ONELOOP4PT

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Abstract

In this document, we caculate scalar One Loop four point function with complex internal mass.

1 The Form of One Loop Four Point in Paralell and Orthogonal Space

In Paralell and Orthogonal Space, the form of One Loop Four Point is

$$D_0 = 2\int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \frac{1}{P_1P_2P_3P_4}$$

Here

$$\begin{aligned} P_1 &= (l_0+q_{10})^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\varepsilon \\ P_2 &= (l_0+q_{20})^2 - (l_1+q_{21})^2 - l_2^2 - l_\perp^2 - m_2^2 + i\varepsilon \\ P_3 &= (l_0+q_{30})^2 - (l_1+q_{31})^2 - (l_2+q_{32})^2 - l_\perp^2 - m_3^2 + i\varepsilon \\ P_4 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon \end{aligned} \tag{1}$$

And

$$\begin{aligned} q_1^2 &= q_{10}^2. \\ q_2^2 &= q_{20}^2 - q_{21}^2 \\ q_3^2 &= q_{30}^2 - q_{31}^2 - q_{32}^2 \\ q_4^2 &= 0. \\ l^2 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2 \end{aligned} \tag{2}$$

$m_i^2 = Re(m_k^2) - i\Gamma_k$ are complex internal mass.

2 The partial fraction

We have

$$
\begin{aligned}
\frac{1}{P_1P_2P_3P_4} &= \frac{1}{P_1(P_2-P_1)(P_3-P_1)(P_4-P_1)} \\
&\quad + \frac{1}{P_2(P_1-P_2)(P_3-P_2)(P_4-P_2)} \\
&\quad + \frac{1}{P_3(P_1-P_3)(P_2-P_3)(P_4-P_3)} \\
&\quad + \frac{1}{P_4(P_1-P_4)(P_2-P_4)(P_3-P_4)} \\
&= \sum_{k=1}^{4} \frac{1}{P_k \prod\limits_{l=1, l\neq k} (P_l - P_k)} \qquad (3)
\end{aligned}
$$

here

$$
\begin{aligned}
P_k &= (l_0 + q_{k0})^2 - (l_1 + q_{k1})^2 - (l_2 + q_{k2})^2 - l_\perp - m_k^2 + i\varepsilon \\
P_l &= (l_0 + q_{l0})^2 - (l_1 + q_{l1})^2 - (l_2 + q_{l2})^2 - l_\perp - m_l^2 + i\varepsilon \\
P_k - P_l &= 2(q_{l0} - q_{k0})l_0 - 2(q_{l1} - q_{k1})l_1 - 2(q_{l2} - q_{k2})l_2 + q_l^2 - q_k^2 - (m_l^2 - m_k^2) \\
&= a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + q_l^2 - q_k^2 - (m_l^2 - m_k^2). \qquad (4)
\end{aligned}
$$

It is important to note that a_{lk}, b_{lk}, c_{lk} in R.

From now, we obtain

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4} \int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \\
&\quad \frac{1}{\left[(l_0 + q_{k0})^2 - (l_1 + q_{k1})^2 - (l_2 + q_{k2})^2 - l_\perp - m_k^2 + i\varepsilon\right]} \\
&\quad \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + q_l^2 - q_k^2 - (m_l^2 - m_k^2)} \qquad (5)
\end{aligned}
$$

We make a shift

$$
\begin{aligned}
l_0 &\to l_0 + q_{k0} \\
l_1 &\to l_1 + q_{k1} \\
l_2 &\to l_2 + q_{k2}
\end{aligned}
\qquad (6)
$$

The Jacobian of this shift is 1. The integration region not change and the form of D_0 now look as

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4} \int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \\
&\quad \frac{1}{\left[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + d_{lk})} \qquad (7)
\end{aligned}
$$

