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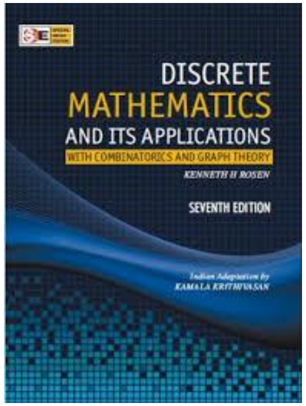

Session 9

Graph Theory

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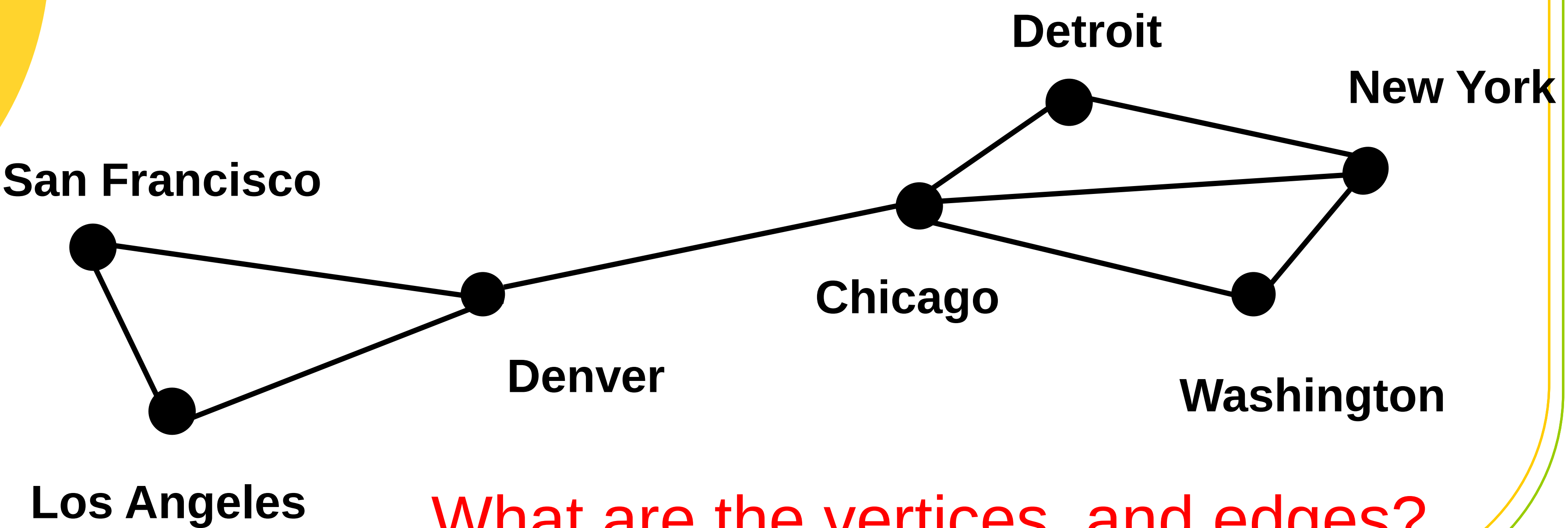
Part 1: Graphs and Graph Models
Basics, Concepts and Terminologies

1. Types of Graphs
2. Basic Terminology
3. Some Special Simple Graphs
4. New Graphs from Old Graphs

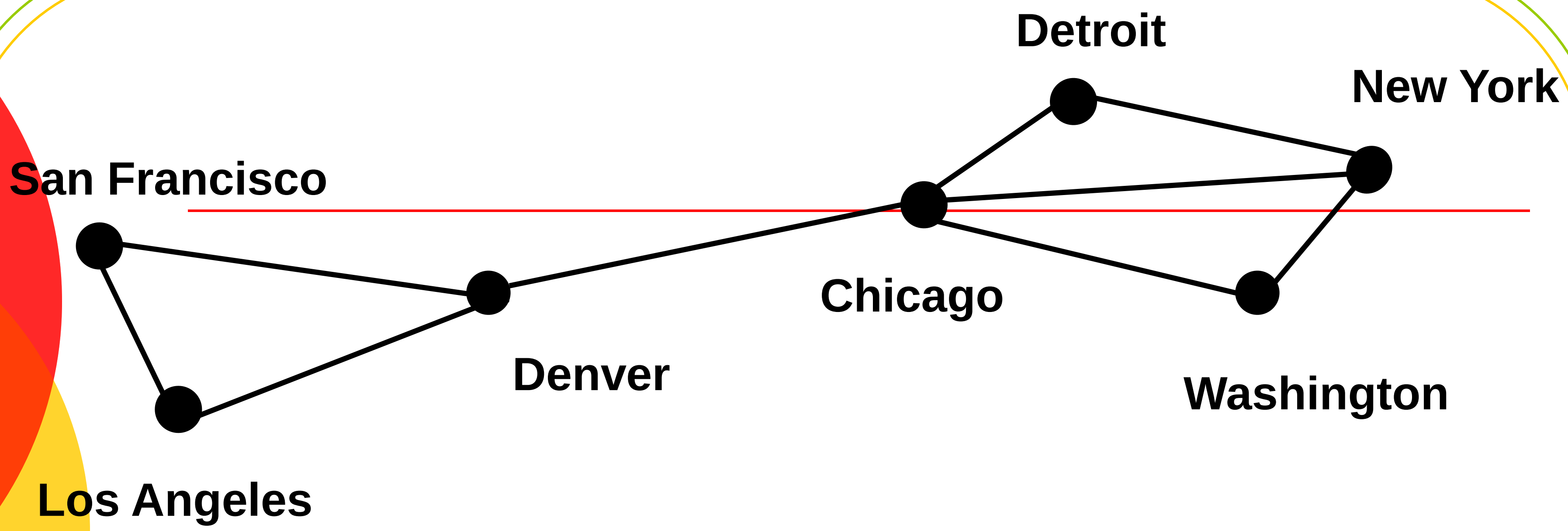
1. Type of Graphs

Simple Graph

Definition 1. A ***simple graph*** $G = (V, E)$ consists of V, a nonempty set of ***vertices***, and E, a set of unordered pairs of distinct elements of V called ***edges***.



What are the vertices and edges?

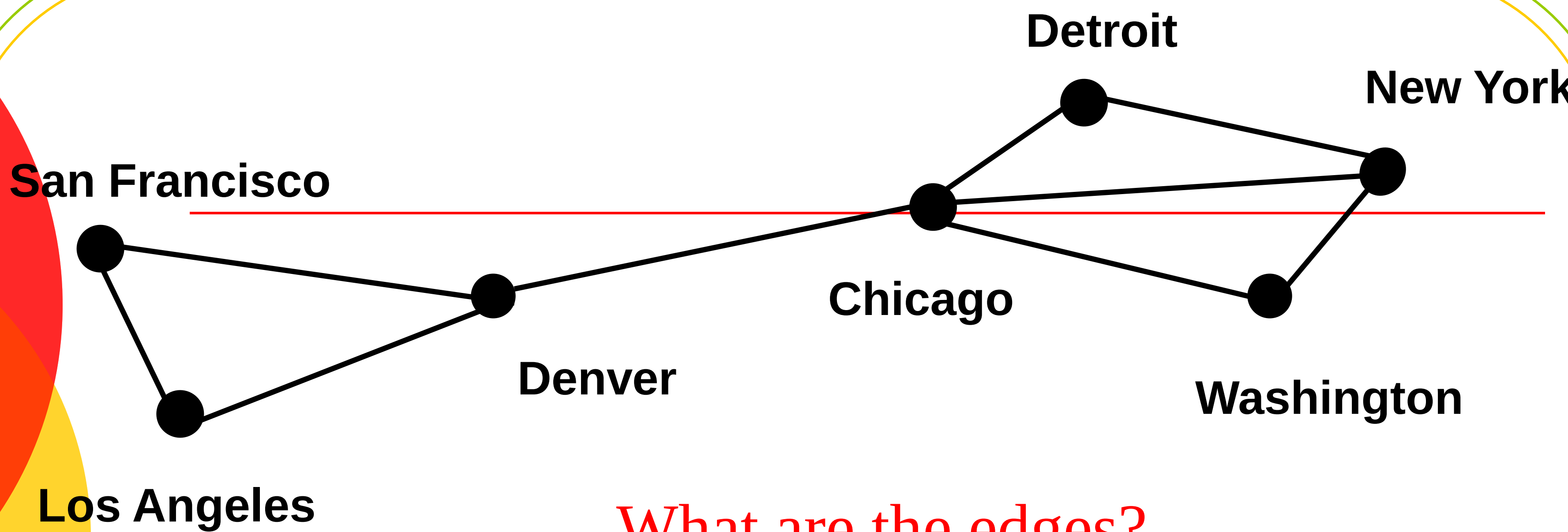


What are the vertices?

SET OF VERTICES

V = {San Francisco, Los Angeles, Denver,

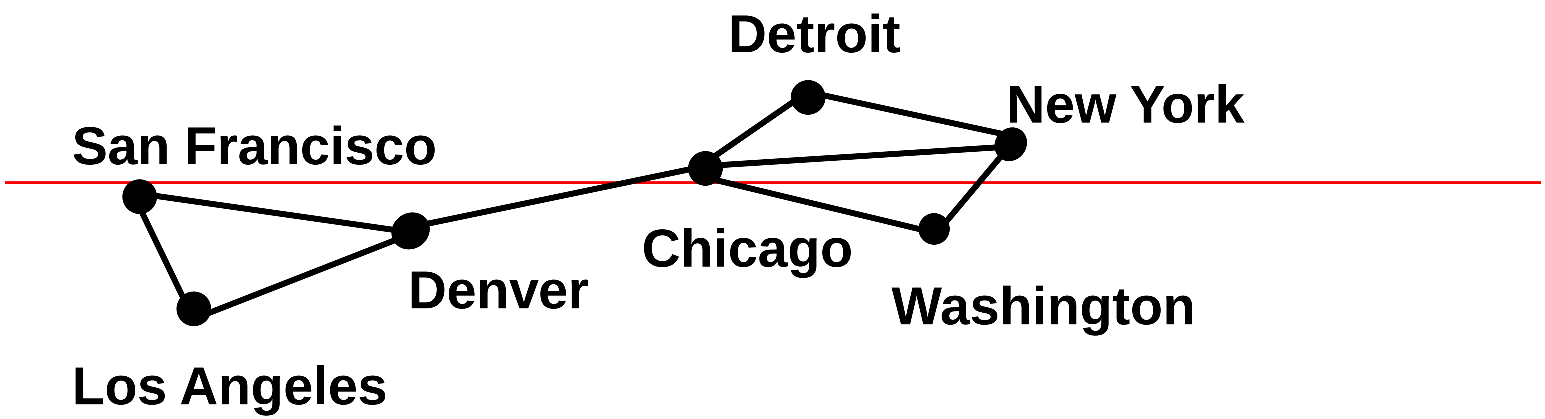
Chicago, Detroit, New York, Washington}



What are the edges?

SET OF EDGES

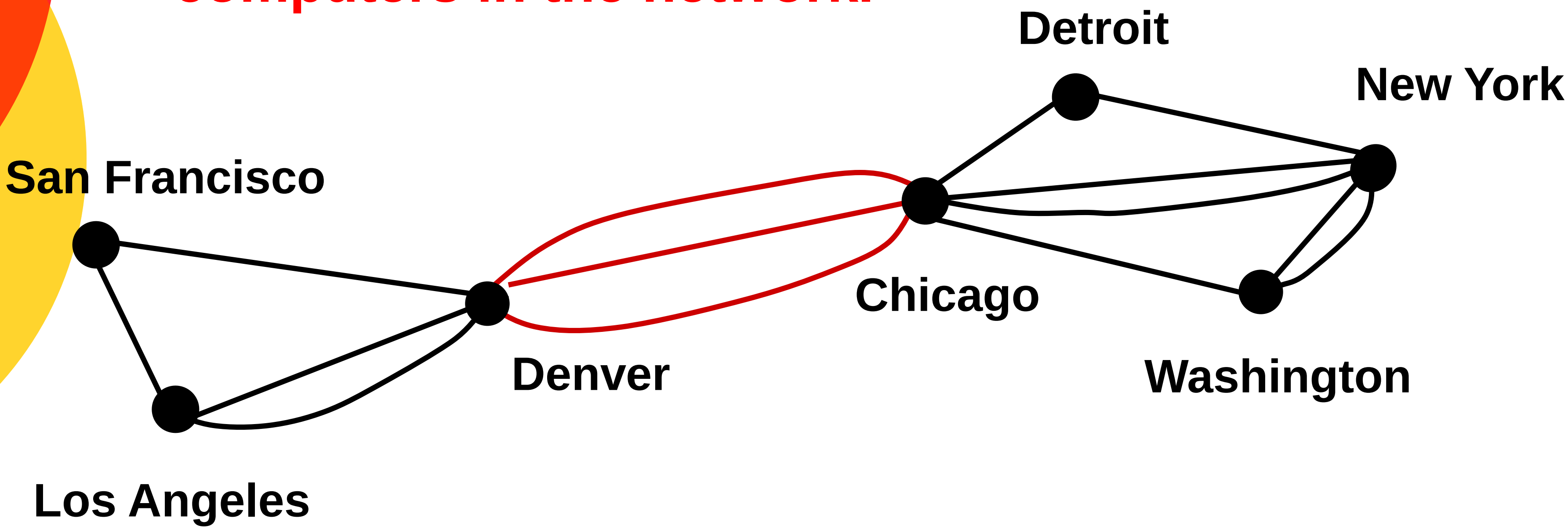
E = { {S F, L A}, {S F, Denver}, {L A, Denver},
{Denver, Chicago}, {Chicago, Detroit}, {Detroit, N Y},
{N Y, Washington}, {Chicago, Washington},
{Chicago, N Y} }



The network is made up of computers and telephone lines between computers.
There is at most 1 telephone line between 2 computers in the network. Each line operates in both directions.
No computer has a telephone line to itself.
These are undirected edges,
each of which connects two distinct vertices, and
no two edges connect the same pair of vertices.

