# Table of Contents

# BrainPy Introduction

In this chapter, we will briefly introduce how to implement computational neuroscience models with BrainPy. For more detailed documents and tutorials, please check our Github repository BrainPy and BrainModels.

`BrainPy` is a Python platform for computational neuroscience and brain-inspired computation. To model with BrainPy, users should follow 3 steps:

1) Define Python classes for neuron and synapse models. BrainPy provides base classes for different kinds of models, users only need to inherit from those base classes, and define specific methods to tell BrainPy what operations they want the models to take during the simulation. In this process, BrainPy will assist users in the numerical integration of differential equations (ODE, SDE, etc.), adaptation of various backends ( `Numpy` , `PyTorch` , etc.), and other functions to simplify code logic.

2) Instantiate Python classes as objects of neuron group and synapse connection groups, pass the instantiated objects to BrainPy class `Network`, and call method `run` to simulate the network.

3) Call BrainPy modules like the `measure` module and the `visualize` module to display the simulation results.

With this overall concept of BrainPy, we will go into more detail about implementations in the following sections. In neural systems, neurons are connected by synapses to build networks, so we will introduce neuron models, synapse models, and network models in order.

# 1. Neuron models

Neuron models can be classified into three types from complex to simple: biophysical models, reduced models and firing rate models.

## 1.1 Biological Background

## 1.2 Biophysical models

## 1.3 Reduced models

## 1.4 Firing rate models

# 1.1 Biological backgrounds

As the basic unit of neural systems, neurons maintain mystique to researchers for a long while. In recent centuries, however, along with the development of experimental techniques, researchers have painted a general figure of those little things working ceaselessly in our neural system.

To achieve our final goal of modeling neurons with computational neuroscience methods, we may start with a patch of real neuron membrane.

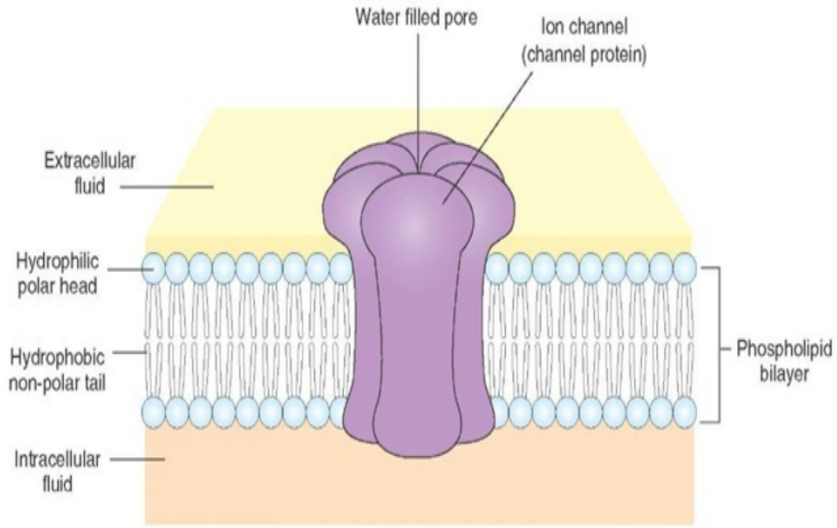



**Fig. 1-1 Neuron membrane diagram | what-when-how.com**

The figure above is a general diagram of neuron membrane with phospholipid bilayer and ion channels. The membrane divides the ions and fluid into intracellular and extracellular, partially prevent them from exchanging, thus generates **membrane potential**---- the difference in electric potential across the membrane.

An ion in the fluid is subjected to two forces. The force of diffusion is caused by the ion concentration difference across the membrane, while the force of electric field is caused by the electric potential difference. When these two forces reach balance, the total forces on ions are 0, and each type of ion meets an equilibrium potential, while the neuron holds a membrane potential lower than 0.

This membrane potential integrated by all those ion equilibrium potentials is the **resting potential**, and the neuron is, in a so-called **resting state**. If the neuron is not disturbed, it will just come to the balanced resting state, and rest.

However, our neural system receives countless inputs every millisecond, from external inputs to recurrent inputs, from specific stimulus inputs to non-specific background inputs. Receiving all these inputs, neurons generate **action potentials** (or **spikes**) to transfer and process information all across the neural system.

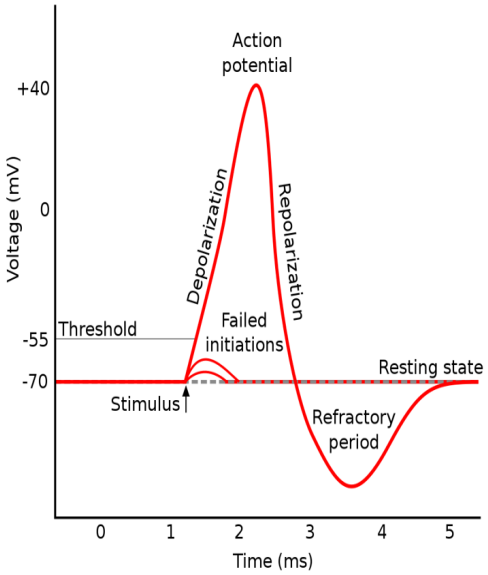



**Fig. 1-2 Action Potential | Wikipedia**

Passing through the ion channels shown in Fig.1-1, ions on both sides of the hydrophobic phospholipid bilayer are exchanged. Due to changes in the environment caused by, for example, an external input, ion channels will switch between their open/close states, therefore allow/prohibit ion exchanges. During the switch, the ion concentrations (mainly Na+ and K+) change, induce a significant change on neuron's membrane potential: the membrane potential will raise to a peak value and then fall back in a short time period. Biologically, when such a series of potential changes happens, we say the neuron generates an **action potential** or a **spike**, or the neuron fires.

An action potential can be mainly divided into three periods: **depolarization**, **repolarization** and **refractory period**. During the depolarization period, Na+ flow into the neuron and K+ flow out of the neuron, however the inflow of Na+ is faster, so the membrane potential raises from a low value $V_{rest}$ to a value much higher called $V_{th}$, then the outflow of K+ becomes faster than Na+, and the membrane potential is reset to a value lower than resting potential during the repolarization period. After that, because of the relatively lower membrane potential, the neuron is unlikely to generate another spike immediately, until the refractory period passes.

A single action potential is complex enough, but in our neural system, one single neuron can generate several action potentials in less than a second. How, exactly, do the neurons fire? Different kinds of neurons may spike when facing different inputs, and the pattern of their spiking can be classified into several firing patterns, some of which are shown in the following figure.

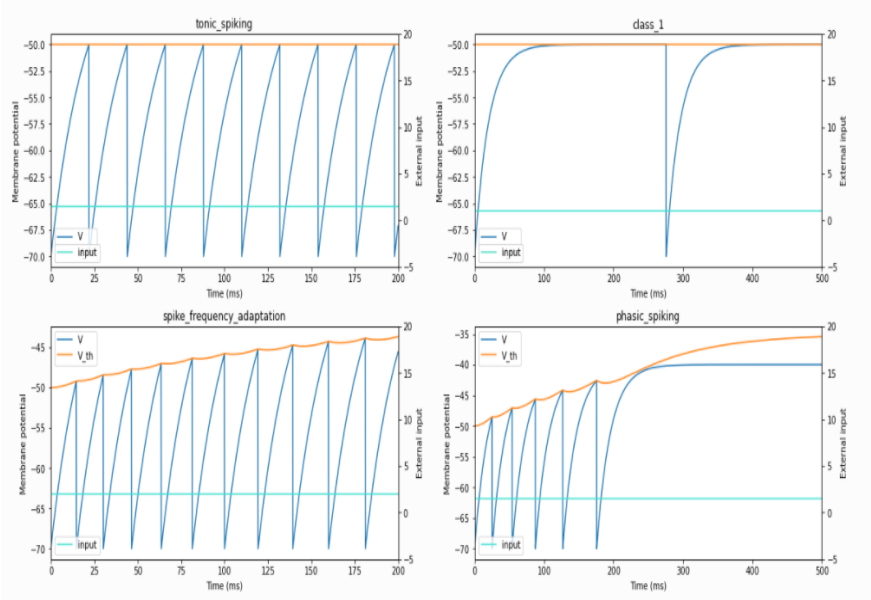



**Figure 1-3 Some firing patterns**

Those firing patterns, together with the shape of action potentials, are what computational neuroscience wants to model at the cellular level.

# 1.2 Biophysical models

## 1.2.1 Hodgkin-Huxley model

Hodgkin and Huxley (1952) recorded the generation of action potential on squid giant axons with voltage clamp technique, and proposed the canonical neuron model called **Hodgin-Huxley model** (**HH model**).

In last section we have introduced a general template for neuron membrane. Computational neuroscientists always model neuron membrane as equivalent circuit like the following figure.

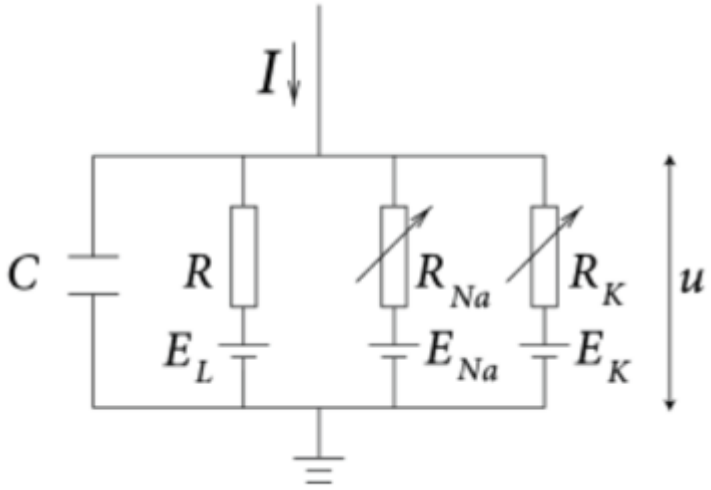



**Fig. 1-4 Equivalent circuit diagram | NeuroDynamics**

The equivalent circuit diagram of Fig.1-1 is shown in Fig. 1-4, in which the patch of neuron membrane is converted into electric components. In Fig.1-4, the capacitance $C$ refers to the hydrophobic phospholipid bilayer with low conductance, and current $I$ refers to the external stimulus.

As Na+ ion channels and K+ ion channels are important in the generation of action potentials, these two ion channels are modeled as the two variable resistances $R_{Na}$ and $R_K$ in parallel on the right side of the circuit diagram, and the resistance $R$ refers to all the non-specific ion channels on the membrane. The batteries $E_{Na}$, $E_K$ and $E_L$ refer to the electric potential differences caused by the concentration differences of corresponding ions.

Consider the Kirchhoff's first law, that is, for any node in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node, Fig. 1-4 can be modeled as differential equations:

$$C\frac{dV}{dt} = -(\bar{g}_{Na}m^3h(V-E_{Na}) + \bar{g}_K n^4(V-E_K) + g_{leak}(V-E_{leak})) + I(t)$$

$$\frac{dx}{dt} = \alpha_x(1-x) - \beta_x, x \in \{Na, K, leak\}$$

That is the HH model. Note that in the first equation above, the first three terms on the right hand are the current go through Na+ ion channels, K+ ion channels and other non-specific ion channels, respectively, while $I(t)$ is an external input. On the left hand, $C\frac{dV}{dt} = \frac{dQ}{dt} = I$ is the current go through the capacitance.

In the computing of ion channel currents, other than the Ohm's law $I = U/R = gU$, HH model introduces three **gating variables** m, n and h to control the open/close state of ion channels. To be precise, variables m and h control the state of Na+ ion channel, variable n controls the state of K+ ion channel, and the real conductance of an ion channel is the product of maximal conductance $\bar{g}$ and the state of gating variables. Gating variables' dynamics can be expressed in a Markov-like form, in which $\alpha_x$ refers to the activation rate of gating variable x, and $\beta_x$ refers to the de-activation rate of x. The expressions of $\alpha_x$ and $\beta_x$ (as shown in equations below) are fitted by experimental data.

$$\alpha_m(V) = \frac{0.1(V+40)}{1-exp(\frac{-(V+40)}{10})}$$

$$\beta_m(V) = 4.0exp(\frac{-(V+65)}{18})$$

$$\alpha_h(V) = 0.07exp(\frac{-(V+65)}{20})$$

$$\beta_h(V) = \frac{1}{1+exp(\frac{-(V+35)}{10})}$$

$$\alpha_n(V) = \frac{0.01(V+55)}{1-exp(\frac{-(V+55)}{10})}$$

$$\beta_n(V) = 0.125exp(\frac{-(V+65)}{80})$$

```
1   class HH(bp.NeuGroup):                                  # bp.NeuGroup class: Group of neurons
2     target_backend = 'general'
3                                                           # Set backend for model.
4     @staticmethod
5     @bp.odeint(method='exponential_euler')                # Call `bp.odeint` to integrate ODEs. Set parameter `method` to choose numerical integration methods.
6     def integral(V, m, h, n, t, C, gNa, ENa, gK, EK, gL, EL, Iext):
7       alpha_m = 0.1*(V+40)/(1-bp.ops.exp(-(V+40)/10))   # α_m(V) = (0.1(V + 40))/(1 − exp(−(V + 40)/10))
8       beta_m = 4.0*bp.ops.exp(-(V+65)/18)               # β_m(V) = 4.0exp(−(V + 65)/18)
9       dmdt = alpha_m * (1 - m) - beta_m * m             # dx/dt = α(1 − x) − βx,   x = m
10
11      alpha_h = 0.07*bp.ops.exp(-(V+65)/20)             # α_h(V) = 0.07 exp(−(V + 65)/20)
12      beta_h = 1/(1+bp.ops.exp(-(V+35)/10))             # β_h(V) = 1/(1 + exp(−(V + 35)/10))
13      dhdt = alpha_h * (1 - h) - beta_h * h             # dx/dt = α(1 − x) − βx,   x = h
14
15      alpha_n = 0.01*(V+55)/(1-bp.ops.exp(-(V+55)/10)) # α_n(V) = (0.01(V + 55))/(1 − exp(−(V + 55)/10))
16      beta_n = 0.125*bp.ops.exp(-(V+65)/80)             # β_n(V) = 0.125exp(−(V + 65)/80)
17      dndt = alpha_n * (1 - n) - beta_n * n             # dx/dt = α(1 − x) − βx,   x = n
18
19      I_Na = (gNa * m ** 3.0 * h) * (V - ENa)           # C dV/dt = −(ḡ_Na m^3 h(V − E_Na) + ḡ_K n^4(V − E_K) +
20      I_K = (gK * n ** 4.0) * (V - EK)                  #           g_leak(V − E_leak)) + I(t)
21      I_leak = gL * (V - EL)
22      dVdt = (- I_Na - I_K - I_leak + Iext) / C
23
24      return dVdt, dmdt, dhdt, dndt
```

```
25
26      def __init__(self, size, ENa=50., gNa=120., EK=-77., gK=36.,
27                   EL=-54.387, gL=0.03, V_th=20., C=1.0, **kwargs):
28        # parameters
29        self.ENa = ENa
30        self.EK = EK
31        self.EL = EL
32        self.gNa = gNa
33        self.gK = gK
34        self.gL = gL
35        self.C = C
36        self.V_th = V_th
37
38        # variables
39        num = bp.size2len(size)
40        self.V = -65. * bp.ops.ones(num)
41        self.m = 0.5 * bp.ops.ones(num)
42        self.h = 0.6 * bp.ops.ones(num)
43        self.n = 0.32 * bp.ops.ones(num)
44        self.spike = bp.ops.zeros(num, dtype=bool)
45        self.input = bp.ops.zeros(num)
46
47        super(HH, self).__init__(size=size, **kwargs)
48
49      def update(self, _t):
50        V, m, h, n = self.integral(self.V, self.m, self.h, self.n, _t,
51                                   self.C, self.gNa, self.ENa, self.gK,
52                                   self.EK, self.gL, self.EL, self.input)
53        self.spike = (self.V < self.V_th) * (V >= self.V_th)
54        self.V = V
55        self.m = m
56        self.h = h
57        self.n = n
58        self.input[:] = 0
```

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.
For example, if size = 100,

| ENa | EK | ..... | V_th |
|---|---|---|---|
| V[0] | V[1] | ..... | V[100] |
| m[0] | m[1] | ..... | m[100] |
| ..... | | ..... | ..... |

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

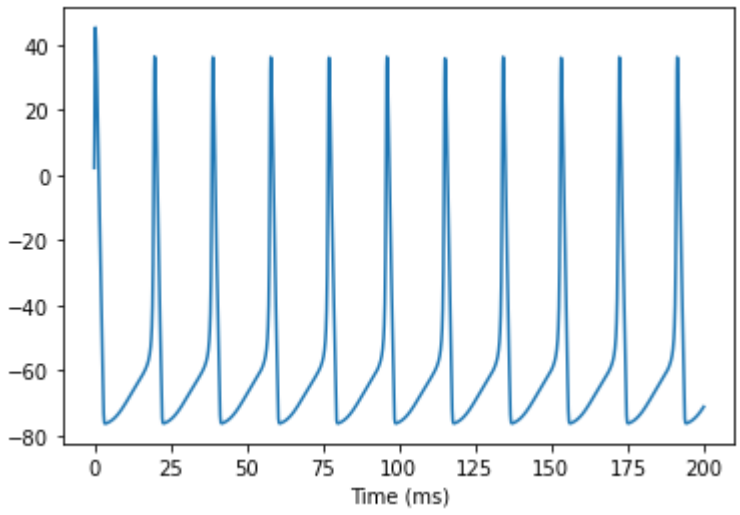
Update variables with numerical integration in vector form.

Judge if the neuron spikes.

Reset external input for this time step.

*Run codes in our github repository: https://github.com/PKU-NIP-Lab/BrainModels*

The V-t plot of HH model simulated by BrainPy is shown below. The three periods, depolarization, repolarization and refractory period of a real action potential can be seen in the V-t plot. In addition, during the depolarization period, the membrane integrates external inputs slowly at first, and increases rapidly once it grows beyond some point, which also reproduces the "shape" of action potentials.



# 1.3 Reduced models

Inspired by biophysical experiments, Hodgkin-Huxley model is precise but costly. Researchers proposed the reduced models to reduce the consumption on computing resources and running time in simulation.

These models are simple and easy to compute, while they can still reproduce the main pattern of neuron behaviors. Although their representation capabilities are not as good as biophysical models, such a loss of accuracy is sometimes acceptable considering their simplicity.

## 1.3.1 Leaky Integrate-and-Fire model

The most typical reduced model is the **Leaky Integrate-and-Fire model** (**LIF model**) presented by Lapicque (1907). LIF model is a combination of integrate process represented by differential equation and spike process represented by conditional judgment:

$$\tau \frac{dV}{dt} = -(V - V_{rest}) + RI(t)$$

If $V > V_{th}$, neuron fires,

$$V \leftarrow V_{reset}$$

$\tau = RC$ is the time constant of LIF model, the larger $\tau$ is, the slower model dynamics is. The equation shown above is corresponding to a simpler equivalent circuit than HH model, for it does not model the Na+ and K+ ion channels any more. Actually, in LIF model, only the consistence $R$, capacitance $C$, battery $E$ and external input $I$ is modeled.

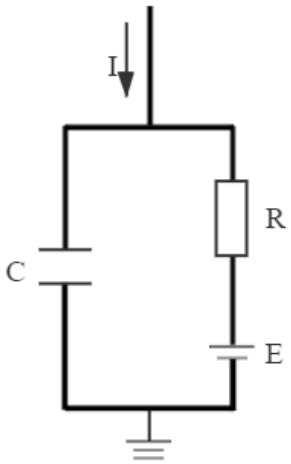



**Fig1-4 Equivalent circuit of LIF model**

Compared with HH model, LIF model does not model the shape of action potentials, which means, the membrane potential does not burst before a spike. Also, the refractory period is overlooked in the original model, and in order to

generate it, another conditional judgment must be added:

If

$$t - t_{lastspike} <= refractoryperiod$$

then neuron is in refractory period, membrane potential $V$ will not be updated.

```python
class LIF(bp.NeuGroup):
  target_backend = ['numpy', 'numba', 'numba-parallel', 'numba-cuda']

  @staticmethod
  def derivative(V, t, Iext, V_rest, R, tau):
    dvdt = (-V + V_rest + R * Iext) / tau
    return dvdt

  def __init__(self, size, t_refractory=1., V_rest=0.,
               V_reset=-5., V_th=20., R=1., tau=10., **kwargs):
    # parameters
    self.V_rest = V_rest
    self.V_reset = V_reset
    self.V_th = V_th
    self.R = R
    self.tau = tau
    self.t_refractory = t_refractory

    # variables
    num = bp.size2len(size)
    self.t_last_spike = bp.ops.ones(num) * -1e7
    self.input = bp.ops.zeros(num)
    self.refractory = bp.ops.zeros(num, dtype=bool)
    self.spike = bp.ops.zeros(num, dtype=bool)
    self.V = bp.ops.ones(num) * V_rest

    self.integral = bp.odeint(self.derivative)
    super(LIF, self).__init__(size=size, **kwargs)

  def update(self, _t):
    for i in prange(self.size[0]):
      spike = 0.
      refractory = (_t - self.t_last_spike[i] <= self.t_refractory)
      if not refractory:
        V = self.integral(self.V[i], _t, self.input[i],
                          self.V_rest, self.R, self.tau)
        spike = (V >= self.V_th)
        if spike:
          V = self.V_reset
          self.t_last_spike[i] = _t
        self.V[i] = V
      self.spike[i] = spike
      self.refractory[i] = refractory or spike
      self.input[i] = 0.
```

bp.NeuGroup class: Group of neurons

Users may also set backend with a list of backend name.

$$\tau\frac{dV}{dt} = -(V - V_{rest}) + RI(t)$$

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.

Call `bp.odeint` to integrate ODEs. Set parameter `method` to choose numerical integration methods.

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

For each neuron in neuron group.