Here

$$
\begin{aligned}
&-a_{lk}q_{k0} - b_{lk}q_{k1} - c_{lk}q_{k2} + q_l^2 - q_k^2 - (m_l^2 - m_k^2) = \\
&-2(q_{l0} - q_{k0})q_{k0} + 2(q_{l1} - q_{k1})q_{k1} + 2(q_{l2} - q_{k2})q_{k2} + q_l^2 - q_k^2 - (m_l^2 - m_k^2) \\
&q_l^2 + q_k^2 - 2q_l q_k - (m_l^2 - m_k^2).
\end{aligned} \tag{9}
$$

SUMMARIZE:

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4}\int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \\
&\quad \frac{1}{\left[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + d_{lk})}. \\
&\text{And} \\
a_{lk} &= 2(q_{l0} - q_{k0}) \\
b_{lk} &= 2(q_{l1} - q_{k1}) \\
a_{lk} &= 2(q_{l2} - q_{k2}) \\
d_{lk} &= (q_l - q_k)^2 - (m_l^2 - m_l^2) \\
&\quad \text{Important note} \\
&\quad a_{lk}, b_{lk}, c_{lk} \text{in} R;\ d_{lk} \text{in} C.
\end{aligned} \tag{9}
$$

3 Linearize in x and the $x-$ integration

In this section, we take $x-$ integration by residuce theorem. To do that, we have to linearize D_0 in x, or take a shift

$$
\begin{aligned}
l_0 &= x + z \\
l_1 &= y \\
l_2 &= x \\
l_\perp &= t.
\end{aligned}
$$

The Jacobian of this shift is

$$
|J| = \left|\frac{\delta(l_0, l_1, l_2, l_\perp)}{\delta(z, y, x, t)}\right| = 1. \tag{10}
$$

For this shift, one obtain

$$D_0 = 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dxdydz \int_0^{\infty} dt \frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}. \tag{12}$$

Here $AC_{lk} = a_{lk} + c_{lk}$

3.1 The x- integration

The poles of the D_0 integrand are

$$\begin{aligned} x_0 &= \frac{z^2 + y^2 + t^2 + m_k^2 - i\varepsilon}{2z} \\ x_l &= \frac{-a_{lk}z - b_{lk}y - d_{lk}}{AC_{lk}} \end{aligned} \tag{12}$$

It is important to note that

$$\begin{aligned} Im(x_0) &= \frac{-\Gamma_k - \varepsilon}{2z} \\ Im(x_l) &= \frac{-d_{lk}}{AC_{lk}} \end{aligned} \tag{13}$$

We now separate D_0 into form

$$D_0 = D_0^+ + D_0^-$$

with

$$\begin{aligned} D_0^+ &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dxdy \int_0^{\infty} dz \int_0^{\infty} dt \\ &\quad \frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}. \\ D_0^- &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dxdy \int_{-\infty}^{0} dz \int_0^{\infty} dt \\ &\quad \frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}. \end{aligned} \tag{14}$$

3.1.1 For D_0^+

We close the uper contour in the x plane and D_0^+ is evaluated

$$D_0^+ = 4\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \;\; Res\Big[F(x,y,z,t), x_l\Big] \tag{15}$$

or

$$D_0^+ = 4\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\Big[2x_l z - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\Big]} \frac{1}{AC_{lk} \prod\limits_{m=1,m\neq l,k} (a_{mk}z + b_{mk}y + AC_{mk}x + d_{mk})} \tag{16}$$

With

$$x_l = \frac{-a_{lk}z - b_{lk}y - d_{lk}}{AC_{lk}} \tag{17}$$

From now we obtain

$$D_0^+ = 2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\varepsilon\Big]}$$

here

$$\begin{aligned} A_{mlk} &= a_{mk} - \frac{a_{lk}AC_{mk}}{AC_{lk}} \\ B_{mlk} &= b_{mk} - \frac{b_{lk}AC_{mk}}{AC_{lk}} \\ C_{mlk} &= d_{mk} - \frac{d_{lk}AC_{mk}}{AC_{lk}} \end{aligned}$$

