

Tutorial 6

keywords: hypothesis testing, t-test, F-test, test statistic, critical value, confidence intervals, R-squared, interpretation of coefficients, multiple linear regression, reparameterisation

estimated reading time: 33 minutes

Quang Bui

August 28, 2018

Question 1

Hypothesis test on a single parameter, the meanings of the size of a test and a confidence interval:

Consider the classical linear regression model

$$y_i = \beta_0 + \beta_1 x_i + u_i \quad i = 1, 2, \dots, n$$

A random sample of size $n = 22$ is drawn and the estimated model based on this sample is:

$$\hat{y}_i = \underset{(3.1)}{5.4} + \underset{(1.5)}{3.2} x_i \quad i = 1, 2, \dots, 22$$
$$R^2 = 0.26$$

(a) Test $H_0 : \beta_1 = 0$ vs $H_1 : \beta_1 \neq 0$ at the 5% significance level.

For the simple linear regression model of y ,

$$y_i = \beta_0 + \beta_1 x_i + u_i \quad i = 1, 2, \dots, 22$$

β_1 measures the true impact that x has on y (not holding any variable(s) constant because there are no other independent variables other than x in this model).

If x does not truly impact y then,

$$\beta_1 = 0$$

but if it does,

$$\beta_1 \neq 0$$

After we estimate our model,

$$\hat{y}_i = \underset{(3.1)}{5.4} + \underset{(1.5)}{3.2} x_i \quad i = 1, 2, \dots, 22$$

we can perform this hypothesis test.

State the null and alternative hypothesis

The test statistic and its distribution under H_0

Calculate the test statistic Critical value and rejection region

5% significance level $\therefore \alpha = 0.05$, two-sided t-test

Since we are performing a t-test, the critical value(s) (which bounds the rejection region) come from a t-distribution. The t-distribution of interest in this hypothesis test, is one with *degrees of freedom* $= n - k - 1 = 22 - 2 = 20$.

TABLE G.2 Critical Values of the t Distribution						
		Significance Level				
1-Tailed:		.10	.05	.025	.01	.005
2-Tailed:		.20	.10	.05	.02	.01
D e g r e e s o f F r e e d o m	1	3.078	6.314	12.706	31.821	63.657
	2	1.886	2.920	4.303	6.965	9.925
	3	1.638	2.353	3.182	4.541	5.841
	4	1.533	2.132	2.776	3.747	4.604
	5	1.476	2.015	2.571	3.365	4.032
	6	1.440	1.943	2.447	3.143	3.707
	7	1.415	1.895	2.365	2.998	3.499
	8	1.397	1.860	2.306	2.896	3.355
	9	1.383	1.833	2.262	2.821	3.250
	10	1.372	1.812	2.228	2.764	3.169
	11	1.363	1.796	2.201	2.718	3.106
	12	1.356	1.782	2.179	2.681	3.055
	13	1.350	1.771	2.160	2.650	3.012
	14	1.345	1.761	2.145	2.624	2.977
	15	1.341	1.753	2.131	2.602	2.947
	16	1.337	1.746	2.120	2.583	2.921
	17	1.333	1.740	2.110	2.567	2.898
	18	1.330	1.734	2.101	2.552	2.878
	19	1.328	1.729	2.093	2.539	2.861
	20	1.325	1.725	2.086	2.528	2.845
	21	1.323	1.721	2.080	2.518	2.831
	22	1.321	1.717	2.074	2.508	2.819
	23	1.319	1.714	2.069	2.500	2.807
	24	1.318	1.711	2.064	2.492	2.797
	25	1.316	1.708	2.060	2.485	2.787
	26	1.315	1.706	2.056	2.479	2.779
	27	1.314	1.703	2.052	2.473	2.771
	28	1.313	1.701	2.048	2.467	2.763
	29	1.311	1.699	2.045	2.462	2.756
	30	1.310	1.697	2.042	2.457	2.750
	40	1.303	1.684	2.021	2.423	2.704
	60	1.296	1.671	2.000	2.390	2.660
	90	1.291	1.662	1.987	2.368	2.632
	120	1.289	1.658	1.980	2.358	2.617
	∞	1.282	1.645	1.960	2.326	2.576

