# Dynamics - Lagrangian

## 1 Center of Mass Positions

The equation used for calculating the positions of the centers of gravity of the links is:

$$ {}^{0}p_{ci} = {}^{0}p_i + {}^{0}R_i {}^{i}r_{ci} \tag{1} $$

where the terms ${}^{0}p_i$ and ${}^{0}R_i$ are extracted from the corresponding transformation matrices ${}^{0}T_i$ (see forward kinematics formulation for these transformations). The c.g. positions are then,

$$ {}^{0}p_{c1} = {}^{0}p_1 + {}^{0}R_1 {}^{1}r_{c1} $$
$$ {}^{0}p_{c1} = \begin{bmatrix} 0 \\ 0 \\ \lambda_1 \end{bmatrix} + \begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_{c1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda_1 - \lambda_{c1} \end{bmatrix} \tag{2a} $$

$$ {}^{0}p_{c2} = {}^{0}p_2 + {}^{0}R_2 {}^{2}r_{c2} $$
$$ {}^{0}p_{c2} = \begin{bmatrix} \lambda_2 c_1 c_2 \\ \lambda_2 s_1 c_2 \\ \lambda_1 - \lambda_2 s_2 \end{bmatrix} + \begin{bmatrix} c_1 c_2 & -c_1 s_2 & -s_1 \\ s_1 c_2 & -s_1 s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix} \begin{bmatrix} -\lambda_{c2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (\lambda_2 - \lambda_{c2}) c_1 c_2 \\ (\lambda_2 - \lambda_{c2}) s_1 c_2 \\ \lambda_1 - (\lambda_2 - \lambda_{c2}) s_2 \end{bmatrix} \tag{2b} $$

$$ {}^{0}p_{c3} = {}^{0}p_3 + {}^{0}R_3 {}^{3}r_{c3} $$
$$ {}^{0}p_{c3} = \begin{bmatrix} \lambda_2 c_1 c_2 \\ \lambda_2 s_1 c_2 \\ \lambda_1 - \lambda_2 s_2 \end{bmatrix} + \begin{bmatrix} c_1 c_{23} & s_1 & -c_1 s_{23} \\ s_1 c_{23} & -c_1 & -s_1 s_{23} \\ -s_{23} & 0 & -c_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \lambda_{c3} \end{bmatrix} = \begin{bmatrix} \lambda_2 c_1 c_2 - \lambda_{c3} c_1 s_{23} \\ \lambda_2 s_1 c_2 - \lambda_{c3} s_1 s_{23} \\ \lambda_1 - \lambda_2 s_2 - \lambda_{c3} c_{23} \end{bmatrix} \tag{2c} $$

$$ {}^{0}p_{c4} = {}^{0}p_4 + {}^{0}R_4 {}^{4}r_{c4} $$
$$ \begin{aligned} {}^{0}p_{c4} &= \begin{bmatrix} \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} \\ \lambda_2 s_1 c_2 - \lambda_3 s_1 s_{23} \\ \lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23} \end{bmatrix} + \begin{bmatrix} c_1 c_{23} c_4 + s_1 s_4 & -c_1 c_{23} s_4 + s_1 c_4 & -c_1 s_{23} \\ s_1 c_{23} c_4 - c_1 s_4 & -s_1 c_{23} s_4 - c_1 c_4 & -s_1 s_{23} \\ -s_{23} c_4 & s_{23} s_4 & -c_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\lambda_{c4} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_2 c_1 c_2 - (\lambda_3 - \lambda_{c4}) c_1 s_{23} \\ \lambda_2 s_1 c_2 - (\lambda_3 - \lambda_{c4}) s_1 s_{23} \\ \lambda_1 - \lambda_2 s_2 - (\lambda_3 - \lambda_{c4}) c_{23} \end{bmatrix} \end{aligned} \tag{2d} $$

# 2 ANGULAR AND LINEAR SPEEDS

The equations used for defining the angular and linear velocities are:

$$
\begin{gathered}
{}^{i}\omega_i = {}^{i}R_{i-1}\left({}^{i-1}\omega_{i-1} + {}^{i-1}z_{i-1}\,\dot{\theta}_i\right) \\
{}^{i}v_i = {}^{i}R_{i-1}{}^{i-1}v_{i-1} + {}^{i}\widetilde{\omega}_i\,{}^{i}r_i \\
{}^{i}v_{ci} = {}^{i}v_i + {}^{i}\widetilde{\omega}_i\,{}^{i}r_{ci}
\end{gathered}
\tag{3}
$$

$$
{}^{i}r_i = \begin{bmatrix} a_i \\ d_i \sin\alpha_i \\ d_i \cos\alpha_i \end{bmatrix}, \quad {}^{i}\widetilde{\omega}_i = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}
$$

