One-loop N-Point Feynman Integrals in $\mathcal{R}$ functions

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Abstract

Explicit calculation of one-loop two-, three- and four-point Feynman integrals in terms of orthogonal and parallel spaces with the help of $\mathcal{R}$ functions is presented. The light-like and space-like configurations of external momenta are treated carefully here.

1 Overview of the method of calculation

In the parallel and orthogonal spaces [**?**], a one-loop N-point integral can be written in the form

$$T^N_{\mu_1\ldots\mu_M} = (-1)^{\frac{p_\perp}{2}} \frac{\Big(g_{\mu_1\mu_2}\ldots g_{\mu_{p_\perp-1}\,\mu_{p_\perp}}\Big)_{sym}}{K}\, T^{(p_0,p_1,\ldots,p_\perp)} \tag{1}$$

with $D = 4-2\epsilon$; J is the number of parallel dimension (spaned by the external momenta);

$$K = \begin{cases} \displaystyle\prod_{i=0}^{(p_\perp-2)/2} (D-J+2i) & \text{when} \quad p_\perp \neq 0 \\ \\ \qquad 1 & \text{when} \quad p_\perp = 0 \end{cases} \tag{2}$$

and

$$T^{(p_0,p_1,\ldots,p_\perp)} = \frac{2\pi^{\frac{D-J}{2}}}{\Gamma\left(\frac{D-J}{2}\right)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \ldots \int_{0}^{\infty} dl_\perp\, l_\perp^{D-J-1} \frac{l_0^{p_0} \ldots l_{J-1}^{p_{j-1}}}{\prod_{k=1}^{N} \left[(l+q_k)^2 - m_k^2 + i\varrho\right]} \tag{3}$$

<u>**Note:**</u>

- p_i are non-negative integer numbers. They are not Lorentz indices.
- As a convention, we always chose $q_N = 0$ and q_1 has minimum amount of nonzero components

This calculation will consider all the cases where $q_1^2 > 0$ (time-like), $q_1^2 = 0$ (light-like) and $q_1^2 < 0$ (space-like). This is the major improvement to the former works [**?**]

2 Scalar one-loop two-point integrals

To treat the cases where $q^2 \leq 0$ we should keep $J = 2$. The general one-loop two-point integral reads

$$\begin{aligned} B_0(q^2, m_1, m_2) &= \frac{2\pi^{\frac{D-2}{2}}}{\Gamma\left(\frac{D-2}{2}\right)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{0}^{\infty} dl_\perp\, l_\perp^{D-3} \\ &\times \frac{1}{[(l_0+q_{10})^2 - (l_1+q_{11})^2 - l_\perp^2 - m_1^2 + i\varrho]\,[l_0^2 - l_1^2 - l_\perp^2 - m_2^2 + i\varrho]} \end{aligned} \tag{4}$$

2.1 The light-like case $q^2 = 0$

This special case is equal to the case where $q^\mu = 0$ and leads to a sum of two tadpol integrals

$$B_0(0, m_1^2, m_2^2) = \frac{1}{m_1^2 - m_2^2}\left[A_0(m_1^2) - A_0(m_2^2)\right] \tag{5}$$

There are two way to check:

- Performing Feynman parametrization of both type $q^\mu = 0$ and $q^2 = 0$ or

- Arguing that when $q^2 = 0$, B_0 is the function of masses only, which should be the same for the case that $q^\mu = 0$.

However, one should be carefull with the tensor reduction procedure for this case. Please check!!!

2.2 The time-like case $q^2 > 0$

When $q^2 > 0$, there exist a boost which transforms

$$q^\mu = (q_{10}, q_{11}, \vec{0}) \to (q'_{10}, 0, \vec{0})$$

In this frame of reference, the integral reads

$$B_0 = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \int\limits_{-\infty}^{\infty} dl_0 \int\limits_{0}^{\infty} dl_\perp \frac{l_\perp^{D-2}}{[(l_0+q_{10})^2 - l_\perp^2 - m_1^2 + i\varrho]\,[l_0^2 - l_\perp^2 - m_2^2 + i\varrho]} \tag{6}$$

Using the identity

$$\frac{1}{P_1\, P_2} = \frac{1}{P_1(P_2-P_1)} + \frac{1}{P_2(P_1-P_2)} \tag{7}$$

$$P_1 - P_2 = 2q_{10}l_0 + q_{10}^2 + m_2^2 - m_1^2 \tag{8}$$

to obtains

$$\begin{aligned} B_0 &= \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \int\limits_{-\infty}^{\infty} dl_0 \int\limits_{0}^{\infty} dl_\perp \left\{ \frac{-\, l_\perp^{D-2}}{[(l_0+q_{10})^2 - l_\perp^2 - m_1^2 + i\varrho]\,[2q_{10}l_0 + q_{10}^2 + m_2^2 - m_1^2]} \right. \\ &\qquad \left. + \frac{l_\perp^{D-3}}{[l_0^2 - l_\perp^2 - m_2^2 + i\varrho]\,[2q_{10}l_0 + q_{10}^2 + m_2^2 - m_1^2]} \right\} \end{aligned}$$

Shift $l_0 \to l_0 + q_{10}$ in the first term

$$\begin{aligned} B_0 &= \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right) 2q_{10}} \int\limits_{-\infty}^{\infty} dl_0 \int\limits_{0}^{\infty} dl_\perp \left\{ \frac{l_\perp^{D-2}}{[l_\perp^2 - l_0^2 + m_1^2 - i\varrho]\left[l_0 - \left(\frac{q_{10}}{2} + M_d\right)\right]} \right. \\ &\qquad \left. - \frac{l_\perp^{D-2}}{[l_\perp^2 - l_0^2 + m_2^2 - i\varrho]\left[l_0 + \left(\frac{q_{10}}{2} - M_d\right)\right]} \right\} \end{aligned}$$

where $M_d = \frac{m_1^2 - m_2^2}{2q_{10}}$.

The integration over $dl_\perp$ can be performed by changing the variable $s = dl_\perp^2 \rightarrow dl_\perp = \frac{ds}{2\sqrt{s}}$ and using the integral formula

$$\int_0^\infty ds\, \frac{s^{\alpha-1}}{s+z} = \Gamma(1-\alpha)\Gamma(\alpha)\; z^{\alpha-1} \tag{9}$$

one obtains

$$\begin{aligned}
B_0 &= \frac{\pi^{\frac{D-1}{2}}\Gamma(\epsilon-\frac{1}{2})}{2q_{10}} \int_{-\infty}^{\infty} dl_0 \left\{ \frac{(-l_0^2+m_1^2-i\varrho)^{\frac{1}{2}-\epsilon}}{l_0-\left(\frac{q_{10}}{2}+M_d\right)} - \frac{(-l_0^2+m_2^2-i\varrho)^{\frac{1}{2}-\epsilon}}{l_0+\left(\frac{q_{10}}{2}-M_d\right)} \right\} \quad (10)\\
&= \frac{\pi^{\frac{D-1}{2}}\Gamma(\epsilon-\frac{1}{2})}{2q_{10}} \int_{-\infty}^{\infty} dl_0 \left\{ \frac{\left[l_0+\left(\frac{q_{10}}{2}+M_d\right)\right](-l_0^2+m_1^2-i\varrho)^{\frac{1}{2}-\epsilon}}{l_0^2-\left(\frac{q_{10}}{2}+M_d\right)^2} \right. \\
&\qquad \left. - \frac{\left[l_0-\left(\frac{q_{10}}{2}-M_d\right)\right](-l_0^2+m_2^2-i\varrho)^{\frac{1}{2}-\epsilon}}{l_0^2-\left(\frac{q_{10}}{2}-M_d\right)^2} \right\}
\end{aligned}$$

Remove the odd integrals w.r.t l_0, change the variable $s = dl_0^2 \rightarrow dl_0 = \frac{ds}{2\sqrt{s}}$, collapse the boundary of integral from $(-\infty,\infty)$ to $(0,\infty)$ one obtains

$$\begin{aligned}
B_0 &= \frac{\pi^{\frac{D-1}{2}}\Gamma(\epsilon-\frac{1}{2})}{2q_{10}} \left\{ \left(\frac{q_{10}}{2}+M_d\right) \int_0^\infty ds\; s^{\frac{1}{2}-1}\left(-s+m_1^2-i\varrho\right)^{\frac{1}{2}-\epsilon} \left[s-\left(\frac{q_{10}}{2}+M_d\right)^2\right]^{-1} \right. \\
&\quad \left. + \left(\frac{q_{10}}{2}-M_d\right) \int_0^\infty ds\; s^{\frac{1}{2}-1}\left(-s+m_2^2-i\varrho\right)^{\frac{1}{2}-\epsilon} \left[s-\left(\frac{q_{10}}{2}-M_d\right)^2\right]^{-1} \right\} \quad (11)
\end{aligned}$$

<u>**Note**</u>:

In general, $(-z)^a \neq (-1)^a\; z^a$ due to a cut in the negative real axes (similar to $\ln(z)$). So $-1 = e^{\pm i\pi}$ has an arbitrary phase. In our integrals, the term $(-s+m_1^2-i\varrho)^{\frac{1}{2}-\epsilon}$ stays in lower half-plan so

$$\begin{aligned}
\left(-s+m_1^2-i\varrho\right)^{\frac{1}{2}-\epsilon} &= e^{-i\pi(\frac{1}{2}-\epsilon)}\left(s-m_1^2+i\varrho\right)^{\frac{1}{2}-\epsilon} \\
&= -ie^{i\pi\epsilon}\left(s-m_1^2+i\varrho\right)^{\frac{1}{2}-\epsilon} \quad (12)
\end{aligned}$$

here $e^{\pm i\pi/2} = \pm\, i$.

Now using the formula

$$\int_0^\infty x^{\alpha-1}\prod_{i=1}^{k}\left(z_i+w_ix\right)^{-b_i}dx = B(\beta-\alpha,\alpha)\mathcal{R}_{\alpha-\beta}\left(b_1,\ldots,b_k;\frac{z_1}{w_1}\ldots,\frac{z_k}{w_k}\right)\prod_{i=1}^{k}w_i^{-b_i} \tag{13}$$

with $\beta = \sum_i b_i$.

Apply relations (13) to Eq.(11) with

$$\begin{aligned} &\alpha = \tfrac{1}{2} && w_i = 1 \\ &b_1 = \epsilon - \tfrac{1}{2} && b_2 = 1 \\ &\beta = \epsilon + \tfrac{1}{2} && \beta - \alpha = -\epsilon \end{aligned} \tag{14}$$

one obtains the result in terms of $\mathcal{R}$ function

$$\begin{aligned} B_0 &= \frac{-ie^{i\pi\epsilon}\pi^{\frac{D-1}{2}}\Gamma(\epsilon-\frac{1}{2})B(\epsilon,\frac{1}{2})}{2q_{10}} \\ &\left\{\left(\frac{q_{10}}{2}+M_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;-m_1^2+i\varrho,-\left(\frac{q_{10}}{2}+M_d\right)^2\right)\right. \\ &+\left.\left(\frac{q_{10}}{2}-M_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;-m_2^2+i\varrho,-\left(\frac{q_{10}}{2}-M_d\right)^2\right)\right\} \end{aligned} \tag{15}$$

or alternative representation which is using in the xloops-GiNaC code

$$\boxed{\begin{aligned} B_0 &= \frac{ie^{i\pi\epsilon}\pi^{\frac{D-1}{2}}\Gamma(\epsilon+\frac{1}{2})B(\epsilon,\frac{1}{2})}{2q_{10}(\frac{1}{2}-\epsilon)} \\ &\left\{\left(\frac{q_{10}}{2}+M_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;-m_1^2+i\varrho,-\left(\frac{q_{10}}{2}+M_d\right)^2\right)\right. \\ &+\left.\left(\frac{q_{10}}{2}-M_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;-m_2^2+i\varrho,-\left(\frac{q_{10}}{2}-M_d\right)^2\right)\right\} \end{aligned}} \tag{16}$$

Here we use the relation

$$\Gamma(\epsilon-\frac{1}{2}) = \frac{\Gamma(\epsilon+\frac{1}{2})}{\epsilon-\frac{1}{2}} \tag{17}$$

2.3 The space-like case $q^2 < 0$

When $q^2 < 0$, there exists a boost which transforms

$$q^\mu = (q_{10}, q_{11}, \vec{0}) \to (0, q_{11}, \vec{0}) \tag{18}$$

To reduce the number of parallel dimensions, we perform a Wick rotation $l_0 \to il_1$. Such a rotation is valid due to the fact that contour deformation does not cross any poles or cut.

