EVALUATING ONE LOOP N POINT INTEGRALS WITH R FUNCTION

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February 23, 2009

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Chapter 1

TEST ANALYTIC CALCULATION

The external momenta have form

$$\begin{aligned} q_1^\mu &= (q_{10}, 0, 0, \vec{0}), \\ q_2^\mu &= (q_{20}, q_{21}, 0, \vec{0}), \\ q_3^\mu &= (q_{30}, q_{31}, q_{32}, \vec{0}), \end{aligned}$$

so the number of parallel dimension is $J = 3$. Scalar one loop four point integrals is

$$D_0 = \int\limits_{-\infty}^{\infty} dl_0 \int\limits_{-\infty}^{\infty} dl_1 \int\limits_{-\infty}^{\infty} dl_2 \int\limits_{0}^{\infty} dl_\perp \frac{1}{P_1 P_2 P_3 P_4} \tag{1.1}$$

with

$$\begin{aligned} P_1 &= (l_0 + q_{10})^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\rho, \\ P_2 &= (l_0 + q_{20})^2 - (l_1 + q_{21})^2 - l_2^2 - l_\perp^2 - m_2^2 + i\rho, \\ P_3 &= (l_0 + q_{30})^2 - (l_1 + q_{31})^2 - (l_2 + q_{32})^2 - l_\perp^2 - m_3^2 + i\rho, \\ P_4 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\rho. \end{aligned} \tag{1.2}$$

With general, we have

$$P_k = (l_0 + q_{k0})^2 - (l_1 + q_{k1})^2 - (l_2 + q_{k2})^2 - l_\perp^2 - m_k^2 + i\rho,$$

and

$$\begin{aligned} P_l - P_k &= 2l_0(q_{l0} - q_{k0}) - 2l_1(q_{l1} - q_{k1}) - 2l_2(q_{l2} - q_{k2} + (q_l^2 - q_k^2) - (m_l^2 - m_k^2), \\ &= l_0 a_{lk} + l_1 b_{lk} + l_2 c_{lk} + (q_l^2 - q_k^2) - (m_l^2 - m_k^2), \end{aligned} \tag{1.3}$$

with

$$\begin{aligned} a_{lk} &= 2(q_{l0} - q_{k0}) \\ b_{lk} &= -2(q_{l1} - q_{k1}) \\ c_{lk} &= -2(q_{l2} - q_{k2}) \end{aligned} \tag{1.4}$$

Partitioning the integrand in Eq.(15) into the form

$$\frac{1}{P_1P_2P_3P_4} = \frac{1}{P_1(P_2-P_1)(P_3-P_1)(P_4-P_1)} + \frac{1}{P_2(P_1-P_2)(P_3-P_2)(P_4-P_2)} + \frac{1}{P_3(P_1-P_3)(P_2-P_3)(P_4-P_3)} + \frac{1}{P_4(P_1-P_4)(P_2-P_4)(P_3-P_4)} \tag{1.5}$$

or

$$\frac{1}{P_1P_2P_3P_4} = \sum_{k=1}^{4} \frac{1}{P_k \prod\limits_{l=1,l\neq k}^{4}(P_l-P_k)} \tag{1.6}$$

We rewrite the integral Eq.(16) as follows

$$D_0 = \sum_{k=1}^{4} \int\limits_{-\infty}^{\infty} dl_0 \int\limits_{-\infty}^{\infty} dl_1 \int\limits_{-\infty}^{\infty} dl_2 \int\limits_{0}^{\infty} dl_\perp \frac{1}{P_k \prod\limits_{l=1,l\neq k}^{4}(P_l-P_k)} \tag{1.7}$$

1. The first integral in Eq.1.7

$$D_0^1 = \int\limits_{-\infty}^{\infty} dl_0 \int\limits_{-\infty}^{\infty} dl_1 \int\limits_{-\infty}^{\infty} dl_2 \int\limits_{0}^{\infty} dl_\perp \frac{1}{P_1 \prod\limits_{l=2}^{4}(P_l-P_1)} \tag{1.8}$$

where

$$\begin{aligned} P_1 &= (l_0+q_{10})^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\rho, \\ \prod_{l=2}^{4}(P_l-P_1) &= \prod_{l=2}^{4}\left[a_{l1}l_0 + b_{l1}l_1 + c_{l1}l_2 + (q_l^2-q_1^2) - (m_l^2-m_1^2)\right] \end{aligned}$$

and

$$\begin{aligned} a_{l1} &= 2(q_{l0}-q_{10}) \\ b_{l1} &= -2(q_{l1}-q_{11}) = -2q_{l1} \\ c_{l1} &= -2(q_{l2}-q_{12}) = -2q_{l2} \end{aligned}$$

Shifft $l_0 \longrightarrow l_0 + q_{10}$ in Eq.1.8

- $P_1 \longrightarrow l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\rho,$

- $\prod_{l=2}^{4}(P_l - P_1) \longrightarrow \prod_{l=2}^{4}\left[a_{l1}l_0 + b_{l1}l_1 + c_{l1}l_2 + (q_l - q_1)^2 - (m_l^2 - m_1^2)\right]$

  Infact

$$\begin{aligned}\prod_{l=2}^{4}(P_l - P_1) &\longrightarrow \prod_{l=2}^{4}\left[a_{l1}(l_0 - q_{10}) + b_{l1}l_1 + c_{l1}l_2 + (q_l^2 - q_1^2) - (m_l^2 - m_1^2)\right] \\ &= \prod_{l=2}^{4}\left[a_{l1}l_0 + b_{l1}l_1 + c_{l1}l_2 - a_{l1}q_{10} + (q_l^2 - q_1^2) - (m_l^2 - m_1^2)\right] \\ &= \prod_{l=2}^{4}\left[a_{l1}l_0 + b_{l1}l_1 + c_{l1}l_2 - 2(q_{l0} - q_{10})q_{10} + (q_l^2 - q_1^2) - (m_l^2 - m_1^2)\right] \\ &= \prod_{l=2}^{4}\left[a_{l1}l_0 + b_{l1}l_1 + c_{l1}l_2 - 2q_{l0}q_{10} + q_l^2 + q_1^2 - (m_l^2 - m_1^2)\right] \\ &= \prod_{l=2}^{4}\left[a_{l1}l_0 + b_{l1}l_1 + c_{l1}l_2 + (q_l - q_1)^2 - (m_l^2 - m_1^2)\right]\end{aligned}$$

2. The second integral in Eq.1.7

$$D_0^2 = \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_2 \int_{0}^{\infty} dl_\perp \frac{1}{P_2 \prod_{l=1,l\neq 2}^{4}(P_l - P_2)} \tag{1.9}$$

where

$$\begin{aligned}P_2 &= (l_0 + q_{20})^2 - (l_1 + q_{21})^2 - l_2^2 - l_\perp^2 - m_2^2 + i\rho, \\ \prod_{l=1,\neq 2}^{4}(P_l - P_2) &= \prod_{l=1,l\neq 2}^{4}\left[a_{l2}l_0 + b_{l2}l_1 + c_{l2}l_2 + (q_l^2 - q_2^2) - (m_l^2 - m_2^2)\right]\end{aligned}$$

and

$$\begin{aligned}a_{l2} &= 2(q_{l0} - q_{20}) \\ b_{l2} &= -2(q_{l1} - q_{21}) \\ c_{l2} &= -2(q_{l2} - q_{22}) = -2q_{l2}\end{aligned}$$

Shifft $l_0 \longrightarrow l_0 + q_{20}; l_1 \longrightarrow l_1 + q_{21}$ in Eq.1.9

- $P_2 \longrightarrow l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_2^2 + i\rho,$
- $\prod_{l=1,\neq 2}^{4}(P_l - P_2) \longrightarrow \prod_{l=1,l\neq 2}^{4}\left[a_{l2}l_0 + b_{l2}l_1 + c_{l2}l_2 + (q_l - q_2)^2 - (m_l^2 - m_2^2)\right]$

To prove

$$\begin{aligned}
\star \quad \prod_{l=1,\neq 2}^{4}(P_l - P_2) &\longrightarrow \prod_{l=1,l\neq 2}^{4}\left[a_{l2}(l_0 - q_{20}) + b_{l2}(l_1 - q_{21}) + c_{l2}l_2 + (q_l^2 - q_2^2) - (m_l^2 - m_2^2)\right] \\
= & \prod_{l=1,l\neq 2}^{4}\left[a_{l2}l_0 - a_{l2}q_{20} + b_{l2}l_1 - b_{l2}q_{21} + c_{l2}l_2 + (q_l^2 - q_2^2) - (m_l^2 - m_2^2)\right] \\
= & \prod_{l=1,l\neq 2}^{4}\left[a_{l2}l_0 + b_{l2}l_1 + c_{l2}l_2 - a_{l2}q_{20} - b_{l2}q_{21} + (q_l^2 - q_2^2) - (m_l^2 - m_2^2)\right] \\
= & \prod_{l=1,l\neq 2}^{4}\left[a_{l2}l_0 + b_{l2}l_1 + c_{l2}l_2 - 2(q_{l0} - q_{20})q_{20}) + 2(q_{l1} - q_{21})q_{21} + (q_l^2 - q_2^2) - (m_l^2 - m_2^2)\right] \\
= & \prod_{l=1,l\neq 2}^{4}\left[a_{l2}l_0 + b_{l2}l_1 + c_{l2}l_2 - (m_l^2 - m_2^2) - 2(q_{l0}q_{20} - q_{l1}q_{21}) + 2(q_{20}^2 - q_{21}^2) - q_2^2 + q_l^2]\right] \\
= & \prod_{l=1,l\neq 2}^{4}\left[a_{l2}l_0 + b_{l2}l_1 + c_{l2}l_2 + (q_l - q_2)^2 - (m_l^2 - m_2^2)\right]
\end{aligned}$$

3. The thirth integral in Eq.1.7

$$D_0^3 = \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_2 \int_{0}^{\infty} dl_\perp \frac{1}{P_3 \prod\limits_{l=1,l\neq 3}^{4}(P_l - P_3)} \tag{1.10}$$

where

$$\begin{aligned}
P_3 &= (l_0 + q_{30})^2 - (l_1 + q_{31})^2 - (l_2 + q_{32})^2 - l_\perp^2 - m_3^2 + i\rho, \\
\prod_{l=1,\neq 3}^{4}(P_l - P_3) &= \prod_{l=1,l\neq 3}^{4}\left[a_{l3}l_0 + b_{l3}l_1 + c_{l3}l_2 + (q_l^2 - q_3^2) - (m_l^2 - m_3^2)\right]
\end{aligned}$$

and

$$\begin{aligned}
a_{l3} &= 2(q_{l0} - q_{30}) \\
b_{l3} &= -2(q_{l1} - q_{31}) \\
c_{l3} &= -2(q_{l2} - q_{32})
\end{aligned}$$

Shifft $l_0 \longrightarrow l_0 + q_{30}, l_1 \longrightarrow l_1 + q_{31}, l_2 \longrightarrow l_2 + q_{32}$

- $P_3 \longrightarrow l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_3^2 + i\rho,$

- $\prod_{l=1,\neq 3}^{4}(P_l - P_3) \mapsto \prod_{l=1,l\neq 3}^{4}[a_{l3}l_0 + b_{l3}l_1 + c_{l3}l_2 + (q_l - q_3)^2 - (m_l^2 - m_3^2)]$

To prove

$$
\begin{aligned}
\mapsto \quad & \prod_{l=1,l\neq 3}^{4}\left[a_{l3}(l_0 - q_{30}) + b_{l3}(l_1 - q_{31}) + c_{l3}(l_2 - q_{32}) + (q_l^2 - q_3^2) - (m_l^2 - m_3^2)\right] \\
= \quad & \prod_{l=1,l\neq 3}^{4}\left[a_{l3}l_0 - a_{l3}q_{30} + b_{l3}l_1 - b_{l3}q_{31} + c_{l3}l_2 - c_{l3}q_{32} + (q_l^2 - q_3^2) - (m_l^2 - m_3^2)\right] \\
= \quad & \prod_{l=1,l\neq 3}^{4}[a_{l3}l_0 + b_{l3}l_1 + c_{l3}l_2 + -(m_l^2 - m_3^2) - \\
& 2(q_{l0} - q_{30})q_{30} + 2(q_{l1} - q_{31})q_{31} + 2(q_{l2} - q_{32})q_{32} + q_l^2 - q_3^2] \\
= \quad & \prod_{l=1,l\neq 3}^{4}[a_{l3}l_0 + b_{l3}l_1 + c_{l3}l_2 + -(m_l^2 - m_3^2) - \\
& 2(q_{l0}q_{30} - q_{l1}q_{31} - q_{l2}q_{32}) + 2(q_{30}^2 - q_{31}^2 - q_{32}^2) - q_3^2 - q_l^2] \\
= \quad & \prod_{l=1,l\neq 3}^{4}\left[a_{l3}l_0 + b_{l3}l_1 + c_{l3}l_2 + (q_l - q_3)^2 - (m_l^2 - m_3^2)\right] \qquad (1.11)
\end{aligned}
$$

4. The four integral in Eq.1.7

$$
D_0^4 = \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_2 \int_{0}^{\infty} dl_\perp \frac{1}{P_4 \prod_{l=1,l\neq 4}^{4}(P_l - P_4)} \qquad (1.12)
$$

where

$$
\begin{aligned}
P_4 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\rho, \\
\prod_{l=1,l\neq 4}^{4}(P_l - P_4) &= \prod_{l=1,l\neq 4}^{4}\left[a_{l4}l_0 + b_{l4}l_1 + c_{l4}l_2 + q_l^2 - (m_l^2 - m_2^2)\right]
\end{aligned}
$$

Summarize:

$$
D_0 = \sum_{k=1}^{4} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_2 \int_{0}^{\infty} dl_\perp \frac{1}{[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_k^2 + i\rho] \prod_{l=1,l\neq k}^{4}[a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + d_{lk}]} \qquad (1.13)
$$

with $d_{lk} = (q_l - q_k)^2 - (m_l^2 - m_k^2)$

1.1 The x integration

To do The l_0- integrattion, we make further shift

$$\begin{aligned} l_0 &= x+z, \\ l_2 &= x, \\ l_1 &= y, \\ l_\perp &= t. \end{aligned}$$

then the Jacobian of this transformation is 1 and

$$\begin{aligned} D_0 &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dx\; dy\; dz \int_{0}^{\infty} \\ &\quad \frac{1}{[2xz+z^2-y^2-t^2-m_k^2+i\rho]\prod_{l=1;l\neq k}^{4}[a_{lk}z+b_{lk}y+(a_{lk}+c_{lk})x+d_{lk}]} \end{aligned}$$

Now we caculate the x-integration first. This can be done with help of the residue theorem.

