# STRONG FORM TO WEAK FORM

# (DEFORMATION PROBLEM)

# 1. Review of the framework of Finite Element Method

## 1.1 Strong form of mechanical problems

The equilibrium equations in the general 3D case:

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial y} + b_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + b_z = 0 \end{cases}$$

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Or in contract form (where $\boldsymbol{b} = (b_x; b_y; b_z)$ is body forces):

$$\nabla_S^T \boldsymbol{\sigma} + \boldsymbol{b} = 0$$

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Constitutive equation:

$$\boldsymbol{\sigma} = \boldsymbol{D} : \boldsymbol{\varepsilon}$$

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Kinematic equation:

$$\boldsymbol{\varepsilon} = \nabla_s \boldsymbol{u}$$

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Thus (Error! **No text of specified style in document.-1)** can be rewritten as:

$$\nabla_S^T \left( \boldsymbol{D} : \nabla_S \boldsymbol{u} \right) + \boldsymbol{b} = 0$$

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in which,

$$\nabla_S^T = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}, \quad \nabla_S = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix}$$

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yx} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \text{ (stress vector), } \boldsymbol{b} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} \text{ (force vector), } \boldsymbol{u} = \begin{Bmatrix} u \\ v \\ \mathrm{w} \end{Bmatrix} \text{ (displacement vector),}$$

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ & & D_{33} & D_{34} & D_{35} & D_{36} \\ & & & D_{44} & D_{45} & D_{46} \\ & \text{sym} & & & D_{55} & D_{56} \\ & & & & & D_{66} \end{bmatrix} \text{ (stiffness matrix)}$$

**Boundary condition:**

We consider the boundary condition $\Gamma = \Gamma_t \cup \Gamma_u$. In which, $\Gamma_t$ is the boundary where the traction is prescribed, and $\Gamma_u$ is the portion of the boundary where the displacement is prescribed. The traction boundary condition is described as:

$$\boldsymbol{n}\boldsymbol{\sigma} = \bar{\boldsymbol{t}} \text{ on the boundary } \Gamma_t$$

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in which $\boldsymbol{n}$ is the normal vector.

The displacement boundary condition is described as:

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on the boundary } \Gamma_u$$ (Error! No text of specified style in document.-7)

## 1.2 Derive weak form from strong form

In Finite Element Method, our purpose is to find an approximate solution of the strong form equation (Error! **No text of specified style in document.-5)**, with boundary equations (Error! **No text of specified style in document.-6)** and (Error! **No text of specified style in document.-7)**. The approximation solution may not satisfy the partial derivative equation exactly at every point inside the domain. The residual of the solution is:

$$\nabla_S^T\left(\boldsymbol{D}:\nabla_S \boldsymbol{u}\right)+\boldsymbol{b}=R\left(x\right)$$ (Error! No text of specified style in document.-8)

We want to minimize the residual $\boldsymbol{R}(x)$ by multiplying the (Error! **No text of specified style in document.-8)** with a weight function $\boldsymbol{v}(x) = [v_1(x) \;\; v_2(x) \;\; v_3(x)]$ and integrating over the domain. By doing that, we obtain a continuous weak form:

$$\int_\Omega R\left(x\right)v(x)dx = 0$$ (Error! No text of specified style in document.-9)

If it satisfies for any $v(x)$ then R(x) will approach zero, and the maximum solution will approach the exact solution. In the above equation, $v(x)$ is an arbitrary function, and equation

(Error! **No text of specified style in document.-9)** has to fulfill for all functions of $v(x)$. The arbitrariness of test function $v(x)$ is crucial as otherwise a weak form is not equivalent to the strong form. Now, (Error! **No text of specified style in document.-9)** becomes:

$$\int_{\Omega}\left(v^{T}:\nabla_{S}^{T}\left(\boldsymbol{D}:\nabla_{S}\boldsymbol{u}\right)\right)d\Omega+\int_{\Omega}\left(v^{T}\cdot\boldsymbol{b}\right)d\Omega=0$$

(Error! No text of specified style in document.-10)