In this frame of reference, and after Wick rotation, the integral reads

$$B_0 = i\,\frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \int\limits_{-\infty}^{\infty} dl_1 \int\limits_{0}^{\infty} dl_\perp \frac{l_\perp^{D-2}}{[(l_1+q_{11})^2 + l_\perp^2 + m_1^2 + i\varrho]\,[l_1^2 + l_\perp^2 + m_2^2 + i\varrho]} \tag{19}$$

Note that, there are two different points compare to Eq(6). First, there is an extra i in the coefficient. Second, the sign of $l_\perp^2$ and m_i^2 and $+i\varrho$. Now, partitioning the integral to obtains

$$\begin{aligned} B_0 &= \frac{2i\,\pi^{\frac{D-1}{2}}}{2q_{11}\Gamma\left(\frac{D-1}{2}\right)} \int\limits_{-\infty}^{\infty} dl_1 \int\limits_{0}^{\infty} dl_\perp \left\{ \frac{-l_\perp^{D-2}}{[(l_1+q_{11})^2 + l_\perp^2 + m_1^2 + i\varrho]\left[l_1 + \left(\frac{q_{11}}{2} + A_d\right)\right]} \right. \\ &\qquad \left. + \frac{l_\perp^{D-2}}{[l_1^2 + l_\perp^2 + m_2^2 + i\varrho]\left[l_1 + \left(\frac{q_{11}}{2} + A_d\right)\right]} \right\} \end{aligned} \tag{20}$$

here $A_d = \dfrac{m_1^2 - m_2^2}{2q_{11}}$. Shift $l_1 \to l_1 + q_{11}$

$$\begin{aligned} B_0 &= \frac{2i\,\pi^{\frac{D-1}{2}}}{2q_{11}\Gamma\left(\frac{D-1}{2}\right)} \int\limits_{-\infty}^{\infty} dl_1 \int\limits_{0}^{\infty} dl_\perp \left\{ \frac{-l_\perp^{D-2}}{[l_1^2 + l_\perp^2 + m_1^2 + i\varrho]\left[l_1 - \left(\frac{q_{11}}{2} - A_d\right)\right]} \right. \\ &\qquad \left. + \frac{l_\perp^{D-2}}{[l_1^2 + l_\perp^2 + m_2^2 + i\varrho]\left[l_1 + \left(\frac{q_{11}}{2} + A_d\right)\right]} \right\} \end{aligned}$$

Integrating over $l_\perp^2$ one obtains

$$\begin{aligned} B_0 &= \frac{i\,\pi^{\frac{D-1}{2}}\Gamma\left(\epsilon - \frac{1}{2}\right)}{2q_{11}} \int\limits_{-\infty}^{\infty} dl_1 \left\{ \frac{-[l_1^2 + m_1^2 + i\varrho]^{\frac{1}{2}-\epsilon}}{\left[l_1 - \left(\frac{q_{11}}{2} - A_d\right)\right]} + \frac{[l_1^2 + m_2^2 + i\varrho]^{\frac{1}{2}-\epsilon}}{\left[l_1 + \left(\frac{q_{11}}{2} + A_d\right)\right]} \right\} \\ &= -\frac{i\,\pi^{\frac{D-1}{2}}\Gamma\left(\epsilon - \frac{1}{2}\right)}{2q_{11}} \int\limits_{-\infty}^{\infty} dl_1 \left\{ \left(\frac{q_{11}}{2} - A_d\right) \frac{[l_1^2 + m_1^2 + i\varrho]^{\frac{1}{2}-\epsilon}}{\left[l_1^2 - \left(\frac{q_{11}}{2} - A_d\right)^2\right]} \right. \\ &\qquad \left. + \left(\frac{q_{11}}{2} + A_d\right) \frac{[l_1^2 + m_2^2 + i\varrho]^{\frac{1}{2}-\epsilon}}{\left[l_1^2 - \left(\frac{q_{11}}{2} + A_d\right)^2\right]} \right\} \\ &= \frac{-i\,\pi^{\frac{D-1}{2}}\Gamma\left(\epsilon - \frac{1}{2}\right)}{2q_{11}} \left\{ \left(\frac{q_{11}}{2} - A_d\right) \int\limits_{0}^{\infty} ds\, s^{\frac{1}{2}-1} \left[s + m_1^2 + i\varrho\right]^{\frac{1}{2}-\epsilon} \left[s - \left(\frac{q_{11}}{2} - A_d\right)^2\right]^{-1} \right. \\ &\qquad \left. + \left(\frac{q_{11}}{2} + A_d\right) \int\limits_{0}^{\infty} ds\, s^{\frac{1}{2}-1} \left[s + m_2^2 + i\varrho\right]^{\frac{1}{2}-\epsilon} \left[s - \left(\frac{q_{11}}{2} + A_d\right)^2\right]^{-1} \right\} \end{aligned} \tag{21}$$

The result in terms of $\mathcal{R}$ functions reads

$$\begin{aligned} B_0 &= \frac{-i\,\pi^{\frac{D-1}{2}}\Gamma\left(\epsilon-\frac{1}{2}\right)}{2q_{11}} B\left(\epsilon,\frac{1}{2}\right) \\ &\left\{\left(\frac{q_{11}}{2}-A_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;m_1^2+i\varrho,-(\frac{q_{11}}{2}-A_d)^2\right)\right. \\ &\left.+\left(\frac{q_{11}}{2}+A_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;m_2^2+i\varrho,-(\frac{q_{11}}{2}+A_d)^2\right)\right\} \end{aligned} \tag{22}$$

Or in the form

$$\boxed{\begin{aligned} B_0 &= \frac{i\,\pi^{\frac{D-1}{2}}\Gamma\left(\epsilon+\frac{1}{2}\right)}{2q_{11}(\frac{1}{2}-\epsilon)} B\left(\epsilon,\frac{1}{2}\right) \\ &\left\{\left(\frac{q_{11}}{2}-A_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;m_1^2+i\varrho,-(\frac{q_{11}}{2}-A_d)^2\right)\right. \\ &\left.+\left(\frac{q_{11}}{2}+A_d\right)\mathcal{R}_{-\epsilon}\left(\epsilon-\frac{1}{2},1;m_2^2+i\varrho,-(\frac{q_{11}}{2}+A_d)^2\right)\right\} \end{aligned}} \tag{23}$$

3 Scalar one-loop three-point integrals

Nothing yet

4 Scalar one-loop four-point integrals

We are not going to deal with IR-divergent so we will set $\epsilon = 0$ to work directly in $D = 4$ dimension

$$\begin{aligned} D_0 &= \frac{2\pi^{\frac{1}{2}-\epsilon}}{\Gamma(\frac{1}{2}-\epsilon)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_2 \int_{0}^{\infty} dl_\perp \\ &\frac{1}{[(l_0+q_{10})^2-l_1^2-l_2^2-l_\perp^2-m_1^2+i\varrho][(l_0+q_{20})^2-(l_1+q_{21})^2-l_2^2-l_\perp^2-m_2^2+i\varrho]} \\ &\frac{1}{[(l_0+q_{30})^2-(l_1+q_{31})^2-(l_2+q_{32})^2-l_\perp^2-m_3^2+i\varrho][l_0^2-l_1^2-l_2^2-l_\perp^2-m_4^2+i\varrho]} \\ &= \frac{2\pi^{\frac{1}{2}-\epsilon}}{\Gamma(\frac{1}{2}-\epsilon)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_2 \int_{0}^{\infty} dl_\perp \frac{1}{P_1P_2P_3P_4} \end{aligned} \tag{24}$$

Partitioning the integrand into the form

$$\frac{1}{P_1P_2P_3P_4} = \frac{1}{P_1(P_2-P_1)(P_3-P_1)(P_4-P_1)} + \frac{1}{P_2(P_1-P_2)(P_3-P_2)(P_4-P_2)} + \frac{1}{P_3(P_1-P_3)(P_2-P_3)(P_4-P_3)} + \frac{1}{P_4(P_1-P_4)(P_2-P_4)(P_3-P_4)}$$

where

$$P_i = (l_0+q_{i0})^2 - (l_1+q_{i1})^2 - (l_2+q_{i2})^2 - l_\perp^2 - m_i^2 + i\varrho$$

and one obtains

$$P_l - P_k = 2l_0(q_{l0}-q_{k0}) - 2l_1(q_{l1}-q_{k1}) - 2l_2(q_{l2}-q_{k2}) + (q_l^2-q_k^2) - (m_i^2-m_k^2) \quad (25)$$

then

$$\frac{1}{P_1P_2P_3P_4} = \sum_{k=1}^{4} \frac{1}{P_k \prod_{\substack{l=1 \\ l\neq k}}^{4} (P_l - P_k)}$$

Now shift on P_k by $l_i \to l_i + q_{ki}$ one obtains

$$D_0 = 2\int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} dl_2 \int_{0}^{\infty} dl_\perp \quad (26)$$
$$\sum_{k=1}^{4} \frac{1}{[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_k^2 + i\varrho] \prod_{\substack{l=1 \\ l\neq k}}^{4} [a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + d_{lk}]}$$

with

$$a_{lk} = 2(q_{l0}-q_{k0}) \quad , \quad b_{lk} = -2(q_{l1}-q_{k1}),$$
$$c_{lk} = -2(q_{l2}-q_{k2}) \quad , \quad d_{lk} = (q_l-q_k)^2 - (m_l^2-m_k^2).$$

Remark:

Only $d_{lk} \in \mathcal{C}$ and $Im[d_{lk}] = \Gamma_l - \Gamma_k \equiv \Delta\Gamma_{lk}$ with Γ_i is the width of particle i $(m_i^2 = \Re[m_i^2] - i\Gamma_i)$

4.1 Linearize and $x-$integration

To integrate over x, we linearize x by a transform

$$l_0 = x+z \quad , \quad l_2 = x$$
$$l_1 = y \quad , \quad l_\perp = t \quad (27)$$

then the Jacobian of this transformation is $\left|\dfrac{\partial(l_0, l_1, l_2, l_\perp)}{\partial(x, y, z, t)}\right| = 1$ and

$$D_0 = 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dx\, dy\, dz \int_{0}^{\infty} dt \frac{1}{[2xz + z^2 - y^2 - t^2 - m_k^2 + i\varrho]} \times \frac{1}{\prod\limits_{\substack{l=1\\ l\neq k}}^{4} [a_{lk}z + b_{lk}y + (a_{lk} + c_{lk})x + d_{lk}]} \tag{28}$$

We will integrate over x by closing the contour over upper plane when $z > 0$ and vise versa. To handle the cases where $AC_{lk} = a_{lk} + c_{lk} = 0$ and d_{lk} complex (complex masses case), we will perform the integration for the upper plane and its couterpart separately. We rewrite

