Minerva Schools At KGI

CS166 - Network Simulations

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Simulation Code

Link: https://gist.github.com/quangntran/fc0d12d2d9fb922770f30dfc8af5e417

Choice of Random Graph

Part 1. Modifications

Description of the Two Modifications

1. Multiple topics: Social agents do not form and base their relationship on only one opinion set. They interact and affect each other's point of view on multiple aspects. Therefore, rather than having only one set of opinion that describes each agent's state, we allow every agent to have opinions on multiple topics, represented by a vector of length K, the number of topics in the universe. In our simulation, we restrict **K = 2**, that is, supposing there are 2 topics in the universe to discuss.
2. Persuasiveness parameter: Each person has a different charisma σ and thus can be more or less convincing than others, affecting how fast people change their mind.

Updating rules

The network dynamics have 4 parts

1. We model new social connection (new edge being formed) randomly: 1% of the time, a new edge with weight 0.5 is formed.
2. A pair of nodes between which there is already an edge is chosen randomly, representing their social interaction. In this interaction, they will have a conversation about a certain topic. The topic is chosen according to their **level of popularity**. We represent relative popularities of the topics by a vector ϕ. For example, $\phi = [2, 1]$ means that topic one is more likely to be chosen as a talking topic than the second one. This makes sense, because certain topic tends to be more popular than others depending on

social situation at the time. For example, in December, plans for Christmas holiday is more likely to be the topic than plans for next year's summer vacation.

3. People change their opinions so that they closely match with the opinions of the people they are talking with. The change in opinion k of Person i when talking to Person j:

$$\Delta o_{ik} = \sigma_j \alpha_k w_{ij}(o_{jk} - o_{ik})$$

Note that we have **a different α for each different topic**. This is motivated by the fact that people are more willing to change their opinions on certain topics than on others. For example, people are not likely to change their political view (much) in just one interaction so the α for a political topic should be low and close to 0. Other easy topics like favorite restaurant should have a higher α.

Note also that we also scale α_k by σ_j - the charisma/persuasiveness of Person j. If σ_j is small, $\alpha_k w_{ij}(o_{jk} - o_{ik})$ is scaled down and thus the opinion gap is bridged at a much smaller pace, so a low σ_j corresponding to an **un**convincing Person j.

Because we scale the change in opinion by the charisma σ of the Person we are talking to, and because two people may have different σ's, we **no longer** have the following relationship as before:

$$\Delta o_{jk} = -\Delta o_{ik}$$

Instead:

$$\Delta o_{jk} = \sigma_i \alpha_k w_{ij}(o_{ik} - o_{jk})$$

4. People strengthen their relationship if they agree on the topic being discussed, and weaken their relationship if they disagree. The change in weight of the edge between Person i and Person j when they discuss topic k is:

$$\Delta w_{ij} = \beta_k w_{ij}(1 - w_{ij})(1 - \gamma|o_{ik} - o_{jk}|)$$

Note that we have **a different β for each different topic**. This is motivated by the fact that some topics are more polarizing and have a more intense effect on people's relationship than others. For example, if two people disagree on their opinions about fascism, then the disagreement leads to a steeper shrink in their relationship than when they disagree on, say, tastes of music.

We also remove the edge if the strength of the relationship drops below 0.05.

Parameters

- K: number of topics, $K \geq 1$

- σ : vector of length N, where N is the number of node. This parameter gives the persuasiveness levels of all agents. $\sigma \in (0, 1]$
- ϕ : vector of length K, the relative probabilities that a topic being chosen.
- α : vector of length K, indicating how fast people change their opinions to match other people's. $\alpha_k \in (0, 0.5]$. This constraint is so that when updating opinions, opinions won't go over 1 or below 0.
- β : vector of length K, indicating how fast relationships among people can shrink (or grow) depending on the topics being discussed. $\beta_k \in (0, 1)$, just like in the original model.
- $\gamma > 0$: This indicates the pickiness. If $\gamma \leq 1$, ten all weights converge to 1 over time as it does not matter whether opinions differ. When $\gamma > 1$, weights decrease if opinions differ enough.