Recall the integration by parts:

$$\int_{a}^{b} v\frac{du}{dx}dx=\int_{a}^{b}\frac{d}{dx}(uv)dx-\int_{a}^{b}\frac{dv}{dx}udx$$

(Error! No text of specified style in document.-11)

Also, recall Divergence theorem:

$$\int_{\Omega}\nabla f d\Omega=\int_{S} nfdS$$

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Firstly, integration by parts is applied to (Error! **No text of specified style in document.-10)**:

$$\int_{\Omega}\left(\nabla_{S}\boldsymbol{v}\right)^{T}:\boldsymbol{\sigma}d\Omega-\int_{\Omega}\left(\nabla_{S}\boldsymbol{v}\right)^{T}:\boldsymbol{D}:\left(\nabla_{S}\boldsymbol{u}\right)d\Omega+\int_{\Omega}\left(\boldsymbol{v}^{T}\cdot\boldsymbol{b}\right)d\Omega=0$$

(Error! No text of specified style in document.-13**)**

Next, divergence theorem is applied to (Error! **No text of specified style in document.-13**):

$$\int_{\Gamma}\left(\boldsymbol{v}^{T}\cdot\boldsymbol{\sigma}\cdot n\right)d\Gamma-\int_{\Omega}\left(\nabla_{S}\boldsymbol{v}\right)^{T}:\boldsymbol{D}:\left(\nabla_{S}\boldsymbol{u}\right)d\Omega+\int_{\Omega}\left(\boldsymbol{v}^{T}\cdot\boldsymbol{b}\right)d\Omega=0$$

(Error! No text of specified style in document.-14**)**

Because $\Gamma = \Gamma_t \cup \Gamma_u$, (Error! **No text of specified style in document.-14**) becomes:

$$\int_{\Gamma_u} \left(\boldsymbol{v}^T \cdot \boldsymbol{\sigma} \cdot n\right) d\Gamma_u + \int_{\Gamma_t} \left(\boldsymbol{v}^T \cdot \boldsymbol{\sigma} \cdot n\right) d\Gamma_t - \int_{\Omega} (\nabla_S \boldsymbol{v})^T : \boldsymbol{D} : (\nabla_S \boldsymbol{u}) d\Omega + \int_{\Omega} \left(\boldsymbol{v}^T \cdot \boldsymbol{b}\right) d\Omega = 0 \qquad \text{(Error! No text of specified style in document.-15)}$$

As $v(x)$ is arbitrary, we choose $\boldsymbol{v}(x)$ that is vanished on the boundary $\Gamma_u$.Also, using (Error! **No text of specified style in document.-7)** condition, (Error! **No text of specified style in document.-15)** is simplified as:

$$\int_{\Gamma_t} \left(\boldsymbol{v}^T \cdot \bar{t}\right) d\Gamma_t - \int_{\Omega} (\nabla_S \boldsymbol{v})^T : \boldsymbol{D} : (\nabla_S \boldsymbol{u}) d\Omega + \int_{\Omega} \left(\boldsymbol{v}^T \cdot \boldsymbol{b}\right) d\Omega = 0 \qquad \text{(Error! No text of specified style in document.-16)}$$

Finally, the continuous weak form is derived as:

$$\underbrace{\int_{\Omega} (\nabla_S \boldsymbol{v})^T : \boldsymbol{D} : (\nabla_S \boldsymbol{u}) d\Omega}_{\alpha(v,u)} = \underbrace{\int_{\Omega} \left(\boldsymbol{v}^T \cdot \boldsymbol{b}\right) d\Omega + \int_{\Gamma_t} \left(\boldsymbol{v}^T \cdot \bar{t}\right) d\Gamma_t}_{f(v)} \qquad \text{(Error! No text of specified style in document.-17)}$$

The name “weak form” comes from the fact that solutions to the weak form need not to be as smooth as solutions of the strong form, i.e. they have weaker continuity requirements. Furthermore, the second derivative equation in strong form (Error! **No text of specified style in document.-5)** is transferred into the first derivative equation in weak form (Error! **No text of specified style in document.-17)**.