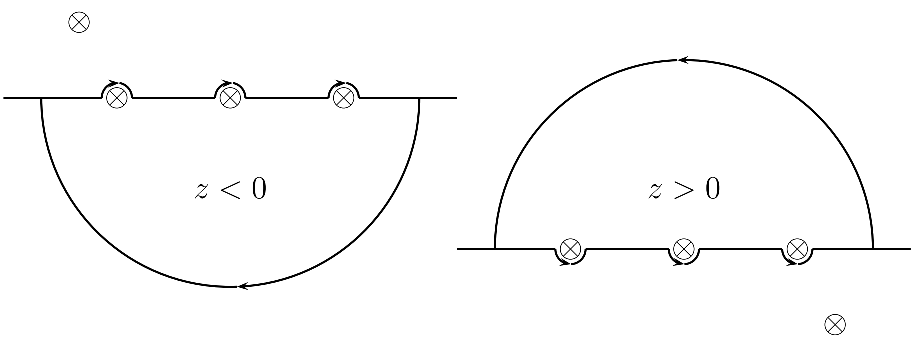
$$D_0 = D_0^+ + D_0^- \tag{29}$$

with

$$D_0^+ = 2\sum_{k=1}^{4}\int_{0}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dt \int_{-\infty}^{\infty} dx\, F(x, y, z, t; m_k, q_k) \tag{30}$$

$$D_0^- = 2\sum_{k=1}^{4}\int_{-\infty}^{0} dz \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dt \int_{-\infty}^{\infty} dx\, F(x, y, z, t; m_k, q_k)$$

For D_0^+ we close contour on the upper plane (a) and lower plane for D_0^- (b) as in the Fig(4.1)



(a) (b)

Only poles stay on the real axes and/or inside the contour will contribute to the integral. The Poles will stay on the real axes when all masses are real or their imaginary part equal. These two cases have difference at a factor 2.

$$x_l = -\left[\frac{a_{lk}}{a_{lk}+c_{lk}} z + \frac{b_{lk}}{a_{lk}+c_{lk}} y + \frac{d_{lk}}{a_{lk}+c_{lk}}\right] \tag{31}$$

Note that

$$Im[x_l] = -Im[\frac{d_{lk}}{AC_{lk}}] = -\frac{\Delta\Gamma_{lk}}{AC_{lk}} \tag{32}$$

so

$$D_0^+ = 2i\pi \sum_{k=1}^{4} \int_0^{\infty} dz \int_{-\infty}^{\infty} dy \int_0^{\infty} dt \sum_{\substack{l=1 \\ l \neq k}}^{4} [1-\delta_{lk}(AC_{lk})] f_{lk} \tag{33}$$

$$\frac{1}{[2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\varrho] \prod_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} [a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk}]}$$

$$D_0^- = -2i\pi \sum_{k=1}^{4} \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} dy \int_0^{\infty} dt \sum_{\substack{l=1 \\ l \neq k}}^{4} [1-\delta_{lk}(AC_{lk})] f_{lk}^- \tag{34}$$

$$\frac{1}{[2zx_l + z^2 - y^2 - t^2 - m_k^2 + i\varrho] \prod_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} [a_{mk}z + b_{mk}y + AC_{mk}x_l + d_{mk}]}$$

where

$$f_{lk} = \begin{cases} 0 & \text{if} \quad Im\left[-\frac{d_{lk}}{AC_{lk}}\right] < 0, \\ 1 & \text{if} \quad Im\left[-\frac{d_{lk}}{AC_{lk}}\right] = 0, \\ 2 & \text{if} \quad Im\left[-\frac{d_{lk}}{AC_{lk}}\right] > 0 \end{cases} \tag{35}$$

$$f_{lk}^- = \begin{cases} 0 & \text{if} \quad Im\left[-\frac{d_{lk}}{AC_{lk}}\right] > 0, \\ 1 & \text{if} \quad Im\left[-\frac{d_{lk}}{AC_{lk}}\right] = 0, \\ 2 & \text{if} \quad Im\left[-\frac{d_{lk}}{AC_{lk}}\right] < 0 \end{cases}$$

Note: The contributions of the poles stay on the real axes is proportional to $i\pi$ and not to $2i\pi$ due to the fact that the integrals along small contours contribute exactly a half of the residue.

Note2: The above result is correct up to two AC_{lk} vanish simultaneously. What if all three AC_{lk} vanish?

Recollect terms and rewrite

$$D_0^+ = 2i\pi \sum_{k=1}^{4} \int_{-\infty}^{\infty} dy \int_0^{\infty} dt\, dz \sum_{\substack{l=1\\ l\neq k}}^{4} \frac{[1-\delta_{lk}(AC_{lk})]AC_{lk}^{-1}\, f_{lk}}{\prod\limits_{\substack{m=1\\ m\neq l\\ m\neq k}}^{4} [A_{mlk}\, z + B_{mlk}\, y + C_{mlk}]} \tag{36}$$

$$\frac{1}{\left[\left(1-\frac{2a_{lk}}{AC_{lk}}\right) z^2 - 2yz\frac{b_{lk}}{AC_{lk}} - 2z\frac{d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\varrho\right]}$$

with

$$\begin{aligned} A_{mlk} &= a_{mk} - \frac{AC_{mk}a_{lk}}{AC_{lk}} \\ B_{mlk} &= b_{mk} - \frac{AC_{mk}b_{lk}}{AC_{lk}} \\ C_{mlk} &= d_{mk} - \frac{AC_{mk}d_{lk}}{AC_{lk}} \end{aligned} \tag{37}$$

4.2 $y-$integration

We first linearize y by a shift $t \to t + y$. To do that t^2 and y^2 must have opposite signs. That can be obtained by a complex rotation.

4.2.1 $t-$rotation

The integration wrt. t has 2 poles locate in the first and the third quarters, or the second and the fourth ones of complex plane, depends on the sign of $Im[2zx_l - m_k^2 + i\varrho]$

$$t = \pm\sqrt{z^2 + 2zx_l - y^2 - m_k^2 + i\varrho} \tag{38}$$

The sign of imaginary part of t depends on the sign of $(2\ z\ Im[-\frac{d_{lk}}{AC_lk}] + \Gamma_k + \varrho)$.

The locations of the t-poles for D^+ and D^- are specified in the Table(4.2.1) and Table(**??**).

From Eq.(36), D^+ vanishes when $Im[-\frac{d_{lk}}{AC_{lk}}] < 0$ and D^- vanishes when $Im[-\frac{d_{lk}}{AC_{lk}}] > 0$ thus t-poles always stay in the first and the third quarters of the complex plane for both D^+ and D^-.

		Poles in
$Im[-\frac{d_{lk}}{AC_{lk}}] \geq 0$	$0 \leq z$	1^{st} and 3^{rd} quarters
$Im[-\frac{d_{lk}}{AC_{lk}}] < 0$	$0 \leq z \leq -\frac{\Gamma_k+\varrho}{2Im[-\frac{d_{lk}}{AC_{lk}}]}$	1^{st} and 3^{rd} quarters
	$0 \leq -\frac{\Gamma_k+\varrho}{2Im[-\frac{d_{lk}}{AC_{lk}}]} < z$	2^{nd} and 4^{th} quarters

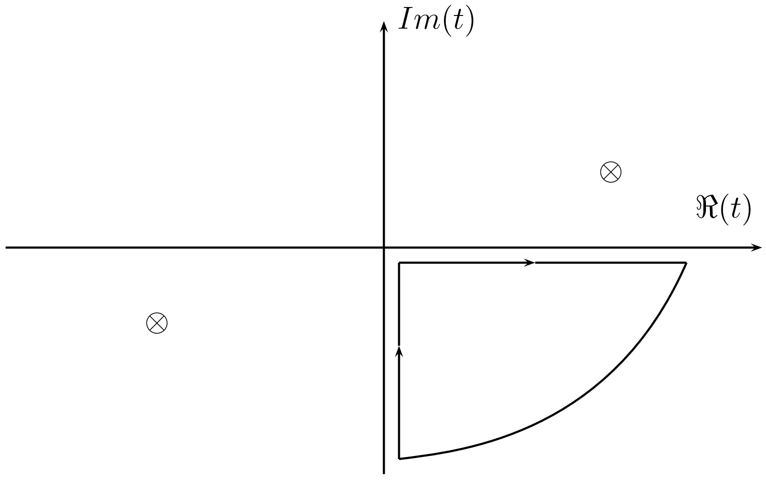
Table 1: t-poles location of D_0^+

		Poles in
$Im[-\frac{d_{lk}}{AC_{lk}}] \leq 0$	$z \leq 0$	1^{st} and 3^{rd} quarters
$Im[-\frac{d_{lk}}{AC_{lk}}] > 0$	$-\frac{\Gamma_k+\varrho}{2Im[-\frac{d_{lk}}{AC_{lk}}]} \leq z < 0$	1^{st} and 3^{rd} quarters
	$z \leq -\frac{\Gamma_k+\varrho}{2Im[-\frac{d_{lk}}{AC_{lk}}]} < 0$	2^{nd} and 4^{th} quarters

Table 2: t-poles location of D_0^-

We now close the contour on the fourth quarter of the $t-$complex plane to obtain

$$\int_0^\infty dt = \frac{1}{2}\int_{-\infty}^{\infty} dt = -\int_{-i\infty}^{0} dt = -\frac{1}{2}\int_{-i\infty}^{i\infty} dt \tag{39}$$



We now rotate $t \to it$ and obtains

$$D_0^+ = \pi \sum_{k=1}^{4} \int_0^{\infty} dz \int_{-\infty}^{\infty} dy\, dt \sum_{\substack{l=1 \\ l \neq k}}^{4} \frac{[1-\delta_{lk}(AC_{lk})] AC_{lk}^{-1}\, f_{lk}}{\prod_{\substack{m=1 \\ m \neq l \\ m \neq k}}^{4} [A_{mlk}\, z + B_{mlk}\, y + C_{mlk}]} \frac{1}{\left[\left(1-\frac{2a_{lk}}{AC_{lk}}\right) z^2 - 2yz\frac{b_{lk}}{AC_{lk}} - 2z\frac{d_{lk}}{AC_{lk}} - y^2 + t^2 - m_k^2 + i\varrho\right]} \tag{40}$$

with $Im[-\frac{d_{lk}}{AC_{lk}}] \geq 0$.

$$D_0^- = -\pi \sum_{k=1}^{4} \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} dy\, dt \sum_{\substack{l=1 \\ l \neq k}}^{4} \frac{[1-\delta_{lk}(AC_{lk})] AC_{lk}^{-1}\, f_{lk}^-}{\prod_{\substack{m=1 \\ m \neq l \\ m \neq k}}^{4} [A_{mlk}\, z + B_{mlk}\, y + C_{mlk}]} \frac{1}{\left[\left(1-\frac{2a_{lk}}{AC_{lk}}\right) z^2 - 2yz\frac{b_{lk}}{AC_{lk}} - 2z\frac{d_{lk}}{AC_{lk}} - y^2 + t^2 - m_k^2 + i\varrho\right]} \tag{41}$$

with $Im[-\frac{d_{lk}}{AC_{lk}}] \leq 0$.

