ONELOOP4PT

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March 12, 2009

Abstract

In this document, we caculate scalar One Loop four point function with complex internal mass.

1 The Form of One Loop Four Point in Paralell and Orthogonal Space

In Paralell and Orthogonal Space, the form of One Loop Four Point is

$$D_0 = 2\int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \frac{1}{P_1 P_2 P_3 P_4}$$

Here

$$\begin{aligned} P_1 &= (l_0+q_{10})^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\varepsilon \\ P_2 &= (l_0+q_{20})^2 - (l_1+q_{21})^2 - l_2^2 - l_\perp^2 - m_2^2 + i\varepsilon \\ P_3 &= (l_0+q_{30})^2 - (l_1+q_{31})^2 - (l_2+q_{32})^2 - l_\perp^2 - m_3^2 + i\varepsilon \\ P_4 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon \end{aligned} \tag{1}$$

And

$$\begin{aligned} q_1^2 &= q_{10}^2. \\ q_2^2 &= q_{20}^2 - q_{21}^2 \\ q_3^2 &= q_{30}^2 - q_{31}^2 - q_{32}^2 \\ q_4^2 &= 0. \\ l^2 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2 \end{aligned} \tag{2}$$

$m_i^2 = Re(m_k^2) - i\Gamma_k$ are complex internal mass.

2 The partial fraction

We have

$$
\begin{aligned}
\frac{1}{P_1P_2P_3P_4} &= \frac{1}{P_1(P_2-P_1)(P_3-P_1)(P_4-P_1)} \\
&\quad+\frac{1}{P_2(P_1-P_2)(P_3-P_2)(P_4-P_2)} \\
&\quad+\frac{1}{P_3(P_1-P_3)(P_2-P_3)(P_4-P_3)} \\
&\quad+\frac{1}{P_4(P_1-P_4)(P_2-P_4)(P_3-P_4)} \\
&= \sum_{k=1}^{4}\frac{1}{P_k\prod\limits_{l=1,l\neq k}(P_l-P_k)} \qquad (3)
\end{aligned}
$$

here

$$
\begin{aligned}
P_k &= (l_0+q_{k0})^2-(l_1+q_{k1})^2-(l_2+q_{k2})^2-l_\perp-m_k^2+i\varepsilon \\
P_l &= (l_0+q_{l0})^2-(l_1+q_{l1})^2-(l_2+q_{l2})^2-l_\perp-m_l^2+i\varepsilon \\
P_k-P_l &= 2(q_{l0}-q_{k0})l_0-2(q_{l1}-q_{k1})l_1-2(q_{l2}-q_{k2})l_2+q_l^2-q_k^2-(m_l^2-m_k^2) \\
&= a_{lk}l_0+b_{lk}l_1+c_{lk}l_2+q_l^2-q_k^2-(m_l^2-m_k^2). \qquad (4)
\end{aligned}
$$

It is important to note that a_{lk}, b_{lk}, c_{lk} in R.

From now, we obtain

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty}dl_0dl_1dl_2\int_0^{\infty}dl_\perp \\
&\quad \frac{1}{\Big[(l_0+q_{k0})^2-(l_1+q_{k1})^2-(l_2+q_{k2})^2-l_\perp-m_k^2+i\varepsilon\Big]} \\
&\quad \frac{1}{\prod\limits_{l=1,l\neq k}(a_{lk}l_0+b_{lk}l_1+c_{lk}l_2+q_l^2-q_k^2-(m_l^2-m_k^2)} \qquad (5)
\end{aligned}
$$

We make a shift

$$
\begin{aligned}
l_0 &\to l_0+q_{k0} \\
l_1 &\to l_1+q_{k1} \\
l_2 &\to l_2+q_{k2}
\end{aligned} \qquad (6)
$$

The Jacobian of this shift is 1. The integration region not change and the form of D_0 now look as

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty}dl_0dl_1dl_2\int_0^{\infty}dl_\perp \\
&\quad \frac{1}{\Big[l_0^2-l_1^2-l_2^2-l_\perp^2-m_4^2+i\varepsilon\Big]}\frac{1}{\prod\limits_{l=1,l\neq k}(a_{lk}l_0+b_{lk}l_1+c_{lk}l_2+d_{lk})} \qquad (7)
\end{aligned}
$$

Here

$$
\begin{aligned}
&-a_{lk}q_{k0} - b_{lk}q_{k1} - c_{lk}q_{k2} + q_l^2 - q_k^2 - (m_l^2 - m_k^2) = \\
&-2(q_{l0} - q_{k0})q_{k0} + 2(q_{l1} - q_{k1})q_{k1} + 2(q_{l2} - q_{k2})q_{k2} + q_l^2 - q_k^2 - (m_l^2 - m_k^2) \\
&q_l^2 + q_k^2 - 2q_l q_k - (m_l^2 - m_k^2). \qquad (9)
\end{aligned}
$$

SUMMARIZE:

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4} \int_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int_0^{\infty} dl_\perp \\
&\quad \frac{1}{\left[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + d_{lk})}. \\
&\text{And} \\
a_{lk} &= 2(q_{l0} - q_{k0}) \\
b_{lk} &= 2(q_{l1} - q_{k1}) \\
a_{lk} &= 2(q_{l2} - q_{k2}) \\
d_{lk} &= (q_l - q_k)^2 - (m_l^2 - m_l^2) \\
&\quad \text{Important note} \\
&\quad a_{lk}, b_{lk}, c_{lk} \text{in} R;\; d_{lk} \text{in} C.
\end{aligned} \qquad (9)
$$

3 Linearize in x and the $x-$ integration

In this section, we take $x-$ integration by residuce theorem. To do that, we have to linearize D_0 in x, or take a shift

$$
\begin{aligned}
l_0 &= x + z \\
l_1 &= y \\
l_2 &= x \\
l_\perp &= t.
\end{aligned}
$$

The Jacobian of this shift is

$$
|J| = \left| \frac{\delta(l_0, l_1, l_2, l_\perp)}{\delta(z, y, x, t)} \right| = 1. \qquad (10)
$$

For this shift, one obtain

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4}\int\limits_{-\infty}^{\infty} dxdydz \int\limits_{0}^{\infty} dt \\
&\quad \frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}.
\end{aligned}
\tag{12}
$$