1. The case of $AC_{lk}=a_{lk}+c_{lk}=0$

$$\begin{aligned} D_0 &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dy\; dz \int_{0}^{\infty}\int_{-\infty}^{\infty} dx \\ &\quad \frac{1}{[2xz+z^2-y^2-t^2-m_k^2+i\rho]\prod_{l=1;l\neq k}^{4}[a_{lk}z+b_{lk}y+(a_{lk}+d_{lk}]} \end{aligned}$$

$\Longrightarrow D_0=\infty$

2. The case of $AC_{lk}=a_{lk}+c_{lk}\neq 0$
The singular points of the x-integrand in Eq are

$$\begin{aligned} x_0 &= \frac{y^2+t^2+m_k^2-i\rho-z^2}{2z} \\ x_l &= -\left[\frac{a_{lk}}{AC_{lk}}z+\frac{b_{lk}}{AC_{lk}}y+\frac{d_{lk}}{AC_{lk}}\right] \end{aligned}$$

Note:$\Im(x_0)=\frac{-\Gamma_k-\rho}{2z}$ and $\Im(x_l)=\frac{-\Delta_{lk}}{AC_{lk}}$. The pole x_0 located in the first or second quarter if $z<0$ and vise versa. So we will integrate over x by closing the contour over upper complex x-plane for $z > 0$ and in the lower complex x-plane for $z < 0$. So we have to split the z-integral

$$D_0 = D_0^+ + D_0^- \tag{1.14}$$

with

$$D_0^+ = 2\sum_{k=1}^{4}\int_0^{\infty} dz \int_{-\infty}^{\infty} dy \int_0^{\infty} dt \int_{-\infty}^{\infty} dx \; F(x,y,z,t,m_k,q_k)$$
$$D_0^- = 2\sum_{k=1}^{4}\int_{-\infty}^{0} dz \int_{-\infty}^{\infty} dy \int_0^{\infty} dt \int_{-\infty}^{\infty} dx \; F(x,y,z,t,m_k,q_k)$$

(a) Caculating the D_0^+
Only pole x_l which have $\Im(x_l) \geq 0$ will contribute to the integral D_0^+. The residuce of $F(x,y,z,t,m_k,q_k)$ at singular point x_l is

$$Res\,[x_l, F(x,...)] = \frac{1}{(2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\rho)} \frac{1}{\prod\limits_{m=1;m\neq l;l\neq k} (a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk})}$$

Following the residuce theorem, D_0^+ have form

$$D_0^+ = 2i\pi \sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4} \int_0^{\infty} dz \int_{-\infty}^{\infty} dy \int_0^{\infty} dt \; \frac{1}{2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\rho} \frac{f_{lk}}{\prod\limits_{m=1;m\neq l;l\neq k} (a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk})}$$

with

$$f_{lk} = \begin{cases} 0, & \text{if } \Im(-\frac{d_{lk}}{AC_{lk}}) < 0; \\ 1, & \text{if } \Im(-\frac{d_{lk}}{AC_{lk}}) = 0; \\ 2, & \text{if } \Im(-\frac{d_{lk}}{AC_{lk}}) > 0. \end{cases}$$

Note: if x_l stay on the real axes[1] then $f_{lk} = 1$. because we have half of contour over these poles.

[1] to evaluate integral whose integrands have poles on the real axis, we use the contour which advoids these singularities by flolowing small semicircles with centers at the singular points

(b) Caculating the D_0^-:
Only pole x_l which have $\Im(x_l) \leq 0$ will contribute to the integral D_0^- Following the residuce theorem, D_0^+ have form

$$D_0^- = -2i\pi \sum_{k=1}^{4} \sum_{l=1;l\neq k}^{4} \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dt \frac{1}{\left[2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\rho\right]} \frac{f_{lk}^-}{\prod\limits_{m=1;m\neq l;l\neq k} \left[a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk}\right]}$$

with

$$f_{lk}^- = \begin{cases} 0, & \text{if } \Im(-\frac{d_{lk}}{AC_{lk}}) > 0; \\ 1, & \text{if } \Im(-\frac{d_{lk}}{AC_{lk}}) = 0; \\ 2, & \text{if } \Im(-\frac{d_{lk}}{AC_{lk}}) < 0. \end{cases} \tag{1.15}$$

Note: The sign "-" in front of D_0^- appear to explain for taking the contour oppsite direct

SUMMARIZE:

$$D_0^+ = 2i\pi \sum_{k=1}^{4} \sum_{l=1;l\neq k}^{4} \int_{0}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dt \{ \frac{1}{2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\rho} \frac{\left[1 - \delta(AC_{lk})\right] f_{lk}}{\prod\limits_{m=1;m\neq l;l\neq k} (a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk})} \}$$

and

$$D_0^- = -2i\pi \sum_{k=1}^{4} \sum_{l=1;l\neq k}^{4} \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dt \{ \frac{1}{2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\rho} \frac{\left[1 - \delta(AC_{lk})\right] f_{lk}^-}{\prod\limits_{m=1;m\neq l;l\neq k} (a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk})} \}$$

with

$$x_l = -\left[\frac{a_{lk}}{AC_{lk}}z + \frac{b_{lk}}{AC_{lk}}y + \frac{d_{lk}}{AC_{lk}}\right]$$

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we have

$$\begin{aligned}
& 2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\rho \\
= & -2z\left[\frac{a_{lk}}{AC_{lk}}z + \frac{b_{lk}}{AC_{lk}}y + \frac{d_{lk}}{AC_{lk}}\right] + z^2 - y^2 - t^2 - m_k^2 + i\rho \\
= & (1 - \frac{2a_{lk}}{AC_{lk}})z^2 - 2yz\frac{b_{lk}}{AC_{lk}} - 2z\frac{d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\rho
\end{aligned}$$

$$\begin{aligned}
& a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk} \\
= & a_{mk}z + b_{mk}y - AC_{mk}\left[\frac{a_{lk}}{AC_{lk}}z + \frac{b_{lk}}{AC_{lk}}y + \frac{d_{lk}}{AC_{lk}}\right] + d_{mk} \\
= & a_{mk}z + b_{mk}y + d_{mk} - z\frac{AC_{mk}}{AC_{lk}}a_{lk} - y\frac{AC_{mk}}{AC_{lk}}b_{lk} - \frac{AC_{mk}}{AC_{lk}}d_{lk} \\
= & \left[a_{mk} - \frac{AC_{mk}}{AC_{lk}}a_{lk}\right]z + \left[b_{mk} - \frac{AC_{mk}}{AC_{lk}}b_{lk}\right]y + d_{mk} - \frac{AC_{mk}}{AC_{lk}}d_{lk} \\
= & A_{mlk}z + B_{mlk}y + C_{mlk} \qquad (1.16)
\end{aligned}$$

Finally, we recollect

$$\begin{aligned}
D_0^+ = & 2i\pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_0^\infty dz\int_{-\infty}^{\infty}dy\int_0^\infty dt\frac{1}{\left[(1-\frac{2a_{lk}}{AC_{lk}})z^2 - 2yz\frac{b_{lk}}{AC_{lk}} - 2z\frac{d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\rho\right]} \\
& \frac{\left[1-\delta(AC_{lk})\right]f_{lk}}{\prod_{m=1;m\neq l;l\neq k}\left[A_{mlk}z + B_{mlk}y + C_{mlk}\right]}
\end{aligned}$$

and

$$\begin{aligned}
D_0^- = & -2i\pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_{-\infty}^0 dz\int_{-\infty}^{\infty}dy\int_0^\infty dt\frac{1}{\left[(1-\frac{2a_{lk}}{AC_{lk}})z^2 - 2yz\frac{b_{lk}}{AC_{lk}} - 2z\frac{d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\rho\right]} \\
& \frac{\left[1-\delta(AC_{lk})\right]f_{lk}}{\prod_{m=1;m\neq l;l\neq k}\left[A_{mlk}z + B_{mlk}y + C_{mlk}\right]}
\end{aligned}$$

with

$$\begin{aligned}
A_{mlk} &= a_{mk} - \frac{AC_{mk}}{AC_{lk}}a_{lk}, \\
B_{mlk} &= b_{mk} - \frac{AC_{mk}}{AC_{lk}}b_{lk}, \\
C_{mlk} &= d_{mk} - \frac{AC_{mk}}{AC_{lk}}d_{lk}.
\end{aligned}$$

1.2 The y integration

We first linearize by shift $t \longrightarrow t+y$. To do that t^2 and y^2 must have oppsite signs. That can be obtained by a complex rotation.

1.2.1 The t rotation

The integration wrt. t has 2 poles locate in the first or the third quarter or the second and the fourth ones of the complex plane, depend on the sign of $\Im[2zx_l - m_k^2 + i\rho]$

$$t = \pm\sqrt{z^2 + 2zx_l - y^2 - m_k^2 + i\rho} \tag{1.17}$$

the sign of imaginary part of depends on the sign of $(2z\Im[-\frac{-d_{lk}}{AC_{lk}}] + \Gamma_k + \rho)$. The locations of the t-poles for D^+ and D^- are specified in the Table4.3 and Table1.2

$\Im[-\frac{d_{lk}}{AC_{lk}}] \geq 0$	$z \geq 0$	$1^{st}, 3^{th}$ quarter
$\Im[-\frac{d_{lk}}{AC_{lk}}] < 0$	$0 \leq z \leq -\frac{\Gamma_k+\rho}{2\Im[\frac{d_{lk}}{AC_{lk}}]}$	$1^{st}, 3^{th}$ quarter
$\Im[-\frac{d_{lk}}{AC_{lk}}] < 0$	$0 \leq -\frac{\Gamma_k+\rho}{2\Im[\frac{d_{lk}}{AC_{lk}}]} < z$	$2^{st}, 4^{th}$ quarter

Table 1.1: t-poles location of D_0^+

$\Im[-\frac{d_{lk}}{AC_{lk}}] \leq 0$	$z \leq 0$	$1^{st}, 3^{th}$ quarter
$\Im[-\frac{d_{lk}}{AC_{lk}}] > 0$	$-\frac{\Gamma_k+\rho}{2\Im[\frac{d_{lk}}{AC_{lk}}]} \leq z \leq 0$	$1^{st}, 3^{th}$ quarter
$\Im[-\frac{d_{lk}}{AC_{lk}}] > 0$	$z \leq -\frac{\Gamma_k+\rho}{2\Im[\frac{d_{lk}}{AC_{lk}}]} < 0$	$2^{st}, 4^{th}$ quarter

Table 1.2: t-poles location of D_0^-

We now close the contour on the fourth quarter of the t-complex plane to obtain

$$\int_0^\infty dt = \frac{1}{2}\int_{-\infty}^{\infty} dt = -\int_{-i\infty}^{0} dt = -\frac{1}{2}\int_{-i\infty}^{i\infty} dt \tag{1.18}$$

After Wick rotation, to obtains

$$D_0^+ = \pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_0^{\infty}dz\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dt\{\frac{1}{[(1-\frac{2a_{lk}}{AC_{lk}})z^2-2yz\frac{b_{lk}}{AC_{lk}}-2z\frac{d_{lk}}{AC_{lk}}-y^2+t^2-m_k^2+i\rho]}$$
$$\frac{[1-\delta(AC_{lk})]\,f_{lk}}{\prod\limits_{m=1;m\neq l;l\neq k}(A_{mlk}z+B_{mlk}y+C_{mlk})}\} \quad (1.19)$$

with $\Im[-\frac{d_{lk}}{AC_{lk}}]\geq 0$ and

$$D_0^- = -\pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_{-\infty}^{0}dz\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dt\{\frac{1}{[(1-\frac{2a_{lk}}{AC_{lk}})z^2-2yz\frac{b_{lk}}{AC_{lk}}-2z\frac{d_{lk}}{AC_{lk}}-y^2+t^2-m_k^2+i\rho]}$$
$$\frac{[1-\delta(AC_{lk})]\,f_{lk}^-}{\prod\limits_{m=1;m\neq l;l\neq k}(A_{mlk}z+B_{mlk}y+C_{mlk})}\} \quad (1.20)$$

with $\Im[-\frac{d_{lk}}{AC_{lk}}]\leq 0$

1.2.2 Evaluate to the y integrate

To do caculation of the integrals in Eq1.19 and Eq1.20 we make further shift $t\longrightarrow t+y$. The Jacobian is 1. We have

$$D_0^+ = \pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_0^{\infty}dz\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dt\frac{[1-\delta(AC_{lk})]\,f_{lk}}{\prod\limits_{m=1;m\neq l;l\neq k}(A_{mlk}z+B_{mlk}y+C_{mlk})}$$
$$\frac{1}{[(1-\frac{2a_{lk}}{AC_{lk}})z^2+2y\left(t-z\frac{b_{lk}}{AC_{lk}}\right)-2z\frac{d_{lk}}{AC_{lk}}-y^2+t^2-m_k^2+i\rho]}$$

and

$$D_0^- = -\pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_{-\infty}^{0}dz\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dt\frac{[1-\delta(AC_{lk})]\,f_{lk}^-}{\prod\limits_{m=1;m\neq l;l\neq k}(A_{mlk}z+B_{mlk}y+C_{mlk})}$$
$$\frac{1}{[(1-\frac{2a_{lk}}{AC_{lk}})z^2-2yz\frac{b_{lk}}{AC_{lk}}-2z\frac{d_{lk}}{AC_{lk}}-y^2+t^2-m_k^2+i\rho]}$$

The singular points of the integrand D_0^+ and D_0^- are

$$\begin{aligned} y_{mlk} &= -\left[\frac{A_{mlk}z + C_{mlk}}{B_{mlk}}\right] \\ y_0 &= \frac{-\left(1-\frac{2a_{lk}}{AC_{lk}}\right)z^2 - t^2 + 2z\frac{d_{lk}}{AC_{lk}} + m_k^2 - i\rho}{2\left(t - z\frac{b_{lk}}{AC_{lk}}\right)} \end{aligned}$$

Note: $\Im(y_0) = \frac{-2z\Im(\frac{-d_{lk}}{AC_{lk}})-\Gamma_k-\rho}{2\left(t-z\frac{b_{lk}}{AC_{lk}}\right)}$**.**

Because z **and** $\Im(\frac{-d_{lk}}{AC_{lk}})$ **have the same sign,**$-2z\Im(\frac{-d_{lk}}{AC_{lk}}) - \Gamma_k - \rho < 0$

- **with** D_0^+ **(**$z > 0$**): the pole** y_0 **locate in** 1^{th} **or** 2^{th} **quarter of the complex plane if** $t < \alpha_{lk}z$ **and vise versa. So we take the** y**-integral by closing the contour on the uper plane if** $t > \alpha_{lk}z$ **and on the lower plane if** $t < \alpha_{lk}z$**.**

$$D_0^+ = D_0^{++} + D_0^{+-} \tag{1.21}$$

with

$$D_0^{++} = \pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_0^\infty dz\int_{-\infty}^{\infty} dy\int_{\alpha_{lk}z}^{\infty} dtG(y,z,t,m_k,q_k) \tag{1.22}$$

and

$$D_0^{+-} = \pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_0^\infty dz\int_{-\infty}^{\infty} dy\int_{-\infty}^{\alpha_{lk}z} dtG(y,z,t,m_k,q_k) \tag{1.23}$$

- **with** D_0^- **(**$z < 0$**): the pole** y_0 **locate in** 1^{th} **or** 2^{th} **quarter of the complex plane if** $t < \alpha_{lk}z$ **and vise versa. So we take the** y**-integral by closing the contour on the uper plane if** $t > \alpha_{lk}z$ **and on the lower plane if** $t < \alpha_{lk}z$**.**

$$D_0^+ = D_0^{-+} + D_0^{--} \tag{1.24}$$

with

$$D_0^{-+} = \pi\sum_{k=1}^{4}\sum_{l=1;l\neq k}^{4}\int_0^\infty dz\int_{-\infty}^{\infty} dy\int_{\alpha_{lk}z}^{\infty} dtG^-(y,z,t,m_k,q_k) \tag{1.25}$$

and

$$D_0^{--} = \pi \sum_{k=1}^{4} \sum_{l=1;l\neq k}^{4} \int_0^\infty dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\alpha_{lk} z} dt G^-(y,z,t,m_k,q_k) \tag{1.26}$$

where

$$G^\pm(y,z,t,m_k,q_k) = \frac{f_{lk}^\pm}{\left[............\right]} \tag{1.27}$$