3.1.2 For D_0^-

We close thelower contour in the x plane and D_0^- is evaluated

$$D_0^- = -2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \frac{f_{lk}^{-}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\varepsilon\Big]}$$

here

$$
\begin{aligned}
A_{mlk} &= a_{mk} - \frac{a_{lk}AC_{mk}}{AC_{lk}} \\
B_{mlk} &= \frac{b_{mk}}{AC_{mk}} - \frac{b_{lk}}{AC_{lk}} \\
C_{mlk} &= \frac{d_{mk}}{AC_{mk}} - \frac{d_{lk}}{AC_{lk}}
\end{aligned}
$$

SUMMARIZE:

$$
D_0 = D_0^+ + D_0^-
$$

and

$$
\begin{aligned}
D_0^+ &= 2\pi i \sum_{k=1}^{4} \sum_{l=1, l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \, \frac{1}{\prod\limits_{m=1, m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\
&\quad \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\left[\left(1-\frac{2a_{lk}}{AC_{lk}}\right)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \\
D_0^- &= -2\pi i \sum_{k=1}^{4} \sum_{l=1, l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dz \int_{0}^{\infty} dt \, \frac{1}{\prod\limits_{m=1, m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\
&\quad \frac{f_{lk}^{-}\Big(1-\delta(AC_{lk})\Big)}{\left[\left(1-\frac{2a_{lk}}{AC_{lk}}\right)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - t^2 - m_k^2 + i\varepsilon\right]}
\end{aligned} \tag{18}
$$

here

$$
\begin{aligned}
A_{mlk} &= a_{mk} - \frac{a_{lk}AC_{mk}}{AC_{lk}} \\
B_{mlk} &= b_{mk} - \frac{b_{lk}AC_{mk}}{AC_{lk}} \\
C_{mlk} &= d_{mk} - \frac{d_{lk}AC_{mk}}{AC_{lk}}
\end{aligned}
$$

4 The y integration

The next we are going to take y integration. To do that we have to perform Wick rotation $t \to it$ then linearize in y.

4.1 t- wick rotation

To linearize in y, the sign of y^2 and t^2 must be opsite. To do that we have to perform t- wick rotation.

The poles of $t-$ integrand are

$$t_{1,2} = \pm\sqrt{\Big(1 - \frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - m_k^2 + i\varepsilon} \tag{19}$$

Because

$$Im\Big[-\frac{2d_{lk}}{AC_{lk}}z - m_k^2 + i\varepsilon\Big] > 0. \tag{20}$$

then $t_{1,2}$ locate in the first or the thirth quarter $t-$ complex plane.

We have

$$\oint f(t^2)dt = \Big\{\int\limits_0^R + \int\limits_{C_k} + \int\limits_{-iR}^0\Big\}f(t^2)dt = 0 \tag{21}$$

When R go to ∞, one obtain

$$\Big\{\int\limits_0^\infty + \int\limits_{-i\infty}^0\Big\}f(t^2)dt = 0. \tag{22}$$

or

$$\int\limits_0^\infty f(t^2)dt = -\int\limits_{-i\infty}^0 f(t^2)dt \tag{23}$$

Making $t-$ rotation, one obtain

$$\int\limits_0^\infty f(t^2)dt = -i\int\limits_0^\infty f(-t^2)dt \tag{24}$$

After $t-$ Wick rotation, We rewrite $D_0^\pm$ to form

$$\boxed{\begin{aligned} D_0^+ &= \pi\sum_{k=1}^4\sum_{l=1,l\neq k}^4 \frac{1}{AC_{lk}}\int\limits_{-\infty}^\infty dy\int\limits_0^\infty dz\int\limits_{-\infty}^\infty dt\,\frac{1}{\prod\limits_{m=1,m\neq l,k}(A_{mlk}z + B_{mlk}y + C_{mlk})} \\ &\qquad \frac{f_{lk}^+\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 + t^2 - m_k^2 + i\varepsilon\Big]} \\ D_0^- &= -\pi\sum_{k=1}^4\sum_{l=1,l\neq k}^4 \frac{1}{AC_{lk}}\int\limits_{-\infty}^\infty dy\int\limits_{-\infty}^0 dz\int\limits_{-\infty}^\infty dt\,\frac{1}{\prod\limits_{m=1,m\neq l,k}(A_{mlk}z + B_{mlk}y + C_{mlk})} \\ &\qquad \frac{f_{lk}^-\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 + t^2 - m_k^2 + i\varepsilon\Big]}\end{aligned}} \tag{25}$$