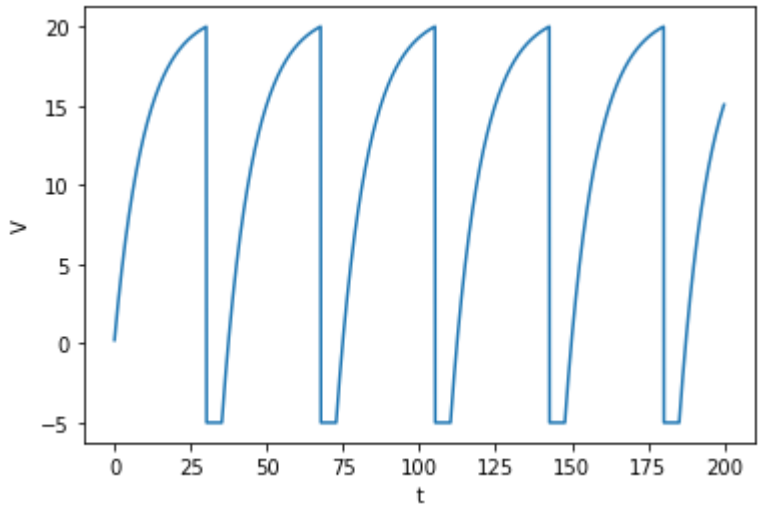
Check if neuron is in refractory period.

Update variables with numerical integration one by one.

Check if neuron spikes.

Reset neuron.

Reset external input for this time step.



### 1.3.2 Quadratic Integrate-and-Fire model

To pursue higher representation capability, Latham et al. (2000) proposed **Quadratic Integrate-and-Fire model** (**QuaIF model**), in which they add a second order term in differential equation so the neurons can generate spike better.

$$\tau \frac{dV}{dt} = a_0(V - V_{rest})(V - V_c) + RI(t)$$

In the equation above, $a_0$ is a parameter controls the slope of membrane potential before a spike, and $V_c$ is the critical potential for action potential initialization.

Below $V_C$, membrane potential $V$ increases slowly, once it grows beyond $V_C$, $V$ turns to rapid increase.

```python
class QuaIF(bp.NeuGroup):
  target_backend = 'general'

  @staticmethod
  def derivative(V, t, I_ext, V_rest, V_c, R, tau, a_0):
    dVdt = (a_0 * (V - V_rest) * (V - V_c) + R * I_ext) / tau
    return dVdt

  def __init__(self, size, V_rest=-65., V_reset=-68.,
               V_th=-30., V_c=-50.0, a_0=.07,
               R=1., tau=10., t_refractory=0., **kwargs):
    # parameters
    self.V_rest = V_rest
    self.V_reset = V_reset
    self.V_th = V_th
    self.V_c = V_c
    self.a_0 = a_0
    self.R = R
    self.tau = tau
    self.t_refractory = t_refractory

    # variables
    num = bp.size2len(size)
    self.V = bp.ops.ones(num) * V_reset
    self.input = bp.ops.zeros(num)
    self.spike = bp.ops.zeros(num, dtype=bool)
    self.refractory = bp.ops.zeros(num, dtype=bool)
    self.t_last_spike = bp.ops.ones(num) * -1e7

    self.integral = bp.odeint(f=self.derivative, method='euler')
    super(QuaIF, self).__init__(size=size, **kwargs)

  def update(self, _t):
    for i in prange(self.size[0]):
      spike = 0.
      refractory = (_t - self.t_last_spike[i] <= self.t_refractory)
      if not refractory:
        V = self.integral(self.V[i], _t, self.input[i],
                          self.V_rest, self.V_c, self.R,
                          self.tau, self.a_0)
        spike = (V >= self.V_th)
        if spike:
          V = self.V_rest
          self.t_last_spike[i] = _t
        self.V[i] = V
      self.spike[i] = spike
      self.refractory[i] = refractory or spike
      self.input[i] = 0.
```

bp.NeuGroup class: Group of neurons

$\tau \frac{dV}{dt} = a_0(V - V_{rest})(V - V_c) + RI(t)$

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.

Call `bp.odeint` to integrate ODEs. Set parameter `method` to choose numerical integration methods.

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

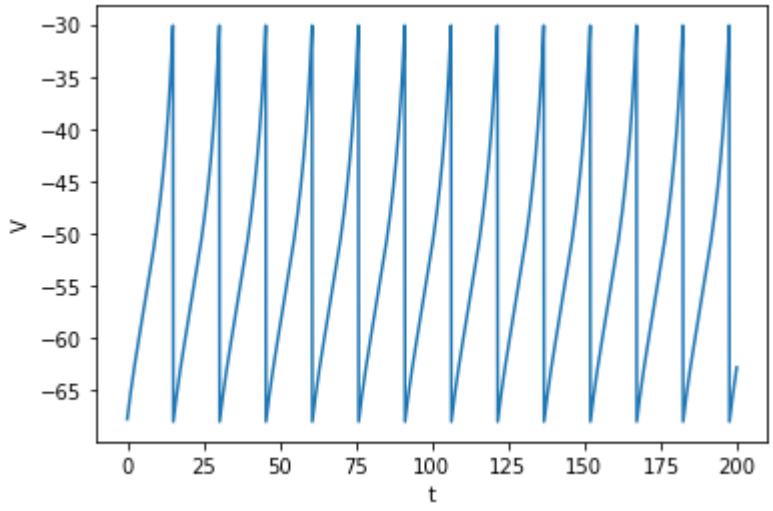
For each neuron in neuron group.

Check if neuron is in refractory period.

Update variables with numerical integration one by one.

Check if neuron spikes.

Reset neuron.




### 1.3.3 Exponential Integrate-and-Fire model

**Exponential Integrate-and-Fire model** (**ExpIF model**) (Fourcaud-Trocme et al., 2003) is more expressive than QuaIF model. With the exponential term added to the right hand of differential equation, the dynamics of ExpIF model can now generates a more realistic action potential.

$$\tau\frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V-V_T}{\Delta_T}} + RI(t)$$

In the exponential term, $V_T$ is the critical potential of generating action potential, below which $V$ increases slowly and above which rapidly. $\Delta_T$ is the slope of action potentials in ExpIF model, and when $\Delta_T \to 0$, the shape of spikes in ExpIF model will be equivalent to the LIF model with $V_{th} = V_T$(Fourcaud-Trocme et al., 2003) .

```python
class ExpIF(bp.NeuGroup):
  target_backend = 'general'

  @staticmethod
  def derivative(V, t, I_ext, V_rest, delta_T, V_T, R, tau):
    exp_term = bp.ops.exp((V - V_T) / delta_T)
    dvdt = (-(V-V_rest) + delta_T*exp_term + R*I_ext) / tau
    return dvdt

  def __init__(self, size, V_rest=-65., V_reset=-68.,
               V_th=-30., V_T=-59.9, delta_T=3.48,
               R=10., C=1., tau=10., t_refractory=1.7,
               **kwargs):
    # parameters
    self.V_rest = V_rest
    self.V_reset = V_reset
    self.V_th = V_th
    self.V_T = V_T
    self.delta_T = delta_T
    self.R = R
    self.C = C
    self.tau = tau
    self.t_refractory = t_refractory

    # variables
    self.V = bp.ops.ones(size) * V_rest
    self.input = bp.ops.zeros(size)
    self.spike = bp.ops.zeros(size, dtype=bool)
    self.refractory = bp.ops.zeros(size, dtype=bool)
    self.t_last_spike = bp.ops.ones(size) * -1e7

    self.integral = bp.odeint(self.derivative)
    super(ExpIF, self).__init__(size=size, **kwargs)

  def update(self, _t):
    for i in prange(self.num):
      spike = 0.
      refractory = (_t - self.t_last_spike[i] <= self.t_refractory)
      if not refractory:
        V = self.integral(
          self.V[i], _t, self.input[i], self.V_rest,
          self.delta_T, self.V_T, self.R, self.tau
        )
        spike = (V >= self.V_th)
        if spike:
          V = self.V_reset
          self.t_last_spike[i] = _t
        self.V[i] = V
      self.spike[i] = spike
      self.refractory[i] = refractory or spike
    self.input[:] = 0.
```

bp.NeuGroup class: Group of neurons

$$\tau \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} + RI(t)$$

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.

Call `bp.odeint` to integrate ODEs. Parameter `method` is set to default value `euler`.

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

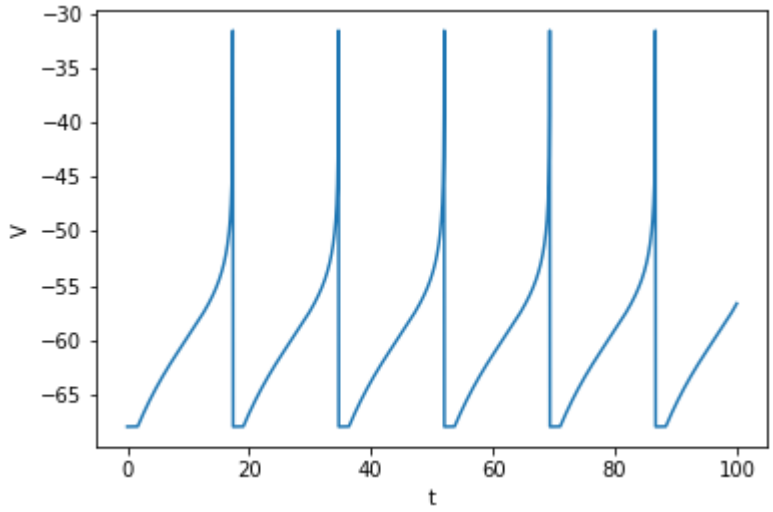
For each neuron in neuron group.

Check if neuron is in refractory period.

Update variables with numerical integration one by one.

Check if neuron spikes.

Reset neuron.




## 1.3.4 Adaptive Exponential Integrate-and-Fire model

While facing a constant stimulus, the response generated by a single neuron will sometimes decreases over time, this phenomenon is called **adaptation** in biology.

To reproduce the adaptation behavior of neurons, researchers add a weight variable $w$ to existing integrate-and-fire models like LIF, QuaIF and ExpIF models. Here we introduce a typical one: **Adaptive Exponential Integrate-and-Fire model** (**AdExIF model**) (Gerstner et al., 2014).

$$\tau_m \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V-V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum \delta(t - t^f))$$

The first differential equation of AdExIF model, as the model's name shows, is similar to ExpIF model we introduced above, except for the term of adaptation, which is shown as $-Rw$ in the equation.

The weight term $w$ is regulated by the second differential equation. $a$ describes the sensitivity of the recovery variable $w$ to the sub-threshold fluctuations of $V$, and $b$ is the increment value of $w$ generated by a spike, and $w$ will also decay over time.

Give AdExIF neuron a constant input, after several spikes, the value of $w$ will increase to a high value, which slows down the rising speed of $V$, thus reduces the neuron's firing rate.

```python
class AdExIF(bp.NeuGroup):
  target_backend = 'general'

  @staticmethod
  def derivative(V, w, t, I_ext, V_rest, delta_T, V_T, R, tau, tau_w, a):
    exp_term = bp.ops.exp((V-V_T)/delta_T)
    dVdt = (-(V-V_rest)+delta_T*exp_term-R*w+R*I_ext)/tau

    dwdt = (a*(V-V_rest)-w)/tau_w

    return dVdt, dwdt

  def __init__(self, size, V_rest=-65., V_reset=-68.,
               V_th=-30., V_T=-59.9, delta_T=3.48,
               a=1., b=1., R=10., tau=10., tau_w=30.,
               t_refractory=0., **kwargs):
    # parameters
    self.V_rest = V_rest
    self.V_reset = V_reset
    self.V_th = V_th
    self.V_T = V_T
    self.delta_T = delta_T
    self.a = a
    self.b = b
    self.R = R
    self.tau = tau
    self.tau_w = tau_w
    self.t_refractory = t_refractory

    # variables
    num = bp.size2len(size)
    self.V = bp.ops.ones(num) * V_reset
    self.w = bp.ops.zeros(size)
    self.input = bp.ops.zeros(num)
    self.spike = bp.ops.zeros(num, dtype=bool)
    self.refractory = bp.ops.zeros(num, dtype=bool)
    self.t_last_spike = bp.ops.ones(num) * -1e7

    self.integral = bp.odeint(f=self.derivative, method='euler')

    super(AdExIF, self).__init__(size=size, **kwargs)

  def update(self, _t):
    for i in prange(self.size[0]):
      spike = 0.
      refractory = (_t - self.t_last_spike[i] <= self.t_refractory)
      if not refractory:
        V, w = self.integral(self.V[i], self.w[i], _t, self.input[i],
                             self.V_rest, self.delta_T,
                             self.V_T, self.R, self.tau, self.tau_w, self.a)
        spike = (V >= self.V_th)
        if spike:
          V = self.V_rest
          w += self.b
          self.t_last_spike[i] = _t
        self.V[i] = V
        self.w[i] = w
      self.spike[i] = spike
      self.refractory[i] = refractory or spike
      self.input[i] = 0.
```

bp.NeuGroup class: Group of neurons

$$\tau \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V-V_T}{\Delta_T}} + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum \delta(t - t^f)$$

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.

Call `bp.odeint` to integrate ODEs. Set parameter `method` to choose numerical integration methods.

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

For each neuron in neuron group.

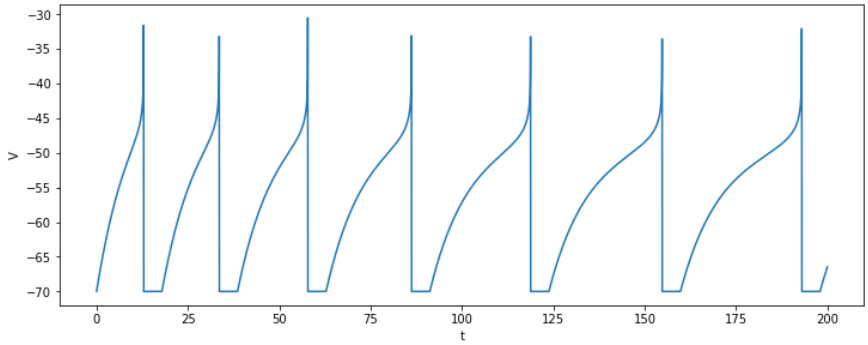
Check if neuron is in refractory period.

Update variables with numerical integration one by one.

Check if neuron spikes.

Reset membrane potential, add the Dirac δ term of w:

$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum \delta(t - t^f)$$




## 1.3.5 Hindmarsh-Rose model

To simulate the bursting spike pattern in neurons (i.e., continuously firing in a short time period), Hindmarsh and Rose (1984) proposed **Hindmarsh-Rose model**, import a third model variable $z$ as slow variable to control the bursting of neuron.

$$\frac{dV}{dt} = y - aV^3 + bV^2 - z + I$$

$$\frac{dy}{dt} = c - dV^2 - y$$

$$\frac{dz}{dt} = r(s(V - V_{rest}) - z)$$

The $V$ variable refers to membrane potential, and $y$, $z$ are two gating variables.

The parameter $b$ in $\frac{dV}{dt}$ equation allows the model to switch between spiking and bursting states, and controls the spiking frequency. $r$ controls slow variable $z$'s variation speed, affects the number of spikes per burst when bursting, and governs the spiking frequency together with $b$. The parameter $s$ governs adaptation, and other parameters are fitted by firing patterns.

```python
class HindmarshRose(bp.NeuGroup):
  target_backend = 'general'

  @staticmethod
  def derivative(V, y, z, t, a, b, I_ext, c, d, r, s, V_rest):
    dVdt = y - a * V * V * V + b * V * V - z + I_ext
    dydt = c - d * V * V - y
    dzdt = r * (s * (V - V_rest) - z)
    return dVdt, dydt, dzdt

  def __init__(self, size, a=1., b=3.,
               c=1., d=5., r=0.01, s=4.,
               V_rest=-1.6, **kwargs):
    # parameters
    self.a = a
    self.b = b
    self.c = c
    self.d = d
    self.r = r
    self.s = s
    self.V_rest = V_rest

    # variables
    num = bp.size2len(size)
    self.z = bp.ops.zeros(num)
    self.input = bp.ops.zeros(num)
    self.V = bp.ops.ones(num) * -1.6
    self.y = bp.ops.ones(num) * -10.
    self.spike = bp.ops.zeros(num, dtype=bool)

    self.integral = bp.odeint(f=self.derivative)
    super(HindmarshRose, self).__init__(size=size, **kwargs)

  def update(self, _t):
    for i in prange(self.num):
      V, self.y[i], self.z[i] = self.integral(
        self.V[i], self.y[i], self.z[i], _t,
        self.a, self.b, self.input[i],
        self.c, self.d, self.r, self.s,
        self.V_rest)
      self.V[i] = V
      self.input[i] = 0.
```

bp.NeuGroup class: Group of neurons

$dV/dt = y - aV^3 + bV^2 - z + I$
$dy/dt = c - dV^2 - y$
$dz/dt = r(s(V - V_{rest}) - z)$

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.

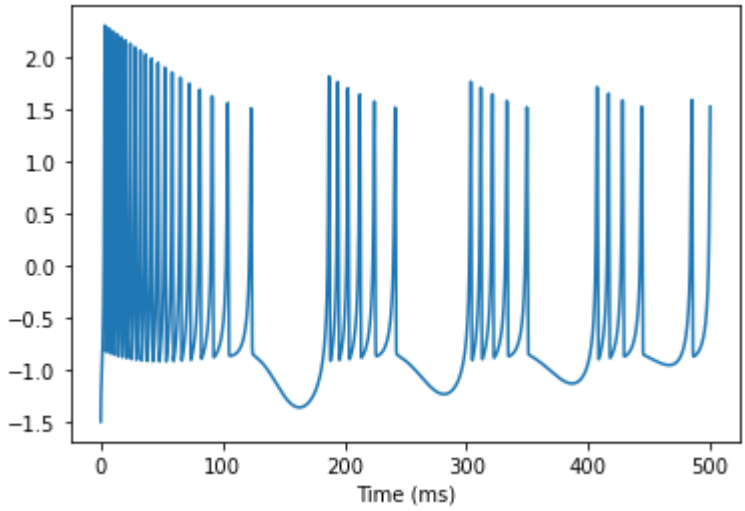
Call `bp.odeint` to integrate ODEs. Parameter `method` is set to default value `euler`.

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

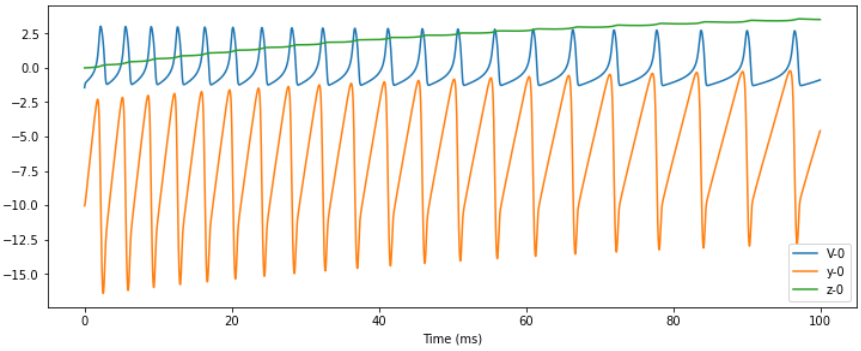
For each neuron in neuron group.

Update variables with numerical integration one by one.

Reset external input for this time step.

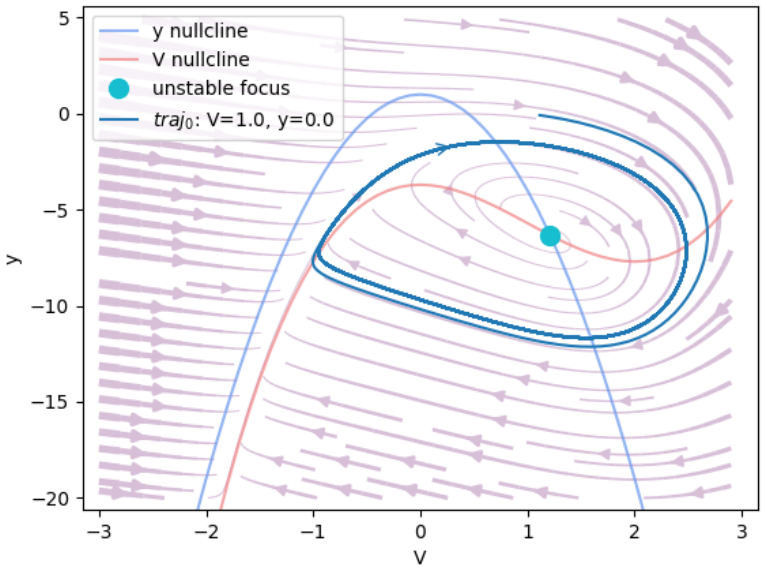



In the variable-t plot painted below, we may see that the slow variable $z$ changes much slower than $V$ and $y$. Also, $V$ and $y$ are changing periodically during the simulation.




With the theoretical analysis module `analysis` of BrainPy, we may explain the existence of this periodicity through theoretical analysis. In Hindmarsh-Rose model, the trajectory of $V$ and $y$ approaches a limit cycle in phase plane, therefore their values change periodically along the limit cycle.

```
68  # Phase plane analysis
69  phase_plane_analyzer = bp.analysis.PhasePlane(
70      neu.integral,                                          → Dynamic system to be analyzed.
71      target_vars={'V': [-3., 3.], 'y': [-20., 5.]},         → Variables to be showed in phase plane.
72      fixed_vars={'z': 0.},                                  → Variables to be fixed in phase plane.
73      pars_update={'I_ext': param[mode][1], 'a': 1., 'b': 3.,
74                   'c': 1., 'd': 5., 'r': 0.01, 's': 4.,     } Other parameters to be fixed.
75                   'V_rest': -1.6}
76  )
77  phase_plane_analyzer.plot_nullcline()                      → Plot nullcline.
78  phase_plane_analyzer.plot_fixed_point()                    → Plot fixed points.
79  phase_plane_analyzer.plot_vector_field()                   → Plot vector field.
80  phase_plane_analyzer.plot_trajectory(                      } Plot trajectory.
81      [{'V': 1., 'y': 0., 'z': -0.}],                        } Define start point of trajectory and
82      duration=100.,                                         } simulation duration.
83      show=True
84  )
```




### 1.3.6 Generalized Integrate-and-Fire model

**Generalized Integrate-and-Fire model** (**GIF model**) (Mihalaş et al., 2009) integrates several firing patterns in one model. With 4 model variables, it can generate more than 20 types of firing patterns, and is able to alternate between patterns by fitting parameters.

$$\frac{dI_j}{dt} = -k_j I_j, j = 1, 2$$

$$\tau \frac{dV}{dt} = (-(V - V_{rest}) + R\sum_j I_j + RI)$$

$$\frac{dV_{th}}{dt} = a(V - V_{rest}) - b(V_{th} - V_{th\infty})$$

When $V$ meets $V_{th}$, Generalized IF neuron fire:

$$I_j \leftarrow R_j I_j + A_j$$

$$V \leftarrow V_{reset}$$

$$V_{th} \leftarrow max(V_{th_{reset}}, V_{th})$$

In the $\frac{dV}{dt}$ differential equation, just like all the integrate-and-fire models, $\tau$ is time constant, $V$ is membrane potential, $V_{rest}$ is resting potential, $R$ is conductance, and $I$ is external input.