Here $AC_{lk} = a_{lk} + c_{lk}$

3.1 The x- integration

The poles of the D_0 integrand are

$$
\begin{aligned}
x_0 &= \frac{z^2 + y^2 + t^2 + m_k^2 - i\varepsilon}{2z} \\
x_l &= \frac{-a_{lk}z - b_{lk}y - d_{lk}}{AC_{lk}}
\end{aligned}
\tag{12}
$$

It is important to note that

$$
\begin{aligned}
Im(x_0) &= \frac{-\Gamma_k - \varepsilon}{2z} \\
Im(x_l) &= \frac{-d_{lk}}{AC_{lk}}
\end{aligned}
\tag{13}
$$

We now separate D_0 into form

$$D_0 = D_0^+ + D_0^-$$

with

$$
\begin{aligned}
D_0^+ &= 2\sum_{k=1}^{4}\int\limits_{-\infty}^{\infty} dxdy \int\limits_{0}^{\infty} dz \int\limits_{0}^{\infty} dt \\
&\quad \frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}. \\
D_0^- &= 2\sum_{k=1}^{4}\int\limits_{-\infty}^{\infty} dxdy \int\limits_{-\infty}^{0} dz \int\limits_{0}^{\infty} dt \\
&\quad \frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}.
\end{aligned}
\tag{14}
$$

3.1.1 For D_0^+

We close the uper contour in the x plane and D_0^+ is evaluated

$$D_0^+ = 4\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \;\; Res\Big[F(x,y,z,t), x_l\Big] \tag{15}$$

or

$$\begin{aligned} D_0^+ &= 4\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{f_{lk}^+ \Big(1-\delta(AC_{lk})\Big)}{\Big[2x_l z - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\Big]} \\ & \qquad \frac{1}{AC_{lk} \prod\limits_{m=1,m\neq l,k} (a_{mk} z + b_{mk} y + AC_{mk} x + d_{mk})} \end{aligned} \tag{16}$$

With

$$x_l = \frac{-a_{lk} z - b_{lk} y - d_{lk}}{AC_{lk}} \tag{17}$$

From now we obtain

$$\begin{aligned} D_0^+ &= 2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk} z + B_{mlk} y + C_{mlk})} \\ & \qquad \frac{f_{lk}^+ \Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big) z^2 - \frac{2b_{lk}}{AC_{lk}} yz - \frac{2d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\varepsilon\Big]} \end{aligned}$$

here

$$\begin{aligned} A_{mlk} &= a_{mk} - \frac{a_{lk} AC_{mk}}{AC_{lk}} \\ B_{mlk} &= b_{mk} - \frac{b_{lk} AC_{mk}}{AC_{lk}} \\ C_{mlk} &= d_{mk} - \frac{d_{lk} AC_{mk}}{AC_{lk}} \end{aligned}$$

3.1.2 For D_0^-

We close thelower contour in the x plane and D_0^- is evaluated

$$\begin{aligned} D_0^- &= -2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk} z + B_{mlk} y + C_{mlk})} \\ & \qquad \frac{f_{lk}^- \Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big) z^2 - \frac{2b_{lk}}{AC_{lk}} yz - \frac{2d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\varepsilon\Big]} \end{aligned}$$

here

$$A_{mlk} = a_{mk} - \frac{a_{lk}AC_{mk}}{AC_{lk}}$$
$$B_{mlk} = \frac{b_{mk}}{AC_{mk}} - \frac{b_{lk}}{AC_{lk}}$$
$$C_{mlk} = \frac{d_{mk}}{AC_{mk}} - \frac{d_{lk}}{AC_{lk}}$$

SUMMARIZE:

$$D_0 = D_0^+ + D_0^-$$

and

$$\begin{aligned} D_0^+ &= 2\pi i \sum_{k=1}^{4}\sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \, \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\ &\quad \frac{f_{lk}^+\Big(1-\delta(AC_{lk})\Big)}{\left[\left(1-\frac{2a_{lk}}{AC_{lk}}\right)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \\ D_0^- &= -2\pi i \sum_{k=1}^{4}\sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dz \int_{0}^{\infty} dt \, \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\ &\quad \frac{f_{lk}^-\Big(1-\delta(AC_{lk})\Big)}{\left[\left(1-\frac{2a_{lk}}{AC_{lk}}\right)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \end{aligned} \tag{18}$$

here

$$A_{mlk} = a_{mk} - \frac{a_{lk}AC_{mk}}{AC_{lk}}$$
$$B_{mlk} = b_{mk} - \frac{b_{lk}AC_{mk}}{AC_{lk}}$$
$$C_{mlk} = d_{mk} - \frac{d_{lk}AC_{mk}}{AC_{lk}}$$

4 The y integration

The next we are going to take y integration. To do that we have to perform Wick rotation $t \to it$ then linearize in y.

4.1 t- wick rotation

To linearize in y, the sign of y^2 and t^2 must be opsite. To do that we have to perform t- wick rotation.

The poles of $t-$ integrand are

$$t_{1,2} = \pm\sqrt{\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - m_k^2 + i\varepsilon} \tag{19}$$

Because

$$Im\Big[-\frac{2d_{lk}}{AC_{lk}}z - m_k^2 + i\varepsilon\Big] > 0. \tag{20}$$

then $t_{1,2}$ locate in the first or the thirth quarter $t-$ complex plane.