Assumptions

From the set-up above, we have assumed:

1. Each topic has its inherent:
    a. level of popularity (some topics are more likely to be discussed than others; why we have a vector ϕ)
    b. level of triggering/ motivating someone to change their opinion (why we have a vector α)
    c. Level of polarizing (why we have a vector β)
2. The pickiness γ is constant across all agents and all topics.
3. When people adjust their opinions and relationships, the only information they have is the opinion of the other person on ONE topic and they are completely agnostic to the other person's opinions on the other topics. For example, one person cannot say, "I would have unfriended him because he supported Trump, but given that he also likes cats, I am still a friend with him", because this person adjusted his/her relationship based on information about two topics, which is not allowed. This assumption is somewhat unrealistic.

Part 2. Local analysis

1. Determine the bounds for γ

We have the updating rule for the change in weight as follows:

$$\Delta w_{ij} = \beta_k w_{ij}(1 - w_{ij})(1 - \gamma|o_{ik} - o_{jk}|)$$

We need the weight after updating $\Delta w_{ij} + w_{ij}$ to be in the range [0,1], and thus:

- Case 1: $\Delta w_{ij} + w_{ij} \geq 0$

This means $\Delta w_{ij} \geq -w_{ij}$

$\Leftrightarrow \ \beta_k w_{ij}(1 - w_{ij})(1 - \gamma|o_{ik} - o_{jk}|) \geq -w_{ij}$

$\Leftrightarrow \ 1 - \gamma|o_{ik} - o_{jk}| \geq \frac{-1}{\beta_k(1-w_{ij})}$

$\Leftrightarrow 1 + \frac{1}{\beta_k(1-w_{ij})} \geq \gamma|o_{ik} - o_{jk}|$

$\Leftrightarrow \ |o_{ik} - o_{jk}|^{-1}\left(1 + \frac{1}{\beta_k(1-w_{ij})}\right) \geq \gamma$

To find the minimum of the LHS, $|o_{ik} - o_{jk}|^{-1}$ reaches its minimum and $\frac{1}{\beta_k(1-w_{ij})}$ reaches its minimum as well. These happen when $|o_{ik} - o_{jk}|$ reaches its maximum, which is 1, and $\beta_k(1 - w_{ij})$ reaches its maximum, which is when $w_{ij} = 0$. To wrap up,

$$\gamma \leq 1 + \frac{1}{\beta_k}$$

Or

$$\gamma \leq 1 + \frac{1}{\beta_{max}}$$

- Case 2. $\Delta w_{ij} + w_{ij} \leq 1$

This means $\Delta w_{ij} \leq 1 - w_{ij}$

$\Leftrightarrow \ \beta_k w_{ij}(1 - w_{ij})(1 - \gamma|o_{ik} - o_{jk}|) \leq 1 - w_{ij}$

$\Leftrightarrow \beta_k w_{ij}(1 - \gamma|o_{ik} - o_{jk}|) \leq 1$

$\Leftrightarrow |o_{ik} - o_{jk}|^{-1}\left(1 - \frac{1}{\beta_k w_{ij}}\right) \leq \gamma$

The LHS reaches its maximum when $w_{ij} = 1$ and $|o_{ik} - o_{jk}| = 1$ so:

$$\gamma \geq 1 - \frac{1}{\beta_k}$$

Or:

$$\gamma \geq 1 - \frac{1}{\beta_{max}}$$

We see that the RHS is negative, while γ is positive, so the inequality above is always true.

Therefore, the range for valid γ is $(0,\ 1 + \frac{1}{\beta_{max}}]$

2. Analyze mathematically for which values of your parameters you should expect opinions to converge or diverge and relationship strengths to increase or decrease

Construct the **expected change in opinion**

This analysis is more challenging than the one with the original model because now we have more parameters. Because each topic is chosen with a particular probability ϕ, and each topic also has an associated α, the **expected change in opinion** of Person i when talking to Person j about topic k is, therefore:

$$E[\Delta o_i] = \sum_{k=1}^{K} \phi_k \sigma_j \alpha_k w_{ij} (o_{jk} - o_{ik})$$

