Introductory Econometrics

Tutorial 3

<u>PART A:</u> To be done before you attend the tutorial. The solutions will be made available at the end of the week.
This homework is a review of statistical concepts of mean, variance, covariance and correlation.

1. If X is a discrete random variable and its possible values are $\{-2, -1, 0, 1, 2\}$, can we determine $E(X)$? If your answer is yes, compute $E(X)$. If your answer is no, what other information do we need to be able to determine $E(X)$?

2. X and Y are random variables. Answer the following questions:

   (a) If $E(X) = 3$ and $E(Y) = 4$, compute $E(2X - Y)$.

   (b) If $Var(X) = 4$ and $Var(Y) = 9$, and X and Y are independent, compute $Var(2X - Y)$.

3. According to an expert, the inflation rate and the cash rate (the interest rate set by the Reserve Bank of Australia) in Australia in the next quarter is governed by the following joint probability density function:

cash rate ↓ ╲ inflation rate ⟶	1%	2%	3%
1.25%	0.15	0.10	0.00
1.50%	0.15	0.30	0.05
1.75%	0.10	0.10	0.05

   According to this expert, what is the expected value of the cash rate next quarter, and what is the expected value of the inflation rate next quarter? What is the expected value of the inflation rate conditional on cash rate being 1.50%?

4. Go over the lesson called "Exercises on Elementary Matrix Algebra" that is under tutorial 3 on Moodle.

Do not forget to bring your answers to PART A and a copy of the tutorial questions to your tutorial.

PART B: You do not need to hand this part in. It will be covered in the tutorial. It is still a good idea to attempt these questions before the tutorial.

1. X and Y are random variables with mean μ_X and μ_Y respectively. The covariance between X and Y is defined as $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$. Show that:

$$Cov(X,Y) = E[(X-\mu_X)Y] = E[X(Y-\mu_Y)] = E(XY) - \mu_X\mu_Y.$$

   Discuss why we cannot simplify $E(XY) - \mu_X\mu_Y$ further to $E(X)E(Y) - \mu_X\mu_Y = \mu_X\mu_Y - \mu_X\mu_Y = 0$.

2. Diversification in everyday life: [A pokie machine is called a slot machine in America and a jackpot machine in some other countries. It is a random device that operates for a price, and draws an outcome randomly, and depending on the outcome, you may win different amounts of money (including zero). The probabilities are controlled such that the expected value of the monetary prize is slightly smaller than the price]. Most pokie machines give you the option of multiplying your bet up. For example, if the machine accepts 25 cents per round for having a go at winnings given by the random variable X, you have the option of paying one dollar to scale up your winnings to $4X$. Farshid's Mum, who is a 98 year old lady with only primary school education and loves pokie machines, told him, "people who scale their bets up are silly because they are at risk of running out of money faster." Suppose you have one dollar only. Compare the expected return and risk of using all of your money at once and betting $4X$, with using it for playing X four times (i.e. $X_1 + X_2 + X_3 + X_4$, where X_i are independent and have distribution identical to X). Do you agree with Farshid's Mum? Discuss.

3. Diversification in econometrics and statistics: Suppose we are interested in estimating the mean of a random variable. We can take a sample of one observation from the random variable and use that as the estimate of the mean, or we can take a sample of 4 observations and take the average of those 4 observations as the estimate of the mean. What is the expected value of each of these estimators and which one is safer (i.e. less risky)? Discuss the similarity of this to Farshid's Mum's Theorem.

4. Diversification in finance + vectors and matrices and how they make our lives simpler: An investment portfolio is a weighted average of assets that we have invested in. For example, suppose we have invested 20% of our savings in Qantas shares, 30% in Telstra shares and the rest (50%) in Wesfarmers shares. The return to these shares are random variables. Let's denote each of these returns by the first letter of the company name. If we denote the return to our portfolio by X, we can write:

$$X = 0.20Q + 0.30T + 0.50W.$$

   Of course this is only a simple made up example. In reality, the portfolios that investment managers manage include a large number of assets. We can use vectors to show this portfolio in a simple way, and more importantly to use vector arithmetic to compute its return. Consider the following two vectors:

$$\mathbf{p} = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.5 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} Q \\ T \\ W \end{pmatrix}.$$

   The first vector is the vector of portfolio weights and the second vector is the vector of assets. Then, using the rules of vector multiplication we can write:

$$X = \mathbf{p}'\mathbf{z}$$

where $\mathbf{p}' = \begin{pmatrix} 0.2 & 0.3 & 0.5 \end{pmatrix}$, is the transpose of the vector $\mathbf{p}$. Since $\mathbf{p}$ is a vector of constants and $\mathbf{z}$ is a vector of random variables, we have:

$$E(X) = \mathbf{p}'E(\mathbf{z})$$

So, if we know the mean return of the assets in the portfolio (or we can estimate the mean return from data), then we can arrange them in the vector $E(\mathbf{z})$ and then get an estimate of the mean return to this portfolio by vector multiplication. For instance, the estimates of mean (and standard deviation) of monthly returns for these three shares based on monthly observations in the last 8 years are given in the table below:

	Q	T	W
Mean	1.0	0.6	0.8
Std. Dev.	9.8	4.5	4.6

from which we can estimate the expected return for the portfolio given by the weight vector $\mathbf{p}$. This part does not save us much time, because calculating $E(X) = 0.20E(Q) + 0.30E(T) + 0.50E(W)$ does not take much time even with a hand calculator. However, mean return is not the only parameter that we are interested in when investing. We want to know the variance of the portfolio, which measures its risk. For a single random variable y and a constant c we know that $Var(cy) = c^2 Var(y)$. For a 3×1 vector random variable such as $\mathbf{z}$, first we need to form its 3×3 variance matrix (sometimes called the variance-covariance matrix):

$$Var(\mathbf{z}) = \begin{bmatrix} Var(Q) & Cov(Q,T) & Cov(Q,W) \\ Cov(Q,T) & Var(T) & Cov(T,W) \\ Cov(Q,W) & Cov(T,W) & Var(W) \end{bmatrix}.$$

Note that the diagonal elements of this matrix are variances of each asset return, and the off diagonal elements are covariances between each pair of asset returns. This matrix is symmetric. The estimated variance covariance matrix of returns based on last 8 years of observed returns is given below (all values are rounded to make life easier):

	Q	T	W
Q	94	1	8
T	1	20	5
W	8	5	21

In contrast with the case of a single random variable, when we have a vector of random variables such as $\mathbf{z}$ and we form $X = \mathbf{p}'\mathbf{z}$, which is a linear combination of elements of $\mathbf{z}$, we have:

$$Var(X) = Var(\mathbf{p}'\mathbf{z}) = \mathbf{p}'Var(\mathbf{z})\,\mathbf{p}.$$

Using the information provided above, compute the variance of the portfolio of Qantas, Telstra and Wesfarmers shares given by the portfolio weights $\mathbf{p}$. Compare the risk of this portfolio with the risk of each individual asset. [The practical importance of this exercise in addition to providing an example of benefit of diversification using real data is that the matrix based formulae for the expected return and variance of return to a portfolio are very easy to compute for a computer, even for portfolios of hundreds of assets. The investment manager can then change the portfolio weights and recompute these to find a portfolio with the highest expected return for a given level of risk, or to find a portfolio with the lowest risk for a given expected return.]