Differential Kinematics

1 FORWARD KINEMATIC EQUATIONS

Recall the forward kinematics formulation f_{FPK}

$$\begin{aligned} P &= f_{FPK}(\Theta) \\ p_x &= \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} \\ p_y &= \lambda_2 s_1 c_2 - \lambda_3 s_1 s_{23} \\ p_z &= \lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23} \\ \gamma &= \phi_4 \end{aligned} \tag{1}$$

where Θ is the vector of the joint states,

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \tag{2}$$

and $\mathbf{p}$ is the position of the end-effector {4} (expressed in base frame {0})

$$\mathrm{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^0p_4 \tag{3}$$

2 INVERSE KINEMATIC FORMULATION

Differentiate the equation set 1, to yield,

$$\begin{aligned} v_x = \frac{dp_x}{dt} &= -\lambda_2 s_1 c_2 \dot{\theta}_1 - \lambda_2 c_1 s_2 \dot{\theta}_2 + \lambda_3 s_1 s_{23} \dot{\theta}_1 - \lambda_3 c_1 c_{23} \dot{\theta}_2 - \lambda_3 c_1 c_{23} \dot{\theta}_3 \\ v_y = \frac{dp_y}{dt} &= \lambda_2 c_1 c_2 \dot{\theta}_1 - \lambda_2 s_1 s_2 \dot{\theta}_2 - \lambda_3 c_1 s_{23} \dot{\theta}_1 - \lambda_3 s_1 c_{23} \dot{\theta}_2 - \lambda_3 s_1 c_{23} \dot{\theta}_3 \\ v_z = \frac{dp_z}{dt} &= \lambda_1 - \lambda_2 c_2 \dot{\theta}_2 + \lambda_3 s_{23} \dot{\theta}_2 + \lambda_3 s_{23} \dot{\theta}_3 \\ \dot{\gamma} = \frac{d\gamma}{dt} &= \dot{\theta}_4 \end{aligned} \tag{4}$$

This can be represented in matrix form,

$$V = {}^{\mathbf{0}}\boldsymbol{J}\,\dot{\Theta}$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23} & -\lambda_2 c_1 s_2 - \lambda_3 c_1 c_{23} & -\lambda_3 c_1 c_{23} & 0 \\ \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} & -\lambda_2 s_1 s_2 - \lambda_3 s_1 c_{23} & -\lambda_3 s_1 c_{23} & 0 \\ 0 & -\lambda_2 c_2 + \lambda_3 s_{23} & \lambda_3 s_{23} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} \tag{5}$$

where ${}^{\mathbf{0}}\boldsymbol{J}$ is the manipulator Jacobian expressed in base frame {0}.

2.1 SINGULARITY ANALYSIS

The determinant of the Jacobian is derived as,

$$\det {}^{\mathbf{0}}\boldsymbol{J} = (-\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23})\big((-\lambda_2 s_1 s_2 - \lambda_3 s_1 c_{23})(\lambda_3 s_{23}) - (-\lambda_2 c_2 + \lambda_3 s_{23})(-\lambda_3 s_1 c_{23})\big) + (\lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23})\big((-\lambda_2 c_2 + \lambda_3 s_{23})(-\lambda_3 c_1 c_{23}) - (-\lambda_2 c_1 s_2 - \lambda_3 c_1 c_{23})(\lambda_3 s_{23})\big)$$

$$\det {}^{\mathbf{0}}\boldsymbol{J} = -\lambda_2\lambda_3\big(s_1(-\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23})(s_2 s_{23} + c_2 c_{23}) - c_1(\lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23})(c_2 c_{23} + s_2 s_{23})\big) \tag{6}$$

$$\det {}^{\mathbf{0}}\boldsymbol{J} = \lambda_2\lambda_3\big(s_1^2(\lambda_2 c_2 - \lambda_3 s_{23})(s_2 s_{23} + c_2 c_{23}) + c_1^2(\lambda_2 c_2 - \lambda_3 s_{23})(c_2 c_{23} + s_2 s_{23})\big)$$

$$\det {}^{\mathbf{0}}\boldsymbol{J} = \lambda_2\lambda_3(\lambda_2 c_2 - \lambda_3 s_{23})(s_2 s_{23} + c_2 c_{23})$$

2.1.1 Singularity Condition 1:

$$\lambda_2 c_2 - \lambda_3 s_{23} = 0 \tag{7}$$

Which corresponds to the end-effector directly above the base.

2.1.2 Singularity Condition 2:

$$s_2 s_{23} + c_2 c_{23} = 0$$
$$s_2 s_2 c_3 + s_2 c_2 s_3 + c_2 c_2 c_3 - c_2 s_2 s_3 = 0 \tag{8}$$
$$c_3 = 0$$

Which corresponds to the arm either folder into itself or completely stretched out.