To simplify our analysis, assume that all topics have the same α. The expected change in opinion is reduced to:

$$E[\Delta o_i] = \sigma_j \alpha \, w_{ij} \sum_{k=1}^{K} \phi_k (o_{jk} - o_{ik})$$

The term $\sum_{k=1}^{K} \phi_k (o_{jk} - o_{ik})$ in the above expression can be seen as the **expected difference in opinion.**

Construct the **expected change in weight**

The **expected change in weight** between Person i and Person j when they interact over topic k is:

$$E[\Delta w_{ij}] = \sum_{k=1}^{K} \phi_k \beta_k w_{ij} (1 - w_{ij})(1 - \gamma |o_{ik} - o_{jk}|)$$

To simplify our analysis, assume that all topics have the same β. The expected change in opinion is reduced to:

$$E[\Delta w_{ij}] = \beta \, w_{ij} (1 - w_{ij}) \sum_{k=1}^{K} \phi_k (1 - \gamma |o_{ik} - o_{jk}|)$$

$$E[\Delta w_{ij}] = \beta \, w_{ij} (1 - w_{ij}) \left[\sum_{k=1}^{K} \phi_k - \gamma \sum_{k=1}^{K} \phi_k |o_{ik} - o_{jk}| \right]$$

Since $\sum_{k=1}^{K} \phi_k$ is just 1:

$$E[\Delta w_{ij}] = \beta \, w_{ij} (1 - w_{ij}) \left[1 - \gamma \sum_{k=1}^{K} \phi_k |o_{ik} - o_{jk}| \right]$$

The term $\sum_{k=1}^{K} \phi_k |o_{ik} - o_{jk}|$ in the equation above can be seen as the **expected absolute difference in opinions.**

We consider two extreme cases. The cases inbetween will be visualized by the vector fields.

A. When two nodes with very different expected opinions interact:

$\sum_{k=1}^{K} \phi_k(o_{jk} - o_{ik}) \approx 1$ and $\sum_{k=1}^{K} \phi_k|o_{jk} - o_{ik}| \approx 1$

The expected change in opinion:

$$E[\Delta o_i] = \sigma_j \alpha\, w_{ij} \sum_{k=1}^{K} \phi_k(o_{jk} - o_{ik})$$

$$E[\Delta o_i] \approx \sigma_j \alpha\, w_{ij}$$

We see that $E[\Delta o_i] > 0,$ so overall the expected opinion increases.

$$E[\Delta o_j] = \sigma_i \alpha w_{ij} \sum_{k=1}^{K} \phi_k(o_{ik} - o_{jk})$$

$$E[\Delta o_j] \approx - \sigma_i \alpha w_{ij}$$

We see that $E[\Delta o_j] > 0,$ so overall the expected opinion decreases for Person j. The expected difference in opinion has decreased by approximately $\sigma_i \alpha\, w_{ij} + \sigma_j \alpha\, w_{ij} = \alpha w_{ij}(\sigma_i + \sigma_j)$. Opinion converges.

The expected change in weight:

$$E[\Delta w_{ij}] = \beta\, w_{ij}(1 - w_{ij}) \left[1 - \gamma \sum_{k=1}^{K} \phi_k|o_{ik} - o_{jk}| \right]$$

$$E[\Delta w_{ij}] \approx \beta\, w_{ij}(1 - w_{ij})(1 - \gamma)$$

If $\gamma < 1,$ then the weight is always expected to increase.
If $\gamma \geq 1,$ then the weight is always expected to decrease.

B. When two nodes with very similar expected opinions interact:

$\sum_{k=1}^{K} \phi_k(o_{jk} - o_{ik}) \approx 0$ and $\sum_{k=1}^{K} \phi_k|o_{jk} - o_{ik}| \approx 0$

The expected change in opinion:

$$E[\Delta o_i] = \sigma_j \alpha\, w_{ij} \sum_{k=1}^{K} \phi_k(o_{jk} - o_{ik})$$

$$E[\Delta o_i] \approx \sigma_j \alpha\, w_{ij} \times 0 \approx 0$$

There is not much change to the opinion, so the opinions are expected to stay the same.