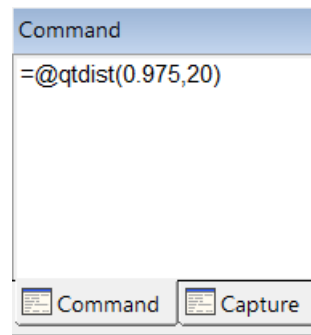
(Note: If the degrees of freedom cannot be found in the statistics table, we take a conservative approach and choose the closest available degrees of freedom less than what we need.) From the statistics table:

$$+t_{crit} = t_{20,0.975} = 2.086$$

$$-t_{crit} = t_{20,0.025} = -2.086$$

To obtain the critical value using EViews,

Command window : = @qtdist(0.975, 20)



(press Enter to execute)

and the value appears in the bottom left corner,

Scalar = 2.08596344727

From EViews:

$$\begin{aligned} +t_{crit} &= t_{20,0.975} = 2.806 \\ -t_{crit} &= t_{20,0.025} = -2.806 \end{aligned}$$

For a two-sided t-test, we reject H_0 if,

$$t_{calc} > +t_{crit}$$

or

$$t_{calc} < -t_{crit}$$

Conclusion

(b) Construct a 95% confidence interval for β_1

$$\hat{\beta}_1 \pm t_{crit} \times se(\hat{\beta}_1)$$

(c) Suppose that you learn that y and x are independent. Would you be surprised? Explain.

If y and x are independent then x should not effect y i.e. $\beta_1 = 0$, but we rejected

$$H_0 : \beta_1 = 0$$

and concluded that x has a statistically significant impact on y at the 5% significance level. Setting our significance level at 5%

$$\alpha = 0.05$$

means that there is only a 5% chance of rejecting the null when it is true \therefore we would be surprised.

Concretely, we have evidence from this sample to reject the null hypothesis that x has no impact on y at the 5% significance level. We could have wrongly rejected H_0 when it is true (Type I error), but since we performed this test at the 5% significance level, this error occurs with only a 5% probability,

$$P(\text{Type I error}) = \alpha = 0.05$$

Given that y and x are independent $\therefore x$ has no true impact on y i.e. $\beta_1 = 0$, if we applied repeated sampling and performed the same hypothesis test across many more samples, we would find that H_0 is wrongly rejected in 5% of these samples.

(d) Suppose that you learn that y and x are independent and many samples of $n = 22$ are drawn, regressions estimated, and (a) and (b) answered. In what fraction of the samples would H_0 from (a) be rejected? In what fraction of samples would the value $\beta_1 = 0$ be included in the confidence interval from (b)?

The significance level, α , is the probability of rejecting the null hypothesis when it is true. Since we set $\alpha = 0.05$, this means there is a 5% chance that we reject $H_0 : \beta_1 = 0$ when it is true. Therefore, we would reject $H_0 : \beta_1 = 0$ in 5% of the samples.

The true value of β_1 will lie in 95% of the confidence intervals. If we learn that y and x are independent, then the true value of β_1 is 0, so $\beta_1 = 0$ would lie in 95% of the confidence intervals.

Question 2

Practice with t-test and F-test: (This is based on problem 3 at the end of Chapter 3 of the textbook): The following multiple regression model was used to study the trade-off between time spent sleeping and working and to look at other factors affecting sleep:

$$sleep = \beta_0 + \beta_1 totwrk + \beta_2 educ + \beta_3 age + u$$

where *sleep* and *totwrk* are measured in minutes per week and *educ* and *age* are measured in years.

(a) If adults trade-off sleep for work, what is the sign of β_1 ?

Keeping education and age constant, if an adult who trades off sleep for work, we would expect β_1 to be negative.

(b) What signs do you think β_2 and β_3 will have?

β_2 measures the impact of education on time spent sleeping after controlling for years of education and age. So, if we take 2 individuals of the same age and who both work the same number of hours each week, would we expect the more educated individual to sleep more or less? If we consider that less educated people perhaps work more physical hours and need more sleep to recover, then we could say that β_2 has a negative sign.

β_3 measures the impact of age on time spent sleeping after controlling for hours spending working each week and years of education. So, if we take 2 individuals with the same years of education and who both work the same number of hours each week, would we expect the older individual to sleep more or less? This is hard to say. All else equal, older people generally need more sleep, but we also know that university students sleep more than 25-35 year olds.

(c) Using data from a random sample of 706 adults, we have estimated the following equation:

$$\widehat{sleep} = 3638.25 - 0.148totwrk - 11.13educ + 2.20age \quad (1)$$

(112.27) (0.017) (5.88) (1.45)

$$R^2 = 0.113 \quad SSR = 123455057$$

Test the hypothesis that adults do not trade-off sleep for work against the alternative that they do at the 1% level of significance.