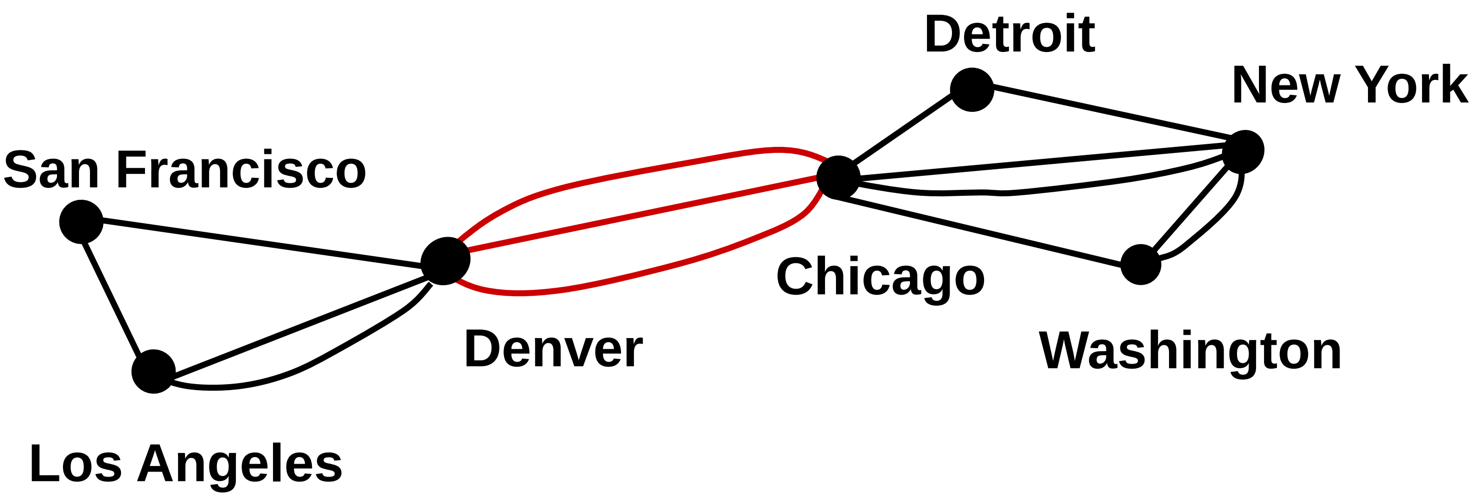
Multigraph, A Non-Simple Graph

There can be multiple telephone lines between two computers in the network.



■In a ***multigraph*** *G* = *(V, E)* two or more edges may connect the same pair of vertices.

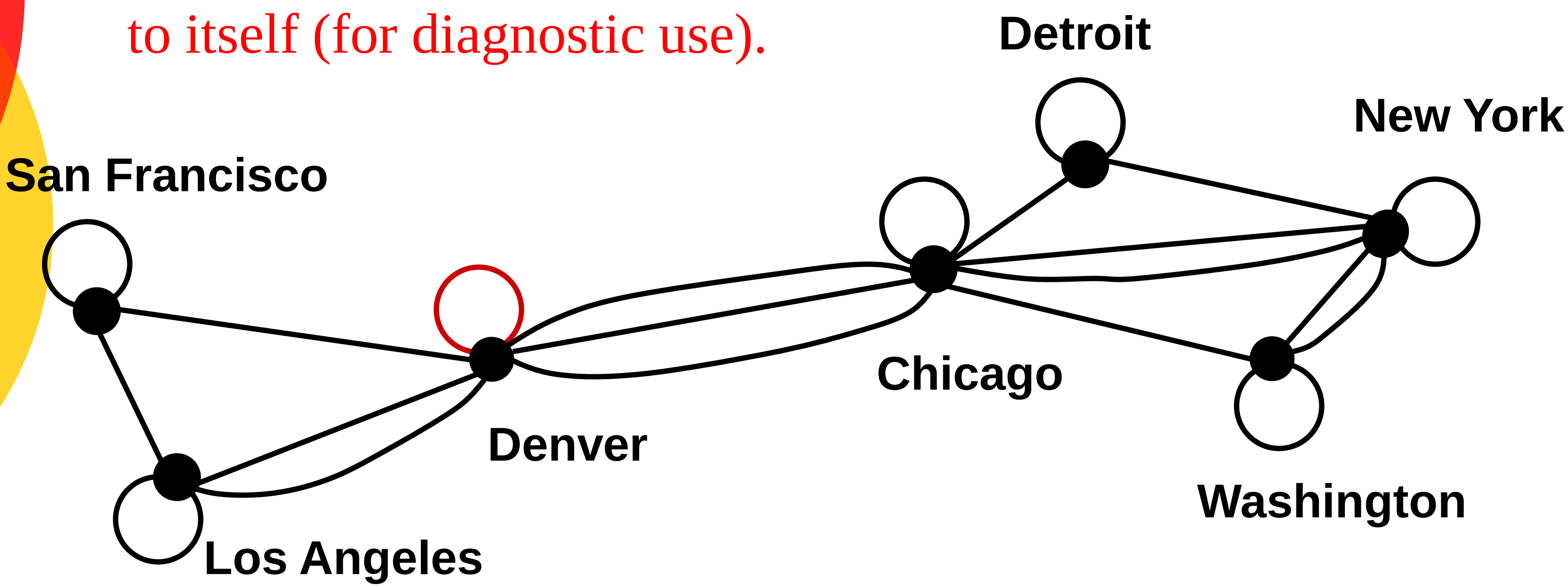
Multiple Edges



Two edges are called ***multiple*** or ***parallel edges*** if they connect the same two distinct vertices.

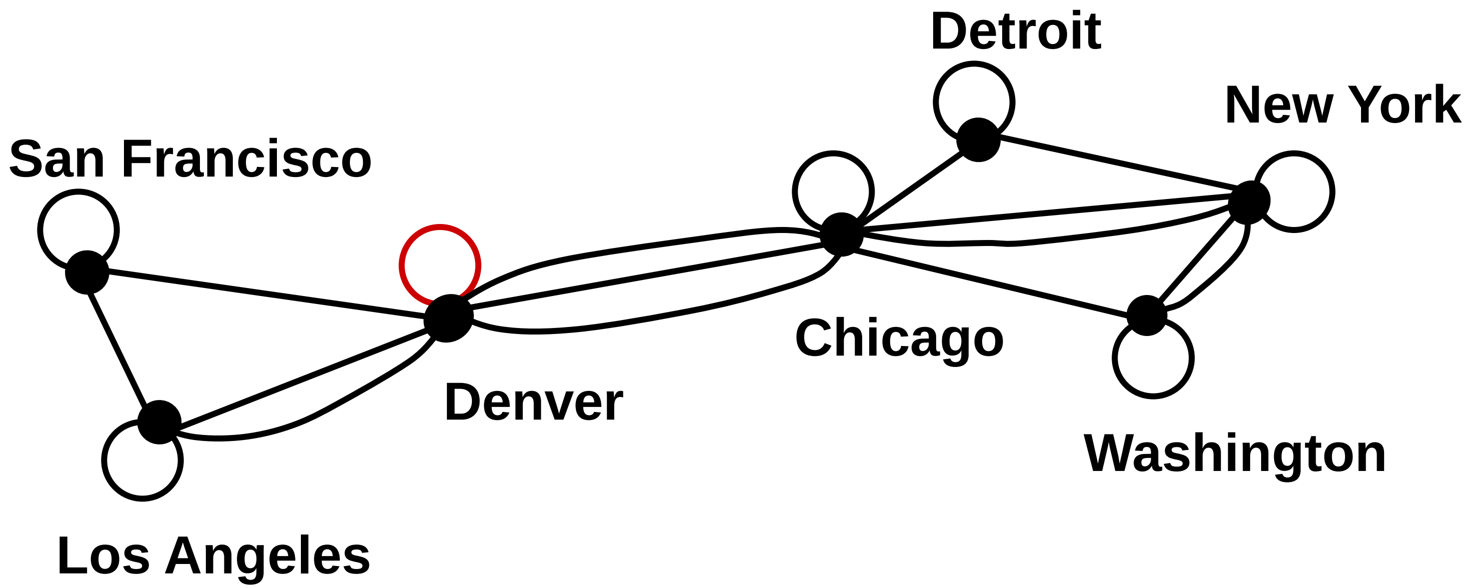
Pseudograph, A Non-Simple Graph

There can be telephone lines in the network from a computer to itself (for diagnostic use).



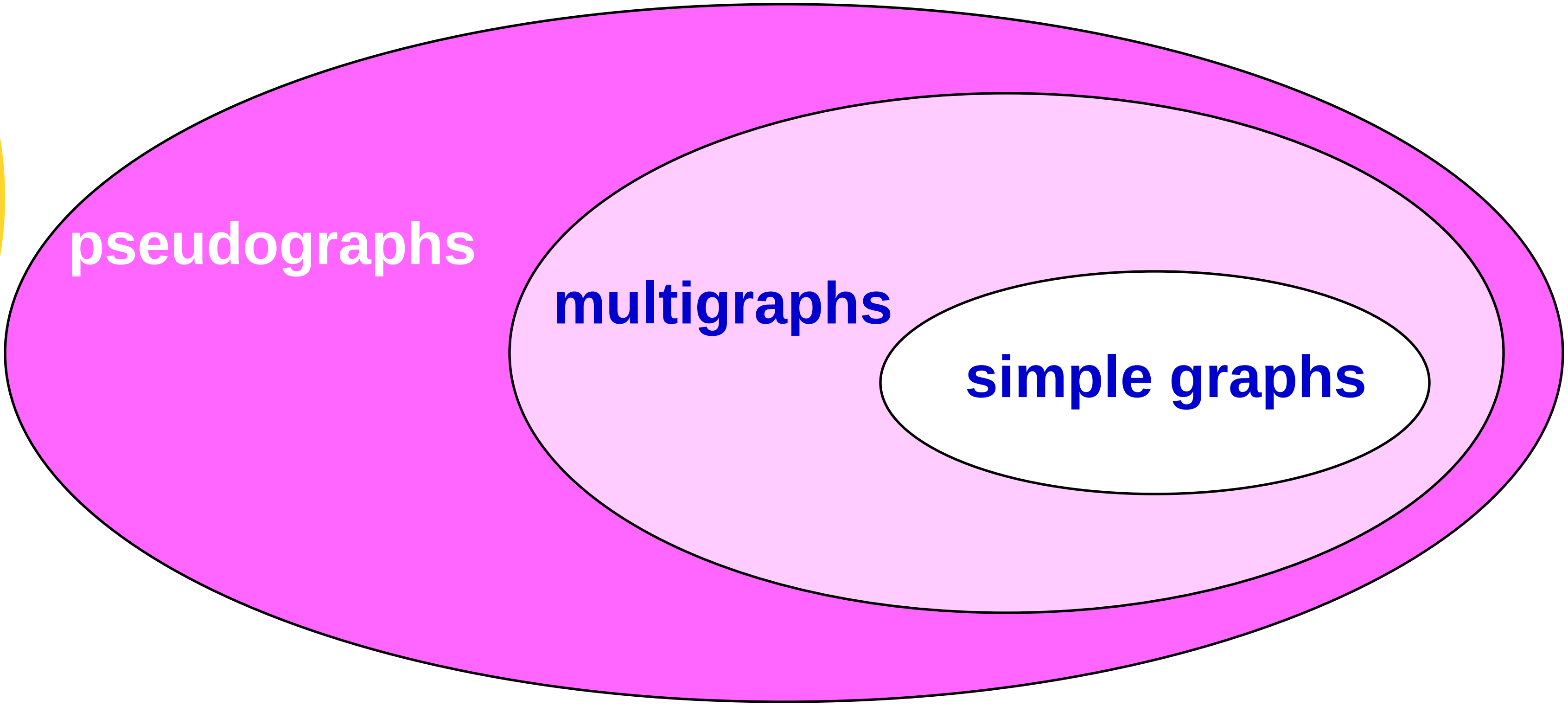
- In a ***pseudograph*** *G = (V, E)* two or more edges may connect the same pair of vertices,
- and in addition, an edge may connect a vertex to itself.