The expected change in weight:

$$E[\Delta w_{ij}] = \beta\, w_{ij}(1 - w_{ij}) \left[1 - \gamma \sum_{k=1}^{K} \phi_k|o_{ik} - o_{jk}| \right]$$

$$E[\Delta w_{ij}] \approx \beta\, w_{ij}(1 - w_{ij})$$

Because $w_{ij}(1-w_{ij}) > 0,\ E[\Delta w_{ij}] \approx \beta\, w_{ij}(1-w_{ij}) > 0$ and the weight is always expected to increase.

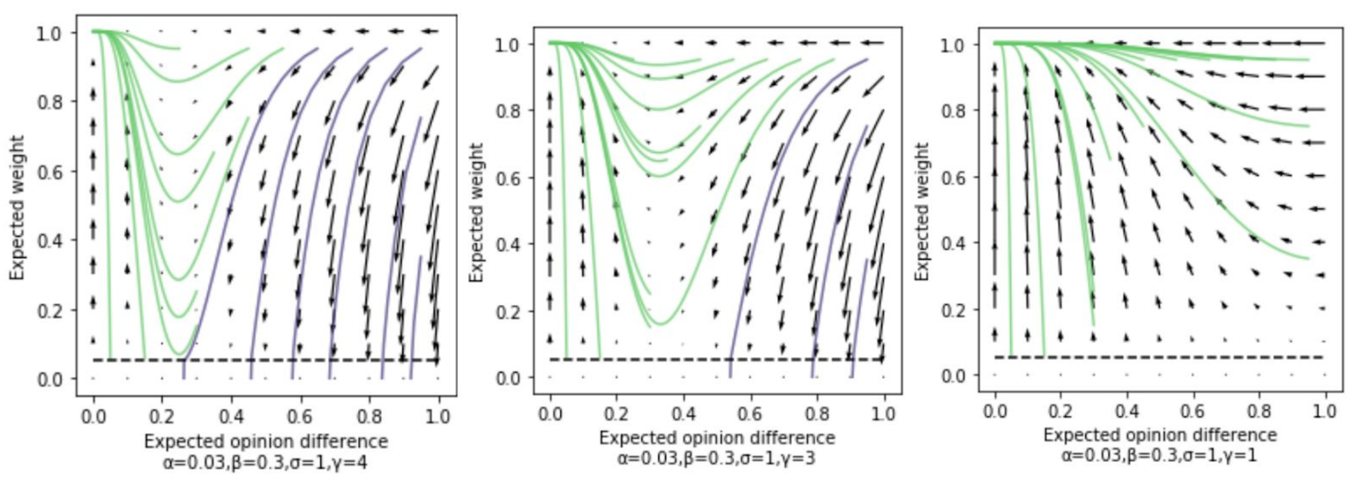
3. Use one or more vector field plots to demonstrate for which opinion and relationship strength values you expect clusters to form or to split apart in the simulation.

Because how the vector field plots would look like depends on specific configurations of the parameters, we will plot several plots corresponding to different parameter settings.

The parameters are chosen so that the constraints are satisfied (e.g., $\gamma \in (0,\ 1+\frac{1}{\beta_{max}}])$
Also note that α and β are vectors, and here we set all elements of each of these vectors to be the same as in our analysis for the sake of simplicity.

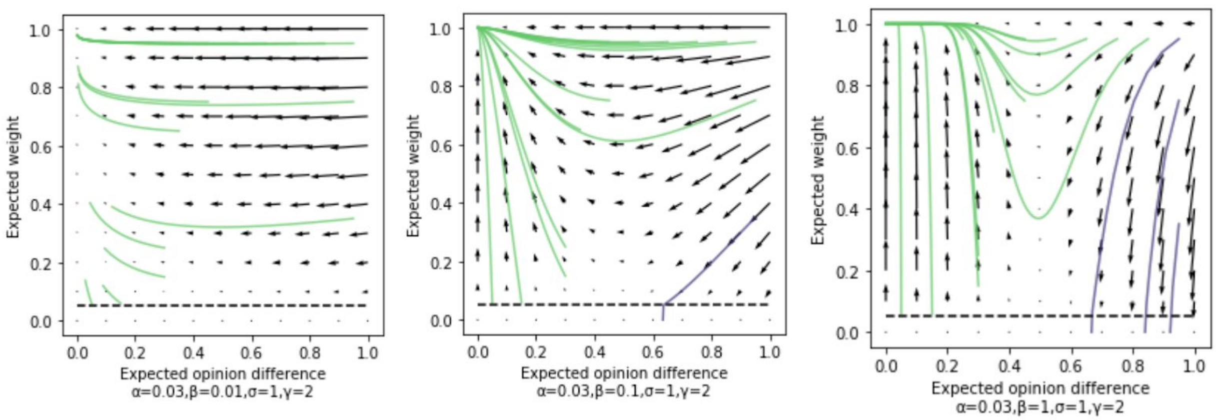
For all the plots, we don't look into cases where $\gamma < 1$ because then we expect the weight to ALWAYS increase.

