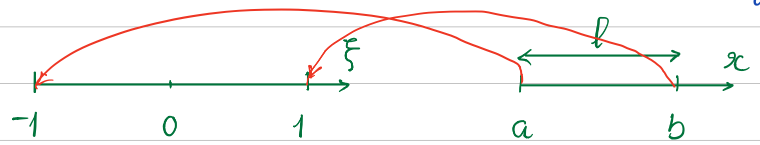
In FEM, numerical integration is needed. Although there are many numerical integration techniques, Gauss quadrature, which is described in this section, is one of the most efficient techniques for functions that are polynomials or nearly polynomials. In FEM, the integrals involves polynomials, so Gauss quadrature is a natural choice

- Consider the following integral: $I = \int_a^b f(x)\, dx = ?$ (1)



- Mapping of the 1D domain from the parent domain $[-1, 1]$ to the physical domain $[a, b]$

$$x = \frac{1}{2}(a+b) + \frac{1}{2}\xi(b-a) \qquad (2)$$

- The above map can also be written directly in terms of the linear shape functions:

$$x = x_1 N_1(\xi) + x_2 N_2(\xi) = a\,\frac{1-\xi}{2} + b\,\frac{1+\xi}{2}$$

$$(1) \Rightarrow \quad dx = \frac{1}{2}(b-a)\,d\xi = \frac{\ell}{2}\,d\xi = \underset{\text{Jacobian}}{J}\,d\xi$$

$$(1) \Leftrightarrow \quad I = J\int_{-1}^{1} f(\xi)\,d\xi = J\hat{I} \quad ; \quad \hat{I} = \int_{-1}^{1} f(\xi)\,d\xi$$

- In the Gauss integration procedure, we approximate the integral by:

points

$$\hat{I} = w_1 f(\xi_1) + w_2 f(\xi_2) + \dots + w_n f(\xi_n)$$

weights

$$\Leftrightarrow \quad \hat{I} = \underbrace{[w_1 \; w_2 \; \dots \; w_n]}_{w^T} \underbrace{\begin{bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_n) \end{bmatrix}}_{f} = w^T f \qquad (3)$$

- The basic idea of the Gauss integration quadrature is to choose the weights and integration points so that the highest possible polynomial is integrated exactly.
- $f(\xi)$ is approximated by a polynomial as:

$$f(\xi) = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \dots + \alpha_m \xi^m = \underbrace{[1\ \xi\ \xi^2 \dots \xi^m]}_{\mathbb{P}} \underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix}}_{\alpha}$$

- Next, we express the values of the coefficients α_i in terms of the function $f(\xi)$ at the integration points:

$$\begin{aligned} f(\xi_1) &= \alpha_1 + \alpha_2 \xi_1 + \alpha_3 \xi_1^2 + \dots + \alpha_m \xi_1^m \\ f(\xi_2) &= \alpha_1 + \alpha_2 \xi_2 + \alpha_3 \xi_2^2 + \dots + \alpha_m \xi_2^m \\ &\vdots \\ f(\xi_n) &= \alpha_1 + \alpha_2 \xi_3 + \alpha_3 \xi_3^2 + \dots + \alpha_m \xi_n^m \end{aligned} \quad \text{or} \quad \underbrace{\begin{bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_n) \end{bmatrix}}_{f} = \underbrace{\begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \dots & \xi_1^m \\ 1 & \xi_2 & \xi_2^2 & \dots & \xi_2^m \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & \xi_n & \xi_n^2 & \dots & \xi_n^m \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}}_{\alpha} \quad (4)$$