However, in GIF model, variable amounts of internal currents are added to the equation, shown as the $\sum_j I_j$ term. Each $I_j$ is an internal current in the neuron, with a decay rate of $k_j$. $R_j$ and $A_j$ are free parameters, $R_j$ describes the

dependence of $I_j$ reset value on the value of $I_j$ before spike, and $A_j$ is a constant value added to the reset value after spike.

The variable threshold potential $V_{th}$ is regulated by two parameters: $a$ describes the dependence of $V_{th}$ on the membrane potential $V$, and $b$ is the rate $V_{th}$ approaches the infinite value of threshold $V_{th_\infty}$. $V_{th_{reset}}$ is the reset value of threshold potential when neuron fires.

```python
class GeneralizedIF(bp.NeuGroup):
  target_backend = 'general'

  @staticmethod
  def derivative(I1, I2, V_th, V, t,
                 k1, k2, a, V_rest, b, V_th_inf,
                 R, I_ext, tau):
    dI1dt = - k1 * I1
    dI2dt = - k2 * I2
    dVthdt = a * (V - V_rest) - b * (V_th - V_th_inf)
    dVdt = (- (V - V_rest) + R * I_ext + R * I1 + R * I2) / tau
    return dI1dt, dI2dt, dVthdt, dVdt

  def __init__(self, size, V_rest=-70., V_reset=-70.,
               V_th_inf=-50., V_th_reset=-60., R=20., tau=20.,
               a=0., b=0.01, k1=0.2, k2=0.02,
               R1=0., R2=1., A1=0., A2=0.,
               **kwargs):
    # params
    self.V_rest = V_rest
    self.V_reset = V_reset
    self.V_th_inf = V_th_inf
    self.V_th_reset = V_th_reset
    self.R = R
    self.tau = tau
    self.a = a
    self.b = b
    self.k1 = k1
    self.k2 = k2
    self.R1 = R1
    self.R2 = R2
    self.A1 = A1
    self.A2 = A2

    # vars
    self.input = bp.ops.zeros(size)
    self.spike = bp.ops.zeros(size, dtype=bool)
    self.I1 = bp.ops.zeros(size)
    self.I2 = bp.ops.zeros(size)
    self.V = bp.ops.ones(size) * -70.
    self.V_th = bp.ops.ones(size) * -50.

    self.integral = bp.odeint(self.derivative)
    super(GeneralizedIF, self).__init__(size=size, **kwargs)
```

bp.NeuGroup class: Group of neurons

$$dI_1/dt = -k_1 I_1$$
$$dI_2/dt = -k_2 I_2$$
$$dV_{th}/dt = a(V - V_{rest}) - b(V_{th} - V_{th_\infty})$$
$$\tau dV/dt = -(V - V_{rest}) + R\sum I_j + RI(t), \quad j = 1,2$$

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.

Call `bp.odeint` to integrate ODEs. Parameter `method` is set to default value `euler`.

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

```
46  def update(self, _t):
47    for i in prange(self.size[0]):
48      I1, I2, V_th, V = self.integral(
49        self.I1[i], self.I2[i], self.V_th[i], self.V[i], _t,
50        self.k1, self.k2, self.a, self.V_rest,
51        self.b, self.V_th_inf,
52        self.R, self.input[i], self.tau
53      )
54      self.spike[i] = self.V_th[i] < V
55      if self.spike[i]:
56        V = self.V_reset
57        I1 = self.R1 * I1 + self.A1
58        I2 = self.R2 * I2 + self.A2
59        V_th = max(V_th, self.V_th_reset)
60      self.I1[i] = I1
61      self.I2[i] = I2
62      self.V_th[i] = V_th
63      self.V[i] = V
64    self.f = 0.
65    self.input[:] = self.f
```

For each neuron in neuron group.

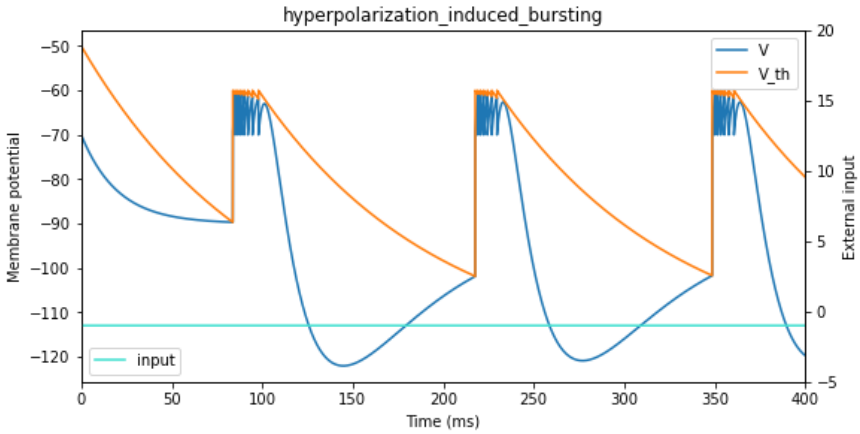
Update variables with numerical integration one by one.

Check if neuron spikes.

Reset membrane potential and threshold potential, update I1, I2.

$V = V_{reset}$
$I_1 = R_1 I_1 + A_1$
$I_2 = R_2 I_2 + A_2$
$V_{th} = \max(V_{th}, V_{th_{reset}})$

Reset external input for this time step.



# 1.4 Firing Rate models

Firing Rate models are simpler than reduced models. In these models, each compute unit represents a neuron group, the membrane potential variable $V$ in single neuron models is replaced by firing rate variable $a$ (or $r$ or $\nu$). Here we introduce a canonical firing rate unit.

## 1.4.1 Firing Rate Unit

Wilson and Cowan (1972) proposed this unit to represent the activities in excitatory and inhibitory cortical neuron columns. Each element of variables $a_e$ and $a_i$ refers to the average activity of a neuron column group contains multiple neurons.

$$\tau_e \frac{da_e(t)}{dt} = -a_e(t) + (k_e - r_e * a_e(t)) * \mathcal{S}(c_1 a_e(t) - c_2 a_i(t) + I_{ext_e}(t))$$

$$\tau_i \frac{da_i(t)}{dt} = -a_i(t) + (k_i - r_i * a_i(t)) * \mathcal{S}(c_3 a_e(t) - c_4 a_i(t) + I_{ext_i}(t))$$

$$\mathcal{S}(x) = \frac{1}{1 + exp(-a(x - \theta))} - \frac{1}{1 + exp(a\theta)}$$

The subscript $x \in \{e, i\}$ points out whether this parameter or variable corresponds to excitatory or inhibitory neuron group. In the differential equations, $\tau_x$ refers to the time constant of neuron columns, parameters $k_x$ and $r_x$ control the refractory periods, $a_x$ and $\theta_x$ refer to the slope factors and phase parameters of sigmoid functions, and external inputs $I_{ext_x}$ are given separately to excitatory and inhibitory neuron groups.

```python
class FiringRateUnit(bp.NeuGroup):
    target_backend = 'general'

    @staticmethod
    def derivative(a_e, a_i, t,
                   k_e, r_e, c1, c2, I_ext_e, slope_e, theta_e, tau_e,
                   k_i, r_i, c3, c4, I_ext_i, slope_i, theta_i, tau_i):
        x_ae = c1 * a_e - c2 * a_i + I_ext_e
        sigmoid_ae_l = 1 / (1 + bp.ops.exp(- slope_e * (x_ae - theta_e)))
        sigmoid_ae_r = 1 / (1 + bp.ops.exp(slope_e * theta_e))
        sigmoid_ae = sigmoid_ae_l - sigmoid_ae_r
        daedt = (- a_e + (k_e - r_e * a_e) * sigmoid_ae) / tau_e

        x_ai = c3 * a_e - c4 * a_i + I_ext_i
        sigmoid_ai_l = 1 / (1 + bp.ops.exp(- slope_i * (x_ai - theta_i)))
        sigmoid_ai_r = 1 / (1 + bp.ops.exp(slope_i * theta_i))
        sigmoid_ai = sigmoid_ai_l - sigmoid_ai_r
        daidt = (- a_i + (k_i - r_i * a_i) * sigmoid_ai) / tau_i
        return daedt, daidt

```

$$S(x) = \frac{1}{1 + \exp(-a(x - \theta))} - \frac{1}{1 + \exp(a\theta)}$$

$$\tau_e \frac{da_e}{dt} = -a_e + (k_e - r_e * a_e) * \mathrm{S_e}(\mathrm{c_1 a_e} - \mathrm{c_2 a_i} + \mathrm{I_{ext_e}})$$

$$\tau_i \frac{da_i}{dt} = -a_i + (k_i - r_i * a_i) * \mathrm{S_i}(\mathrm{c_1 a_e} - \mathrm{c_2 a_i} + \mathrm{I_{ext_i}})$$

```
21      def __init__(self, size, c1=12., c2=4., c3=13., c4=11.,
22                   k_e=1., k_i=1., tau_e=1., tau_i=1., r_e=1., r_i=1.,
23                   slope_e=1.2, slope_i=1., theta_e=2.8, theta_i=4.,
24                   **kwargs):
25          # params
26          self.c1 = c1
27          self.c2 = c2
28          self.c3 = c3
29          self.c4 = c4
30          self.k_e = k_e
31          self.k_i = k_i
32          self.tau_e = tau_e
33          self.tau_i = tau_i
34          self.r_e = r_e
35          self.r_i = r_i
36          self.slope_e = slope_e
37          self.slope_i = slope_i
38          self.theta_e = theta_e
39          self.theta_i = theta_i
40
41          # vars
42          self.input_e = bp.backend.zeros(size)
43          self.input_i = bp.backend.zeros(size)
44          self.a_e = bp.backend.ones(size) * 0.1
45          self.a_i = bp.backend.ones(size) * 0.05
46
47          self.integral = bp.odeint(self.derivative)
48          super(FiringRateUnit, self).__init__(size=size, **kwargs)
49
50      def update(self, _t):
51          self.a_e, self.a_i = self.integral(
52              self.a_e, self.a_i, _t,
53              self.k_e, self.r_e, self.c1, self.c2,
54              self.input_e, self.slope_e,
55              self.theta_e, self.tau_e,
56              self.k_i, self.r_i, self.c3, self.c4,
57              self.input_i, self.slope_i,
58              self.theta_i, self.tau_i)
59          self.input_e[:] = 0.
60          self.input_i[:] = 0.
```

Model parameters saved as floating point numbers.

Model variables saved as vectors of floating point numbers.

Call `bp.odeint` to integrate ODEs. Parameter `method` is set to default value `euler`.

Pass `size` and `**kwargs` to superclass bp.NeuGroup's constrctor.

Update variables with numerical integration in vector form.

Reset external input for this time step.

# 2. Synapse models

When we model the firing of neurons, we need to connect them. Synapse is very important for the communication between neurons, and it is an essential component of the formation of the network. Therefore, we need to model the synapse.

We will first introduce how to implement synaptic dynamics with BrainPy, then introduce synapse plasticity.

## 2.1 Synaptic Models

## 2.2 Plasticity Models

# 2.1 Synaptic Models

In the previous section, we learned how to model neurons and their action potentials. In this section, we will focus on how neurons communicate.

## 2.1.1 Chemical Synapses

## Biological Background

Fig. 2-1 shows the biological process of information transmission between neurons. The action potential of the presynaptic neuron makes the axon terminal release **neurotransmitters** (also called transmitters) into the synaptic cleft, and then the membrane potential of the postsynaptic cell changes after a brief delay. These changes are called postsynaptic potentials (PSP), and they can be either excitatory or inhibitory depending on the type of transmitter. **Glutamate** is one of the important excitatory neurotransmitters, and Gamma-aminobutyric acid (**GABA**) is one of the important inhibitory neurotransmitters.

Neurotransmitters affect their targets by interacting with receptors on the postsynaptic membrane. When the transmitter binds to the receptor, it would either open an ion channel (**ionotropic** receptors) or alter chemical reactions within the target cell (**metabotropic** receptors).

In this section, we will introduce how to model some common synapses and their implementations with `BrainPy`:

- **AMPA** and **NMDA** receptors are both ionotropic receptors of Glutamate, but the NMDA receptor are typically blocked by magnesium ions ($Mg^{2+}$) and cannot respond to the glutamate. With repeated activation of AMPA receptors, the change in postsynaptic potential drives $Mg^{2+}$ out of NMDA channel, then the NMDA receptors are able to respond to glutamate. Therefore, the dynamics of NMDA is much slower than that of AMPA.
- **$GABA_A$** and **$GABA_B$** are two classes of GABA receptors. $GABA_A$ receptors are ionotropic, typically producing fast inhibitory postsynaptic potential; while $GABA_B$ receptors are metabotropic receptors, typically producing a slow-occurring inhibitory postsynaptic potential.

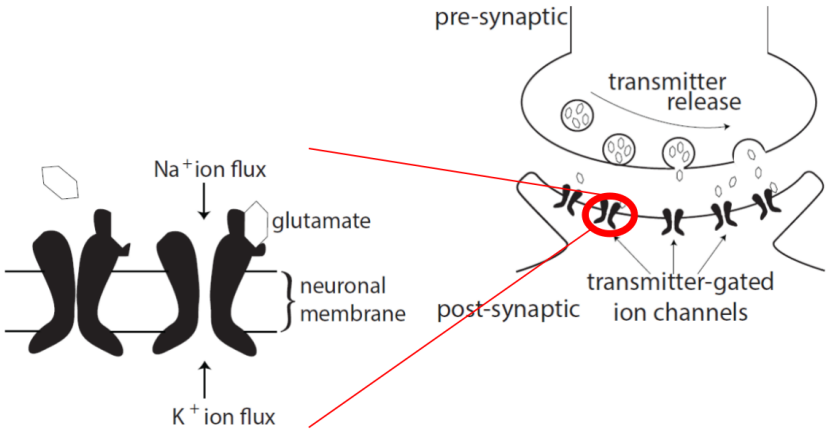



**Fig. 2-1 Biological Synapse** (Adapted from *Gerstner et al., 2014* [1])

In order to keep things simple, we use gating variable `s` to describe how many portion of ion channels will open whenever a presynaptic spike arrives while modeling. We will first introduce AMPA receptor as an example to show how to develop synapse models and implement them with `BrainPy`.

## AMPA Synapse

As we mentioned before, the AMPA receptor is an ionotropic receptor, that is, when a neurotransmitter binds to it, the ion channel will be opened immediately to allow $Na^+$ and $K^+$ ions flux.

We can use Markov process to describe the opening and closing process of ion channels. As shown in Fig. 2-2, $s$ represents the probability of channel opening, $1-s$ represents the probability of ion channel closing, and $\alpha$ and $\beta$ are the transition probability. Because neurotransmitters can open ion channels, the transfer probability from $1-s$ to $s$ is affected by the concentration of neurotransmitters. We denote the concentration of neurotransmitters as [T].

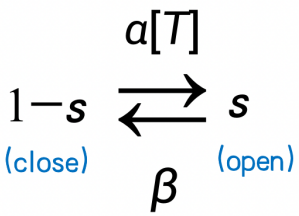



**Fig. 2-2 Markov process of channel dynamics**

We obtained the following formula when describing the process by a differential equation.

$$\frac{ds}{dt} = \alpha[T](1-s) - \beta s$$

Where $\alpha[T]$ denotes the transition probability from state $(1-s)$ to state $(s)$; and $\beta$ represents the transition probability of the other direction.

Now let's see how to implement such a model with BrainPy. First of all, we need to define a class that inherits from `bp.TwoEndConn`, because synapses connect two neurons. Within the class, we can define the differential equation with `derivative` function, this is the same as the definition of neuron models. Then we use the `__ init__` Function to initialize the required parameters and variables.

```
import brainpy as bp


class AMPA(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(s, t, TT, alpha, beta):
        ds = alpha * TT * (1 - s) - beta * s
        return ds

    def __init__(self, pre, post, conn, alpha=0.98, beta=0.18, T=0.5,
                 T_duration=0.5, **kwargs):
        # parameters
        self.alpha = alpha
        self.beta = beta
        self.T = T
        self.T_duration = T_duration

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.s = bp.ops.zeros(self.size)
        self.t_last_pre_spike = -1e7 * bp.ops.ones(self.size)

        self.int_s = bp.odeint(f=self.derivative, method='exponential_euler')
        super(AMPA, self).__init__(pre=pre, post=post, **kwargs)

```

bp.TwoEndConn class:
Connections between two neuron groups

$$\frac{ds}{dt} = \alpha[T](1-s) - \beta s$$

Regard [T] as a constant T, last for T_duration

Specify how the presynaptic and postsynaptic neuron groups connect to each other

| | | | | |
|---|---|---|---|---|
| pre ids | 0 | 0 | 0 | 1 |
| post ids | 3 | 5 | 7 | 0 |
| syn ids | 0 | 1 | 2 | 3 |

We update $s$ by an `update` function.

```
    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]
            post_id = self.post_ids[i]

            if self.pre.spike[pre_id]:
                self.t_last_pre_spike[pre_id] = _t
            TT = ((_t - self.t_last_pre_spike[pre_id])
                  < self.T_duration) * self.T
            self.s[i] = self.int_s(self.s[i], _t, TT, self.alpha, self.beta)

```

For each single synapse

TT denotes [T], TT=T if pre neuron spikes within T_duration, otherwise TT=0.

After the implementation, we can plot the graph of $s$ changing with time. We would first write a `run_syn` function to run and plot the graph. To run a synapse, we need neuron groups, so we use the LIF neuron provided by `brainmodels` package.

```

import brainmodels as bm

bp.backend.set(backend='numba', dt=0.1)


def run_syn(syn_model, **kwargs):
    neu1 = bm.neurons.LIF(2, monitors=['V'])
    neu2 = bm.neurons.LIF(3, monitors=['V'])

    syn = syn_model(pre=neu1, post=neu2, conn=bp.connect.All2All(),
                    monitors=['s'], **kwargs)

    net = bp.Network(neu1, syn, neu2)
    net.run(30., inputs=(neu1, 'input', 35.))
    bp.visualize.line_plot(net.ts, syn.mon.s, ylabel='s', show=True)


run_syn(AMPA, T_duration=3.)

```

Set numba backend

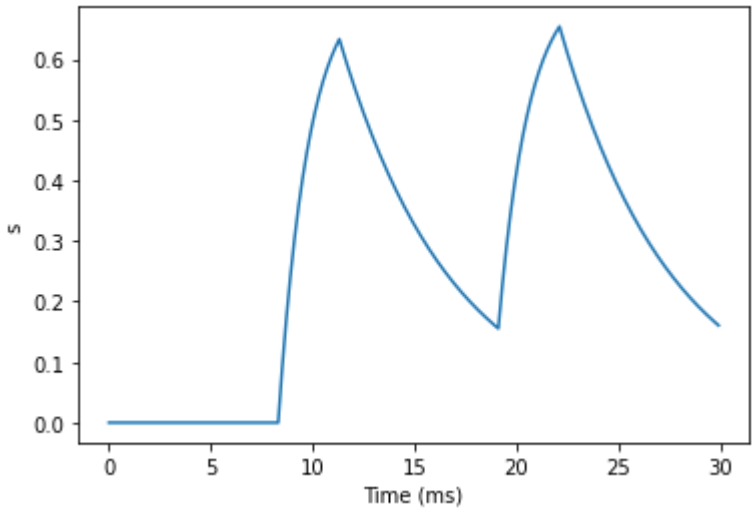
A method to run synapse

Get two LIF neuron groups from brainmodels package

Specify pre and post neurons and there connections. Here we use all to all connections provided by BrainPy.

Run AMPA synapse and set parameter T_duration to be 3.

Then we would expect to see the following result:




As can be seen from the above figure, when the presynaptic neurons fire, the value of $s$ will first increase, and then decay.

## Alpha、Exponential Synapses

Because many synaptic models have the same dynamic characteristics as AMPA synapses, sometimes we don't need to use models that specifically correspond to biological synapses. Therefore, some abstract synaptic models have been proposed. Here, we will introduce the implementation of four abstract models on BrainPy. These models can also be found in the `Brain-Models` package.

**(1) Differences of two exponentials**

Let's first introduce the `Differences of two exponentials` model, its dynamics is given by,

$$s = \frac{\tau_1 \tau_2}{\tau_1 - \tau_2}(\exp(-\frac{t - t_s}{\tau_1}) - \exp(-\frac{t - t_s}{\tau_2}))$$

Where $t_s$ denotes the spike timing of the presynatic neuron, $\tau_1$ and $\tau_2$ are time constants.