If adults do not trade-off sleep for work then,

$$\beta_1 = 0$$

We test this against the alternative hypothesis that they do trade sleep for work,

$$\beta_1 < 0$$

Here we are performed a 1-sided t-test at the 1% level of significance.

State the null and alternative hypothesis

The test statistic and its distribution under H_0

Calculate the test statistic

Critical value and rejection region

1% *significance level* $\therefore \alpha = 0.01$, one-sided t-test (one the left side because $H_1 : \beta_1 < 0$)

Since we are performing a t-test, the critical value(s) (which bounds the rejection region) come from a t-distribution. The t-distribution of interest in this hypothesis test, is one with *degrees of freedom* $= n - k - 1 = 706 - 3 - 1 = 702$. Since $d.o.f = 702$ is not in the statistics table, we take a conservative approach and choose the closest available degrees of freedom less than 702 i.e. $d.o.f = 120$.

TABLE G.2 Critical Values of the <i>t</i> Distribution						
		Significance Level				
1-Tailed:		.10	.05	.025	.01	.005
2-Tailed:		.20	.10	.05	.02	.01
D e g r e e s o f F r e e d o m	1	3.078	6.314	12.706	31.821	63.657
	2	1.886	2.920	4.303	6.965	9.925
	3	1.638	2.353	3.182	4.541	5.841
	4	1.533	2.132	2.776	3.747	4.604
	5	1.476	2.015	2.571	3.365	4.032
	6	1.440	1.943	2.447	3.143	3.707
	7	1.415	1.895	2.365	2.998	3.499
	8	1.397	1.860	2.306	2.896	3.355
	9	1.383	1.833	2.262	2.821	3.250
	10	1.372	1.812	2.228	2.764	3.169
	11	1.363	1.796	2.201	2.718	3.106
	12	1.356	1.782	2.179	2.681	3.055
	13	1.350	1.771	2.160	2.650	3.012
	14	1.345	1.761	2.145	2.624	2.977
	15	1.341	1.753	2.131	2.602	2.947
	16	1.337	1.746	2.120	2.583	2.921
	17	1.333	1.740	2.110	2.567	2.898
	18	1.330	1.734	2.101	2.552	2.878
	19	1.328	1.729	2.093	2.539	2.861
	20	1.325	1.725	2.086	2.528	2.845
	21	1.323	1.721	2.080	2.518	2.831
	22	1.321	1.717	2.074	2.508	2.819
	23	1.319	1.714	2.069	2.500	2.807
	24	1.318	1.711	2.064	2.492	2.797
	25	1.316	1.708	2.060	2.485	2.787
	26	1.315	1.706	2.056	2.479	2.779
	27	1.314	1.703	2.052	2.473	2.771
	28	1.313	1.701	2.048	2.467	2.763
	29	1.311	1.699	2.045	2.462	2.756
	30	1.310	1.697	2.042	2.457	2.750
	40	1.303	1.684	2.021	2.423	2.704
	60	1.296	1.671	2.000	2.390	2.660
	90	1.291	1.662	1.987	2.368	2.632
	120	1.289	1.658	1.980	2.358	2.617
	∞	1.282	1.645	1.960	2.326	2.576

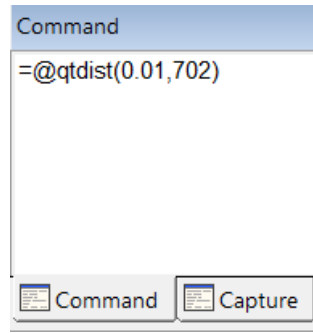
From the statistics table:

$$-t_{crit} = -2.358$$

(We are only concerned with the negative critical value because we are performing a one-sided t-test on the left side.)

To obtain the critical value using EViews,

$$\text{Command window : } = @qtdist(0.01, 702)$$



(press *Enter* to execute)

and the value appears in the bottom left corner,

Scalar = -2.33167157159

From EViews:

$$+t_{crit} = t_{702,0.01} = -2.3317$$

For a one-sided t-test on the negative side, we reject H_0 if,

$$t_{calc} < -t_{crit}$$

Conclusion

(d) Construct a 95% confidence interval for β_3 . Interpret the confidence interval.

$$\begin{aligned} & \hat{\beta}_3 \pm t_{crit} \times se(\hat{\beta}_3) \\ & \hat{\beta}_3 \pm t_{n-k-1, 1-\frac{\alpha}{2}} \times se(\hat{\beta}_3) \\ & \hat{\beta}_3 \pm t_{702, 0.975} \times se(\hat{\beta}_3) \\ & 2.20 \pm 1.980 \times 1.45 \\ & [-0.671, 5.071] \end{aligned}$$

In repeated samples, 95% of the sample will contain β_3 . The range from our confidence interval is an *interval estimate* of β_3 . Since the interval estimate includes 0, this means that *age* is not a statistically significant predictor of sleep time after controlling for work and education. ($\hat{\beta}_3$ is not statistically significantly different from 0 at the 5% significance level.)