Following the residuce theorem, we have

$$\begin{aligned} D_0^{++} &= i\pi^2 \sum_{k=1}^{4} \int_0^\infty dz \sum_{l=1;l\neq k}^{4} \int_{\alpha lkz}^{\infty} dt\,[1-\delta(AC_{lk})]\,f_{lk} \\ & \sum_{m=1,m\neq k,l}^{4} \frac{[1-\delta(B_{mlk})]\,g_{mlk}}{(A_{nlk}z + B_{nlk}y_{mlk} + C_{nlk})} \\ & \frac{1}{\left[2y_{mlk}\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\rho\right]} \end{aligned} \tag{1.28}$$

and

$$\begin{aligned} D_0^{+-} &= -i\pi^2 \sum_{k=1}^{4} \int_0^\infty dz \sum_{l=1;l\neq k}^{4} \int_{-\infty}^{\alpha lkz} dt\,[1-\delta(AC_{lk})]\,f_{lk} \\ & \sum_{m=1,m\neq k,l}^{4} \frac{[1-\delta(B_{mlk})]\,g_{mlk}^-}{(A_{nlk}z + B_{nlk}y_{mlk} + C_{nlk})} \\ & \frac{1}{\left[2y_{mlk}\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\rho\right]} \end{aligned} \tag{1.29}$$

and

$$\begin{aligned} D_0^{-+} &= -i\pi^2 \sum_{k=1}^{4} \int_{-\infty}^{0} dz \sum_{l=1;l\neq k}^{4} \int_{\alpha lkz}^{\infty} dt\,[1-\delta(AC_{lk})]\,f_{lk}^- \\ & \sum_{m=1,m\neq k,l}^{4} \frac{[1-\delta(B_{mlk})]\,g_{mlk}}{(A_{nlk}z + B_{nlk}y_{mlk} + C_{nlk})} \\ & \frac{1}{\left[2y_{mlk}\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\rho\right]} \end{aligned} \tag{1.30}$$

and

$$
\begin{aligned}
D_0^{--} &= i\pi^2 \sum_{k=1}^{4} \int_{-\infty}^{0} dz \sum_{l=1;l\neq k}^{4} \int_{-\infty}^{\alpha lkz} dt \left[1-\delta(AC_{lk})\right] f_{lk}^{-} \\
& \sum_{m=1,m\neq k,l}^{4} \frac{\left[1-\delta(B_{mlk})\right] g_{mlk}^{-}}{(A_{nlk}z + B_{nlk}y_{mlk} + C_{nlk})} \\
& \frac{1}{\left[2y_{mlk}\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\rho\right]} \qquad (1.31)
\end{aligned}
$$

with

$$
g_{mlk} = \begin{cases} 0, & \text{if } \Im(\frac{-C_{mlk}}{B_{mlk}}) < 0; \\ 1, & \text{if } \Im(\frac{-C_{mlk}}{B_{mlk}}) = 0; \\ 2, & \text{if } \Im(\frac{-C_{mlk}}{B_{mlk}}) > 0. \end{cases}
$$

and

$$
g_{mlk}^{-} = \begin{cases} 0, & \text{if } \Im(\frac{-C_{mlk}}{B_{mlk}}) > 0; \\ 1, & \text{if } \Im(\frac{-C_{mlk}}{B_{mlk}}) = 0; \\ 2, & \text{if } \Im(\frac{-C_{mlk}}{B_{mlk}}) < 0. \end{cases}
$$

---

with $y_{mlk} = -\left[\frac{A_{mlk}z + C_{mlk}}{B_{mlk}}\right]$

$$
\begin{aligned}
& A_{nlk}z + B_{nlk}y_{mlk} + C_{nlk} \\
= \; & A_{nlk}z - B_{nlk}\left[\frac{A_{mlk}z + C_{mlk}}{B_{mlk}}\right] + C_{nlk} \\
= \; & \frac{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}}{B_{mlk}} z - \frac{B_{nlk}C_{mlk} - B_{mlk}C_{nlk}}{B_{mlk}} \\
= \; & \frac{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}}{B_{mlk}} \left[z - \frac{B_{nlk}C_{mlk} - B_{mlk}C_{nlk}}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& 2y_{mlk}\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\rho \\
= \;& -2\left[\frac{A_{mlk}z + C_{mlk}}{B_{mlk}}\right]\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\rho \\
= \;& -2\frac{A_{mlk}}{B_{mlk}}zt + 2\frac{A_{mlk}}{B_{mlk}}\frac{b_{lk}}{AC_{lk}}z^2 - 2\frac{C_{mlk}}{B_{mlk}}t + 2\frac{C_{mlk}}{B_{mlk}}\frac{b_{lk}}{AC_{lk}}z \\
& + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\rho \\
= \;& \left(1 - \frac{2a_{lk}}{AC_{lk}} + 2\frac{A_{mlk}}{B_{lmk}}\frac{b_{lk}}{AC_{lk}}\right)z^2 + \left(2\frac{C_{mlk}}{B_{lmk}}\frac{b_{lk}}{AC_{lk}} - 2\frac{d_{lk}}{AC_{lk}}\right)z \\
& -2\frac{A_{mlk}}{B_{mlk}}zt - 2\frac{C_{mlk}}{B_{mlk}}t + t^2 - m_k^2 + i\rho \\
= \;& D'_{mlk}z^2 + E_{mlk}z - 2\frac{A_{mlk}}{B_{mlk}}zt - 2\frac{C_{mlk}}{B_{mlk}}t + t^2 - m_k^2 + i\rho
\end{aligned} \tag{1.32}
$$

So we can rewrite

$$
\begin{aligned}
D_0^{++} &= +\bigoplus_{nlmk}\int_0^\infty dz \int_{\alpha_{lk}z}^\infty dt\; f_{lk}\; g_{nlk} I'_{nmlk}(z,t) \\
D_0^{+-} &= -\bigoplus_{nlmk}\int_0^\infty dz \int_{-\infty}^{\alpha_{lk}z} dt\; f_{lk}\; g^-_{nlk} I'_{nmlk}(z,t) \\
D_0^{-+} &= -\bigoplus_{nlmk}\int_{-\infty}^0 dz \int_{\alpha_{lk}z}^\infty dt\; f^-_{lk}\; g_{nlk} I'_{nmlk}(z,t) \\
D_0^{--} &= +\bigoplus_{nlmk}\int_{-\infty}^0 dz \int_{-\infty}^{\alpha_{lk}z} dt\; f^-_{lk}\; g^-_{nlk} I'_{nmlk}(z,t)
\end{aligned} \tag{1.33}
$$

with

$$
\begin{aligned}
\bigoplus_{nmlk} &= i\pi^2 \sum_{k=1}^4 \sum_{l=1,l\neq k}^4 \sum_{m=1,m\neq l,k}^4 \frac{B_{mlk}}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \left[1 - \delta_{lk}(AC_{lk})\right]\left[1 - \delta(B_{mlk})\right] \\
I'_{nmlk}(z,t) &= \frac{1}{[z - F_{nmlk}]} \frac{1}{\left[D'_{mlk}z^2 + E_{mlk}z - 2\frac{A_{mlk}}{B_{mlk}}tz - 2\frac{C_{mlk}}{B_{lmk}}t + t^2 - m_k^2 + i\rho\right]}
\end{aligned}
$$

where

$$\begin{aligned} D'_{mlk} &= 1 - \frac{2a_{lk}}{AC_{lk}} + \frac{2b_{lk}A_{mlk}}{AC_{lk}B_{mlk}} \\ E'_{mlk} &= \frac{2b_{lk}C_{mlk}}{AC_{lk}B_{mlk}} - \frac{2d_{lk}}{AC_{lk}} \\ F_{nmlk} &= \frac{C_{nlk}B_{mlk} - B_{nlk}C_{mlk}}{A_{nlk}B_{mlk} - B_{nlk}A_{mlk}} \end{aligned} \tag{1.34}$$

Changing the variable $t' = t - \alpha_{lk}z$, then the Jacobian is 1 and

$$\begin{aligned} \star \quad & D^{'}_{mlk}z^2 + E_{mlk}z - 2\frac{A_{mlk}}{B_{mlk}}zt - 2\frac{C_{mlk}}{B_{mlk}}t + t^2 - m_k^2 + i\rho \\ = \quad & D^{'}_{mlk}z^2 + E_{mlk}z - 2\frac{A_{mlk}}{B_{mlk}}z\left(t + \alpha_{lk}z\right)z - 2\frac{C_{mlk}}{B_{mlk}}\left(t + \alpha_{lk}z\right) + \left(t + \alpha_{lk}z\right)^2 - m_k^2 + i\rho \\ = \quad & \left(D^{'}_{mlk} - 2\frac{A_{mlk}}{B_{mlk}}\alpha_{lk} + \alpha_{lk}^2\right)z^2 + 2\left(\alpha_{lk} - \frac{A_{mlk}}{B_{mlk}}\right)zt + \left(E_{mlk} - 2\frac{C_{mlk}}{B_{mlk}}\alpha_{lk}\right)z \\ & -2\frac{C_{mlk}}{B_{mlk}}t + t^2 - m_k^2 + i\rho \end{aligned} \tag{1.35}$$

After changing the variable, we obtains

$$\begin{aligned} D_0^{++} &= +\bigoplus_{nlmk}\int_0^\infty dz \int_0^\infty dt\ f_{lk}\ g_{nlk}I_{nmlk}(z,t) \\ D_0^{+-} &= -\bigoplus_{nlmk}\int_0^\infty dz \int_{-\infty}^0 dt\ f_{lk}\ g_{nlk}^{-}I_{nmlk}(z,t) \\ D_0^{-+} &= -\bigoplus_{nlmk}\int_{-\infty}^0 dz \int_0^\infty dt\ f_{lk}^{-}\ g_{nlk}I_{nmlk}(z,t) \\ D_0^{--} &= +\bigoplus_{nlmk}\int_{-\infty}^0 dz \int_{-\infty}^0 dt\ f_{lk}^{-}\ g_{nlk}^{-}I_{nmlk}(z,t) \end{aligned} \tag{1.36}$$

Here

$$\begin{aligned}
D_{mlk} &= D'_{mlk} + \alpha_{lk}^2 - 2\frac{A_{mlk}}{B_{mlk}}\alpha_{lk} \\
&= 1 - \frac{2a_{lk}}{AC_{lk}} + \frac{2b_{lk}A_{mlk}}{AC_{lk}B_{mlk}} + \alpha_{lk}^2 - 2\frac{A_{mlk}}{B_{mlk}}\alpha_{lk} \\
&= 1 - \frac{2a_{lk}}{AC_{lk}} + \alpha_{lk}^2 \\
&= -\frac{(q_l - q_k)^2}{AC_{lk}}. \\
I_{nmlk}(z,t) &= \frac{1}{[z - F_{nmlk}]} \frac{1}{\left[D_{mlk}z^2 - 2\frac{d_{lk}}{AC_{lk}}z - 2\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)tz - 2\frac{C_{mlk}}{B_{lmk}}t + t^2 - m_k^2 + i\rho\right]}
\end{aligned}$$

1.3 Linearize t and t-integral

To linearize t we now make a shift

$$\begin{aligned}
z &= z' + \beta_{mlk}t' \\
t &= t' + \varphi_{mlk}z'.
\end{aligned} \tag{1.37}$$

The jacobian of this shift is

$$J = |1 - \beta\varphi|. \tag{1.38}$$

$$
\begin{aligned}
Dz^2 &\rightarrow D(z^2 + 2\beta zt + \beta^2 t^2) \\
-2\frac{d}{AC}z &\rightarrow -2\frac{d}{AC}(z + \beta t) \\
-2zt(\frac{A}{B} - \alpha) &\rightarrow -2(\frac{A}{B} - \alpha)(zt + \varphi z^2 + \beta t^2 + \beta\varphi zt) \\
-2\frac{C}{B}t &\rightarrow -2\frac{C}{B}(t + \varphi z) \\
t^2 &\rightarrow t^2 + 2\varphi zt + \varphi^2 z^2. \\
&= \Big(D - 2\varphi(\frac{A}{B} - \alpha) + \varphi^2\Big)z^2 \\
&\quad +\Big(D\beta^2 - 2\beta(\frac{A}{B} - \alpha) + 1\Big)t^2 \\
&\quad -2\Big(\frac{d}{AC}\beta + \frac{C}{B}\Big)t \\
&\quad -2\Big[(\frac{A}{B} - \alpha)(1 + \beta\varphi) - D\beta - \varphi\Big]zt
\end{aligned}
\tag{1.39}
$$

To remove the quardratic tern of t^2, we choice β as

$$
\beta_{mlk} = \frac{\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big) + \sqrt{\Big(\frac{A_{mlk}}{B_{mlk} - \alpha_{lk}}\Big)^2 - D_{mlk} + i\eta}}{D_{mlk}} < 0 \tag{1.40}
$$

We now obtain

$$
\begin{aligned}
Q_{mlk} &= -2\Big(\frac{C_{mlk}}{B_{mlk}} + \frac{d_{lk}}{AC_{lk}}\beta_{mlk}\Big) \\
P_{mlk} &= -2\Big[\Big(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\Big)(1 + \beta_{mlk}\varphi_{mlk}) - D_{mlk}\beta_{mlk} - \varphi_{mlk}\Big] \\
E_{mlk} &= -2\Big(\frac{d_{lk}}{AC_{lk}} + \frac{C_{mlk}}{B_{mlk}}\varphi_{mlk}\Big) \\
F_{nmlk} &= \frac{C_{nlk}B_{mlk} - C_{mlk}B_{nlk}}{A_{nlk}B_{mlk} - A_{mlk}B_{nlk}}
\end{aligned}
\tag{1.41}
$$

and

$$
I_{mlk} = \frac{1}{\Big[z + \beta_{mlk}t + F_{mlk}\Big]\Big[Q_{mlk}t + P_{mlk}tz + E_{mlk}z - m_k^2 + i\rho\Big]} \tag{1.42}
$$

The quardratic tern of z^2 is vanish by chosing $\varphi_{mlk} = \left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right) + \sqrt{\left(\frac{A_{mlk}}{B_{mlk} - \alpha_{lk}}\right)^2 - D_{mlk} + i\eta} > 0$. " In Npoint.ps" ??????

The region we take D_0 now look as

After performing this shift, one obtain

$$
\begin{aligned}
D_0 \;=\; & i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1\\ k\neq l}}^{4} \sum_{\substack{m=1\\ m\neq l\\ m\neq k}}^{4} \frac{B_{mlk}}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \Big(1 - \delta_{lk}(AC_{lk})\Big)\Big(1 - \delta_{lk}(B_{mlk})\Big) |1 - \beta_{mlk}\varphi_{mlk}| \times \\
& \Bigg[\quad \int_0^\infty dz\; G(z) \Bigg\{ (f_{lk}g_{mlk} + f_{lk}^- g_{mlk}) \ln\left(\frac{F}{\beta}\right) \\
& -f_{lk}g_{mlk} \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) - f_{lk}g_{mlk}^- \ln\left(-\frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& -(f_{lk}g_{mlk} + f_{lk}^- g_{mlk}) \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& +f_{lk}g_{mlk} \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& +f_{lk}g_{mlk}^- \ln\left(-\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \\
& \quad + \int_{-\infty}^0 dz\; G(z) \Bigg\{ -f_{lk}^- g_{mlk}^- \ln\left(\frac{F}{\beta}\right) - f_{lk}g_{mlk}^- \ln\left(-\frac{F}{\beta}\right) \\
& +(f_{lk}^- g_{mlk}^- + f_{lk}^- g_{mlk}) \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& +f_{lk}^- g_{mlk}^- \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& +f_{lk}g_{mlk}^- \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& -(f_{lk}^- g_{mlk}^- + f_{lk}^- g_{mlk}) \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \Bigg\} \qquad \Bigg]
\end{aligned} \tag{1.43}
$$

In the next chapter, we will represent this integral to R function.

Chapter 2

CALCULATION D_0 FOR THE CASE OF REAL MASS

in this chapter, we continue to caculation $z-$ integral.