While implementing with BrainPy, we use the following differential equation form,

$$\frac{ds}{dt} = x$$

$$\frac{dx}{dt} = -\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}x - \frac{s}{\tau_1 \tau_2}$$

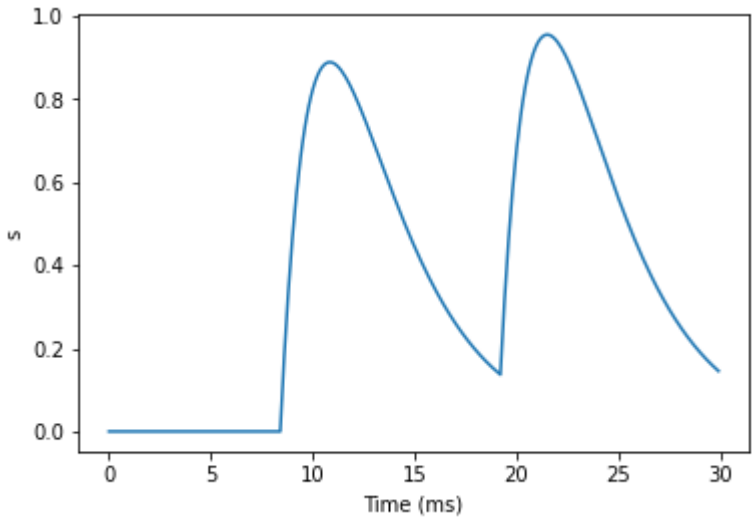
$$\text{if } (\text{fire}), \text{then } x \leftarrow x + 1$$

Here we specify the logic of increment of $x$ in the `update` function when the presynaptic neurons fire. The code is as follows:

```python
class Two_exponentials(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(s, x, t, tau1, tau2):
        dxdt = (-(tau1 + tau2) * x - s) / (tau1 * tau2)
        dsdt = x
        return dsdt, dxdt

    def __init__(self, pre, post, conn, tau1=1.0, tau2=3.0, **kwargs):
        # parameters
        self.tau1 = tau1
        self.tau2 = tau2

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.s = bp.ops.zeros(self.size)
        self.x = bp.ops.zeros(self.size)

        self.integral = bp.odeint(f=self.derivative, method='rk4')

        super(Two_exponentials, self).__init__(pre=pre, post=post, **kwargs)

    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]

            self.s[i], self.x[i] = self.integral(self.s[i], self.x[i], _t,
                                                 self.tau1, self.tau2)
            self.x[i] += self.pre.spike[pre_id]
                                                 # if (pre spike), then x = x + 1

run_syn(Two_exponentials, tau1=2.)
```

$$\frac{dx}{dt} = (-(\tau_1 + \tau_2)x - s)/(\tau_1\tau_2)$$
$$\frac{ds}{dt} = x$$

Then we expect to see the following result:




**(2) Alpha synapse**

Dynamics of `Alpha synapse` is given by,

$$s = \frac{t - t_s}{\tau} \exp(-\frac{t - t_s}{\tau})$$

As the dual exponential synapse, $t_s$ denotes the spike timing of the presynaptic neuron, with a time constant $\tau$.

The differential equation form of alpha synapse is also very similar with the dual exponential synapses, with $\tau = \tau_1 = \tau_2$, as shown below:

$$\frac{ds}{dt} = x$$

$$\frac{dx}{dt} = -\frac{2x}{\tau} - \frac{s}{\tau^2}$$

$$\text{if } (\text{fire}), \text{ then } x \leftarrow x + 1$$

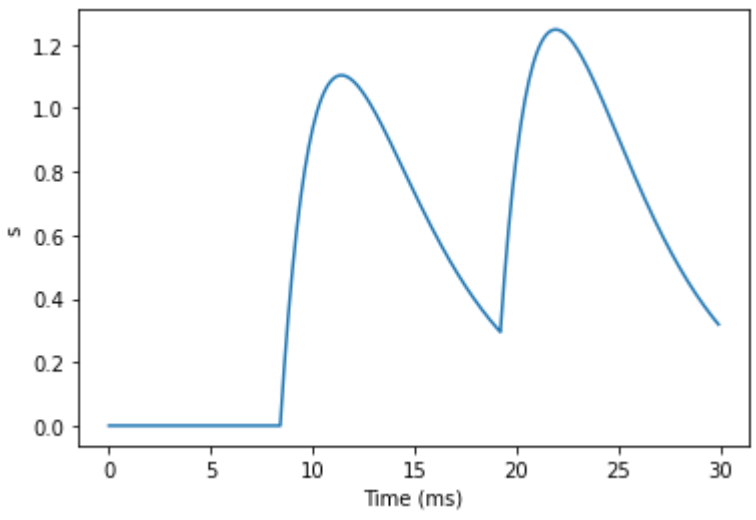
Code implementation is similar:

```python
class Alpha(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(s, x, t, tau):
        dxdt = (-2 * tau * x - s) / (tau ** 2)
        dsdt = x
        return dsdt, dxdt

    def __init__(self, pre, post, conn, tau=3.0, **kwargs):
        # parameters
        self.tau = tau

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.s = bp.ops.zeros(self.size)
        self.x = bp.ops.zeros(self.size)

        self.integral = bp.odeint(f=self.derivative, method='rk4')

        super(Alpha, self).__init__(pre=pre, post=post, **kwargs)

    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]

            self.s[i], self.x[i] = self.integral(self.s[i], self.x[i], _t,
                                                 self.tau)
            self.x[i] += self.pre.spike[pre_id]

run_syn(Alpha)
```

$\frac{dx}{dt} = (-2\tau x - s)/\tau^2$, $\frac{ds}{dt} = x$

if (pre spike), then x = x + 1

Then we expect to see the following result:




**(3) Single exponential decay**

Sometimes we can ignore the rising process in modeling, and only need to model the decay process. Therefore, the formula of `single exponential decay` model is more simplified:

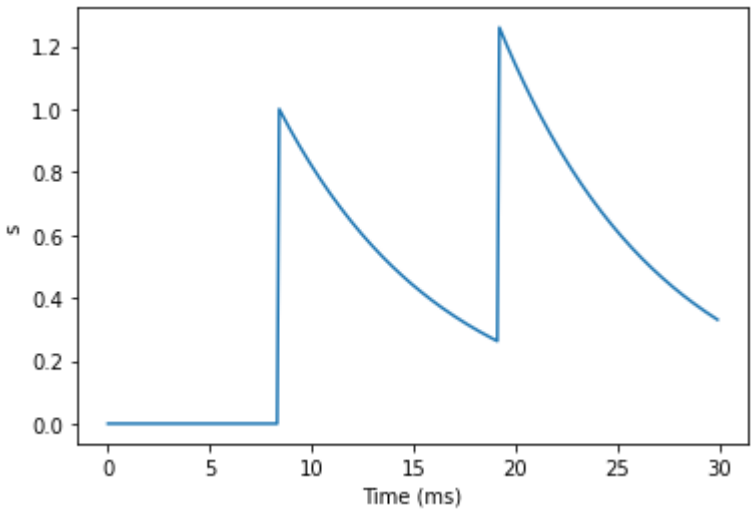
$$\frac{ds}{dt} = -\frac{s}{\tau_{decay}}$$

$$\text{if } (\text{fire}), \text{ then } s \leftarrow s + 1$$

The implementing code is given by:

```
class Exponential(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(s, t, tau):
        ds = -s / tau      ---------->  ds/dt = -s/τ
        return ds

    def __init__(self, pre, post, conn, tau=8.0, **kwargs):
        # parameters
        self.tau = tau

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.s = bp.ops.zeros(self.size)

        self.integral = bp.odeint(f=self.derivative, method='exponential_euler')

        super(Exponential, self).__init__(pre=pre, post=post, **kwargs)

    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]

            self.s[i] = self.integral(self.s[i], _t, self.tau)
            self.s[i] += self.pre.spike[pre_id]   ----------> if (pre spike), then s = s + 1


run_syn(Exponential)
```

Then we expect to see the following result:




**(4) Voltage jump**

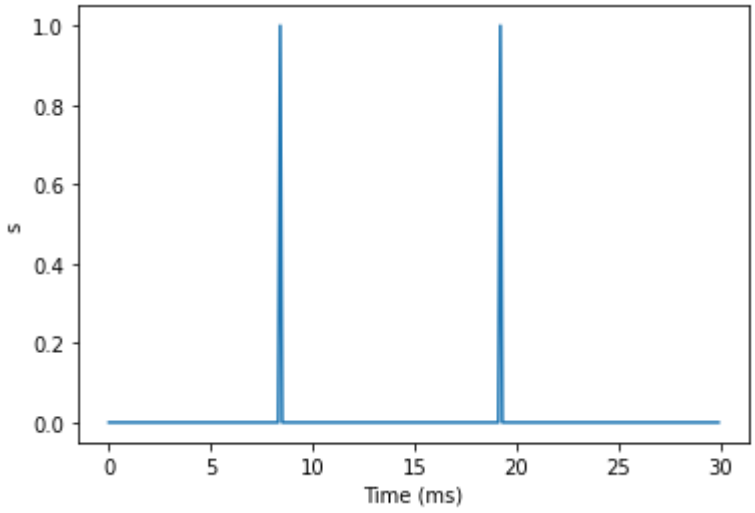
Sometimes even the decay process can be ignored, so there is a `voltage jump` model, which is given by:

$$\text{if } (\text{fire}), \text{ then } s \leftarrow s + 1$$

The code is as follows:

```python
class Voltage_jump(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    def __init__(self, pre, post, conn, **kwargs):
        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.s = bp.ops.zeros(self.size)

        super(Voltage_jump, self).__init__(pre=pre, post=post, **kwargs)

    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]
            self.s[i] = self.pre.spike[pre_id] ————————————→ if (pre spike), then s = s + 1


run_syn(Voltage_jump)
```

Then we expect to see the following result:




## Current-based and Conductance-based synapses

So far, we have modeled the gating variable $s$, now let's see how to model the effect of the gating variables on the synaptic current. The current that passes through a synaptic channel is denoted as $I$. There are two different methods to model the relationships between $s$ and $I$: **current-based** and **conductance-based**. The main difference between them is whether the synaptic current is influenced by the membrane potential of postsynaptic neurons.

#### (1) Current-based

The formula of the current-based model is as follow:

$$I \propto s$$

While coding, we usually multiply $s$ by a weight $w$. We can implement excitatory and inhibitory synapses by adjusting the positive and negative values of the weight $w$.

The delay of synapses is implemented by applying the delay time to the `I_syn` variable using the `register_constant_delay` function provided by BrainPy.

```python
def __init__(self, pre, post, conn, delay, **kwargs):
    # ...
    self.s = bp.ops.zeros(self.size)
    self.w = bp.ops.ones(self.size) * .2
    self.I_syn = self.register_constant_delay('I_syn', size=self.size,
                                              delay_time=delay)

def update(self, _t):
    for i in range(self.size):
        # ...
        self.I_syn.push(i, self.w[i] * self.s[i])
        self.post.input[post_id] += self.I_syn.pull(i)
```

set delay time before changing synaptic current by using the register_constant_delay method

use push and pull to set delay before applying synaptic current into postsynatic input.

**(2) Conductance-based**

In the conductance-based model, the conductance is $g = \bar{g}s$. Therefore, according to Ohm's law, the formula is given by:

$$I = \bar{g}s(V - E)$$

Here $E$ is a reverse potential, which can determine whether the effect of $I$ is inhibition or excitation. For example, when the resting potential is about -65, subtracting a lower $E$, such as -75, will become positive, thus will change the direction of the current in the formula and produce the suppression current. The $E$ value of excitatory synapses is relatively high, such as 0.

In terms of implementation, you can apply a synaptic delay to the variable `g`.

```python
def __init__(self, pre, post, conn, g_max, E, delay, **kwargs):
    self.g_max = g_max
    self.E = E
    # ...
    self.s = bp.ops.zeros(self.size)
    self.g = self.register_constant_delay('g', size=self.size, delay_time=delay)

def update(self, _t):
    for i in range(self.size):
        # ...
        self.g.push(i, self.g_max * self.s[i])
        self.post.input[post_id] -= self.g.pull(i) * (self.post.V[post_id] -
                                                      self.E)
```

set delay time before changing the conductance

$g = \bar{g}s$

$I = g(V-)$

## 2.1.2 Electrical Synapses

In addition to the chemical synapses described earlier, electrical synapses are also common in our neural system.

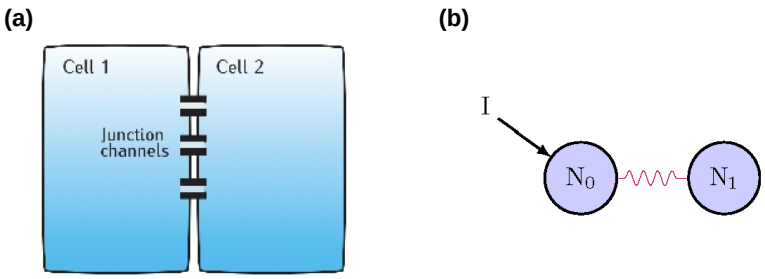



**Fig. 2-3 (a)** Gap junction connection between two cells. **(b)** An equivalent diagram.
(From *Sterratt et al., 2011* [2])

As shown in the Fig. 2-3a, two neurons are connected by junction channels and can conduct electricity directly. Therefore, it can be seen that two neurons are connected by a constant resistance, as shown in the Fig. 2-3b.

According to Ohm's law, the current of one neuron is given by,

$$I_1 = w(V_0 - V_1)$$

where $V_0$ and $V_1$ are the membrane potentials of the two neurons, and the synaptic weight $w$ is equivalent with the conductance.

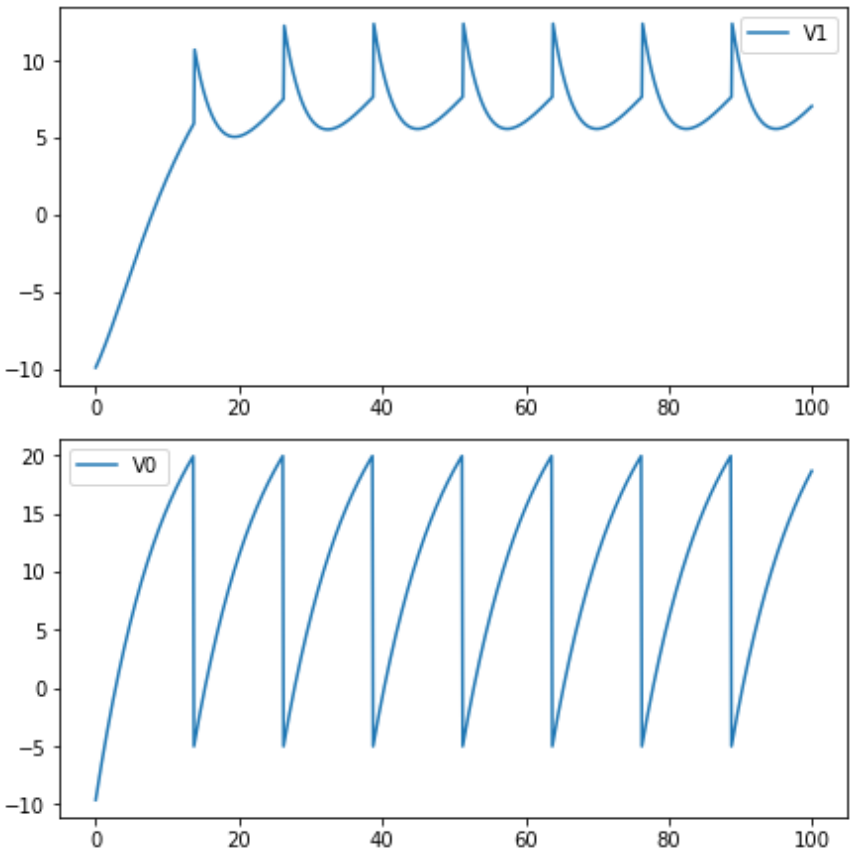
While implementing with BrainPy, you only need to specify the equation in the `update` function.

```
class Gap_junction(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    def __init__(self, pre, post, conn, delay=0., k_spikelet=0.1,
                 post_refractory=False, **kwargs):
        self.delay = delay
        self.k_spikelet = k_spikelet
        self.post_has_refractory = post_refractory

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.w = bp.ops.ones(self.size)
        self.spikelet = self.register_constant_delay('spikelet', size=self.size,
                                                     delay_time=self.delay)

        super(Gap_junction, self).__init__(pre=pre, post=post, **kwargs)

    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]
            post_id = self.post_ids[i]

            self.post.input[post_id] += self.w[i] * (self.pre.V[pre_id] -
                                                     self.post.V[post_id])

            self.spikelet.push(i, self.w[i] * self.k_spikelet *
                               self.pre.spike[pre_id])

            out = self.spikelet.pull(i)
            if self.post_has_refractory:
                self.post.V[post_id] += out * (1. -
                                               self.post.refractory[post_id])
            else:
                self.post.V[post_id] += out

```

$I_{post} = w\,(V_{pre} - V_{post})$

Then we can run a simulation.

```
import matplotlib.pyplot as plt
import numpy as np

neu0 = bm.neurons.LIF(2, monitors=['V'], t_refractory=0)
neu0.V = np.ones(neu0.V.shape) * -10.
neu1 = bm.neurons.LIF(3, monitors=['V'], t_refractory=0)
neu1.V = np.ones(neu1.V.shape) * -10.
syn = Gap_junction(k_spikelet=5., pre=neu0, post=neu1,
                   conn=bp.connect.All2All())
syn.w = np.ones(syn.w.shape) * .5

net = bp.Network(neu0, neu1, syn)
net.run(100., inputs=(neu0, 'input', 30.))

fig, gs = bp.visualize.get_figure(row_num=2, col_num=1, )

fig.add_subplot(gs[1, 0])
plt.plot(net.ts, neu0.mon.V[:, 0], label='V0')
plt.legend()

fig.add_subplot(gs[0, 0])
plt.plot(net.ts, neu1.mon.V[:, 0], label='V1')
plt.legend()
plt.show()
```




## References

[1]. Gerstner, Wulfram, et al. Neuronal dynamics: From single neurons to networks and models of cognition. Cambridge University Press, 2014. ↩

> [2]. Sterratt, David, et al. Principles of computational modeling in neuroscience. Cambridge University Press, 2011. ↩

# 2.2 Plasticity Models

We just talked about synaptic dynamics, but we haven't talked about synaptic plasticity. Next, let's see how to use BrainPy to implement synaptic plasticity.

Plasticity mainly distinguishes short-term plasticity from long-term plasticity. We will first introduce short-term plasticity (STP), and then introduce several different models of long-term synaptic plasticity (also known as learning rules).

The introduction is as follows:

- Short-term plasticity
- Long-term plasticity
  - Spike-timing dependent plasticity
  - Rate-based Hebb rule
    - Oja's rule
    - BCM rule

## 2.2.1 Short-term plasticity (STP)

Let's first look at short-term plasticity. We will start with the results of the experiment. Fig. 2-1 shows the changes of the membrane potential of postsynaptic neurons as the firing of presynaptic neurons. We can see that when the presynaptic neurons repeatedly firing with short intervals, the response of the postsynaptic neurons becomes weaker and weaker, showing a short term depression. But the response recovers after a short period of time, so this plasticity is short-term.

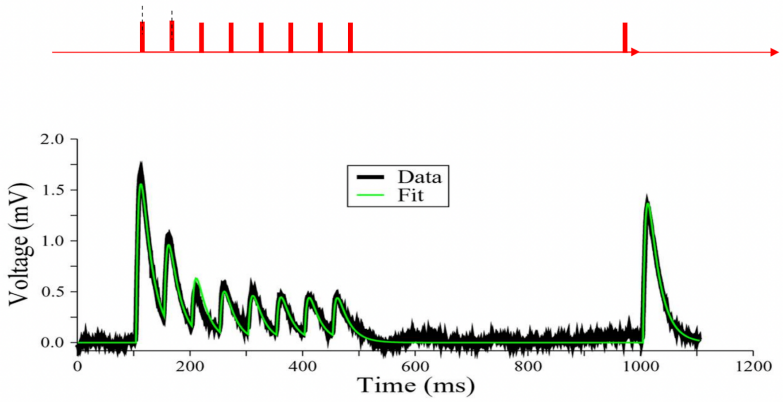



**Fig. 2-1 Short-term plasticity.** (Adapted from *Gerstner et al., 2014* [1])

Now let's turn to the model. The short term plasticity can be described by two variables, $u$ and $x$. Where $u$ represents the probability of neurotransmitter release, and $x$ represents the residual amount of neurotransmitters. The dynamic of a synapse with short term plasticity is given by,

$$\frac{dI}{dt} = -\frac{I}{\tau}$$

$$\frac{du}{dt} = -\frac{u}{\tau_f}$$

$$\frac{dx}{dt} = \frac{1-x}{\tau_d}$$

$$\text{if (pre fire), then } \begin{cases} u^+ = u^- + U(1-u^-) \\ I^+ = I^- + Au^+x^- \\ x^+ = x^- - u^+x^- \end{cases}$$

where the dynamics of the synaptic current $I$ can be one of the dynamics we introduced in the previous section (i.e., the dynamic of gating variable $s$ under current-based condition). $U$ and $A$ are two constants representing the increments of $u$ and $I$ after a presynaptic spike, respectively. $\tau_f$ and $\tau_d$ are time constants of $u$ and $x$, respectively.

In this model, $u$ contributes to the short-term facilitation (STF) by increasing from 0 whenever there is a spike on the presynaptic neuron; while $x$ contributes to the short-term depression (STD) by decreasing from 1 after the presynaptic spike. The two directions of facilitation and depression occur simultaneously, and the value of $\tau_f$ and $\tau_d$ determines which direction of plasticity plays a dominant role.