A. Varying γ



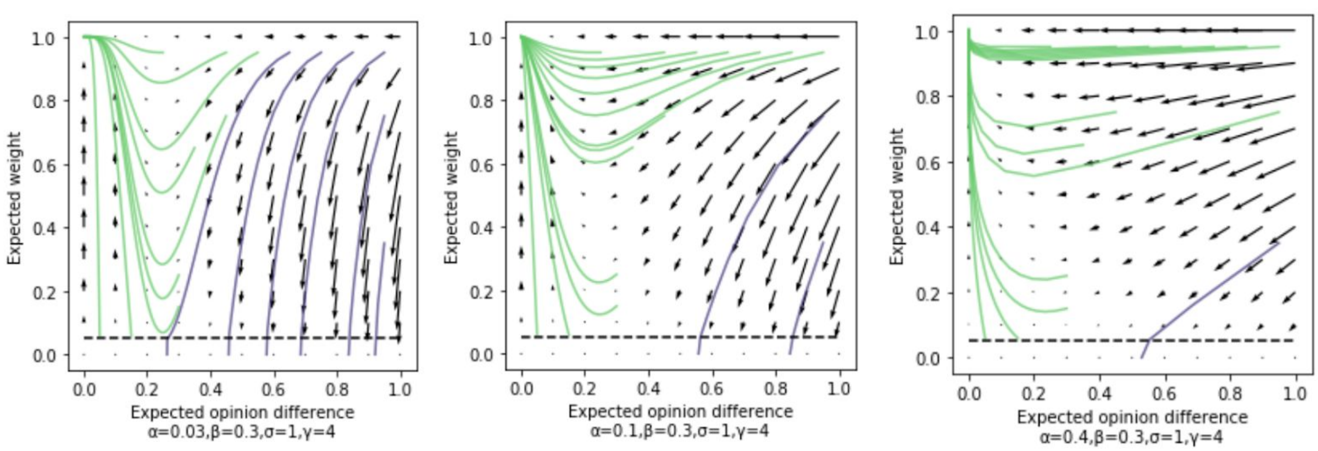
We see that the larger γ is, the more similar in opinion it requires (small opinion difference) in order for the relationship to strengthens over time. With small γ, meaning that the agents are not picky, there relationship strengthens anyway regardless of the initial weights and the initial expected difference in opinions.

B. Varying β



We see that as we increase β, difference in opinions matters. A small enough β means that the weights always strengthen over time towards 1, while a large β means if we start off with high difference in opinion, the graph will be split because the relationship goes to 0.

