^2p^2 \\
=&e^4\tan^4\theta\left[16A_1 + 8A_4 + 8A_4'\right]E^2p^2.
\end{aligned} \tag{A.48}$$

A.2.2 $\mathcal{L}_{T,2}, \mathcal{L}_{T,7}, \mathcal{L}_{T,9}$

Next, consider $\mathcal{L}_{T,2}$,

$$\begin{aligned}
&\mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}\right] \\
\supset &\frac{g^4\cos^4\theta}{4} Z_{\alpha\mu}Z^{\mu\beta}Z_{\beta\nu}Z^{\nu\alpha} \qquad (A.49)\\
\supset &\frac{g^4\cos^4\theta}{4}(\partial_\alpha Z_\mu \partial^\beta Z^\mu \partial_\beta Z_\nu \partial^\alpha Z^\nu + \partial_\mu Z_\alpha \partial^\mu Z^\beta \partial_\nu Z_\beta \partial^\nu Z^\alpha). \qquad (A.50)
\end{aligned}$$

In A.50, we also dropped all 14 terms contain $Z_\gamma \partial^\gamma$. The two above term differ only by swapping $\alpha \leftrightarrow \mu, \nu \leftrightarrow \beta$, and hence yield the same result (x2). Let's investigate the first one. There are also 6 cases of extracting $: a^\dagger a^\dagger a a :$. First, consider:

$\partial_\alpha Z_\mu$	$\partial^\beta Z^\mu$	$\partial_\beta Z_\nu$	$\partial^\alpha Z^\nu$
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
1	2	2	1
2	1	1	2

The first term read:

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^\beta \epsilon_2^\mu p_{\beta 1}\epsilon^*_{\nu 1}p_2^\alpha \epsilon_2^{\nu *} = (E^2+p^2)^2(a_1b_1 + a_2b_2)(a_1^*b_1^* + a_2^*b_2^*). \qquad (A.51)$$

The second term yield the same result(x2). Let's consider the third one,

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^\beta \epsilon_2^\mu p_{\beta 2}\epsilon^*_{\nu 2}p_1^\alpha \epsilon_1^{\nu *} = m^4(a_1b_1 + a_2b_2)(a_1^*b_1^* + a_2^*b_2^*). \qquad (A.52)$$

This is also the result for the final term (x2).

Now, since swapping $a \leftrightarrow a^\dagger$ is equivalent to swapping the indices pairwise, we have a second case that yield exactly the same results as the first one (x2).

Next, consider the third case of

$\partial_\alpha Z_\mu$	$\partial^\beta Z^\mu$	$\partial_\beta Z_\nu$	$\partial^\alpha Z^\nu$
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
2	2	1	1
1	2	2	1
2	1	1	2

The first term (and also the second) (x2) yields,

$$p_{\alpha 1}\epsilon_{\mu 1}p_1^{\beta}\epsilon_1^{\mu *}p_{\beta 2}\epsilon_{\nu 2}p_2^{\alpha}\epsilon_2^{\nu *}=(E^2+p^2)^2(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2). \tag{A.53}$$

The third (and also fourth) (x2) yields,

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^{\beta}\epsilon_2^{\mu *}p_{\beta 2}\epsilon_{\nu 2}p_1^{\alpha}\epsilon_1^{\nu *}=m^4(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2). \tag{A.54}$$

Swaping $a \leftrightarrow a^\dagger$ does not change the result, hence, we have the fourth case that also yield the same result (x2). Now for the last 2 cases. First, consider,

$\partial_\alpha Z_\mu$	$\partial^\beta Z^\mu$	$\partial_\beta Z_\nu$	$\partial^\alpha Z^\nu$
a	$a^\dagger$	$a^\dagger$	a
1	1	2	2
2	2	1	1
1	2	1	2
2	1	2	1

