Tutorial 3

keywords: covariance, expected value, diversification, variance, risk, vector, matrix

estimated reading time: 30 minutes

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August 5, 2018

Question 1

Covariance of two random variables X and Y

X and Y are random variables with mean μ_X and μ_Y respectively. The covariance between X and Y is defined as

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

Show that:

$$Cov(X,Y) = E[(X-\mu_X)Y] = E[X(Y-\mu_Y)] = E(XY) - \mu_X\mu_Y$$

Discuss why we cannot simplify $E(XY) - \mu_X\mu_Y$ further to $E(X)E(Y) - \mu_X\mu_Y = \mu_X\mu_Y - \mu_X\mu_Y = 0$.

Background

Covariance

The covariance between X and Y is a measure of linear association between them and is define as the expected value of $(X-\mu_X)(Y-\mu_Y)$,

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

So, if X is above its mean, is Y more likely to be below or above its mean?

- If X is above its mean and Y is above its mean, then $(X-\mu_X)(Y-\mu_Y) > 0$ and this data point falls in quadrant 1.
- If X is below its mean and Y is below its mean, then $(X-\mu_X)(Y-\mu_Y) > 0$ and this data point falls in quadrant 3.
- If X is above its mean and Y is below its mean, then $(X-\mu_X)(Y-\mu_Y) > 0$ and this data point falls in quadrant 4.
- If X is below its mean and Y is above its mean, then $(X-\mu_X)(Y-\mu_Y) > 0$ and this data point falls in quadrant 2.

Since the covariance between X and Y is defined as the expected value (or population average) of $(X-\mu_X)(Y-\mu_Y)$,

$$
\begin{aligned}
Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_X)(y_i-\mu_Y) \\
&= \frac{(x_1-\mu_X)(y_1-\mu_Y)+(x_2-\mu_X)(y_2-\mu_Y)+\cdots+(x_N-\mu_X)(y_N-\mu_Y)}{N}
\end{aligned}
$$

it then follows that,

- If $Cov(X,Y) > 0$, then, on average, when X is above its mean, Y is also above its mean.
- If $Cov(X,Y) < 0$, then, on average, when X is above its mean, Y is below its mean.

The sign of the covariance coefficient is directly interpretable, but the magnitude is not because the covariance depends on the units of measurement of X and Y. Scaling the covariance by the standard deviations of the variables eliminates the unit of measurement, and defines the correlation between X and Y,

$$Corr(X,Y) = \frac{Cov(X,Y)}{sd(X)sd(Y)}$$

Unlike the covariance, the correlation must lie between -1 and 1.

(Discuss in class)

Question 2

Diversification in everyday life

Most pokie machines give you the option of multiplying your bet up. For example,

- If the machine accepts 25 cents per round for having a go at winnings given by the random variable X...
- ...you also have the option of paying \$1 to scale up your winnings to $4X$

Farshid's Mum, who is a 98 year old lady with primary school education and loves pokie machines, told him,

> "people who scale their bets up are silly because they are at risk of running out of money faster"

Suppose you have \$1 only. Compare the expected return and risk of using all of your money at once and betting $4X$, with using it for playing X four times (i.e. $X_1 + X_2 + X_3 + X_4$, where X_i are independent and have distributions identical to X). Do you agree with Farshid's Mum? Discuss.

Background

The Expected Value

- For any constant c, $E(c) = c$
- For any constants a and b, $E(aX + b) = aE(X) + b$
- If $\{a_1, a_2, \ldots, a_n\}$ are constants and $\{X_1, X_2, \ldots, X_n\}$ are random variables then,

$$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n)$$

The Variance

- For any constant c, $Var(c) = 0$
- For any constants a and b, $Var(aX + b) = a^2Var(X)$
- For any constants a and b,

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$$

$$X : \textit{winnings by betting } 25 \textit{ cents}$$
$$4X : \textit{winnings by betting } \$1$$

Let $E(X) = \mu$ and $Var(X) = \sigma^2$.