### 1.3 Shape function matrix of elements

Displacement components of element $\boldsymbol{u}^e$ are interpolated from the node displacement $\boldsymbol{d}^e$ through shape function matrix of elements $\boldsymbol{N}^e(x)$:

$$\boldsymbol{u}^e = \boldsymbol{N}^e(x)\boldsymbol{d}^e$$

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in which:

$$\boldsymbol{u}^e = \begin{bmatrix} u_1^e(x) \\ u_2^e(x) \\ \vdots \\ u_{n_d}^e(x) \end{bmatrix}$$

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In the calculation in FEM, we usually arrange the displacement vector $\boldsymbol{d}^e$ of element $\Omega_e$ in the nodal order:

$$\boldsymbol{d}^e = \begin{bmatrix} d_{11}^e \\ d_{21}^e \\ \vdots \\ d_{n_d 1}^e \\ d_{11}^e \\ d_{21}^e \\ \vdots \\ d_{n_d 1}^e \\ \vdots \\ d_{11}^e \\ d_{21}^e \\ \vdots \\ d_{n_d 1}^e \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n_d \text{ displacement component of node 1} \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n_d \text{ displacement component of node 2} \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n_d \text{ displacement component of node } n_n \end{matrix}$$

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Shape function $\boldsymbol{N}^e(\boldsymbol{x})$ matrix of element $\Omega_e$ is described as:

$$N^e(x)=\begin{bmatrix} \underbrace{\begin{matrix} N_1^e(\boldsymbol{x}) & 0 & 0 & 0 \\ 0 & N_1^e(\boldsymbol{x}) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & N_1^e(\boldsymbol{x}) \end{matrix}}_{\text{Node 1 } (\mathrm{n_d} \text{ components})} & \underbrace{\begin{matrix} N_2^e(\boldsymbol{x}) & 0 & 0 & 0 \\ 0 & N_2^e(\boldsymbol{x}) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & N_2^e(\boldsymbol{x}) \end{matrix}}_{\text{Node 1 } (\mathrm{n_d} \text{ components})} & \begin{matrix} \ldots \\ \ldots \\ \vdots \\ \ldots \end{matrix} & \underbrace{\begin{matrix} N_{n_n}^e(\boldsymbol{x}) & 0 & 0 & 0 \\ 0 & N_{n_n}^e(\boldsymbol{x}) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & N_{n_n}^e(\boldsymbol{x}) \end{matrix}}_{\text{Node } n_n \ (\mathrm{n_d} \text{ components})} \end{bmatrix}$$

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Or in concise form:

$$\boldsymbol{N}^e(x)=\left[\boldsymbol{N}_1^e(x)\ \ \boldsymbol{N}_2^e(x)\ \ \cdots\ \boldsymbol{N}_{n_n}^e(x)\right]$$

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in which $\boldsymbol{N}_I^e(x)$, $I=1,\ 2,\ ...,\ n_n$ is shape function matrix of element $\boldsymbol{\Omega}_e$ corresponding to node I:

$$\boldsymbol{N}_I^e(x)=\begin{bmatrix} \boldsymbol{N}_I^e(\boldsymbol{x}) & 0 & \cdots & 0 \\ 0 & \boldsymbol{N}_I^e(\boldsymbol{x}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{N}_I^e(\boldsymbol{x}) \end{bmatrix}$$

(Error! No text of specified style in document.-23)

Strain-displacement matrix of element $\boldsymbol{\varepsilon}^e=\dfrac{d\boldsymbol{u}^e}{d\boldsymbol{x}}=\dfrac{\partial \boldsymbol{N}^e(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{d}^e=\boldsymbol{B}^e(\boldsymbol{x})\boldsymbol{d}^e$:

$$\begin{aligned}\boldsymbol{B}^e(x)&=\nabla_S\boldsymbol{N}^e(x)=\left[\nabla_S\boldsymbol{N}_1^e(x)\ \ \nabla_S\boldsymbol{N}_2^e(x)\ \ \cdots\ \nabla_S\boldsymbol{N}_{n_n}^e(x)\right]\\ &=\left[\boldsymbol{B}_1^e(x)\ \ \boldsymbol{B}_2^e(x)\ \ \cdots\ \boldsymbol{N}_{n_n}^e(x)\right]\end{aligned}$$

(Error! No text of specified style in document.-24)

in which $\boldsymbol{B}_I^e(x)$ is strain-displacement matrix of the element corresponding to node I.