The first (and also second) term (x2) read,

$$p_{\alpha 1}\epsilon_{\mu 1}p_1^{\beta}\epsilon_1^{\mu *}p_{\beta 2}\epsilon_{\nu 2}^{*}p_2^{\alpha}\epsilon_2^{\nu}=(E^2+p^2)^2(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2). \tag{A.55}$$

The third (and fourth) (x2) read,

$$p_{\alpha 1}\epsilon_{\mu 1}p_2^{\beta}\epsilon_2^{\mu *}p_{\beta_1}\epsilon_{\nu 1}^{*}p_2^{\alpha}\epsilon_2^{\nu}=(E^2+p^2)^2(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2). \tag{A.56}$$

Swaping $a \leftrightarrow a^\dagger$ does not change the result, hence, we have the sixth case that also yield the same result (x2).

We have completed calculating all the posibilities for this vetices. Summing up and then subtracting the m^4 term we have the result for $\mathcal{L}_{T,2}$:

$$\begin{aligned}
&\frac{e^4}{4}\cot\theta^4\Big[(E^2+p^2)\big[2.2.2(a_1b_1+a_2b_2)(a_1^*b_1^*+a_2^*b_2^*)\\
&+2.2.2.2(|a_1|^2+|a_2|^2)(|b_1|^2+|b_2|^2)+2.2.2(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2)\big]\\
&+2.2.2m^4(a_1b_1^*+a_2b_2^*)(a_1^*b_1+a_2^*b_2)\Big]\\
&\supset e^4\cot\theta^4\Big[32\left(|a_1|^2|b_1|^2+|a_2|^2|b_2|^2\right)+32\left(|a_1|^2|b_2|^2+|a_2|^2|b_1|^2\right) &\text{(A.57)}\\
&\quad+8\left(a_1a_2^*b_1^*b_2+c.c.\right)+8(a_1a_2^*b_1b_2^*+c.c)\Big]E^2p^2 &\text{(A.58)}\\
&\equiv e^4\cot^4\theta\left[32A_1+16A_2+8A_4+8A_4'\right]E^2p^2. &\text{(A.59)}
\end{aligned}$$

From there, the $\mathcal{L}_{T,7}$ can be derived,

$$\frac{\tan^4\theta}{2}e^4\cot^4\theta\left[32A_1+16A_2+8A_4+8A_4'\right]E^2p^2 \tag{A.60}$$
$$=e^4\left[16A_1+8A_2+4A_4+4A_4'\right]E^2p^2. \tag{A.61}$$

And also, the $\mathcal{L}_{T,9}$

$$\frac{\tan^4\theta}{2}e^4\left[32A_1+16A_2+8A_4+8A_4'\right]E^2p^2 \tag{A.62}$$
$$=e^4\tan^4\theta\left[8A_1+4A_2+2A_4+2A_4'\right]E^2p^2. \tag{A.63}$$

A.3 M-type operators

We have,

$$D_\mu\Phi\supset\frac{v}{\sqrt{2}}\begin{pmatrix}0\\ -i\frac{g}{2}W_\mu^3+i\frac{g'}{2}B_\mu\end{pmatrix} \tag{A.64}$$
$$\supset-\frac{im_Z}{\sqrt{2}}\begin{pmatrix}0\\ Z_\mu\end{pmatrix}. \tag{A.65}$$

Also,

$$\hat{W}_{\mu\nu}\supset\frac{ie\cot\theta}{2}\begin{pmatrix}Z^{\mu\nu} & 0\\ 0 & -Z^{\mu\nu}\end{pmatrix}. \tag{A.66}$$

Hence,

$$[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\mu\Phi]\times\hat{B}^{\beta\nu}\supset\frac{im}{\sqrt{2}}\frac{-ie\cot\theta}{2}\frac{-im}{\sqrt{2}}\frac{-ie\tan\theta}{2}Z_\mu Z_{\beta\nu}Z^\mu Z^{\beta\nu} \tag{A.67}$$
$$=-\frac{m^2e^2}{8}Z_\mu Z_{\beta\nu}Z^\mu Z^{\beta\nu}. \tag{A.68}$$