Now let's see how to implement the STP model with BrainPy. We can see that the plasticity also occurs in synapses, so we will define the class by inheriting from the `bp.TwoEndConn` class like synaptic models. The code is as follows:

```
class STP(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(s, u, x, t, tau, tau_d, tau_f):
        dsdt = -s / tau
        dudt = - u / tau_f
        dxdt = (1 - x) / tau_d
        return dsdt, dudt, dxdt

    def __init__(self, pre, post, conn, delay=0., U=0.15, tau_f=1500.,
                 tau_d=200., tau=8., **kwargs):
        # parameters
        self.tau_d = tau_d
        self.tau_f = tau_f
        self.tau = tau
        self.U = U
        self.delay = delay

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.s = bp.ops.zeros(self.size)
        self.x = bp.ops.ones(self.size)
        self.u = bp.ops.zeros(self.size)
        self.w = bp.ops.ones(self.size)
        self.I_syn = self.register_constant_delay('I_syn', size=self.size,
                                                  delay_time=delay)

        self.integral = bp.odeint(f=self.derivative, method='exponential_euler')

        super(STP, self).__init__(pre=pre, post=post, **kwargs)

```

$\frac{ds}{dt} = -s/\tau$
$\frac{du}{dt} = -/\tau_f$
$\frac{dx}{dt} = (1-x)/\tau_d$

```
    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]

            self.s[i], u, x = self.integral(self.s[i], self.u[i], self.x[i], _t,
                                            self.tau, self.tau_d, self.tau_f)

            if self.pre.spike[pre_id] > 0:
                u += self.U * (1 - self.u[i])
                self.s[i] += self.w[i] * u * self.x[i]
                x -= u * self.x[i]
            self.u[i] = u
            self.x[i] = x

            # output
            post_id = self.post_ids[i]
            self.I_syn.push(i, self.s[i])
            self.post.input[post_id] += self.I_syn.pull(i)

```

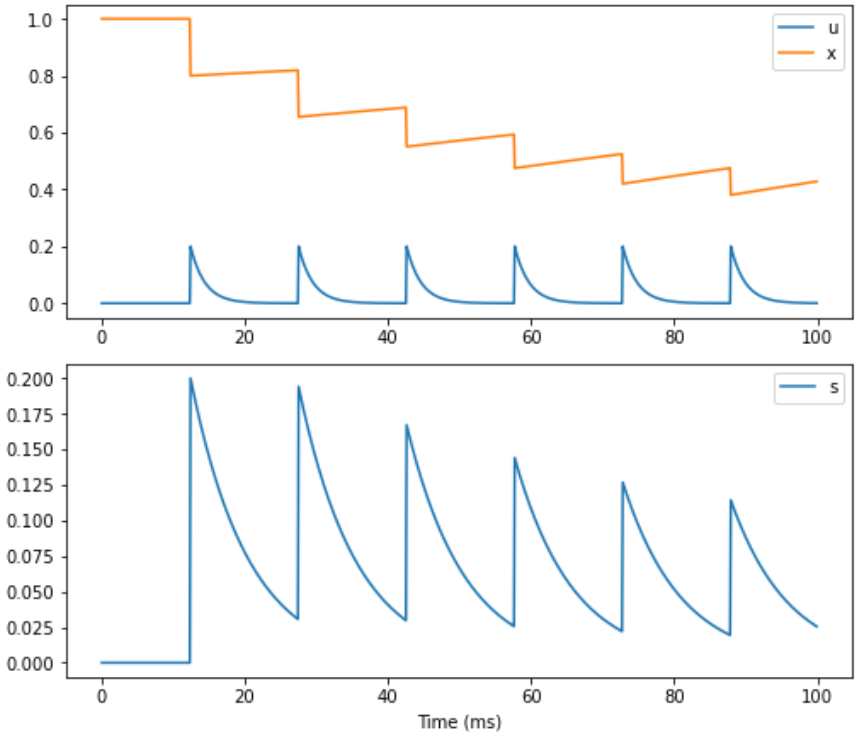
$$\text{if (pre fire), then } \begin{cases} u^+ = u^- + U(1-u^-) \\ I^+ = I^- + Au^+x^- \\ x^+ = x^- - u^+x^- \end{cases}$$

Then let's define a function to run the code. Like synapse models, we need two neuron groups to be connected. Besides the dynamic of $s$, we also want to see how $u$ and $x$ changes over time, so we monitor 's', 'u' and 'x' and plot them.

```
def run_stp(**kwargs):
    neu1 = bm.neurons.LIF(1, monitors=['V'])
    neu2 = bm.neurons.LIF(1, monitors=['V'])

    syn = STP(pre=neu1, post=neu2, conn=bp.connect.All2All(),
              monitors=['s', 'u', 'x'], **kwargs)
    net = bp.Network(neu1, syn, neu2)
    net.run(100., inputs=(neu1, 'input', 28.))

    # plot
    fig, gs = bp.visualize.get_figure(2, 1, 3, 7)

    fig.add_subplot(gs[0, 0])
    plt.plot(net.ts, syn.mon.u[:, 0], label='u')
    plt.plot(net.ts, syn.mon.x[:, 0], label='x')
    plt.legend()

    fig.add_subplot(gs[1, 0])
    plt.plot(net.ts, syn.mon.s[:, 0], label='s')
    plt.legend()

    plt.xlabel('Time (ms)')
    plt.show()
```

Let's first set `tau_d` > `tau_f`.

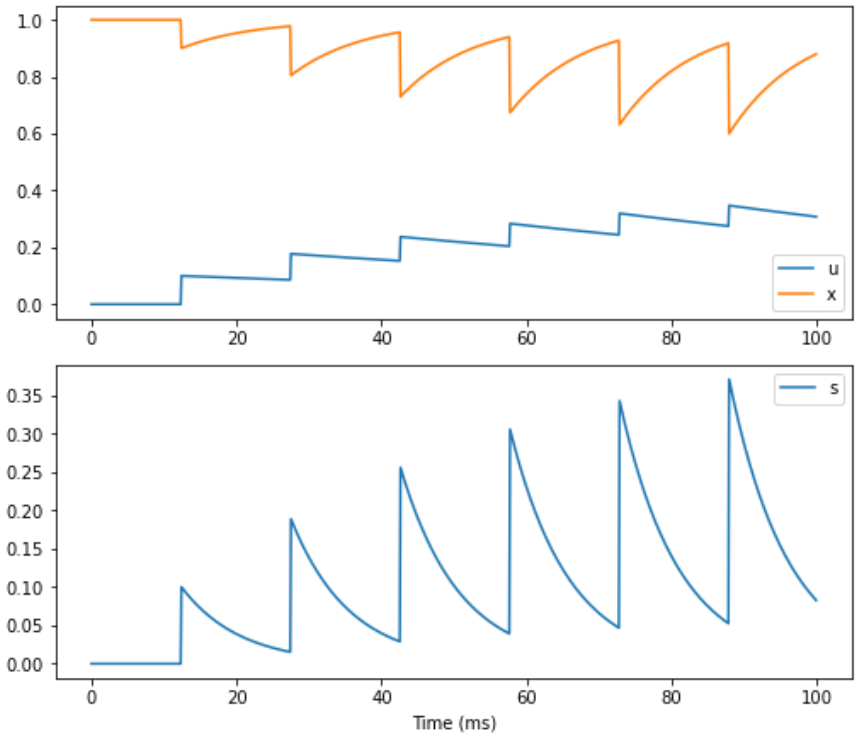
```
run_stp(U=0.2, tau_d=150., tau_f=2.)
```




The plots show that when we set the parameters $\tau_d > \tau_f$, $x$ recovers very slowly, and $u$ decays very quickly, so in the end, the transmitter is not enough to open the receptors, showing STD dominants.

Then let's set `tau_f` > `tau_d`.

```
run_stp(U=0.1, tau_d=10, tau_f=100.)
```



We can see from the figure that when we set $\tau_f > \tau_d$, on the contrary, every time $x$ is used, it will be added back quickly. There are always enough transmitters available. At the same time, the decay of $u$ is very slow, so the probability of releasing transmitters is getting higher and higher, showing STF dominants.

### 2.2.2 Long-term Plasticity

## Spike-timing dependent plasticity (STDP)

Fig. 2-2 shows the spiking timing dependent plasticity (STDP) of experimental results. The x-axis is the time difference between the spike of the presynaptic neuron and the postsynaptic neuron. The left part of the zero represents the spike timing of the presynaptic neuron earlier than that of the postsynaptic neuron, which shows long term potentiation (LTP); and the right side of the zero represents the postsynaptic neuron fires before the presynaptic neuron does, showing long term depression (LTD).

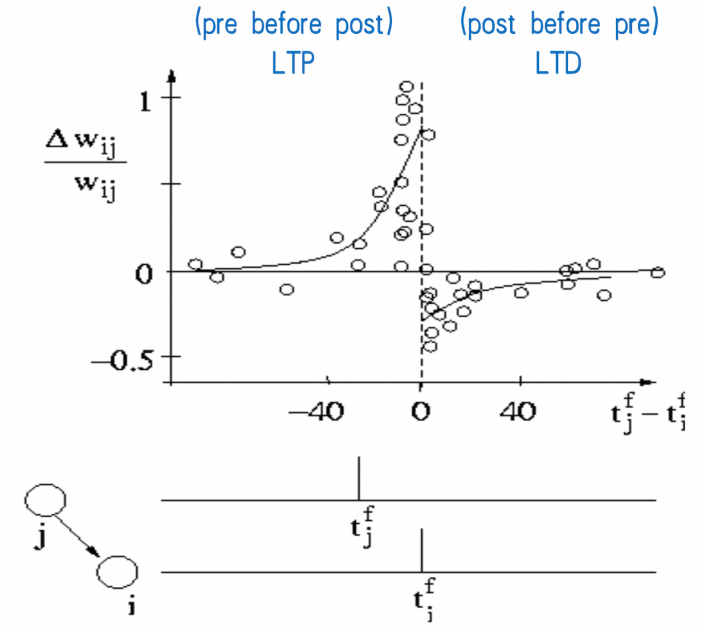



**Fig. 2-2 Spike timing dependent plasticity.** (Adapted from *Bi & Poo, 2001* [2])

The computational model of STDP is given by,

$$\frac{dA_s}{dt} = -\frac{A_s}{\tau_s}$$

$$\frac{dA_t}{dt} = -\frac{A_t}{\tau_t}$$

$$\text{if (pre fire), then} \begin{cases} s \leftarrow s + w \\ A_s \leftarrow A_s + \Delta A_s \\ w \leftarrow w - A_t \end{cases}$$

$$\text{if (post fire), then} \begin{cases} A_t \leftarrow A_t + \Delta A_t \\ w \leftarrow w + A_s \end{cases}$$

Where $w$ is the synaptic weight, and $s$ is the same gating variable as we mentioned in the previous section. Like the STP model, LTD and LTP are controlled by two variables $A_s$ and $A_t$, respectively. $\Delta A_s$ and $\Delta A_t$ are the increments of $A_s$ and $A_t$, respectively. $\tau_s$ and $\tau_t$ are time constants.

According to this model, when a presynaptic neuron fire before the postsynaptic neuron, $A_s$ increases everytime when there is a spike on the presynaptic neuron, and $A_t$ will stay on 0 until the postsynaptic neuron fire, so $w$ will not change for the time being. When there is a spike in the postsynaptic neuron, the increment of $w$ will be an amount of $A_s - A_t$, since $A_s > A_t$ in this situation, LTP will be presented, and vice verse.

Now let's see how to use BrainPy to implement this model. Here we use the single exponential decay model to implement the dynamics of $s$.

```python
class STDP(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(s, A_s, A_t, t, tau, tau_s, tau_t):
        dsdt = -s / tau
        dAsdt = - A_s / tau_s
        dAtdt = - A_t / tau_t
        return dsdt, dAsdt, dAtdt

    def __init__(self, pre, post, conn, delay=0., delta_A_s=0.5,
                 delta_A_t=0.5, w_min=0., w_max=20., tau_s=10., tau_t=10.,
                 tau=10., **kwargs):
        # parameters
        self.tau_s = tau_s
        self.tau_t = tau_t
        self.tau = tau
        self.delta_A_s = delta_A_s
        self.delta_A_t = delta_A_t
        self.w_min = w_min
        self.w_max = w_max
        self.delay = delay

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = self.conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.s = bp.ops.zeros(self.size)
        self.A_s = bp.ops.zeros(self.size)
        self.A_t = bp.ops.zeros(self.size)
        self.w = bp.ops.ones(self.size) * 1.
        self.I_syn = self.register_constant_delay('I_syn', size=self.size,
                                                  delay_time=delay)
        self.integral = bp.odeint(f=self.derivative, method='exponential_euler')

        super(STDP, self).__init__(pre=pre, post=post, **kwargs)

```

$$\frac{ds}{dt} = -s/\tau$$
$$\frac{dA_s}{dt} = -A_s/\tau_s$$
$$\frac{dA_t}{dt} = -A_t/\tau_t$$

```
    def update(self, _t):
        for i in range(self.size):
            pre_id = self.pre_ids[i]
            post_id = self.post_ids[i]

            self.s[i], A_s, A_t = self.integral(self.s[i], self.A_s[i],
                                                self.A_t[i], _t, self.tau,
                                                self.tau_s, self.tau_t)

            w = self.w[i]
            if self.pre.spike[pre_id] > 0:
                self.s[i] += w
                A_s += self.delta_A_s
                w -= A_t

            if self.post.spike[post_id] > 0:
                A_t += self.delta_A_t
                w += A_s

            self.A_s[i] = A_s
            self.A_t[i] = A_t

            self.w[i] = bp.ops.clip(w, self.w_min, self.w_max)

            # output
            self.I_syn.push(i, self.s[i])
            self.post.input[post_id] += self.I_syn.pull(i)
```

if (pre fire), then $\begin{cases} s \leftarrow s + w \\ A_s \leftarrow A_s + \Delta A_s \\ w \leftarrow w - A_t \end{cases}$

if (post fire), then $\begin{cases} A_t \leftarrow A_t + \Delta A_t \\ w \leftarrow w + A_s \end{cases}$

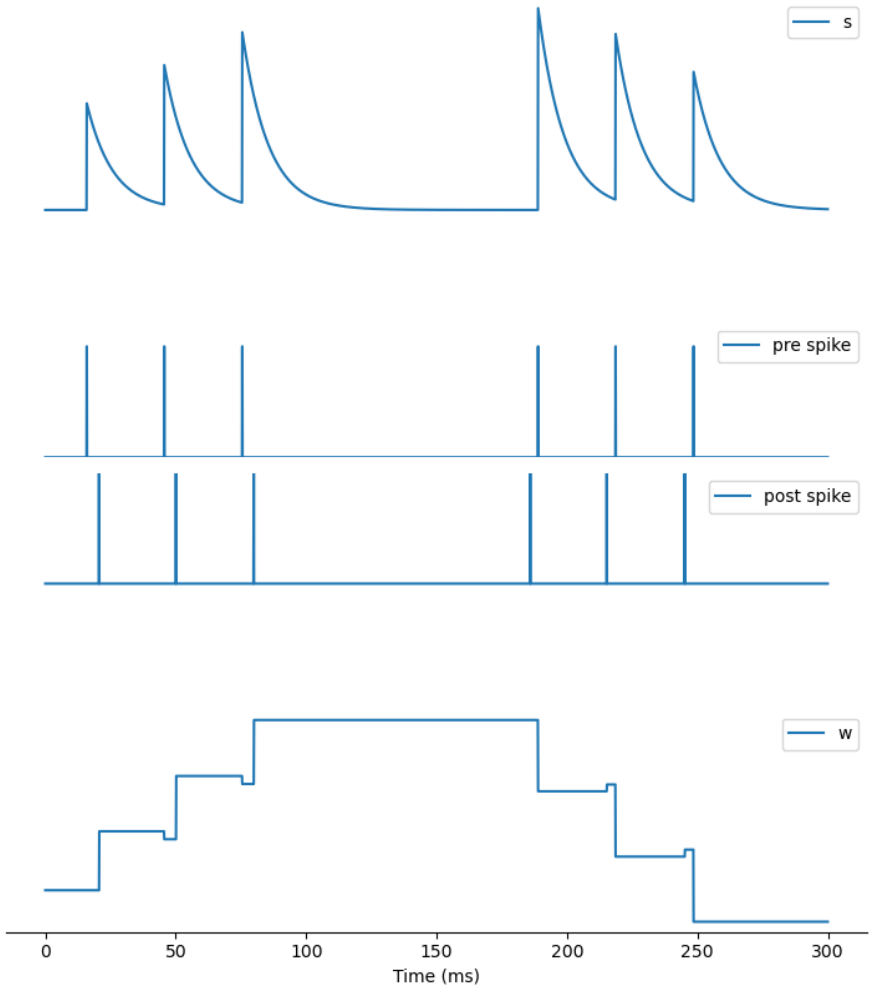
limit w

We control the spike timing by varying the input current of the presynaptic group and postsynaptic group. We apply the first input to the presynaptic group starting at $t = 5ms$ (with amplitude of 30 $\mu A$, lasts for 15 ms to ensure to induce a spike with LIF neuron model), then start to stimulate the postsynaptic group at $t = 10ms$. The intervals between each two inputs are $15ms$. We keep those $t_{post} = t_{pre} + 5$ during the first 3 spike-pairs. Then we set a long interval before switching the stimulating order to be $t_{post} = t_{pre} - 3$ since the 4th spike.

```
duration = 300.
(I_pre, _) = bp.inputs.constant_current([(0, 5), (30, 15),   # pre at 5ms
                 (0, 15), (30, 15),
                 (0, 15), (30, 15),
                 (0, 98), (30, 15),  # switch order: t_interval=98ms
                 (0, 15), (30, 15),
                 (0, 15), (30, 15),
                 (0, duration-155-98)])
(I_post, _) = bp.inputs.constant_current([(0, 10), (30, 15), # post at 10
                  (0, 15), (30, 15),
                  (0, 15), (30, 15),
                  (0, 90), (30, 15), # switch order: t_interval=98-8=90(ms)
                  (0, 15), (30, 15),
                  (0, 15), (30, 15),
                  (0, duration-160-90)])
```

Then let's run the simulation.

```
pre = bm.neurons.LIF(1, monitors=['spike'])
post = bm.neurons.LIF(1, monitors=['spike'])

syn = STDP(pre=pre, post=post, conn=bp.connect.All2All(),
           monitors=['s', 'w'])
net = bp.Network(pre, syn, post)
net.run(duration, inputs=[(pre, 'input', I_pre), (post, 'input', I_post)])

# plot
fig, gs = bp.visualize.get_figure(4, 1, 2, 7)

def hide_spines(my_ax):
    plt.legend()
    plt.xticks([])
    plt.yticks([])
    my_ax.spines['left'].set_visible(False)
    my_ax.spines['right'].set_visible(False)
    my_ax.spines['bottom'].set_visible(False)
    my_ax.spines['top'].set_visible(False)

ax=fig.add_subplot(gs[0, 0])
plt.plot(net.ts, syn.mon.s[:, 0], label="s")
hide_spines(ax)

ax1=fig.add_subplot(gs[1, 0])
plt.plot(net.ts, pre.mon.spike[:, 0], label="pre spike")
plt.ylim(0, 2)
hide_spines(ax1)
plt.legend(loc = 'center right')

ax2=fig.add_subplot(gs[2, 0])
plt.plot(net.ts, post.mon.spike[:, 0], label="post spike")
plt.ylim(-1, 1)
hide_spines(ax2)

ax3=fig.add_subplot(gs[3, 0])
plt.plot(net.ts, syn.mon.w[:, 0], label="w")
plt.legend()
# hide spines
plt.yticks([])
ax3.spines['left'].set_visible(False)
ax3.spines['right'].set_visible(False)
ax3.spines['top'].set_visible(False)

plt.xlabel('Time (ms)')
plt.show()
```




The simulation result shows that weights $w$ increase when the presynaptic neuron fire before the postsynaptic neuron (before 150ms); and decrease when the order switched (after 150ms).

## Oja's rule

Next, let's look at the rate model based on Hebbian learning. Because Hebbian learning is "fire together, wire together", regardless of the order before and after, spiking time can be ignored, so it can be simplified as a rate-based model. Let's first look at the general form of Hebbian learning. For the $j$ to $i$ connection, $r_j, r_i$ denotes the firing rate of pre- and post-neuron groups, respectively. According to the locality characteristic of Hebbian learning, The change of $w_{ij}$ is affected by $w$ itself and $r_j, r_i$, we get the following differential equation.

$$\frac{d}{dt}w_{ij} = F(w_{ij}; r_i, r_j)$$

The following formula is obtained by Taylor expansion on the right side of the above formula.

$$\frac{d}{dt}w_{ij} = c_{00}w_{ij} + c_{10}w_{ij}r_j + c_{01}w_{ij}r_i + c_{20}w_{ij}r_j^2 + c_{02}w_{ij}r_i^2 + c_{11}w_{ij}r_ir_j + O(r^3)$$

The 6th term contains $r_i r_j$, only if $c_{11}$ is not zero can the "fire together" of Hebbian learning be satisfied. For example, the formula of `Oja's rule` is as follows, which corresponds to the 5th and 6th terms of the above formula.

$$\frac{d}{dt}w_{ij} = \gamma[r_i r_j - w_{ij} r_i^2]$$

$\gamma$ represents the learning rate.

Now let's see how to use BrainPy to implement `Oja's rule`.

```python
class Oja(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(w, t, gamma, r_pre, r_post):
        dwdt = gamma * (r_post * r_pre - r_post * r_post * w)
        return dwdt

    def __init__(self, pre, post, conn, delay=0.,
                 gamma=0.005, w_max=1., w_min=0.,
                 **kwargs):
        # params
        self.gamma = gamma
        self.w_max = w_max
        self.w_min = w_min
        # no delay in firing rate models

        # conns
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = self.conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # data
        self.w = bp.ops.ones(self.size) * 0.05

        self.integral = bp.odeint(f=self.derivative)
        super(Oja, self).__init__(pre=pre, post=post, **kwargs)


    def update(self, _t):
        post_r = bp.ops.zeros(self.post.size[0])
        for i in range(self.size):
            pre_id = self.pre_ids[i]
            post_id = self.post_ids[i]
            add = self.w[i] * self.pre.r[pre_id]
            post_r[post_id] += add
            self.w[i] = self.integral(
                self.w[i], _t, self.gamma,
                self.pre.r[pre_id], self.post.r[post_id])
        self.post.r = post_r
```

bp.TwoEndConn class: Learning rule occur between two neuron groups

$\frac{dw}{dt} = \gamma(r_i r_j - w\, r_i^2)$

rate model: $r_i = \sum_j w_{ij}\ r_j$

$let\ add_j = w_{ij} r_j$

then $r_i = \sum_j add_j$

use self.post.r, not post_r, see next line for the reason.

$r_i = \sum_j add_j$

Note that the effects of pre groups are simultaneously, so we update post_r after the for loop

Since Oja's rule is a rate-based model, we need a rate-based neuron model to see this learning rule of two groups of neurons.

```python
class neu(bp.NeuGroup):
    target_backend = ['numpy', 'numba']

    def __init__(self, size, **kwargs):
        self.r = bp.ops.zeros(size)
        super(neu, self).__init__(size=size, **kwargs)

    def update(self, _t):
        self.r = self.r
```

we need a firing rate neuron with parameter r

we only need a simple unit to test oja's rule.

