ONELOOP4PT

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February 28, 2009

Abstract

In this document, we caculate scalar One Loop four point function with complex internal mass.

1 The Form of One Loop Four Point in Paralell and Orthogonal Space

In Paralell and Orthogonal Space, the form of One Loop Four Point is

$$D_0 = 2\int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \frac{1}{P_1P_2P_3P_4}$$

Here

$$\begin{aligned}
P_1 &= (l_0+q_{10})^2 - l_1^2 - l_2^2 - l_\perp^2 - m_1^2 + i\varepsilon \\
P_2 &= (l_0+q_{20})^2 - (l_1+q_{21})^2 - l_2^2 - l_\perp^2 - m_2^2 + i\varepsilon \\
P_3 &= (l_0+q_{30})^2 - (l_1+q_{31})^2 - (l_2+q_{32})^2 - l_\perp^2 - m_3^2 + i\varepsilon \\
P_4 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon
\end{aligned} \tag{1}$$

And

$$\begin{aligned}
q_1^2 &= q_{10}^2. \\
q_2^2 &= q_{20}^2 - q_{21}^2 \\
q_3^2 &= q_{30}^2 - q_{31}^2 - q_{32}^2 \\
q_4^2 &= 0. \\
l^2 &= l_0^2 - l_1^2 - l_2^2 - l_\perp^2
\end{aligned} \tag{2}$$

$m_i^2 = Re(m_k^2) - i\Gamma_k$ are complex internal mass.

2 The partial fraction

We have

$$
\begin{aligned}
\frac{1}{P_1P_2P_3P_4} &= \frac{1}{P_1(P_2-P_1)(P_3-P_1)(P_4-P_1)} \\
&+ \frac{1}{P_2(P_1-P_2)(P_3-P_2)(P_4-P_2)} \\
&+ \frac{1}{P_3(P_1-P_3)(P_2-P_3)(P_4-P_3)} \\
&+ \frac{1}{P_4(P_1-P_4)(P_2-P_4)(P_3-P_4)} \\
&= \sum_{k=1}^{4} \frac{1}{P_k \prod\limits_{l=1,l\neq k} (P_l - P_k)}
\end{aligned}
\tag{3}
$$

here

$$
\begin{aligned}
P_k &= (l_0+q_{k0})^2 - (l_1+q_{k1})^2 - (l_2+q_{k2})^2 - l_\perp - m_k^2 + i\varepsilon \\
P_l &= (l_0+q_{l0})^2 - (l_1+q_{l1})^2 - (l_2+q_{l2})^2 - l_\perp - m_l^2 + i\varepsilon \\
P_k - P_l &= 2(q_{l0}-q_{k0})l_0 - 2(q_{l1}-q_{k1})l_1 - 2(q_{l2}-q_{k2})l_2 + q_l^2 - q_k^2 - (m_l^2 - m_k^2) \\
&= a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + q_l^2 - q_k^2 - (m_l^2 - m_k^2).
\end{aligned}
\tag{4}
$$

It is important to note that a_{lk}, b_{lk}, c_{lk} in R.

From now, we obtain

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4} \int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \\
&\quad \frac{1}{\left[(l_0+q_{k0})^2 - (l_1+q_{k1})^2 - (l_2+q_{k2})^2 - l_\perp - m_k^2 + i\varepsilon\right]} \\
&\quad \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + q_l^2 - q_k^2 - (m_l^2 - m_k^2)}
\end{aligned}
\tag{5}
$$

We make a shift

$$
\begin{aligned}
l_0 &\to l_0 + q_{k0} \\
l_1 &\to l_1 + q_{k1} \\
l_2 &\to l_2 + q_{k2}
\end{aligned}
\tag{6}
$$

The Jacobian of this shift is 1. The integration region not change and the form of D_0 now look as

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4} \int\limits_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int\limits_{0}^{\infty} dl_\perp \\
&\quad \frac{1}{\left[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + d_{lk})}
\end{aligned}
\tag{7}
$$