The expected return of betting \$1 all at once,

$$\begin{aligned} E(4X) &= 4E(X) \\ &= 4\mu \end{aligned}$$

and the risk of betting \$1 all at once (variance is a measure of risk),

$$\begin{aligned} Var(4X) &= 16Var(X) \\ &= 16\sigma^2 \end{aligned}$$

The expected return of betting 25 cents four times,

$$\begin{aligned} E(X_1 + X_2 + X_3 + X_4) &= E(X_1) + E(X_2) + E(X_3) + E(X_4) \\ &= \mu + \mu + \mu + \mu \\ &= 4\mu \end{aligned}$$

and the risk of betting 25 cents four times,

$$\begin{aligned} Var(X_1 + X_2 + X_3 + X_4) &= Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) \\ &+ cov(X_1, X_2) + cov(X_1, X_3) + \cdots + cov(X_2, X_1) + cov(X_2, X_3) \\ &+ \cdots + cov(\ldots, \ldots) \\ &= \sigma^2 + \sigma^2 + \sigma^2 + \sigma^2 + 0 + 0 + \cdots + 0 \\ &= 4\sigma^2 \end{aligned}$$

Farshid's Mum is right! Both strategies have the same expected return but betting \$1 has a much larger risk.

Question 3

Diversification in econometrics and statistics

Suppose we are interested in estimating the mean of a random variable X,

$$population\ mean\ of\ X = \mu\ (unknown)$$

$$we\ want\ to\ estimate\ \mu$$

We have two estimators for μ,

- Estimator 1 ($\tilde{\mu}$): The formula of Estimator 1 is given by,

  $$\tilde{\mu} = X$$

  this estimator takes one observation from the random variable X and uses this as the estimate of μ.

- Estimator 2 ($\hat{\mu}$): The formula of Estimator 2 is given by,

  $$\hat{\mu} = \frac{X_1 + X_2 + X_3 + X_4}{4} = \bar{X}$$

  this estimator takes a random sample of 4 observations from the random variable X, then averages them and uses this as the estimate of μ. A random sample of 4 observations can be denote by $\{X_1, X_2, X_3, X_4\}$.

What is the expected value of each of these estimators and which one is safer (i.e. less risky)? Discuss the similarity of this to Farshid's Mum's Theorem.

The expected value and variance of Estimator 1,

$$E(\tilde{\mu}) = E(X) = \mu$$
$$Var(\tilde{\mu}) = Var(X) = \sigma^2$$

The expected value and variance of Estimator 2,

$$\begin{aligned} E(\hat{\mu}) &= E\Big(\frac{X_1 + X_2 + X_3 + X_4}{4}\Big) \\ &= \frac{1}{4}E(X_1 + X_2 + X_3 + X_4) \\ &= \frac{1}{4}E(X_1) + E(X_2) + E(X_3) + E(X_4) \\ &= \frac{1}{4} \times 4\mu \end{aligned}$$

$$
\begin{aligned}
&= \mu \\
Var(\hat{\mu}) &= Var(\bar{X}) \\
&= Var\Big(\frac{X_1 + X_2 + X_3 + X_4}{4}\Big) \\
&= \frac{1}{16} Var(X_1 + X_2 + X_3 + X_4) \\
&= \frac{1}{16}[Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) \\
&+ cov(X_1, X_2) + cov(X_1, X_3) + \cdots + cov(X_2, X_1) + cov(X_2, X_3) + \cdots + cov(\ldots, \ldots)] \\
&= \frac{1}{16}[Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) + 0 + 0 + \cdots + 0] \\
&= \frac{1}{16}[\sigma^2 + \sigma^2 + \sigma^2 + \sigma^2] \\
&= \frac{\sigma^2}{4}
\end{aligned}
$$

Both estimators have the same expected value $E(X) = E(\bar{X}) = \mu$ but the 2nd estimator (average of 4 observations) is less risky $Var(X) = \sigma^2 > Var(\bar{X}) = \dfrac{\sigma^2}{4}$.

This is very similar to Farshid's Mum's Theorem. There we were comparing 4X and $(X_1+X_2+X_3+X_4)$, here we are comparing X and $\frac{1}{4}(X_1+X_2+X_3+X_4)$. Same principle: it is safer to diversity and not depend only on a single draw from the distribution.