4.2 The $y-$ integration

To linearize in y, we make a shift $t = t' + y$. The Jacobian of this shift is 1. The $t-$ integration region not change and one obtain

$$
\begin{aligned}
D_0^+ &= \pi \sum_{k=1}^{4} \sum_{l=1, l\neq k}^{4} \frac{1}{AC_{lk}} \int\limits_{-\infty}^{\infty} dy \int\limits_{0}^{\infty} dz \int\limits_{-\infty}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\
&\quad \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 + 2\Big(t - \frac{b_{lk}}{AC_{lk}}z\Big)y - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon\Big]} \\
D_0^- &= -\pi \sum_{k=1}^{4} \sum_{l=1, l\neq k}^{4} \frac{1}{AC_{lk}} \int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{0} dz \int\limits_{-\infty}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\
&\quad \frac{f_{lk}^{-}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 + 2\Big(t - \frac{b_{lk}}{AC_{lk}}z\Big)y - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon\Big]}
\end{aligned}
$$

The poles of The $y-$ integrand are

$$
\begin{aligned}
y_0 &= -\frac{\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon}{2\Big(t - \frac{b_{lk}}{AC_{lk}}z\Big)} \\
y_{mlk} &= -\frac{A_{mlk}z + C_{mlk}}{B_{mlk}}
\end{aligned}
\tag{26}
$$

Apply the residue theorem, we obtain

$$
D_0 = D_0^{++} + D_0^{+-} + D_0^{-+} + D_0^{--} \tag{27}
$$

with

$$
\begin{aligned}
D_0^{++} &= +i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{0}^{\infty} dz \int\limits_{\alpha_{lk}z}^{\infty} dt \quad f_{lk}^{+} g_{mlk}^{+} I'_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{0}^{\infty} dz \int\limits_{-\infty}^{\alpha_{lk}z} dt \quad f_{lk}^{+} g_{mlk}^{-} I'_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{-\infty}^{0} dz \int\limits_{\alpha_{lk}z}^{\infty} dt \quad f_{lk}^{-} g_{mlk}^{+} I'_{nmlk} \\
D_0^{--} &= i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{-\infty}^{0} dz \int\limits_{-\infty}^{\alpha_{lk}z} dt \quad f_{lk}^{-} g_{mlk}^{-} I'_{nmlk}
\end{aligned}
$$

Here

$$
\begin{aligned}
I'_{nmlk} &= \frac{1}{AC_{lk}} \frac{\Big[1-\delta(AC_{lk})\Big]\Big[1-\delta(B_{mlk})\Big]}{\Big[A_{nlk}B_{mlk}-A_{mlk}B_{nlk}\Big]} \\
&\quad \frac{1}{\Big[z+F_{nmlk}\Big]} \frac{1}{\Big[D'_{mlk}z^2-2\frac{A_{mlk}}{B_{mlk}}zt-2\frac{C_{mlk}}{B_{mlk}}t+E'_{mlk}z+t^2-m_k^2+i\varepsilon\Big]} \qquad (28)
\end{aligned}
$$

and

$$
\begin{aligned}
F_{nmlk} &= \frac{C_{nlk}B_{mlk}-B_{nlk}C_{mlk}}{A_{nlk}B_{mlk}-B_{nlk}A_{mlk}} \\
D'_{mlk} &= 1-\frac{2a_{lk}}{AC_{lk}}+2\frac{b_{lk}}{AC_{lk}}\frac{A_{mlk}}{B_{mlk}} \\
E'_{mlk} &= -2\Big(\frac{d_{lk}}{AC_{lk}}-\frac{b_{lk}}{AC_{lk}}\frac{C_{mlk}}{B_{mlk}}\Big) \qquad (29)
\end{aligned}
$$