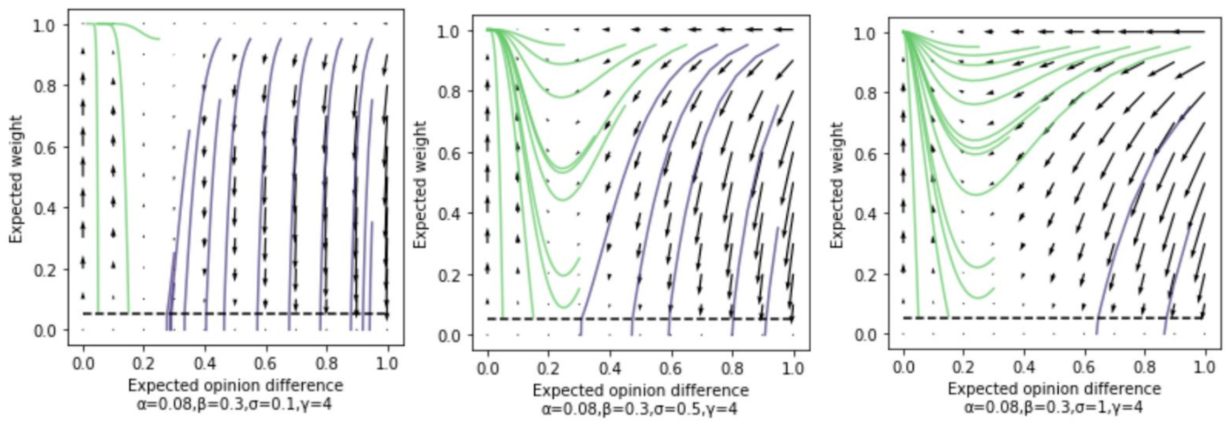
C. Varying α



As we increase α, meaning as people are less stubborn in adapting their opinion and thus move faster towards each other's opinion, the opinions become homogeneous faster and the relationship strenthens accordingly. This is observed in the plots: with larger α, even with an initial large polarization in opinion, if the weight is large enough, then the cluster is expected to form as the weights are expected to converge to 1.

When α is small ($\alpha = 0.03$), even with a small expected opinion difference like 0.4 (meaning the opinions are somewhat similar) and a neutral initial weight of 0.5, the social network is still expected to be split as the weights go to 0.

D. Varying σ, the persuasiveness.



A large σ means that the person has high charisma and is more convincing. As a result, this agent (who is interacting with this charismatic person) is more willing to change his opinion so that it matches closer to the charismatic person's opinion, paving the way for the relationship to be strengthened faster in turn. We see this in the plots: With high σ ($\sigma = 1$ for example), there is a larger area for initial starting points at which the weight will approach 1 over time, meaning that we would expect the social cluster to be formed more easily. When σ is low, it requires the very small initial expected opinion difference in order for the network not to be split.

Part 3. Implementation and results

Please see my attached notebook for the Python code.

1. Visualization strategy

Because the theoretical analysis concerns with **expected opinion** as well as **expected opinion difference,** in the visualization, the average of the opinion set of each node is taken to be a holistic state for that node to visualize (see the method *observe).*

2. The graph used

I chose the Barabasi Albert graph as it has the following properties:

1. The small-world property: this speaks to the fact that any two agents in the network are 6 degrees of separation.
2. The power law in the degree distribution: this speaks to the fact that “the rich get richer" which is also applicable in real life.

On the other hand, the Wattz-Strogatz model only harbours the small-world property but not the power law in degree distribution.

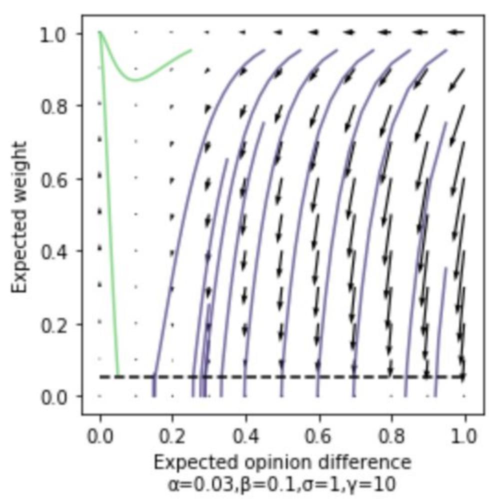
3. How experimental results confirm theoretical results

As shown in the vector field analysis, whether a social cluster forms or is split depend on the initial position in the plot which makes up of the **expected opinion difference** (the x-axis) and **the expected weight.**