2.1 The analytic calculation

We apply the formular

$$\begin{aligned}
\ln\Big(\frac{S(\sigma,z)}{Pz+Q}\Big) &= \ln(\sigma z-\sigma z_1)+\ln(z-z_2)-\ln\Big(z+\frac{Q}{P}+\frac{i\rho}{P}\Big)+\eta^+(\sigma,z_1,z_2)\\
\ln\Big(\frac{-S(\sigma,z)}{Pz+Q}\Big) &= \ln(-\sigma z+\sigma z_1)+\ln(z-z_2)-\ln\Big(z+\frac{Q}{P}+\frac{i\rho}{P}\Big)+\eta^-(\sigma,z_1,z_2)
\end{aligned} \tag{2.1}$$

here z_1 and z_2 are solution of equation

$$S(\sigma,z)=0$$

and

$$\begin{aligned}
\eta^+(\sigma,z_1,z_2) &= 2\pi i\Big(\theta[Im(\sigma z_1)]\theta[Im(z_2)]\theta[P]-\theta[-Im(\sigma z_1)]\theta[-Im(z_2)]\theta[-P]\Big)\\
\eta^-(\sigma,z_1,z_2) &= 2\pi i\Big(\theta[-Im(\sigma z_1)]\theta[Im(z_2)]\theta[-P]-\theta[Im(\sigma z_1)]\theta[-Im(z_2)]\theta[P]\Big)
\end{aligned}$$

Using the formular (2.1), We recolect D_0 to form

$$
\begin{aligned}
D_0 \;=\; & \bigoplus_{nmlk}\int_0^\infty \Big\{\Omega^+_{nmlk} - fg\ln\Big(\frac{1-\beta\varphi}{\beta}z + \frac{F+I\beta\rho}{\beta}\Big) \\
& -fg^-\ln\Big(-\frac{1-\beta\varphi}{\beta}z - \frac{F+I\beta\rho}{\beta}\Big) - (fg+f^-g)\ln\Big(-\frac{z}{\beta}+\frac{z_{1\beta}}{\beta}\Big) \\
& -(fg+f^-g)\ln\Big(z - z_{2\beta}\Big) + fg\ln(-\varphi z + \varphi z_{1\varphi}) + fg\ln(z-\varphi z_{2\varphi}) \\
& +fg^-\ln(\varphi z - \varphi z_{1\varphi}) + fg^-\ln(z - z_{2\varphi}) + f^-g\ln\Big(z+\frac{Q}{P}+\frac{i\rho}{P}\Big) \\
& -fg^-\ln\Big(z+\frac{Q}{P}-\frac{i\rho}{P}\Big)\Big\}(z-T_1)^{-1}(z-T_2)^{-1} \\
+\; & \bigoplus_{nmlk}\int_0^\infty \Big\{\Omega^-_{nmlk} + (f^-g^- + f^-g)\ln\Big(\frac{1-\beta\varphi}{\beta}z + \frac{F+I\beta\rho}{\beta}\Big) \\
& +f^-g^-\ln\Big(-\frac{z}{\beta}+\frac{z_{3\beta}}{\beta}\Big) + f^-g^-\ln\Big(z-z_{4\beta}\Big) \\
& +fg^-\ln\Big(\frac{z}{\beta}-\frac{z_{3\beta}}{\beta}\Big) + fg^-\ln\Big(z-z_{4\beta}\Big) \\
& -(f^-g^- + f^-g)\ln(-\varphi z + \varphi z_{3\varphi}) - (f^-g^- + f^-g)\ln(z - z_{4\varphi}) \\
& -f^-g\ln\Big(-z+\frac{Q}{P}+\frac{i\rho}{P}\Big) - fg^-\ln\Big(-z+\frac{Q}{P}-\frac{i\rho}{P}\Big)\Big\}(z-T_3)^{-1}(z-T_4)^{-1}
\end{aligned}
\tag{2.2}
$$

2.2 TO BUILD CODE ONELOOP4T

We defind some function as

$$
\begin{aligned}
z_{1\varphi}^{mlk} &= \frac{-(E_{mlk}-Q_{mlk}\varphi_{mlk})+\sqrt{(E_{mlk}-Q_{mlk}\varphi_{mlk})^2-4P_{mlk}\varphi_{mlk}(m_k^2-i\rho)}}{-2P_{mlk}\varphi_{mlk}}\\
z_{2\varphi}^{mlk} &= \frac{-(E_{mlk}-Q_{mlk}\varphi_{mlk})-\sqrt{(E_{mlk}-Q_{mlk}\varphi_{mlk})^2-4P_{mlk}\varphi_{mlk}(m_k^2-i\rho)}}{-2P_{mlk}\varphi_{mlk}}\\
z_{3\varphi}^{mlk} &= \frac{(E_{mlk}-Q_{mlk}\varphi_{mlk})+\sqrt{(E_{mlk}-Q_{mlk}\varphi_{mlk})^2-4P_{mlk}\varphi_{mlk}(m_k^2-i\rho)}}{-2P_{mlk}\varphi_{mlk}}\\
z_{4\varphi}^{mlk} &= \frac{(E_{mlk}-Q_{mlk}\varphi_{mlk})-\sqrt{(E_{mlk}-Q_{mlk}\varphi_{mlk})^2-4P_{mlk}\varphi_{mlk}(m_k^2-i\rho)}}{-2P_{mlk}\varphi_{mlk}}\\
z_{1\beta}^{mlk} &= \frac{-(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})+\sqrt{(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})^2-4\frac{P_{mlk}}{\beta_{mlk}}(m_k^2-i\rho)}}{\frac{-2P_{mlk}}{\beta_{mlk}}}\\
z_{2\beta}^{mlk} &= \frac{-(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})-\sqrt{(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})^2-4\frac{P_{mlk}}{\beta_{mlk}}(m_k^2-i\rho)}}{\frac{-2P_{mlk}}{\beta_{mlk}}}\\
z_{3\beta}^{mlk} &= \frac{(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})+\sqrt{(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})^2-4\frac{P_{mlk}}{\beta_{mlk}}(m_k^2-i\rho)}}{\frac{-2P_{mlk}}{\beta_{mlk}}}\\
z_{4\beta}^{mlk} &= \frac{(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})-\sqrt{(E_{mlk}-\frac{Q_{mlk}}{\beta_{mlk}})^2-4\frac{P_{mlk}}{\beta_{mlk}}(m_k^2-i\rho)}}{\frac{-2P_{mlk}}{\beta_{mlk}}}\\
T_1^{nmlk} &= \frac{(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})}{-2P_{mlk}}\\
&+\frac{\sqrt{(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})^2-4P_{mlk}(Q_{mlk}F_{nmlk}+\beta_{mlk}m_k^2-i\beta_{mlk}\rho)}}{-2P_{mlk}}\\
T_2^{nmlk} &= \frac{(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})}{-2P_{mlk}}\\
&-\frac{\sqrt{(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})^2-4P_{mlk}(Q_{mlk}F_{nmlk}+\beta_{mlk}m_k^2-i\beta_{mlk}\rho)}}{-2P_{mlk}}\\
T_3^{nmlk} &= \frac{-(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})}{-2P_{mlk}}\\
&+\frac{\sqrt{(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})^2-4P_{mlk}(Q_{mlk}F_{nmlk}+\beta_{mlk}m_k^2-i\beta_{mlk}\rho)}}{-2P_{mlk}}\\
T_4^{nmlk} &= \frac{-(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})}{-2P_{mlk}}\\
&-\frac{\sqrt{(Q_{mlk}+P_{mlk}F_{nmlk}-\beta_{mlk}E_{mlk})^2-4P_{mlk}(Q_{mlk}F_{nmlk}+\beta_{mlk}m_k^2-i\beta_{mlk}\rho)}}{-2P_{mlk}}
\end{aligned}
$$

$$\begin{aligned}
\eta^+(\sigma, z_1, z_2) &= 2\pi i\Big(\theta[Im(\sigma z_1)]\theta[Im(z_2)]\theta[P] - \theta[-Im(\sigma z_1)]\theta[-Im(z_2)]\theta[-P]\Big) \\
\eta^-(\sigma, z_1, z_2) &= 2\pi i\Big(\theta[-Im(\sigma z_1)]\theta[Im(z_2)]\theta[-P] - \theta[Im(\sigma z_1)]\theta[-Im(z_2)]\theta[P]\Big)
\end{aligned}$$

Here $\sigma = -\frac{1}{\beta}, -\varphi$

$$\begin{aligned}
\Omega^-_{nmlk} &= -f^-_{lk}g^-_{mlk}\ln\Big(\frac{F_{nmlk} + I\rho\beta_{nmlk}}{\beta_{nmlk}}\Big) - f_{lk}g_{mlk}\ln\Big(\frac{F_{nmlk} + I\rho\beta_{nmlk}}{-\beta_{nmlk}}\Big) \\
&\quad + f^-_{lk}g^-_{mlk}\,\eta^+\Big(-\frac{1}{\beta_{mlk}}, z_{3\beta}, z_{4\beta}\Big) + f_{lk}g^-_{mlk}\,\eta^-\Big(-\frac{1}{\beta_{mlk}}, z_{3\beta}, z_{4\beta}\Big) \\
&\quad - (f^-_{lk}g^-_{mlk} + f^-_{lk}g_{mlk})\,\eta^+\Big(-\varphi_{mlk}, z_{3\varphi}, z_{4\varphi}\Big) \\
\Omega^+_{nmlk} &= (f_{lk}g_{mlk} + f^-_{lk}g_{mlk})\ln\Big(\frac{F_{nmlk} + I\rho\beta_{mlk}}{\beta_{mlk}}\Big) \\
&\quad - (f_{lk}g_{mlk} + f^-_{lk}g_{mlk})\,\eta^+\Big(-\frac{1}{\beta_{mlk}}, z_{1\beta}, z_{2\beta}\Big) \\
&\quad + f_{lk}g_{mlk}\eta^+\Big(-\varphi_{mlk}, z_{1\varphi}, z_{2\varphi}\Big) + f_{lk}g^-_{mlk}\eta^-\Big(-\varphi_{mlk}, z_{1\varphi}, z_{2\varphi}\Big) \qquad (2.3)
\end{aligned}$$

2.2.1 Rfunction

$$Rfunction(x, y) = \frac{\ln x - \ln y}{x - y} \qquad (2.4)$$

2.2.2 The LogACG function

$$LogACG(a, b, x, y) = \int_0^\infty \ln(az + b)(z + x)^{-1}(z + y)^{-1}dz \qquad (2.5)$$

with $t = \frac{b}{a}$ and $a > 0$.

$$A = \sqrt{(Ret)^2 + (Imt)^2} + \sqrt{(Rex)^2 + (Imx)^2} + \sqrt{(Rey)^2 + (Imy)^2} \qquad (2.6)$$

and

$$x_0 = \frac{x}{A}; \qquad y_0 = \frac{y}{A}; \qquad z_0 = \frac{t}{A} \qquad (2.7)$$

so one obtain

$$
\begin{aligned}
LogACG(a,b,x,y) &= \frac{\ln(x_0)-\ln(y_0)}{A(x_0-y_0)}\ln(a*A) \\
-\frac{1}{A(x_0-y_0)}\Big\{ &-\frac{1}{2}(\ln x_0)^2+\frac{1}{2}(\ln y_0)^2+Li_2(1-\frac{z_0}{y_0})-Li_2(1-\frac{z_0}{x_0}) \\
&+\ln(y_0)\Big[\eta(z_0-y_0,\frac{1}{1-y_0})-\eta(z_0-y_0,\frac{1}{-y_0})\Big] \\
&-\ln(x_0)\Big[\eta(z_0-x_0,\frac{1}{1-x_0})-\eta(z_0-x_0,\frac{1}{-x_0})\Big] \\
&+\ln\Big(1-\frac{z_0}{y_0}\Big)\eta(z_0,\frac{1}{y_0})-\ln\Big(1-\frac{z_0}{x_0}\Big)\eta(z_0,\frac{1}{x_0}) \quad \Big\}
\end{aligned}
\tag{2.8}
$$

2.2.3 The LogARG function

$$
LogARG(a,b,x,y)=\int_0^\infty \ln(az+b)(z+x)^{-1}(z+y)^{-1}dz \tag{2.9}
$$

with $a<0$. Return to

$$
GiNaCLogARG=\ln(b)\frac{\ln(x)-\ln(y)}{x-y}+GiNaCLogACG(a/b,1.0,x,y). \tag{2.10}
$$

2.2.4 The LogAG function

$LogAG(a,b,x,y)$ is defind as

- If $a>0$.

$$
LogAG(a,b,x,y)=LogACG(a,b,x,y) \tag{2.11}
$$

- If $a<0$

$$
LogAG(a,b,x,y)=LogARG(a,b,x,y) \tag{2.12}
$$

2.2.5 THE FORM OF D_0

Using the Rfunction and LogAG function, we repesent D_0 to form

$$
\begin{aligned}
D_0 \;=\; & i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1\\ k\neq l}}^{4} \sum_{\substack{m=1\\ m\neq l\\ m\neq k}}^{4} \frac{B_{mlk}}{B_{mlk}A_{nlk} - B_{nlk}A_{mlk}} \Big(1-\delta_{lk}(AC_{lk})\Big)\Big(1-\delta_{lk}(B_{mlk})\Big) \times \\
& |1-\beta_{mlk}\varphi_{mlk}| \left(\frac{-1}{P_{mlk}}\right) \times \Big\{ \qquad \Omega^{+}_{nmlk} Rfunction(-T_1^{nmlk}, -T_2^{nmlk}) \\
& -f_{lk} g_{mlk} LogAG\Big(\frac{1-\beta_{mlk}\varphi_{mlk}}{\beta_{mlk}}, \frac{F_{nmlk} + I\rho\beta_{mlk}}{\beta_{mlk}}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& -f_{lk} g^{-}_{mlk} LogAG\Big(\frac{1-\beta_{mlk}\varphi_{mlk}}{-\beta_{mlk}}, \frac{F_{nmlk} + I\rho\beta_{mlk}}{-\beta_{mlk}}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& -(f_{lk} g^{-}_{mlk} + f_{lk} g_{mlk}) LogAG\Big(\frac{-1}{\beta_{mlk}}, \frac{z_{1\beta}}{\beta_{mlk}}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& -(f_{lk} g^{-}_{mlk} + f_{lk} g_{mlk}) LogAG\Big(1, -z_{2\beta}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& +f_{lk} g_{mlk} LogAG\Big(-\varphi_{mlk}, \varphi_{mlk} z_{1\varphi}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& +f_{lk} g_{mlk} LogAG\Big(1, -z_{2\varphi}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& +f_{lk} g^{-}_{mlk} LogAG\Big(\varphi_{mlk}, -\varphi_{mlk} z_{1\varphi}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& +f_{lk} g^{-}_{mlk} LogAG\Big(1, -z_{2\varphi}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& +f^{-}_{lk} g_{mlk} LogAG\Big(1, \frac{Q_{mlk} + I\rho}{P_{mlk}}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& -f_{lk} g^{-}_{mlk} LogAG\Big(1, \frac{Q_{mlk} - I\rho}{P_{mlk}}, -T_1^{nmlk}, -T_2^{nmlk}\Big) \\
& +\Omega^{-}_{nmlk} Rfunction(-T_3^{nmlk}, -T_4^{nmlk}) \\
& +(f^{-}_{lk} g^{-}_{mlk} + f^{-}_{lk} g_{mlk}) LogAG\Big(-\frac{1-\beta_{mlk}\varphi_{mlk}}{\beta_{mlk}}, \frac{F_{nmlk}}{\beta_{mlk}}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& +f^{-}_{lk} g^{-}_{mlk} LogAG\Big(-\frac{1}{\beta_{mlk}}, \frac{z_{3\beta}}{\beta_{mlk}}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& +f^{-}_{lk} g^{-}_{mlk} LogAG\Big(1, z_{4\beta}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& +f_{lk} g^{-}_{mlk} LogAG\Big(\frac{1}{\beta_{mlk}}, -\frac{z_{3\beta}}{\beta_{mlk}}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& +f_{lk} g^{-}_{mlk} LogAG\Big(1, z_{4\beta}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& -(f^{-}_{lk} g^{-}_{mlk} + f_{lk} g^{-}_{mlk}) LogAG\Big(-\varphi_{mlk}, \varphi_{mlk} z_{3\varphi}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& -(f^{-}_{lk} g^{-}_{mlk} + f_{lk} g^{-}_{mlk}) LogAG\Big(1, -z_{4\varphi}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& +f^{-}_{lk} g_{mlk} LogAG\Big(-1, \frac{Q_{mlk} + I\rho}{P_{mlk}}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \\
& -f_{lk} g^{-}_{mlk} LogAG\Big(-1, \frac{Q_{mlk} - I\rho}{P_{mlk}}, -T_3^{nmlk}, -T_4^{nmlk}\Big) \qquad \Big\}
\end{aligned}
$$

2.3 The numerical result

In this section, we compare Oneloop4pt code to LoopTools

2.3.1 Input in Looptools

We call function $D_0(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2)$.