Loops



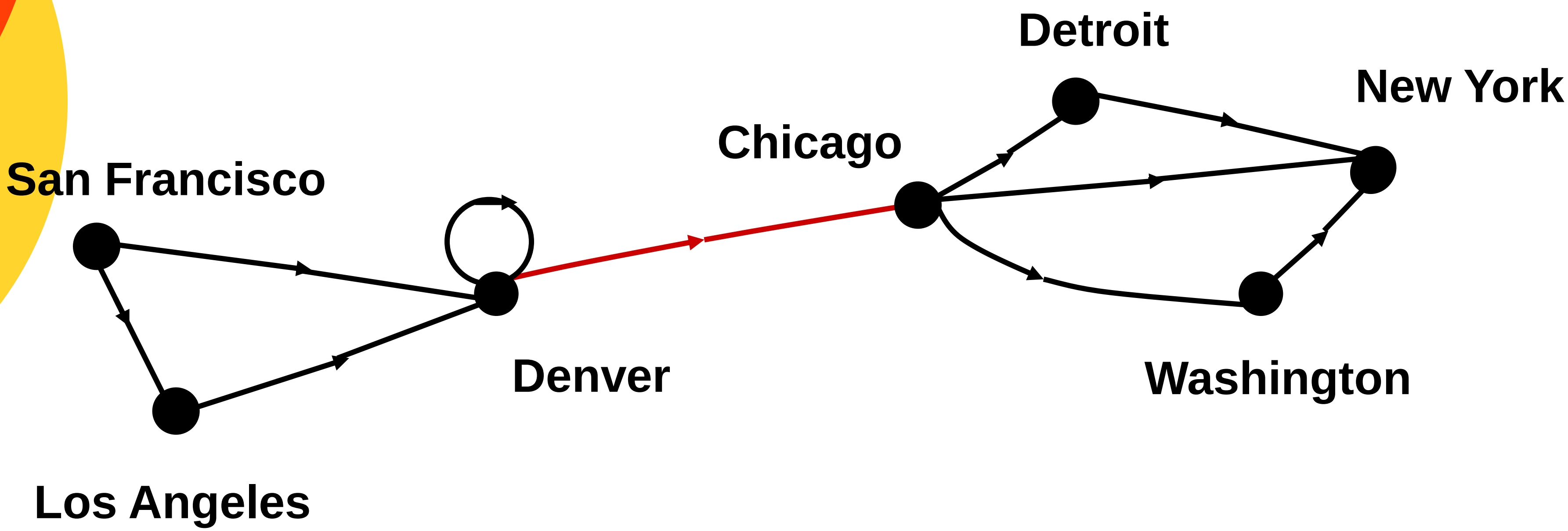
An edge is called a ***loop***
if it connects a vertex to itself.

Undirected Graphs



A Directed Graph

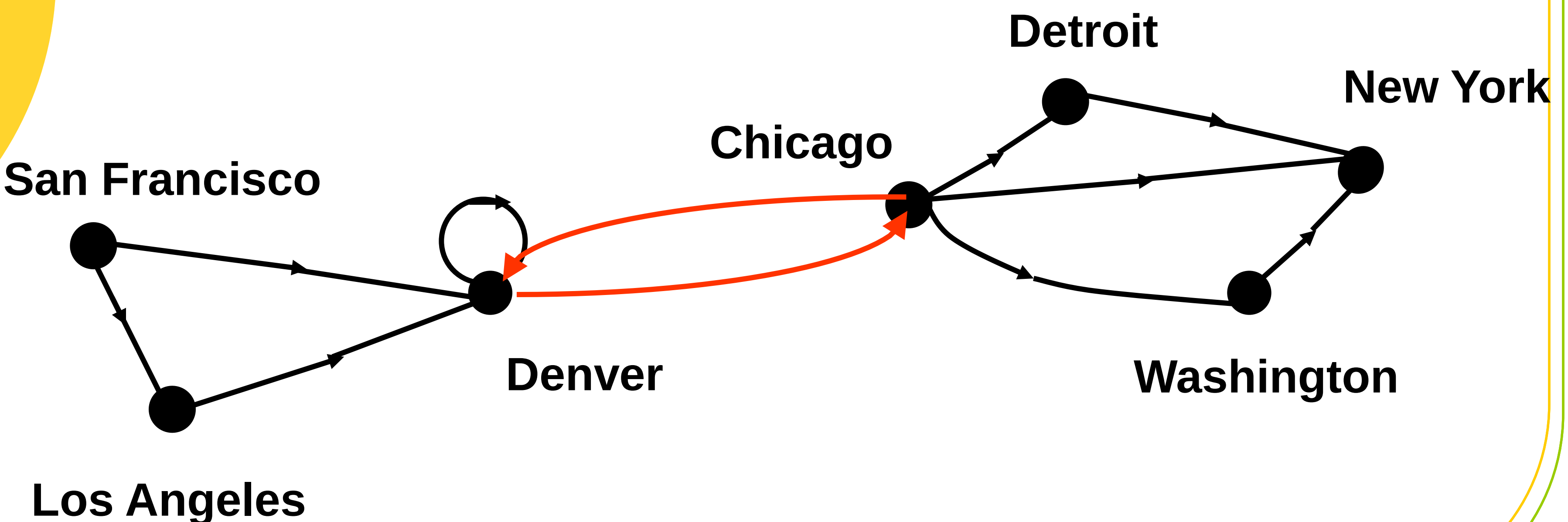
- In a ***directed graph*** $G = (V, E)$ the edges are ordered pairs of (not necessarily distinct) vertices.



Some telephone lines in the network may operate in only one direction .

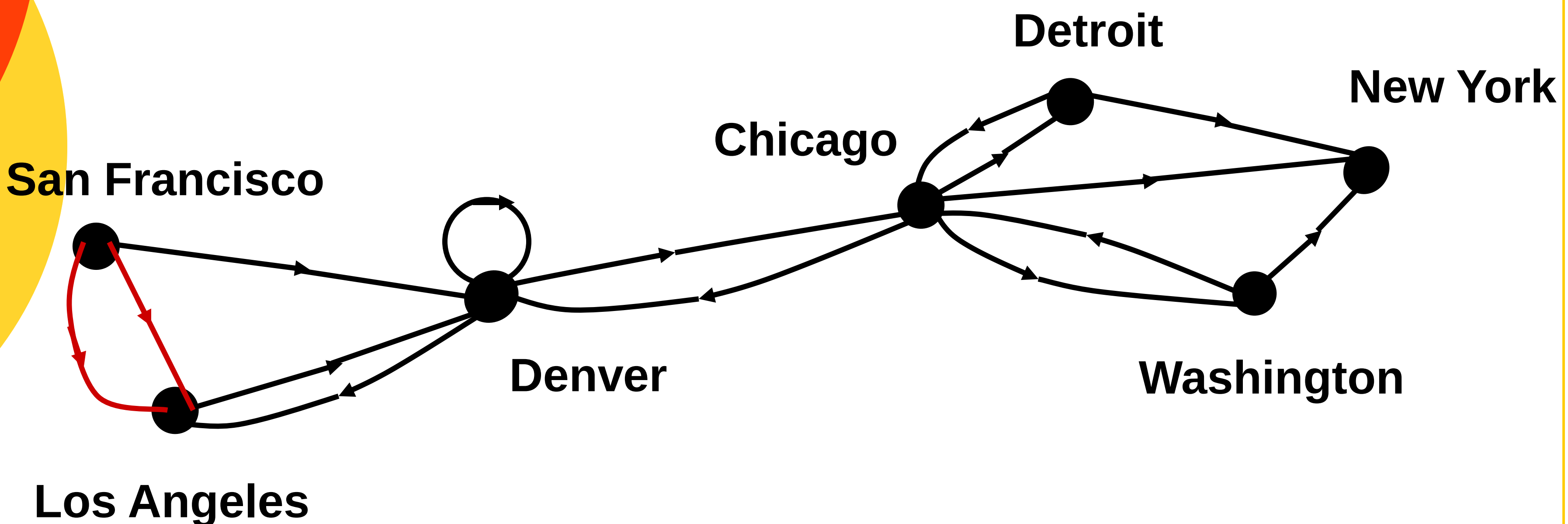
A Directed Graph

The telephone lines in the network that operate in two directions are represented by pairs of edges in opposite directions.



A Directed Multigraph

In a ***directed multigraph*** $G = (V, E)$ the edges are ordered pairs of (not necessarily distinct) vertices, and in addition there may be multiple edges.



There may be several one-way lines in the same direction from one computer to another in the network.

Types of Graphs

Graph Terminology			
Type	Edges	Multiple Edges?	Loops Allowed?
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Driected multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

2. Basic Terminology Adjacent Vertices

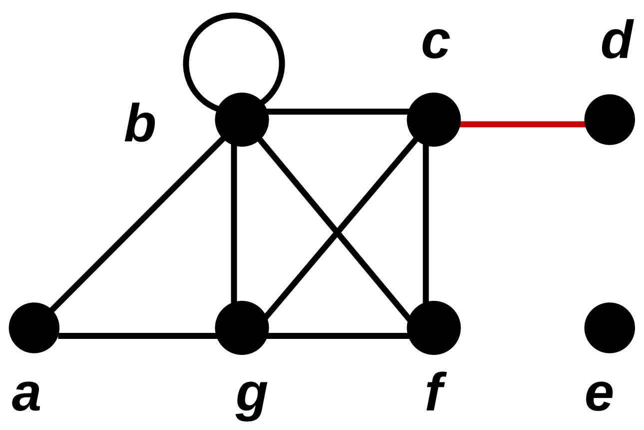
Definition 1. Two vertices, *u* and *v* in an undirected graph G are called ***adjacent*** (or ***neighbors***) in G, if {*u*, *v*} is an edge of G.

An edge *e* connecting *u* and *v* is called ***incident with vertices u and v***, or is said to connect *u* and *v*.

The vertices *u* and *v* are called ***endpoints*** of the edge {*u*, *v*}.

Degree of a vertex

Definition 1. The ***degree of a vertex*** in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.



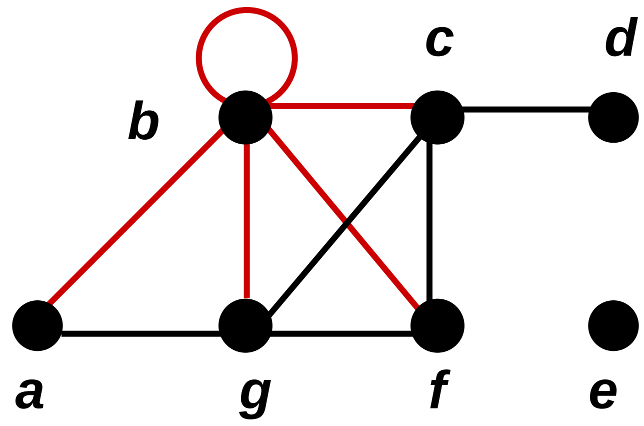
deg(*d*) = 1

deg(*e*) = 0

Degree of a vertex

Definition 1. The ***degree of a vertex*** in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

deg(*b*) = 6

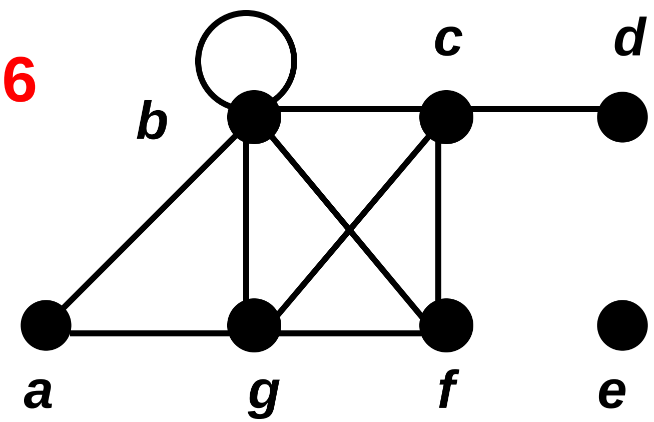


Degree of a vertex

Find the degree of all the other vertices.