### 1.4 Derivation of system equations

From the continuous weak form, we will change it to a discrete one. In other words, instead of finding an unknown function, we want to find "*n*" unknowns. We will need a system of discrete equations, and eventually obtain the system equation in the form: KU = F. K is the stiffness of the system, U is the displacement vector of nodes. F is the vector of forces applied to the systems. The following will describe the process in detail.

Interpolation of displacements by using shape function $N(x)$ and nodal displacement, $d$:

$$\boldsymbol{u} = \boldsymbol{N}(\boldsymbol{x})\boldsymbol{d} = [\boldsymbol{N}_1(\boldsymbol{x})\ \ \boldsymbol{N}_2(\boldsymbol{x})...\boldsymbol{N}_{N_n}(\boldsymbol{x})]\begin{Bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_{N_n} \end{Bmatrix}$$

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Interpolation of strain by using strain-displacement matrix $\boldsymbol{B}(\boldsymbol{x})$:

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{u}}{d\boldsymbol{x}} = \frac{\partial \boldsymbol{N}(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{d} = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{d} = \left[\boldsymbol{B}_1(x)\ \ \boldsymbol{B}_2(x)\ \ ...\ \ \boldsymbol{B}_{N_n}(x)\right]\begin{Bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_{N_n} \end{Bmatrix}$$

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From continuous weak form, we choose $N_n$ test functions $v_1(x), v_2(x), ..., v_{Nn}(x)$. Each function gives one equation, thus, we obtain $N_n$ equations. In Galerkin FEM method, we simply choose the test functions $v_1(x), v_2(x), ..., v_N(x)$ the same as shape functions $N_1(x), N_2(x), ..., N_{N_n}(x)$. Substituting these $N_n$ functions of $v(x)$ into (Error! **No text of specified style in document.-17)**:

$$\int_{\Omega} \left(\nabla_S \boldsymbol{N}_I\right)^T : \boldsymbol{D} : \left(\nabla_S \boldsymbol{u}\right) d\Omega = \int_{\Omega} \left(\boldsymbol{N}_I^T \cdot \boldsymbol{b}\right) d\Omega + \int_{\Gamma_t} \left(\boldsymbol{N}_I^T \cdot \bar{\boldsymbol{t}}\right) d\Gamma$$

(Error! No text of specified style in document.-27) ni which , $\mathrm{I}=1,\ 2,\ ...,\ \mathrm{N_n}$.

Using $\boldsymbol{B}(\boldsymbol{x})=\nabla_S \boldsymbol{N}(\boldsymbol{x})$ and substituting (Error! **No text of specified style in document.-26)** into (Error! **No text of specified style in document.-27)**, we obtain,

$$\left(\int_{\Omega} \boldsymbol{B}_I^T : \boldsymbol{D} : \boldsymbol{B} d\Omega\right) d = \int_{\Omega} \left(\boldsymbol{N}_I^T \cdot \boldsymbol{b}\right) d\Omega + \int_{\Gamma_t} \left(\boldsymbol{N}_I^T \cdot \bar{\boldsymbol{t}}\right) d\Gamma, \quad \mathrm{I}=1,\ 2,\ ...,\ \mathrm{N_I}$$

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in which the transpose of the global strain-displacement matrix is:

$$\boldsymbol{B}^T = \begin{bmatrix} \boldsymbol{B}_1^T \\ \boldsymbol{B}_2^T \\ \vdots \\ \boldsymbol{B}_{N_n}^T \end{bmatrix}$$

(Error! No text of specified style in document.-29)