2.3.2 Input in xLoops

From $p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2$, we obtain

$$\begin{aligned}
p_1p_2 &= \frac{(p_1+p_2)^2 - p_1^2 - p_2^2}{2}\\
p_2p_3 &= \frac{(p_2+p_3)^2 - p_2^2 - p_3^2}{2}\\
p_1p_3 &= p_1(-p_1 - p_2 - p_4)\\
&= -p_1^2 - p_1p_2 - p_1p_4\\
&= \frac{p_2^2 + p_4^2 - (p_1+p_2)^2 - (p_1+p_4)^2}{2}\\
&= \frac{p_2^2 + p_4^2 - (p_1+p_2)^2 - (p_2+p_3)^2}{2} \qquad (2.13)
\end{aligned}$$

We have

$$\begin{aligned}
q_1 &= (q_{10}, 0, 0, 0) = p_1\\
q_2 &= (q_{20}, q_{21}, 0, 0) = p_1 + p_2\\
q_3 &= (q_{30}, q_{31}, q_{32}, 0) = p_1 + p_2 + p_3
\end{aligned}$$

and

$$
\begin{aligned}
q_1^2 &= q_{10}^2 = p_1^2 \\
\Longleftrightarrow q_{10} &= \sqrt{p_1^2} \\
q_1 q_2 &= q_{10} q_{20} = p_1(p_2 + p_1) \\
\Longleftrightarrow q_{20} &= \frac{p_1(p_2 + p_1)}{\sqrt{p_1^2}} \\
q_2^2 &= q_{20}^2 - q_{21}^2 = (p_1 + p_2)^2 \\
\Longleftrightarrow q_{21} &= \sqrt{\left(\frac{p_1(p_2 + p_1)}{\sqrt{p_1^2}}\right)^2 - (p_1 + p_2)^2} \\
q_1 q_3 &= q_{10} q_{30} = p_1(p_3 + p_2 + p_1) \\
\Longleftrightarrow q_{30} &= \frac{p_1 p_3}{\sqrt{p_1^2}} + \frac{p_1 p_2}{\sqrt{p_1^2}} + \sqrt{p_1^2} \\
q_2 q_3 &= q_{30} q_{20} - q_{21} q_{31} = (p_1 + p_2)(p_3 + p_2 + p_1) \\
\Longleftrightarrow q_{31} &= \frac{q_{30} q_{20} - (p_1 + p_2)^2 - p_1 p_3 - p_2 p_3}{q_{21}} \\
q_3^2 &= q_{30}^2 - q_{31}^2 - q_{32}^2 = (p_1 + p_2 + p_3)^2 = p_4^2 \\
\Longleftrightarrow q_{32} &= \sqrt{q_{30}^2 - q_{31}^2 - p_4^2}
\end{aligned} \tag{2.14}
$$

From (2.14), we obtain

$$
\begin{aligned}
q_{10} &= \sqrt{p_1^2} \\
q_{20} &= \frac{p_1 p_2}{\sqrt{p_1^2}} + \sqrt{p_1^2} \\
q_{21} &= \sqrt{\left(\frac{p_1 p_2}{\sqrt{p_1^2}} + \sqrt{p_1^2}\right)^2 - (p_1 + p_2)^2} \\
q_{30} &= \frac{p_1 p_3}{\sqrt{p_1^2}} + \frac{p_1 p_2}{\sqrt{p_1^2}} + \sqrt{p_1^2} \\
q_{31} &= \frac{q_{30} q_{20} - (p_1 + p_2)^2 - p_1 p_3 - p_2 p_3}{q_{21}} \\
q_{32} &= \sqrt{q_{30}^2 - q_{31}^2 - p_4^2}
\end{aligned} \tag{2.15}
$$

Finally, We call $D_0(q_{10}, q_{20}, q_{21}, q_{30}, q_{31}, q_{32}, m_2^2, m_3^2, m_4^2, m_1^2)$

2.3.3 The result

We input

$$\begin{aligned} p_1^2 &= 1 \\ p_2^2 &= 1 \\ p_3^2 &= 1 \\ p_4^2 &= 1 \\ m_1^2 &= m_3^2 = 81 * 81 \\ m_2^2 &= m_4^2 = 91 * 91 \end{aligned} \quad (2.16)$$

Compare Oneloop4pt to LoopTools

$(p_1^2, p_2^2, p_3^2, p_4^2)$	$(p_1+p_2)^2$	$(p_2+p_3)^2$	$LoopTools$	$XLOOPS$
$(1, 5, 1, 7)$	15	1	$3.05231662E^{-09}$	???
$(1, 5, 1, 8)$	20	1	$3.05258996E^{-09}$	???
$(1, 6, 1, 9)$	20	1	$3.05267256E^{-09}$	???

Table 2.1: $m_1s = m_3s = 81 * 81; m_2s = m_4s = 91 * 91$

Compare LoopTools To Vegas

$(p_1^2, p_2^2, p_3^2, p_4^2)$	$(p_1+p_2)^2$	$(p_2+p_3)^2$	$LoopTools$	$Vegas$
$(1, 5, 1, 7)$	15	1	$3.05231662E^{-09}$	$3.0524E^{-09} \pm 5.37E^{-15}$
$(1, 5, 1, 8)$	20	1	$3.05258996E^{-09}$	$3.05272E^{-09} \pm 5.37E^{-15}$
$(1, 6, 1, 9)$	20	1	$3.05267256E^{-09}$	$3.0528E^{-09} \pm 5.37E^{-15}$

Table 2.2: $m_1s = m_3s = 81 * 81; m_2s = m_4s = 91 * 91$

Chapter 3

CALCULATION D_0 FOR THE CASE OF COMPLEX MASS

3.1 The analytic calculation

3.1.1 Decompose Log function

We have

$$log\Big(\frac{S(\sigma,z)}{\pm(Pz+Q)}\Big) = log(S(\sigma,z)) - log(\pm(Pz+Q)) + \eta\Big(S(\sigma,z), \pm\frac{1}{(Pz+Q)}\Big) \tag{3.1}$$

Here

$$\begin{aligned} S(\sigma,z) &= P\sigma z^2 + (E+Q\sigma)z - m_k^2 + i\rho \\ &= P\sigma(z-z_{1\sigma})(z-z_{2\sigma}) \\ & \quad Im\Big(S(\sigma,z)\Big) \geq 0. \end{aligned} \tag{3.2}$$

and

$$\eta\Big(S(\sigma,z), \pm\frac{1}{(Pz+Q)}\Big) = -2\pi i\theta[\pm Im(Q^*)]\theta\Big[Im\Big(\frac{\mp S}{Pz+Q}\Big)\Big] \tag{3.3}$$

and

$$log(S(\sigma,z)) = log(P\sigma z - P\sigma z_{1\sigma}) + log(z - z_{2\sigma}) + 2\pi i\theta[Im(P\sigma z_{1\sigma})]\theta[Im(z_{2\sigma})]$$

with the help of these formular, We now obtain

$$\begin{aligned} log\Big(\frac{S(\sigma,z)}{\pm(Pz+Q)}\Big) &= log(P\sigma z - P\sigma z_{1\sigma}) + log(z - z_{2\sigma}) - log(\pm(Pz+Q)) \\ &+2\pi i\theta[Im(P\sigma z_{1\sigma})]\theta[Im(z_{2\sigma})] \\ &-2\pi i\theta[\pm Im(Q^*)]\theta\Big[Im\Big(\frac{\mp S(\sigma,z)}{Pz+Q}\Big)\Big] \end{aligned} \tag{3.4}$$

Because

$$\begin{aligned} Im\left(\frac{S(\sigma,z)}{Pz+Q}\right) &= Im\left[S(\sigma,z)(Pz+Q^*)\right] \\ &= Im\left[(P\sigma z^2 + (E+Q\sigma)z - m_k^2 + i\rho)(Pz+Q^*)\right] \\ &= Im\left[P^2\sigma z^3 + PEz^2 + (Q+Q^*)P\sigma z^2 - QQ^*\sigma z + EQ^*z - m_k^2 Pz - m_k^2 Q^*\right] \\ &= Im\Big[PEz^2 + \Big(EQ^* - m_k^2 P + i\rho P\Big)z - m_k^2 Q^* + i\rho Q^*\Big] \\ &= Im\Big[\,(Pz+Q^*)\left(Ez - m_k^2 + i\rho\right)\Big] \end{aligned}$$

and

$$\begin{aligned} Re\Big(Pz+Q^*\Big) &= Pz + Re(Q^*) = Pz + Re(Q) \\ Im\Big(Pz+Q^*\Big) &= Im(Q^*) = -Im(Q) \\ Re\Big(Ez - m_k^2 + i\rho\Big) &= Re(E)z - Re(m_k^2) \\ Im\Big(Ez - m_k^2 + i\rho\Big) &= Im(E)z + \Gamma_k + \rho \end{aligned} \tag{3.5}$$

so

$$\begin{aligned} Im\left(\frac{S(\sigma,z)}{Pz+Q}\right) &= \Big[Pz + Re(Q)\Big]\Big[Im(E)z + \Gamma_k + \rho\Big] - Im(Q)\Big[Re(E)z - Re(m_k^2)\Big] \\ &= P\ Im(E)\ z^2 + \Big[P\Gamma_k + \rho\ P + Re(Q)Im(E) - Im(Q)Re(E)\Big]z \\ &\quad +Im(Q)Re(m_k^2) + Re(Q)(\Gamma_k + \rho) \\ &= A_0 z^2 + B_0 z + C_0 \end{aligned}$$

Here

$$\begin{aligned} A_0 &= P\ Im(E) \\ B_0 &= Im(EQ^* - Pm_k^2 + iP\rho) = P\Gamma_k + \rho\ P + Re(Q)Im(E) - Im(Q)Re(E) \\ C_0 &= Im(i\rho Q^* - m_k^2 Q^*) = Im(Q)Re(m_k^2) + Re(Q)(\Gamma_k + \rho) \end{aligned} \tag{3.6}$$

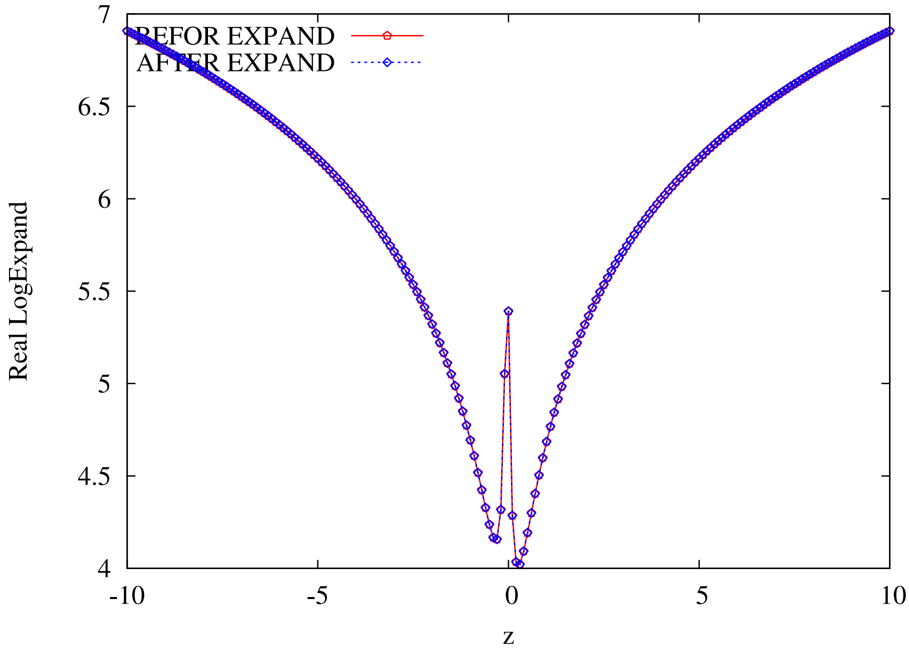

Figure 3.1: For real part

so we now obtain

$$\begin{aligned} log\Big(\frac{S(\sigma,z)}{\pm(Pz+Q)}\Big) &= log(P\sigma z - P\sigma z_{1\sigma}) + log(z - z_{2\sigma}) - log(\pm(Pz+Q)) \\ &\quad +2\pi i\theta[Im(P\sigma z_{1\sigma})]\theta[Im(z_{2\sigma})] \\ &\quad -2\pi i\theta[\mp Im(Q)]\theta\Big[\mp(A_0z^2 + B_0z + C_0)\Big] \end{aligned} \tag{3.7}$$

We now go to test the fomular (3.7).