Starting with the initial conditions,

$$
{}^{0}\omega_0 = {}^{0}v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}
$$

The velocities for the first link are,

$$
{}^{1}\omega_1 = {}^{1}R_0\left({}^{0}\omega_0 + {}^{0}z_0\,\dot{\theta}_1\right)
$$

$$
{}^{1}\omega_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & -1 \\ -s_1 & c_1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\theta}_1\right) = \begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} \tag{5a}
$$

$$
{}^{1}v_1 = {}^{1}R_0{}^{0}v_0 + {}^{1}\widetilde{\omega}_1\,{}^{1}r_1
$$

$$
{}^{1}v_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & -1 \\ -s_1 & c_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\dot{\theta}_1 \\ 0 & 0 & 0 \\ \dot{\theta}_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -\lambda_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5b}
$$

$$
{}^{1}v_{c1} = {}^{1}v_1 + {}^{1}\widetilde{\omega}_1\,{}^{1}r_{c1}
$$

$$
{}^{1}v_{c1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\dot{\theta}_1 \\ 0 & 0 & 0 \\ \dot{\theta}_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \lambda_{c1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5c}
$$

Repeating this for the second link,

$$
{}^{2}\omega_2 = {}^{2}R_1\left({}^{1}\omega_1 + {}^{1}z_1\,\dot{\theta}_2\right)
$$

$$
{}^{2}\omega_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\theta}_2\right) = \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{6a}
$$

$$
{}^{2}v_2 = {}^{2}R_1{}^{1}v_1 + {}^{2}\widetilde{\omega}_2\,{}^{2}r_2
$$

$$
{}^{2}v_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\,\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\,\dot{\theta}_1 \\ c_2\,\dot{\theta}_1 & -s_2\,\dot{\theta}_1 & 0 \end{bmatrix}\begin{bmatrix} \lambda_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2\dot{\theta}_2 \\ \lambda_2 c_2\dot{\theta}_1 \end{bmatrix} \tag{6b}
$$

$$^2v_{c2} = {}^2v_2 + {}^2\widetilde{\omega}_2\, {}^2r_{c2}$$

$$^2v_{c2} = \begin{bmatrix} 0 \\ \lambda_2\dot{\theta}_2 \\ \lambda_2 c_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta}_2 & -c_2\,\dot{\theta}_1 \\ \dot{\theta}_2 & 0 & s_2\,\dot{\theta}_1 \\ c_2\,\dot{\theta}_1 & -s_2\,\dot{\theta}_1 & 0 \end{bmatrix}\begin{bmatrix} -\lambda_{c2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ (\lambda_2 - \lambda_{c2})\dot{\theta}_2 \\ (\lambda_2 - \lambda_{c2})c_2\dot{\theta}_1 \end{bmatrix} \tag{6c}$$

Repeating this for the third link,

$$^3\omega_3 = {}^3R_2\left({}^2\omega_2 + {}^2z_2\,\dot{\theta}_3\right)$$

$$^3\omega_3 = \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & 0 \end{bmatrix}\left( \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ \dot{\theta}_3 \right) = \begin{bmatrix} -s_{23}\dot{\theta}_1 \\ -\dot{\theta}_2 - \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 \end{bmatrix} \tag{7a}$$

$$^3v_3 = {}^3R_2\,{}^2v_2 + {}^3\widetilde{\omega}_3\, {}^3r_3$$

$$^3v_3 = \begin{bmatrix} c_3 & s_3 & 0 \\ 0 & 0 & -1 \\ -s_3 & c_3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \lambda_2\dot{\theta}_2 \\ \lambda_2 c_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 & c_{23}\dot{\theta}_1 & -\dot{\theta}_2 - \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 & 0 & s_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 & -s_{23}\dot{\theta}_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_2 s_3\dot{\theta}_2 \\ -\lambda_2 c_2\dot{\theta}_1 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix} \tag{7b}$$

$$^3v_{c3} = {}^3v_3 + {}^3\widetilde{\omega}_3\, {}^3r_{c3}$$

$$^3v_{c3} = \begin{bmatrix} \lambda_2 s_3\dot{\theta}_2 \\ -\lambda_2 c_2\dot{\theta}_1 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & c_{23}\dot{\theta}_1 & -\dot{\theta}_2 - \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 & 0 & s_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 & -s_{23}\dot{\theta}_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \lambda_{c3} \end{bmatrix} = \begin{bmatrix} (\lambda_2 s_3 - \lambda_{c3})\dot{\theta}_2 - \lambda_{c3}\dot{\theta}_3 \\ -(\lambda_2 c_2 - \lambda_{c3}s_{23})\dot{\theta}_1 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix} \tag{7c}$$