We make a change $t'=t+\alpha_{lk}z$, the jacobian is 1. The $t-$ integrand move to $[0,\pm\infty]$ and one obtain

$$
\begin{aligned}
D_0^{++} &= +i\pi^2\sum_{m,l,k=1}^{4}\int_0^{\infty}dz\int_0^{\infty}dt \quad f_{lk}^{+}g_{mlk}^{+}I_{nmlk} \\
D_0^{+-} &= -i\pi^2\sum_{m,l,k=1}^{4}\int_0^{\infty}dz\int_{-\infty}^{0}dt \quad f_{lk}^{+}g_{mlk}^{-}I_{nmlk} \\
D_0^{+-} &= -i\pi^2\sum_{m,l,k=1}^{4}\int_{-\infty}^{0}dz\int_0^{\infty}dt \quad f_{lk}^{-}g_{mlk}^{+}I_{nmlk} \\
D_0^{--} &= i\pi^2\sum_{m,l,k=1}^{4}\int_{-\infty}^{0}dz\int_{-\infty}^{0}dt \quad f_{lk}^{-}g_{mlk}^{-}I_{nmlk}
\end{aligned}
$$

Here

$$
\begin{aligned}
I_{nmlk} &= \frac{1}{AC_{lk}} \frac{\Big[1-\delta(AC_{lk})\Big]\Big[1-\delta(B_{mlk})\Big]}{\Big[A_{nlk}B_{mlk}-A_{mlk}B_{nlk}\Big]} \\
&\quad \frac{1}{\Big[z+F_{nmlk}\Big]} \frac{1}{\Big[D_{mlk}z^2-2\Big(\frac{A_{mlk}}{B_{mlk}}-\alpha_{lk}\Big)zt-2\frac{C_{mlk}}{B_{mlk}}t-\frac{2d_{lk}}{AC_{lk}}z+t^2-m_k^2+i\varepsilon\Big]} \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
D_{mlk} &= D'_{mlk} + \alpha_{lk}^2 - 2\frac{A_{mlk}}{B_{mlk}}\alpha_{lk} \\
&= 1 - \frac{2\alpha_{lk}}{AC_{lk}} + \frac{b_{lk}^2}{AC_{lk}^2} \\
&= -\frac{a_{lk}^2 - b_{lk}^2 - c_{lk}^2}{AC_{lk}^2} \\
&= -4\frac{(q_l - q_k)^2}{AC_{lk}^2}
\end{aligned}
$$

SUMMARIZE

$$
\begin{aligned}
D_0^{++} &= +i\pi^2 \sum_{m,l,k=1}^{4} \int_0^{\infty} dz \int_0^{\infty} dt \quad f_{lk}^{+} g_{mlk}^{+} I_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int_0^{\infty} dz \int_{-\infty}^{0} dt \quad f_{lk}^{+} g_{mlk}^{-} I_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int_{-\infty}^{0} dz \int_0^{\infty} dt \quad f_{lk}^{-} g_{mlk}^{+} I_{nmlk} \\
D_0^{--} &= i\pi^2 \sum_{m,l,k=1}^{4} \int_{-\infty}^{0} dz \int_{-\infty}^{0} dt \quad f_{lk}^{-} g_{mlk}^{-} I_{nmlk}
\end{aligned}
$$

Here

$$
\begin{aligned}
I_{nmlk} &= \frac{1}{AC_{lk}} \frac{\Big[1 - \delta(AC_{lk})\Big]\Big[1 - \delta(B_{mlk})\Big]}{\Big[A_{nlk}B_{mlk} - A_{mlk}B_{nlk}\Big]} \\
&\quad \frac{1}{\Big[z + F_{nmlk}\Big]} \frac{1}{\Big[D_{mlk}z^2 - 2\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big)zt - 2\frac{C_{mlk}}{B_{mlk}}t - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon\Big]}
\end{aligned}
$$