3.2 The numerical result

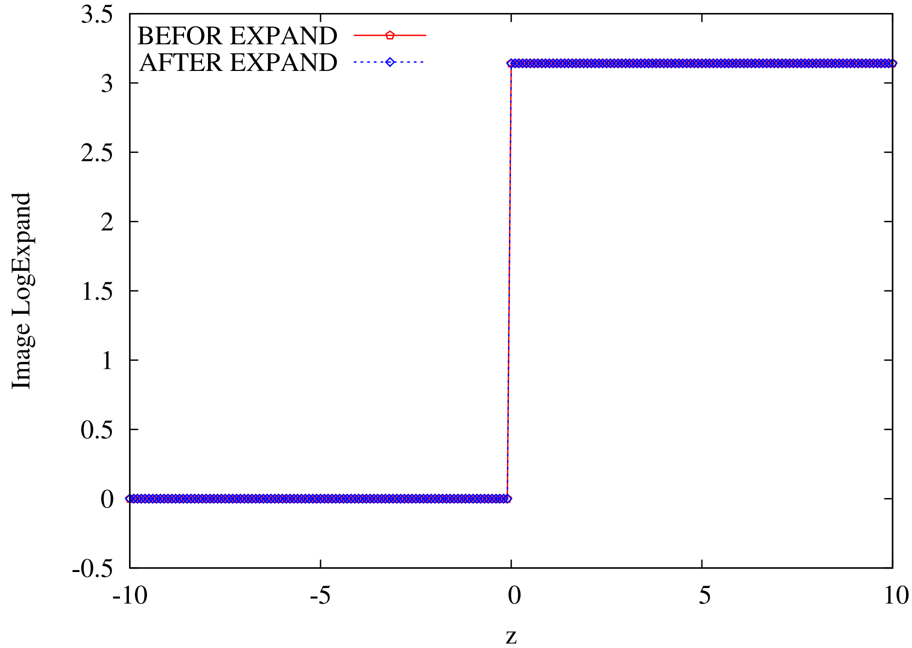

Figure 3.2: For Imge part

$(p_1^2, p_2^2, p_3^2, p_4^2)$	$(p_1+p_2)^2$	$(p_2+p_3)^2$	$RealXloopsVegas$	$RealFeynmanVegas$
$(1,5,1,7)$	15	1	$3.05256e^{-09} \pm 3.92962e^{-13}$	$3.05229e^{-09} \pm 1.55193e^{-14}$
$(1,5,1,8)$	20	1	$3.05278e^{-09} \pm 3.92316e^{-13}$	$3.05261e^{-09} \pm 1.55212e^{-14}$
$(1,6,1,9)$	20	1	$3.05287e^{-09} \pm 3.92293e^{-13}$	$3.05269e^{-09} \pm 1.55216e^{-14}$

Table 3.1: $m_1 s = m_3 s = 81 * 81 - 20I; m_2 s = m_4 s = 91 * 91 - 30I$

$(p_1^2, p_2^2, p_3^2, p_4^2)$	$(p_1+p_2)^2$	$(p_2+p_3)^2$	$ImageXloopsVegas$	$ImageFeynmanVegas$
$(1,5,1,7)$	15	1	$2.10355e^{-11} \pm 1.92669e^{-}14$	$2.10299e^{-11} \pm 2.48297e^{-15}$
$(1,5,1,8)$	20	1	$2.10307e^{-11} \pm 2.4639e^{-}14$	$2.10334e^{-11} \pm 2.48342e^{-15}$
$(1,6,1,9)$	20	1	$2.10308e^{-11} \pm 2.44831e^{-}14$	$2.10343e^{-11} \pm 2.48353e^{-15}$

Table 3.2: $m_1 s = m_3 s = 81 * 81 - 20I; m_2 s = m_4 s = 91 * 91 - 30I$

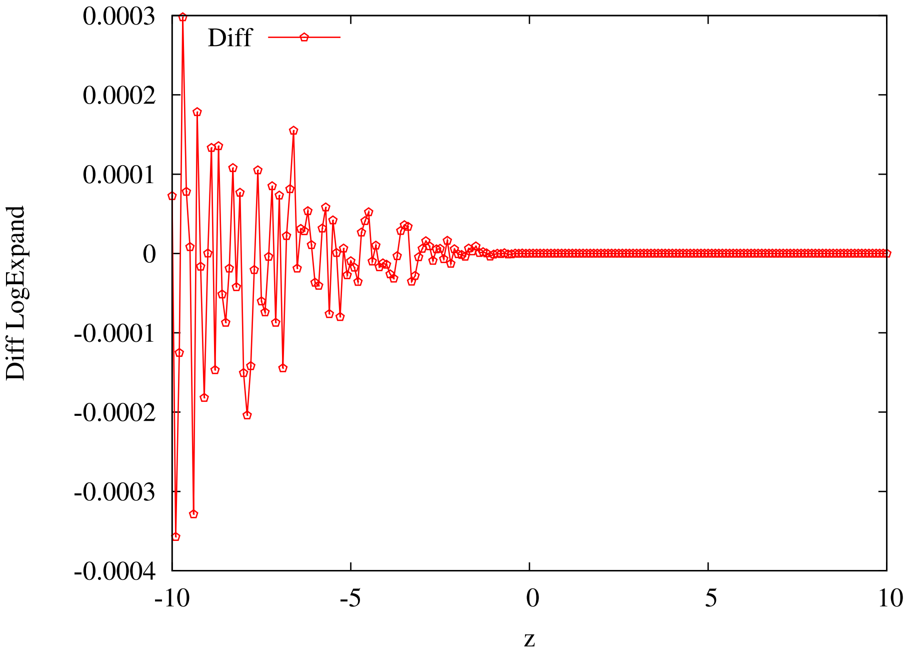

Figure 3.3: For real part

Chapter 4

APENDIX

4.1 Compare Vegas to Looptools

4.1.1 Feyman Parametrization

We have

$$D_0 = \int d^4l \frac{1}{P_1\ P_2\ P_3\ P_4} \tag{4.1}$$

Here

$$\begin{aligned} P_1 &= l^2 - m_1^2 \\ P_2 &= (l + p_1)^2 - m_2^2 \\ P_3 &= (l + p_1 + p_2)^2 - m_3^2 \\ P_4 &= (l + p_1 + p_2 + p_3)^2 - m_4^2 \end{aligned}$$

Using the formula

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^1 dx_1 dx_2 ... dx_n \delta(\sum x_i - 1) \frac{(n-1)!}{\Big[x_1 A_1 + x_2 A_2 + ... x_n A_n\Big]^n} \tag{4.2}$$

one obtain

$$D_0 = 3! \int_0^1 dx\ dy\ dz\ dt \int \frac{\delta(x + y + z + t - 1)}{\Big[tP_1 + xP_2 + yP_3 + zP_4\Big]^4} d^4l \tag{4.3}$$

or

$$D_0 = 3! \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \int \frac{1}{\Big[(1 - x - y - z)P_1 + xP_2 + yP_3 + zP_4\Big]^4} d^4l \tag{4.4}$$

here

$$\begin{aligned}
(1-x-y-z)P_1 &= (1-x-y-z)l^2-(1-x-y-z)m_1^2 \\
xP_2 &= xl^2+2lp_1x+xp_1^2-xm_2^2 \\
yP_3 &= yl^2+2l(p_1+p_2)y+(p_1+p_2)^2y-ym_3^2 \\
zP_4 &= zl^2+2l(p_1+p_2+p_3)z+(p_1+p_2+p_3)^2z-zm_4^2. \\
&\overline{} \\
&= l^2+2l\Big[p_1x+(p_1+p_2)y+(p_1+p_2+p_3)z\Big] \\
&\quad +(p_1^2+m_1^2-m_2^2)x+\Big[(p_1+p_2)^2+m_1^2-m_3^2\Big]y \\
&\quad +\Big[(p_1+p_2+_3)^2+m_1^2-m_4^2\Big]-m_1^2
\end{aligned}$$

We make a shift $l \longrightarrow l+p_1x+(p_1+p_2)y+(p_1+p_2+p_3)z$, one obtain

$$D_0 = 3!\int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} dz \int \frac{1}{\Big[l^2+\Delta\Big]^4} d^4l \tag{4.5}$$

here

$$\begin{aligned}
\Delta &= -\Big[p_1x+(p_1+p_2)y+(p_1+p_2+p_3)z\Big]^2 \\
&\quad +(p_1^2+m_1^2-m_2^2)x+\Big[(p_1+p_2)^2+m_1^2-m_3^2\Big]y \\
&\quad +\Big[(p_1+p_2+_3)^2+m_1^2-m_4^2\Big]-m_1^2
\end{aligned}$$

Applying the formular

$$\frac{1}{(2\pi)^4}\int \frac{1}{\Big[l^2+\Delta\Big]^4} d^4l = \frac{i\pi^2}{3!}\frac{1}{\Delta^2} \tag{4.6}$$

So

$$D_0 = i\pi^2 \int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} dz \frac{1}{\Delta^2} \tag{4.7}$$

with

$$\begin{aligned}
\Delta &= -p_1^2x^2-(p_1+p_2)^2y^2-(p_1+p_2+p_3)^2z^2 \\
&\quad -2p_1(p_1+p_2)xy-2p_1(p_1+p_2+p_3)xz-2(p_1+p_2)(p_1+p_2+p_3)yz \\
&\quad +(p_1^2+m_1^2-m_2^2)x+\Big[(p_1+p_2)^2+m_1^2-m_3^2\Big]y \\
&\quad +\Big[(p_1+p_2+_3)^2+m_1^2-m_4^2\Big]z-m_1^2
\end{aligned}$$

we have

$$\begin{aligned}
2p_1p_2 &= (p_1+p_2)^2 - p_1^2 - p_2^2 \\
2p_2p_3 &= (p_2+p_3)^2 - p_2^2 - p_3^3 \\
2p_1p_3 &= 2p_1(-p_1 - p_2 - p_4) \\
&= -2p_1^2 - 2p_1p_2 - 2p_1p_4 \\
&= -2p_1^2 - (p_1+p_2)^2 + p_1^2 + p_2^2 - (p_1+p_4)^2 + p_1^2 + p_4^2 \\
&= p_2^2 + p_4^2 - (p_1+p_2)^2 - (p_1+p_4)^2 \\
&= p_2^2 + p_4^2 - (p_1+p_2)^2 - (p_2+p_3)^2
\end{aligned} \tag{4.8}$$

we obtain

$$\begin{aligned}
\Delta &= -p_1^2x^2 - (p_1+p_2)^2y^2 - p_4^2z^2 \\
&\quad -\Big[(p_1+p_2)^2 + p_1^2 - p_2^2\Big]xy \\
&\quad -\Big[p_1^2 + p_4^2 - (p_2+p_3)^2\Big]xz \\
&\quad -\Big[p_2^2 - p_3^2 + (p_1+p_2)^2\Big]yz \\
&\quad +(p_1^2 + m_1^2 - m_2^2)x \\
&\quad +\Big[(p_1+p_2)^2 + m_1^2 - m_3^2\Big]y \\
&\quad +\Big[p_4^2 + m_1^2 - m_4^2\Big]z - m_1^2
\end{aligned}$$

SUMMARIZE:

$$\frac{D_0}{i\pi^2} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{1}{\Delta^2} \tag{4.9}$$

with

$$\begin{aligned}
\Delta &= \Delta(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2) \\
\Delta &= -p_1^2 x^2 - (p_1+p_2)^2 y^2 - p_4^2 z^2 \\
&\quad -\Big[(p_1+p_2)^2 + p_1^2 - p_2^2\Big] xy \\
&\quad -\Big[p_1^2 + p_4^2 - (p_2+p_3)^2\Big] xz \\
&\quad -\Big[p_2^2 - p_3^2 + (p_1+p_2)^2\Big] yz \\
&\quad +(p_1^2 + m_1^2 - m_2^2) x \\
&\quad +\Big[(p_1+p_2)^2 + m_1^2 - m_3^2\Big] y \\
&\quad +\Big[p_4^2 + m_1^2 - m_4^2\Big] z - m_1^2
\end{aligned}$$

Or

$$\frac{D_0}{i\pi^2} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{(1-x)^2(1-y)}{\Delta_1^2} \tag{4.10}$$

with

$$\begin{aligned}
\Delta &= \Delta_1(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2) \\
\Delta &= -p_1^2 x^2 - (p_1+p_2)^2 (1-x)^2 y^2 - p_4^2 (1-x)^2 (1-y)^2 z^2 \\
&\quad -\Big[(p_1+p_2)^2 + p_1^2 - p_2^2\Big] x(1-x)y \\
&\quad -\Big[p_1^2 + p_4^2 - (p_2+p_3)^2\Big] x(1-x)(1-y)z \\
&\quad -\Big[p_2^2 - p_3^2 + (p_1+p_2)^2\Big] (1-x)^2(1-y)yz \\
&\quad +(p_1^2 + m_1^2 - m_2^2) x \\
&\quad +\Big[(p_1+p_2)^2 + m_1^2 - m_3^2\Big] (1-x)y \\
&\quad +\Big[p_4^2 + m_1^2 - m_4^2\Big] (1-x)(1-y)z - m_1^2
\end{aligned}$$

4.1.2 The result

The notation:

$$
\begin{aligned}
p_1^2 &= p1s; \quad m_1^2 = m1s \\
p_2^2 &= p2s; \quad m_2^2 = m2s \\
p_3^2 &= p3s; \quad m_3^2 = m3s \\
p_4^2 &= p4s; \quad m_4^2 = m4s \\
(p_1+p_2)^2 &= p12s \\
(p_2+p_3)^2 &= p23s
\end{aligned}
\tag{4.11}
$$

In Vegas code, we input

$$
\begin{aligned}
&voidval(double\, x[DIMENSION], double\; f[FUNCTIONS])\\
&\{\\
&\qquad\qquad double \quad f1, f2, f3, f4, f5, f6;\\
&\qquad\qquad double \quad Pi = 3.1415926535897932385;\\
&\qquad\qquad double \quad p1s = 1;\\
&\qquad\qquad double \quad p2s = 1;\\
&\qquad\qquad double \quad p3s = 1;\\
&\qquad\qquad double \quad p4s = 1;\\
&\qquad\qquad double \quad p12s = 100;\\
&\qquad\qquad double \quad p23s = 50;\\
&\qquad\qquad double \quad m1s = 81 * 81;\\
&\qquad\qquad double \quad m2s = 91 * 91;\\
&\qquad\qquad double \quad m3s = 81 * 81;\\
&\qquad\qquad double \quad m4s = 91 * 91;\\
&\qquad\qquad double \quad f1 = -(p12s + p1s - p2s);\\
&\qquad\qquad double \quad f2 = -(-p23s + p1s + p4s);\\
&\qquad\qquad double \quad f3 = -(p12s + p2s - p3s);\\
&\qquad\qquad double \quad f4 = (p1s - m2s + m1s);\\
&\qquad\qquad double \quad f5 = (p12s - m3s + m1s);\\
&\qquad\qquad double \quad f6 = (p4s - m4s + m1s);\\
&\qquad\qquad \text{//we input } some - t \text{ here}\\
&\qquad\qquad If(x[2] <= 1 - x[0] - x[1] \text{and } \; x[1] <= 1 - x[0])\\
&\qquad\qquad f[0] = 1/pow\Big(- p1s * pow(x[0], 2) - p12s * pow(x[1], 2) - p4s * pow(x[2], 2)\\
&\qquad\qquad\qquad +f1 * x[0] * x[1] + f2 * x[0] * x[2] + f3 * x[1] * x[2] + f4 * x[0]\\
&\qquad\qquad\qquad +f5 * x[1] + f6 * x[2] - m1s, \quad 2\Big) \;\; ;\\
&\qquad\qquad else\\
&\qquad\qquad f[0] = 0;\\
&\}
\end{aligned}
\tag{4.12}
$$

And

$(p_1^2, p_2^2, p_3^2, p_4^2)$	$(p_1+p_2)^2$	$(p_2+p_3)^2$	Vegas	LoopTools
(1,1,1,1)	100	50	$3.05757E^{-09} \pm 5.05325E^{-13}$	$3.05766235E^{-09}$
(1,1,1,1)	100	60	$3.05794E^{-09} \pm 5.05019E^{-13}$	$3.05803094E^{-09}$
(1,1,1,1)	100	70	$3.05831E^{-09} \pm 5.052E^{-13}$	$3.05839976E^{-09}$
(1,1,1,1)	1000	200	$3.10604e^{-09} \pm 5.26804e^{-13}$	$3.10606296E^{-09}$
(1,1,1,1)	5	1	$3.05144e^{-09} \pm 8.61833e^{-16}$	$3.05143955E^{-09}$

Table 4.1: $m_1 s = m3s = 81 * 81; m_2 s = m4s = 91 * 91$

$(p_1^2, p_2^2, p_3^2, p_4^2)$	$(p_1+p_2)^2$	$(p_2+p_3)^2$	$LoopTools$	vegas	$xLoop$
$(1,1,1,1)$	200	-50	$3.05865044E^{-09}$	$3.05865e^{-09} \pm 8.69883e^{-16}$	$xLoop$
$(1,1,1,1)$	1000	-50	$3.0967473E^{-09}$	$3.09675e^{-09} \pm 8.57708e^{-16}$	$xLoop$
$(1,1,1,1)$	10000	-50	$3.64967589E^{-09}$	$3.64968e^{-09} \pm 1.11451e^{-15}$	$xLoop$
$(1,1,1,1)$	10000	-100	$3.64760089E^{-09}$	$3.6476e^{-09} \pm 1.11072e^{-15}$	$xLoop$

Table 4.2: $m_1 s = m_3 s = 81 * 81; m_2 s = m_4 s = 91 * 91$

4.2 Check the Lorentz transformation

The form of D_0 in Paralell and orgonal space is

$$D_0 = 2 \int_{-\infty}^{\infty} dl_0\, dl_1\, dl_2 \int_0^{\infty} dl_\perp \tag{4.13}$$
$$\frac{1}{[(l_0+q_{10})^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\rho][(l_0+q_{20})^2 - (l_1+q_{21})^2 - l_2^2 - l_\perp^2 - m_2^2 + i\rho]}$$
$$\frac{1}{[(l_0+q_{30})^2 - (l_1+q_{31})^2 - (l_2+q_{32})^2 - l_\perp^2 - m_3^2 + i\rho][l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\rho]}$$

Because

$$(l_0+q_{10})^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\rho = 0 \tag{4.14}$$

have two pole

$$l_0 = -q_{10} \pm \sqrt{l_1^2 + l_2^2 + l_\perp^2 + m_1^2 - i\rho} \tag{4.15}$$

here $q_{10} << m_1$, so these pole locate across in real l_0 line.
Performing Wick rotaion $l_0 \to il_0$, we obtain

$$\begin{aligned}
D_0 &= 2i \int\limits_{-\infty}^{\infty} dl_0\, dl_1\, dl_2 \int\limits_{0}^{\infty} dl_\perp \\
&\frac{1}{[-l_0^2 + 2il_0 q_{10} + q_{10}^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\rho]} \\
&\frac{1}{[-l_0^2 + 2il_0 q_{20} + q_{20}^2 - (l_1 + q_{21})^2 - l_2^2 - l_\perp^2 - m_2^2 + i\rho]} \\
&\frac{1}{[-l_0 + 2il_0 q_{30} + q_{30}^2 - (l_1 + q_{31})^2 - (l_2 + q_{32})^2 - l_\perp^2 - m_3^2 + i\rho]} \\
&\frac{1}{[-l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\rho]}
\end{aligned}$$

We make a change variable

$$\begin{aligned}
x &= tg(l_0) \\
y &= tg(l_1) \\
z &= tg(l_2) \\
t &= tg(l_3)
\end{aligned} \tag{4.16}$$

and run Vegas.