We have

$$\oint f(t^2)dt = \Big\{\int\limits_0^R + \int\limits_{C_k} + \int\limits_{-iR}^0\Big\} f(t^2)dt = 0 \tag{21}$$

When R go to ∞, one obtain

$$\Big\{\int\limits_0^\infty + \int\limits_{-i\infty}^0\Big\} f(t^2)dt = 0. \tag{22}$$

or

$$\int\limits_0^\infty f(t^2)dt = -\int\limits_{-i\infty}^0 f(t^2)dt \tag{23}$$

Making $t-$ rotation, one obtain

$$\int\limits_0^\infty f(t^2)dt = -i\int\limits_0^\infty f(-t^2)dt \tag{24}$$

After $t-$ Wick rotation, We rewrite $D_0^\pm$ to form

$$\boxed{\begin{aligned} D_0^+ &= \pi\sum_{k=1}^4\sum_{l=1,l\neq k}^4 \frac{1}{AC_{lk}}\int\limits_{-\infty}^\infty dy\int\limits_0^\infty dz\int\limits_{-\infty}^\infty dt\,\frac{1}{\prod\limits_{m=1,m\neq l,k}(A_{mlk}z+B_{mlk}y+C_{mlk})} \\ &\quad \frac{f_{lk}^+\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2-\frac{2b_{lk}}{AC_{lk}}yz-\frac{2d_{lk}}{AC_{lk}}z-y^2+t^2-m_k^2+i\varepsilon\Big]} \\ D_0^- &= -\pi\sum_{k=1}^4\sum_{l=1,l\neq k}^4 \frac{1}{AC_{lk}}\int\limits_{-\infty}^\infty dy\int\limits_{-\infty}^0 dz\int\limits_{-\infty}^\infty dt\,\frac{1}{\prod\limits_{m=1,m\neq l,k}(A_{mlk}z+B_{mlk}y+C_{mlk})} \\ &\quad \frac{f_{lk}^-\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2-\frac{2b_{lk}}{AC_{lk}}yz-\frac{2d_{lk}}{AC_{lk}}z-y^2+t^2-m_k^2+i\varepsilon\Big]} \end{aligned}} \tag{25}$$

4.2 The $y-$ integration

To linearize in y, we make a shift $t = t' + y$. The Jacobian of this shift is 1. The $t-$ integration region not change and one obtain

$$
\begin{aligned}
D_0^+ &= \pi \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int\limits_{-\infty}^{\infty} dy \int\limits_{0}^{\infty} dz \int\limits_{-\infty}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\
&\qquad \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 + 2\Big(t - \frac{b_{lk}}{AC_{lk}}z\Big)y - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon\Big]} \\
D_0^- &= -\pi \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{AC_{lk}} \int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{0} dz \int\limits_{-\infty}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\
&\qquad \frac{f_{lk}^{-}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 + 2\Big(t - \frac{b_{lk}}{AC_{lk}}z\Big)y - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon\Big]}
\end{aligned}
$$

The poles of The $y-$ integrand are

$$
\begin{aligned}
y_0 &= -\frac{\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon}{2\Big(t - \frac{b_{lk}}{AC_{lk}}z\Big)} \\
y_{mlk} &= -\frac{A_{mlk}z + C_{mlk}}{B_{mlk}}
\end{aligned} \tag{26}
$$

Apply the residue theorem, we obtain

$$
D_0 = D_0^{++} + D_0^{+-} + D_0^{-+} + D_0^{--} \tag{27}
$$

with

$$
\begin{aligned}
D_0^{++} &= +i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{0}^{\infty} dz \int\limits_{\alpha_{lk}z}^{\infty} dt \quad f_{lk}^{+} g_{mlk}^{+} I'_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{0}^{\infty} dz \int\limits_{-\infty}^{\alpha_{lk}z} dt \quad f_{lk}^{+} g_{mlk}^{-} I'_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{-\infty}^{0} dz \int\limits_{\alpha_{lk}z}^{\infty} dt \quad f_{lk}^{-} g_{mlk}^{+} I'_{nmlk} \\
D_0^{--} &= i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{-\infty}^{0} dz \int\limits_{-\infty}^{\alpha_{lk}z} dt \quad f_{lk}^{-} g_{mlk}^{-} I'_{nmlk}
\end{aligned}
$$

Here

$$
\begin{aligned}
I'_{nmlk} &= \frac{1}{AC_{lk}} \frac{\Big[1-\delta(AC_{lk})\Big]\Big[1-\delta(B_{mlk})\Big]}{\Big[A_{nlk}B_{mlk}-A_{mlk}B_{nlk}\Big]} \\
&\quad \frac{1}{\Big[z+F_{nmlk}\Big]} \frac{1}{\Big[D'_{mlk}z^2-2\frac{A_{mlk}}{B_{mlk}}zt-2\frac{C_{mlk}}{B_{mlk}}t+E'_{mlk}z+t^2-m_k^2+i\varepsilon\Big]}
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
F_{nmlk} &= \frac{C_{nlk}B_{mlk}-B_{nlk}C_{mlk}}{A_{nlk}B_{mlk}-B_{nlk}A_{mlk}} \\
D'_{mlk} &= 1-\frac{2a_{lk}}{AC_{lk}}+2\frac{b_{lk}}{AC_{lk}}\frac{A_{mlk}}{B_{mlk}} \\
E'_{mlk} &= -2\Big(\frac{d_{lk}}{AC_{lk}}-\frac{b_{lk}}{AC_{lk}}\frac{C_{mlk}}{B_{mlk}}\Big)
\end{aligned} \tag{29}
$$

We make a change $t'=t+\alpha_{lk}z$, the jacobian is 1. The $t-$ integrand move to $[0,\pm\infty]$ and one obtain

$$
\begin{aligned}
D_0^{++} &= +i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_0^\infty dz \int\limits_0^\infty dt \quad f_{lk}^+ g_{mlk}^+ I_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_0^\infty dz \int\limits_{-\infty}^0 dt \quad f_{lk}^+ g_{mlk}^- I_{nmlk} \\
D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{-\infty}^0 dz \int\limits_0^\infty dt \quad f_{lk}^- g_{mlk}^+ I_{nmlk} \\
D_0^{--} &= i\pi^2 \sum_{m,l,k=1}^{4} \int\limits_{-\infty}^0 dz \int\limits_{-\infty}^0 dt \quad f_{lk}^- g_{mlk}^- I_{nmlk}
\end{aligned}
$$

