ETC2410 Tutorial 12

2018 Sem 2

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Question 1.

Assume that $\{y_t\}$ is a stationary $AR(1)$ process given by,

$$y_t = c + \varphi_1 y_{t-1} + e_t \quad (1)$$

where,

$$e_t \sim i.i.d(0, \sigma^2)$$

(a) Show that,

$$E(y_t) = \mu = \frac{c}{(1-\varphi_1)} \quad \forall t \quad (2)$$

$$\forall t: for\ all\ t$$

Since $\{y_t\}$ is stationary this implies,

$$E(y_t) = \mu \qquad for\ all\ t$$

$$Var(y_t) = \gamma_0 \qquad for\ all\ t$$

$$Cov(y_t, y_{t-j}) = \gamma_j \qquad for\ all\ t\ and\ j \neq 0$$

$$(\mu\ and\ \gamma_0\ are\ constants\ and\ \gamma_j\ depends\ only\ only\ on\ j)$$

Therefore taking the expectation of (1) we get,

$$E(y_t) = E(c + \varphi_1 y_{t-1} + e_t)$$

$$Expectation\ of\ Sum = Sum\ of\ Expectation$$

$$\therefore E(y_t) = E(c) + E(\varphi_1 y_{t-1}) + E(e_t)$$

since $e_t \sim i.i.d(0, \sigma^2)$, $E(e_t) = 0$,

$$\therefore E(y_t) = E(c) + E(\varphi_1 y_{t-1})$$

$c\ and\ \varphi_1$ are constants so we can take it out of the expectation operator,

$$E(y_t) = c + \varphi_1 E(y_{t-1})$$

$\{y_t\}$ is stationary $\therefore E(y_t) = E(y_{t-1}) = \mu$,

$$\therefore \mu = c + \varphi_1 \mu$$

$$\mu - \varphi_1 \mu = c$$

$$\mu(1 - \varphi_1) = c$$

$$\mu = \frac{c}{(1-\varphi_1)}$$

This means that if our $AR(1)$ model of y_t does not have an intercept,

$$c = 0$$

then the $E(y_t)$ will equal to 0,

$$E(y_t) = \mu = \frac{0}{(1-\varphi_1)} = 0$$

This results generalises for the cases of an $AR(p)$ model.

(b) Show that,

$$Var(y_t) = \gamma_0 = \frac{\sigma^2}{(1-\varphi_1^2)} \quad \forall t \qquad (3)$$

Taking the variance of (1) we get,

$$Var(y_t) = Var(c + \varphi_1 y_{t-1} + e_t)$$

c is a constant so we can remove it from the variance operator,

$$Var(y_t) = Var(\varphi_1 y_{t-1} + e_t)$$

we can expand $Var(y_t)$ using the variance formula,

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

$$(where\ a\ and\ b\ are\ constants)$$

Expand $Var(y_t)$,

$$Var(y_t) = \varphi_1^2 Var(y_{t-1}) + Var(e_t) + 2\varphi_1 Cov(y_{t-1}, e_t)$$

$$(where\ \varphi_1\ is\ a\ constant)$$

from $e_t \sim i.i.d(0, \sigma^2)$, $Var(e_t) = \sigma^2$,

$$Var(y_t) = \varphi_1^2 Var(y_{t-1}) + \sigma^2 + 2\varphi_1 Cov(y_{t-1}, e_t)$$

$\{y_t\}$ is stationary, $\therefore Var(y_t) = Var(y_{t-1}) = \gamma_0$,

$$\gamma_0 = \varphi_1^2 \gamma_0 + \sigma^2 + 2\varphi_1 Cov(y_{t-1}, e_t)$$

e_t is uncorrelated with regressor y_{t-1},

$$Cov(y_{t-1}, e_t) = 0$$

Therefore,

$$\begin{aligned}
\gamma_0 &= \varphi_1^2\gamma_0 + \sigma^2 + \cancel{2\varphi_1 Cov(y_{t-1}, e_t)} \\
\gamma_0 - \varphi_1^2\gamma_0 &= \sigma^2 \\
\gamma_0(1-\varphi_1^2) &= \sigma^2 \\
\gamma_0 &= \frac{\sigma^2}{(1-\varphi_1^2)}
\end{aligned}$$

