STRONG FORM TO WEAK FORM

(DEFORMATION PROBLEM)

## Review of the framework of Finite Element Method

### 4.1.1 Strong form of mechanical problems

The equilibrium equations in the general 3D case:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑1) |

Or in contract form (where ***b* =** (*bx*; *by*; *bz*)is body forces):

|  |  |  |
| --- | --- | --- |
|  |  | (4‑2) |

Constitutive equation:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑3) |

Kinematic equation:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑4) |

Thus (4‑1) can be rewritten as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑5) |

in which,

, 

 (stress vector),  (force vector),  (displacement vector),  (stiffness matrix)

**Boundary condition:**

We consider the boundary condition****. In which, **** is the boundary where the traction is prescribed, and **** is the portion of the boundary where the displacement is prescribed. The traction boundary condition is described as:

|  |  |  |
| --- | --- | --- |
|  | on the boundary | (4‑6) |

in which  is the normal vector.

The displacement boundary condition is described as:

|  |  |  |
| --- | --- | --- |
|  | on the boundary | (4‑7) |

### 4.1.2 Derive weak form from strong form

In Finite Element Method, our purpose is to find an approximate solution of the strong form equation (4‑5), with boundary equations (4‑6) and (4‑7). The approximation solution may not satisfy the partial derivative equation exactly at every point inside the domain. The residual of the solution is:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑8) |

We want to minimize the residual by multiplying the (4‑8) with a weight function and integrating over the domain. By doing that, we obtain a continuous weak form:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑9) |

If it satisfies for any then R(x) will approach zero, and the maximum solution will approach the exact solution. In the above equation, is an arbitrary function, and equation (4‑9) has to fulfill for all functions of . The arbitrariness of test function  is crucial as otherwise a weak form is not equivalent to the strong form. Now, (4‑9) becomes:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑10) |

Recall the integration by parts:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑11) |

Also, recall Divergence theorem:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑12) |

Firstly, integration by parts is applied to (4‑10):

|  |  |
| --- | --- |
|  | (4‑13) |

Next, divergence theorem is applied to (4‑13):

|  |  |
| --- | --- |
|  | (4‑14) |

Because**,** (4‑14) becomes:

|  |  |
| --- | --- |
|  | (4‑15) |

As is arbitrary, we choose that is vanished on the boundary .Also, using (4‑7) condition, (4‑15) is simplified as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑16) |

Finally, the continuous weak form is derived as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑17) |

The name “weak form” comes from the fact that solutions to the weak form need not to be as smooth as solutions of the strong form, i.e. they have weaker continuity requirements. Furthermore, the second derivative equation in strong form (4‑5) is transferred into the first derivative equation in weak form (4‑17).

### 4.1.3 Shape function matrix of elements

Displacement components of element **** are interpolated from the node displacement  through shape function matrix of elements:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑18) |

in which:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑19) |

In the calculation in FEM, we usually arrange the displacement vector  of element  in the nodal order:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑20) |

Shape function  matrix of element  is described as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑21) |

Or in concise form:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑22) |

in which ,  is shape function matrix of element corresponding to node I:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑23) |

Strain-displacement matrix of element **:**

|  |  |  |
| --- | --- | --- |
|  |  | (4‑24) |

in which is strain-displacement matrix of the element corresponding to node I.

### 4.1.4 Derivation of system equations

From the continuous weak form, we will change it to a discrete one. In other words, instead of finding an unknown function, we want to find “*n*” unknowns. We will need a system of discrete equations, and eventually obtain the system equation in the form: KU = F. K is the stiffness of the system, U is the displacement vector of nodes. F is the vector of forces applied to the systems. The following will describe the process in detail.

Interpolation of displacements by using shape function **** and nodal displacement, **:**

|  |  |  |
| --- | --- | --- |
|  |  | (4‑25) |

Interpolation of strain by using strain-displacement matrix ****:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑26) |

From continuous weak form, we choose  test functions . Each function gives one equation, thus, we obtain  equations. In Galerkin FEM method, we simply choose the test functions the same as shape functions . Substituting thesefunctions of  into (4‑17):

|  |  |  |
| --- | --- | --- |
|  |  | (4‑27) |

ni which .

Using ****and substituting (4‑26) into (4‑27), we obtain,

|  |  |  |
| --- | --- | --- |
|  |  | (4‑28) |

in which the transpose of the global strain-displacement matrix is:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑29) |

We can expand (4‑28) into a system of equations:

|  |  |
| --- | --- |
|  | (4‑30) |

Or we can write the matrix form of (4‑30):

|  |  |
| --- | --- |
|  | (4‑31) |

Eq. (4‑31) can be further simply written as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑32) |

in which, global stiffness matrix is expressed as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑33) |

Force vector is described as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑34) |

In the calculation of FEM:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑35) |

In FEM, we will calculate the components of stiffness matrix in which based on the elements and assemble them together:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑36) |

The calculation of  is only based on the element , thus, we only consider the component inside the element in the integration, and ignored the others outside. Thus, the element stiffness matrix becomes:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑37) |

in which is the portion of  in the element ,

|  |  |  |
| --- | --- | --- |
|  |  | (4‑38) |

Similarly, the force vector is calculated as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑39) |

The element force vector is expressed as:

|  |  |  |
| --- | --- | --- |
|  |  | (4‑40) |
|  |  | (4‑41) |

## 