Tutorial 3

keywords: covariance, expected value, diversification, variance, risk, vector, matrix

estimated reading time: 30 minutes

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Question 1

Covariance of two random variables X and Y

X and Y are random variables with mean μ_X and μ_Y respectively. The covariance between X and Y is defined as

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Show that:

$$Cov(X, Y) = E[(X - \mu_X)Y] = E[X(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

Discuss why we cannot simplify $E(XY) - \mu_X \mu_Y$ further to $E(X)E(Y) - \mu_X \mu_Y = \mu_X \mu_Y - \mu_X \mu_Y = 0$.

Question 2

$$X : winnings\ by\ betting\ 25\ cents$$
$$4X : winnings\ by\ betting\ \$1$$

Let $E(X) = \mu$ and $Var(X) = \sigma^2$.

The expected return of betting \$1 all at once,

$$E(4X) = 4E(X)$$
$$= 4\mu$$

and the risk of betting \$1 all at once (variance is a measure of risk),

$$Var(4X) = 16Var(X)$$
$$= 16\sigma^2$$

The expected return of betting 25 cents four times,

$$E(X_1 + X_2 + X_3 + X_4) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$$

2

$$= \mu + \mu + \mu + \mu$$
$$= 4\mu$$

and the risk of betting 25 cents four times,

$$\begin{aligned}
Var(X_1 + X_2 + X_3 + X_4) &= Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) \\
&\quad + cov(X_1, X_2) + cov(X_1, X_3) + \cdots + cov(\ldots, \ldots) \\
&= \sigma^2 + \sigma^2 + \sigma^2 + \sigma^2 + 0 + 0 + \cdots + 0 \\
&= 4\sigma^2
\end{aligned}$$

Farshid's Mum is right! Both strategies have the same expected return but betting $1 has a much larger risk.

Question 3

Suppose we are interested in estimating the mean of a random variable X,

$$population\ mean\ of\ X = \mu$$

$$we\ want\ to\ estimate\ the\ population\ mean\ of\ X$$

We have two estimators for the mean,

- Estimator 1 ($\tilde{\mu}$): The formula of Estimator 1 is given by,

$$\tilde{\mu} = X$$

  X is the random variable and this estimator takes one observation from the random variable X and uses this as the estimate of μ.

- Estimator 2 ($\hat{\mu}$): The formula of Estimator 2 is given by,

$$\hat{\mu} = \frac{X_1 + X_2 + X_3 + X_4}{4} = \bar{X}$$

  It takes random sample of 4 observations from the random variable X, then averages them and uses this as the estimate of μ. A random sample of 4 observations can be denote by $\{X_1, X_2, X_3, X_4\}$.

What is the expected value of each of these estimators and which one is safer (i.e. less risky)? Discuss the similarity of this to Farshid's Mum's Theorem.

The expected value and variance of Estimator 1,

$$E(\tilde{\mu}) = E(X) = \mu$$
$$Var(\tilde{\mu}) = Var(X) = \sigma^2$$

The expected value and variance of Estimator 2,

$$
\begin{aligned}
E(\hat{\mu}) &= E\left(\frac{X_1 + X_2 + X_3 + X_4}{4}\right) \\
&= \frac{1}{4}E(X_1 + X_2 + X_3 + X_4) \\
&= \frac{1}{4}E(X_1) + E(X_2) + E(X_3) + E(X_4) \\
&= \frac{1}{4} \times 4\mu
\end{aligned}
$$

$$= \mu$$

$$Var(\hat{\mu}) = Var(\bar{X})$$

$$= Var\left(\frac{X_1 + X_2 + X_3 + X_4}{4}\right)$$

$$= \frac{1}{16}Var(X_1 + X_2 + X_3 + X_4)$$

$$= \frac{1}{16}[Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4)$$

$$+ cov(X_1, X_2) + cov(X_1, X_3) + \cdots + cov(\ldots, \ldots)]$$

$$= \frac{1}{16}[Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) + 0 + 0 + \cdots + 0]$$

$$= \frac{1}{16}[\sigma^2 + \sigma^2 + \sigma^2 + \sigma^2]$$

$$= \frac{\sigma^2}{4}$$

Both estimators have the same expected value $E(X) = E(\bar{X}) = \mu$ but the 2nd estimator (average of 4 observations) is less risky $Var(X) = \sigma^2 > Var(\bar{X}) = \frac{\sigma^2}{4}$.

This is very similar to Farshid's Mum's Theorem. There we were comparing 4X and $(X_1 + X_2 + X_3 + X_4)$, here we are comparing X and $\frac{1}{4}(X_1 + X_2 + X_3 + X_4)$. Same principle: it is safer to diversity and not depend only on a single draw from the distribution.

Question 4

<u>Diversification in finance</u>

Suppose we have invested 60% of our money in shares of company 1 which have mean return of 10% and standard deviation of 20% and the remaining 40% in share of company 2 that have mean return of 15% and standard deviation of 25%. The correlation coefficient between these returns is -0.1. Compute the mean and the standard deviation of this portfolio. Show your work.