4.2.2 Linearize y

We are going to linearize the integration wrt. y by perform a transformation $t \to t + y$, then

$$D_0^+ = +\pi \sum_{k=1}^{4} \int_0^{\infty} dz \int_{-\infty}^{\infty} dy\, dt \sum_{\substack{l=1 \\ l \neq k}}^{4} \frac{[1-\delta_{lk}(AC_{lk})] AC_{lk}^{-1}\, f_{lk}}{\prod_{\substack{m=1 \\ m \neq l \\ m \neq k}}^{4} [A_{mlk}\, z + B_{mlk}\, y + C_{mlk}]} \frac{1}{\left[2y\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1-\frac{2a_{lk}}{AC_{lk}}\right) z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\varrho\right]} \tag{42}$$

and

$$D_0^- = -\pi \sum_{k=1}^{4} \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} dy\, dt \sum_{\substack{l=1 \\ l \neq k}}^{4} \frac{[1-\delta_{lk}(AC_{lk})] AC_{lk}^{-1}\, f_{lk}^-}{\prod_{\substack{m=1 \\ m \neq l \\ m \neq k}}^{4} [A_{mlk}\, z + B_{mlk}\, y + C_{mlk}]} \frac{1}{\left[2y\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1-\frac{2a_{lk}}{AC_{lk}}\right) z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\varrho\right]}$$

4.2.3 Perform $y-$integration

The location of y-poles are more complicated than in the case of x-integration. Both two terms have poles on the complex plane. However acording to Eq.(36) and Table.(4.2.1), for both D_0^+ and D_0^-, the imaginary part of the second terms are always positive

$$Im\left[2y\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\varrho\right] \geq 0 \qquad (43)$$

The imaginary part of the first terms become complicated. It will contribute to the residue if it stays inside the contour which we are going to define by cutting the integration range of t as below.

Similar to the $x-$integration, we cut the integartion of t into two segments $t > \alpha_{lk}z$ and $t < \alpha_{lk}z$ with $\alpha_{lk} = \dfrac{b_{lk}}{a_{lk} + c_{lk}}$. Close the contour on the upper plane of y if $t > \alpha_{lk}z$ and vice versa one obtains

$$D_0^+ = D_0^{++} + D_0^{+-} \qquad (44)$$

with

$$\begin{aligned} D_0^{++} &= +i\pi^2 \sum_{k=1}^{4} \int_0^\infty dz \sum_{\substack{l=1 \\ l\neq k}}^{4} \int_{\alpha_{lk}z}^{\infty} dt\, [1 - \delta_{lk}(AC_{lk})]AC_{lk}^{-1}\, f_{lk} \\ &\quad \sum_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} \frac{[1 - \delta(B_{mlk})]\, g_{mlk}}{[A_{nlk}\, z + B_{nlk}\, y_{mlk} + C_{nlk}]} \\ &\quad \frac{1}{\left[2y_{mlk}\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\varrho\right]} \end{aligned} \qquad (45)$$

$$\begin{aligned} D_0^{+-} &= -i\pi^2 \sum_{k=1}^{4} \int_0^\infty dz \sum_{\substack{l=1 \\ l\neq k}}^{4} \int_{-\infty}^{\alpha_{lk}z} dt\, [1 - \delta_{lk}(AC_{lk})]AC_{lk}^{-1}\, f_{lk} \\ &\quad \sum_{\substack{m=1 \\ m\neq l \\ m\neq k}}^{4} \frac{[1 - \delta(B_{mlk})]\, g_{mlk}^{-}}{[A_{nlk}\, z + B_{nlk}\, y_{mlk} + C_{nlk}]} \\ &\quad \frac{1}{\left[2y_{mlk}\left(t - z\frac{b_{lk}}{AC_{lk}}\right) + \left(1 - \frac{2a_{lk}}{AC_{lk}}\right)z^2 - 2z\frac{d_{lk}}{AC_{lk}} + t^2 - m_k^2 + i\varrho\right]} \end{aligned} \qquad (46)$$

where $n \neq m \neq l \neq k$ and the poles

$$y_{mlk} = -\left[\frac{A_{mlk}z + C_{mlk}}{B_{mlk}}\right] \tag{47}$$

and

$$g_{mlk} = \begin{cases} 0 & \text{if} \quad Im\left[-\frac{C_{mlk}}{B_{mlk}}\right] < 0 \\ 1 & \text{if} \quad Im\left[-\frac{C_{mlk}}{B_{mlk}}\right] = 0 \\ 2 & \text{if} \quad Im\left[-\frac{C_{mlk}}{B_{mlk}}\right] > 0 \end{cases} \tag{48}$$

$$g^{-}_{mlk} = \begin{cases} 0 & \text{if} \quad Im\left[-\frac{C_{mlk}}{B_{mlk}}\right] > 0 \\ 1 & \text{if} \quad Im\left[-\frac{C_{mlk}}{B_{mlk}}\right] = 0 \\ 2 & \text{if} \quad Im\left[-\frac{C_{mlk}}{B_{mlk}}\right] < 0 \end{cases} \tag{49}$$

Note: The above result is correct up to one of the two B_{mlk} vanish simultaneously. What if both are vanish?

From now on

$$D_0 = D_0^{++} + D_0^{+-} + D_0^{-+} + D_0^{--} \tag{50}$$

Expanding and collecting terms one obtains

$$D_0^{++} = + \bigoplus_{nmlk} \int_0^{\infty} dz \int_{\alpha_{lk}z}^{\infty} dt \;\; f_{lk}\, g_{mlk}\, I'_{nmlk}(z,t)$$

$$D_0^{+-} = - \bigoplus_{nmlk} \int_0^{\infty} dz \int_{-\infty}^{\alpha_{lk}z} dt \;\; f_{lk}\, g^{-}_{mlk}\, I'_{nmlk}(z,t)$$

$$D_0^{-+} = - \bigoplus_{nmlk} \int_{-\infty}^{0} dz \int_{\alpha_{lk}z}^{\infty} dt \;\; f^{-}_{lk}\, g_{mlk}\, I'_{nmlk}(z,t)$$

$$D_0^{--} = + \bigoplus_{nmlk} \int_{-\infty}^{0} dz \int_{-\infty}^{\alpha_{lk}z} dt \; f^{-}_{lk}\, g^{-}_{mlk}\, I'_{nmlk}(z,t)$$

with

$$\bigoplus_{nmlk} = i\pi^2 \sum_{k=1}^{4} \sum_{\substack{l=1 \\ l \neq k}}^{4} \sum_{\substack{m=1 \\ m \neq l \\ m \neq k}}^{4} \frac{[1-\delta_{lk}(AC_{lk})][1-\delta(B_{mlk})]}{AC_{lk}[B_{mlk}A_{nlk} - B_{nlk}A_{mlk}]}$$

$$I'_{nmlk}(z,t) = \frac{1}{[z+F_{nmlk}]} \frac{1}{\left[D'_{mlk}\, z^2 + E'_{mlk}\, z - 2\frac{A_{mlk}}{B_{mlk}}\, t\, z - 2\frac{C_{mlk}}{B_{mlk}}\, t + t^2 - m_k^2 + i\varrho\right]}$$

where

$$\begin{aligned} a_{lk} &= 2(q_{l0} - q_{k0}) \quad , \quad b_{lk} = -2(q_{l1} - q_{k1}), \\ c_{lk} &= -2(q_{l2} - q_{k2}) \quad , \quad d_{lk} = (q_l - q_k)^2 - (m_l^2 - m_k^2). \end{aligned}$$

$$\begin{aligned} A_{mlk} &= a_{mk} - \tfrac{AC_{mk}a_{lk}}{AC_{lk}} \quad , \quad AC_{lk} = a_{lk} + c_{lk} \\ B_{mlk} &= b_{mk} - \tfrac{AC_{mk}b_{lk}}{AC_{lk}} \quad , \quad C_{mlk} = d_{mk} - \tfrac{AC_{mk}d_{lk}}{AC_{lk}} \end{aligned} \qquad (51)$$

$$\begin{aligned} D'_{mlk} &= 1 - \frac{2a_{lk}}{a_{lk}+c_{lk}} + \frac{2b_{lk}A_{mlk}}{(a_{lk}+c_{lk})B_{mlk}} \\ E'_{mlk} &= \frac{2b_{lk}C_{mlk}}{(a_{lk}+c_{lk})B_{mlk}} - \frac{2d_{lk}}{a_{lk}+c_{lk}} \\ F_{nmlk} &= \frac{C_{nlk}B_{mlk} - B_{nlk}C_{mlk}}{A_{nlk}B_{mlk} - B_{nlk}A_{mlk}} \end{aligned} \qquad (52)$$

Now we make a shift $t \to t' = t - \alpha_{lk} z$. The Jacobian is 1 and the $t-$integrals change the border to $[0, \pm\infty]$. The Integrals reads

$$D_0^{++} = + \bigoplus_{nmlk} \int_0^\infty dz \int_0^\infty dt \;\; f_{lk}\, g_{mlk}\, I_{nmlk}(z,t)$$

$$D_0^{+-} = - \bigoplus_{nmlk} \int_0^\infty dz \int_{-\infty}^0 dt \;\; f_{lk}\, g^-_{mlk}\, I_{nmlk}(z,t)$$

$$D_0^{-+} = - \bigoplus_{nmlk} \int_{-\infty}^0 dz \int_0^\infty dt \;\; f^-_{lk}\, g_{mlk}\, I_{nmlk}(z,t)$$

$$D_0^{--} = + \bigoplus_{nmlk} \int_{-\infty}^0 dz \int_{-\infty}^0 dt \; f^-_{lk}\, g^-_{mlk}\, I_{nmlk}(z,t)$$

with

$$I_{nmlk}(z,t) \;=\; \frac{1}{[z+F_{nmlk}]}\,\frac{1}{\left[D_{mlk}\;z^2-2\frac{d_{lk}}{AC_{lk}}\;z-2zt\left(\frac{A_{mlk}}{B_{mlk}}-\alpha_{lk}\right)\;-2\frac{C_{mlk}}{B_{mlk}}\;t+t^2-m_k^2+i\varrho\right]}$$

and

$$D_{mlk}=1-2\frac{a_{lk}}{AC_{lk}}+\alpha_{lk}^2=-4\frac{(q_l-q_k)^2}{AC_{lk}^2}=-4\frac{t_{lk}}{AC_{lk}^2}\quad\leq\;\;0 \tag{53}$$

4.3 Linearize t and $t-$integration

To linearize t, we perform a shift

$$\begin{array}{lcl} z=z'+\beta_{mlk}t' & & z'=\dfrac{z-\beta_{mlk}t}{1-\beta_{mlk}\varphi_{mlk}} \\ & \Longrightarrow & \\ t=t'+\varphi_{mlk}z' & & t'=\dfrac{t-\varphi_{mlk}z}{1-\beta_{mlk}\varphi_{mlk}} \end{array} \tag{54}$$

The jacobian of the shift is

$$\left\|\frac{\partial(z,t)}{\partial(z',t')}\right\|=|1-\beta_{mlk}\varphi_{mlk}| \tag{55}$$

Because $D_{mlk}\leq 0$ then we chose

$$\beta_{mlk}=\frac{\left(\frac{A_{mlk}}{B_{mlk}}-\alpha_{lk}\right)+\sqrt{\left(\frac{A_{mlk}}{B_{mlk}}-\alpha_{lk}\right)^2-D_{mlk}+i\eta}}{D_{mlk}}\leq 0\;, \qquad \eta\rightarrow 0 \tag{56}$$