Similarly,

$$[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\nu\Phi]\times\hat{B}^{\beta\mu}\supset-\frac{m^2e^2}{8}Z_\mu Z_{\beta\nu}Z^\nu Z^{\beta\mu}. \tag{A.69}$$

With this result, we will see later that $\mathcal{L}_{M,4}$ (and $\mathcal{L}_{M,5}$) only differs from $\mathcal{L}_{M,0}$ (and $\mathcal{L}_{M,1}$) by a factor of $-\frac{\tan^2\theta}{2}$. We also have,

$$\hat{W}_{\beta\nu}\hat{W}^{\beta\mu}\supset-\frac{e^2\cot^2\theta}{4}\begin{pmatrix}Z_{\beta\nu}Z^{\beta\mu} & 0\\ 0 & Z_{\beta\nu}Z^{\beta\mu}\end{pmatrix} \tag{A.70}$$

Hence,

$$[(D_\mu\Phi)^\dagger \hat{W}_{\beta\nu}\hat{W}^{\beta\mu}D^\nu\Phi] \supset \frac{im}{\sqrt{2}}\frac{-e^2\cot^2\theta}{4}\frac{-im}{\sqrt{2}}Z_\mu Z_{\beta\nu}Z^{\beta\mu}Z^\nu \tag{A.71}$$

$$= -\frac{e^2}{8}\cot^2\theta m^2 Z_\mu Z_{\beta\nu}Z^{\beta\mu}Z^\nu, \tag{A.72}$$

from which we see that $\mathcal{L}_{M,7}$ differs from $\mathcal{L}_{M,1}$ by a factor of $-\frac{1}{2}$.

A.3.1 $\mathcal{L}_{M,0}, \mathcal{L}_{M,2}, \mathcal{L}_{M,4}$

First, consider $\mathcal{L}_{M,0}$,

$$\mathrm{Tr}\Big[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\Big]\left[(D_\beta\Phi)^\dagger D^\beta\Phi\right] \tag{A.73}$$

$$\supset \frac{e^2\cot^2\theta}{2}Z_{\mu\nu}Z^{\mu\nu}\frac{m^2}{2}Z_\beta Z^\beta \tag{A.74}$$

$$\supset \frac{e^2}{4}\cot^2\theta m^2\left(\partial_\mu Z_\nu\partial^\mu Z^\nu + \partial_\nu Z_\mu \partial^\nu Z^\mu\right)Z_\beta Z^\beta. \tag{A.75}$$

Those 2 terms also yield same results (x2). Consider one of them ($\partial_\mu Z_\nu \partial^\mu Z^\nu Z_\beta Z^\beta$), there are 6 ways of extracting $:a^\dagger a^\dagger a a:$. Consider the first way,

$\partial_\mu Z_\nu$	$\partial^\mu Z^\nu$	Z_β	Z^β
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
1	2	2	1
2	1	1	2
2	1	2	1

First term reads,

$$p_{1\mu}\epsilon_{1\nu}p_2^\mu\epsilon_2^\nu\epsilon_{1\beta}^*\epsilon_2^{\beta *} = \frac{(E^2+p^2)^2}{m^2}(a_1b_1+a_2b_2)a_3^*b_3^*. \tag{A.76}$$

All the rest 3 terms read the same result (x4) since swaping "1" $\leftrightarrow$ "2" with a Lorenzt contracted term does not change the result.

Consider the next case of swaping all $a \leftrightarrow a^\dagger$, we have the result is the complex conjugate of the first case.