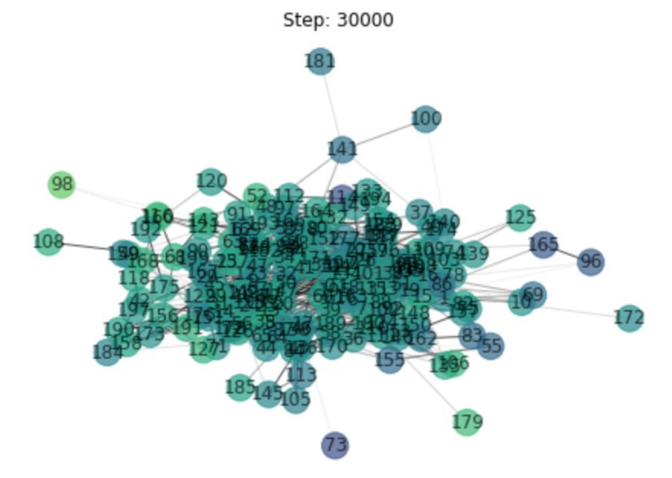
As I can't directly manipulate the initial expected opinion difference, I will only manipulate the mean weight (by setting the argument `weight_init` in the simulation.)

I first computed the expected opinion difference and observed that the expected opinion difference is around 0.3 (see my notebook.)

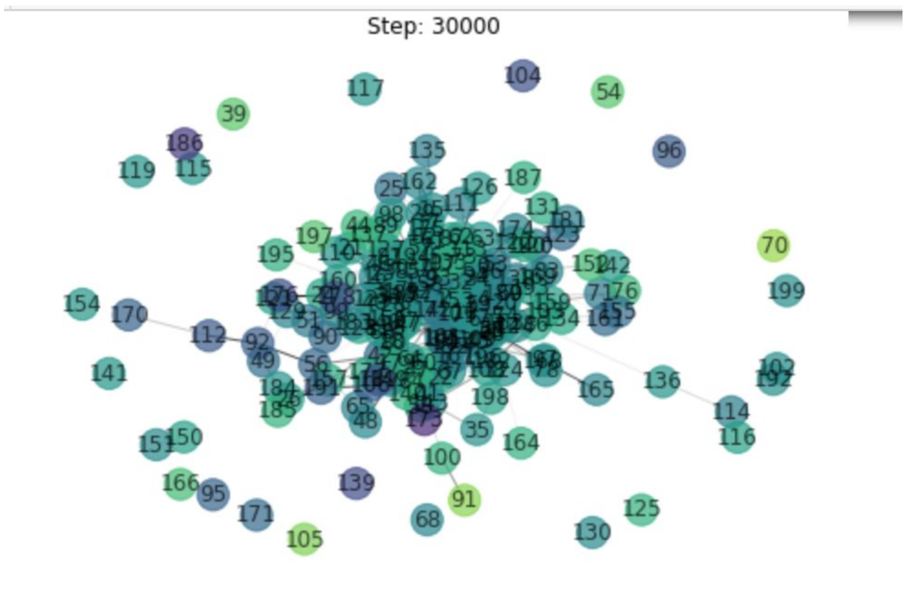
Because in all the vector fields above, there is no setting of parameters that can lead to an expected opinion difference of 0.3 to form a split, I create a new vector field with a new setting of parameters where there is some initial point at x=0.3 that network can form a split:



This setting shows that for expected opinion difference at around 0.3, it requires a very high expected weight in order for a social cluster to form (around higher than 0.9). So in order to have a cluster, I just need to set weight_init=0.95:



If I set weight_init to a value slightly smaller than that, say, weight_init=0, as predicted by the theoretical results, the network is split badly:



Interpret the results:

1. The simulation confirms the theoretical results.
2. We see that in the second plot where network is split, the color of the nodes are still similar (not polarizing). This is because we are taking the average of the opinions of each node. This suggests that nodes can differ by some certain opinion of certain topics extremely, and the extremity in one topic is cancelled out by the extremity in another topic.
3. Despite the similar colors of the nodes in the second visualization above where the network is split, those nodes that are isolated from the big cluster do have a slightly different color. This suggests that there is synergy between topics: difference in the

opinion in one topic can affect the opinion in another topic as well, leading to a change in holistic/ average opinion. This is applicable to real life as well: when I disagree with some person on one topic, I tend to hate the things that he/she loves in another, unrelated topic as well, affecting the overall relationship between him/her and me and, in turn, affecting my average opinions across topics.