deg(*a*)   deg(c)   deg(f)   deg(*g*)

deg(*b*) = 6



deg(*d*) = 1

deg(*e*) = 0

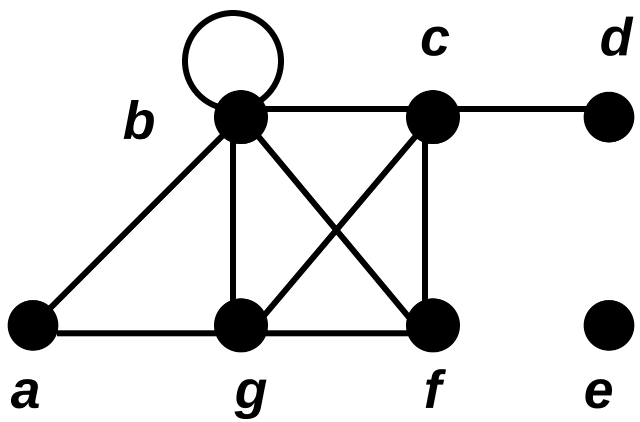
Degree of a vertex

Find the degree of all the other vertices.

deg(a) = 2 deg(c) = 4

deg(f) = 3 deg(g) = 4

deg(*b*) = 6



deg(*d*) = 1

deg(*e*) = 0

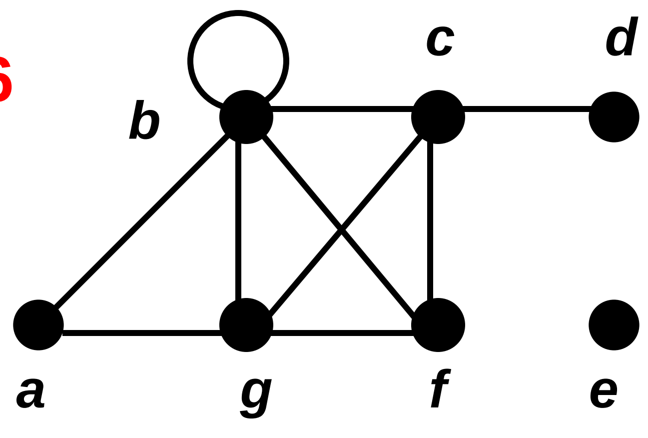
Degree of a vertex

Find the degree of all the other vertices.

deg(a) = 2 deg(c) = 4 deg(f) = 3 deg(g) = 4

TOTAL of degrees = 2 + 4 + 3 + 4 + 6 + 1 + 0 = 20

deg(*b*) = 6



deg(*d*) = 1

deg(*e*) = 0

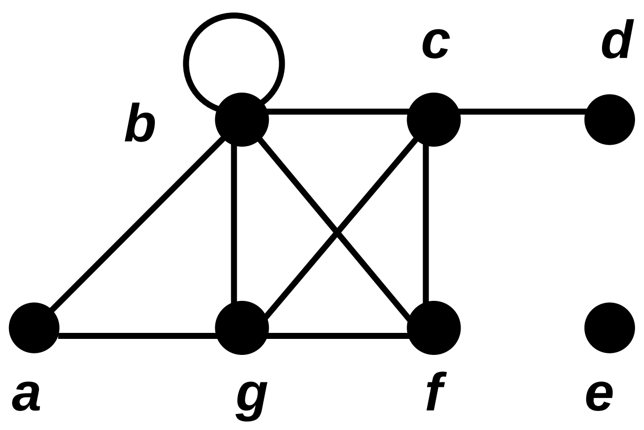
Degree of a vertex

Find the degree of all the other vertices.

deg(a) = 2 deg(c) = 4 deg(f) = 3 deg(g) = 4

TOTAL of degrees = 2 + 4 + 3 + 4 + 6 + 1 + 0 = 20

TOTAL NUMBER OF EDGES = 10



deg(*d*) = 1

deg(*e*) = 0

Handshaking Theorem

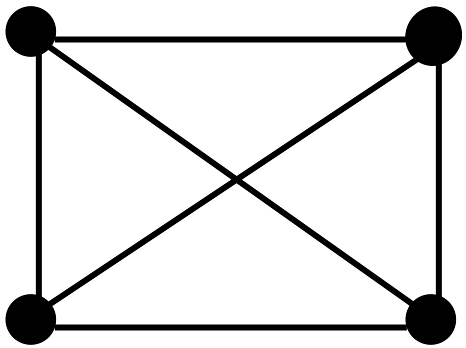
Theorem 1. Let $G = (V, E)$ be an undirected graph G with e edges. Then

$$\sum_{v \in V} \deg(v) = 2e$$

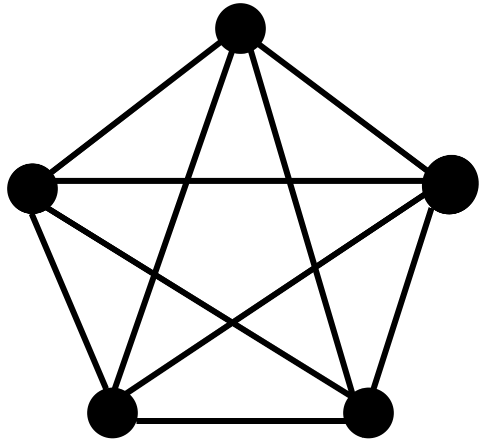
"The sum of the degrees over all the vertices equals twice the number of edges."

NOTE: This applies even if multiple edges and loops are present.

3. Some Special Simple Graphs



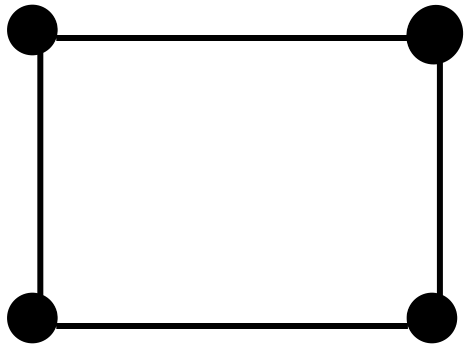
K_4



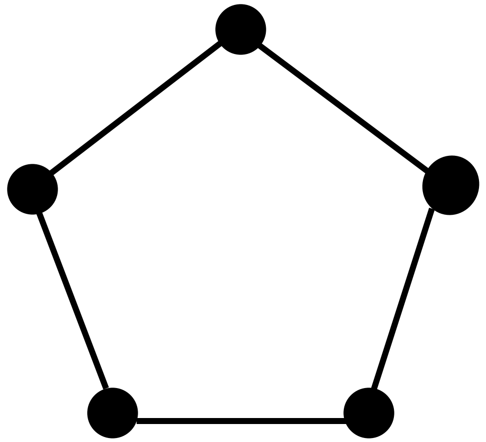
K_5

Complete graph K_n

3. Some Special Simple Graphs



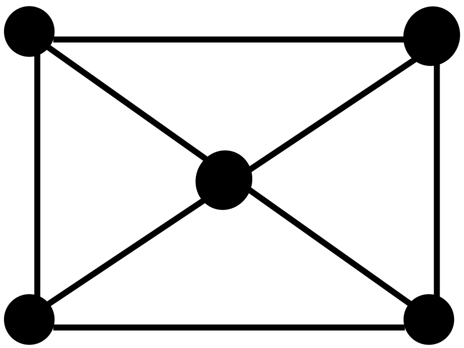
C_4



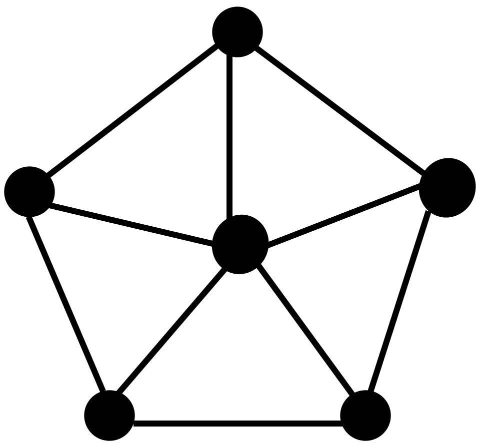
C_5

Cycle C_n

3. Some Special Simple Graphs



W_4

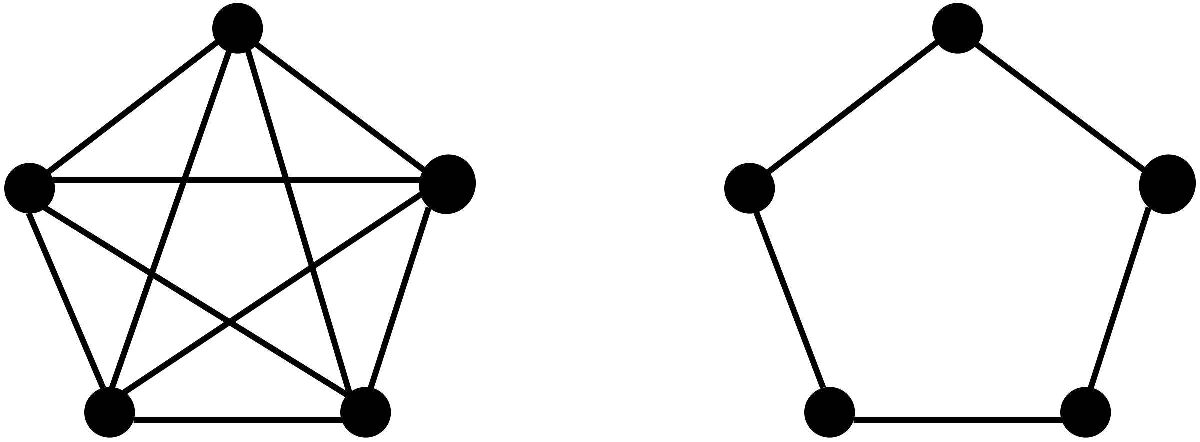


W_5

Wheele W_n

4. New Graphs from Old Graphs

Definition. A ***subgraph*** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.

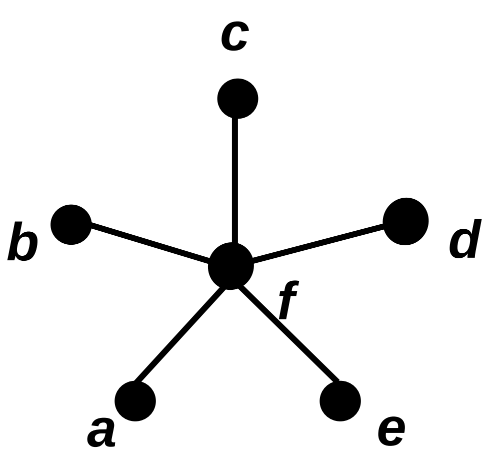


K_5 C_5

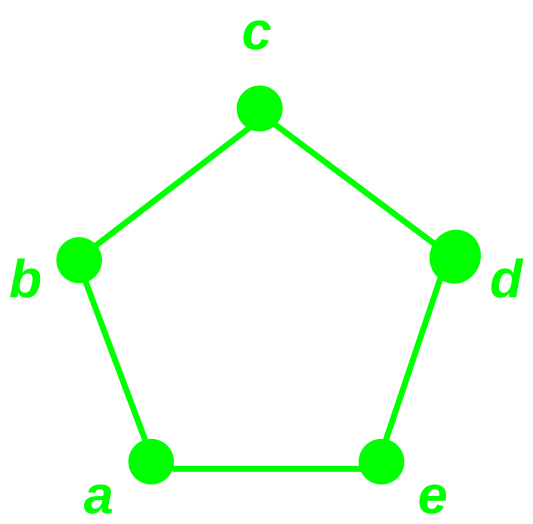
C_5 is a subgraph of K_5

4. New Graphs from Old Graphs

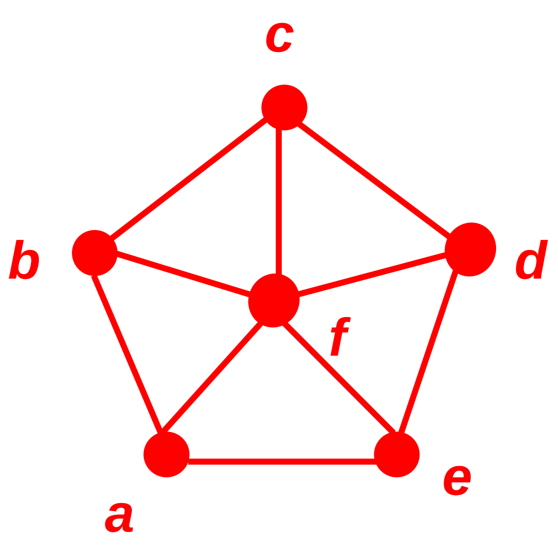
Definition 7. The ***union*** of 2 simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \cup G_2$, is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.