Here

$$
\begin{aligned}
I_{nmlk} &= \frac{1}{AC_{lk}} \frac{\Big[1-\delta(AC_{lk})\Big]\Big[1-\delta(B_{mlk})\Big]}{\Big[A_{nlk}B_{mlk}-A_{mlk}B_{nlk}\Big]} \\
&\quad \frac{1}{\Big[z+F_{nmlk}\Big]} \frac{1}{\Big[D_{mlk}z^2-2\Big(\frac{A_{mlk}}{B_{mlk}}-\alpha_{lk}\Big)zt-2\frac{C_{mlk}}{B_{mlk}}t-\frac{2d_{lk}}{AC_{lk}}z+t^2-m_k^2+i\varepsilon\Big]}
\end{aligned} \tag{30}
$$

$$\begin{aligned} D_{mlk} &= D'_{mlk} + \alpha_{lk}^2 - 2\frac{A_{mlk}}{B_{mlk}}\alpha_{lk} \\ &= 1 - \frac{2\alpha_{lk}}{AC_{lk}} + \frac{b_{lk}^2}{AC_{lk}^2} \\ &= -\frac{a_{lk}^2 - b_{lk}^2 - c_{lk}^2}{AC_{lk}^2} \\ &= -4\frac{(q_l - q_k)^2}{AC_{lk}^2} \end{aligned}$$

SUMMARIZE

$$\begin{aligned} D_0^{++} &= +i\pi^2 \sum_{m,l,k=1}^{4} \int_0^{\infty} dz \int_0^{\infty} dt \quad f_{lk}^{+} g_{mlk}^{+} I_{nmlk} \\ D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int_0^{\infty} dz \int_{-\infty}^{0} dt \quad f_{lk}^{+} g_{mlk}^{-} I_{nmlk} \\ D_0^{+-} &= -i\pi^2 \sum_{m,l,k=1}^{4} \int_{-\infty}^{0} dz \int_0^{\infty} dt \quad f_{lk}^{-} g_{mlk}^{+} I_{nmlk} \\ D_0^{--} &= i\pi^2 \sum_{m,l,k=1}^{4} \int_{-\infty}^{0} dz \int_{-\infty}^{0} dt \quad f_{lk}^{-} g_{mlk}^{-} I_{nmlk} \end{aligned}$$

Here

$$\begin{aligned} I_{nmlk} &= \frac{1}{AC_{lk}} \frac{\Big[1 - \delta(AC_{lk})\Big]\Big[1 - \delta(B_{mlk})\Big]}{\Big[A_{nlk}B_{mlk} - A_{mlk}B_{nlk}\Big]} \\ & \frac{1}{\Big[z + F_{nmlk}\Big]} \frac{1}{\Big[D_{mlk}z^2 - 2\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big)zt - 2\frac{C_{mlk}}{B_{mlk}}t - \frac{2d_{lk}}{AC_{lk}}z + t^2 - m_k^2 + i\varepsilon\Big]} \end{aligned}$$

and

$$D_{mlk} = -4\frac{(q_l - q_k)^2}{AC_{lk}^2}$$

5 t- integration

To linear in t, we make a shift

$$z = z' + \beta t'$$
$$t = t' + \varphi z'$$

The Jacobian of this shift is

$$J = \Big|1 - \beta\varphi\Big| \tag{31}$$

For this shift, we have

$$z + F_{nmlk} \longrightarrow z + F_{nmlk} + \beta_{mlk} t$$

$$\begin{aligned}
&D_{mlk} z^2 - 2\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big) zt - 2\frac{C_{mlk}}{B_{mlk}} t - \frac{2d_{lk}}{AC_{lk}} z + t^2 - m_k^2 + i\varepsilon \\
&\longrightarrow \Big[D_{mlk} - 2(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})\varphi_{mlk} + \varphi_{mlk}^2\Big] z^2 \\
&+ \Big[D_{mlk}\beta_{mlk}^2 - 2(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})\beta_{mlk} + 1\Big] t^2 \\
&+ \Big[2D_{mlk}\beta_{mlk} + 2\varphi_{mlk} - 2(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk})(1 - \beta_{mlk}\varphi_{mlk})\Big] zt \\
&+ \Big[-2\frac{C_{mlk}}{B_{mlk}}\varphi_{mlk} - 2\frac{d_{lk}}{AC_{lk}}\Big] z \\
&+ \Big[-2\frac{C_{mlk}}{B_{mlk}} - 2\frac{d_{lk}}{AC_{lk}}\beta_{mlk}\Big] t \\
&- m_k^2 + i\varepsilon \\
&\longrightarrow P_{mlk} zt + E_{mlk} z + Q_{mlk} t - m_k^2 + i\varepsilon
\end{aligned}$$

Here we choice

$$\begin{aligned}
\beta_{mlk} &= \frac{\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk} + \sqrt{\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big)^2 - D_{mlk} + i\eta}}{D_{mlk}} \\
\varphi_{mlk} &= \frac{A_{mlk}}{B_{mlk}} - \alpha_{lk} + \sqrt{\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big)^2 - D_{mlk} + i\eta}
\end{aligned} \tag{32}$$

The $t-$integration now look as

5.1 For D_0^{++}

$$\begin{aligned}
&z > 0 \longrightarrow z' + \beta t' > 0 \longrightarrow t' < -\frac{z'}{\beta} \\
&t > 0 \longrightarrow t' + \varphi z' > 0 \longrightarrow t' > -\varphi z'
\end{aligned}$$

So

$$D_0^{++} \longrightarrow \int_0^\infty dz \int_{-\varphi z}^{-\frac{z}{\beta}} dt \tag{33}$$

5.2 For D_0^{+-}

$$z>0 \longrightarrow z'+\beta t'>0 \longrightarrow t'<-\frac{z'}{\beta}$$
$$t<0 \longrightarrow t'+\varphi z'<0 \longrightarrow t'<-\varphi z'$$

So

$$D_0^{+-} \longrightarrow \int\limits_0^{\infty} dz \int\limits_{-\infty}^{-\varphi z} dt + \int\limits_{-\infty}^{0} dz \int\limits_{-\infty}^{-\frac{z}{\beta}} dt \tag{34}$$

5.3 For D_0^{-+}

$$z<0 \longrightarrow z'+\beta t'<0 \longrightarrow t'>-\frac{z'}{\beta}$$
$$t>0 \longrightarrow t'+\varphi z'>0 \longrightarrow t'>-\varphi z'$$

So

$$D_0^{-+} \longrightarrow \int\limits_{-\infty}^{0} dz \int\limits_{-\varphi z}^{\infty} dt + \int\limits_{0}^{\infty} dz \int\limits_{-\frac{z}{\beta}}^{\infty} dt \tag{35}$$