and

$$
D_{mlk} = -4\frac{(q_l - q_k)^2}{AC_{lk}^2}
$$

5 t- integration

To linear in t, we make a shift

$$
\begin{aligned}
z &= z' + \beta t' \\
t &= t' + \varphi z'
\end{aligned}
$$

The Jacobian of this shift is

$$J = \Big|1 - \beta\varphi\Big| \tag{31}$$

For this shift, we have

$$z + F_{nmlk} \longrightarrow z + F_{nmlk} + \beta_{mlk} t$$

$$\begin{aligned}
& D_{mlk} z^2 - 2\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big) zt - 2\frac{C_{mlk}}{B_{mlk}} t - \frac{2d_{lk}}{AC_{lk}} z + t^2 - m_k^2 + i\varepsilon \\
& \longrightarrow \Big[D_{mlk} - 2(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})\varphi_{mlk} + \varphi_{mlk}^2\Big] z^2 \\
& + \Big[D_{mlk}\beta_{mlk}^2 - 2(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})\beta_{mlk} + 1\Big] t^2 \\
& + \Big[2D_{mlk}\beta_{mlk} + 2\varphi_{mlk} - 2(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})(1 - \beta_{mlk}\varphi_{mlk})\Big] zt \\
& + \Big[-2\frac{C_{mlk}}{B_{mlk}}\varphi_{mlk} - 2\frac{d_{lk}}{AC_{lk}}\Big] z \\
& + \Big[-2\frac{C_{mlk}}{B_{mlk}} - 2\frac{d_{lk}}{AC_{lk}}\beta_{mlk}\Big] t \\
& - m_k^2 + i\varepsilon \\
& \longrightarrow P_{mlk} zt + E_{mlk} z + Q_{mlk} t - m_k^2 + i\varepsilon
\end{aligned}$$

Here we choice

$$\begin{aligned}
\beta_{mlk} &= \frac{\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk} + \sqrt{\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big)^2 - D_{mlk} + i\eta}}{D_{mlk}} \\
\varphi_{mlk} &= \frac{A_{mlk}}{B_{mlk}} - \alpha_{lk} + \sqrt{\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big)^2 - D_{mlk} + i\eta}
\end{aligned} \tag{32}$$

The $t-$integration now look as

5.1 For D_0^{++}

$$\begin{aligned}
& z > 0 \longrightarrow z' + \beta t' > 0 \longrightarrow t' < -\frac{z'}{\beta} \\
& t > 0 \longrightarrow t' + \varphi z' > 0 \longrightarrow t' > -\varphi z'
\end{aligned}$$

So

$$D_0^{++} \longrightarrow \int_0^\infty dz \int_{-\varphi z}^{-\frac{z}{\beta}} dt \tag{33}$$

5.2 For D_0^{+-}

$$z > 0 \longrightarrow z' + \beta t' > 0 \longrightarrow t' < -\frac{z'}{\beta}$$
$$t < 0 \longrightarrow t' + \varphi z' < 0 \longrightarrow t' < -\varphi z'$$

So

$$D_0^{+-} \longrightarrow \int\limits_0^{\infty} dz \int\limits_{-\infty}^{-\varphi z} dt + \int\limits_{-\infty}^{0} dz \int\limits_{-\infty}^{-\frac{z}{\beta}} dt \tag{34}$$

5.3 For D_0^{-+}

$$z < 0 \longrightarrow z' + \beta t' < 0 \longrightarrow t' > -\frac{z'}{\beta}$$
$$t > 0 \longrightarrow t' + \varphi z' > 0 \longrightarrow t' > -\varphi z'$$

So

$$D_0^{-+} \longrightarrow \int\limits_{-\infty}^{0} dz \int\limits_{-\varphi z}^{\infty} dt + \int\limits_{0}^{\infty} dz \int\limits_{-\frac{z}{\beta}}^{\infty} dt \tag{35}$$

5.4 For D_0^{--}

$$z < 0 \longrightarrow z' + \beta t' < 0 \longrightarrow t' > -\frac{z'}{\beta}$$
$$t < 0 \longrightarrow t' + \varphi z' < 0 \longrightarrow t' < -\varphi z'$$

So

$$D_0^{--} \longrightarrow \int\limits_{-\infty}^{0} dz \int\limits_{-\frac{z}{\beta}}^{-\varphi z} dt \tag{36}$$