Repeating this for the last link,

$$^4\omega_4 = {}^4R_3\left({}^3\omega_3 + {}^3z_3\,\dot{\theta}_4\right)$$

$$^4\omega_4 = \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left( \begin{bmatrix} -s_{23}\dot{\theta}_1 \\ -\dot{\theta}_2 - \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ \dot{\theta}_4 \right) = \begin{bmatrix} -s_{23}c_4\dot{\theta}_1 - s_4\dot{\theta}_2 - s_4\dot{\theta}_3 \\ s_{23}s_4\dot{\theta}_1 - c_4\dot{\theta}_2 - c_4\dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 + \dot{\theta}_4 \end{bmatrix} \tag{8a}$$

$$\begin{aligned} ^4v_4 &= {}^4R_3\,{}^3v_3 + {}^4\widetilde{\omega}_4\, {}^4r_4 \\ ^4v_4 &= \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \lambda_2 s_3\dot{\theta}_2 \\ -\lambda_2 c_2\dot{\theta}_1 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & c_{23}\dot{\theta}_1 - \dot{\theta}_4 & s_{23}s_4\dot{\theta}_1 - c_4\dot{\theta}_2 - c_4\dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 + \dot{\theta}_4 & 0 & s_{23}c_4\dot{\theta}_1 + s_4\dot{\theta}_2 + s_4\dot{\theta}_3 \\ -s_{23}s_4\dot{\theta}_1 + c_4\dot{\theta}_2 + c_4\dot{\theta}_3 & -s_{23}c_4\dot{\theta}_1 - s_4\dot{\theta}_2 - s_4\dot{\theta}_3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \lambda_3 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_2\, s_3c_4\dot{\theta}_2 - \lambda_2 c_2 s_4\dot{\theta}_1 \\ -\lambda_2\, s_3s_4\dot{\theta}_2 - \lambda_2 c_2 c_4\dot{\theta}_1 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \lambda_3 s_{23}s_4\dot{\theta}_1 - \lambda_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_3 \\ \lambda_3 s_{23}c_4\dot{\theta}_1 + \lambda_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_3 \\ 0 \end{bmatrix} \end{aligned} \tag{8b}$$

$$^4v_4 = \begin{bmatrix} -\lambda_2 c_2 s_4\dot{\theta}_1 + \lambda_3 s_{23}s_4\dot{\theta}_1 + \lambda_2\, s_3c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_3 \\ -\lambda_2 c_2 c_4\dot{\theta}_1 + \lambda_3 s_{23}c_4\dot{\theta}_1 - \lambda_2\, s_3s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_3 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix}$$

$$
\begin{aligned}
{}^{4}v_{c4} &= {}^{4}v_{4} + {}^{4}\widetilde{\omega}_{4}\, {}^{4}r_{c4} \\
{}^{4}v_{c4} &= \begin{bmatrix} -\lambda_2 c_2 s_4 \dot{\theta}_1 + \lambda_3 s_{23} s_4 \dot{\theta}_1 + \lambda_2\, s_3 c_4 \dot{\theta}_2 - \lambda_3 c_4 \dot{\theta}_2 - \lambda_3 c_4 \dot{\theta}_3 \\ -\lambda_2 c_2 c_4 \dot{\theta}_1 + \lambda_3 s_{23} c_4 \dot{\theta}_1 - \lambda_2\, s_3 s_4 \dot{\theta}_2 + \lambda_3 s_4 \dot{\theta}_2 + \lambda_3 s_4 \dot{\theta}_3 \\ \lambda_2 c_3 \dot{\theta}_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & c_{23}\dot{\theta}_1 - \dot{\theta}_4 & s_{23} s_4 \dot{\theta}_1 - c_4 \dot{\theta}_2 - c_4 \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 + \dot{\theta}_4 & 0 & s_{23} c_4 \dot{\theta}_1 + s_4 \dot{\theta}_2 + s_4 \dot{\theta}_3 \\ -s_{23} s_4 \dot{\theta}_1 + c_4 \dot{\theta}_2 + c_4 \dot{\theta}_3 & -s_{23} c_4 \dot{\theta}_1 - s_4 \dot{\theta}_2 - s_4 \dot{\theta}_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\lambda_{c4} \end{bmatrix}
\end{aligned} \tag{8c}
$$

$$
{}^{4}v_{c4} = \begin{bmatrix} -\lambda_2 c_2 s_4 \dot{\theta}_1 + \lambda_3 s_{23} s_4 \dot{\theta}_1 - \lambda_{c4} s_{23} s_4 \dot{\theta}_1 + \lambda_2\, s_3 c_4 \dot{\theta}_2 - \lambda_3 c_4 \dot{\theta}_2 + \lambda_{c4} c_4 \dot{\theta}_2 - \lambda_3 c_4 \dot{\theta}_3 + \lambda_{c4} c_4 \dot{\theta}_3 \\ -\lambda_2 c_2 c_4 \dot{\theta}_1 + \lambda_3 s_{23} c_4 \dot{\theta}_1 - \lambda_{c4} s_{23} c_4 \dot{\theta}_1 - \lambda_2\, s_3 s_4 \dot{\theta}_2 + \lambda_3 s_4 \dot{\theta}_2 - \lambda_{c4} s_4 \dot{\theta}_2 + \lambda_3 s_4 \dot{\theta}_3 - \lambda_{c4} s_4 \dot{\theta}_3 \\ \lambda_2 c_3 \dot{\theta}_2 \end{bmatrix}
$$