5.4 For D_0^{--}

$$z<0 \longrightarrow z'+\beta t'<0 \longrightarrow t'>-\frac{z'}{\beta}$$
$$t<0 \longrightarrow t'+\varphi z'<0 \longrightarrow t'<-\varphi z'$$

So

$$D_0^{--} \longrightarrow \int\limits_{-\infty}^{0} dz \int\limits_{-\frac{z}{\beta}}^{-\varphi z} dt \tag{36}$$

To be more compact, we rewrite I_{nmlk} to form

$$I_{nmlk} = G(z)\Big[\frac{1}{t+\frac{z+F_{nmlk}}{\beta_{mlk}}} - \frac{1}{t+\frac{E_{mlk}z-m_k^2+i\varepsilon}{P_{mlk}z+Q_{mlk}}}\Big] \tag{37}$$

with

$$G(z) = \frac{1}{\beta_{mlk}(E_{mlk}z-m_k^2+i\varepsilon)-(P_{mlk}z+Q_{mlk})(z+F_{nmlk})} \tag{38}$$

Apply the formular

$$\int_{-\infty}^{a} f(z)dz = \sum_{k=1} Res\Big\{log(z-a)f(z); z_k\Big\}$$
$$\int_{-a}^{\infty} f(-z)dz = \sum_{k=1} Res\Big\{log(z-a)f(z); z_k\Big\}$$

We obtain

$$\begin{aligned}
D_0 &= i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1\\k\neq l}}^{4} \sum_{\substack{m=1\\m\neq l\\m\neq k}}^{4} \frac{1}{AC_{lk}} \frac{1}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \times \\
&\Big(1-\delta_{lk}(AC_{lk})\Big)\Big(1-\delta_{lk}(B_{mlk})\Big)\,|1-\beta_{mlk}\varphi_{mlk}| \times \\
&\Bigg[\quad \int_0^\infty dz\; G(z)\Bigg\{(f_{lk}g_{mlk} + f_{lk}^- g_{mlk})\ln\left(\frac{F}{\beta}\right) \\
&-f_{lk}g_{mlk}\ln\left(\frac{(1-\beta\varphi)z+F}{\beta}\right) - f_{lk}g_{mlk}^-\ln\left(-\frac{(1-\beta\varphi)z+F}{\beta}\right) \\
&-(f_{lk}g_{mlk} + f_{lk}^- g_{mlk})\ln\left(\frac{-\frac{P}{\beta}z^2+(E-\frac{Q}{\beta})z-m_k^2+i\varrho}{Q+Pz}\right) \\
&+f_{lk}g_{mlk}\ln\left(\frac{-P\varphi z^2+(E-Q\varphi)z-m_k^2+i\varrho}{Q+Pz}\right) \\
&+f_{lk}g_{mlk}^-\ln\left(-\frac{-P\varphi z^2+(E-Q\varphi)z-m_k^2+i\varrho}{Q+Pz}\right)\Bigg\} \\
&\quad + \int_{-\infty}^0 dz\; G(z)\Bigg\{-f_{lk}^- g_{mlk}^-\ln\left(\frac{F}{\beta}\right) - f_{lk}g_{mlk}^-\ln\left(-\frac{F}{\beta}\right) \\
&+(f_{lk}^- g_{mlk}^- + f_{lk}^- g_{mlk})\ln\left(\frac{(1-\beta\varphi)z+F}{\beta}\right) \\
&+f_{lk}^- g_{mlk}^-\ln\left(\frac{-\frac{P}{\beta}z^2+(E-\frac{Q}{\beta})z-m_k^2+i\varrho}{Q+Pz}\right) \\
&+f_{lk}g_{mlk}^-\ln\left(-\frac{-\frac{P}{\beta}z^2+(E-\frac{Q}{\beta})z-m_k^2+i\varrho}{Q+Pz}\right) \\
&-(f_{lk}^- g_{mlk}^- + f_{lk}^- g_{mlk})\ln\left(\frac{-P\varphi z^2+(E-Q\varphi)z-m_k^2+i\varrho}{Q+Pz}\right)\Bigg\} \quad \Bigg]
\end{aligned} \tag{39}$$

6 Summarize

From now, we summarize the result D_0 and compare to (69) in Npoint.ps

$$
\boxed{
\begin{aligned}
D_0 \quad = \quad & i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1\\k\neq l}}^{4} \sum_{\substack{m=1\\m\neq l\\m\neq k}}^{4} \frac{1}{AC_{lk}} \frac{1}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \times \\
& \Big(1 - \delta_{lk}(AC_{lk})\Big)\Big(1 - \delta_{lk}(B_{mlk})\Big) \left|1 - \beta_{mlk}\varphi_{mlk}\right| \times \\
& \Bigg[\qquad \int\limits_{0}^{\infty} dz\; G(z) \Bigg\{ (f_{lk}g_{mlk} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{F}{\beta}\right) \\
& -f_{lk}g_{mlk} \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) - f_{lk}g_{mlk}^{-} \ln\left(-\frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& -(f_{lk}g_{mlk} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& +f_{lk}g_{mlk} \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& +f_{lk}g_{mlk}^{-} \ln\left(-\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \\
& \qquad + \int_{-\infty}^{0} dz\; G(z) \Bigg\{ -f_{lk}^{-}g_{mlk}^{-} \ln\left(\frac{F}{\beta}\right) - f_{lk}g_{mlk}^{-} \ln\left(-\frac{F}{\beta}\right) \\
& +(f_{lk}^{-}g_{mlk}^{-} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& +f_{lk}^{-}g_{mlk}^{-} \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& +f_{lk}g_{mlk}^{-} \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& -(f_{lk}^{-}g_{mlk}^{-} + f_{lk}^{-}g_{mlk}) \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \qquad \Bigg]
\end{aligned}
}
\tag{40}
$$

Conclusion: This result is different $\frac{1}{AC_{lk}}\frac{1}{B_{mlk}}$ to (69) in Npoint.ps