- m: order of f function
- n: the number of Gauss Point

$$(3)(4) \Rightarrow \hat{I} = w^T M \alpha$$

- Gauss quadrature provides the weights & integration points that yield an exact integral of a polynomial of a given order. To detect what the weights and quadrature points should be, we integrate the polynomial $f(\xi)$

$$\begin{cases} \hat{I} = \displaystyle\int_{-1}^{1} f(\xi)\, d\xi = \int_{-1}^{1} [1\ \xi\ \xi^2 \dots \xi^m] \begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix} d\xi = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix} \int_{-1}^{1} [1\ \xi\ \xi^2 \dots \xi^m]\, d\xi \\ \hat{I} = [w_1\ \ w_2\ \dots\ w_n] \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \dots & \xi_1^m \\ 1 & \xi_2 & \xi_2^2 & \dots & \xi_2^m \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & \xi_n & \xi_n^2 & \dots & \xi_n^m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \end{cases}$$

$$\int_{-1}^{1} [1\ \xi\ \xi^2 \dots \xi^m]\, d\xi = [w_1\ w_2 \dots w_n] \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \dots & \xi_1^m \\ 1 & \xi_2 & \xi_2^2 & \dots & \xi_2^m \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & \xi_n & \xi_n^2 & \dots & \xi_n^m \end{bmatrix}$$

$\Rightarrow$ Solve this equation for w_i, ξ_i

$\Rightarrow n$ Gauss Point $\longrightarrow$ ($2n$ unknowns, $\underbrace{(m+1) \text{ equations}}_{m:\ \text{highest order of } f \text{ function}}$)

. n Gauss Points $\Rightarrow$ can estimate m^{th} order function with 1 condition:

$$2n = m+1$$
$$n = \frac{m+1}{2}$$

Example ① $m=3 \Rightarrow n=2$. This means 2 Gauss Points can estimate exactly a function of order 3^{rd}. Of course, this also mean that 2 Gauss points can estimate a function of order less than 3.

② $m=4 \Rightarrow n=2,5 \Rightarrow$ 2.5 Gauss Point can estimate exactly 4^{th} order function. But the number of Gauss Point should be integer $\Rightarrow n=3$!

$$\int_{-1}^{1}[1\ \xi\ \xi^2 \ldots \xi^m]d\xi = [w_1\ w_2 \ldots w_n]\begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \ldots & \xi_1^m \\ 1 & \xi_2 & \xi_2^2 & \ldots & \xi_2^m \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & \xi_n & \xi_n^2 & \ldots & \xi_n^m \end{bmatrix}$$

1^{st} equation:

$$\int_{-1}^{1} 1\, d\xi = w_1 + w_2 + \ldots + w_n$$

2^{nd} equation:

$$\int_{-1}^{1} \xi\, d\xi = w_1\xi_1 + w_2\xi_2 + \ldots + w_n\xi_n$$

$\vdots$

$(m+1)^{th}$ equation

$$\int_{-1}^{1} \xi^m d\xi = w_1\xi_1^m + w_2\xi_2^m + \ldots + w_n\xi_n^m$$

- 1 Gauss Point : $\Rightarrow$ 2 equations:

$$\int_{-1}^{1} 1\, d\xi = w_1 \qquad \Rightarrow w_1 = 2$$

$$\int_{-1}^{1} \xi\, d\xi = w_1 \xi_1 \qquad \Rightarrow \xi = 0$$

- 2 Gauss Points $\Rightarrow$ 4 equations:

$$\int_{-1}^{1} 1\, d\xi = w_1 + w_2 \Rightarrow w_1 + w_2 = 2 \qquad (1)$$

$$\int_{-1}^{1} \xi\, d\xi = w_1 \xi_1 + w_2 \xi_2 \Rightarrow w_1 \xi_1 + w_2 \xi_2 = 0 \qquad (2)$$

$$\int_{-1}^{1} \xi^2\, d\xi = w_1 \xi_1^2 + w_2 \xi_2^2 \Rightarrow w_1 \xi_1^2 + w_2 \xi_2^2 = 2/3 \qquad (3)$$

$$\int_{-1}^{1} \xi^3\, d\xi = w_1 \xi_1^3 + w_2 \xi_2^3 \Rightarrow w_1 \xi_1^3 + w_2 \xi_2^3 = 0 \qquad (4)$$

