Euler's Constant to 1271 Places

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Main result

The value of Euler's or Mascheroni's constant
$\gamma = \lim_{n\to\infty} (1+\frac{1}{2}+...+\frac{1}{n}+\ln n)$

It is defined as the limiting difference between the harmonic series and the natural logarithm and is now determined to 1271 decimal places based on using as Euler's summation formula. In addition, we still use Determination of Partial Quotients in order to find best rational approximations to γ, we represent it as a continued fraction.

Historical Background

At that time, Euler's constant was frist evaluated by Leonhard Euler, and he obtained the value 0.577218 in 1735. By 1781 he had calculated it more exactly as 0.5772156649015325. The calculations were carried out more precisely by several later mathematicians, among them Gauss, who obtained

$$\gamma = 0.577215664901532860606.$$

Adams' result stood until 1952, when Wrench found 328 decimal places. Although much work has been done trying to decide whether γ is rational or not? The evaluation has not been carried out any more precisely. With the use of high-speed computers, the constants π and ϵ have been accurately evaluated to many thousands of decimal places. The evaluation of γ to many places is considerably more difficult.

Evalution of γ

We use Euler's summation formula in the form

$$\sum_{i=1}^{n} f(i) = \int_1^n f(x)dx + \frac{1}{2}(f(n)+f(1))$$
$$+\sum_{j=1}^{k}\frac{B_2j}{(2j)!}[f^{(2j-1)}(n) - f^{(2j-1)}(1)] + R_k$$

R_k: Remainder
B_m: Bernoulli Numbers
$f^{(2j-1)}(n)$: (2j-1) Derivative at (n)
$\rightarrow$ *The purpose is to assign value to string* $\sum_{i=1}^{n} f(i)$

- B_m are defined symbolically by $e^{B_x} = \frac{x}{e^x-1}$
- R_k is given by

$$R_k = \frac{1}{(2k+1)!}\int_1^n P_{2k+1}(x) f^{(2k+1)}(x)dx$$

- $P_{2k+1}(x)$ is a periodic Bernoulli polynomial

$$P_{2k+1}(x) = (\{x\}+B)^{2k+1} = (-1)^{k-1}(2k+1)!\sum_{r=1}^{\infty}\frac{2sin(2r\pi x)}{(2r\pi)^{2k+1}}$$

where {x} is the fractional part of x

- Periodic Bernoulli polynomial level 2k+1

$$|P_{2k+1}(x)| \leq \frac{2(2k+1)!}{(2\pi)^{2k+1}}\sum_{r=1}^{\infty}\frac{1}{r^{2k+1}}$$

- The range of values of $\int_n^\infty \frac{P_{2k+1}(x)}{x^{2k+2}}$

$$\left|\int_n^\infty \frac{P_{2k+1}(x)dx}{x^{2k+2}}\right| \leq \frac{4}{n}\sqrt{\frac{k}{\pi}}(\frac{k}{n\pi e})^{2k}$$

Determination of Partial Quotients

This formula found best rational approximations to γ

$$\gamma = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}.$$

Matrix representation of the sequence P_i, Q_i

$$\begin{pmatrix} P_{i+1} & P_i \\ Q_{i+1} & Q_i \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}...\begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$$

Formula determine Khintchine's constant applies to all real numbers with a partial quotients, where K $\approx$ 2.685

$$\lim \sqrt[n]{a_2 a_3 ... a_{n+1}} = K$$

Formula determine γ but don't need Bernoulli numbers to avoid the bottleneck, which was used by Dura W.Sweenet to calculate 3566 decimal digits

$$\gamma + \ln n = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k}{k!k} - \int_n^\infty \frac{e^{-x}}{x}dx$$

Details of Illustrative Example

We use method of Knopp to determine γ

$$\gamma = 1+\frac{1}{2}+...+\frac{1}{n} - \ln n - \frac{1}{2n} + \frac{B_2}{2n^2} + ... + \frac{B_{2k}}{2kn^{2k}} - \int_n^\infty \frac{P_{2k+1}(x)}{x^{2k+2}}dx$$

Step 1: Determine sum of $S_{1000} = 1+\frac{1}{2}+...+\frac{1}{1000}$

- Combine two adjacent terms to reduce the number of times division is performed
- $S_{10000} = (1+\frac{1}{2}) + (\frac{1}{3}+\frac{1}{4}) + ... + (\frac{1}{9999}+\frac{1}{10000}) =$ 9.787606036...

Step 2: Determine $\ln(10000)$

- Find small values of (x, y, z) so that $x + y\log_2 3 + \log_2 5 \approx 0$. We find (-1,5,-3),(-4,4,-1) and (6,5,-6)
- Find the numbers a, b, c so that $(2^{-1}3^5 5^{-3})^a.(2^{-4}3^4 5^{-1})^b.(2^6 3^5 5^{-6})^c = 10000$. After calculating, we get a=-292, b=200, c=92
- Finally, we have $\ln(10000) = -292\ln(1-0.028) + 200\ln(1+0.0125) + 92\ln(1-0.004672)$

Step 3: Find sum expressions of Bernoulli numbers

- $B_{2k}^{'} = 10^{-8k}B_{2k}$ were evaluated using the recursion

$$\binom{2k+1}{2k}B_{2k}^{'} + 10^{-8}\binom{2k+1}{2k-2}B_{2k-2}^{'} + ... + 10^{8-8k}\binom{2k+1}{2}B_2^{'} = \frac{2k-1}{2.10^{8k}}$$

- $S = \frac{B_2}{2n^2} + \frac{B_4}{4n^4} + ... + \frac{B_2 50}{250n^2 50}$

Step 4: Combine all values in steps, give the result of γ