# 3 Kinetic and Potential Link Energies

Next, we will derive the kinetic energy $\mathrm{T}_i$ and potential energy $\mathrm{P}_i$ of each link using the equations,

$$
\mathrm{T}_i = \tfrac{1}{2} m_i \left({}^{i}v_{ci}\right)^T \left({}^{i}v_{ci}\right) + \tfrac{1}{2} \left({}^{i}\omega_i\right)^T {}^{i}I_{ci} \left({}^{i}\omega_i\right) \tag{9a}
$$

$$
\mathrm{P}_i = -m_i ({}^{0}g)^T\, {}^{0}p_{ci} \tag{9b}
$$

Where $m_i$ is the mass of link $i$ and ${}^{i}I_{ci}$ is the second mass moment of inertia matrix of each link about the center of mass and expressed in the frame $i$. The matrices for the links are,

$$
\begin{aligned}
{}^{1}I_{c1} &= \begin{bmatrix} I_{1L} & 0 & 0 \\ 0 & I_{1A} & 0 \\ 0 & 0 & I_{1L} \end{bmatrix} & \quad {}^{2}I_{c2} &= \begin{bmatrix} I_{2A} & 0 & 0 \\ 0 & I_{2L} & 0 \\ 0 & 0 & I_{2L} \end{bmatrix} \\
{}^{3}I_{c3} &= \begin{bmatrix} I_{3L} & 0 & 0 \\ 0 & I_{3L} & 0 \\ 0 & 0 & I_{3A} \end{bmatrix} & \quad {}^{4}I_{c4} &= \begin{bmatrix} I_{4L} & 0 & 0 \\ 0 & I_{4L} & 0 \\ 0 & 0 & I_{4A} \end{bmatrix}
\end{aligned} \tag{10}
$$

There are a few assumptions made here,

1. The moments of inertia are principally aligned in the corresponding frame of reference $i$. This makes the non-diagonal terms disappear.
2. Each link is treated as a cylindrical rod. Thus, the diagonal terms of the moment of inertia matrix are $I_{iA}$ along the principle major axis, and symmetric about the other two axes $I_{iL}$.

The kinetic and potential energies are then,

$$
\begin{aligned}
\mathrm{T}_1 &= \frac{1}{2} m_1 ({}^{1}v_{c1})^T ({}^{1}v_{c1}) + \frac{1}{2} {}^{1}\omega_1{}^{T}\, {}^{1}I_{c1} ({}^{1}\omega_1) \\
\mathrm{T}_1 &= \frac{1}{2} m_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} I_{1L} & 0 & 0 \\ 0 & I_{1A} & 0 \\ 0 & 0 & I_{1L} \end{bmatrix} \begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} = \frac{1}{2} I_{1A} \dot{\theta}_1^2
\end{aligned} \tag{11}
$$

$$
\mathrm{P}_1 = -m_1 ({}^{0}g)^T\, {}^{0}p_{c1} = -m_1 \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \lambda_1 - \lambda_{c1} \end{bmatrix} = m_1 g\, (\lambda_1 - \lambda_{c1})
$$

$$
\begin{aligned}
\mathrm{T}_2 &= \frac{1}{2} m_2\, ({}^2v_{c2})^T ({}^2v_{c2}) + \frac{1}{2}\, {}^2\omega_2{}^T\, {}^2I_{c2} ({}^2\omega_2) \\
\mathrm{T}_2 &= \frac{1}{2} m_2 \begin{bmatrix} 0 \\ (\lambda_2 - \lambda_{c2})\dot{\theta}_2 \\ (\lambda_2 - \lambda_{c2}) c_2 \dot{\theta}_1 \end{bmatrix}^T \begin{bmatrix} 0 \\ (\lambda_2 - \lambda_{c2})\dot{\theta}_2 \\ (\lambda_2 - \lambda_{c2}) c_2 \dot{\theta}_1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \begin{bmatrix} I_{2A} & 0 & 0 \\ 0 & I_{2L} & 0 \\ 0 & 0 & I_{2L} \end{bmatrix} \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
\mathrm{T}_2 &= \left(\frac{1}{2}\, m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + \frac{1}{2}\, I_{2A} s_2^2 + \frac{1}{2}\, I_{2L} c_2^2\right)\dot{\theta}_1^2 + \left(\frac{1}{2}\, m_2(\lambda_2 - \lambda_{c2})^2 + \frac{1}{2}\, I_{2L}\right)\dot{\theta}_2^2
\end{aligned} \tag{12}
$$