Let,

$$X : portfolio\ return$$
$$C1 : company\ 1's\ return$$
$$C2 : company\ 2's\ return$$

Therefore our portfolio return is given by,

$$X = 0.6C1 + 0.4C2$$

The expected value and standard deviation of our portfolio return,

$$
\begin{aligned}
E(X) &= E(0.6C1 + 0.4C2) \\
&= 0.6E(C1) + 0.4E(C2) \\
&= 0.6 \times 10 + 0.4 \times 15 \\
&= 12\% \\
Var(X) &= Var(0.6C1 + 0.4C2) \\
&= 0.6^2 Var(C1) + 0.4^2 Var(C2) + 2 \times 0.6 \times 0.4[Cov(C1, C2)] \\
&= 0.6^2 Var(C1) + 0.4^2 Var(C2) + 2 \times 0.6 \times 0.4[Corr(C1, C2) \times SD(C1) \times SD(C2)] \\
&= 0.6^2 20^2 + 0.4^2 25^2 + 2 \times 0.6 \times 0.4[\times(-0.1) \times 20 \times 25] \\
&= 0.6^2 20^2 + 0.4^2 25^2 + 2 \times 0.6 \times 0.4[-50] \\
&= 220 \\
\implies SD(X) &= \sqrt{Var(X)} = \sqrt{220} = 14.8\%
\end{aligned}
$$

This portfolio has a higher expected return and a smaller risk than holding the shares of C1.

Question 5

Vectors and matrices and how they make our lives simpler

An investment portfolio is an weighted average of assets that we have invested in. For example, suppose we have invested:

- 10% of our savings in Qantas shares

- 40% in BHP shares

- 20% in Telstra shares

- 30% in Wesfarmers shares

The returns of these shares are random variables. Let's denote each of these returns by the first letter of the company name. If we denote the return to our portfolio by X, we can write:

$$X = 0.10Q + 0.40B + 0.20T + 0.30W$$

Of course this is only a simple made up example. In reality, the portfolios that investment managers manage include a large number of assets. We can use vectors to show this portfolio in a simple way, and more importantly to use vector arithmetic to compute its return. Consider the following two vectors:

$$\mathbf{p}_{4\times 1} = \begin{bmatrix} 0.1 \\ 0.4 \\ 0.2 \\ 0.3 \end{bmatrix}, \mathbf{z}_{4\times 1} = \begin{bmatrix} Q \\ B \\ T \\ W \end{bmatrix}$$

- $\mathbf{p}$ is the vector of portfolio weights

- $\mathbf{z}$ is the vector of assets

Using the rules of vector multiplication we can write:

$$X = \mathbf{p}'\mathbf{z}$$

where $\mathbf{p}' = \begin{bmatrix} 0.1 & 0.4 & 0.2 & 0.3 \end{bmatrix}$, is the transpose of the vector $\mathbf{p}$. Since $\mathbf{p}$ is a vector of constants and $\mathbf{z}$ is a vector of random variables, we have:

$$E(X) = \mathbf{p}'E(\mathbf{z})$$

So, if we know the mean return of the assets in the portfolio (or we can estimate the mean return from data), then we can arrange them in the vector $E(\mathbf{z})$ and then get an estimate of the mean return to this portfolio by vector multiplication. This part does not save us much time, because calculating

$$E(X) = 0.10E(Q) + 0.40E(B) + 0.20E(T) + 0.30E(W)$$

does not take much time even with a hand calculator. However, mean return is not the only parameter that we are interested in when investing. We want to know the variance of the portfolio, which measures its risk. For a single random variable y and a constant c we know that

$$Var(cy) = c^2 Var(y)$$

For a 4×1 vector random variable such as $\mathbf{z}$, first we need to form its 4×4 variance matrix (sometimes called the variance-covariance matrix):

$$\underset{4\times 4}{Var(\mathbf{z})} = \begin{bmatrix} Var(Q) & Cov(Q,B) & Cov(Q,T) & Cov(Q,W) \\ Cov(Q,B) & Var(B) & Cov(B,T) & Cov(B,W) \\ Cov(Q,T) & Cov(B,T) & Var(T) & Cov(T,W) \\ Cov(Q,W) & Cov(B,W) & Cov(T,W) & Var(W) \end{bmatrix}$$

Note that the diagonal elements of this matrix are variances of each asset return, and the off diagonal elements are covariances between each pair of asset returns. This matrix is symmetric (a symmetric matrix is a square matrix that is equal to its transpose). Then, we have:

$$Var(X) = Var(\mathbf{p'z}) = \mathbf{p'} Var(\mathbf{z}) \mathbf{p}$$

These two formulae for the expected return and variance of return to a portfolio is very easy for a computer to compute. The investment manager can then change the portfolio weights and recompute these to find a portfolio with the highest expected return for a given level of risk, or find a portfolio with the lowest risk for a given expected return. For practice, re-do question 4 by forming the variance-covariance matrix of the two asset returns and then computing the expected return and variance of the return to the portfolios using $\mathbf{p'}E(\mathbf{z})$ and $\mathbf{p'}Var(\mathbf{z})\mathbf{p}$.