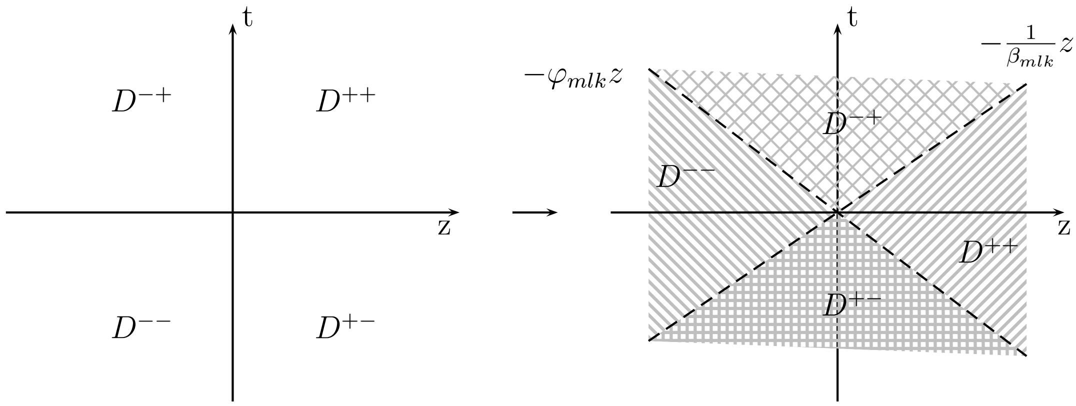
to remove the quadratic terms of t and obtain

$$I_{nmlk}(z,t)=\frac{1}{[z+\beta_{mlk}t+F_{nmlk}]\,[Q_{mlk}t+P_{mlk}tz+E_{mlk}z+Z_{mlk}z^2-m_k^2+i\varrho]} \tag{57}$$

with

$$
\begin{aligned}
Q_{mlk} &= -2\left(\frac{C_{mlk}}{B_{mlk}} + \frac{d_{lk}}{AC_{lk}}\beta_{mlk}\right) \\
P_{mlk} &= -2\left[\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)(1+\beta_{mlk}\varphi_{mlk}) - D_{mlk}\beta_{mlk} - \varphi_{mlk}\right] \\
E_{mlk} &= -2\left(\frac{d_{lk}}{AC_{lk}} + \frac{C_{mlk}}{B_{mlk}}\varphi_{mlk}\right) \\
Z_{mlk} &= D_{mlk} - 2\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)\varphi_{mlk} + \varphi^2_{mlk} \\
D_{mlk} &= -4\frac{(q_l-q_k)^2}{AC^2_{lk}} = -4\frac{s_{lk}}{AC^2_{lk}} \quad \leq \quad 0 \\
F_{nmlk} &= \frac{C_{nlk}B_{mlk} - B_{nlk}C_{mlk}}{A_{nlk}B_{mlk} - B_{nlk}A_{mlk}}
\end{aligned}
\tag{58}
$$

The integration region now looks



Here, we (will) chose $\varphi_{mlk} > 0$. The new integration regions is devided by the two lines

$$
\begin{aligned}
t &= -\frac{1}{\beta_{mlk}}z \\
t &= -\varphi_{mlk}z
\end{aligned}
\tag{59}
$$

Note: These two lines are determined by

$$t \;=\; 0 \quad\Longrightarrow\quad \begin{cases} z' &= \dfrac{z}{1-\beta_{mlk}\varphi_{mlk}} \\ t' &= -\dfrac{\varphi_{mlk}\;z}{1-\beta_{mlk}\varphi_{mlk}} \end{cases} \quad\Longrightarrow\quad t' = -\varphi_{mlk} z'$$

$$z \;=\; 0 \quad\Longrightarrow\quad \begin{cases} z' &= -\dfrac{\beta_{mlk}\;t}{1-\beta_{mlk}\varphi_{mlk}} \\ t' &= \dfrac{t}{1-\beta_{mlk}\varphi_{mlk}} \end{cases} \quad\Longrightarrow\quad t' = -\frac{1}{\beta_{mlk}} z'$$

And so

$$D^{++} \;\to \int_0^\infty dz \int_{-\varphi_{mlk}\,z}^{-1/\beta_{mlk}\,z} dt \quad\to\quad \int_0^\infty dz \int_{-\varphi_{mlk}\,z}^{\infty} dt - \int_0^\infty dz \int_{-1/\beta_{mlk}\,z}^{\infty} dt$$

$$D^{+-} \quad\to\quad \int_0^\infty dz \int_{-\infty}^{-\varphi_{mlk}\,z} dt + \int_{-\infty}^{0} dz \int_{-\infty}^{-1/\beta_{mlk}\,z} dt$$

$$D^{-+} \quad\to\quad \int_0^\infty dz \int_{-1/\beta_{mlk}\,z}^{\infty} dt + \int_{-\infty}^{0} dz \int_{-\varphi_{mlk}\,z}^{\infty} dt$$

$$D^{--} \;\to \int_{-\infty}^{0} dz \int_{-1/\beta_{mlk}\,z}^{-\varphi_{mlk}\,z} dt \quad\to\quad \int_{-\infty}^{0} dz \int_{-1/\beta_{mlk}\,z}^{\infty} dt - \int_{-\infty}^{0} dz \int_{-\varphi_{mlk}\,z}^{\infty} dt$$

We are going to use the following formular

$$\int_{-\infty}^{a} f(z)dz \;=\; +\sum_k Res\{f(z)\ln(z-a); z_k\} = \int_{-a}^{\infty} f(-z)dz \tag{60}$$

where $f(z)$ is a rational function with $deg(f) \le 2$ and has not poles on the negative real axes. z_k are poles of $f(z)$.

Prove:

Chose a cut along the negative real axis of z, close the contour around the cut then

$$
\begin{aligned}
\oint f(z)\ln(z) &= 2i\pi \sum_k Res\{f(z)\ln(z); z_k\} \\
&= \left\{ \int_{-\infty}^{0} + \int_{0}^{-\infty} + \int_{\Gamma_1} + \int_{C} \right\} f(z)\ln(z)dz \\
&= \int_{-\infty}^{0} f(x)\ln(x)dx + \int_{0}^{-\infty} f(xe^{-2i\pi})(\ln(x) - 2i\pi)dx \\
&= 2i\pi \int_{-\infty}^{0} f(z)dz
\end{aligned}
$$

To be more compact, we rewrite

$$
I_{nmlk}(t,z) = \frac{1}{\beta(Q+Pz)} \frac{1}{\left[\frac{Zz^2+Ez-m_k^2+i\varrho}{Q+Pz} - \frac{F+z}{\beta}\right]} \left\{ \frac{1}{t+\frac{F+z}{\beta}} - \frac{1}{t+\frac{Zz^2+Ez-m_k^2+i\varrho}{Q+Pz}} \right\}
$$

and perform $t-$integration to obtain

$$
\begin{aligned}
\int_{-\infty}^{\sigma z} dt\ I_{nmlk}(t,z) &= \frac{1}{\beta(Q+Pz)} \frac{1}{\left[\frac{Zz^2+Ez-m_k^2+i\varrho}{Q+Pz} - \frac{F+z}{\beta}\right]} \left\{ \ln\left(-\frac{z+\beta\sigma z+F}{\beta}\right) \right. \\
&\left. - \ln\left(-\frac{Zz^2+Ez-m_k^2+i\varrho+(Q+Pz)\sigma z}{Q+Pz}\right) \right\}
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
\int_{\sigma z}^{\infty} dt\ I_{nmlk}(t+\sigma,z) &= \frac{1}{\beta(Q+Pz)} \frac{1}{\left[\frac{Zz^2+Ez-m_k^2+i\varrho}{Q+Pz} - \frac{F+z}{\beta}\right]} \left\{ -\ln\left(\frac{z+\beta\sigma z+F}{\beta}\right) \right. \\
&\left. + \ln\left(\frac{Zz^2+Ez-m_k^2+i\varrho+(Q+Pz)\sigma z}{Q+Pz}\right) \right\}
\end{aligned} \tag{62}
$$

and

In Eq(67), for $\sigma = -\varphi_{mlk} z$ or $\sigma = -1/\beta_{mlk} z$ if we chose

$$
Z_{mlk} + P_{mlk}\sigma = 0 \tag{63}
$$

then

$$
\varphi_{mlk} = (\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}) - \sqrt{(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}) - D_{mlk} + i\eta} \tag{64}
$$

which makes the Jacobian of the shift vanish. Thus we can not linearize the argument of the ln functions as Franzkowski did. The best is to chose

$$Z_{mlk} = 0 \Longleftrightarrow \varphi = (\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}) + \sqrt{\left(\frac{A_{mlk}}{B_{mlk}} - \alpha_{lk}\right)^2 - D_{mlk} + i\eta} = \beta_{mlk} D_{mlk} \geq 0 \tag{65}$$

With this choice, we can rewrite

$$\int_{-\infty}^{\sigma z} dt\ I_{nmlk}(t,z) = \frac{1}{\beta(Q+Pz)} \frac{1}{\left[\frac{Ez-m_k^2+i\varrho}{Q+Pz} - \frac{F+z}{\beta}\right]} \left\{ \ln\left(-\frac{(1+\beta\sigma)z+F}{\beta}\right) - \ln\left(-\frac{P\sigma z^2 + (E+Q\sigma)z - m_k^2 + i\varrho}{Q+Pz}\right)\right\} \tag{66}$$

$$\int_{\sigma z}^{\infty} dt\ I_{nmlk}(t+\sigma,z) = \frac{1}{\beta(Q+Pz)} \frac{1}{\left[\frac{Ez-m_k^2+i\varrho}{Q+Pz} - \frac{F+z}{\beta}\right]} \left\{ -\ln\left(\frac{(1+\beta\sigma)z+F}{\beta}\right) + \ln\left(\frac{P\sigma z^2 + (E+Q\sigma)z - m_k^2 + i\varrho}{Q+Pz}\right)\right\} \tag{67}$$

We now obtain

$$\begin{aligned} D^{++} &= \bigoplus_{nmlk} f_{lk} g_{mlk} \int_0^\infty dz \left\{ \int_{-\varphi z}^{\infty} dt - \int_{-1/\beta z}^{\infty} dt \right\} I_{nmlk}(t,z) \qquad (68)\\ &= \bigoplus_{nmlk} f_{lk} g_{mlk} \int_0^\infty dz\ G(z) \left\{ -\ln\left(\frac{(1-\beta\varphi)z+F}{\beta}\right) + \ln\left(\frac{F}{\beta}\right) \right. \\ &\quad + \ln\left(\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz}\right) \\ &\quad \left. - \ln\left(\frac{-\frac{P}{\beta} z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
D^{+-} &= -\bigoplus_{nmlk} f_{lk} g^{-}_{mlk} \left\{ \int_0^{\infty} dz \int_{-\infty}^{-\varphi z} dt + \int_{-\infty}^{0} dz \int_{-\infty}^{-1/\beta z} dt \right\} I_{nmlk}(t,z) \\
&= -\bigoplus_{nmlk} f_{lk} g^{-}_{mlk} \left\{ \int_0^{\infty} dz\, G(z) \left[\ln\left(-\frac{(1-\beta\varphi)z+F}{\beta} \right) \right. \right. \\
&\qquad \left. - \ln\left(-\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz} \right) \right] + \\
&\qquad \left. \int_{-\infty}^{0} dz\, G(z) \left[\ln\left(-\frac{F}{\beta} \right) - \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz} \right) \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
D^{-+} &= -\bigoplus_{nmlk} f^{-}_{lk} g_{mlk} \left\{ \int_0^{\infty} dz \int_{-1/\beta z}^{\infty} dt + \int_{-\infty}^{0} dz \int_{-\varphi z}^{\infty} dt \right\} I_{nmlk}(t,z) \\
&= -\bigoplus_{nmlk} f^{-}_{lk} g_{mlk} \left\{ \int_0^{\infty} dz\, G(z) \left[-\ln\left(\frac{F}{\beta} \right) + \ln\left(\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz} \right) \right] + \right. \\
&\qquad \left. \int_{-\infty}^{0} dz\, G(z) \left[-\ln\left(\frac{(1-\beta\varphi)z+F}{\beta} \right) + \ln\left(\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz} \right) \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
D^{--} &= \bigoplus_{nmlk} f^{-}_{lk} g^{-}_{mlk} \int_{-\infty}^{0} dz \left\{ \int_{-1/\beta z}^{\infty} dt - \int_{-\varphi z}^{\infty} dt \right\} I_{nmlk}(t,z) \\
&= \bigoplus_{nmlk} f^{-}_{lk} g^{-}_{mlk} \int_{-\infty}^{0} dz\, G(z) \left\{ -\ln\left(\frac{F}{\beta} \right) + \ln\left(\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz} \right) \right. \\
&\qquad \left. + \ln\left(\frac{(1-\beta\varphi)z+F}{\beta} \right) - \ln\left(\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz} \right) \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
G(z) &= \frac{1}{\beta(Q+Pz)} \frac{1}{\left[\frac{Ez - m_k^2 + i\varrho}{Q+Pz} - \frac{F+z}{\beta} \right]} \\
&= +\frac{1}{\beta(Ez - m_k^2 + i\varrho) - (Q+Pz)(F+z)}
\end{aligned}
$$