S_5



C_5



W_5

W_5 is the union of S_5 and C_5

Part 2: Representing Graphs, Connectivity, and Euler Paths

1. Representing Graphs
2. Adjacency Matrices
3. Incidence Matrices
4. Isomorphism of Graphs
5. Paths
6. Connectedness
7. Euler Paths and Circuits

1. Representing Graphs

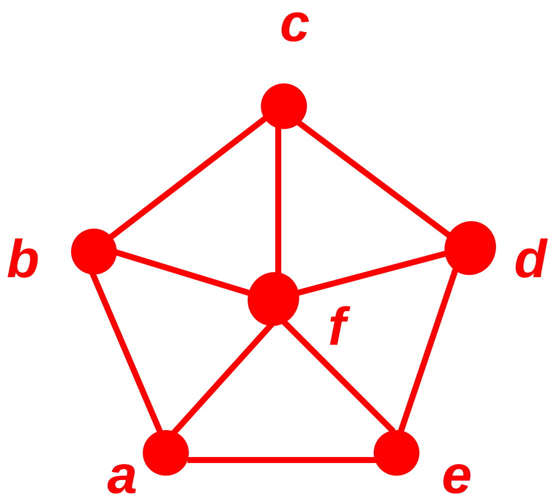
- One way to represent a graph without multiple edges is to list all the edges of this graph.
- Another way to represent a graph with no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph.

2. Adjacency Matrix

A simple graph $G = (V, E)$ with n vertices can be represented by its ***adjacency matrix***,
A, where entry a_{ij} in row i and column j is

$$a_{ij} = \begin{cases} 1 & \text{if } \{ v_i, v_j \} \text{ is an edge in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Finding the adjacency matrix

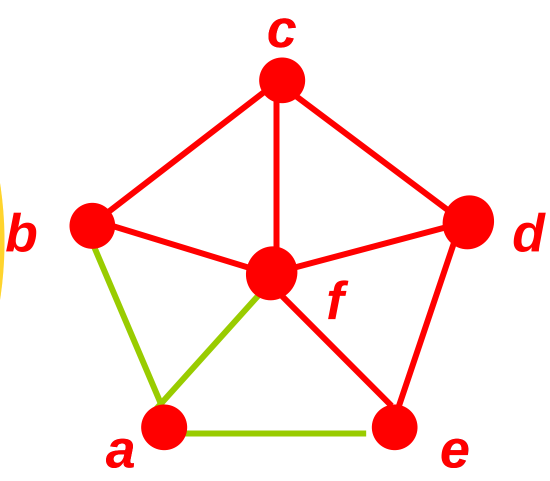


W_5

This graph has 6 vertices a, b, c, d, e, f.

We can arrange them in that order.

Finding the adjacency matrix



W_5

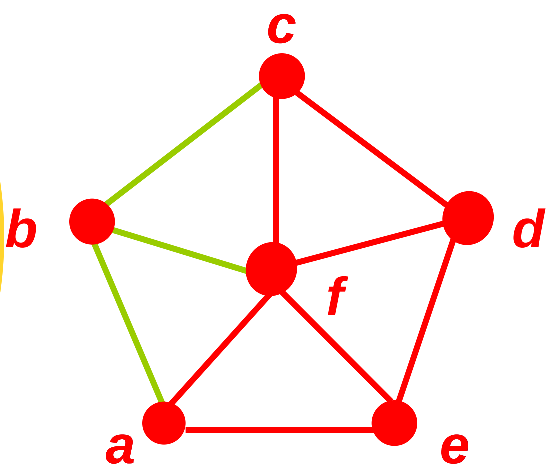
TO

FROM

$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right] \end{array}$$

There are edges from a to b, e, and f

Finding the adjacency matrix



W_5

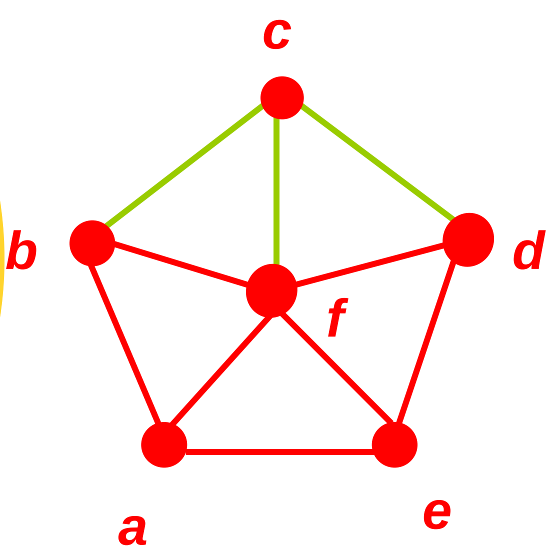
TO

FROM

$$\begin{array}{cc} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

There are edges from b to a, c, and f

Finding the adjacency matrix



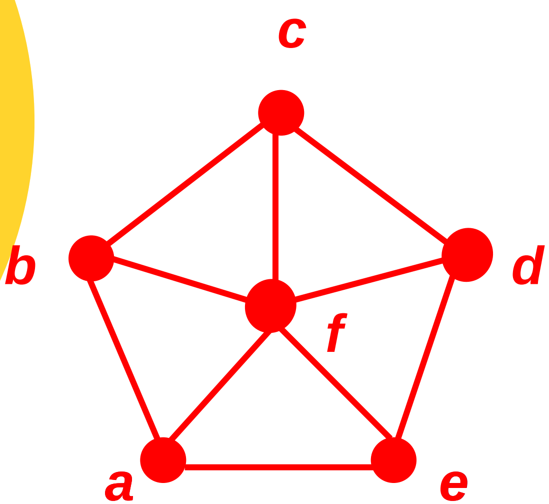
W_5

$$
\begin{array}{c|c}
 & \text{TO} \\
\text{FROM} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right]
\end{array}
$$

There are edges from c to b, d, and f

Finding the adjacency matrix

COMPLETE THE ADJACENCY MATRIX . . .



W_5

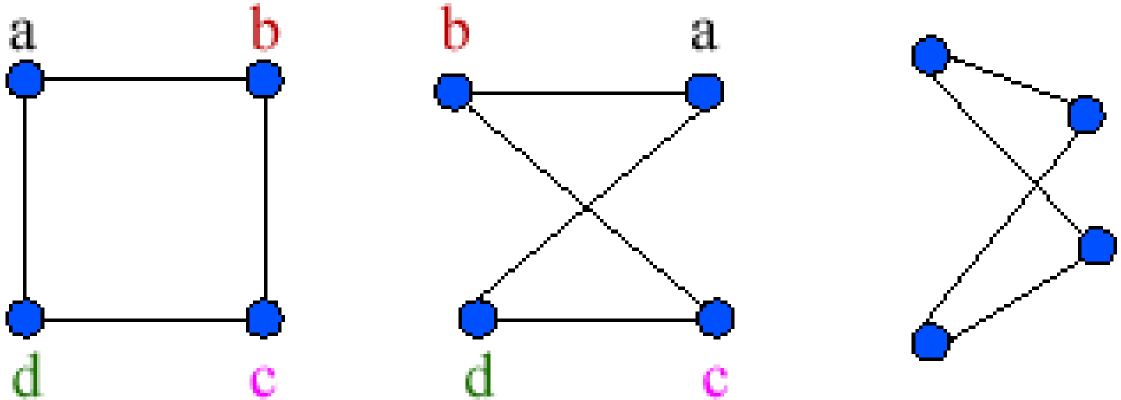
TO

$$
\begin{array}{c c}
\textbf{FROM} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{e} \\ \mathbf{f} \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

Notice that this matrix is symmetric. i.e. $a_{ij} = a_{ji}$

4. Graph Isomorphism

Definition 1. **An** ***isomorphism*** between simple graphs G_1 and G_2 is a bijection between the set of vertices that preserves the adjacency list



4. Graph Isomorphism

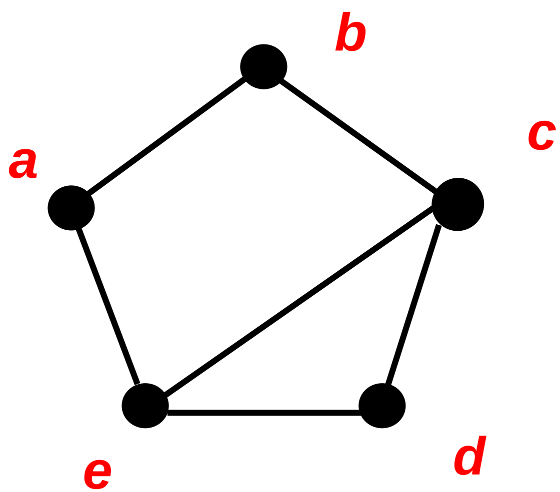
Note. ***Isomporphic*** simple graphs must have the same ***invariants***:

✓The number of vertices

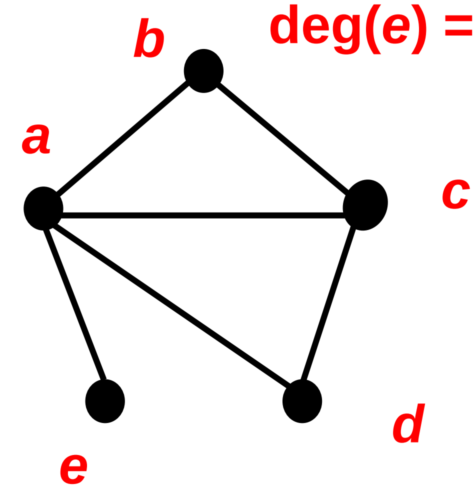
✓The number of edges

✓The degrees of the vertices

No vertex of deg 1



deg(*e*) = 1

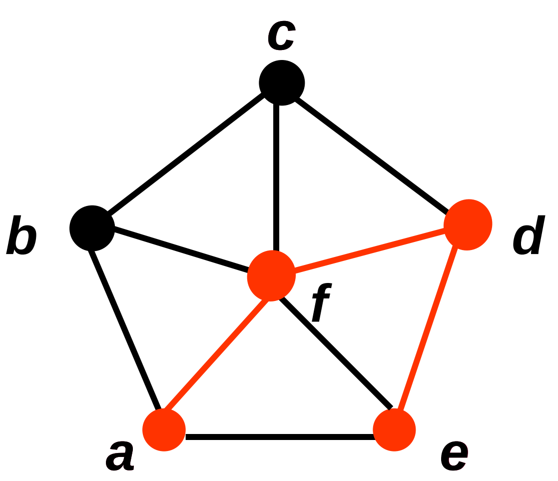


Non-isomorphic graphs

5. Connectivity-Paths

Definition 1. A ***path of length n*** from u to v in an undirected graph is a sequence of edges $e_1 = \{u, x_1\}, e_2 = \{x_1, x_2\}, \ldots, e_n = \{x_{n-1}, v\}$.

We also denote this path by: $u\, x_1\, x_2 \ldots x_{n-1}\, v$



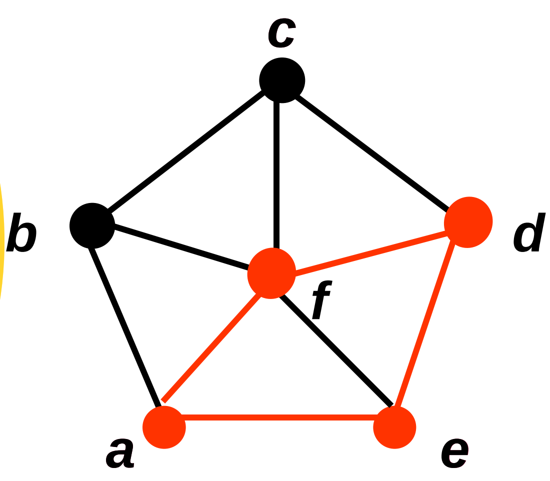
W_5

One path from *a* to *e*

This path passes through vertices f and d in that order.

The length of this path is 3

One path from *a* to *a*



W_5

This path passes through vertices f, d, e, in that order. It has length 4.

It is a circuit because it begins and ends at the same vertex.

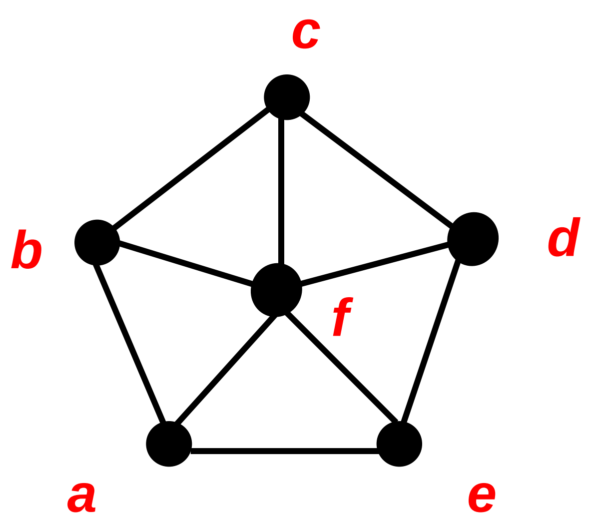
It is called simple because it does not contain the same edge more than once.