$$
\mathrm{P}_2 = -m_2({}^0g)^T\, {}^0p_{c2} = -m_2 \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}^T \begin{bmatrix} (\lambda_2 - \lambda_{c2}) c_1 c_2 \\ (\lambda_2 - \lambda_{c2}) s_1 c_2 \\ \lambda_1 - (\lambda_2 - \lambda_{c2}) s_2 \end{bmatrix} = m_2 g\, (\lambda_1 - (\lambda_2 - \lambda_{c2}) s_2)
$$

$$
\begin{aligned}
\mathrm{T}_3 &= \frac{1}{2} m_3\, ({}^3v_{c3})^T ({}^3v_{c3}) + \frac{1}{2}\, {}^3\omega_3{}^T\, {}^3I_{c3} ({}^3\omega_3) \\
\mathrm{T}_3 &= \frac{1}{2} m_3 \begin{bmatrix} (\lambda_2 s_3 - \lambda_{c3})\dot{\theta}_2 - \lambda_{c3}\dot{\theta}_3 \\ -(\lambda_2 c_2 - \lambda_{c3} s_{23})\dot{\theta}_1 \\ \lambda_2 c_3 \dot{\theta}_2 \end{bmatrix}^T \begin{bmatrix} (\lambda_2 s_3 - \lambda_{c3})\dot{\theta}_2 - \lambda_{c3}\dot{\theta}_3 \\ -(\lambda_2 c_2 - \lambda_{c3} s_{23})\dot{\theta}_1 \\ \lambda_2 c_3 \dot{\theta}_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -s_{23}\dot{\theta}_1 \\ -\dot{\theta}_2 - \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 \end{bmatrix}^T \begin{bmatrix} I_{3L} & 0 & 0 \\ 0 & I_{3L} & 0 \\ 0 & 0 & I_{3A} \end{bmatrix} \begin{bmatrix} -s_{23}\dot{\theta}_1 \\ -\dot{\theta}_2 - \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 \end{bmatrix} \\
\mathrm{T}_3 &= \frac{1}{2}\, m_3 \left((\lambda_2 s_3 - \lambda_{c3})\dot{\theta}_2 - \lambda_{c3}\dot{\theta}_3\right)^2 + \frac{1}{2}\, m_3(\lambda_2 c_2 - \lambda_{c3} s_{23})^2 \dot{\theta}_1^2 + \frac{1}{2}\, m_3(\lambda_2 c_3)^2 \dot{\theta}_2^2 + \frac{1}{2}\, I_{3L} s_{23}^2 \dot{\theta}_1^2 \\
&\quad + \frac{1}{2}\, I_{3L}\left(\dot{\theta}_2 + \dot{\theta}_3\right)^2 + \frac{1}{2}\, I_{3A} c_{23}^2 \dot{\theta}_1^2 \\
\mathrm{T}_3 &= \frac{1}{2}\, m_3 \lambda_2^2 s_3^2 \dot{\theta}_2^2 + \frac{1}{2}\, m_3 \lambda_{c3}^2 \dot{\theta}_2^2 - m_3 \lambda_2 \lambda_{c3} s_3 \dot{\theta}_2^2 + \frac{1}{2} m_3 \lambda_{c3}^2 \dot{\theta}_3^2 - m_3 \lambda_{c3} \lambda_2 s_3 \dot{\theta}_2 \dot{\theta}_3 + m_3 \lambda_{c3}^2 \dot{\theta}_2 \dot{\theta}_3 \\
&\quad + \frac{1}{2}\, m_3 \lambda_2^2 c_2^2 \dot{\theta}_1^2 + \frac{1}{2}\, m_3 \lambda_{c3}^2 s_{23}^2 \dot{\theta}_1^2 - m_3 \lambda_2 \lambda_{c3} c_2 s_{23} \dot{\theta}_1^2 + \frac{1}{2}\, m_3 \lambda_2^2 c_3^2 \dot{\theta}_2^2 + \frac{1}{2}\, I_{3L} s_{23}^2 \dot{\theta}_1^2 \\
&\quad + \frac{1}{2}\, I_{3L}(\dot{\theta}_2 + \dot{\theta}_3)^2 + \frac{1}{2}\, I_{3A} c_{23}^2 \dot{\theta}_1^2 \\
\mathrm{T}_3 &= \left(\frac{1}{2}\, m_3 \lambda_2^2 c_2^2 + \frac{1}{2}\, m_3 \lambda_{c3}^2 s_{23}^2 - m_3 \lambda_2 \lambda_{c3} c_2 s_{23} + \frac{1}{2}\, I_{3L} s_{23}^2 + \frac{1}{2}\, I_{3A} c_{23}^2\right)\dot{\theta}_1^2 \\
&\quad + \left(\frac{1}{2}\, m_3 \lambda_2^2 + \frac{1}{2}\, m_3 \lambda_{c3}^2 - m_3 \lambda_2 \lambda_{c3} s_3 + \frac{1}{2}\, I_{3L}\right)\dot{\theta}_2^2 + \left(\frac{1}{2} m_3 \lambda_{c3}^2 + \frac{1}{2}\, I_{3L}\right)\dot{\theta}_3^2 \\
&\quad + (m_3 \lambda_{c3}^2 - m_3 \lambda_{c3} \lambda_2 s_3 + I_{3L})\dot{\theta}_2 \dot{\theta}_3
\end{aligned} \tag{13}
$$