7 $z-$ integration

The notetation

$$
\begin{aligned}
G(z) &= \frac{1}{\beta(Ez - m_k^2 + i\varepsilon) - (Pz+Q)(z+F)} \\
&= \frac{1}{-P(z-T_1)(z-T_2)}
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
S(\sigma, z) &= P\sigma z^2 + (E + Q\sigma)z - m_k^2 + i\varepsilon \\
&= P\sigma(z - z_{1\sigma})(z - z_{2\sigma})
\end{aligned} \tag{42}
$$

$$Im\Big[S(\sigma, z)\Big] > 0 \tag{43}$$

Using the decompose log function formular

$$
\begin{aligned}
log(a.b) &= log(a) + log(b) + \eta(a, b) \\
log(a/b) &= log(a) - log(b) + \eta(a, \frac{1}{b})
\end{aligned} \tag{44}
$$

$$\eta(a, b) = 2\pi i\Big\{\theta[-Ima]\theta[-Imb]\theta[Imab] - \theta[Ima]\theta[Imb]\theta[-Imab]\Big\} \tag{45}$$

Apply these formular, we obtain

$$
\boxed{
\begin{aligned}
log\Big(\frac{S(\sigma, z)}{Pz+Q}\Big) &= log(P\sigma z - P\sigma z_{1\sigma}) + log(z - z_{2\sigma}) - log(Pz+Q) \\
&\quad +2\pi i\theta[Im(P\sigma z_{1\sigma})]\theta[Im(z_{2\sigma})] - 2\pi i\theta[-Im(Q)]\theta\Big[Im\frac{S(\sigma, z)}{Pz+Q}\Big] \\
log\Big(\frac{-S(\sigma, z)}{Pz+Q}\Big) &= log(-P\sigma z + P\sigma z_{1\sigma}) + log(z - z_{2\sigma}) - log(Pz+Q) \\
&\quad -2\pi i\theta[Im(P\sigma z_{1\sigma})]\theta[-Im(z_{2\sigma})] + 2\pi i\theta[Im(Q)]\theta\Big[-Im\frac{S(\sigma, z)}{Pz+Q}\Big]
\end{aligned}
} \tag{46}
$$

To be more compact, We now represent D_0 in (41) to form

$$\frac{D_0}{i\pi^2} = Coff * (posTerm + negTerm) \tag{47}$$

here

$$
\begin{aligned}
Coff &= i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1 \\ k\neq l}}^{4} \sum_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} \frac{1}{AC_{lk}} \frac{1}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \times \\
&\quad \Big(1 - \delta_{lk}(AC_{lk})\Big)\Big(1 - \delta_{lk}(B_{mlk})\Big)\, |1 - \beta_{mlk}\varphi_{mlk}| \\
posTerm &= \int_0^{\infty} dz \Big\{ ... \Big\} \\
negTerm &= \int_{-\infty}^{0} dz \Big\{ ... \Big\}
\end{aligned}
$$

With the help of (47), one obtain

$$
\begin{aligned}
posTerm = \int_0^{\infty} dz G(z) \Big\{ \; & Oplus_{nmlk} - fg\; log\Big(\frac{1-\beta\varphi}{\beta} z + \frac{F}{\beta}\Big) \\
& - fg^{-}\; log\Big(\frac{-(1-\beta\varphi)}{\beta} z - \frac{F}{\beta}\Big) \\
& - (fg + f^{-}g) log\Big(\frac{-Pz}{\beta} + \frac{Pz_{1\beta}}{\beta}\Big) \\
& - (fg + f^{-}g) log\Big(z - z_{2\beta}\Big) \\
& + fg\; log\Big(-P\varphi z + P\varphi z_{1\varphi}\Big) + fg\; log\Big(z + z_{2\varphi}\Big) \\
& + fg^{-}\; log\Big(P\varphi z - P\varphi z_{1\varphi}\Big) + fg\; log\Big(z + z_{2\varphi}\Big) \\
& + (f^{-}g - fg^{-})\; log(Pz + Q) \\
& + 2\pi\; i\; f^{-}g\theta[-ImQ]\theta\Big[Im\frac{S(\beta, z)}{Pz + Q}\Big] \\
& + 2\pi\; i\; fg^{-}\theta[ImQ]\theta\Big[Im\frac{-S(\varphi, z)}{Pz + Q}\Big] \qquad \Big\}
\end{aligned}
$$

here

$$
\begin{aligned}
Oplus_{nmlk} &= (fg + f^{-}g) log\Big(\frac{F}{\beta}\Big) \\
&\quad - 2\pi\; i\; (fg + f^{-}g)\theta\Big[Im(\frac{-Pz_{1\beta}}{\beta})\Big]\theta\Big[Im(z_{2\beta})\Big] \\
&\quad + 2\pi\; i\; fg\theta\Big[-Im(P\varphi z_{1\varphi})\Big]\;\theta\Big[Im(z_{2\varphi})\Big] \\
&\quad + 2\pi\; i\; fg^{-}\theta\Big[-Im(P\varphi z_{1\varphi})\Big]\;\theta\Big[-Im(z_{2\varphi})\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
negTerm \;\; = \;\; \int\limits_{-\infty}^{0} dzG(z)\Big\{ \;& Ominus_{nmlk} + (f^- g^- + f^- g)\; log\Big(\frac{1-\beta\varphi}{\beta} + \frac{F}{\beta}\Big) \\
& + f^- g^- \; log\Big(- \frac{-Pz}{\beta} + \frac{Pz_{1\beta}}{\beta}\Big) + f^- g^- \; log\Big(z - z_{2\beta}\Big) \\
& + fg^- \; log\Big(\frac{-Pz}{\beta} - \frac{Pz_{1\beta}}{\beta}\Big) + f^- g^- \; log\Big(z - z_{2\beta}\Big) \\
& -(f^- g^- + f^- g)\; log(-P\varphi z + P\varphi z_{1\varphi}) - (f^- g^- + f^- g)\; log(z - z_{2\varphi}) \\
& +(f^- g - fg^-)\; log(Pz + Q) \\
& +2\pi \; i \; f^- g\theta[-ImQ]\theta\Big[Im\frac{S(\varphi,z)}{Pz+Q}\Big] + 2\pi \; i \; fg^- \theta[ImQ]\theta\Big[Im\frac{-S(\beta,z)}{Pz+Q}\Big] \quad \Big\}
\end{aligned}
$$

here

$$
\begin{aligned}
Ominus_{nmlk} \;\; = \;\; & -f^- g^- \; log\Big(\frac{F}{\beta}\Big) - fg^- \; log\Big(\frac{-F}{\beta}\Big) \\
& +2\pi \; i \; f^- g^- \theta\Big[Im(\frac{-Pz_{1\beta}}{\beta})\Big]\theta\Big[Im(z_{2\beta})\Big] \\
& -2\pi \; i \; fg^- \theta\Big[Im(\frac{-Pz_{1\beta}}{\beta})\Big]\theta\Big[-Im(z_{2\beta})\Big] \\
& -2\pi \; i \; (f^- g^- + f^- g\theta\Big[-Im(P\varphi z_{1\varphi})\Big] \; \theta\Big[Im(z_{2\varphi})\Big]
\end{aligned}
$$