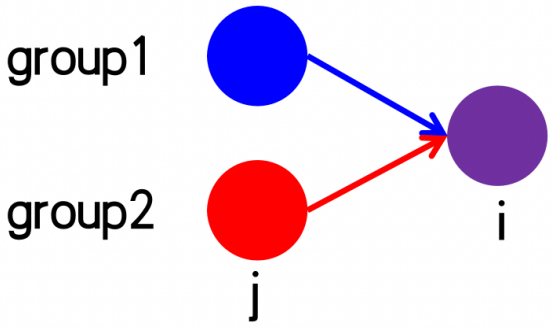



**Fig. 2-3 Connection of neuron groups.**

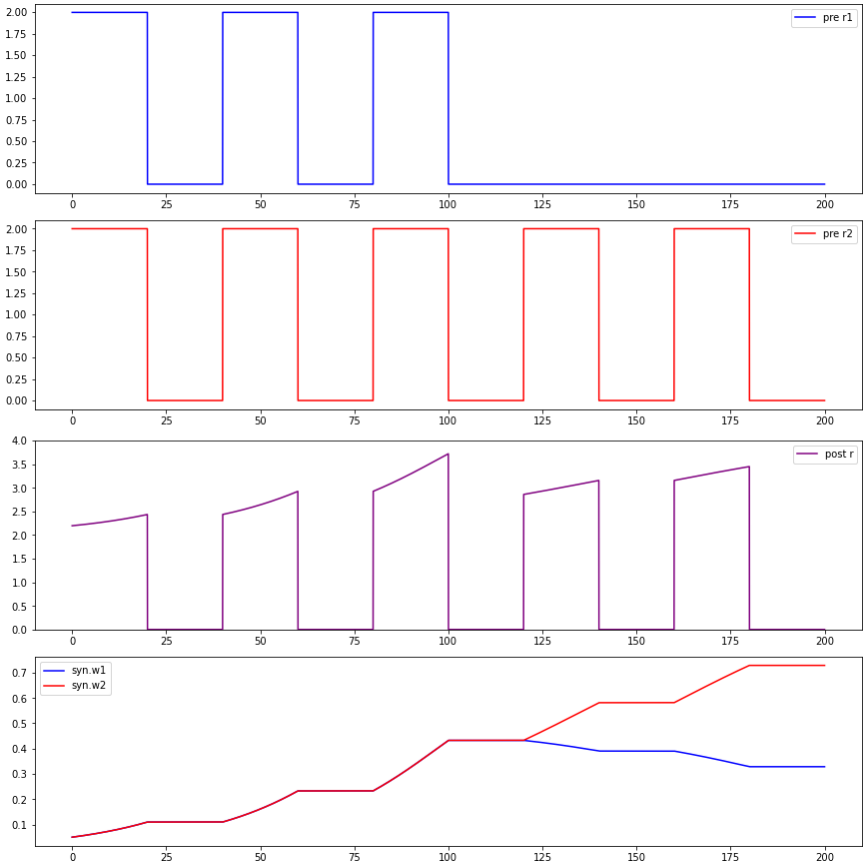
We aim to implement the connection as shown in Fig. 2-3. The purple neuron group receives inputs from the blue and red groups. The external input to the post group is exactly the same as the red one, while the blue one is the same at first, but not later.

The simulation code is as follows.

```
# create input
current1, _ = bp.inputs.constant_current(
                      [(2., 20.), (0., 20.)] * 3 + [(0., 20.), (0., 20.)] * 2)
current2, _ = bp.inputs.constant_current([(2., 20.), (0., 20.)] * 5)
current3, _ = bp.inputs.constant_current([(2., 20.), (0., 20.)] * 5)
current_pre = np.vstack((current1, current2))
current_post = np.vstack((current3, current3))

# simulate
neu_pre = neu(2, monitors=['r'])
neu_post = neu(2, monitors=['r'])
syn = Oja(pre=neu_pre, post=neu_post, conn=bp.connect.All2All(), monitors=['w'])
net = bp.Network(neu_pre, syn, neu_post)
net.run(duration=200., inputs=[(neu_pre, 'r', current_pre.T, '='),
                               (neu_post, 'r', current_post.T)])

# plot
fig, gs = bp.visualize.get_figure(4, 1)

fig.add_subplot(gs[0, 0])
plt.plot(net.ts, neu_pre.mon.r[:, 0], 'b', label='pre r1') ——→ index [:, 0], group 1
plt.legend()

fig.add_subplot(gs[1, 0])
plt.plot(net.ts, neu_pre.mon.r[:, 1], 'r', label='pre r2') ——→ index [:, 0], group 2
plt.legend()

fig.add_subplot(gs[2, 0])
plt.plot(net.ts, neu_post.mon.r[:, 0], color='purple', label='post r')
plt.ylim([0, 4])
plt.legend()

fig.add_subplot(gs[3, 0])
plt.plot(net.ts, syn.mon.w[:, 0], 'b', label='syn.w1')
plt.plot(net.ts, syn.mon.w[:, 1], 'r', label='syn.w2')
plt.legend()
plt.show()
```

current1: input to pre-group1
current2: input to pre-group2
current3: input to post group, the same as current2.(fire together)

simulation is like synapse model. Here we give inputs to both pre-group and post-group.




It can be seen from the results that at the beginning, when the two groups of neurons were given input at the same time, their weights increased simultaneously, and the response of post became stronger and stronger, showing LTP. After 100ms, the blue group is no longer fire together, only the red group still fire together, and only the weights of the red group are increased. The results accord with the "fire together, wire together" of Hebbian learning.

## BCM rule

Now let's see other example of Hebbian learning, the BCM rule. It's given by,

$$\frac{d}{dt}w_{ij} = \eta r_i(r_i - r_\theta)r_j$$

where $\eta$ represents the learning rate, and $r_\theta$ represents the threshold of learning (see Fig. 2-4). Fig. 2-4 shows the right side of the formula. When the firing rate is greater than the threshold, there is LTP, and when the firing rate is lower than the threshold, there is LTD. Therefore, the selectivity can be achieved by adjusting the threshold $r_\theta$.

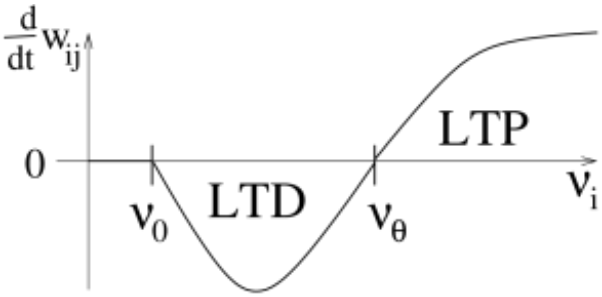



**Fig. 2-4 BCM rule** (From *Gerstner et al., 2014* [1])

We will implement the same connections as the previous Oja's rule (Fig. 2-3), with different firing rates. Here the two groups of neurons are alternately firing. Among them, the blue group is always stronger than the red one. We adjust the threshold by setting it as the time average of $r_i$, that is $r_\theta = f(r_i)$. The code implemented by BrainPy is as follows.

```python
class BCM(bp.TwoEndConn):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(w, t, lr, r_pre, r_post, r_th):
        dwdt = lr * r_post * (r_post - r_th) * r_pre
        return dwdt

    def __init__(self, pre, post, conn, lr=0.005, w_max=2., w_min=0., **kwargs):
        # parameters
        self.lr = lr
        self.w_max = w_max
        self.w_min = w_min
        self.dt = bp.backend._dt

        # connections
        self.conn = conn(pre.size, post.size)
        self.pre_ids, self.post_ids = self.conn.requires('pre_ids', 'post_ids')
        self.size = len(self.pre_ids)

        # variables
        self.w = bp.ops.ones(self.size)
        self.sum_post_r = bp.ops.zeros(post.size[0])

        self.int_w = bp.odeint(f=self.derivative, method='rk4')

        super(BCM, self).__init__(pre=pre, post=post, **kwargs)

    def update(self, _t):
        # update threshold
        self.sum_post_r += self.post.r
        r_th = self.sum_post_r / (_t / self.dt + 1)

        # update w and post_r
        post_r = bp.ops.zeros(self.post.size[0])

        for i in range(self.size):
            pre_id = self.pre_ids[i]
            post_id = self.post_ids[i]

            post_r[post_id] += self.w[i] * self.pre.r[pre_id]

            w = self.int_w(self.w[i], _t, self.lr, self.pre.r[pre_id],
                           self.post.r[post_id], r_th[post_id])

            self.w[i] = bp.ops.clip(w, self.w_min, self.w_max)

        self.post.r = post_r
```

$\frac{dw}{dt} = \eta r_i (r_i - r_\theta) r_j$

$r_\theta = \frac{\sum_t dt r_i}{T}$

self.dt +1 here since self.dt begin with 0

rate model: $r_i = \sum_j w_{ij} r_j$

let $add_j = w_{ij} r_j$

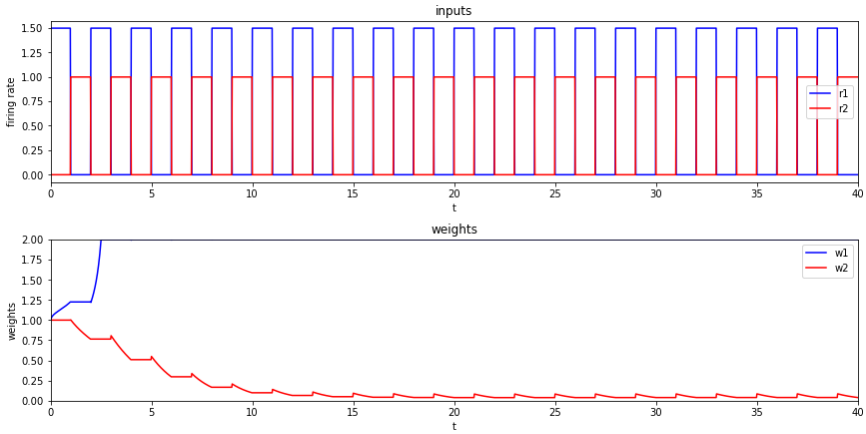
then $r_i = \sum_j add_j$

limit w

$r_i = \sum_j add_j$

Then we can run the simulation with the following code.

```
n_post = 1
n_pre = 20

# group selection
group1, duration = bp.inputs.constant_current(([1.5, 1], [0, 1]) * 20)
group2, duration = bp.inputs.constant_current(([0, 1], [1., 1]) * 20)
group1 = bp.ops.vstack(((group1,)*10))
group2 = bp.ops.vstack(((group2,)*10))
input_r = bp.ops.vstack((group1, group2))

pre = neu(n_pre, monitors=['r'])
post = neu(n_post, monitors=['r'])
bcm = BCM(pre=pre, post=post,conn=bp.connect.All2All(),
          monitors=['w'])

net = bp.Network(pre, bcm, post)
net.run(duration, inputs=(pre, 'r', input_r.T, "="))

w1 = bp.ops.mean(bcm.mon.w[:, :10, 0], 1)
w2 = bp.ops.mean(bcm.mon.w[:, 10:, 0], 1)

r1 = bp.ops.mean(pre.mon.r[:, :10], 1)
r2 = bp.ops.mean(pre.mon.r[:, 10:], 1)

fig, gs = bp.visualize.get_figure(2, 1, 3, 12)
fig.add_subplot(gs[1, 0], xlim=(0, duration), ylim=(0, w_max))
plt.plot(net.ts, w1, 'b', label='w1')
plt.plot(net.ts, w2, 'r', label='w2')
plt.title("weights")
plt.ylabel("weights")
plt.xlabel("t")
plt.legend()

fig.add_subplot(gs[0, 0], xlim=(0, duration))
plt.plot(net.ts, r1, 'b', label='r1')
plt.plot(net.ts, r2, 'r', label='r2')
plt.title("inputs")
plt.ylabel("firing rate")
plt.xlabel("t")
plt.legend()

plt.show()
```




The results show that the blue group with stronger input demonstrating LTP, while the red group with weaker input showing LTD, so the blue group is being chosen.

## References


[1]. Gerstner, Wulfram, et al. Neuronal dynamics: From single neurons to networks and models of cognition. Cambridge University Press, 2014. ↩

[2]. Bi, Guo-qiang, and Mu-ming Poo. "Synaptic modification by correlated activity: Hebb's postulate revisited." Annual review of neuroscience 24.1 (2001): 139-166. ↩

# 3. Network models

With the neuron and synapse models we have realized with BrainPy, users can now build networks of there own. In this section, we will introduce two main types of network models as examples: 1) spiking neural networks that model and compute each neuron or synapse separately; 2) firing rate networks that simplify neuron groups in the network as firing rate units and compute each neuron group as one unit.

## 3.1 Spiking Neural Networks

## 3.2 Firing Rate Networks

# 3.1 Spiking Neural Network

## 3.1.1 E/I balanced network

In 1990s, biologists found in experiments that neuron activities in brain cortex show a temporal irregular spiking pattern. This pattern exists widely in brain areas, but researchers knew few about its mechanism or function.

Vreeswijk and Sompolinsky (1996) proposed **E/I balanced network** to explain this irregular spiking pattern. The feature of this network is the strong, random and sparse synapse connections between neurons. Because of this feature and corresponding parameter settings, each neuron in the network will receive great excitatory and inhibitory input from within the network. However, these two types of inputs will cancel each other, and maintain the total internal input at a relatively small order of magnitude, which is only enough to generate action potentials.

The randomness and noise in E/I balanced network give each neuron in the network an internal input which varies with time and space at the order of threshold potential. Therefore, the firing of neurons also has randomness, ensures that E/I balanced network can generate temporal irregular firing pattern spontaneously.

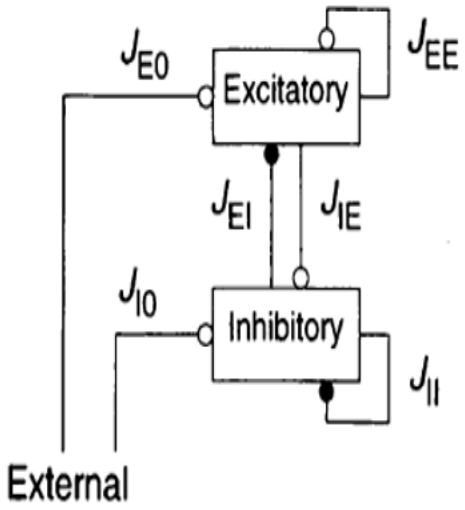



**Fig.3-1 Structure of E/I balanced network | Vreeswijk and Sompolinsky, 1996**

Vreeswijk and Sompolinsky also suggested a possible function of this irregular firing pattern: E/I balanced network can respond to the changes of external stimulus quickly.

As shown in Fig. 3-3, when there is no external input, the distribution of neurons' membrane potentials in E/I balanced network follows a relatively uniform random distribution between resting potential $V_0$and threshold potential $\theta$.

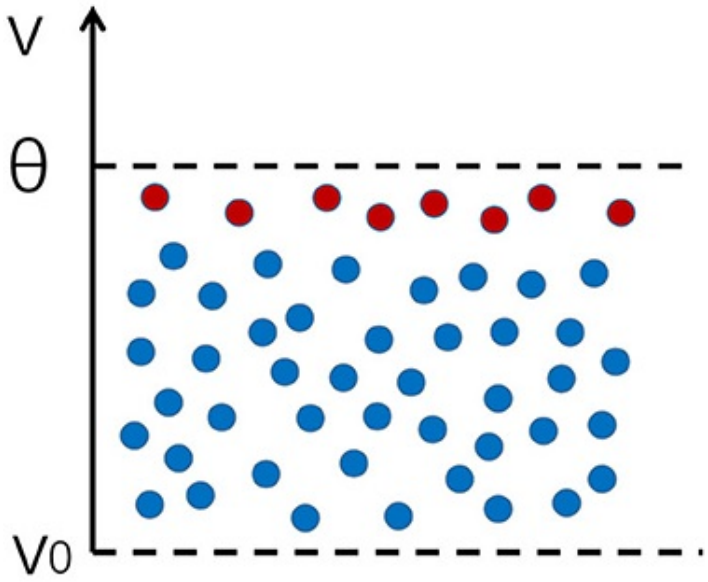



**Fig.3-2 Distribution of neuron membrane potentials in E/I balanced network | Tian et al., 2020**

When we give the network a small constant external stimulus, those neurons whose membrane potentials fall near the threshold potential will soon meet the threshold, therefore spike rapidly. On the network scale, the firing rate of the network can adjust rapidly once the input changes.

Simulation suggests that the delay of network response to input and the delay of synapses have the same time scale, and both are significantly smaller than the delay of a single neuron from resting potential to generating a spike. So E/I balanced network may provide a fast response mechanism for neural networks.

Fig. 3-1 shows the structure of E/I balanced network:

1) Neurons: Neurons are realized with LIF neuron model. The neurons can be divided into excitatory neurons and inhibitory neurons, the ratio of the two types of neurons is $N_E$: $N_I$ = 4:1.

2) Synapses: Synapses are realized with exponential synapse model. 4 groups of synapse connections are generated between the two groups of neurons, that is, excitatory-excitatory connection (E2E conn), excitatory-inhibitory connection (E2I conn), inhibitory-excitatory connection (I2E conn) and inhibitory-inhibitory connection (I2I conn). For excitatory or inhibitory synapse connections, we define synapse weights with different signal.

3) Inputs: All neurons in the network receive a constant external input current.

See above section 1 and 2 for definition of LIF neuron and exponential synapse. After simulation, we visualize the raster plot and firing rate-t plot of E/I balanced network. the network firing rate changes from strong synchronization to irregular fluctuation.

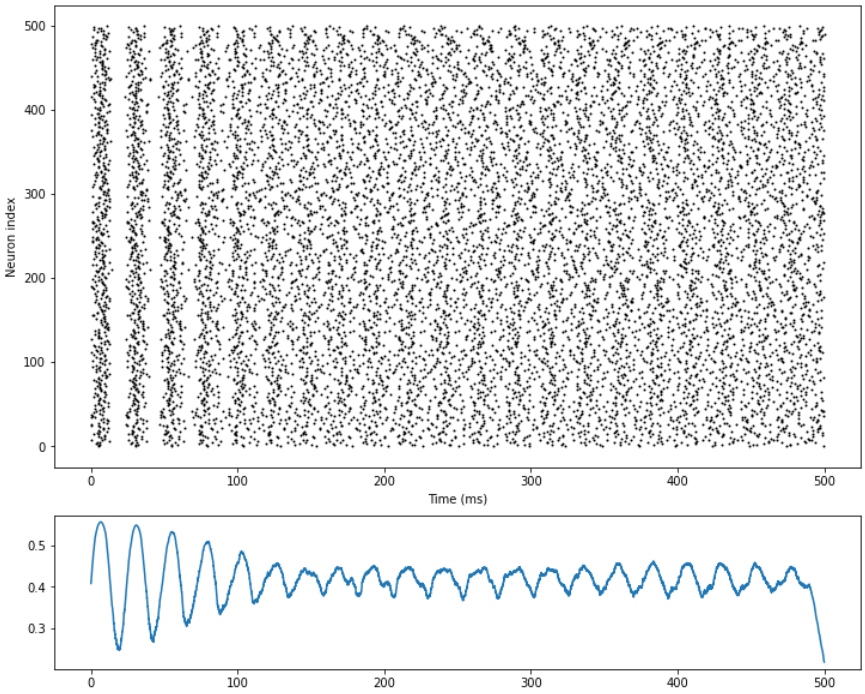



**Fig.3-3 E/I balanced net raster plot**

## 3.1.2 Decision Making Network

The modeling of computational neuroscience networks can correspond to specific physiological tasks.

For example, in the visual motion discrimination task (Roitman and Shadlen, 2002), rhesus watch a video in which random dots move towards left or right with definite coherence. Rhesus are required to choose the direction that most dots move to and give their answer by saccade. At the meantime, researchers record the activity of their LIP neurons by implanted electrode.

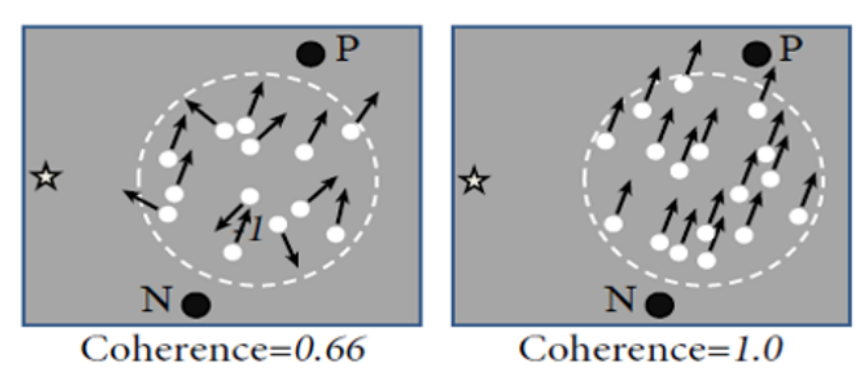



**Fig.3-4 Experimental Diagram**

Wang (2002) proposed a decision making network to model the activity of rhesus LIP neurons during decision making period in the visual motion discrimination task.

As shown in Fig. 3-5, this network is based on E/I balanced network. The ratio of excitatory neurons and inhibitory neurons is $N_E : N_I = 4 : 1$, and parameters are adjusted to maintain the balanced state.

To accomplish the decision making task, among the excitatory neuron group, two selective subgroup A and B are chosen, both with a size of $N_A = N_B = 0.15N_E$. These two subgroups are marked as A and B in Fig. 3-5, and we call other excitatory neurons as non-selective neurons, $N_{non} = (1 - 2 * 0.15)N_E$.