(Covariances equal to 0 because the sample is random, so $X_1, X_2, X_3, and\ X_4$ are unrelated to each other.)

Question 4

Diversification in finance and vectors & matrices and how they make our lives simpler

An investment portfolio is as weighted average of assets that we have invested in. For example, suppose we have invested:

- 20% of our savings in Qantas shares
- 30% in Telstra shares
- 50% in Wesfarmers shares

The returns of these shares are random variables. Let's denote each of these returns by the first letter of the company name. If we denote the return to our portfolio by X, we can write:

$$X = 0.20Q + 0.30T + 0.50W$$

Of course this is only a simple made up example. In reality, the portfolios that investment managers manage include a large number of assets. We can use vectors to show this portfolio in a simple way, and more importantly to use vector arithmetic to compute its return. Consider the following two vectors:

$$\underset{3\times 1}{\mathbf{p}} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}, \underset{3\times 1}{\mathbf{z}} = \begin{bmatrix} Q \\ T \\ W \end{bmatrix}$$

- $\mathbf{p}$ is the vector of portfolio weights
- $\mathbf{z}$ is the vector of assets

Using the rules of vector multiplication we can write:

$$X = \mathbf{p}'\mathbf{z}$$

where $\mathbf{p}' = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix}$, is the transpose of the vector $\mathbf{p}$. Since $\mathbf{p}$ is a vector of constants and $\mathbf{z}$ is a vector of random variables, we have:

$$E(X) = \mathbf{p}'E(\mathbf{z})$$

So, if we know the mean return of the assets in the portfolio (or we can estimate the mean return from data), then we can arrange them in the vector $E(\mathbf{z})$ and then get an estimate of the mean return to this portfolio by vector multiplication. For instance, the estimates of mean (and standard deviation) of monthly returns for these three shares based on monthly observations in the last 8 years are given in the table below:

	Q	T	B
mean	1.0	0.6	0.8
std. dev	9.8	4.5	4.6

This part does not save us much time, because calculating

$$E(X) = 0.20E(Q) + 0.30E(T) + 0.50E(W)$$

does not take much time even with a hand calculator. However, mean return is not the only parameter that we are interested in when investing. We want to know the variance of the portfolio, which measures its risk. For a single random variable y and a constant c we know that

$$Var(cy) = c^2 Var(y)$$

For a 3×1 vector random variable such as $\mathbf{z}$, first we need to form its 3×3 variance matrix (sometimes called the variance-covariance matrix):

$$\underset{3\times 3}{Var(\mathbf{z})} = \begin{bmatrix} Var(Q) & Cov(Q,T) & Cov(Q,W) \\ Cov(Q,T) & Var(T) & Cov(T,W) \\ Cov(Q,W) & Cov(T,W) & Var(W) \end{bmatrix}$$

Note that the diagonal elements of this matrix are variances of each asset return, and the off diagonal elements are covariances between each pair of asset returns. This matrix is symmetric (a symmetric matrix is a square matrix that is equal to its transpose). The estimated variance covariance matrix of returns based on the last 8 years of observed returns is given below (all values rounded to make life easier):

	Q	T	B
Q	94	1	8
T	1	20	5
W	8	5	21

Then, we have:

$$Var(X) = Var(\mathbf{p}'\mathbf{z}) = \mathbf{p}'Var(\mathbf{z})\mathbf{p}$$

Using the information provided above, compute the variance of the portfolio of Qantas, Telstra and Wesfarmers shares given by the portfolio weights $\mathbf{p}$. Compare the risk of this portfolio with the risk of each individual asset. [The practical importance of this exercise in addition to providing an example of the benefit of diversification using real data is that the matrix based formulae for the expected return and variance of return to a portfolio are very easy to compute for a computer, even for portfolios of hundreds of assets. The investment manager can then change the portfolio weights and recompute these to find a portfolio with the highest expected return for a given level of risk, or find a portfolio with the lowest risk for a given expected return.]

(Discuss in class)