Sum up the terms to obtain

$$
\begin{aligned}
D_0 \quad = \quad & \bigoplus_{nmlk} |1-\beta_{mlk}\varphi_{mlk}| \times \qquad (69) \\
& \Bigg[\quad \int_0^\infty dz\; G(z) \Bigg\{ (f_{lk}g_{mlk} + f_{lk}^- g_{mlk}) \ln\left(\frac{F}{\beta}\right) \\
& -f_{lk}g_{mlk} \ln\left(\frac{(1-\beta\varphi)z+F}{\beta}\right) - f_{lk}g_{mlk}^- \ln\left(-\frac{(1-\beta\varphi)z+F}{\beta}\right) \\
& -(f_{lk}g_{mlk} + f_{lk}^- g_{mlk}) \ln\left(\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& +f_{lk}g_{mlk} \ln\left(\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& +f_{lk}g_{mlk}^- \ln\left(-\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz}\right) \Bigg\} \\
& + \quad \int_{-\infty}^0 dz\; G(z) \Bigg\{ -f_{lk}^- g_{mlk}^- \ln\left(\frac{F}{\beta}\right) - f_{lk}g_{mlk}^- \ln\left(-\frac{F}{\beta}\right) \\
& +(f_{lk}^- g_{mlk}^- + f_{lk}^- g_{mlk}) \ln\left(\frac{(1-\beta\varphi)z+F}{\beta}\right) \\
& +f_{lk}^- g_{mlk}^- \ln\left(\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& +f_{lk}g_{mlk}^- \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E-\frac{Q}{\beta})z - m_k^2 + i\varrho}{Q+Pz}\right) \\
& -(f_{lk}^- g_{mlk}^- + f_{lk}^- g_{mlk}) \ln\left(\frac{-P\varphi z^2 + (E-Q\varphi)z - m_k^2 + i\varrho}{Q+Pz}\right) \Bigg\} \Bigg]
\end{aligned}
$$

Note: Up to now, this result is correct with many restrictions

- In the sum, there are **at most** 2 AC_{lk} and one B_{mlk} vanish. If more than that, we have to investigate those cases in more detailed.
- If P_{mlk} vanishes then the integral becomes simple. All log functions will be linearized in this simple case. But we need to work on detail.

4.4 $z-$integration

4.4.1 The case $P_{mlk} \neq 0$

Define

$$GZ(x, y, a) \equiv \frac{\ln(y-a) - \ln(x-a)}{(x-y)} \tag{70}$$

Using Eq.(60) we write

$$\int_{-\infty}^{a} \bar{G}(z)dz = \frac{\ln(T_2 - a) - \ln(T_1 - a)}{(T_1 - T_2)} = GZ(T_1, T_2, a) \tag{71}$$

and

$$\int_{a}^{\infty} \bar{G}(z)dz = \int_{-\infty}^{-a} \bar{G}(-z)dz = \frac{\ln(T_4 + a) - \ln(T_3 + a)}{(T_3 - T_4)} = GZ(T_3, T_4, -a) \tag{72}$$

where

$$T_1 = \frac{-(Q + FP - \beta E) + \sqrt{(Q + FP - \beta E)^2 - 4P(QF + \beta(m_k^2 - i\rho))}}{2P}$$
$$T_2 = \frac{-(Q + FP - \beta E) - \sqrt{(Q + FP - \beta E)^2 - 4P(QF + \beta(m_k^2 - i\rho))}}{2P}$$
$$T_3 = \frac{(Q + FP - \beta E) + \sqrt{(Q + FP - \beta E)^2 - 4P(QF + \beta(m_k^2 - i\rho))}}{2P}$$
$$T_4 = \frac{(Q + FP - \beta E) - \sqrt{(Q + FP - \beta E)^2 - 4P(QF + \beta(m_k^2 - i\rho))}}{2P}$$

Note that, T_1, T_2 are of $\bar{G}^{-1}(z) = 0$ and T_3, T_4 are zeros of $\bar{G}^{-1}(-z) = 0$. i.e

$$G(z) = \frac{-1}{P(z - T_1)(z - T_2)} \equiv \frac{\bar{G}^{-1}(z)}{P}$$
$$G(-z) = \frac{-1}{P(z - T_3)(z - T_4)} \equiv \frac{\bar{G}^{-1}(-z)}{P}$$

We rewrite Eq.(69)

$$
\begin{aligned}
D_0 \quad = \quad & \bigoplus_{nmlk} \frac{|1-\beta_{mlk}\varphi_{mlk}|}{P_{mlk}} \times \qquad (73) \\
& \Bigg[(f_{lk} g_{mlk} + f^-_{lk} g_{mlk}) \ln\left(\frac{F}{\beta}\right) GZ(T_3, T_4; 0) \\
& - \left(f^-_{lk} g^-_{mlk} \ln\left(\frac{F}{\beta}\right) + f_{lk} g^-_{mlk} \ln\left(-\frac{F}{\beta}\right)\right) GZ(T_1, T_2; 0) \\
& + \int\limits_0^\infty dz\ \bar{G}(z) \Bigg\{ - f_{lk} g_{mlk} \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) - f_{lk} g^-_{mlk} \ln\left(-\ \frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& -(f_{lk} g_{mlk} + f^-_{lk} g_{mlk}) \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& + f_{lk} g_{mlk} \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& + f_{lk} g^-_{mlk} \ln\left(-\ \frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right)\Bigg\} \\
& + \qquad \int\limits_{-\infty}^0 dz\ \bar{G}(z) \Bigg\{ (f^-_{lk} g^-_{mlk} + f^-_{lk} g_{mlk}) \ln\left(\frac{(1-\beta\varphi)z + F}{\beta}\right) \\
& + f^-_{lk} g^-_{mlk} \ln\left(\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& + f_{lk} g^-_{mlk} \ln\left(-\frac{-\frac{P}{\beta}z^2 + (E - \frac{Q}{\beta})z - m_k^2 + i\varrho}{Q + Pz}\right) \\
& -(f^-_{lk} g^-_{mlk} + f^-_{lk} g_{mlk}) \ln\left(\frac{-P\varphi z^2 + (E - Q\varphi)z - m_k^2 + i\varrho}{Q + Pz}\right)\Bigg\} \Bigg]
\end{aligned}
$$

Now we turn to the integrals of the form

$$\mathcal{L}^{+}(a,b) \equiv \int_0^\infty \bar{G}(z)\ln(az+b)dz \tag{74}$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\infty [1-(az+b)^{-\varepsilon}]\bar{G}(z)dz$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_0^\infty \bar{G}(z)dz + \int_0^\infty z^{(1-1)}(az+b)^{-\varepsilon}(z-T_1)^{-1}(z-T_2)^{-1}dz\right]$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[GZ(T_3,T_4;0) + B(1+\varepsilon,1)a^{-\varepsilon}\mathcal{R}_{-\varepsilon-1}(1,1,\varepsilon;-T_1,-T_2,\frac{b}{a})\right]$$

And

$$\mathcal{L}^{-}(a,b) \equiv \int_{-\infty}^0 \bar{G}(z)\ln(az+b)dz \tag{75}$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\infty}^0 [1-(az+b)^{-\varepsilon}]\bar{G}(z)dz$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{-\infty}^0 \bar{G}(z)dz - \int_0^\infty (b-az)^{-\varepsilon}\bar{G}(-z)dz\right]$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[GZ(T_1,T_2;0) + B(1+\varepsilon,1)(-a)^{-\varepsilon}\mathcal{R}_{-\varepsilon-1}(1,1,\varepsilon;-T_3,-T_4,-\frac{b}{a})\right]$$

Here we used

$$\ln(z) = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(z^{-\varepsilon}-1) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(z^{\varepsilon}-1)$$
$$\int_0^\infty x^{\alpha-1}\prod_{i=1}^{k}(z_i+w_ix)^{-b_i}dx = B(\beta-\alpha,\alpha)\mathcal{R}_{\alpha-\beta}\left(b_1\ldots,b_k;\frac{z_1}{w_1}\ldots,\frac{z_k}{w_k}\right)\prod_{i=1}^{k}w_i^{-b_i}$$

with $\beta=\sum_1^k b_i$, and $(z_i+w_ix)\neq 0$ with $\forall x\in \mathrm{R}_+$.

Once again, we rewrite

$$
\begin{aligned}
D_0 \quad = \quad & \bigoplus_{nmlk} \frac{|1-\beta_{mlk}\varphi_{mlk}|}{P_{mlk}} \times \qquad (76) \\
& \Bigg[(f_{lk}+f^-_{lk}) g_{mlk} \ln\left(\frac{F}{\beta}\right) GZ(T_3,T_4;0) \\
& -g^-_{mlk}\left(f^-_{lk}\ln\left(\frac{F}{\beta}\right)+f_{lk}\ln\left(-\frac{F}{\beta}\right)\right) GZ(T_1,T_2;0) \\
& -f_{lk}g_{mlk}\ \mathcal{L}^+\left(\frac{1-\beta\phi}{\beta},\frac{F}{\beta}\right) - f_{lk}g^-_{mlk}\ \mathcal{L}^+\left(-\frac{1-\beta\phi}{\beta},-\frac{F}{\beta}\right) \\
& +f^-_{lk}(g_{mlk}+g^-_{mlk})\ \mathcal{L}^-\left(\frac{1-\beta\phi}{\beta},\frac{F}{\beta}\right) \\
& +\int_0^\infty dz\ \bar{G}(z)\Bigg\{-(f_{lk}g_{mlk}+f^-_{lk}g_{mlk})\ln\left(\frac{-\frac{P}{\beta}z^2+(E-\frac{Q}{\beta})z-m_k^2+i\varrho}{Q+Pz}\right) \\
& +f_{lk}g_{mlk}\ln\left(\frac{-P\varphi z^2+(E-Q\varphi)z-m_k^2+i\varrho}{Q+Pz}\right) \\
& +f_{lk}g^-_{mlk}\ln\left(-\frac{-P\varphi z^2+(E-Q\varphi)z-m_k^2+i\varrho}{Q+Pz}\right)\Bigg\} \\
& + \qquad \int_{-\infty}^0 dz\ \bar{G}(z)\Bigg\{f^-_{lk}g^-_{mlk}\ln\left(\frac{-\frac{P}{\beta}z^2+(E-\frac{Q}{\beta})z-m_k^2+i\varrho}{Q+Pz}\right) \\
& +f_{lk}g^-_{mlk}\ln\left(-\frac{-\frac{P}{\beta}z^2+(E-\frac{Q}{\beta})z-m_k^2+i\varrho}{Q+Pz}\right) \\
& -(f^-_{lk}g^-_{mlk}+f^-_{lk}g_{mlk})\ln\left(\frac{-P\varphi z^2+(E-Q\varphi)z-m_k^2+i\varrho}{Q+Pz}\right)\Bigg\}\ \Bigg]
\end{aligned}
$$

Note: Maybe usefull for the next steps.

$$
\begin{aligned}
Im\{Q\} &= \begin{cases} \geq 0 & \text{for} \quad ++ \\ ?? & \text{for} \quad +- \\ ?? & \text{for} \quad -+ \\ \leq 0 & \text{for} \quad -- \end{cases} \\
Im\{E - Q\varphi\} &= \begin{cases} \geq 0 & \text{for} \quad ++, +- \\ \leq 0 & \text{for} \quad -+, -- \end{cases} \\
Im\{E - Q/\beta\} &= \begin{cases} \geq 0 & \text{for} \quad ++, -+ \\ \leq 0 & \text{for} \quad +-, -- \end{cases}
\end{aligned} \tag{77}
$$

Important remark, from Eq(77) and Eq(69)

$$Im\left\{P\sigma z^2 + (E + Q\sigma)z - m_k^2 + i\varrho\right\} \geq 0 \tag{78}$$

with any z and σ cases.