Here

$$
\begin{aligned}
&-a_{lk}q_{k0} - b_{lk}q_{k1} - c_{lk}q_{k2} + q_l^2 - q_k^2 - (m_l^2 - m_k^2) = \\
&-2(q_{l0} - q_{k0})q_{k0} + 2(q_{l1} - q_{k1})q_{k1} + 2(q_{l2} - q_{k2})q_{k2} + q_l^2 - q_k^2 - (m_l^2 - m_k^2) \\
&q_l^2 + q_k^2 - 2q_l q_k - (m_l^2 - m_k^2).
\end{aligned} \tag{9}
$$

SUMMARIZE:

$$
\begin{aligned}
D_0 &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int_0^{\infty} dl_\perp \\
&\quad \frac{1}{\left[l_0^2 - l_1^2 - l_2^2 - l_\perp^2 - m_4^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1, l\neq k} (a_{lk}l_0 + b_{lk}l_1 + c_{lk}l_2 + d_{lk})}. \\
&\text{And} \\
a_{lk} &= 2(q_{l0} - q_{k0}) \\
b_{lk} &= 2(q_{l1} - q_{k1}) \\
a_{lk} &= 2(q_{l2} - q_{k2}) \\
d_{lk} &= (q_l - q_k)^2 - (m_l^2 - m_l^2) \\
&\text{Important note} \\
&a_{lk}, b_{lk}, c_{lk} \text{in} R;\ d_{lk} \text{in} C.
\end{aligned} \tag{9}
$$

3 Linearize in x and the $x-$ integration

In this section, we take $x-$ integration by residuce theorem. To do that, we have to linearize D_0 in x, or take a shift

$$
\begin{aligned}
l_0 &= x + z \\
l_1 &= y \\
l_2 &= x \\
l_\perp &= t.
\end{aligned}
$$

The Jacobian of this shift is

$$
|J| = \left|\frac{\delta(l_0, l_1, l_2, l_\perp)}{\delta(z, y, x, t)}\right| = 1. \tag{10}
$$

For this shift, one obtain

$$D_0 = 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dxdydz \int_{0}^{\infty} dt \frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}. \quad (12)$$

Here $AC_{lk} = a_{lk} + c_{lk}$

3.1 The x- integration

The poles of the D_0 integrand are

$$\begin{aligned} x_0 &= \frac{z^2 + y^2 + t^2 + m_k^2 - i\varepsilon}{2z} \\ x_l &= \frac{-a_{lk}z - b_{lk}y - d_{lk}}{AC_{lk}} \end{aligned} \quad (12)$$

It is important to note that

$$\begin{aligned} Im(x_0) &= \frac{-\Gamma_k - \varepsilon}{2z} \\ Im(x_l) &= \frac{-d_{lk}}{AC_{lk}} \end{aligned} \quad (13)$$

We now separate D_0 into form

$$D_0 = D_0^+ + D_0^-$$

with

$$\begin{aligned} D_0^+ &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dxdy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \\ &\frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}. \\ D_0^- &= 2\sum_{k=1}^{4}\int_{-\infty}^{\infty} dxdy \int_{-\infty}^{0} dz \int_{0}^{\infty} dt \\ &\frac{1}{\left[2xz - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\right]} \frac{1}{\prod\limits_{l=1,l\neq k} (a_{lk}z + b_{lk}y + AC_{lk}x + d_{lk})}. \end{aligned} \quad (14)$$

3.1.1 For D_0^+

We close the uper contour in the x plane and D_0^+ is evaluated

$$D_0^+ = 4\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \;\; Res\Big[F(x,y,z,t), x_l\Big] \tag{15}$$

or

$$D_0^+ = 4\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\Big[2x_l z - z^2 - y^2 - t^2 - m_k^2 + i\varepsilon\Big]} \frac{1}{AC_{lk} \prod\limits_{m=1,m\neq l,k} (a_{mk}z + b_{mk}y + AC_{mk}x + d_{mk})} \tag{16}$$

With

$$x_l = \frac{-a_{lk}z - b_{lk}y - d_{lk}}{AC_{lk}} \tag{17}$$

From now we obtain

$$D_0^+ = 2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{\prod\limits_{l=1,l\neq l,k}^{4} AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\varepsilon\Big]}$$

here

$$A_{mlk} = \frac{a_{mk}}{AC_{mk}} - \frac{a_{lk}}{AC_{lk}}$$
$$B_{mlk} = \frac{b_{mk}}{AC_{mk}} - \frac{b_{lk}}{AC_{lk}}$$
$$C_{mlk} = \frac{d_{mk}}{AC_{mk}} - \frac{d_{lk}}{AC_{lk}}$$