We can expand (Error! **No text of specified style in document.-28)** into a system of equations:

$$\begin{cases} \left(\int_{\Omega} \boldsymbol{B}_1^T \boldsymbol{D}[\boldsymbol{B}_1\ \boldsymbol{B}_2\ ...\ \boldsymbol{B}_{N_n}] \mathrm{d}\Omega\right) \boldsymbol{d} = \int_{\Omega} \boldsymbol{N}_1^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_1^T \boldsymbol{t}\, d\Gamma \\ \left(\int_{\Omega} \boldsymbol{B}_2^T \boldsymbol{D}[\boldsymbol{B}_1\ \boldsymbol{B}_2\ ...\ \boldsymbol{B}_{N_n}] \mathrm{d}\Omega\right) \boldsymbol{d} = \int_{\Omega} \boldsymbol{N}_2^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_2^T \boldsymbol{t}\, d\Gamma \\ \vdots \\ \left(\int_{\Omega} \boldsymbol{B}_{N_n}^T \boldsymbol{D}[\boldsymbol{B}_1\ \boldsymbol{B}_2\ ...\ \boldsymbol{B}_{N_n}] \mathrm{d}\Omega\right) \boldsymbol{d} = \int_{\Omega} \boldsymbol{N}_{N_n}^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_{N_n}^T \boldsymbol{t}\, d\Gamma \end{cases}$$

(Error! No text of specified style in document.-30)

Or we can write the matrix form of (Error! **No text of specified style in document.-30)**:

$$\left\{\begin{bmatrix} \int_\Omega \boldsymbol{B}_1^T \boldsymbol{D}\boldsymbol{B}_1 \,\mathrm{d}\Omega & \int_\Omega \boldsymbol{B}_1^T \boldsymbol{D}\boldsymbol{B}_2 \,\mathrm{d}\Omega & \cdots & \int_\Omega \boldsymbol{B}_1^T \boldsymbol{D}\boldsymbol{B}_{N_n} \,\mathrm{d}\Omega \\ \int_\Omega \boldsymbol{B}_2^T \boldsymbol{D}\boldsymbol{B}_1 \,\mathrm{d}\Omega & \int_\Omega \boldsymbol{B}_2^T \boldsymbol{D}\boldsymbol{B}_2 \,\mathrm{d}\Omega & \cdots & \int_\Omega \boldsymbol{B}_2^T \boldsymbol{D}\boldsymbol{B}_{N_n} \,\mathrm{d}\Omega \\ \vdots & \vdots & \ddots & \vdots \\ \int_\Omega \boldsymbol{B}_{N_n}^T \boldsymbol{D}\boldsymbol{B}_1 \,\mathrm{d}\Omega & \int_\Omega \boldsymbol{B}_{N_n}^T \boldsymbol{D}\boldsymbol{B}_2 \,\mathrm{d}\Omega & \ldots & \int_\Omega \boldsymbol{B}_{N_n}^T \boldsymbol{D}\boldsymbol{B}_{N_n} \,\mathrm{d}\Omega \end{bmatrix}\right. \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_N \end{bmatrix} = \begin{bmatrix} \int_\Omega \boldsymbol{N}_1^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_1^T \boldsymbol{t}\, d\Gamma \\ \int_\Omega \boldsymbol{N}_2^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_2^T \boldsymbol{t}\, d\Gamma \\ \vdots \\ \int_\Omega \boldsymbol{N}_N^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_N^T \boldsymbol{t}\, d\Gamma \end{bmatrix}$$

(Error! No text of specified style in document.-31)

Eq. (Error! **No text of specified style in document.-31)** can be further simply written as:

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$$

(Error! No text of specified style in document.-32)

in which, global stiffness matrix is expressed as:

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \ldots & \boldsymbol{K}_{1N_n} \\ \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \ldots & \boldsymbol{K}_{1N_n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \ldots & \boldsymbol{K}_{1N_n} \end{bmatrix} = \left\{\begin{bmatrix} \int_\Omega \boldsymbol{B}_1^T \boldsymbol{D}\boldsymbol{B}_1 \,\mathrm{d}\Omega & \int_\Omega \boldsymbol{B}_1^T \boldsymbol{D}\boldsymbol{B}_2 \,\mathrm{d}\Omega & \cdots & \int_\Omega \boldsymbol{B}_1^T \boldsymbol{D}\boldsymbol{B}_{N_n} \,\mathrm{d}\Omega \\ \int_\Omega \boldsymbol{B}_2^T \boldsymbol{D}\boldsymbol{B}_1 \,\mathrm{d}\Omega & \int_\Omega \boldsymbol{B}_2^T \boldsymbol{D}\boldsymbol{B}_2 \,\mathrm{d}\Omega & \cdots & \int_\Omega \boldsymbol{B}_2^T \boldsymbol{D}\boldsymbol{B}_{N_n} \,\mathrm{d}\Omega \\ \vdots & \vdots & \ddots & \vdots \\ \int_\Omega \boldsymbol{B}_{N_n}^T \boldsymbol{D}\boldsymbol{B}_1 \,\mathrm{d}\Omega & \int_\Omega \boldsymbol{B}_{N_n}^T \boldsymbol{D}\boldsymbol{B}_2 \,\mathrm{d}\Omega & \ldots & \int_\Omega \boldsymbol{B}_{N_n}^T \boldsymbol{D}\boldsymbol{B}_{N_n} \,\mathrm{d}\Omega \end{bmatrix}\right.$$

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Force vector is described as:

$$\boldsymbol{f} = \begin{bmatrix} \int_\Omega \boldsymbol{N}_1^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_1^T \boldsymbol{t}\, d\Gamma \\ \int_\Omega \boldsymbol{N}_2^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_2^T \boldsymbol{t}\, d\Gamma \\ \vdots \\ \int_\Omega \boldsymbol{N}_N^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_N^T \boldsymbol{t}\, d\Gamma \end{bmatrix}$$

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In the calculation of FEM:

$$\boldsymbol{K} = \int_\Omega \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} d\Omega = \sum_{e=1}^{N_e} \int_{\Omega_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} d\Omega$$

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In FEM, we will calculate the components of stiffness matrix $K_{ij}$ in which $I, J = 1,\ 2,\ ...\ , N_n$ based on the elements $\Omega_e$ and assemble them together:

$$\boldsymbol{K}_{IJ} = \int_{\Omega} \boldsymbol{B}_I^T \boldsymbol{D} \boldsymbol{B}_J d\Omega = \sum_{e=1}^{N_e} \underbrace{\int_{\Omega_e} \boldsymbol{B}_I^T \boldsymbol{D} \boldsymbol{B}_J d\Omega}_{K_{IJ}^e}$$

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The calculation of $\boldsymbol{K}_{IJ}^e$ is only based on the element $\Omega_e$, thus, we only consider the component inside the element in the integration, and ignored the others outside. Thus, the element stiffness matrix becomes:

$$\boldsymbol{K}_{IJ}^e = \int_{\Omega_e} \left(\boldsymbol{B}_I^e\right)^T \boldsymbol{D} \boldsymbol{B}_J^e d\Omega$$

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in which $\boldsymbol{B}_I^e$ is the portion of $\boldsymbol{B}_I$ in the element $\Omega_e$,

$$\boldsymbol{B}_I^e = \nabla_S \boldsymbol{N}_I^e$$

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Similarly, the force vector is calculated as:

$$\boldsymbol{f}_J = \int_{\Omega} \boldsymbol{N}_J^T \boldsymbol{b}\ d\Omega + \int_{\Gamma_t} \boldsymbol{N}_J^T \boldsymbol{t}\ d\Gamma = \sum_{e=1}^{N_e} \int_{\Omega} \boldsymbol{N}_J^T \boldsymbol{b}\ d\Omega + \int_{\Gamma_t} \boldsymbol{N}_J^T \boldsymbol{t}\ d\Gamma$$

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The element force vector is expressed as:

$$\boldsymbol{f}_J^e = \int_{\Omega} \boldsymbol{N}_J^T(\boldsymbol{x})\boldsymbol{b}\, d\Omega + \int_{\Gamma_t} \boldsymbol{N}_J^T(\boldsymbol{x})\boldsymbol{t}\, d\Gamma$$

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$$\boldsymbol{f}_J^e = \int_{\Omega_e} \left(\boldsymbol{N}_J^e(\boldsymbol{x})\right)^T \boldsymbol{b}\, d\Omega + \int_{\Gamma_t^e} \left(\boldsymbol{N}_J^e(\boldsymbol{x})\right)^T \boldsymbol{t}\, d\Gamma$$

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