Moving on to the next case, we consider,

$\partial_\mu Z_\nu$	$\partial^\mu Z^\nu$	Z_β	Z^β
a	$a^\dagger$	a	$a^\dagger$
1	1	2	2
1	2	2	1
2	2	1	1
2	1	1	2

First term read,

$$p_{1\mu}\epsilon_{1\nu}p_1^{\mu}\epsilon_1^{\nu *}\epsilon_{2\beta}\epsilon_2^{\beta *} = m^2(|a_1|^2 + |a_2|^2)|b_3|^2. \tag{A.77}$$

Second term read,

$$p_{1\mu}\epsilon_{1\nu}p_2^{\mu}\epsilon_2^{\nu *}\epsilon_{2\beta}\epsilon_1^{\beta *} = -\frac{(E^2+p^2)^2}{m^2}(a_1 b_1^* + a_2 b_2^*)a_3^* b_3. \tag{A.78}$$

Third term read,

$$p_{2\mu}\epsilon_{2\nu}p_2^{\mu}\epsilon_2^{\nu *}\epsilon_{1\beta}\epsilon_1^{\beta *} = m^2(|b_1|^2 + |b_2|^2)|a_3|^2. \tag{A.79}$$

Last term read,

$$p_{2\mu}\epsilon_{2\nu}p_1^{\mu}\epsilon_1^{\nu *}\epsilon_{1\beta}\epsilon_2^{\beta *} = -\frac{(E^2+p^2)^2}{m^2}(a_1^* b_1 + a_2^* b_2)a_3 b_3^*. \tag{A.80}$$

All the rest 3 cases yield exactly the same results since they only swap $a \leftrightarrow a^\dagger$ within Lorent invariant terms (x4).

To sum up, then to drop the m^4 term, we have the result for $\mathcal{L}_{M,0}$:

$$\begin{aligned}
&\frac{e^2}{4}\cot^2\theta m^2\Bigg[\frac{(E^2+p^2)^2}{m^2}\Big[4(a_1b_1 + a_2b_2)a_3^*b_3^* + c.c. \\
&- 4(a_1b_1^* + a_2b_2^*)a_3^*b_3 - c.c.\Big] \\
&+ 4m^2\Big[(|a_1|^2 + |a_2|^2)|b_3|^2 + (|b_1|^2 + |b_2|^2)|a_3|^2\Big]\Bigg] \\
\supset & e^2\cot^2\theta\Big[4(a_1b_1 + a_2b_2)a_3^*b_3^* + c.c. - 4(a_1b_1^* + a_2b_2^*)a_3^*b_3 - c.c.\Big]E^2p^2 && \text{(A.81)} \\
\equiv & e^2\cot^2\theta\,[4A_5 + 4A_5']\,E^2p^2. && \text{(A.82)}
\end{aligned}$$

We can also derive the result for $\mathcal{L}_{M,2}$ which is

$$\frac{\tan^4\theta}{2}e^2\cot^2\theta\,[4A_5 + 4A_5']\,E^2p^2 = e^2\tan^2\theta\,[2A_5 + 2A_5']\,E^2p^2. \tag{A.83}$$

And the result for $\mathcal{L}_{M,4}$,

$$-\frac{\tan^2\theta}{2}e^2\cot^2\theta\,[4A_5 + 4A_5']\,E^2p^2 = -e^2\,[2A_5 + 2A_5']\,E^2p^2. \tag{A.84}$$

A.3.2 $\mathcal{L}_{M,1}, \mathcal{L}_{M,3}, \mathcal{L}_{M,5}, \mathcal{L}_{M,7}$

First, consider $\mathcal{L}_{M,1}$

$$\mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\nu\beta}\right]\left[(D_\beta\Phi)^\dagger D^\mu\Phi\right] \tag{A.85}$$

$$\supset \frac{e^2\cot^2\theta}{2}\cos^2\theta Z_{\mu\nu}Z^{\nu\beta}\frac{m^2}{2}\cos^2\theta Z_\beta Z^\mu \tag{A.86}$$

$$\supset \frac{e^2}{4}\cot^2\theta m^2\left(-\partial_\mu Z_\nu\partial^\beta Z^\nu - \partial_\nu Z_\mu\partial^\nu Z^\beta\right)Z_\beta Z^\mu. \tag{A.87}$$