$$
\mathrm{P}_3 = -m_3({}^0g)^T\, {}^0p_{c3} = -m_3 \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}^T \begin{bmatrix} \lambda_2 c_1 c_2 - \lambda_{c3} c_1 s_{23} \\ \lambda_2 s_1 c_2 - \lambda_{c3} s_1 s_{23} \\ \lambda_1 - \lambda_2 s_2 - \lambda_{c3} c_{23} \end{bmatrix} = m_3 g\, (\lambda_1 - \lambda_2 s_2 - \lambda_{c3} c_{23})
$$

$$
\begin{aligned}
\mathrm{T}_4 &= \frac{1}{2} m_4\, ({}^4v_{c4})^T ({}^4v_{c4}) + \frac{1}{2}\, {}^4\omega_4{}^T\, {}^4I_{c4} ({}^4\omega_4) \\
\mathrm{T}_4 &= \frac{1}{2} m_4 \begin{bmatrix} ((\lambda_3 - \lambda_{c4}) s_{23} s_4 - \lambda_2 c_2 s_4)\dot{\theta}_1 + (\lambda_2 s_3 c_4 - (\lambda_3 - \lambda_{c4}) c_4)\, \dot{\theta}_2 - (\lambda_3 - \lambda_{c4}) c_4 \dot{\theta}_3 \\ ((\lambda_3 - \lambda_{c4}) s_{23} c_4 - \lambda_2 c_2 c_4)\dot{\theta}_1 + ((\lambda_3 - \lambda_{c4}) s_4 - \lambda_2 s_3 s_4)\, \dot{\theta}_2 + (\lambda_3 - \lambda_{c4}) s_4 \dot{\theta}_3 \\ \lambda_2 c_3\, \dot{\theta}_2 \end{bmatrix}^T \begin{bmatrix} ((\lambda_3 - \lambda_{c4}) s_{23} s_4 - \lambda_2 c_2 s_4)\dot{\theta}_1 + (\lambda_2 s_3 c_4 - (\lambda_3 - \lambda_{c4}) c_4)\, \dot{\theta}_2 - (\lambda_3 - \lambda_{c4}) c_4 \dot{\theta}_3 \\ ((\lambda_3 - \lambda_{c4}) s_{23} c_4 - \lambda_2 c_2 c_4)\dot{\theta}_1 + ((\lambda_3 - \lambda_{c4}) s_4 - \lambda_2 s_3 s_4)\, \dot{\theta}_2 + (\lambda_3 - \lambda_{c4}) s_4 \dot{\theta}_3 \\ \lambda_2 c_3\, \dot{\theta}_2 \end{bmatrix} \\
&\quad + \frac{1}{2} \begin{bmatrix} -s_{23} c_4 \dot{\theta}_1 - s_4 \dot{\theta}_2 - s_4 \dot{\theta}_3 \\ s_{23} s_4 \dot{\theta}_1 - c_4 \dot{\theta}_2 - c_4 \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 + \dot{\theta}_4 \end{bmatrix}^T \begin{bmatrix} I_{4L} & 0 & 0 \\ 0 & I_{4L} & 0 \\ 0 & 0 & I_{4A} \end{bmatrix} \begin{bmatrix} -s_{23} c_4 \dot{\theta}_1 - s_4 \dot{\theta}_2 - s_4 \dot{\theta}_3 \\ s_{23} s_4 \dot{\theta}_1 - c_4 \dot{\theta}_2 - c_4 \dot{\theta}_3 \\ -c_{23}\dot{\theta}_1 + \dot{\theta}_4 \end{bmatrix} \\
\mathrm{T}_4 &= \frac{1}{2} m_4 \left(((\lambda_3 - \lambda_{c4}) s_{23} s_4 - \lambda_2 c_2 s_4)\dot{\theta}_1 + (\lambda_2 s_3 c_4 - (\lambda_3 - \lambda_{c4}) c_4)\, \dot{\theta}_2 - (\lambda_3 - \lambda_{c4}) c_4 \dot{\theta}_3\right)^2 + \frac{1}{2} m_4 \left(((\lambda_3 - \lambda_{c4}) s_{23} c_4 - \lambda_2 c_2 c_4)\dot{\theta}_1 + ((\lambda_3 - \lambda_{c4}) s_4 - \lambda_2 s_3 s_4)\, \dot{\theta}_2 + (\lambda_3 - \lambda_{c4}) s_4 \dot{\theta}_3\right)^2 \\
&\quad + \frac{1}{2} m_4 (\lambda_2 c_3\, \dot{\theta}_2)^2 + \frac{1}{2} I_{4L}(-s_{23} c_4 \dot{\theta}_1 - s_4 \dot{\theta}_2 - s_4 \dot{\theta}_3)^2 + \frac{1}{2} I_{4L}(s_{23} s_4 \dot{\theta}_1 - c_4 \dot{\theta}_2 - c_4 \dot{\theta}_3)^2 + \frac{1}{2} I_{4A}(-c_{23}\dot{\theta}_1 + \dot{\theta}_4)^2 \\
\mathrm{T}_4 &= \frac{1}{2} m_4 ((\lambda_3 - \lambda_{c4}) s_{23}\, s_4 - \lambda_2 c_2\, s_4)^2 \dot{\theta}_1^2 + \frac{1}{2} m_4 (\lambda_2\, s_3\, c_4 - (\lambda_3 - \lambda_{c4}) c_4)^2 \dot{\theta}_2^2 + \frac{1}{2} m_4 ((\lambda_3 - \lambda_{c4}) c_4)^2 \dot{\theta}_3^2 + m_4 \left(((\lambda_3 - \lambda_{c4}) s_{23}\, s_4 - \lambda_2 c_2\, s_4)(\lambda_2\, s_3\, c_4 - (\lambda_3 - \lambda_{c4}) c_4)\right)\dot{\theta}_1 \dot{\theta}_2 \\