(c) Show that (1) can be written in mean deviation form as,

$$y_t - \mu = \varphi_1(y_{t-1} - \mu) + e_t \quad (4)$$

From (a),

$$c = \mu(1-\varphi_1)$$

Through substitution the AR(1) model can be rewritten as,

$$\begin{aligned}
y_t &= c + \varphi_1 y_{t-1} + e_t \\
y_t &= \mu(1-\varphi_1) + \varphi_1 y_{t-1} + e_t \\
y_t &= \mu - \mu\varphi_1 + \varphi_1 y_{t-1} + e_t \\
y_t - \mu &= \varphi_1(y_{t-1} - \mu) + e_t
\end{aligned}$$

(d) Use (4) to show that,

$$\gamma_1 = Cov(y_t, y_{t-1}) = \frac{\sigma^2}{(1-\varphi_i^2)}\varphi_1 \quad (5)$$

$$\gamma_1 = Cov(y_t, y_{t-1}) = E\big[\big(y_t - E(y_t)\big)\big(y_{t-1} - E(y_{t-1})\big)\big] = E[(y_t - \mu)(y_{t-1} - \mu)]$$

Multiplying both sides of (4) by $(y_{t-1} - \mu)$,

$$(y_t - \mu)(y_{t-1} - \mu) = (\varphi_1(y_{t-1} - \mu) + e_t)(y_{t-1} - \mu)$$

Taking the expectation on both side,

$$\begin{aligned}
E[(y_t - \mu)(y_{t-1} - \mu)] &= E[(\varphi_1(y_{t-1} - \mu) + e_t)(y_{t-1} - \mu)] \\
&= E(\varphi_1(y_{t-1} - \mu)^2) + E\big(e_t(y_{t-1} - \mu)\big) \\
&= \varphi_1 E((y_{t-1} - \mu)^2) + E\big(e_t(y_{t-1} - \mu)\big)
\end{aligned}$$

$\{y_t\}$ is stationary, $\therefore Var(y_t) = Var(y_{t-1}) = E\left(\big(y_{t-1} - E(y_{t-1})\big)^2\right) = E((y_{t-1} - \mu)^2) = \gamma_0$

$$\begin{aligned}
E[(y_t - \mu)(y_{t-1} - \mu)] &= \varphi_1\gamma_0 + E\big(e_t(y_{t-1} - \mu)\big) \\
&= \varphi_1\gamma_0 + E(y_{t-1}e_t - \mu e_t) \\
&= \varphi_1\gamma_0 + E(y_{t-1}e_t) - E(y_{t-1})E(e_t) \\
&= \varphi_1\gamma_0 + Cov(y_{t-1}, e_t) \\
&= \varphi_1\gamma_0 + 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma_1 &= Cov(y_t, y_{t-1}) \\
&= E[(y_t - \mu)(y_{t-1} - \mu)]
\end{aligned}$$

$$
\begin{aligned}
&= \varphi_1 \gamma_0 \\
&= \varphi_1 \frac{\sigma^2}{(1-\varphi_1^2)}
\end{aligned}
$$

(e) Show that,

$$\rho_1 = Corr(y_t, y_{t-1}) = \varphi_1 \quad (6)$$

$$
\begin{aligned}
\rho_1 &= Corr(y_t, y_{t-1}) \\
&= \frac{Cov(y_t, y_{t-1})}{\sqrt{Var(y_t)}\sqrt{Var(y_{t-1})}} \\
&= \frac{Cov(y_t, y_{t-1})}{\sqrt{Var(y_t)}\sqrt{Var(y_t)}} \\
&= \frac{Cov(y_t, y_{t-1})}{Var(y_t)} \\
&= \frac{\gamma_1}{\gamma_0} \\
&= \frac{\varphi_1 \gamma_0}{\gamma_0} \\
&= \varphi_1
\end{aligned}
$$