(e) We have also estimated the following regression:

$$\widehat{sleep} = 3586.38 - 0.151totwrk \quad (2)$$

(38.91) (0.017)

$$SSR = 124858119$$

Test the joint hypothesis given work time, education & age have no effect on sleep time versus the alternative that at least one of them does (when the others are constant). Perform this test at the 5% level of significance.

If, after controlling for *work*, *educ* & *age* jointly have no effect on sleep time then,

$$sleep = \beta_0 + \beta_1 totwrk + \cancel{\beta_2 educ} + \cancel{\beta_3 age} + u$$

$$\beta_2 = 0, \beta_3 = 0$$

but if at least one of *educ* and *age* does effect sleep then,

$$\beta_2 \neq 0 \text{ and/or } \beta_3 \neq 0$$

Unlike the t-test, which testings a single linear restriction, the F-test is used when we wish to perform a hypothesis test of multiple linear restrictions. Since we have 2 linear restrictions to test, we use the F-test.

The null and alternative hypothesis are stated as,

$$H_0 : \beta_2 = \beta_3 = 0$$

$$H_1 : \text{at least one of } \beta_2 \text{ or } \beta_3 \text{ does not equal } 0$$

The model when H_0 is true i.e without *educ* and *age*,

$$sleep = \beta_0 + \beta_1 totwrk + u$$

is called the *restricted model* because it imposes the 2 linear restrictions that $\beta_2 = \beta_3 = 0$, while the model without any restrictions imposed,

$$sleep = \beta_0 + \beta_1 totwrk + \beta_2 educ + \beta_3 age + u$$

is called the *unrestricted model*.

Unrestricted model (the model before imposing restrictions):

$$sleep = \beta_0 + \beta_1 totwrk + \beta_2 educ + \beta_3 age + u$$

Restricted model (the model after imposing restrictions):

$$sleep = \beta_0 + \beta_1 totwrk + u$$

State the null and alternative hypothesis

Calculate the test statistic

Critical value and rejection region

5% *significance level* $\rightarrow \alpha = 0.05$

To obtain the critical value using the Stats Table, locate the F distribution table at the 5% significance level,

$$Numerator\ d.o.f = 2$$

$$Denominator\ d.o.f = 702$$

Since 702 is not in the table, we take a conservative approach and choose the closest available degrees of freedom less than 702 i.e. $d.o.f = 120$.

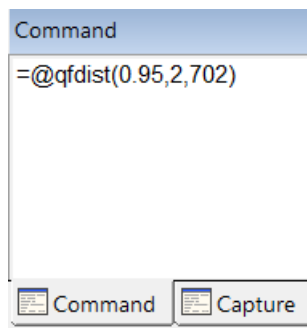
TABLE G.3b 5% Critical Values of the F Distribution											
		Numerator Degrees of Freedom									
		1	2	3	4	5	6	7	8	9	10
D e n o m i n a t o r D e g r e e s o f F r e e d o m	10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
	11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
	12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75
	13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67
	14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60
	15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54
	16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49
	17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45
	18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41
	19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38
	20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35
	21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32
	22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30
	23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27
	24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25
	25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24
	26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22
	27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20
	28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19
	29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18
	30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16
	40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08
	60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99
	90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94
	120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91
	∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83

Example: The 5% critical value for numerator $df = 4$ and large denominator $df(\infty)$ is 2.37.

Source: This table was generated using the Stata® function invFtail.

To obtain the critical value using EViews,

Command window := @qfdist(0.95,2,702)



Press Enter to execute and the cv appears at the bottom left corner

$$\text{Scalar} = 3.00855278378$$

$$F_{crit} (\text{from Stat Table}) = 3.07$$

$$F_{crit} (\text{from EViews}) = 3.0086$$

Rejection rule:

Comparing the calculated test statistic with the critical value, we reject H_0 if,

$$F_{calc} > F_{crit}$$

Conclusion

(f) Compute the R^2 of regression (2).

(Discuss in class)

(g) Suppose that someone suggests that one year of education keeping all else constant has the same effect but with opposite sign of the effect of one more year of age keeping all else constant. That is, $\beta_2 = -\beta_3$. Explain how you would test this hypothesis with an F-test. You need to state the alternative hypothesis that can be tested with an F-test, specify any extra regression that you need to estimate, and explain how you would use the results of that regression to test this hypothesis.