$(p_1^2, p_2^2, p_3^2, p_4^2)$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$	$LoopTools$	$D_0 XLOOPS$
$(1, 5, 1, 7)$	15	1	$3.05231662E^{-09}$	$3.05262e^{-09} \pm 3.88742e^{-13}$
$(1, 5, 1, 8)$	20	1	$3.05258996E^{-09}$	$3.05276e^{-09} \pm 3.91644e^{-13}$
$(1, 6, 1, 9)$	20	1	$3.05267256E^{-09}$	$3.05289e^{-09} \pm 3.90594e^{-13}$

Table 4.3: $m_1 s = m_3 s = 81 * 81; m_2 s = m_4 s = 91 * 91$

4.3 Epand R function

The R function have form

$$R_{-1-\varepsilon}\Big(1, 1, \varepsilon; x, y, z\Big) \tag{4.17}$$

we expand this function as the series of ε

$$R_{-1-\varepsilon}\Big(1,1,\varepsilon;x,y,z\Big) = \frac{B(\varepsilon,1)}{x-y}\Big[R_{-\varepsilon}(1,\varepsilon;y,z) - R_{-\varepsilon}(1,\varepsilon;x,z)\Big] \tag{4.18}$$

and

$$R_{-\varepsilon}(1,\varepsilon;y,z) = \sum_{n=0}^{\infty} \frac{(\varepsilon,n)}{(1+\varepsilon,n)} \sum_{m=0}^{n} \frac{(\varepsilon,m)}{(1,m)}(1-y)^{n-m}(1-z)^m \tag{4.19}$$

because

$$\frac{(\varepsilon,n)}{(1+\varepsilon,n)} \quad = \quad \frac{\varepsilon(\varepsilon+1)...(\varepsilon+n-1)}{(\varepsilon+1)(\varepsilon+2)...(\varepsilon+n)} = \frac{\varepsilon}{\varepsilon+n}$$

one obtain

$$\begin{aligned}
R_{-\varepsilon}(1,\varepsilon;y,z) &= \sum_{n=0}^{\infty} \frac{\varepsilon}{\varepsilon+n} \sum_{m=0}^{n} \frac{(\varepsilon,m)}{(1,m)}(1-y)^{n-m}(1-z)^m \\
&= 1 + \varepsilon \sum_{n=1}^{\infty} \frac{1}{\varepsilon+n}(1-y)^n \\
&\quad +\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{\varepsilon+n} \sum_{m=1}^{n} \frac{1}{m}(1-y)^{n-m}(1-z)^m + \Theta(\varepsilon^3) \\
&\quad = 1 + \varepsilon \sum_{n=1}^{\infty} \frac{1}{n}(1-y)^n - \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n^2}(1-y)^2 \\
&\quad +\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{m}(1-y)^{n-m}(1-z)^m + \Theta(\varepsilon^3)
\end{aligned} \tag{4.20}$$

For the case $|1-z_i| < 0$, this series converge to log or dilog function

$$\begin{aligned}
R_{-\varepsilon}(1,\varepsilon;y,z) \quad = \quad & 1 - \varepsilon\ln(y) - \varepsilon^2 Li_2(1-y) \\
& +\varepsilon^2\Big\{ \quad Li_2(1-y) + Li_2(1-\frac{z}{y}) + \frac{1}{2}\Big(\ln(y)\Big)^2 \\
& \qquad + \ln(y)\Big[\eta(z-y,\frac{1}{1-y}) - \eta(z-y,\frac{1}{-y})\Big] \\
& \qquad + \ln\Big(1-\frac{z}{y}\Big)\eta(z,\frac{1}{y}) \qquad \Big\} + \Theta(\varepsilon^3)
\end{aligned}$$

now we obtain

$$
\begin{aligned}
R_{-1-\varepsilon}\Big(1,1,\varepsilon;x,y,z\Big) &= \frac{\ln x-\ln y}{x-y}+\varepsilon\frac{\ln x-\ln y}{x-y} \\
&+\frac{\varepsilon}{x-y}\Big\{\frac{1}{2}\ln^2 y-\frac{1}{2}\ln^2 x+Li_2(1-\frac{z}{y})-Li_2(1-\frac{z}{x}) \\
&\quad +\ln(y)\Big[\eta(z-y,\frac{1}{1-y})-\eta(z-y,\frac{1}{-y})\Big] \\
&\quad -\ln(x)\Big[\eta(z-x,\frac{1}{1-x})-\eta(z-x,\frac{1}{-x})\Big] \\
&\quad +\ln\Big(1-\frac{z}{y}\Big)\eta(z,\frac{1}{y})-\ln\Big(1-\frac{z}{x}\Big)\eta(z,\frac{1}{x}) \quad \Big\}+\Theta(\varepsilon^2)
\end{aligned}
$$

The conditional: We know that $|1-z_i|<0$, $z\in C$, the series (4.20) converge to log or dilog function $\Longleftrightarrow 0<|z_i|<2$

4.3.1 THE SOLUTION FOR LogACG FUNCTION

We have

$$
LogACG(a,b,x,y)=\int_0^\infty (az+b)^{-\varepsilon}(z+x)^{-1}(z+y)^{-1}dz \tag{4.21}
$$

with $z=\frac{b}{a}$ and $a>0$, one obtain

$$
LogACG(a,b,x,y)=a^{-\varepsilon}\int_0^\infty (z+z_0)^{-\varepsilon}(z+x)^{-1}(z+y)^{-1}dz \tag{4.22}
$$

To obtain the converge series, we make $z=Az'$

$$
LogACG(a,b,x,y)=a^{-\varepsilon}A\int_0^\infty (Az+z_0)^{-\varepsilon}(Az+x)^{-1}(Az+y)^{-1}dz \tag{4.23}
$$

here

$$
A=|z_0|+|x|+|y|>0 \tag{4.24}
$$

$$
\begin{aligned}
LogACG(a,b,x,y) &= \frac{(aA)^{-\varepsilon}}{A}\int_0^\infty (z+\frac{z_0}{A})^{-\varepsilon}(z+\frac{x}{A})^{-1}(z+\frac{y}{A})^{-1}dz \\
&= \frac{(aA)^{-\varepsilon}}{A}B(1+\varepsilon,1)R_{-1-\varepsilon}\Big(1,1,\varepsilon;\frac{x}{A},\frac{y}{A},\frac{z_0}{A}\Big)
\end{aligned} \tag{4.25}
$$

Now we have

$$0 < |\frac{z_i}{A}| < 2 \tag{4.26}$$

So the series of this function is converge to log or dilog function.

4.3.2 REWRITE THE LogACG FUNCTION

$$LogACG(a,b,x,y) = \int_0^\infty (az+b)^{-\varepsilon}(z+x)^{-1}(z+y)^{-1}dz \tag{4.27}$$

with $t = \frac{b}{a}$ and $a > 0$.

$$A = \sqrt{(Rez_0)^2 + (Imz_0)^2} + \sqrt{(Rex)^2 + (Imx)^2} + \sqrt{(Rey)^2 + (Imy)^2} \tag{4.28}$$

and

$$x_0 = \frac{x}{A}; \qquad y_0 = \frac{y}{A}; \qquad z_0 = \frac{t}{A} \tag{4.29}$$

so one obtain

$$\begin{aligned}
LogACG(a,b,x,y) &= \frac{\ln(x_0) - \ln(y_0)}{A(x_0 - y_0)}\ln(a * A) \\
-\frac{1}{A(x_0 - y_0)}\Big\{ &- \frac{1}{2}(\ln x_0)^2 + \frac{1}{2}(\ln y_0)^2 + Li_2(1 - \frac{z_0}{y_0}) - Li_2(1 - \frac{z_0}{x_0}) \\
&+ \ln(y_0)\Big[\eta(z_0 - y_0, \frac{1}{1 - y_0}) - \eta(z_0 - y_0, \frac{1}{-y_0})\Big] \\
&- \ln(x_0)\Big[\eta(z_0 - x_0, \frac{1}{1 - x_0}) - \eta(z_0 - x_0, \frac{1}{-x_0})\Big] \\
&+ \ln\Big(1 - \frac{z_0}{y_0}\Big)\eta(z_0, \frac{1}{y_0}) - \ln\Big(1 - \frac{z_0}{x_0}\Big)\eta(z_0, \frac{1}{x_0}) \quad \Big\}
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
LogARG(a,b,x,y) &= \int_0^\infty \ln(az+b)(z+x)^{-1}(z+y)^{-1} \\
&= \ln(b)\Big(\frac{\ln(x) - \ln(y)}{x - y}\Big) + LogACG(\frac{a}{b}, 1, x, y)
\end{aligned} \tag{4.31}$$

4.4 The result LogARG function

$$LogARG(a,b,x,y) = \int_0^\infty \ln(az+b)(z+x)^{-1}(z+y)^{-1} \tag{4.32}$$

Here $a < 0$.

4.4.1 For x, y are complex

We test for $y = -10 + 10I$ and $x = -50 + (Imx) * I$, Imx run -50 to 50. see fig.1,fig.2,3

4.4.2 For x, y are complex have small image

We test for $y = -10 + 10^{-5}I$ and $x = -50 + (Imx) * I$, Imx run -5.10^{-6} to 5.10^{-6}, see fig4,5,6

4.4.3 The solution for small image

We test for $y = -10 + 10^{-5}I$ and $x = -50 + (Imx) * I$, Imx run -5.10^{-6} to 5.10^{-6} and $y = -10 + 10^{-15}I$ and $x = -50 + (Imx) * I$, Imx run -5.10^{-16} to 5.10^{-16}