The remain non-linearized logarithms in Eq(69) are of the form

$$\ln\left(\frac{S(\sigma, z)}{\pm(Pz + Q)}\right) \tag{79}$$

where

$$
\begin{aligned}
S(\sigma, z) &= P\sigma z^2 + (E + Q\sigma)z - m_k^2 + i\varrho && (80) \\
&= P\sigma(z - Z_1)(z - Z_2) && (81)
\end{aligned}
$$

and $\sigma = -\varphi$ or $\sigma = -1/\beta$. and

$$
\begin{aligned}
Z_1 &= \frac{-(E + Q\sigma) + \sqrt{(E + Q\sigma)^2 + 4P\sigma(m_k^2 - i\rho)}}{2P\sigma} \\
Z_2 &= \frac{-(E + Q\sigma) - \sqrt{(E + Q\sigma)^2 + 4P\sigma(m_k^2 - i\rho)}}{2P\sigma}
\end{aligned} \tag{82}
$$

Using the relation

$$\ln(ab) = log(a) + log(b) + \eta(a, b) \tag{83}$$

We can rewrite

$$\begin{aligned}\ln\left(\frac{S(\sigma,z)}{\pm(Pz+Q)}\right) &= \ln(S(\sigma,z)) + \ln\left(\frac{1}{\pm(Pz+Q)}\right) + \eta\left(S(\sigma,z), \pm(Pz+Q)^{-1}\right) \\ &= \ln(P\sigma(z-Z_1)) + \ln(z-Z_2) + \ln\left(\frac{1}{\pm(Pz+Q)}\right) \\ &+ \eta\left(S(\sigma,z), \pm(Pz+Q)^{-1}\right) + \eta(P\sigma(z-Z_1), z-Z_2)\end{aligned} \tag{84}$$

where η-functions is defined as following

η- and θ-functions

- We define the step function as following

$$\theta(x) = \begin{cases} 1 \quad , & x \geq 0 \\ 0 \quad , & x < 0. \end{cases} \tag{85}$$

  With this definition of $\theta-$function, we define the $\eta(a, b)$ as following

- For $\forall a \in \mathcal{C}$ and $\forall b \in \mathcal{C}$

  $$\eta(a,b) = \quad 2i\pi \left\{\theta[-Im(a)]\ \theta[-Im(b)]\ \theta[Im(ab)] - \theta[Im(a)]\ \theta[Im(b)]\ \theta[-Im(ab)]\right\}$$

  In this case, when $Im(ab) = 0$ then $\theta[\pm\ Im(ab)] = \theta[0] = 1$

- Special case, where $a \in \mathcal{R}$ and $b \in \mathcal{C}$ then

$$\eta(a,b) = \begin{cases} 0 & , \quad a > 0 \\ -2i\pi\ \theta[Im(b)] & , \quad a < 0 \end{cases} \tag{86}$$

  Do keep in mind that, when $a < 0$ and $Im(b) < 0$, $\eta(a, b) = 0$

- When $Im(a) = 0$ and $Im(b) = 0$, a and b must be $\in \mathcal{R}^+$ and $\eta(a, b) = 0$, else $\eta(a, b)$ is undefined.

Taking into account that, $Im[Z_i] \neq 0$ in any cases of D_0 (i.e in both complex and real masses problem) then we obtain

$$\begin{aligned}
\eta(P\sigma(z-Z_1), z-Z_2) &= 2i\pi\, \theta\left(-Im[P\sigma(z-Z_1)]\right)\theta\left(-Im[z-Z_2]\right) \\
&= 2i\pi\, \theta\left(Im[P\sigma Z_1]\right)\ \theta\left(Im[Z_2]\right) \\
&\equiv Eta(P\sigma Z_1, Z_2; \sigma)
\end{aligned}$$

For the other $\eta-$function, we consider two cases

1. $\underline{Im[Q] = 0}$:
   In this case, $(Pz+Q) \in \mathcal{R}$ and $Im[S(\sigma, z)] > 0$, using Eq.(86) we obtain

$$\eta\left(S(\sigma,z), \pm(Pz+Q)^{-1}\right) = \begin{cases} 0 & \text{if} \quad \pm(Pz+Q) > 0 \\ -2i\pi & \text{if} \quad \pm(Pz+Q) < 0 \end{cases} \tag{87}$$

   so

$$\eta\left(S(\sigma,z), \pm(Pz+Q)^{-1}\right) = -2i\pi\ \theta[\mp(Pz+Q)] \tag{88}$$

   What if $(Pz+Q) = 0$? We have to investigate this case. However, it seems not to happen.

2. $\underline{Im[Q] \neq 0}$:

$$\begin{aligned}
\eta\left(S(\sigma,z), \pm(Pz+Q)^{-1}\right) &= -2i\pi\theta\left(Im[\pm(Pz+Q)^{-1}]\right)\theta\left(-Im\left[\frac{S(\sigma,z)}{\pm(Pz+Q)}\right]\right) \\
&= -2i\pi\theta\left(Im[\pm Q^*]\right)\theta\left(-Im\left[\frac{S(\sigma,z)}{\pm(Pz+Q)}\right]\right)
\end{aligned}$$

   we have to work a bit more on θ-function. Define

$$\theta\left(\mp \mathsf{H}(z)\right) \equiv \theta\left(-Im\left[\frac{S(\sigma,z)}{\pm(Pz+Q)}\right]\right) = \theta\left(\mp Im\left[S(\sigma,z)(Pz+Q^*)\right]\right) \tag{89}$$

   where

$$\mathsf{H}(z) \equiv Im[E_{mlk}P_{mlk}]z^2 + Im[(Q^*E - P(m_k^2 - i\rho))]z - Im[Q^*(m_k^2 - i\rho)]$$

   Note: $\eta\left(S(\sigma,z), \pm(Pz+Q)^{-1}\right)$ does not depend on σ. From now on, we drop σ in S

Define

$$\Delta = (Im[Q^*E - P(m_k^2 - i\rho)])^2 + 4Im[E_{mlk}P_{mlk}]Im[Q^*(m_k^2 - i\rho)] \tag{90}$$

and

$$X_1 = \frac{-Im[Q^*E - P(m_k^2 - i\rho)] + \sqrt{\Delta}}{2Im[E_{mlk}P_{mlk}]}$$
$$X_2 = \frac{-Im[Q^*E - P(m_k^2 - i\rho)] - \sqrt{\Delta}}{2Im[E_{mlk}P_{mlk}]}$$

One obtains

$$\eta\left(S(z), \pm\,(Pz+Q)^{-1}\right) = -2i\pi\theta\left(Im[\pm\,Q^*]\right)\theta\left(\mp\mathsf{H}(z)\right) \tag{91}$$

(a) If $Im[E_{mlk}P_{mlk}] > 0$, then $X_1 - X_2 > 0$

$$\theta\left(-\mathsf{H}(z)\right) = 1 \Longleftrightarrow z \in [X_2, X_1], \qquad \text{and} \;\; \Delta \geq 0 \tag{92}$$

$$\theta\left(\mathsf{H}(z)\right) = 1 \Longleftrightarrow \begin{cases} z \notin [X_2, X_1] & , \quad \Delta > 0 \\ \forall z & , \quad \Delta \leq 0 \end{cases} \tag{93}$$

(b) If $Im[E_{mlk}P_{mlk}] < 0$:

$$\theta\left(-\mathsf{H}(z)\right) = 1 \Longleftrightarrow \begin{cases} z \notin [X_1, X_2] & , \quad \Delta > 0 \\ \forall z & , \quad \Delta \leq 0 \end{cases} \tag{94}$$

$$\theta\left(\mathsf{H}(z)\right) = 1 \Longleftrightarrow z \in [X_1, X_2], \qquad \text{and} \;\; \Delta \geq 0 \tag{95}$$

(c) If $Im[E_{mlk}P_{mlk}] = 0$ and $Im[Q^*E - P(m_k^2 - i\rho)] \neq 0$:

- if $Im[Q^*E - P(m_k^2 - i\rho)] > 0$

$$\theta\left(-\mathsf{H}(z)\right) = 1 \Longleftrightarrow z \leq \frac{Im[Q^*(m_k^2 - i\rho)]}{Im[Q^*E - P(m_k^2 - i\rho)]} \tag{96}$$

$$\theta\left(\mathsf{H}(z)\right) = 1 \Longleftrightarrow z \geq \frac{Im[Q^*(m_k^2 - i\rho)]}{Im[Q^*E - P(m_k^2 - i\rho)]} \tag{97}$$

- if $Im[Q^*E - P(m_k^2 - i\rho)] < 0$

$$\theta\left(-\mathsf{H}(z)\right) = 1 \Longleftrightarrow z \geq \frac{Im[Q^*(m_k^2 - i\rho)]}{Im[Q^*E - P(m_k^2 - i\rho)]} \tag{98}$$

$$\theta\left(\mathsf{H}(z)\right) = 1 \Longleftrightarrow z \leq \frac{Im[Q^*(m_k^2 - i\rho)]}{Im[Q^*E - P(m_k^2 - i\rho)]} \tag{99}$$

(d) If $Im[E_{mlk}P_{mlk}] = 0$ and $Im[Q^*E - P(m_k^2 - i\rho)] = 0$:

$$\eta\left(S(z), \pm(Pz + Q)^{-1}\right) = -2i\pi\theta\left(Im[\pm\, Q^*]\right)\theta\left(\pm Im[Q^*(m_k^2 - i\rho)]\right) \tag{100}$$