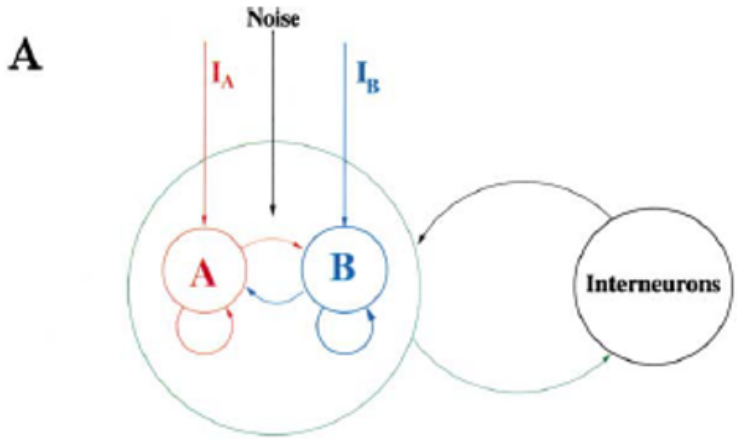
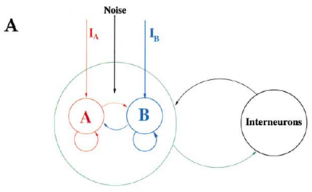



**Fig.3-5 structure of decision makingnetwork**

```
# def neurons
# def E neurons/pyramid neurons
neu_A = LIF(N_A, monitors=['spike', 'input', 'V'])
neu_A.V_rest = V_rest_E
neu_A.V_reset = V_reset_E
neu_A.V_th = V_th_E
neu_A.R = R_E
neu_A.tau = tau_E
neu_A.t_refractory = t_refractory_E
neu_A.V = bp.ops.ones(N_A) * V_rest_E

neu_B = LIF(N_B, monitors=['spike', 'input', 'V'])
neu_B.V_rest = V_rest_E
neu_B.V_reset = V_reset_E
neu_B.V_th = V_th_E
neu_B.R = R_E
neu_B.tau = tau_E
neu_B.t_refractory = t_refractory_E
neu_B.V = bp.ops.ones(N_B) * V_rest_E

neu_non = LIF(N_non, monitors=['spike', 'input', 'V'])
neu_non.V_rest = V_rest_E
neu_non.V_reset = V_reset_E
neu_non.V_th = V_th_E
neu_non.R = R_E
neu_non.tau = tau_E
neu_non.t_refractory = t_refractory_E
neu_non.V = bp.ops.ones(N_non) * V_rest_E

# def I neurons/interneurons
neu_I = LIF(N_I, monitors=['input', 'V'])
neu_I.V_rest = V_rest_I
neu_I.V_reset = V_reset_I
neu_I.V_th = V_th_I
neu_I.R = R_I
neu_I.tau = tau_I
neu_I.t_refractory = t_refractory_I
neu_I.V = bp.ops.ones(N_I) * V_rest_I

```



As it is in E/I balanced network, 4 groups of synapses ---- E2E connection, E2I connection, I2E connection and I2I connection ---- are built in decision making network. Excitatory connections are realized with AMPA synapse, inhibitory connections are realized with GABAa synapse.

Decision making network needs to make a decision among the two choice, i.e., among the two subgroups A and B in this task. To achieve this, network must discriminate between these two groups. Excitatory neurons in the same subgroup should self-activate, and inhibit neurons in another selective subgroup.

Therefore, E2E connections are structured in the network. As shown in Sheet 3-1, $w+ > 1 > w-$. In this way, a relative activation is established within the subgroups by stronger excitatory synapse connections, and relative inhibition is established between two subgroups or between selective and non-selective subgroups by weaker excitatory synapse connections.

**Sheet 3-1 Weight of synapse connections between E-neurons**

| w | A | B | non |
|---|---|---|---|
| A | w+ | w- | 1. |
| B | w- | w+ | 1. |
| non | w- | w- | 1. |

```
# set syn weights (only used in recurrent E connections)
w_pos = 1.7
w_neg = 1. - f * (w_pos - 1.) / (1. - f)
print(f"the structured weight is: w_pos = {w_pos}, w_neg = {w_neg}")
# inside select group: w = w+
# between group / from non-select group to select group: w = w-
# A2A B2B w+, A2B B2A w-, non2A non2B w-
weight = np.ones((N_E, N_E), dtype=np.float)
for i in range(N_A):
  weight[i, 0: N_A] = w_pos
  weight[i, N_A: N_A + N_B] = w_neg
for i in range(N_A, N_A + N_B):
  weight[i, N_A: N_A + N_B] = w_pos
  weight[i, 0: N_A] = w_neg
for i in range(N_A + N_B, N_E):
  weight[i, 0: N_A + N_B] = w_neg
print(f"Check constraints: Weight sum {weight.sum(axis=0)[0]} \
        should be equal to N_E = {N_E}")
```

$w+ = 1.7$

$w- = 1 - f(x + -1)/1 - f$

Define structured weights for E2E conns. Strong connection inside selective group, weak connection for other synapses.

| w | A | B | non |
|---|---|---|---|
| A | w+ | w- | 1. |
| B | w- | w+ | 1. |
| non | w- | w- | 1. |

We give two types of external inputs to the decision making network:

1) Background inputs from other brain areas without specific meaning. Represented as high frequency Poisson input mediated by AMPA synapse.

```python
# def background poisson input
class PoissonInput(bp.NeuGroup):
  target_backend = 'general'

  def __init__(self, size, freqs, dt, **kwargs):
    self.freqs = freqs
    self.dt = dt

    self.spike = bp.ops.zeros(size, dtype=bool)

    super(PoissonInput, self).__init__(size=size, **kwargs)

  def update(self, _t):
    self.spike = np.random.random(self.size) \
                 < self.freqs * self.dt / 1000.

# poisson_freq = 2400Hz
neu_poisson_A = PoissonInput(N_A, freqs=poisson_freq, dt=dt)
neu_poisson_B = PoissonInput(N_B, freqs=poisson_freq, dt=dt)
neu_poisson_non = PoissonInput(N_non, freqs=poisson_freq, dt=dt)
neu_poisson_I = PoissonInput(N_I, freqs=poisson_freq, dt=dt)

syn_back2A_AMPA = AMPA(pre=neu_poisson_A, post=neu_A,
                       conn=bp.connect.One2One())
syn_back2B_AMPA = AMPA(pre=neu_poisson_B, post=neu_B,
                       conn=bp.connect.One2One())
syn_back2non_AMPA = AMPA(pre=neu_poisson_non, post=neu_non,
                         conn=bp.connect.One2One())

syn_back2I_AMPA = AMPA(pre=neu_poisson_I, post=neu_I,
                       conn=bp.connect.One2One())
```

Generate spikes with a given Poisson frequency.

Define neurons that generate Poisson background inputs for every neuron in the network.

Define AMPA synapses that send the Poisson background inputs to the neurons.

2) Stimulus inputs from outside the brain, which are given only to the two selective subgroup A and B. Represented as lower frequency Poisson input mediated by AMPA synapse.

The frequency of Poisson input given to A and B subgroup have a certain difference, simulate the difference in the number of dots moving to left and right in physiological experiments, induce the network to make a decision among these two subgroups.

$$\rho_A = \rho_B = \mu_0/100$$

$$\mu_A = \mu_0 + \rho_A * c$$

$$\mu_B = \mu_0 + \rho_B * c$$

Every 50ms, the Poisson frequencies $f_x$ change once, follows a Gaussian distribution defined by mean $\mu_x$ and variance $\delta^2$.

$$f_A \sim N(\mu_A, \delta^2)$$

$$f_B \sim N(\mu_B, \delta^2)$$

```python
## def stimulus input
# Note: inputs only given to A and B group
mu_0 = 40.
coherence = 25.6
rou_A = mu_0 / 100.
rou_B = mu_0 / 100.
mu_A = mu_0 + rou_A * coherence
mu_B = mu_0 - rou_B * coherence
print(f"coherence = {coherence}, mu_A = {mu_A}, mu_B = {mu_B}")

class PoissonStim(bp.NeuGroup):
  """
  from time <t_start> to <t_end> during the simulation, the neuron
  generates a poisson spike with frequency <self.freq>. however,
  the value of <self.freq> changes every <t_interval> ms and obey
  a Gaussian distribution defined by <mean_freq> and <var_freq>.
  """
  target_backend = 'general'

  def __init__(self, size, dt=0., t_start=0., t_end=0., t_interval=0.,
               mean_freq=0., var_freq=20., **kwargs):
    self.dt = dt
    self.stim_start_t = t_start
    self.stim_end_t = t_end
    self.stim_change_freq_interval = t_interval
    self.mean_freq = mean_freq
    self.var_freq = var_freq

    self.freq = 0.
    self.t_last_change_freq = -1e7
    self.spike = bp.ops.zeros(size, dtype=bool)

    super(PoissonStim, self).__init__(size=size, **kwargs)

  def update(self, _t):
    if self.stim_start_t < _t < self.stim_end_t:
      if self.stim_change_freq_interval <= _t - self.t_last_change_freq:
        self.freq = np.random.normal(self.mean_freq, self.var_freq)
        self.freq = max(self.freq, 0)
        self.t_last_change_freq = _t
      self.spike = np.random.random(self.size) < (self.freq * self.dt / 1000)
    else:
      self.freq = 0.
      self.spike[:] = False


neu_input2A = PoissonStim(N_A, dt=dt, t_start=pre_period,
                          t_end=pre_period + stim_period,
                          t_interval=50., mean_freq=mu_A, var_freq=10.,
                          monitors=['freq'])
neu_input2B = PoissonStim(N_B, dt=dt, t_start=pre_period,
                          t_end=pre_period + stim_period,
                          t_interval=50., mean_freq=mu_B, var_freq=10.,
                          monitors=['freq'])

syn_input2A_AMPA = AMPA(pre=neu_input2A, post=neu_A,
                        conn=bp.connect.One2One())

syn_input2B_AMPA = AMPA(pre=neu_input2B, post=neu_B,
                        conn=bp.connect.One2One())
```

Compute basic Poisson frequencies for stimuli given to neuron group A and B.

$$\mu_A = \mu_0 + \rho_A * c$$
$$\mu_B = \mu_0 - \rho_B * c$$

Save model parameters and variables.

Stimulus are only generated in simulation between time period [stim_start_t, stim_end_t].

Generate new Poisson frequency obeys Gaussian distribution for this neuron every 50ms.

$$f_A \sim N(\mu_A, \delta^2)$$
$$f_B \sim N(\mu_B, \delta^2)$$

Generate Poisson inputs with the Poisson frequency of this neuron.

Define neurons that generate Poisson stimulus inputs for every neuron in the network.

Define AMPA synapses that send the Poisson stimulus inputs to the neurons.

During the simulation, subgroup A receives a larger stimulus input than B, after a definite delay period, the activity of group A is significantly higher than group B, which means, the network chooses the right direction.

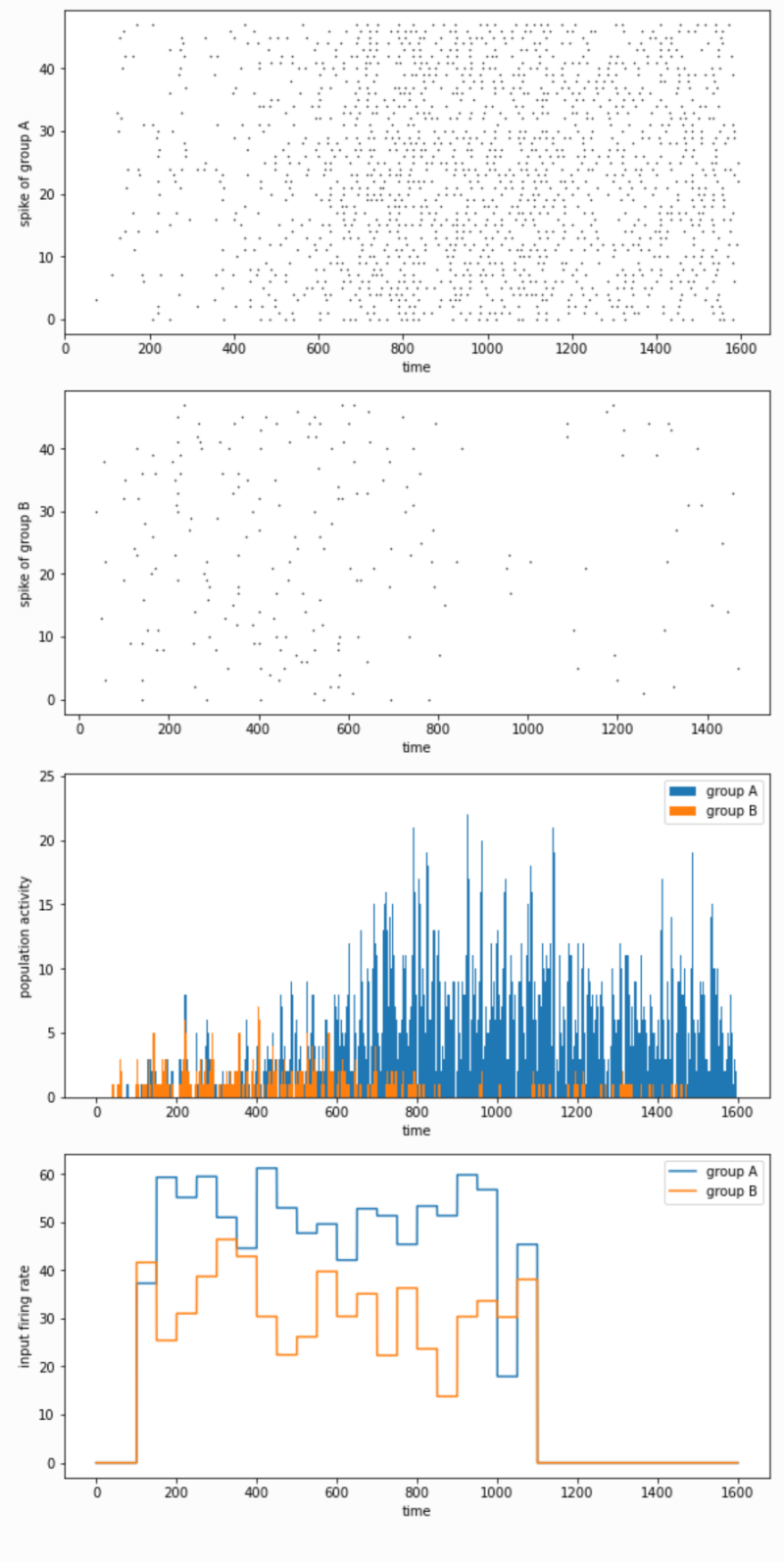



**Fig.3-6 decision making network**

# 3.2 Firing rate networks

## 3.2.1 Decision model

In addition to spiking models, BrainPy can also implement Firing rate models. Let's first look at the implementation of a simplified version of the decision model. The model was simplified by the researcher (Wong & Wang, 2006)[1] through a series of means such as mean field approach. In the end, there are only two variables, $S_1$ and $S_2$, which respectively represent the state of two neuron groups and correspond to two options.

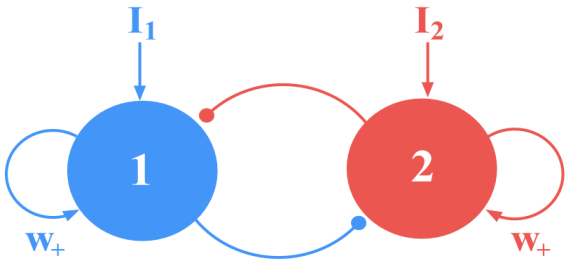



**Fig. 3-1 Reduced decision model.** (From *Wong & Wang, 2006* [1])

The model is given by,

$$\frac{dS_1}{dt} = -\frac{S_1}{\tau} + (1 - S_1)\gamma r_1$$

$$\frac{dS_2}{dt} = -\frac{S_2}{\tau} + (1 - S_2)\gamma r_2$$

where $r_1$ and $r_2$ is the firing rate of two neuron groups, which is given by the input-output function,

$$r_i = f(I_{syn,i})$$

$$f(I) = \frac{aI - b}{1 - \exp[-d(aI - b)]}$$

where $I_{syn,i}$ is given by the model structure (Fig. 3-1),

$$I_{syn,1} = J_{11}S_1 - J_{12}S_2 + I_0 + I_1$$

$$I_{syn,2} = J_{22}S_2 - J_{21}S_1 + I_0 + I_2$$

where $I_0$ is the background current, and the external inputs $I_1, I_2$ are determined by the total input strength $\mu_0$ and a coherence $c'$. The higher the coherence, the more definite $S_1$ is the correct answer, while the lower the coherence, the more random it is. The formula is as follows:

$$I_1 = J_{\mathrm{A,\,ext}}\mu_0(1+\frac{c'}{100\%})$$

$$I_2 = J_{\mathrm{A,\,ext}}\mu_0(1-\frac{c'}{100\%})$$

The code implementation is as follows: we can create a neuron group class, and use $S_1$ and $S_2$ to store the two states of the neuron group. The dynamics of the model can be implemented by a `derivative` function for dynamics analysis.

```python
from collections import OrderedDict
import brainpy as bp
bp.backend.set(backend='numba', dt=0.1)

class Decision(bp.NeuGroup):
    target_backend = ['numpy', 'numba']

    @staticmethod
    def derivative(s1, s2, t, I, coh,
                   JAext, J_rec, J_inh, I_0,
                   a, b, d, tau_s, gamma):
        I1 = JAext * I * (1. + coh)
        I2 = JAext * I * (1. - coh)

        I_syn1 = J_rec * s1 - J_inh * s2 + I_0 + I1
        r1 = (a * I_syn1 - b) / (1. - bp.ops.exp(-d * (a * I_syn1 - b)))
        ds1dt = - s1 / tau_s + (1. - s1) * gamma * r1

        I_syn2 = J_rec * s2 - J_inh * s1 + I_0 + I2
        r2 = (a * I_syn2 - b) / (1. - bp.ops.exp(-d * (a * I_syn2 - b)))
        ds2dt = - s2 / tau_s + (1. - s2) * gamma * r2

        return ds1dt, ds2dt

    def __init__(self, size, coh, JAext=.00117, J_rec=.3725, J_inh=.1137,
                 I_0=.3297, a=270., b=108., d=0.154, tau_s=.06, gamma=0.641,
                 **kwargs):
        # parameters
        self.coh = coh
        self.JAext = JAext
        self.J_rec = J_rec
        self.J_inh = J_inh
        self.I0 = I_0
        self.a = a
        self.b = b
        self.d = d
        self.tau_s = tau_s
        self.gamma = gamma

        # variables
        self.s1 = bp.ops.ones(size) * .06
        self.s2 = bp.ops.ones(size) * .06
        self.input = bp.ops.zeros(size)

        self.integral = bp.odeint(f=self.derivative, method='rk4', dt=0.01)

        super(Decision, self).__init__(size=size, **kwargs)

    def update(self, _t):
        for i in range(self.size):
            self.s1[i], self.s2[i] = self.integral(self.s1[i], self.s2[i], _t,
                                                   self.input[i], self.coh,
                                                   self.JAext, self.J_rec,
                                                   self.J_inh, self.I0,
                                                   self.a, self.b, self.d,
                                                   self.tau_s, self.gamma)
            self.input[i] = 0.
```

$I_1 = J_{Aext}\mu_0(1+c')$
$I_2 = J_{Aext}\mu_0(1-c')$

$$I_{syn,1} = J_{11}S_1 - J_{12}S_2 + I_0 + I_1$$
$$r_1 = \frac{aI_{syn,1} - b}{1 - exp(-d(aI_{syn,1} - b))}$$
$$\frac{dS_1}{dt} = -S_1/\tau + (1 - S_1)\gamma r_1$$

$$I_{syn,2} = J_{22}S_2 - J_{21}S_1 + I_0 + I_2$$
$$r_2 = \frac{aI_{syn,2} - b}{1 - exp(-d(aI_{syn,2} - b))}$$
$$\frac{dS_2}{dt} = -S_2/\tau + (1 - S_2)\gamma r_2$$

Then we can define a function to perform phase plane analysis.

```python
def phase_analyze(I, coh):
    decision = Decision(1, coh=coh)

    phase = bp.analysis.PhasePlane(decision.integral,
                                   target_vars=OrderedDict(s2=[0., 1.],
                                                           s1=[0., 1.]),
                                   fixed_vars=None,
                                   pars_update=dict(I=I, coh=coh,
                                                    JAext=.00117, J_rec=.3725,
                                                    J_inh=.1137, I_0=.3297,
                                                    a=270., b=108., d=0.154,
                                                    tau_s=.06, gamma=0.641),
                                   numerical_resolution=.001,
                                   options={'escape_sympy_solver': True})

    phase.plot_nullcline()
    phase.plot_fixed_point()
    phase.plot_vector_field(show=True)
```

Functions defining the differential equations (from decision model)

Variables to be showed in phase plane.

Specify parameters.

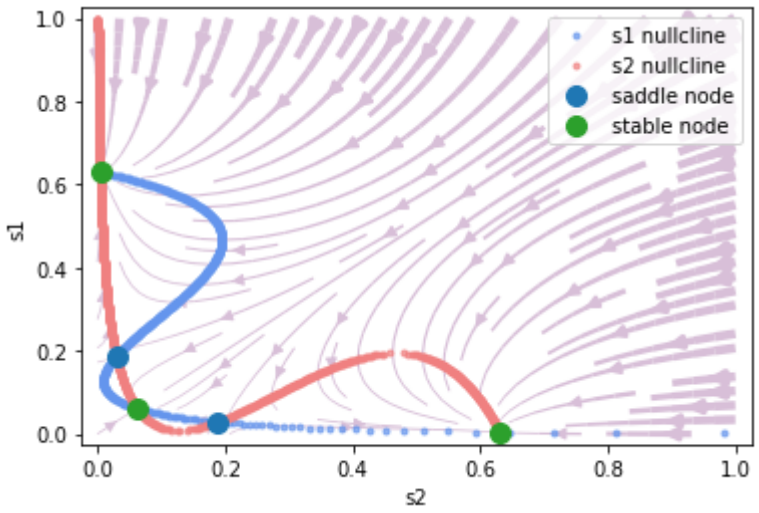
We use numerical solution rather than analytic solution because the equations are so complex. Thus, we don't need to use the sympy solver .