$(1)(4) \Rightarrow \xi_1^2 = \xi_2^2 \Rightarrow \xi_1 = -\xi_2$, plug into 2

$\Rightarrow w_1 - w_2 = 0$

$\Rightarrow w_1 = w_2 = 1$

$(3) \Rightarrow \xi_1^2 = 1/3 \Rightarrow \xi_1 = -1/\sqrt{3}$; $\xi_2 = 1/\sqrt{3}$

- 3 Gauss points:

$$\begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \\ \hline w_1 & 0 & w_2 \end{array}$$

Symmetric properties:
$\xi_2 = 0$; $\xi_1 = -\xi_3$

$$\begin{cases} w_1 + w_2 + w_3 = 2 \\ w_1 \xi_1 + \cancel{w_2 \xi_2}^{0} + w_3 \xi_3 = 0 \xrightarrow{\xi_1 = -\xi_3} w_1 = w_3 \\ w_1 \xi_1^2 + \cancel{w_2 \xi_2^2}^{0} + w_3 \xi_3^2 = 2/3 \xrightarrow[w_1 = w_3]{\xi_1 = -\xi_3} w_1 \xi_1^2 = 1/3 \\ w_1 \xi_1^4 + \cancel{w_2 \xi_2^4}^{0} + w_3 \xi_3^4 = 2/5 \longrightarrow w_1 \xi_1^4 = 1/5 \end{cases}$$

$\Rightarrow \xi_1^2 = \frac{3}{5}$

$\Rightarrow \xi_1 = -\sqrt{\frac{3}{5}}$; $\xi_3 = \sqrt{\frac{3}{5}}$

$\Rightarrow w_1 \cdot \frac{3}{5} = \frac{1}{3} \Rightarrow w_1 = \frac{5}{9}$

$\Rightarrow w_3 = \frac{5}{9}$

$\Rightarrow w_2 = 2 - 2 \cdot \frac{5}{9} = \frac{8}{9}$

Topic: Gauss Integration Notebook

Example: Evaluate $I = \int_2^5 (x^3 + x^2)\,dx$

. $2n_{gp} - 1 = 3 \Rightarrow n_{gp} = 2 \Rightarrow \begin{cases} w_1 = w_2 = 1 \\ \xi_1 = -\frac{1}{\sqrt{3}} ; \ \xi_2 = \frac{1}{\sqrt{3}} \end{cases}$

. $x = \frac{1}{2}(a+b) + \frac{1}{2}\xi(b-a) = 3.5 + 1.5\xi$.

. $f(\xi) = (3.5 + 1.5\xi)^3 + (3.5 + 1.5\xi)^2$

. $\hat{I} = J\hat{I} \frac{\ell}{2} \int_{-1}^{1} \left[(3.5 + 1.5\xi)^3 + (3.5 + 1.5\xi)^2 \right] d\xi$

$$= \frac{\ell}{2}\left[w_1\left[(3.5 + 1.5\xi_1)^3 + (3.5 + 1.5\xi_1)^2 \right] + w_2\left[(3.5 + 1.5\xi_1)^3 + (3.5 + 1.5\xi_1)^2 \right] \right]$$

$= 191.25$

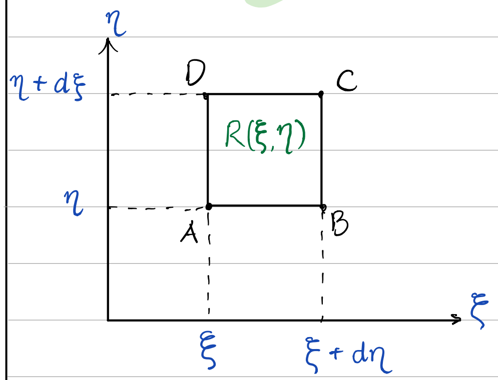
- In this case, as Gauss integration is exact, we can check the result by performing analytical integration

$$\int_2^5 (x^3 + x^2)\,dx = \left(\frac{x^4}{4} + \frac{x^3}{3} \right)\Big|_2^5 = 191,25$$