Now, put all back to the main formula to obtain

$$
\begin{aligned}
D_0 \quad = \quad & \bigoplus_{nmlk} \frac{|1-\beta_{mlk}\varphi_{mlk}|}{P_{mlk}} \times \qquad (101) \\
& \Bigg\{ (f_{lk}+f_{lk}^{-}) g_{mlk} \ln\left(\frac{F}{\beta}\right) GZ(T_3,T_4;0) \\
& - g_{mlk}^{-} \left(f_{lk}^{-} \ln\left(\frac{F}{\beta}\right) + f_{lk} \ln\left(-\frac{F}{\beta}\right)\right) GZ(T_1,T_2;0) \\
& - f_{lk} g_{mlk}\, \mathcal{L}^{+}\left(\frac{1-\beta\phi}{\beta}, \frac{F}{\beta}\right) - f_{lk} g_{mlk}^{-}\, \mathcal{L}^{+}\left(-\frac{1-\beta\phi}{\beta}, -\frac{F}{\beta}\right) \\
& + f_{lk}^{-} (g_{mlk}+g_{mlk}^{-})\, \mathcal{L}^{-}\left(\frac{1-\beta\phi}{\beta}, \frac{F}{\beta}\right) \\
& - (f_{lk}+f_{lk}^{-}) g_{mlk}\, \left\{ \mathcal{L}^{+}\left(P\sigma, -P\sigma Z_1|_\beta\right) + \mathcal{L}^{+}\left(1, -Z_2|_\beta\right) \right\} \\
& + f_{lk} (g_{mlk}+g_{mlk}^{-})\, \left\{ \mathcal{L}^{+}\left(P\sigma, -P\sigma Z_1|_\phi\right) + \mathcal{L}^{+}\left(1, -Z_2|_\phi\right) \right\} \\
& + f_{lk}^{-} g_{mlk}\, \left\{ \mathcal{L}^{+}(P,Q) + \mathcal{L}^{-}(P,Q) \right\} - f_{lk} g_{mlk}^{-}\, \left\{ \mathcal{L}^{+}(-P,-Q) + \mathcal{L}^{-}(-P,-Q) \right\} \\
& + (f_{lk}+f_{lk}^{-}) g_{mlk}^{-}\, \left\{ \mathcal{L}^{-}\left(P\sigma, -P\sigma Z_1|_\beta\right) + \mathcal{L}^{-}\left(1, -Z_2|_\beta\right) \right\} \\
& - f_{lk}^{-} (g_{mlk}+g_{mlk}^{-})\, \left\{ \mathcal{L}^{-}\left(P\sigma, -P\sigma Z_1|_\phi\right) + \mathcal{L}^{-}\left(1, -Z_2|_\phi\right) \right\} \\
& + \int_0^\infty dz\, \bar{G}(z) \Bigg\{ -(f_{lk}+f_{lk}^{-}) g_{mlk}\, \eta\left(S(z), \frac{1}{Pz+Q}\right) \\
& - (f_{lk}+f_{lk}^{-}) g_{mlk}\, \mathit{Eta}\left(P\sigma Z_1|_\beta, Z_2|_\beta\right) + f_{lk}(g_{mlk}+g_{mlk}^{-})\, \mathit{Eta}\left(P\sigma Z_1|_\phi, Z_2|_\phi\right) \\
& + f_{lk} g_{mlk}\, \eta\left(S(z), \frac{1}{Pz+Q}\right) + f_{lk} g_{mlk}^{-}\, \eta\left(S(z), -\frac{1}{Pz+Q}\right) \Bigg\} \\
& + \int_{-\infty}^0 dz\, \bar{G}(z) \Bigg\{ f_{lk}^{-} g_{mlk}^{-}\, \eta\left(S(z), \frac{1}{Pz+Q}\right) \\
& + (f_{lk}+f_{lk}^{-}) g_{mlk}^{-}\, \mathit{Eta}\left(P\sigma Z_1|_\beta, Z_2|_\beta\right) - f_{lk}^{-}(g_{mlk}+g_{mlk}^{-})\, \mathit{Eta}\left(P\sigma Z_1|_\phi, Z_2|_\phi\right) \\
& + f_{lk} g_{mlk}^{-}\, \eta\left(S(z), -\frac{1}{Pz+Q}\right) - f_{lk}^{-}(g_{mlk}+g_{mlk}^{-})\, \eta\left(S(z), \frac{1}{Pz+Q}\right) \Bigg\} \Bigg\}
\end{aligned}
$$

Note that, $\eta\left(P\sigma z - P\sigma Z_1|_\beta, z - Z_2|_\beta\right)$ does not depend on z so we can rewrite

$$
\begin{aligned}
D_0 \;=\; & \bigoplus_{nmlk} \frac{|1-\beta_{mlk}\varphi_{mlk}|}{P_{mlk}} \times \qquad (102) \\
& \Bigg\{ \; (f_{lk}+f_{lk}^{-})g_{mlk}\left(\ln\left(\frac{F}{\beta}\right) - Eta\left(P\sigma Z_1, Z_2;\beta\right)\right)\; GZ(T_3,T_4;0) \\
& + f_{lk}(g_{mlk}+g_{mlk}^{-})\; Eta\left(P\sigma Z_1, Z_2;\phi\right)\; GZ(T_3,T_4;0) \\
& + g_{mlk}^{-}\left[(f_{lk}+f_{lk}^{-})\; Eta\left(P\sigma Z_1, Z_2;\beta\right) - f_{lk}^{-}\ln\left(\frac{F}{\beta}\right) - f_{lk}\ln\left(-\frac{F}{\beta}\right)\right] GZ(T_1,T_2;0) \\
& - f_{lk}^{-}(g_{mlk}+g_{mlk}^{-})\; Eta\left(P\sigma Z_1, Z_2;\phi\right) GZ(T_1,T_2;0) \\
& - f_{lk}g_{mlk}\;\mathcal{L}^{+}\left(\frac{1-\beta\phi}{\beta},\frac{F}{\beta}\right) - f_{lk}g_{mlk}^{-}\;\mathcal{L}^{+}\left(-\frac{1-\beta\phi}{\beta},-\frac{F}{\beta}\right) \\
& + f_{lk}^{-}(g_{mlk}+g_{mlk}^{-})\;\mathcal{L}^{-}\left(\frac{1-\beta\phi}{\beta},\frac{F}{\beta}\right) \\
& - (f_{lk}+f_{lk}^{-})g_{mlk}\;\left\{\mathcal{L}^{+}\left(P\sigma,-P\sigma Z_1|_\beta\right)+\mathcal{L}^{+}\left(1,-Z_2|_\beta\right)\right\} \\
& + f_{lk}(g_{mlk}+g_{mlk}^{-})\;\left\{\mathcal{L}^{+}\left(P\sigma,-P\sigma Z_1|_\phi\right)+\mathcal{L}^{+}\left(1,-Z_2|_\phi\right)\right\} \\
& + f_{lk}^{-}g_{mlk}\;\left\{\mathcal{L}^{+}\left(P,Q\right)+\mathcal{L}^{-}\left(P,Q\right)\right\} - f_{lk}g_{mlk}^{-}\;\left\{\mathcal{L}^{+}\left(-P,-Q\right)+\mathcal{L}^{-}\left(-P,-Q\right)\right\} \\
& + (f_{lk}+f_{lk}^{-})g_{mlk}^{-}\;\left\{\mathcal{L}^{-}\left(P\sigma,-P\sigma Z_1|_\beta\right)+\mathcal{L}^{-}\left(1,-Z_2|_\beta\right)\right\} \\
& - f_{lk}^{-}(g_{mlk}+g_{mlk}^{-})\;\left\{\mathcal{L}^{-}\left(P\sigma,-P\sigma Z_1|_\phi\right)+\mathcal{L}^{-}\left(1,-Z_2|_\phi\right)\right\} \\
& + \int_0^\infty dz\;\bar{G}(z)\left\{f_{lk}g_{mlk}^{-}\;\eta\left(S(z),-\frac{1}{Pz+Q}\right) - f_{lk}^{-}g_{mlk}\;\eta\left(S(z),\frac{1}{Pz+Q}\right)\right\} \\
& + \int_{-\infty}^0 dz\;\bar{G}(z)\left\{f_{lk}g_{mlk}^{-}\;\eta\left(S(z),-\frac{1}{Pz+Q}\right) - f_{lk}^{-}g_{mlk}\;\eta\left(S(z),\frac{1}{Pz+Q}\right)\right\}\;\Bigg\}
\end{aligned}
$$

This section is a detailed description of the GSP() and GSM() functions

1. First, consider the case of GSP(+)

$$GSP^+ \equiv \int_0^\infty \bar{G}(z)\eta\left(S(z),(Pz+Q)^{-1}\right) \tag{103}$$

(a) If $Im[Q]=0$

$$GSP^+ = -2i\pi \int_0^\infty \bar{G}(z)\theta[-(Pz+Q)] \tag{104}$$

$GSP^+ \neq 0$ only if $Pz+Q \leq 0$, i.e.

$$z \leq -Q/P \;\text{ when }\; P>0 \;\text{ or}$$
$$z \geq -Q/P \;\text{ when }\; P<0$$

i. If $P>0$ and $-Q/P \leq 0$:

$$GSP^+ = 0 \tag{105}$$

ii. If $P>0$ and $-Q/P>0$:

$$\begin{aligned} GSP^+ &= -2i\pi \int_0^{-Q/P} \bar{G}(z)dz \\ &= -2i\pi(GZ(T_3,T_4,0) - GZ(T_3,T_4,Q/P)) \end{aligned}$$

iii. If $P<0$ and $-Q/P \leq 0$:

$$GSP^+ = -2i\pi GZ(T_3,T_4,0) \tag{106}$$

iv. If $P<0$ and $-Q/P>0$:

$$GSP^+ = -2i\pi GZ(T_3,T_4,Q/P) \tag{107}$$

(b) If $Im[Q] \neq 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)] \int_0^\infty \bar{G}(z)\;\theta[-\mathsf{H}(z)] \tag{108}$$

The integral is not zero only if $\mathsf{H}(z) \leq 0$ that leads to 4 cases

i. <u>If $Im(EP) = 0$ and $Im(Q^*E - P(m_k^3 - i\rho) = 0$</u>

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ \theta[Im(Q^*(m_k^2 - i\rho))]\ GZ(T_3, T_4, 0) \quad (109)$$

ii. <u>If $Im(EP) > 0$ and $\Delta \geq 0$</u> (then $X_1 \geq X_2$)

- If $X_1 > 0$ and $X_2 \leq 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ (GZ(T_3, T_4, 0) - GZ(T_3, T_4, -X_1)) \quad (110)$$

- If $X_1 > 0$ and $X_2 > 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ (GZ(T_3, T_4, -X_2) - GZ(T_3, T_4, -X_1)) \quad (111)$$

iii. <u>If $Im(EP) < 0$</u> (then $X_1 < X_2$)

- If $\Delta \leq 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ GZ(T_3, T_4, 0) \quad (112)$$

- If $\Delta > 0$ and $X_1 > 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ (GZ(T_3, T_4, -X_2) + GZ(T_3, T_4, 0) - GZ(T_3, T_4, -X_1)) \quad (113)$$

- If $\Delta > 0$ and $X_1 \leq 0$ and $X_2 > 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ GZ(T_3, T_4, -X_2) \quad (114)$$

- If $\Delta > 0$ and $X_2 \leq 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ GZ(T_3, T_4, 0) \quad (115)$$

iv. <u>If $Im(EP) = 0$ and $Im(Q^*E - P(m_k^3 - i\rho)) \neq 0$</u>

- If $Im(Q^*E - P(m_k^2 - i\rho)) > 0$ and $\dfrac{Im(Q^*(m_k^2 - i\rho))}{Im(Q^*E - P(m_k^2 - i\rho))} > 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ \left(GZ(T_3, T_4, 0) - GZ\left(T_3, T_4, -\frac{Im(Q^*(m_k^2 - i\rho))}{Im(Q^*E - P(m_k^2 - i\rho))}\right)\right) \quad (116)$$

- If $Im(Q^*E - P(m_k^2 - i\rho)) < 0$

$$GSP^+ = -2i\pi\theta[Im(Q^*)]\ GZ\left(T_3, T_4, -\mathrm{Max}\left(0, \frac{Im(Q^*(m_k^2 - i\rho))}{Im(Q^*E - P(m_k^2 - i\rho))}\right)\right) \quad (117)$$