6. Connectedness in Undirected Graphs

Definition 3. An undirected graph is called ***connected*** if there is a path between every pair of distinct vertices of the graph.

IS THIS GRAPH CONNECTED?

YES



W_5

6. Connectedness in Undirected Graphs

Theorem . There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof. Let $x_0\, x_1 \ldots x_n$ be a path of shortest length joining the vertices $x_o = u$ and $x_1 = v$.

If this path is not simple, then there are $0 \leq i < j$ such that $x_i = x_j$.

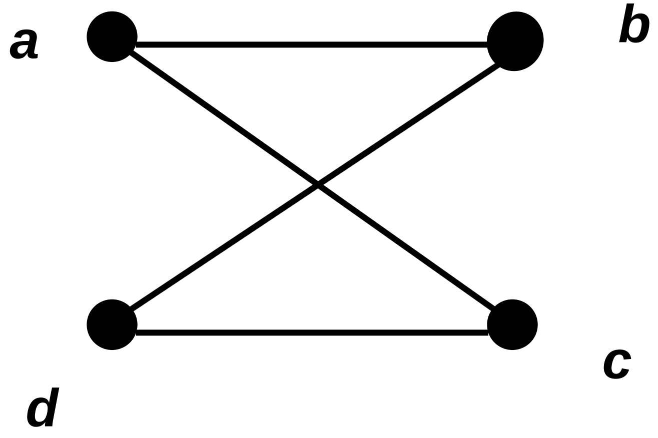
Now deleting the edges $x_i\, x_{i+1}, x_{i+1}\, x_{i+2}, \ldots, x_{j-1}\, x_j$, we will obtain a path of shorter length joining u and v: ***contradiction***.

Counting Paths between Vertices

Theorem. Let G be a graph with adjacency matrix **A** with respect to the ordering $v_1, v_2, \ldots, v_n$.
The number of different paths of length r from v_i to v_j, where r is a postive integer, equals the entry in row i and column j of $\mathbf{A}^r$.

NOTE. This applies with directed or undirected edges, with multiple edges and loops allowed.

Example



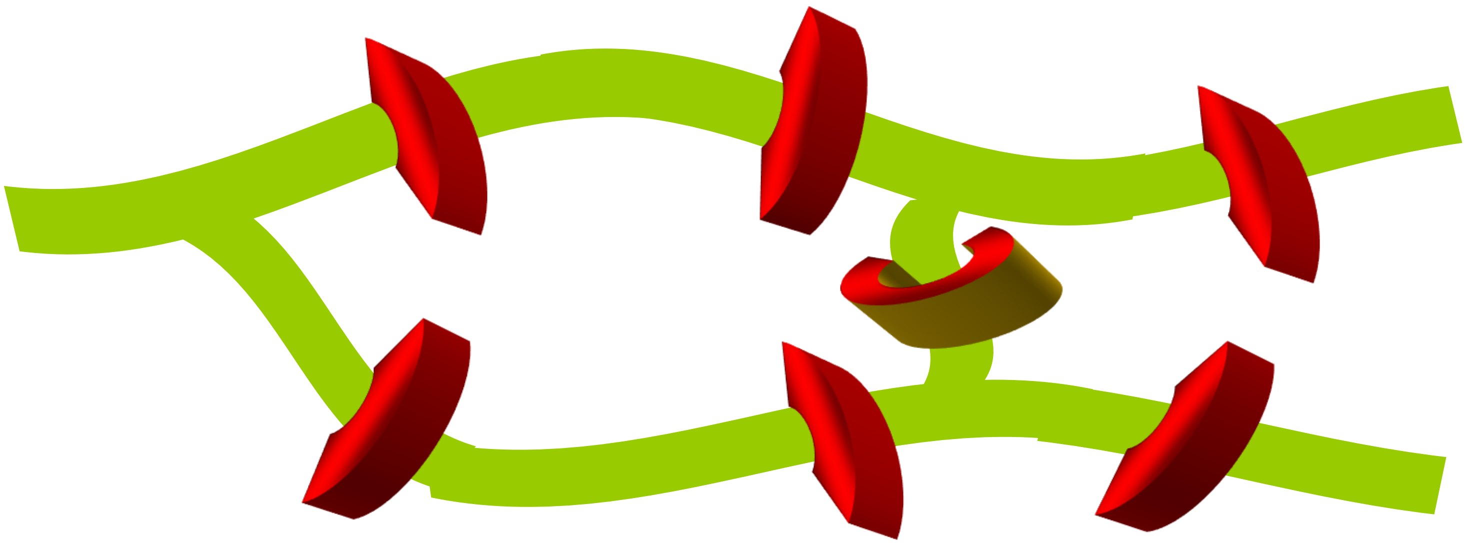
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Thus there are 8 different paths of length 4 joining *a* and *d*

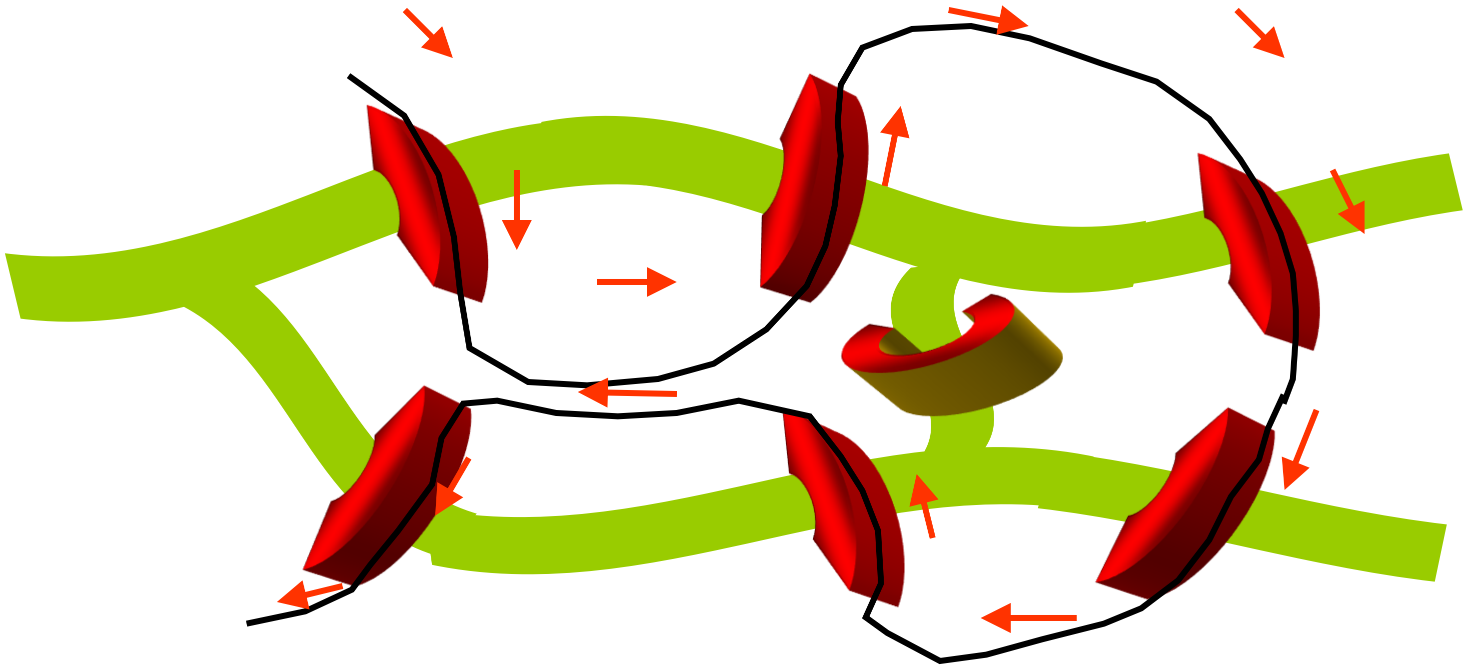
7. Euler Paths

Problem. The town of Königsberg was divided into four sections by the branch of the Pregel River



These four sections are connected by seven bridges

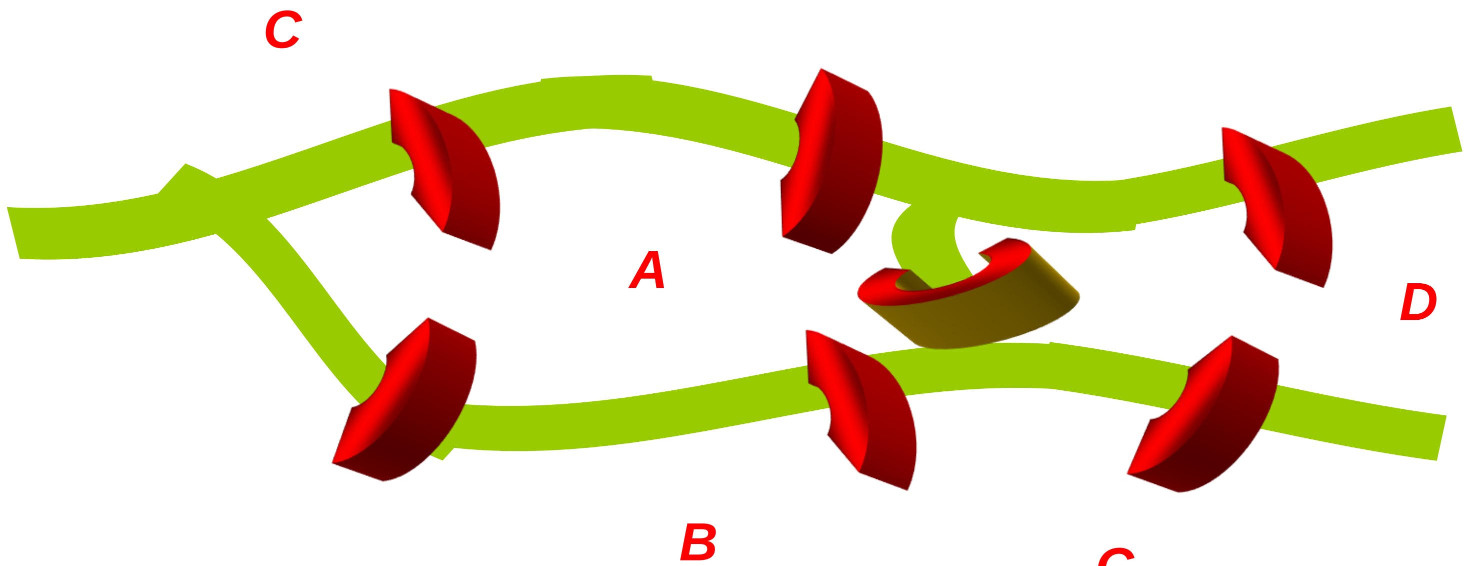
7. Euler Paths



Question. Can one cross seven bridges and return to the starting point without crossing any bridge twice?

In the eighteenth century, Euler solved this problem using Graph Theory

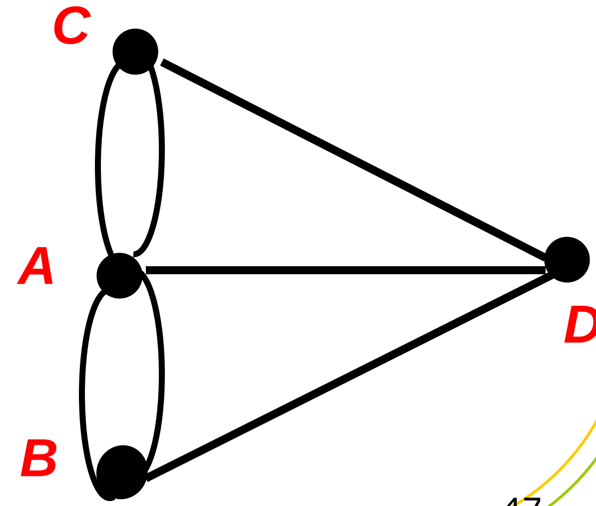
7. Euler Paths



Euler modeled this problem using the multigraph:

✓ four sections correspond to four vertices *A*, *B*, *C*, *D*.

✓ each bridge corresponds to an edge

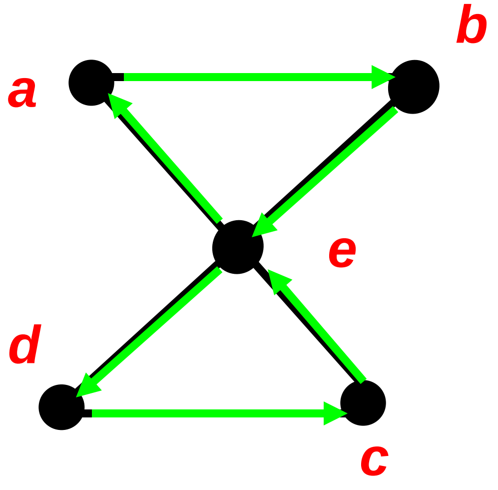


7. Euler Paths

Definition. In a graph *G* an ***Euler path*** is a simple path containing every edge of *G*. A simple circuit which is also an Euler path is called an ***Euler circuit***.