&\quad - m_4 \left(((\lambda_3 - \lambda_{c4}) s_{23}\, s_4 - \lambda_2 c_2\, s_4)(\lambda_3 - \lambda_{c4}) c_4\right)\dot{\theta}_1 \dot{\theta}_3 - m_4 ((\lambda_2\, s_3\, c_4 - (\lambda_3 - \lambda_{c4}) c_4)\, (\lambda_3 - \lambda_{c4}) c_4)\dot{\theta}_2 \dot{\theta}_3 + \frac{1}{2} m_4 ((\lambda_3 - \lambda_{c4}) s_{23}\, c_4 - \lambda_2 c_2\, c_4)^2 \dot{\theta}_1^2 \\
&\quad + \frac{1}{2} m_4 ((\lambda_3 - \lambda_{c4}) s4 - \lambda_2\, s_3\, s_4)^2 \dot{\theta}_2^2 + \frac{1}{2} m_4 ((\lambda_3 - \lambda_{c4}) s_4)^2 \dot{\theta}_3^2 + m_4 \left(((\lambda_3 - \lambda_{c4}) s_{23}\, c_4 - \lambda_2 c_2\, c_4)((\lambda_3 - \lambda_{c4}) s4 - \lambda_2\, s_3\, s_4)\right)\dot{\theta}_1 \dot{\theta}_2 \\
&\quad + m_4 \left(((\lambda_3 - \lambda_{c4}) s_{23}\, c_4 - \lambda_2 c_2\, c_4)(\lambda_3 - \lambda_{c4}) s_4\right)\dot{\theta}_1 \dot{\theta}_3 + m_4 \left(((\lambda_3 - \lambda_{c4}) s_4 - \lambda_2\, s_3\, s_4)\, (\lambda_3 - \lambda_{c4}) s_4\right)\dot{\theta}_2 \dot{\theta}_3 + \frac{1}{2} m_4 \lambda_2^2 c_3^2 \dot{\theta}_2^2 + \frac{1}{2} I_{4L} s_{23}^2 c_4^2 \dot{\theta}_1^2 + \frac{1}{2} I_{4L} s_4^2 \dot{\theta}_2^2 + \frac{1}{2} I_{4L} s_4^2 \dot{\theta}_3^2 \\
&\quad + I_{4L} s_{23} c_4 s_4 \dot{\theta}_1 \dot{\theta}_2 + I_{4L} s_{23} c_4 s_4 \dot{\theta}_1 \dot{\theta}_3 + I_{4L} s_4{}^2 \dot{\theta}_2\, \dot{\theta}_3 + \frac{1}{2} I_{4L} s_{23}^2 s_4^2 \dot{\theta}_1^2 + \frac{1}{2} I_{4L} c_4^2 \dot{\theta}_2^2 + \frac{1}{2} I_{4L} c_4^2 \dot{\theta}_3^2 - I_{4L} s_{23} s_4 c_4 \dot{\theta}_1 \dot{\theta}_2 - I_{4L} s_{23} s_4 c_4 \dot{\theta}_1 \dot{\theta}_3 + I_{4L} c_4^2 \dot{\theta}_2\, \dot{\theta}_3 + \frac{1}{2} I_{4A} c_{23}^2 \dot{\theta}_1^2 \\
&\quad + \frac{1}{2} I_{4A} \dot{\theta}_4^2 - I_{4A} c_{23} \dot{\theta}_1 \dot{\theta}_4 \\
\mathrm{T}_4 &= \frac{1}{2} m_4 ((\lambda_3 - \lambda_{c4})^2 s_{23}^2 + \lambda_2^2 c_2^2 - 2(\lambda_3 - \lambda_{c4}) \lambda_2 c_2 s_{23})\dot{\theta}_1^2 + \frac{1}{2} m_4 (\lambda_2 + \lambda_3 - \lambda_{c4})^2 \dot{\theta}_2^2 + \frac{1}{2} m_4 (\lambda_{c4} - \lambda_3)^2 \dot{\theta}_3^2 + m_4 (\lambda_3 - \lambda_{c4} - \lambda_2 s_3)\, (\lambda_3 - \lambda_{c4})\dot{\theta}_2 \dot{\theta}_3 + \frac{1}{2} I_{4L} s_{23}^2 \dot{\theta}_1^2 + \frac{1}{2} I_{4L} \dot{\theta}_2^2 + \frac{1}{2} I_{4L} \dot{\theta}_3^2 + I_{4L} \dot{\theta}_2\, \dot{\theta}_3 \\
&\quad + \frac{1}{2} I_{4A} c_{23}^2 \dot{\theta}_1^2 + \frac{1}{2} I_{4A} \dot{\theta}_4^2 - I_{4A} c_{23}\, \dot{\theta}_1 \dot{\theta}_4 \\
\mathrm{T}_4 &= \frac{1}{2}(m_4 (\lambda_3 - \lambda_{c4})^2 s_{23}^2 + m_4 \lambda_2^2 c_2^2 - 2 m_4 (\lambda_3 - \lambda_{c4}) \lambda_2 c_2 s_{23} + I_{4L} s_{23}^2 + I_{4A} c_{23}^2)\dot{\theta}_1^2 + \frac{1}{2}(m_4 (\lambda_2 + \lambda_3 - \lambda_{c4})^2 + I_{4L})\dot{\theta}_2^2 + \frac{1}{2}(m_4 (\lambda_{c4} - \lambda_3)^2 + I_{4L})\dot{\theta}_3^2 + \frac{1}{2} I_{4A} \dot{\theta}_4^2 - I_{4A} c23\, \dot{\theta}_1 \dot{\theta}_4 \\
&\quad + (m_4 (\lambda_3 - \lambda_{c4} - \lambda_2 s_3)\, (\lambda_3 - \lambda_{c4}) + I_{4L})\dot{\theta}_2 \dot{\theta}_3
\end{aligned} \tag{14}
$$