To be more compact, we rewrite I_{nmlk} to form

$$I_{nmlk} = G(z)\Big[\frac{1}{t + \frac{z+F_{nmlk}}{\beta_{mlk}}} - \frac{1}{t + \frac{E_{mlk}z - m_k^2 + i\varepsilon}{P_{mlk}z + Q_{mlk}}}\Big] \tag{37}$$

with

$$G(z) = \frac{1}{\beta_{mlk}(E_{mlk}z - m_k^2 + i\varepsilon) - (P_{mlk}z + Q_{mlk})(z + F_{nmlk})} \tag{38}$$

Apply the formular

$$\int_{-\infty}^{a} f(z)dz = \sum_{k=1} Res\Big\{log(z-a)f(z); z_k\Big\}$$
$$\int_{-a}^{\infty} f(-z)dz = \sum_{k=1} Res\Big\{log(z-a)f(z); z_k\Big\}$$

We obtain

$$\begin{aligned} D_0 \;=\; & i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1 \\ k\neq l}}^{4} \sum_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} \frac{1}{AC_{lk}} \frac{1}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \times \\ & \Big(1 - \delta_{lk}(AC_{lk})\Big)\Big(1 - \delta_{lk}(B_{mlk})\Big) \left|1 - \beta_{mlk}\varphi_{mlk}\right| \times \\ & \Bigg[\int_0^{\infty} dz\, G(z) \Bigg\{ (f_{lk}g_{mlk} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{F}{\beta}\right) \\ & -f_{lk}g_{mlk} \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) - f_{lk}g_{mlk}^{-} \ln\left(-\frac{(1-\beta\varphi)z + F}{\beta}\right) \\ & -(f_{lk}g_{mlk} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\ & +f_{lk}g_{mlk} \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \\ & +f_{lk}g_{mlk}^{-} \ln\left(-\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \\ & + \int_{-\infty}^{0} dz\, G(z) \Bigg\{ -f_{lk}^{-}g_{mlk}^{-} \ln\left(\frac{F}{\beta}\right) - f_{lk}g_{mlk}^{-} \ln\left(-\frac{F}{\beta}\right) \\ & +(f_{lk}^{-}g_{mlk}^{-} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) \\ & +f_{lk}^{-}g_{mlk}^{-} \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\ & +f_{lk}g_{mlk}^{-} \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\ & -(f_{lk}^{-}g_{mlk}^{-} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \Bigg] \end{aligned} \tag{39}$$

6 Summarize

From now, we summarize the result D_0 to (69) in Npoint.ps

$$
\begin{aligned}
D_0 \quad = \quad & i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1 \\ k\neq l}}^{4} \sum_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} \frac{1}{AC_{lk}} \frac{1}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \times \\
& \Big(1 - \delta_{lk}(AC_{lk})\Big)\Big(1 - \delta_{lk}(B_{mlk})\Big)\,|1 - \beta_{mlk}\varphi_{mlk}| \times \\
& \Bigg[\qquad \int_0^\infty dz\; G(z) \Bigg\{ (f_{lk}g_{mlk} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{F}{\beta}\right) \\
& - f_{lk}g_{mlk} \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) - f_{lk}g_{mlk}^{-} \ln\left(-\frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& - (f_{lk}g_{mlk} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& + f_{lk}g_{mlk} \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& + f_{lk}g_{mlk}^{-} \ln\left(-\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \\
& \qquad + \int_{-\infty}^{0} dz\; G(z) \Bigg\{ -f_{lk}^{-}g_{mlk}^{-} \ln\left(\frac{F}{\beta}\right) - f_{lk}g_{mlk}^{-} \ln\left(-\frac{F}{\beta}\right) \\
& + (f_{lk}^{-}g_{mlk}^{-} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& + f_{lk}^{-}g_{mlk}^{-} \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& + f_{lk}g_{mlk}^{-} \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& - (f_{lk}^{-}g_{mlk}^{-} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \qquad \Bigg]
\end{aligned}
$$

Conclusion: This result is different $\frac{1}{AC_{lk}} \frac{1}{B_{mlk}}$ to (69) in Npoint.ps