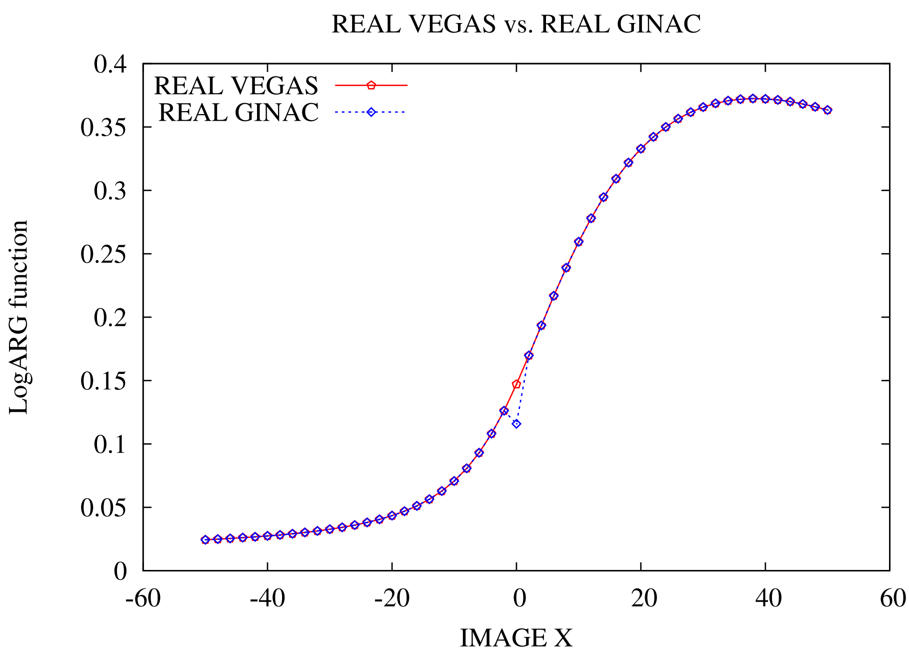

Figure 4.1: For real part

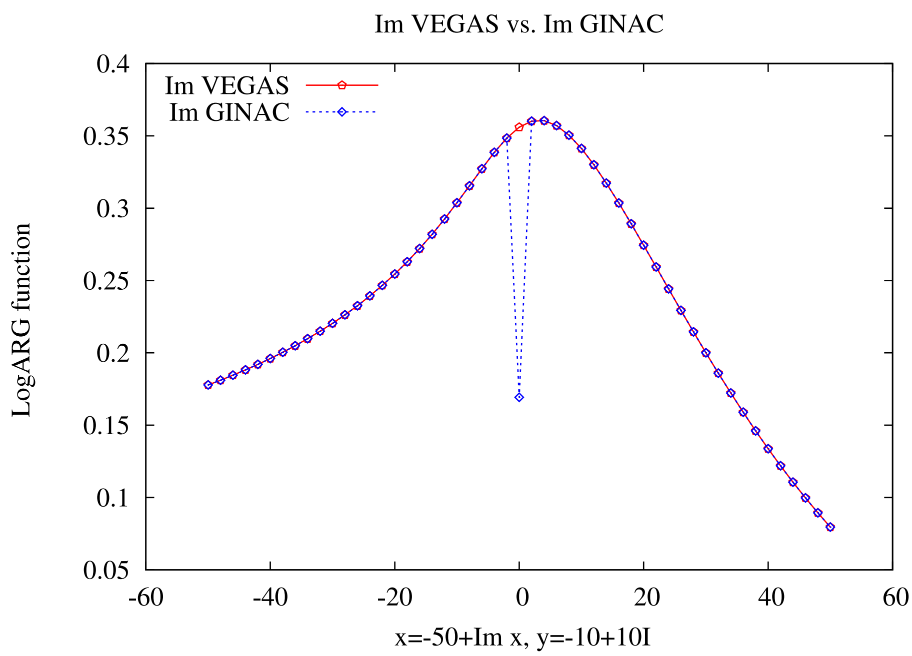

Figure 4.2: For Image part

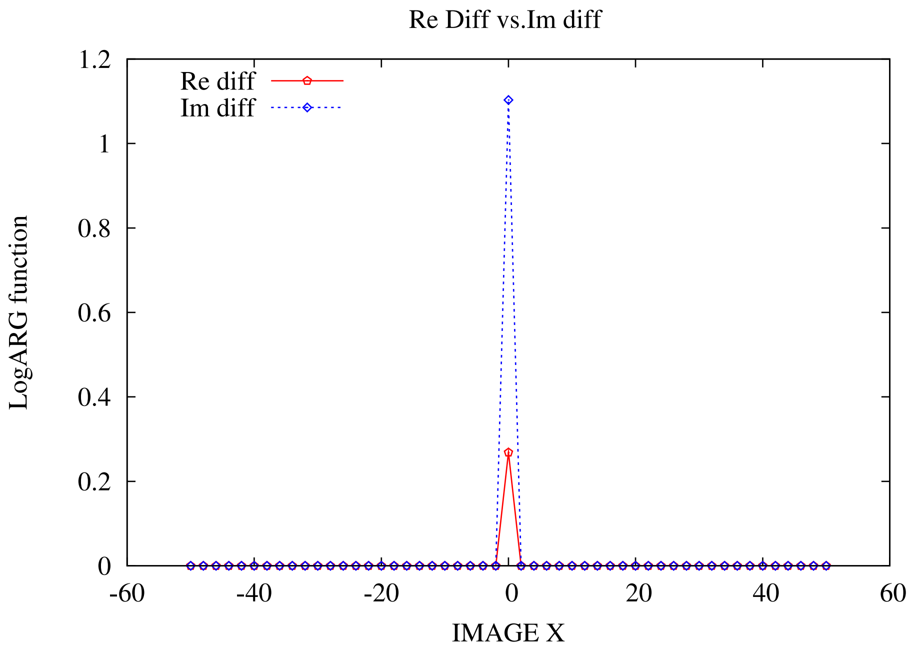

Figure 4.3: Diff 1

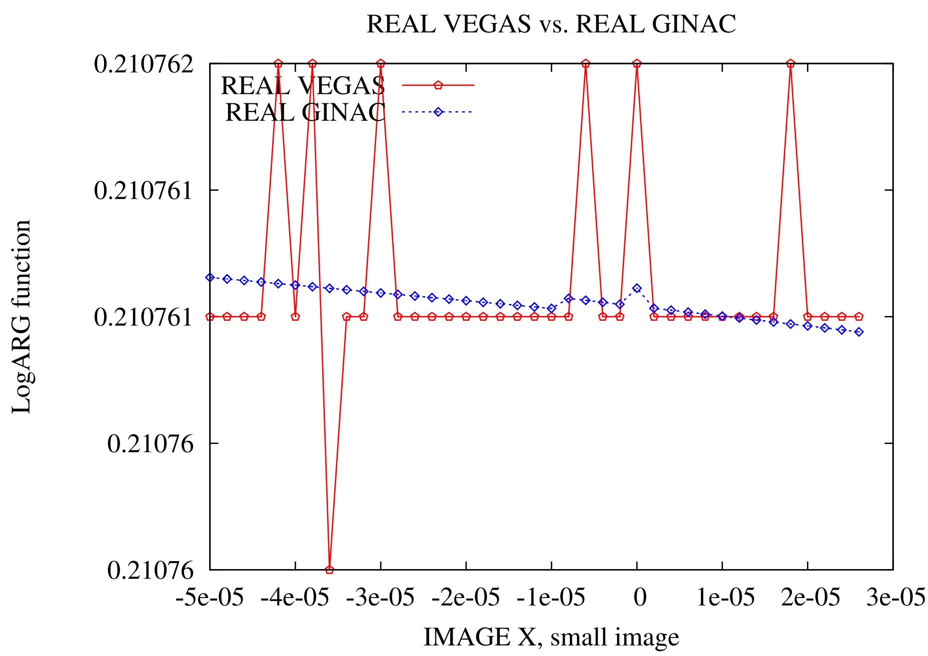

Figure 4.4: For real part (small image)

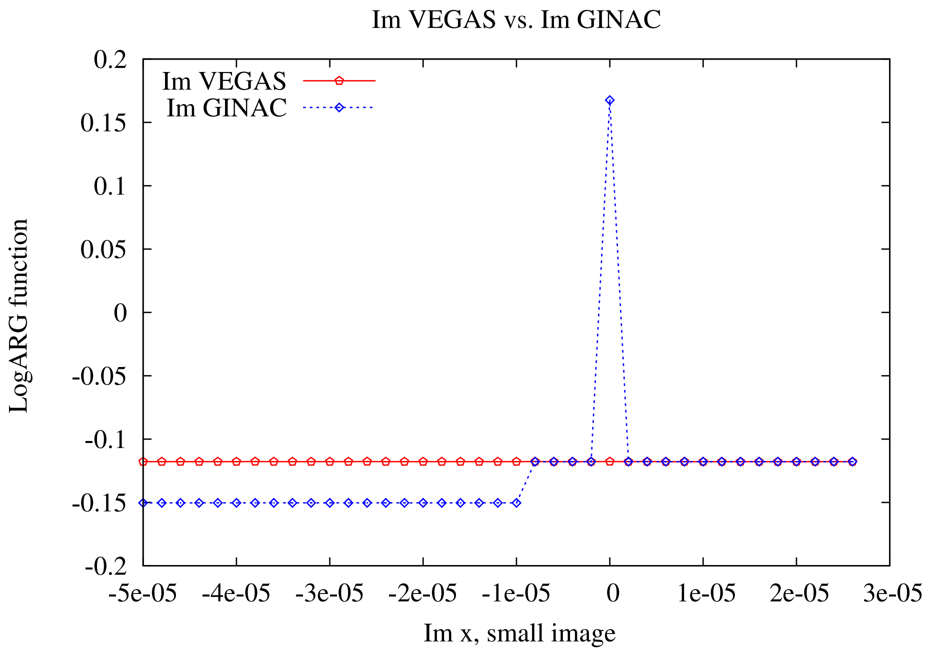

Figure 4.5: For Image part (small image)

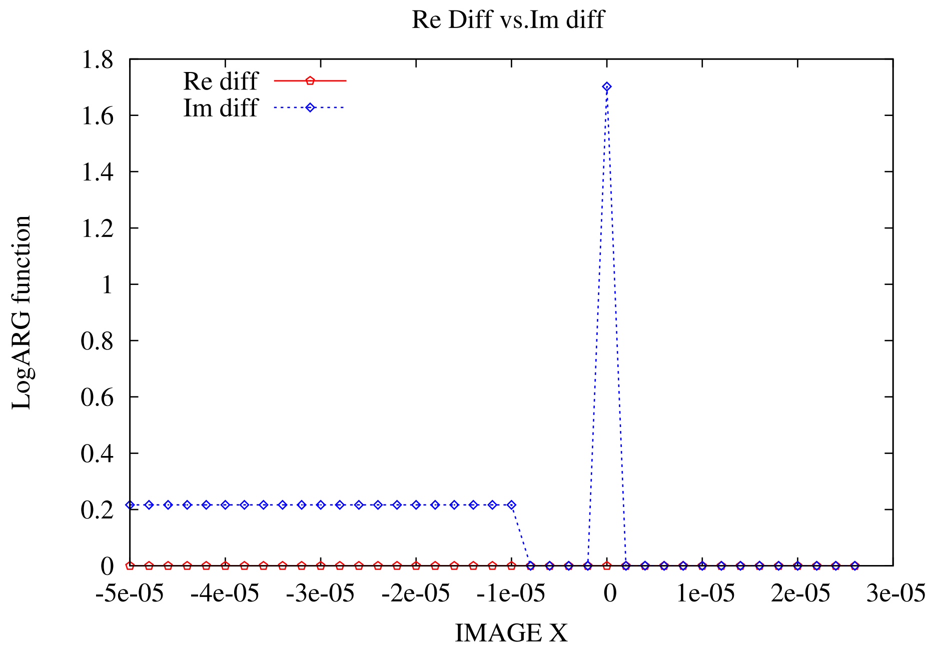

Figure 4.6: Diff (small image)

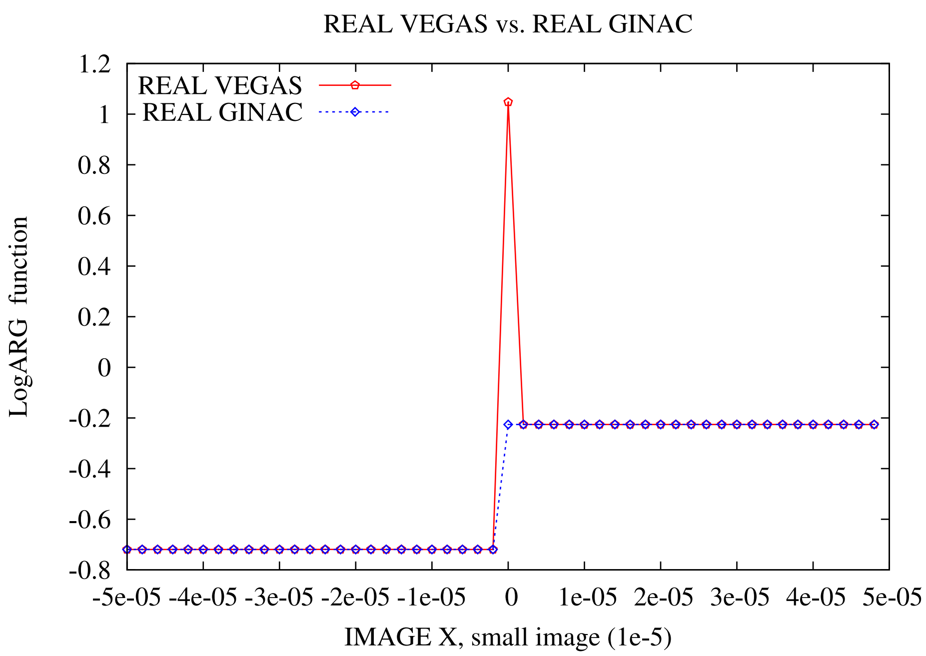

Figure 4.7: Compare real part

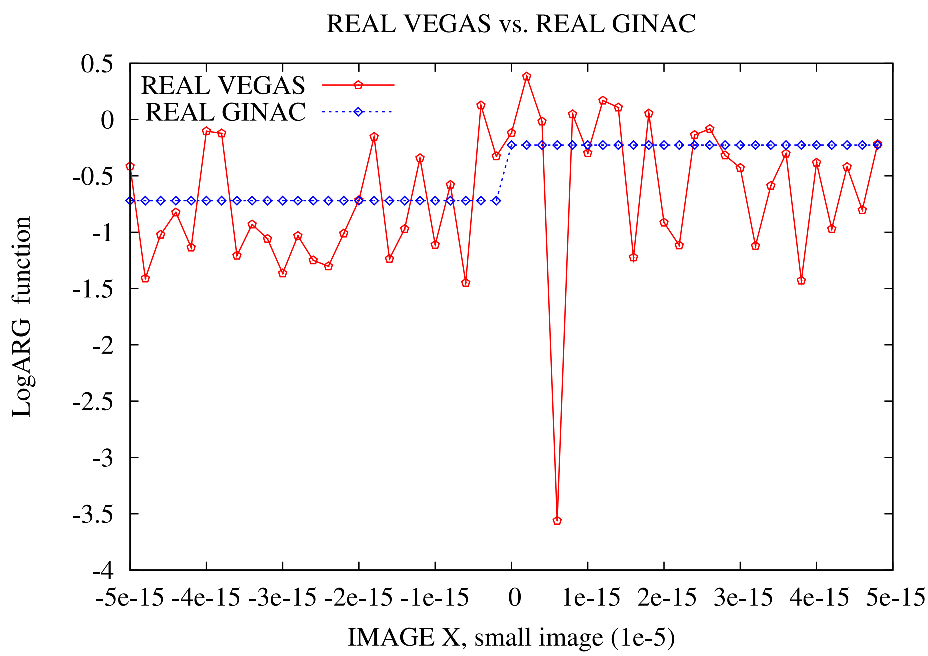

Figure 4.8: Compare real part

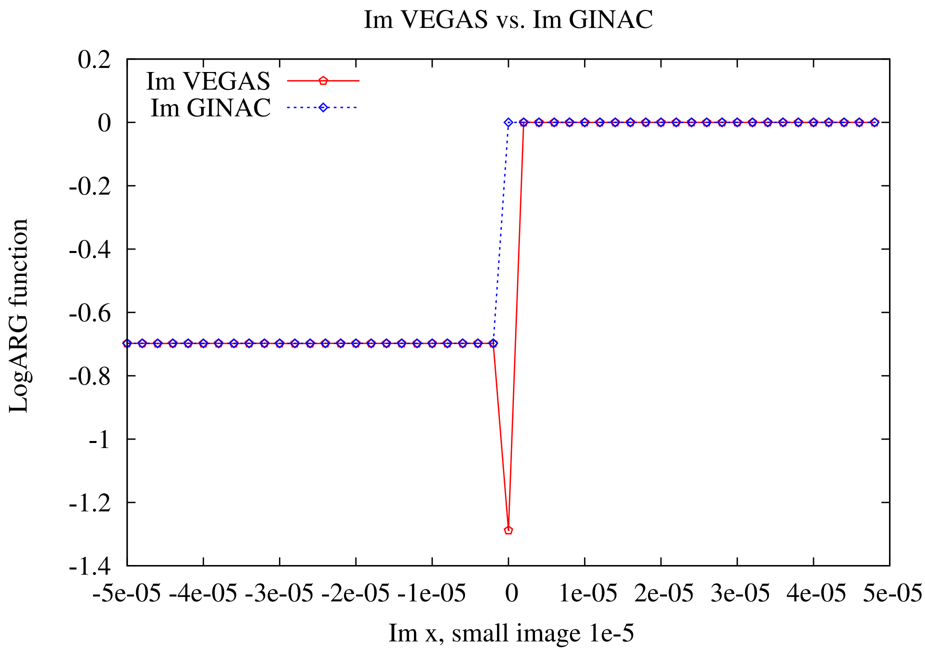

Figure 4.9: Compare Image part

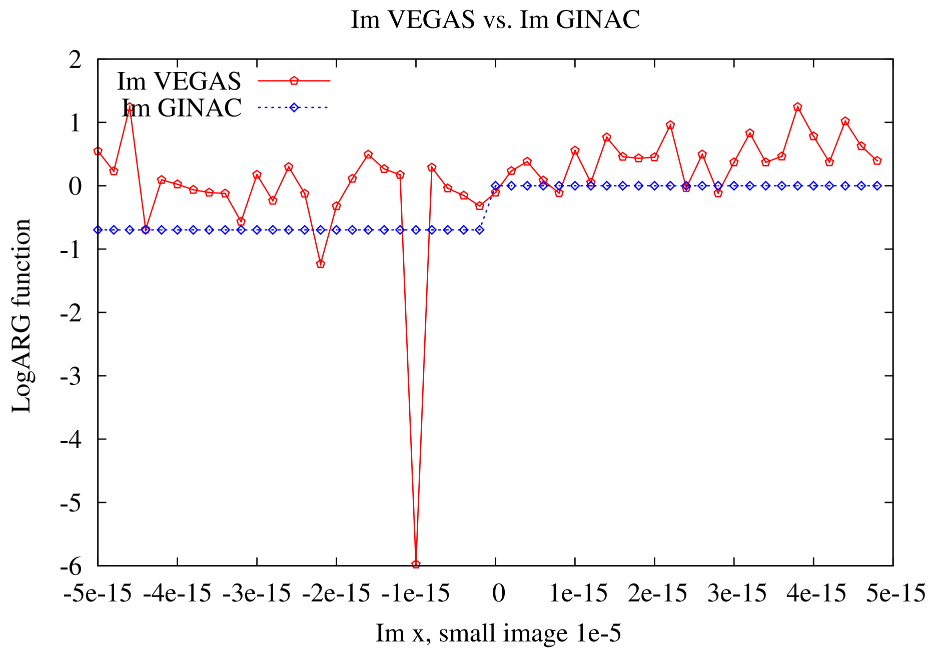

Figure 4.10: Compare Image part