Because

$$
\theta\Big[Im\frac{S(\sigma,z)}{Pz+Q}\Big] = \theta\Big[A_0 z^2 + B_0 z + C_0\Big]
$$

independent to σ then one obtain

SUMMARIZE:

$$\frac{D_0}{i\pi^2} = Coff * (posTerm + negTerm + extraTerm)$$

here

$$\begin{aligned}
Coff &= \sum_{k=1}^{4} \sum_{\substack{l=1 \\ k\neq l}}^{4} \sum_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} \frac{1}{AC_{lk}} \frac{1}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \times \\
&\quad \Big(1 - \delta_{lk}(AC_{lk})\Big)\Big(1 - \delta_{lk}(B_{mlk})\Big) \left|1 - \beta_{mlk}\varphi_{mlk}\right| \\
posTerm &= \int_0^{\infty} dzG(z)\Big\{ \; Oplus_{nmlk} - fg\; log\Big(\frac{1-\beta\varphi}{\beta}z + \frac{F}{\beta}\Big) \\
&\quad -fg^- \; log\Big(\frac{-(1-\beta\varphi)}{\beta}z - \frac{F}{\beta}\Big) \\
&\quad -(fg + f^-g) log\Big(\frac{-Pz}{\beta} + \frac{Pz_{1\beta}}{\beta}\Big) \\
&\quad -(fg + f^-g) log\Big(z - z_{2\beta}\Big) \\
&\quad +fg\; log\Big(- P\varphi z + P\varphi z_{1\varphi}\Big) + fg\; log\Big(z + z_{2\varphi}\Big) \\
&\quad +fg^- \; log\Big(P\varphi z - P\varphi z_{1\varphi}\Big) + fg\; log\Big(z + z_{2\varphi}\Big) \\
&\quad +(f^-g - fg^-)\; log(Pz + Q) \quad \Big\} \\
negTerm &= \int_{-\infty}^{0} dzG(z)\Big\{ \; Ominus_{nmlk} + (f^-g^- + f^-g)\; log\Big(\frac{1-\beta\varphi}{\beta} + \frac{F}{\beta}\Big) \\
&\quad +f^-g^- \; log\Big(- \frac{-Pz}{\beta} + \frac{Pz_{1\beta}}{\beta}\Big) + f^-g^- \; log\Big(z - z_{2\beta}\Big) \\
&\quad +fg^- \; log\Big(\frac{-Pz}{\beta} - \frac{Pz_{1\beta}}{\beta}\Big) + f^-g^- \; log\Big(z - z_{2\beta}\Big) \\
&\quad -(f^-g^- + f^-g)\; log(-P\varphi z + P\varphi z_{1\varphi}) - (f^-g^- + f^-g)\; log(z - z_{2\varphi}) \\
&\quad +(f^-g - fg^-)\; log(Pz + Q) \quad \Big\}
\end{aligned} \tag{48}$$

$$
\begin{aligned}
extraTerm &= 2\pi\, if^{-}g\,\theta[-ImQ]\int_{-\infty}^{\infty} dz\,\theta\Big[A_0z^2+B_0z+C_0\Big]G(z)\\
&\quad +2\pi\, ifg^{-}\,\theta[ImQ]\int_{-\infty}^{\infty} dz\,\theta\Big[-A_0z^2-B_0z-C_0\Big]G(z)\\
A_0 &= Im(PE)\\
B_0 &= Im\Big(E-Pm_k^2+i\rho P\Big)\\
C_0 &= Im\Big((-m_k^2+i\rho)Q^*\Big)
\end{aligned}
$$

7.1 Rfunction

Rfunction is a name of integral

$$\int_0^{\infty}\frac{1}{(z+x)(z+y)}dz=\frac{log(x)-log(y)}{x-y} \tag{49}$$

7.2 ThetaG function

ThetaG function is a name of integral

$$\int_{-\infty}^{\infty} dz\,\theta\Big[A_0z^2+B_0z+C_0\Big]\frac{1}{(z+x)(z+y)}=ThetaG(A_0,B_0,C_0,x,y) \tag{50}$$

8 LogAG function

LogAG function is a name of integral

$$\int_0^{\infty} dz\frac{log(az+b)}{(z+x)(z+y)}=LogAG(a,b,x,y) \tag{51}$$

here a in Real, b in complex.

With the help of these function, we present D_0 to form

$$\frac{D_0}{i\pi^2}=Coff*(posTerm+negTerm+extraTerm)$$

with

$$
\begin{aligned}
posTerm &= Oplus_{nmlk} * Rfunction(-T_1, -T_2) - fg\ LogAG\Big(\frac{1-\beta\varphi}{\beta}, \frac{F}{\beta}, -T_1, -T_2\Big) \\
&\quad -fg^-\ LogAG\Big(-\frac{1-\beta\varphi}{\beta}, -\frac{F}{\beta}, -T_1, -T_2\Big) - (fg + f^-g)\ LogAG\Big(\frac{-P}{\beta}, \frac{Pz_{1\beta}}{\beta}, -T_1, -T_2\Big) \\
&\quad -(fg + f^-g)\ log\Big(1, -z_{2\beta}, -T_1, -T_2\Big) + fg\ LogAG(-P\varphi, P\varphi z_{1\varphi}, -T_1, -T_2\Big) \\
&\quad +fg\ LogAG(1, -z_{2\varphi}, -T_1, -T_2\Big) + fg^-\ LogAG(P\varphi, -P\varphi z_{1\varphi}, -T_1, -T_2\Big) \\
&\quad +fg^-\ LogAG(1, -z_{2\varphi}, -T_1, -T_2\Big) + (f^-g - fg^-)\ LogAG(P, Q, -T_1, -T_2)
\end{aligned}
$$