Let's first look at the case when there is no external input. At this time, $\mu_0 = 0$.

```
phase_analyze(I=0., coh=0.)
```

Output:

```
plot nullcline ...
plot fixed point ...
Fixed point #1 at s2=0.06176109215560733, s1=0.061761097890810475 is a stable n
Fixed point #2 at s2=0.029354239100062428, s1=0.18815448592736211 is a saddle n
Fixed point #3 at s2=0.0042468423702408655, s1=0.6303045696241589 is a stable n
Fixed point #4 at s2=0.6303045696241589, s1=0.004246842370235128 is a stable no
Fixed point #5 at s2=0.18815439944520335, s1=0.029354240536530615 is a saddle n
plot vector field ...
```




It can be seen that it is very convenient to use BrainPy for dynamics analysis. The vector field and fixed point indicate which option will fall in the end under different initial values.

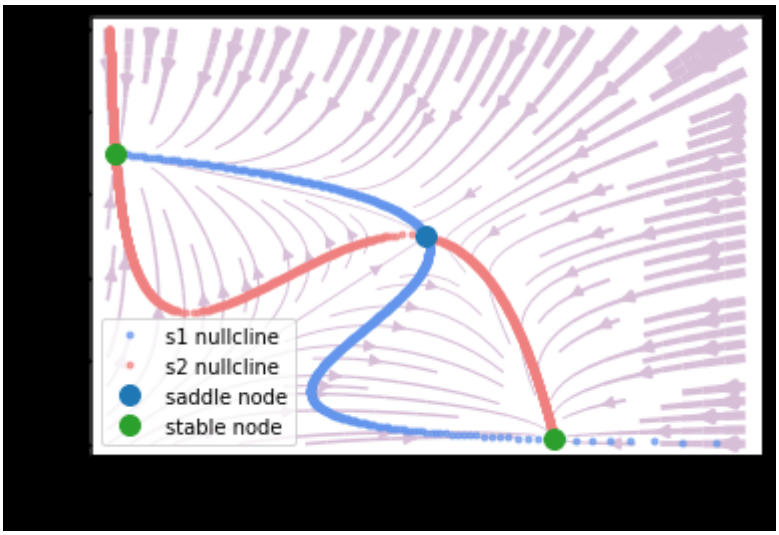
Here, the x-axis is $S_2$ which represents choice 2, and the y-axis is $S_1$, which represents choice 1. As you can see, the upper-left fixed point represents choice 1, the lower-right fixed point represents choice 2, and the lower-left fixed point represents no choice.

Now let's see which option will eventually fall under different initial values with different coherence, and we fix the external input strength to 30.

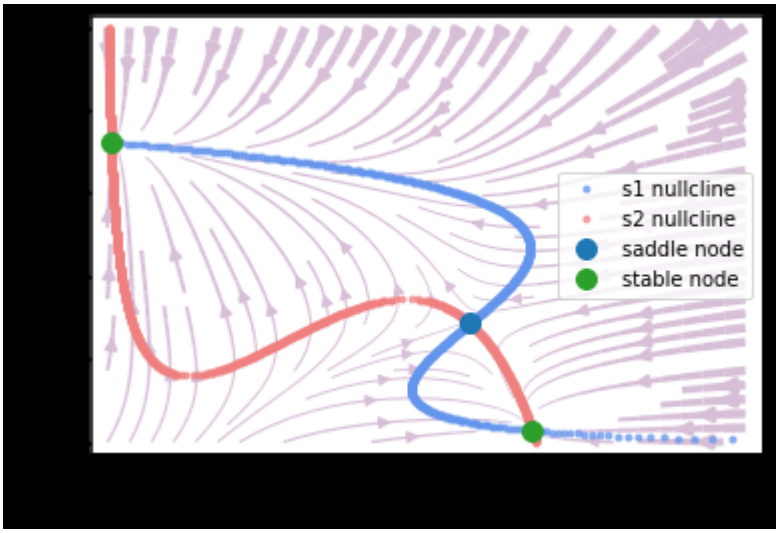
Now let's look at the phase plane under different coherences when we fix the external input strength to 30.

```
# coherence = 0%
print("coherence = 0%")
phase_analyze(I=30., coh=0.)

# coherence = 51.2%
print("coherence = 51.2%")
phase_analyze(I=30., coh=0.512)

# coherence = 100%
print("coherence = 100%")
phase_analyze(I=30., coh=1.)
```

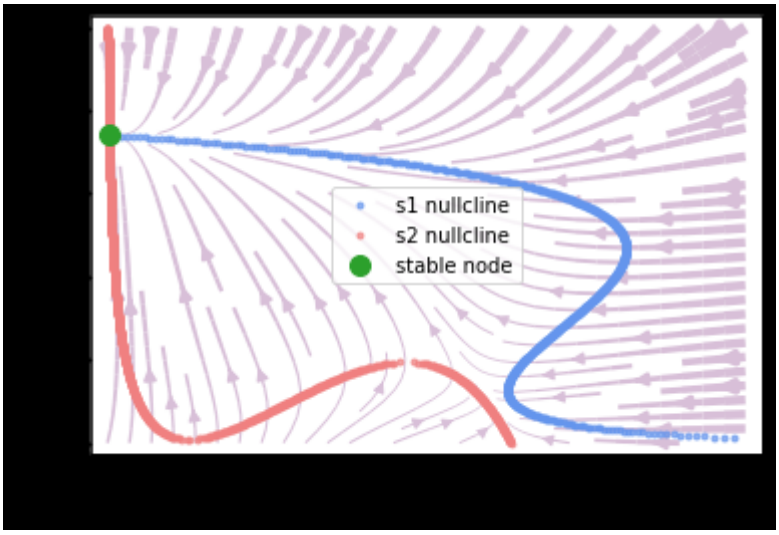
```
coherence = 0%
plot nullcline ...
plot fixed point ...
Fixed point #1 at s2=0.6993504413889349, s1=0.011622049526766405 is a stable no
Fixed point #2 at s2=0.49867489858358865, s1=0.49867489858358865 is a saddle no
Fixed point #3 at s2=0.011622051540013889, s1=0.6993504355529329 is a stable no
plot vector field ...
```




```
coherence = 51.2%
plot nullcline ...
plot fixed point ...
Fixed point #1 at s2=0.5673124813731691, s1=0.2864701069327971 is a saddle node
Fixed point #2 at s2=0.6655747347157656, s1=0.027835279565912054 is a stable no
Fixed point #3 at s2=0.005397687847426814, s1=0.7231453520305031 is a stable no
plot vector field ...
```




```
coherence = 100%
plot nullcline ...
plot fixed point ...
Fixed point #1 at s2=0.0026865954387078755, s1=0.7410985604497689 is a stable r
plot vector field ...
```




### 3.2.2 CANN

Let's see another example of firing rate model, a continuous attractor neural network (CANN)[2]. Fig. 3-2 demonstrates the structure of one-dimensional CANN.

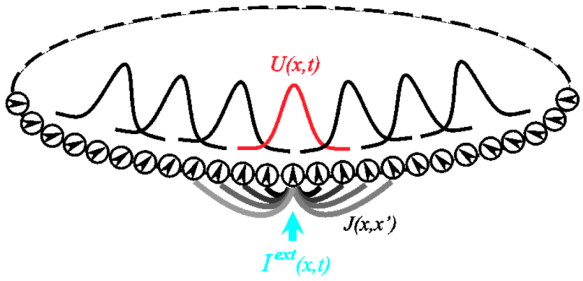



**Fig. 3-2 Structure of CANN.** (From *Wu et al., 2008* [2])

We denote (x) as the parameter space site of the neuron group, and the dynamics of the total synaptic input of neuron group (x) $u(x)$ is given by:

$$\tau \frac{du(x,t)}{dt} = -u(x,t) + \rho \int dx' J(x,x') r(x',t) + I_{ext}$$

Where $r(x',t)$ is the firing rate of the neuron group (x'), which is given by:

$$r(x,t) = \frac{u(x,t)^2}{1 + k\rho \int dx' u(x',t)^2}$$

The intensity of excitatory connection between (x) and (x') $J(x,x')$ is given by a Gaussian function:

$$J(x,x') = \frac{1}{\sqrt{2\pi}a} \exp(-\frac{|x-x'|^2}{2a^2})$$

The external input $I_{ext}$ is related to position $z(t)$:

$$I_{ext} = A \exp\left[-\frac{|x-z(t)|^2}{4a^2}\right]$$

While implementing with BrainPy, we create a class of `CANN1D` by inheriting `bp.NeuGroup`.

```python
import brainpy as bp
import numpy as np
bp.backend.set(backend='numpy', dt=0.1)

class CANN1D(bp.NeuGroup):
    target_backend = ['numpy', 'numba']

    def __init__(self, num, tau=1., k=8.1, a=0.5, A=10., J0=4.,
                 z_min=-np.pi, z_max=np.pi, **kwargs):
        # parameters
        self.tau = tau  # The synaptic time constant
        self.k = k      # Degree of the rescaled inhibition
        self.a = a      # Half-width of the range of excitatory connections
        self.A = A      # Magnitude of the external input
        self.J0 = J0    # maximum connection value

        # feature space
        self.z_min = z_min
        self.z_max = z_max
        self.z_range = z_max - z_min
        self.x = np.linspace(z_min, z_max, num)  # The encoded feature values

        # variables
        self.u = np.zeros(num)
        self.input = np.zeros(num)

        # The connection matrix
        self.conn_mat = self.make_conn(self.x)

        super(CANN1D, self).__init__(size=num, **kwargs)

        self.rho = num / self.z_range   # The neural density
        self.dx = self.z_range / num    # The stimulus density

```

Distances on the ring:
$\rho \int dx'$
*z_range denotes the range of $x$*

Then we define the functions.

```python
    @staticmethod
    @bp.odeint(method='rk4', dt=0.05)
    def int_u(u, t, conn, k, tau, Iext):
        r1 = np.square(u)
        r2 = 1.0 + k * np.sum(r1)
        r = r1 / r2
        Irec = np.dot(conn, r)
        du = (-u + Irec + Iext) / tau
        return du

    def dist(self, d):
        d = np.remainder(d, self.z_range)
        d = np.where(d > 0.5 * self.z_range, d - self.z_range, d)
        return d

    def make_conn(self, x):
        assert np.ndim(x) == 1
        x_left = np.reshape(x, (-1, 1))
        x_right = np.repeat(x.reshape((1, -1)), len(x), axis=0)
        d = self.dist(x_left - x_right)
        Jxx = self.J0 * np.exp(-0.5 * np.square(d / self.a)) / (
                                              np.sqrt(2 * np.pi) * self.a)
        return Jxx

    def get_stimulus_by_pos(self, pos):
        return self.A * np.exp(-0.25 * np.square(self.dist(self.x -
                                                           pos) / self.a))

    def update(self, _t):
        self.u = self.int_u(self.u, _t, self.conn_mat, self.k, self.tau,
                            self.input)
        self.input[:] = 0.
```

$r(x,t) = \frac{u(x,t)^2}{1+k\rho\int dx' u(x',t)^2}$

$\frac{du(x,t)}{dt} = -u(x,t) + \rho\int dx' J(x,x') r(x',t) + I_{ext}$

Distances on the ring:
x is the position of the ring, so we have:
$x \in [-\pi, \pi]$
z_range denotes the range of $x$ ($= 2\pi$).
distance on the ring: $d = |x - x'| < \pi$

*d=|x-x'|*
Get a matrix of d for all positions

| | $x_1$ | $x_2$ | ... |
|---|---|---|---|
| $x_1$ | | $d_{12}$ | |
| $x_2$ | $d_{21}$ | | |
| ... | | | |

$J(x,x') = \frac{\exp(-0.5(\frac{d}{a})^2)}{\sqrt{2\pi}a}$

$I_{ext} = A\exp(\frac{|x - z(t)|^2}{4a^2})$

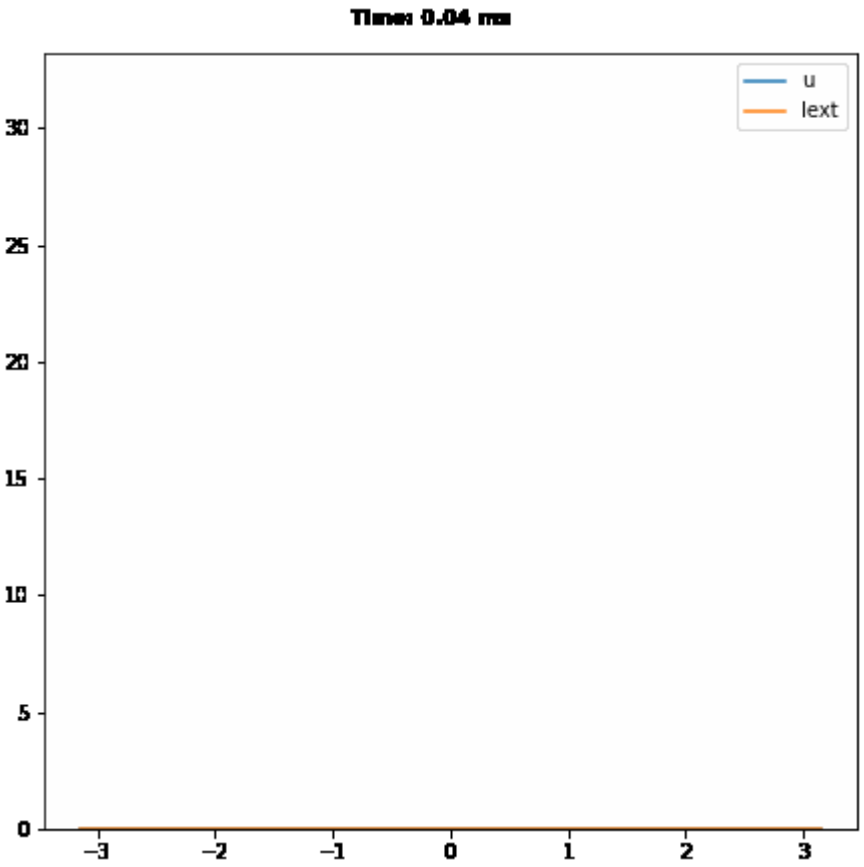
Where the functions `dist` and `make_conn` are designed to get the connection strength $J$ between each of the two neuron groups. In the `make_conn` function, we first calculate the distance matrix between each of the two $x$. Because neurons are arranged in rings, the value of $x$ is between $-\pi$ and $\pi$, so the range of $|x - x'|$ is $2\pi$, and -$\pi$ and $\pi$ are the same points (the actual maximum value is $\pi$, that is, half of `z_range`, the distance exceeded needs to be subtracted from a `z_range`). We use the `dist` function to handle the distance on the ring.

The `get_stimulus_by_pos` function processes external inputs based on position `pos`, which allows users to get input current by setting target positions. For example, in a simple population coding, we give an external input of `pos=0`, and we run in this way:

```
cann = CANN1D(num=512, k=0.1, monitors=['u'])

I1 = cann.get_stimulus_by_pos(0.)
Iext, duration = bp.inputs.constant_current([(0., 1.), (I1, 8.), (0., 8.)])
cann.run(duration=duration, inputs=('input', Iext))
```

Then lets plot an animation by calling the `bp.visualize.animate_1D` function.

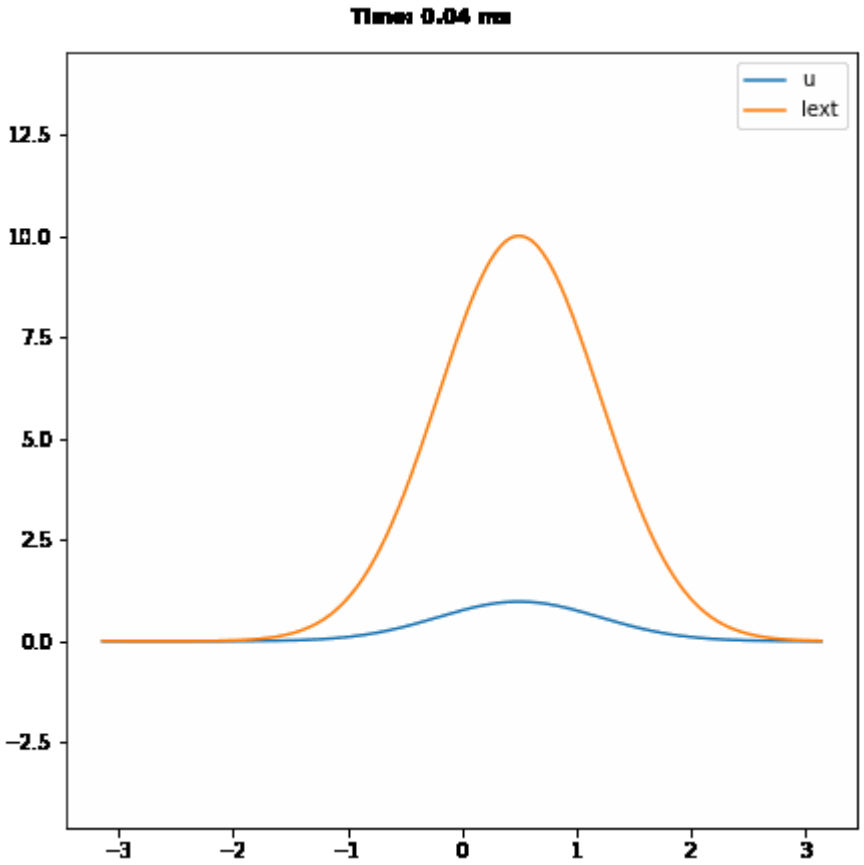
```
# define function
def plot_animate(frame_step=5, frame_delay=50):
    bp.visualize.animate_1D(dynamical_vars=[{'ys': cann.mon.u, 'xs': cann.x,
                                             'legend': 'u'}, {'ys': Iext,
                                             'xs': cann.x, 'legend': 'Iext'}],
                            frame_step=frame_step, frame_delay=frame_delay,
                            show=True)


# call the function
plot_animate(frame_step=1, frame_delay=100)
```




We can see that the shape of $u$ encodes the shape of external input.

Now we add random noise to the external input to see how the shape of $u$ changes.

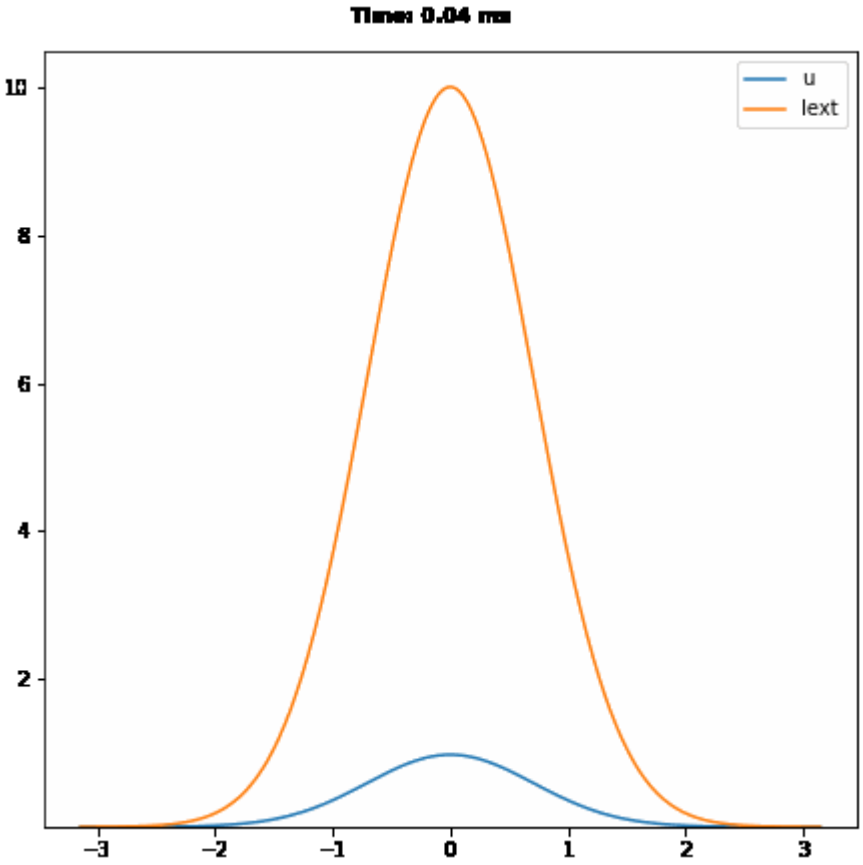
```
cann = CANN1D(num=512, k=8.1, monitors=['u'])

dur1, dur2, dur3 = 10., 30., 0.
num1 = int(dur1 / bp.backend.get_dt())
num2 = int(dur2 / bp.backend.get_dt())
num3 = int(dur3 / bp.backend.get_dt())
Iext = np.zeros((num1 + num2 + num3,) + cann.size)
Iext[:num1] = cann.get_stimulus_by_pos(0.5)
Iext[num1:num1 + num2] = cann.get_stimulus_by_pos(0.)
Iext[num1:num1 + num2] += 0.1 * cann.A * np.random.randn(num2, *cann.size)
cann.run(duration=dur1 + dur2 + dur3, inputs=('input', Iext))

plot_animate()
```




We can see that the shape of $u$ remains like a bell shape, which indicates that it can perform template matching based on the input.

Now let's give a moving input, we vary the position of the input with `np.linspace`, we will see that the $u$ will follow the input, i.e., smooth tracking.

```
cann = CANN1D(num=512, k=8.1, monitors=['u'])

dur1, dur2, dur3 = 20., 20., 20.
num1 = int(dur1 / bp.backend.get_dt())
num2 = int(dur2 / bp.backend.get_dt())
num3 = int(dur3 / bp.backend.get_dt())
position = np.zeros(num1 + num2 + num3)
position[num1: num1 + num2] = np.linspace(0., 12., num2)
position[num1 + num2:] = 12.
position = position.reshape((-1, 1))
Iext = cann.get_stimulus_by_pos(position)
cann.run(duration=dur1 + dur2 + dur3, inputs=('input', Iext))

plot_animate()
```




## Reference


[1]. Wong, K.-F. & Wang, X.-J. A Recurrent Network Mechanism of Time Integration in Perceptual Decisions. J. Neurosci. 26, 1314–1328 (2006). ↩

[2]. Si Wu, Kosuke Hamaguchi, and Shun-ichi Amari. "Dynamics and computation of continuous attractors." Neural computation 20.4 (2008): 994-1025. ↩