Example. Does this graph have an Euler circuit?

Solution. An Euler circuit is ***abedcea***

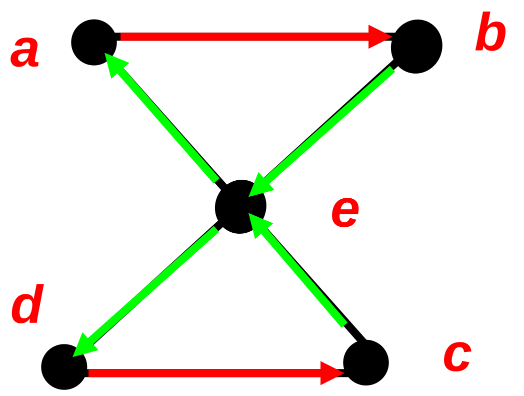


7. Euler Paths

Theorem. A connected multigraph has an Euler circuit if and only if each vertex has even degree.

Proof. When an Euler circuit passes by a vertex, it arrives by an edge and leaves by a different edge.

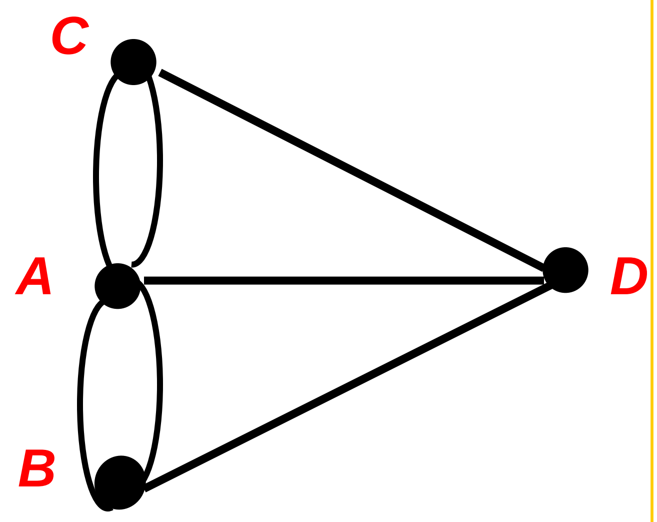
Therefore the degree of the vertex increases by 2 units.



deg(e) = 0 + 2 + … = 2 + 2 = 4

7. Euler Paths

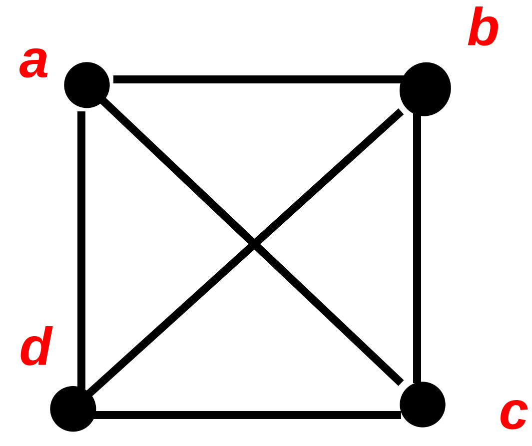
Example. All vertices of the multigraph corresponding to the Königsberg problem have odd degree so there is no Euler circuit. Thus the Problem has no solution



Example. All four vertices of this graph have dgree 3

So there is no Euler circuit.

Thus we cannot draw it without removing the pen head



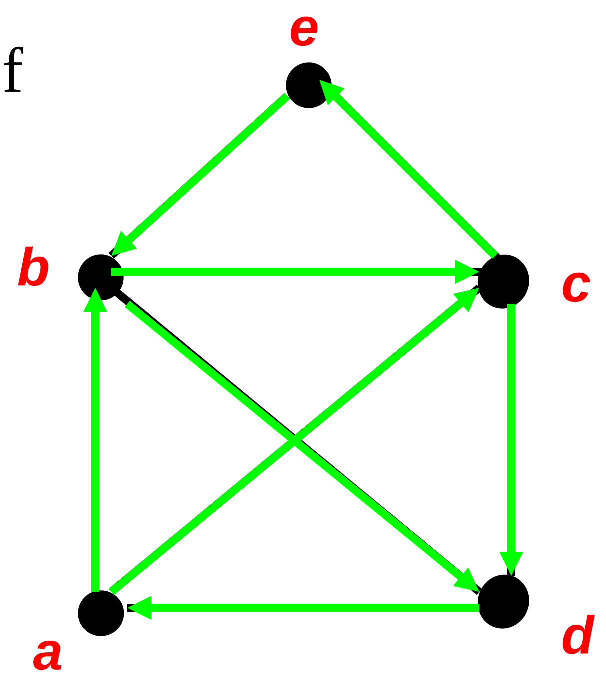
7. Euler Paths

Example. This graph has two vertices of degree 3, namely **a** and **d**

So there is no Euler circuit

However there is an Euler path

abcdaceb



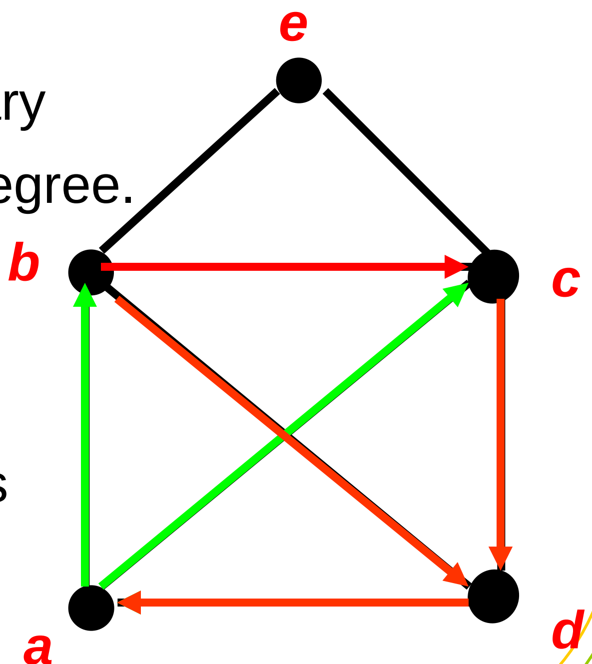
Therefore e can draw this without removing the pen head

The fact that this graph has exactly two vertices of odd degree is not an incidence. In fact we have

7. Euler Paths

Theorem. A connected multigraph has an Euler path that is not an Euler circuit if and only if there are exactly two vertices of odd degree.

Proof. As in the last Theorem, the intermediary vertices passed by an Euler path has even degree.



At the starting vertex **a**, the degree increases by 1 units when it leaves.

In any subsequent visit to **a**, deg(**a**) increases by 2. So deg(**a**) is odd.

Similarly degree of ending vertex **d** is odd.

7. Algorithm for finding Euler Circuit

Step 1: Start from any v to find Circuit C

Step 2: Delete the edges in C out of G and alone vertices.

Step 3: If G=Ø then C is Euler Circuit of G.

Step 4: Find the sharing vertices v of C and G:

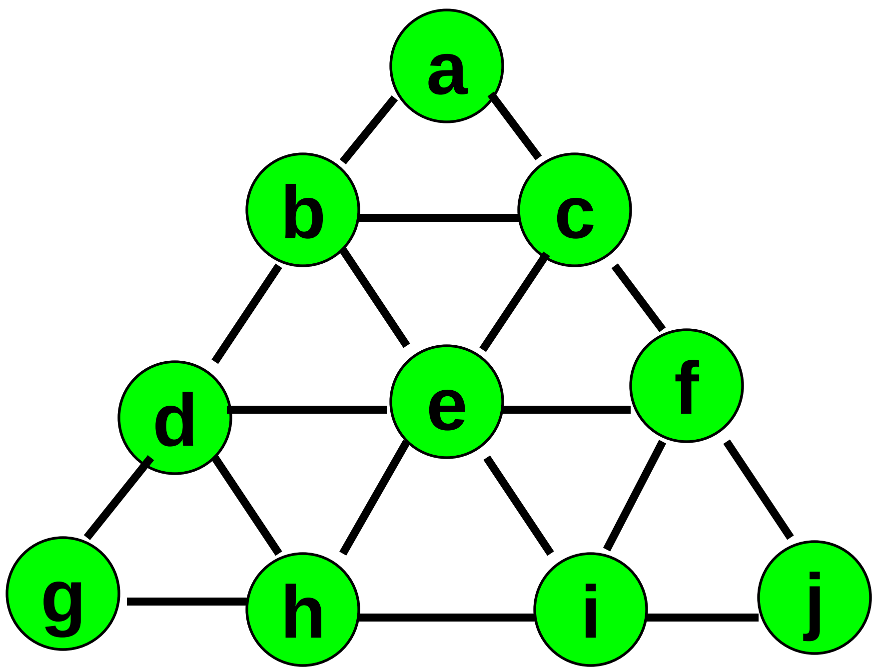
- Sort C starting from v.
- Start from v, find Circuit C'.
- Connect C’ into C.
- Return Step 2.

7. Algorithm for finding Euler Circuit

Example: Given G=(V, E) as follows:

a) Prove G is an Euler graph.

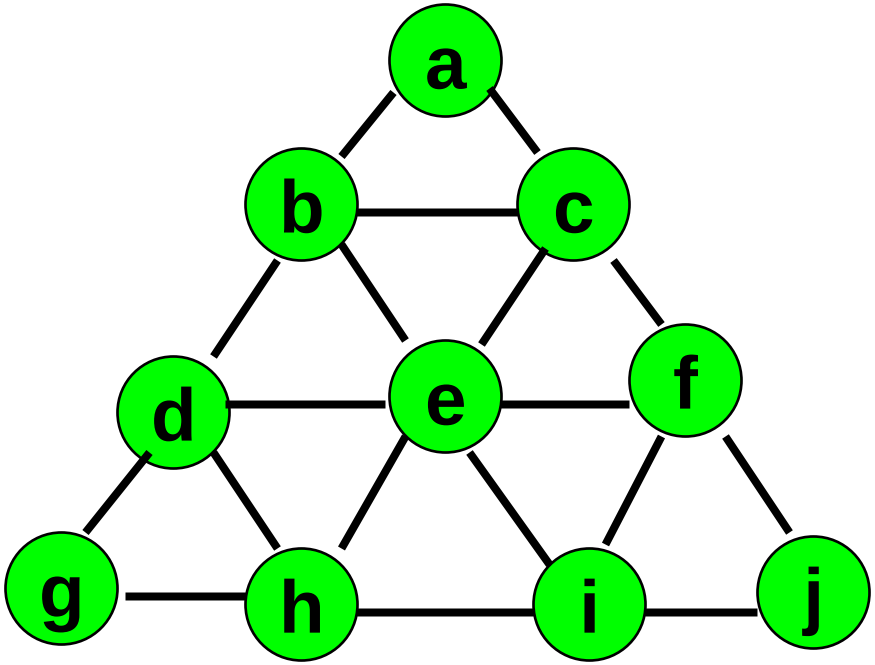
b) Find Euler Circuit C of G



7. Algorithm for finding Euler Circuit

Solution:

a) G is connected and all vertices have even degree.