Note that the contraction in the first term $(Z_\beta Z_\mu\partial^\beta\partial^\mu)$ doesn't varnish. However, the second term varnish since it contracts the pure longitudinal with transversal polarised Z-boson. Hence, we only consider the first term. There are some relation for contracting the longitudinal polarization vectors $\epsilon_{1\mu} = (a_3\frac{p}{m}, 0, 0, -a_3\frac{p}{m}), \epsilon_{2\mu} = (-b_3\frac{p}{m}, 0, 0, -b_3\frac{p}{m})$ with the 4-momentum $p_1^\mu = (E, 0, 0, p), p_2^\mu = (E, 0, 0, -p)$ as,

$$p_1\epsilon_1^{(*)} = p_2\epsilon_2^{(*)} = 0, \tag{A.88}$$

$$p_2\epsilon_1^{(*)} = 2a_3^{(*)}\frac{Ep}{m}, \tag{A.89}$$

$$p_1\epsilon_2^{(*)} = -2b_3^{(*)}\frac{Ep}{m}. \tag{A.90}$$

For the first term, there are also 6 cases, we calculate the first case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
$a^\dagger$	a	a	$a^\dagger$
1	1	2	2
2	2	1	1
2	1	2	1
1	2	1	2

The first term read,

$$p_{1\mu}\epsilon_{1\nu}^* p_1^\beta \epsilon_1^\nu \epsilon_{2\beta}\epsilon_2^{\mu*} = -4\frac{E^2p^2}{m^2}|b_3|^2(|a_1|^2 + |a_2|^2). \tag{A.91}$$

The second term read,

$$p_{2\mu}\epsilon_{2\nu}^* p_2^\beta \epsilon_2^\nu \epsilon_{1\beta}\epsilon_1^{\mu*} = -4\frac{E^2p^2}{m^2}|a_3|^2(|b_1|^2 + |b_2|^2). \tag{A.92}$$

The third term read:

$$p_{2\mu}\epsilon_{2\nu}^* p_1^\beta \epsilon_1^\nu \epsilon_{2\beta}\epsilon_1^{\mu*} = 4\frac{E^2p^2}{m^2}a_3^* b_3(a_1 b_1^* + a_2 b_2^*). \tag{A.93}$$

The fourth term read:

$$p_{1\mu}\epsilon^*_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon_{1\beta}\epsilon^{\mu*}_2 = 4\frac{E^2p^2}{m^2}a_3b^*_3(a^*_1b_1 + a^*_2b_2). \tag{A.94}$$

Swaping $a \leftrightarrow a^\dagger$ yields the same result, we got the second case (x2). Consider the third case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
$a^\dagger$	a	$a^\dagger$	a
1	1	2	2
2	2	1	1
1	2	2	1
2	1	1	2

The first term read,

$$p_{1\mu}\epsilon^*_{1\nu}p^\beta_1\epsilon^\nu_1\epsilon^*_{2\beta}\epsilon^\mu_2 = -4\frac{E^2p^2}{m^2}|a_3|^2(|b_1|^2 + |b_2|^2). \tag{A.95}$$

The second term read,

$$p_{2\mu}\epsilon^*_{2\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{1\beta}\epsilon^\mu_1 = -4\frac{E^2p^2}{m^2}|b_3|^2(|a_1|^2 + |a_2|^2). \tag{A.96}$$

For the third term, since $p_1\epsilon_1 = p_2\epsilon_2 = 0$, we have,

$$p_{1\mu}\epsilon^*_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{2\beta}\epsilon^\mu_1 = 0 \tag{A.97}$$

Since swaping $a \leftrightarrow a^\dagger$ yields the same result, we derive the same thing for the fourth term as well (x2). Consider the fifth case,

$\partial_\mu Z_\nu$	$\partial^\beta Z^\nu$	Z_β	Z^μ
a	a	$a^\dagger$	$a^\dagger$
1	2	1	2
2	1	2	1
1	2	2	1
2	1	1	2