3.1.2 For D_0^-

We close thelower contour in the x plane and D_0^- is evaluated

$$D_0^- = -2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{\prod\limits_{l=1,l\neq l,k}^{4} AC_{lk}} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dz \int_{0}^{\infty} dt \; \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \frac{f_{lk}^{-}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}} - y^2 - t^2 - m_k^2 + i\varepsilon\Big]}$$

here

$$\begin{aligned} A_{mlk} &= \frac{a_{mk}}{AC_{mk}} - \frac{a_{lk}}{AC_{lk}} \\ B_{mlk} &= \frac{b_{mk}}{AC_{mk}} - \frac{b_{lk}}{AC_{lk}} \\ C_{mlk} &= \frac{d_{mk}}{AC_{mk}} - \frac{d_{lk}}{AC_{lk}} \end{aligned}$$

SUMMARIZE:

$$D_0 = D_0^+ + D_0^-$$

and

$$\begin{aligned} D_0^+ &= 2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{\prod\limits_{l=1,l\neq l,k}^{4} AC_{lk}} \int\limits_{-\infty}^{\infty} dy \int\limits_{0}^{\infty} dz \int\limits_{0}^{\infty} dt \, \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\ & \frac{f_{lk}^{+}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - t^2 - m_k^2 + i\varepsilon\Big]} \\ D_0^- &= -2\pi i \sum_{k=1}^{4} \sum_{l=1,l\neq k}^{4} \frac{1}{\prod\limits_{l=1,l\neq l,k}^{4} AC_{lk}} \int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{0} dz \int\limits_{0}^{\infty} dt \, \frac{1}{\prod\limits_{m=1,m\neq l,k} (A_{mlk}z + B_{mlk}y + C_{mlk})} \\ & \frac{f_{lk}^{-}\Big(1-\delta(AC_{lk})\Big)}{\Big[\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - t^2 - m_k^2 + i\varepsilon\Big]} \end{aligned} \tag{18}$$

here

$$\begin{aligned} A_{mlk} &= \frac{a_{mk}}{AC_{mk}} - \frac{a_{lk}}{AC_{lk}} \\ B_{mlk} &= \frac{b_{mk}}{AC_{mk}} - \frac{b_{lk}}{AC_{lk}} \\ C_{mlk} &= \frac{d_{mk}}{AC_{mk}} - \frac{d_{lk}}{AC_{lk}} \end{aligned}$$

4 The y integration

The next we are going to take y integration. To do that we have to perform Wick rotation $t \longrightarrow it$ then linearize in y.

4.1 t- wick rotation

To linearize in y, the sign of y^2 and t^2 must be opsite. To do that we have to perform t- wick rotation.

The poles of $t-$ integrand are

$$t_{1,2} = \pm\sqrt{\Big(1-\frac{2a_{lk}}{AC_{lk}}\Big)z^2 - \frac{2b_{lk}}{AC_{lk}}yz - \frac{2d_{lk}}{AC_{lk}}z - y^2 - m_k^2 + i\varepsilon} \tag{19}$$

Because

$$Im\Big[-\frac{2d_{lk}}{AC_{lk}}z - m_k^2 + i\varepsilon\Big] > 0. \tag{20}$$

then $t_{1,2}$ locate in the first or the thirth quarter $t-$ complex plane.

We have

$$\oint f(t^2)dt = \Big\{\int\limits_0^R + \int\limits_{C_k} + \int\limits_{-iR}^0\Big\} f(t^2)dt = 0 \tag{21}$$

When R go to ∞, one obtain

$$\Big\{\int\limits_0^\infty + \int\limits_{-i\infty}^0\Big\} f(t^2)dt = 0. \tag{22}$$

or

$$\int\limits_0^\infty f(t^2)dt = -\int\limits_{-i\infty}^0 f(t^2)dt \tag{23}$$

Making $t-$ rotation, one obtain

$$\int\limits_0^\infty f(t^2)dt = -i\int\limits_0^\infty f(-t^2)dt \tag{24}$$