and

$$
\begin{aligned}
negTerm &= Ominus_{nmlk} * Rfunction(T_1, T_2) + (f^-g^- + f^-g)LogAG\Big(-\frac{1-\beta\varphi}{\beta}, \frac{F}{\beta}, T_1, T_2\Big) \\
&\quad +f^-g^-LogAG\Big(\frac{P}{\beta}, \frac{Pz_{1\beta}}{\beta}, T_1, T_2\Big) + f^-g^-LogAG\Big(-1, -z_{2\beta}, T_1, T_2\Big) \\
&\quad +fg^-LogAG\Big(\frac{-P}{\beta}, \frac{-Pz_{1\beta}}{\beta}, T_1, T_2\Big) + fg^-LogAG\Big(-1, -z_{2\beta}, T_1, T_2\Big) \\
&\quad -(f^-g^- + f^-g)\ LogAG\Big(P\varphi, P\varphi z_{1\varphi}, T_1, T_2\Big) - (f^-g^- + f^-g)\ LogAG\Big(-1, -z_{2\varphi}, T_1, T_2\Big) \\
&\quad +(f^-g - fg^-)\ LogAG(-P, Q, T_1, T_2)
\end{aligned}
$$

$$
\begin{aligned}
extraTerm &= 2\pi\ if^-g\ \theta[-ImQ]ThetaG(A_0, B_0, C_0, -T_1, -T_2) \\
&\quad +2\pi\ ifg^-\ \theta[ImQ]ThetaG(-A_0, -B_0, -C_0, -T_1, -T_2)
\end{aligned} \tag{52}
$$

9 APPENDIX A-LogAG(a,b,x,y) functionVersion 1

9.1 The LogACG function

$$
LogACG(a, b, x, y) = \int_0^\infty \ln(az + b)(z + x)^{-1}(z + y)^{-1} dz \tag{53}
$$

with $t = \frac{b}{a}$ and $a > 0$.

$$
A = \sqrt{(Ret)^2 + (Imt)^2} + \sqrt{(Rex)^2 + (Imx)^2} + \sqrt{(Rey)^2 + (Imy)^2} \tag{54}
$$

and

$$
x_0 = \frac{x}{A}; \qquad y_0 = \frac{y}{A}; \qquad z_0 = \frac{t}{A} \tag{55}
$$

so one obtain

$$
\begin{aligned}
LogACG(a,b,x,y) = \frac{\ln(x_0)-\ln(y_0)}{A(x_0-y_0)} \ln(a*A) \\
-\frac{1}{A(x_0-y_0)}\Big\{ -\frac{1}{2}(\ln x_0)^2 + \frac{1}{2}(\ln y_0)^2 + Li_2(1-\frac{z_0}{y_0}) - Li_2(1-\frac{z_0}{x_0}) \\
+\ln(y_0)\Big[\eta(z_0-y_0,\frac{1}{1-y_0}) - \eta(z_0-y_0,\frac{1}{-y_0})\Big] \\
-\ln(x_0)\Big[\eta(z_0-x_0,\frac{1}{1-x_0}) - \eta(z_0-x_0,\frac{1}{-x_0})\Big] \\
+\ln\Big(1-\frac{z_0}{y_0}\Big)\eta(z_0,\frac{1}{y_0}) - \ln\Big(1-\frac{z_0}{x_0}\Big)\eta(z_0,\frac{1}{x_0}) \quad \Big\}
\end{aligned} \tag{56}
$$

9.2 The LogARG function

$$
LogARG(a,b,x,y) = \int_0^\infty \ln(az+b)(z+x)^{-1}(z+y)^{-1}dz \tag{57}
$$

with $a < 0$. Return to

$$
GiNaCLogARG = \ln(b)\frac{\ln(x)-\ln(y)}{x-y} + GiNaCLogACG(a/b, 1.0, x, y). \tag{58}
$$

9.3 The LogAG function

$LogAG(a,b,x,y)$ is defind as

- If $a > 0$.

$$
LogAG(a,b,x,y) = LogACG(a,b,x,y) \tag{59}
$$

- If $a < 0$

$$
LogAG(a,b,x,y) = LogARG(a,b,x,y) \tag{60}
$$

10 APPENDIX B-ThetaG(a,b,c,x,y) function

ThetaG(a,b,c,x,y) is a name of integral

$$
ThetaG(a,b,c,x,y) = \int_{-\infty}^{\infty} \Theta[az^2+bz+c](z+x)^{-1}(z+y)^{-1}dz \tag{61}
$$

we have

$$
\begin{aligned}
\Delta &= b^2 - 4*ac \\
z^0_{1,2} &= \frac{-b\pm\sqrt{\Delta}}{2a}
\end{aligned} \tag{62}
$$

1. If $a = b = 0$ and $c >= 0$

$$\Longrightarrow Rfunction(-x, -y) + Rfunction(x, y) \tag{63}$$

2. If $a = b = 0$ and $c < 0$

$$\Longrightarrow 0 \tag{64}$$

3. $a = 0$ and $b > 0$

$$\Longrightarrow Rfunction(-\frac{b}{c} + x, -\frac{b}{c} + y) \tag{65}$$

4. $a = 0$ and $b < 0$

$$\Longrightarrow Rfunction(\frac{b}{c} - x, \frac{b}{c} - y) \tag{66}$$

5. $a > 0$ and $\Delta <= 0$

$$\Longrightarrow Rfunction(-x, -y) + Rfunction(x, y) \tag{67}$$

6. $a > 0$ and $\Delta > 0$

$$\Longrightarrow Rfunction(-z_2^0 - x, -z_2^0 - y) + Rfunction(z_1^0 + x, z_2^0 + y) \tag{68}$$

7. $a < 0$ and $\Delta <= 0$

$$\Longrightarrow 0 \tag{69}$$

8. $a < 0$ and $\Delta > 0$

$$\Longrightarrow Rfunction(z_2^0 + x, z_2^0 + y) + Rfunction(z_1^0 + x, z_2^0 + y) \tag{70}$$

11 Plan to test D_0

1. Check formular decompose log function 47.
2. Check formular 41 and 48.
3. Check formular 52.
4. LogAG function, ThetaG function.