A Jacobian is required for integrals in more than one variables. Suppose that:

$$x = f(\xi, \eta) \quad ; \quad y = g(\xi, \eta)$$

Let's see what happens to a small infinisimal box in the uv plane:



Since the side-lengths are infinisimal each side of the box in the uv plane is transformed into a straight line in the xy plane. The result is that the box in the uv plane is transformed into a parallelogram into the xy plane

Suppose:

1. The point (ξ, η) is transformed into the point
$$(x = f(\xi, \eta) \;,\; y = g(\xi, \eta))$$

2. The point $(\xi + d\xi, \eta)$ is transformed into the point:

Taylor series:

$$\begin{cases} f(\xi + d\xi, \eta) = \overbrace{f(\xi, \eta)}^{x} + f_\xi(\xi, \eta)\, d\xi \\ g(\xi + d\xi, \eta) = \underbrace{g(\xi, \eta)}_{y} + g_\xi(\xi, \eta)\, d\xi \end{cases}$$

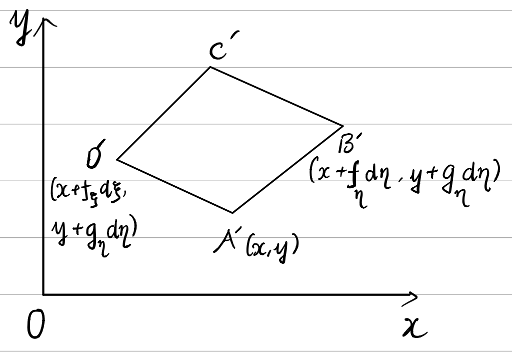
3. The point $(\xi, \eta + d\eta)$ is transformed into:

$$\begin{cases} f(\xi, \eta + d\eta) = x + \overbrace{f_\eta}^{\frac{\partial f}{\partial \eta}}(\xi, \eta)\, d\eta \\ g(\xi, \eta + d\eta) = y + g_\eta(\xi, \eta)\, d\eta \end{cases}$$

$$\overrightarrow{A'B'} = (f_\xi\, d\xi, g_\xi\, d\xi) \quad ; \quad \overrightarrow{A'D'} = (f_\eta\, d\eta, g_\eta\, d\eta)$$

- The area of R in the x, y plane is $\overrightarrow{A'B'} \times \overrightarrow{A'D'}$

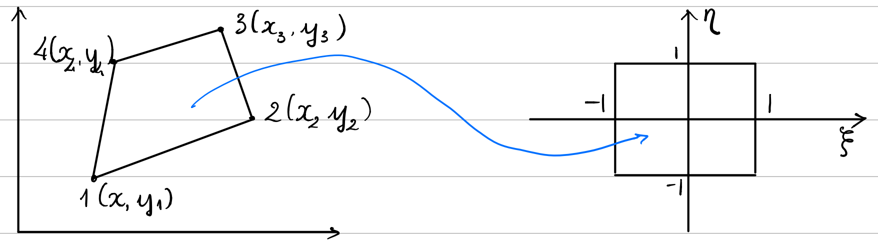
$$\text{Area of } R(x,y) = \underbrace{|f_\xi g_\eta - f_\eta g_\xi|}_{\text{Jacobian}}\, d\xi\, d\eta$$



The quantity dudv is the area of the box $R(\xi, \eta)$

$$\Rightarrow \text{Area of } R(x,y) = J \,.\, \text{Area of } R(\xi, \eta)$$

Topic: *Gauss Integration - 2D - Arbitrary domain* **Notebook**



The relationship between convex quadrilateral in the physical coordinate Oxy and the standard domain $[-1,1]\times[-1,1]$ in the natural coordinate $(O\xi\eta)$ is given by:

$$x = N_1(\xi,\eta)x_1 + N_2(\xi,\eta)x_2 + N_3(\xi,\eta)x_3 + N_4(\xi,\eta)x_4$$
$$y = N_1(\xi,\eta)y_1 + N_2(\xi,\eta)y_2 + N_3(\xi,\eta)y_3 + N_4(\xi,\eta)y_4$$

in which (x_i, y_i), $i = 1,2,3,4$ are the coordinates of 4 nodes in physical coordinate system Oxy.