The first term reads,

$$p_{1\mu}\epsilon_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{1\beta}\epsilon^{\mu*}_2 = 4\frac{E^2p^2}{m^2}a^*_3b^*_3(a_1b_1 + a_2b_2). \tag{A.98}$$

The second term reads the same result (x2). The third term reads,

$$p_{1\mu}\epsilon_{1\nu}p^\beta_2\epsilon^\nu_2\epsilon^*_{2\beta}\epsilon^{\mu*}_1 = 0. \tag{A.99}$$

This result also hold for the fourth term (x2). Finally, we get the sixth case by swaping $a \leftrightarrow a^\dagger$. It yields the complex conjugate of the fifth case (+c.c.). To sum up and then to subtract the m^4 term, we have,

$$
\begin{aligned}
&-\frac{e^2}{4}\cot^2\theta m^2 4\frac{E^2p^2}{m^2}\Big[2.2|a_3|^2(|b_1|^2+|b_2|^2)+2.2|b_3|^2(|a_1|^2+|a_2|^2) \\
&-2a_3b_3^*(a_1^*b_1+a_2^*b_2)+c.c.+2a_3^*b_3^*(a_1b_1+a_2b_2)+c.c.\Big] && \text{(A.100)} \\
=&-e^2\cot^2\theta\Big[4|a_3|^2(|b_1|^2+|b_2|^2)+4|b_3|^2(|a_1|^2+|a_2|^2) \\
&-2a_3b_3^*(a_1^*b_1+a_2^*b_2)+c.c.+2a_3^*b_3^*(a_1b_1+a_2b_2)+c.c.\Big]E^2p^2. && \text{(A.101)} \\
\equiv&-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2. && \text{(A.102)}
\end{aligned}
$$

We can also derive the $\mathcal{L}_{M,3}$

$$
\begin{aligned}
&\frac{\tan^4\theta}{2}\Big[-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2\Big] \\
=&-e^2\tan^2\theta(2A_3+2A_3'+A_5'+A_5)E^2p^2. && \text{(A.103)}
\end{aligned}
$$

and $\mathcal{L}_{M,5}$,

$$
\begin{aligned}
&-\frac{\tan^2\theta}{2}\Big[-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2\Big] && \text{(A.104)} \\
=&e^2(2A_3+2A_3'+A_5'+A_5)E^2p^2. && \text{(A.105)}
\end{aligned}
$$

and $\mathcal{L}_{M,7}$,

$$
\begin{aligned}
&-\frac{1}{2}\Big[-e^2\cot^2\theta(4A_3+4A_3'+2A_5'+2A_5)E^2p^2\Big] && \text{(A.106)} \\
=&+e^2\cot^2\theta(2A_3+2A_3'+A_5'+A_5)E^2p^2. && \text{(A.107)}
\end{aligned}
$$

A.4 Results

Summing up all the operators, we get:
ZZ:

$$
\begin{aligned}
&16A_6(\mathcal{L}_{S,0}+\mathcal{L}_{S,1}+\mathcal{L}_{S,2}) \\
&+[16A_1+8(A_4+A_4')]\left[4\cot^4\theta(\mathcal{L}_{T,0}+\mathcal{L}_{T,1})+2(\mathcal{L}_{T,5}+\mathcal{L}_{T,6})+\tan^4\theta\mathcal{L}_{T,8}\right] \\
&+(8A_1+4A_2+2A_4+2A_4')(4\cot^4\theta\mathcal{L}_{T,2}+2\mathcal{L}_{T,7}+\tan^4\theta\mathcal{L}_{T,9}) \\
&+(4A_5+4A_5')(2\cot^2\theta\mathcal{L}_{M,0}+\tan^2\theta\mathcal{L}_{M,2}-\mathcal{L}_{M,4}) \\
&+(2A_3+2A_3'+A_5'+A_5)(-2\cot^2\theta\mathcal{L}_{M,1}-\tan^2\theta\mathcal{L}_{M,3}+\mathcal{L}_{M,5}+\cot^2\theta\mathcal{L}_{M,7})\geq 0. \\
& && \text{(A.108)}
\end{aligned}
$$