(Discuss in class)

(h) Suppose the alternative hypothesis of interest was $\beta_2 < -\beta_3$. Explain how you would test $H_0 : \beta_2 = -\beta_3$ against this one-sided alternative.

The null hypothesis is,

$$H_0 : \beta_2 + \beta_3 = 0$$

and although it involves two parameters, it tests only one restriction. The alternative is,

$$H_1 : \beta_2 + \beta_3 < 0$$

so we cannot use the F test because the F test provides inference against $\beta_2 + \beta_3 \neq 0$. In such cases that we have only one restriction about a linear combination, we use a reparameterisation trick:

$$\text{Define } \delta = \beta_2 + \beta_3 \implies \beta_2 = \delta - \beta_3$$

Substitute for β_2 in the population model and re-arrange,

$$\text{sleep} = \beta_0 + \beta_1 \text{totwrk} + \beta_2 \text{educ} + \beta_3 \text{age} + u$$

$$\begin{aligned}
&= \beta_0 + \beta_1 totwrk + (\delta - \beta_3)educ + \beta_3 age + u \\
&= \beta_0 + \beta_1 totwrk + \delta educ - \beta_3 educ + \beta_3 age + u \\
&= \beta_0 + \beta_1 totwrk + \delta educ + \beta_3 (age - educ) + u
\end{aligned}$$

you will see that δ becomes the coefficient of one of the explanatory variables in the reparameterised model.

Dependent Variable: SLEEP
Method: Least Squares
Date: 08/27/18 Time: 22:01
Sample: 1 706
Included observations: 706

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	3638.245	112.2751	32.40474	0.0000
TOTWRK	-0.148373	0.016694	-8.888075	0.0000
EDUC	-8.933928	6.420423	-1.391486	0.1645
AGE-EDUC	2.199885	1.445717	1.521657	0.1285
R-squared	0.113364	Mean dependent var	3266.356	
Adjusted R-squared	0.109575	S.D. dependent var	444.4134	
S.E. of regression	419.3589	Akaike info criterion	14.92098	
Sum squared resid	1.23E+08	Schwarz criterion	14.94681	
Log likelihood	-5263.106	Hannan-Quinn criter.	14.93096	
F-statistic	29.91889	Durbin-Watson stat	1.942609	
Prob(F-statistic)	0.000000			

$$\widehat{sleep} = \hat{\beta}_0 + \hat{\beta}_1 totwrk + \hat{\delta} educ + \hat{\beta}_3 (age - educ)$$

where

$$\hat{\delta} = -8.9339 \quad se(\hat{\delta}) = 6.4204$$

You can see that testing $\delta = 0$ against $\delta < 0$ can be performed with a simple t test in this reparameterised model,

$$H_0 : \delta = 0 \quad (\text{same as } \beta_2 + \beta_3 = 0)$$

$$H_1 : \delta < 0 \quad (\text{same as } \beta_2 + \beta_3 < 0)$$

(one – sided t test)

The test statistic and its distribution under H_0

$$t = \frac{\hat{\delta} - \delta}{se(\hat{\delta})} = \frac{\hat{\delta}}{se(\hat{\delta})} \sim t_{n-k-1} \quad \text{under } H_0$$

$$n = \text{sample size} = 706$$

$$k = \text{number of regressors in the model} = 3$$

Calculate the test statistic

$$t_{calc} = -\frac{8.9339}{6.4204} = -1.3915$$

Critical value and rejection region

5% *significance level* $\rightarrow \alpha = 0.05$

$$\text{degrees of freedom} = n - k - 1 = 702$$

Since 702 is not in the table, we take a conservative approach and choose the closest available degrees of freedom less than 702 i.e. $d.o.f = 120$.

Note: This gives us $+t_{crit}$ but we need $-t_{crit}$ since we are performing a one side test on the negative side.

From Stat Table:

$$-t_{crit} = -1.658$$

Rejection rule:

Comparing the calculated test statistic with the critical value, we reject H_0 if,

$$t_{calc} < -t_{crit}$$

Comparing the p-value with the significance level, we reject H_0 if,

$$\frac{p - value}{2} < \alpha = 0.05$$

(divide p value by 2 for one sided t test)

Conclusion

Since $t_{calc} = -1.3915 > -t_{crit} = -1.658$, we do not reject the null at the 5% significance level and conclude that there is insufficient evidence from our sample to suggest that $\beta_2 + \beta_3 < 0$.