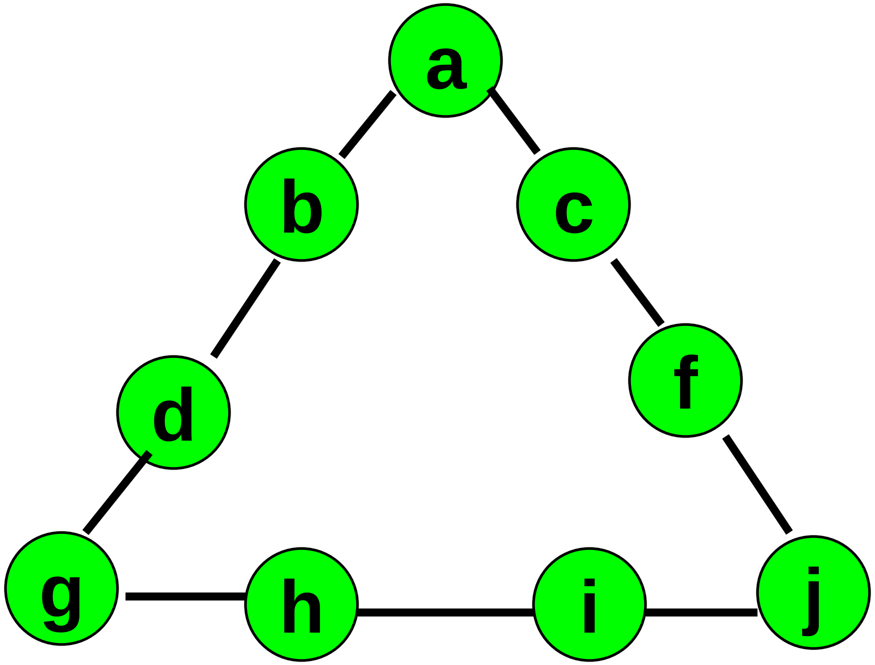


7. Algorithm for finding Euler Circuit

Solution:

b) Starting from vertex a,

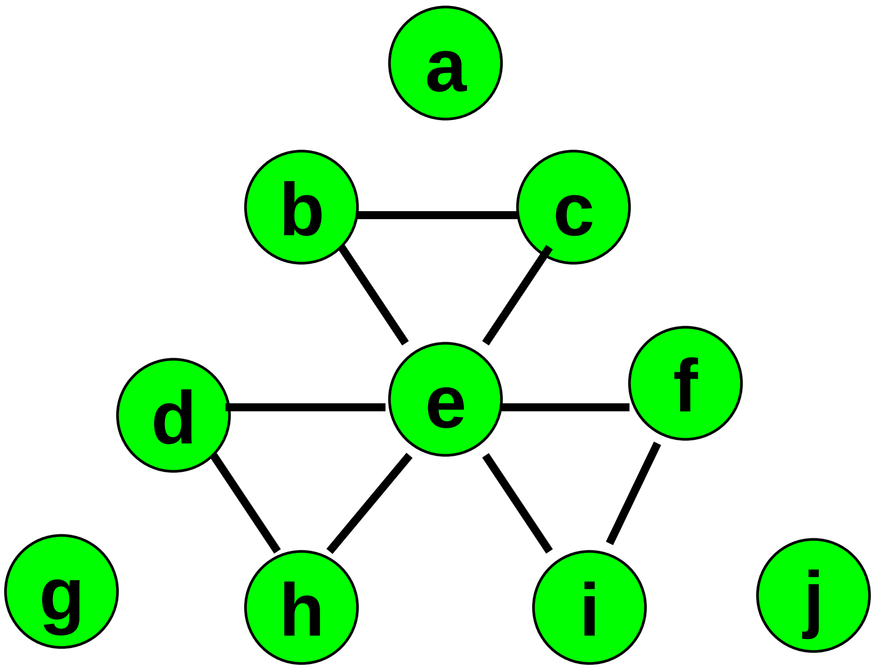
C = (a, b, d, g, h, i, j, f, c, a).



7. Algorithm for finding Euler Circuit

Solution:

- Delete edge on C out of G



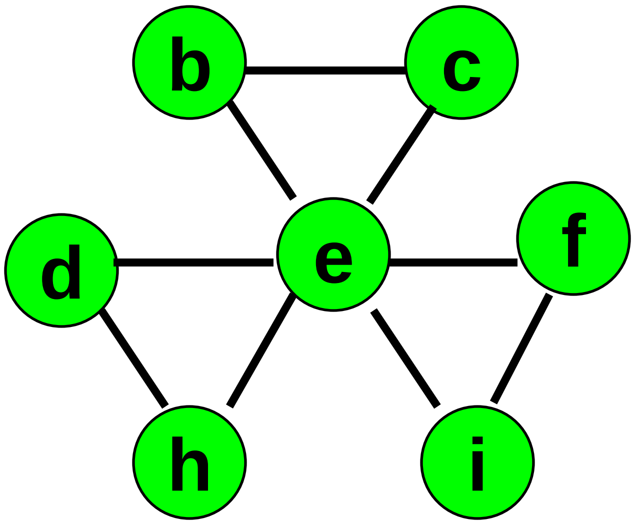
- Delete the alone vertices: *a, g, j*.

7. Algorithm for finding Euler Circuit

Solution:

After deleting the alone vertices: *a*, *g*, *j*.

The remaining graph is G_1



7. Algorithm for finding Euler Circuit

C and G_1 have the same vertex b.

Sort C starting from b

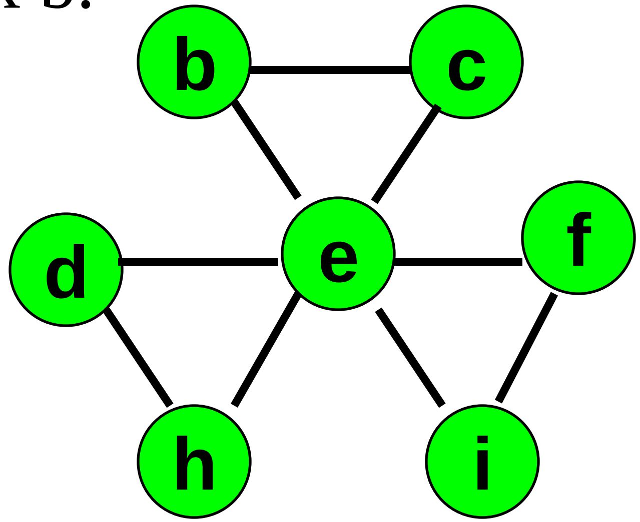
C=(b, d, g, h, i, j, f, c, a, b).

G_1 has circuit starting from b,

C_1=(b, c, e, b).

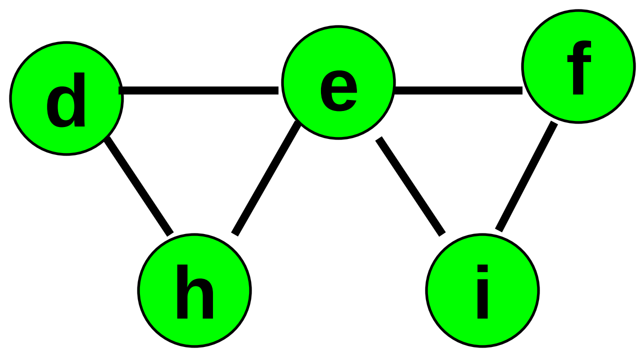
Connect C_1 into C,

C=(b, d, g, h, i, j, f, c, a, b, c, e, b).



7. Algorithm for finding Euler Circuit

The remaining graph is G_2



7. Algorithm for finding Euler Circuit

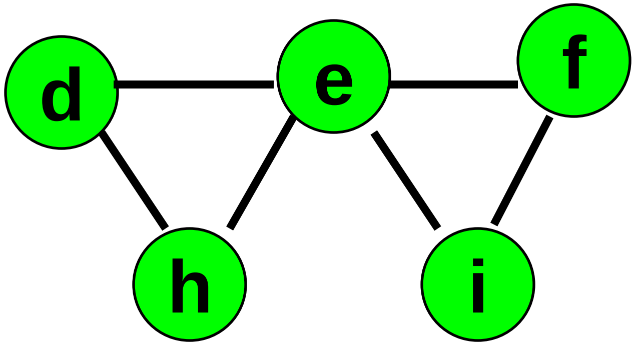
C and G_2 have sharing vertex d.

Sort C starting from d,

C= (d, g, h, i, j, f, c, a, b, c, e, b, d).

On G_2 has circuit starting from d,

C_2 = (d, e, f, i, e, h, d).



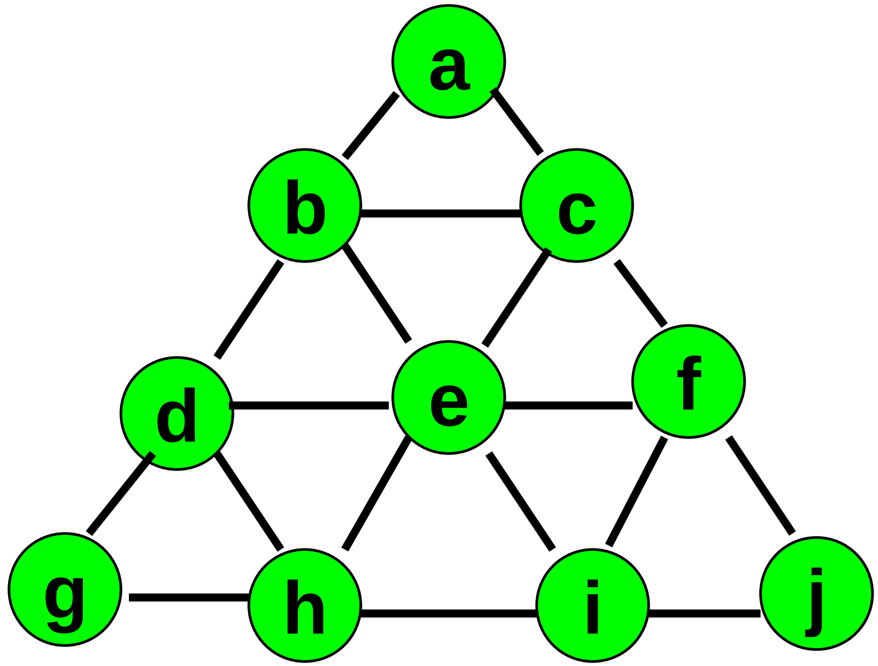
Connect C_2 into C,

C=(d, g, h, i, j, f, c, a, b, c, e, b, d, e, f, i, e, h, d).

G_3=Ø, C is the final Euler circuit that we want to find.

7. Algorithm for finding Euler Circuit

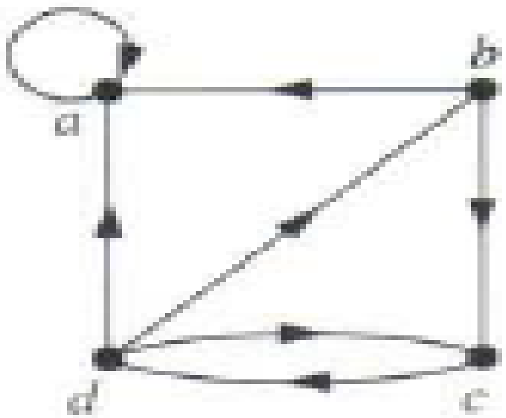
C=(a, b, c, e, b, d, e, f, i, e, h, d , g, h, i, j, f, c, a).



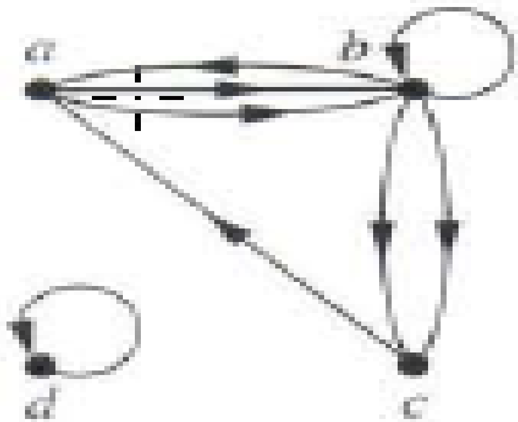
CW Assignment. Time: from 10h to 11h

- 1. Find the number of edges, number of vertices, and degree of each vertex:

7.

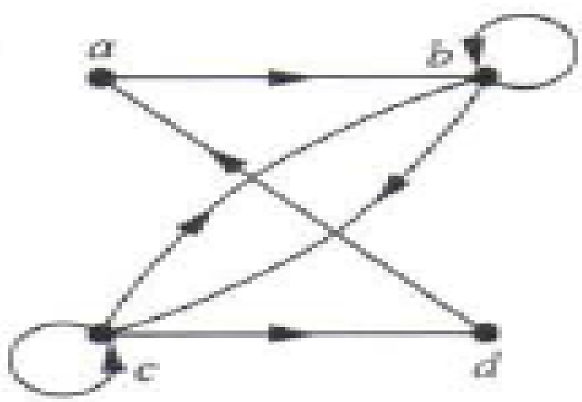


8.

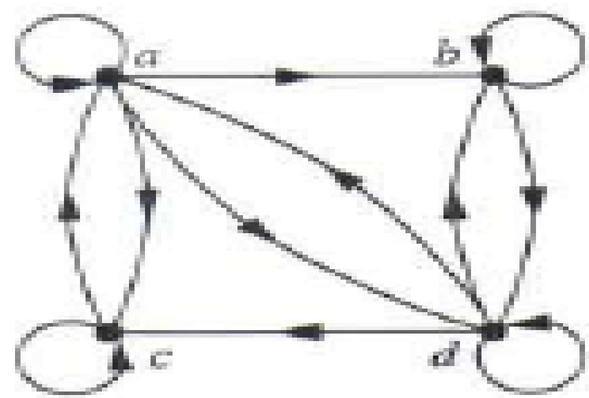


- 2. Find the adjacency matrix of multigraphs:

19.



20.



Homework.

- Pages 608-609: 4, 6
- Pages 618-619: 4, 8, 10, 12, 22, 24, 57, 58
- Pages 644-645: 8, 10, 18, 20
- Implement the algorithm EulerCircuit.C/C++ following the description from slides 53 to 62.
- Deadline: before 9PM Sat, Nov, 22th