With the convention of,

$$\begin{aligned}
&A_1 \equiv |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2, && A_4 \equiv a_1 a_2^* b_1 b_2^* + c.c., \\
&A_2 \equiv |a_1|^2|b_2|^2 + |a_2|^2|b_1|^2, && A_4' \equiv a_1 a_2^* b_1^* b_2 + c.c., \\
&A_3 \equiv (|b_1|^2 + |b_2|^2)|a_3|^2, && A_5 \equiv (a_1 b_1 + a_2 b_2) a_3^* b_3^* + c.c., \\
&A_3' \equiv (|a_1|^2 + |a_2|^2)|b_3|^2, && A_5' \equiv -(a_1 b_1^* + a_2 b_2^*) a_3^* b_3 + c.c. \\
&A_3'' \equiv |b_1|^2|a_3|^2 && A_6 \equiv |a_3|^2|b_3|^2,
\end{aligned} \tag{A.109}$$

When we consider only real polarizations, $(A_5 + A_5')$ varnishes and we get,

$$\begin{aligned}
&ZZ: \\
&8At_W^4\left(F_{S,0} + F_{S,1} + F_{S,2}\right) + Dt_W^2\left(-t_W^4 F_{M,3} + t_W^2 F_{M,5} - 2F_{M,1} + F_{M,7}\right) \\
&+ (B + C)\left(2t_W^8 F_{T,9} + 4t_W^4 F_{T,7} + 8F_{T,2}\right) + 8B\left[t_W^4\left(t_W^4 F_{T,8} + 2F_{T,5} + 2F_{T,6}\right) + 4F_{T,0} + 4F_{T,1}\right].
\end{aligned} \tag{A.110}$$

with the convention of

$$\begin{aligned}
&A \equiv a_3^2 b_3^2, && E \equiv a_3 b_3 \left(a_1 b_1 + a_2 b_2\right), \\
&B \equiv \left(a_1 b_1 + a_2 b_2\right)^2, && F \equiv \left(a_1 b_3 - a_3 b_1\right)^2 + \left(a_2 b_3 - a_3 b_2\right)^2, \\
&C \equiv \left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right), && G \equiv \left(a_3 b_1 + a_1 b_3\right)^2 + \left(a_3 b_2 + a_2 b_3\right)^2, \\
&D \equiv a_3^2\left(b_1^2 + b_2^2\right) + \left(a_1^2 + a_2^2\right) b_3^2, && H \equiv a_3^2\left(b_1^2 + b_2^2\right).
\end{aligned} \tag{A.111}$$

B Proof of Optical theorem

Consider a process of $p_1 + p_2 \to \{k_f\}$, with $f = 1, .., n$, the total cross-section reads:

$$\sigma_t = \sum_n \mathrm{d}\Pi_n (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_f k_f) \frac{M^2(p_1 p_2 \to \{k_f\})}{2E_1 2E_2 |v_1 - v_2|}, \tag{B.1}$$

with

$$\mathrm{d}\Pi_n \equiv \prod_{f=1}^{n} \int \frac{\mathrm{d}k_f}{(2\pi)^3} \frac{1}{2E_f}. \tag{B.2}$$

The interacting T-matrix is defined from S-matrix as:

$$S = \mathbb{1} + iT. \tag{B.3}$$

Unitarity of S-matrix implies the following property of T:

$$S^\dagger S \Rightarrow (1 - iT^\dagger)(1 + iT^\dagger) = 1 \tag{B.4}$$

$$\Rightarrow T^\dagger T = i(T^\dagger - T). \tag{B.5}$$

The interaction reads:

$$i\langle\{k_f\}|\,T\,|p_1,p_2\rangle = iM(p_1p_2\to k_f)(2\pi)^4\delta^{(4)}(p_1+p_2-\sum_f k_f). \tag{B.6}$$