- N_i, $i = 1,2,3,4$ are shape functions of the quadrilateral in the physical coordinate Oxy:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad ; \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) \quad ; \quad N_4 = \frac{1}{4}(1-\xi)(1-\eta)$$

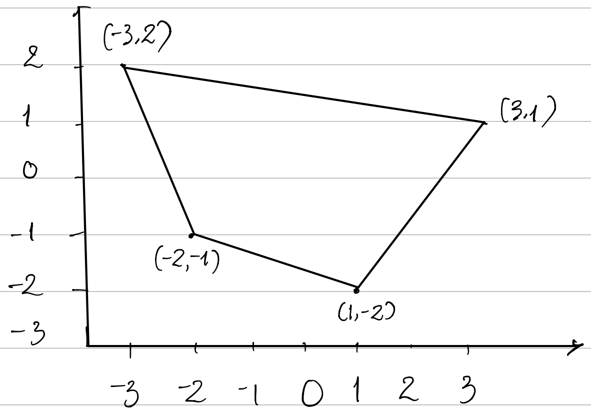
- Now: $$I = \iint_{\Omega_{xy}} f(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\det J\,d\xi\,d\eta$$

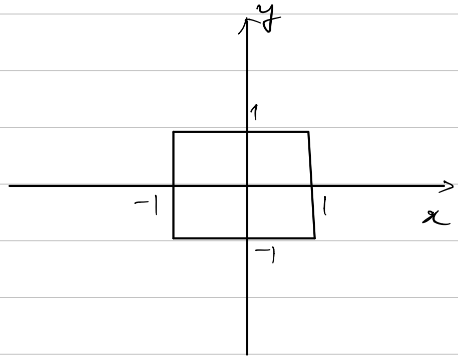
- In which $\det J$ is the determinant of Jacobian matrix, which relate the convex quadrilateral Ω_{xy} in the physical coordinate system Oxy with the standard domain $[-1,1]\times[-1,1]$ in the natural coordinate system $O\xi\eta$:

$$J = \left[\frac{\partial(x,y)}{\partial(\xi,\eta)}\right] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Example: Calculate integration on an arbitrary domain of convex quadrilateral :

$$I = \iint_{\Omega_{xy}} f(x,y)\,dx\,dy = \iint_{\Omega_{xy}} (1+2xy)\,dx\,dy$$





- $x = [N_1\ N_2\ N_3\ N_4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = [N_1\ N_2\ N_3\ N_4]\begin{bmatrix} -2 \\ 1 \\ 3 \\ -3 \end{bmatrix} = -\frac{1}{4} - \frac{3}{4}\xi + \frac{\eta}{4} - \frac{9}{4}\xi\eta$

- $y = [N_1\ N_2\ N_3\ N_4]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = [N_1\ N_2\ N_3\ N_4]\begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \end{bmatrix} = \frac{3}{2}\eta + \frac{1}{2}\xi\eta$

- Jacobian matrix:

$$J = \begin{bmatrix} -\frac{3}{4} - \frac{9}{4}\eta & \frac{1}{4} - \frac{9}{4}\xi \\ \frac{1}{2}\eta & \frac{3}{2} + \frac{1}{2}\xi \end{bmatrix}$$

$$\Rightarrow \det J = -\frac{3}{8}\xi - \frac{7}{2}\eta - \frac{9}{8}$$

- $I = \iint_{\Omega_{xy}} ($