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Strong form:

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = -\begin{Bmatrix} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z \end{Bmatrix} = 0$$

where $\mathbb{b} = [b_x \quad b_y \quad b_z]^T$ stands for the body forces acting per unit volume.

- In solid mechanic, the six stress components will be some general functions of the 6 components of strain (ε) which are computed by the displacement

$$\mathbb{u} = [u \quad v \quad w]^T$$

Weak form:

- Arbitrary weighting function vector:

$$v = [v_1 \quad v_2 \quad v_3]^T$$

$$\int_\Omega v^T A \, d\Omega = -\int_\Omega v_1^T \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\tau_{xz}}{\partial z} + b_x \right) + v_2 A_2 + v_3 A_3 \, d\Omega$$

$$= 0$$

where Ω is the volume of the domain

- Integration by part:

$$\int_\Omega (ab)' d\Omega = \int_\Omega a' b \, d\Omega + \int_\Omega b' a \, d\Omega$$

$$\Rightarrow -\int_\Omega v_1 \frac{\partial \sigma_x}{\partial x} d\Omega = \int \frac{\partial v_1}{\partial x} \sigma_{xx} \, d\Omega - \int_\Omega \frac{\partial}{\partial x}(v_1 \sigma_{xx}) \, d\Omega$$

$$-\int_\Omega v_1 \frac{\partial \tau_{xy}}{\partial y} d\Omega = \int \frac{\partial v_1}{\partial y} \tau_{xy} \, d\Omega - \int_\Omega \frac{\partial}{\partial y}(v_1 \tau_{xy}) \, d\Omega$$

$$-\int_\Omega v_1 \frac{\partial \tau_{xz}}{\partial z} d\Omega = \int \frac{\partial v_1}{\partial z} \tau_{xz} \, d\Omega - \int_\Omega \frac{\partial}{\partial z}(v_1 \tau_{xz}) \, d\Omega$$

...

Rearanging:

$$\int_\Omega \Big[\Big\{ \frac{\partial v_1}{\partial x}\sigma_{xx} + \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right)\tau_{xy} + \frac{\partial v_2}{\partial y}\sigma_{yy} + \left(\frac{\partial v_2}{\partial z} + \frac{\partial v_3}{\partial y}\right)\tau_{yz} + \frac{\partial v_3}{\partial z}\sigma_{zz} + \left(\frac{\partial v_3}{\partial x} + \frac{\partial v_1}{\partial z}\right)\tau_{zx} \Big\}$$

$$- \left\{ u_1 b_x + u_2 b_y + u_3 b_z \right\}$$

$$- \Big\{ \frac{\partial}{\partial x}(v_1\sigma_{xx}) + \frac{\partial}{\partial y}(v_1\tau_{xy}) + \frac{\partial}{\partial z}(v_1\tau_{xz}) + \frac{\partial}{\partial x}(v_2\tau_{yx}) + \frac{\partial}{\partial y}(v_2\sigma_{yy}) + \frac{\partial}{\partial z}(v_3\tau_{yz}) + \frac{\partial}{\partial x}(v_3\tau_{zx}) + \frac{\partial}{\partial y}(v_3\tau_{zy}) + \frac{\partial}{\partial z}(v_3\sigma_{zz}) \Big\} \Big] d\Omega$$

$$\Rightarrow \int_\Omega (\nabla_s v)^T d\Omega = \int_\Omega u^T b \, d\Omega + \int_\Gamma v^T t \, d\Gamma \quad : \text{Weak Form}$$

(*) Note: Divergence theorem: $\oint_\Omega \nabla F d\Omega = \oint_s n.F ds$

$$- \Big[v_1 (n_x\sigma_{xx} + n_y\tau_{xy} + n_z\tau_{xz}) + v_2 (n_x\tau_{yx} + n_y\sigma_{yy} + n_z\tau_{yz}) + v_3 (n_x\tau_{zx} + n_y\tau_{zy} + n_z\sigma_{zz}) \Big]$$

$$\rightarrow -[v_1 t_1 + v_2 t_2 + v_3 t_3]$$

$t_1 = n_x\sigma_{xx} + n_y\tau_{xy} + n_z\tau_{xz}$

$t_2 = n_x\tau_{yx} + n_y\sigma_{yy} + n_z\tau_{yz}$

$t_3 = n_x\tau_{zx} + n_y\tau_{zy} + n_z\sigma_{zz}$

Tractions acting per unit area of external boundary surface Γ of the solid →

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