The Completeness relation in Hilbert space reads:

$$\mathbb{1} = \sum_n \mathrm{d}\Pi_n\,|\{k_f\}\rangle\langle\{k_f\}|\,. \tag{B.7}$$

Using the Completeness relation, we have,

$$\langle q_1,q_2|\,T^\dagger T\,|p_1,p_2\rangle = \sum_n \mathrm{d}\Pi_n\,\langle q_1,q_2|\,T^\dagger\,|\{k_f\}\rangle\langle\{k_f\}|\,T\,|p_1,p_2\rangle \tag{B.8}$$

Using Eq. B.5, the L.H.S of Eq. B.8 can be rewrite as

$$\begin{aligned}
&\langle q_1,q_2|\,T^\dagger T\,|p_1,p_2\rangle\\
=&i\left(\langle q_1,q_2|\,T^\dagger\,|p_1,p_2\rangle - \langle q_1,q_2|\,T\,|p_1,p_2\rangle\right) &\text{(B.9)}\\
=&i(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)\left[M^*(q_1q_2\to p_1p_2)-M(p_1p_2\to q_1q_2)\right]. &\text{(B.10)}
\end{aligned}$$

Using Eq. B.6, the R.H.S of Eq. B.5 reads:

$$\begin{aligned}
&\sum_n \mathrm{d}\Pi_n\,\langle q_1,q_2|\,T^\dagger\,|\{k_f\}\rangle\langle\{k_f\}|\,T\,|p_1,p_2\rangle\\
=&\sum_n \mathrm{d}\Pi_n(2\pi)^8\delta^{(4)}(q_1+q_2-k_f)\delta^{(4)}(p_1+p_2-k_f)M^*(q_1q_2\to k_f)M(p_1p_2\to k_f).
\end{aligned} \tag{B.11}$$

Consolidate Eq. B.10 and Eq. B.11, we get:

$$M(p_1p_2\to q_1q_2)-M^*(q_1q_2\to p_1p_2) = \sum_n \mathrm{d}\Pi_n\,(2\pi)^4 i\delta^{(4)}(p_1+p_2-\sum_f k_f)|M(p_1p_2\to k_f)|^2 \tag{B.12}$$

Or,

$$2\mathrm{Im}\ M(p_1p_2\to p_1p_2) = 2E_1 2E_2|v_1-v_2|\sigma_t. \tag{B.13}$$

When we have the same incoming particles, it becomes the standard form of

$$\mathrm{Im}\ M(p_1p_2\to p_1p_2) = 2E_{\mathrm{CM}}|\mathbf{p}_{\mathrm{CM}}|\sigma_t. \tag{B.14}$$

B.1 A

$$iM = \frac{(i\lambda)^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{(\frac{p}{2}-k)^2-m^2+i\epsilon}\frac{i}{(\frac{p}{2}+k)^2-m^2+i\epsilon} \tag{B.15}$$

$$= \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(\frac{p_0}{2}-k_0)^2-E_k^2+i\epsilon}\frac{1}{(\frac{p_0}{2}+k_0)^2-E_k^2+i\epsilon}. \tag{B.16}$$

In the C.M. frame, $p=(p_0,\mathbf{0})$, $k=(k_0,\mathbf{k})$, $E_k^2=|\mathbf{k}|^2+m^2$.
The inegration has poles at:

$$k_0 = \pm(E_k - i\epsilon) - \frac{p_0}{2} \tag{B.17}$$

$$k_0 = \pm(E_k - i\epsilon) + \frac{p_0}{2}. \tag{B.18}$$

$$\frac{1}{\left(\frac{p_0}{2}+k\right)^2-m^2+i\epsilon} \to -2\pi i\delta\left(\left(\frac{p_0}{2}+k\right)^2-m^2\right) \tag{B.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}k_0}\left(\left(\frac{p_0}{2}-k_0\right)^2+E_k^2\right)\Bigg|_{k_0=E_k-\frac{p_0}{2}} = 2E_k. \tag{B.20}$$

$$\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4} \tag{B.21}$$

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