$$
\mathrm{P}_4 = -m_4({}^0g)^T\, {}^0p_{c4} = -m_4 \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}^T \begin{bmatrix} \lambda_2 c_1 c_2 - (\lambda_3 - \lambda_{c4}) c_1 s_{23} \\ \lambda_2 s_1 c_2 - (\lambda_3 - \lambda_{c4}) s_1 s_{23} \\ \lambda_1 - \lambda_2 s_2 - (\lambda_3 - \lambda_{c4}) c_{23} \end{bmatrix} = m_4 g\, (\lambda_1 - \lambda_2 s_2 - (\lambda_3 - \lambda_{c4}) c_{23})
$$

Lastly, we must also find the kinetic and potential energies of a point load picked up by the end-effector. We will assume that the load is a point mass $m_L$, and thus, has no inertia.

$$\mathrm{T}_L = \frac{1}{2}m_L\,({}^4v_4)^T({}^4v_4)$$

$$\mathrm{T}_L = \frac{1}{2}m_L \left[\begin{bmatrix} -\lambda_2 c_2 s_4\dot{\theta}_1 + \lambda_3 s_{23} s_4\dot{\theta}_1 + \lambda_2\, s_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_3 \\ -\lambda_2 c_2 c_4\dot{\theta}_1 + \lambda_3 s_{23} c_4\dot{\theta}_1 - \lambda_2\, s_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_3 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix}\right]^T \left[\begin{bmatrix} -\lambda_2 c_2 s_4\dot{\theta}_1 + \lambda_3 s_{23} s_4\dot{\theta}_1 + \lambda_2\, s_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_3 \\ -\lambda_2 c_2 c_4\dot{\theta}_1 + \lambda_3 s_{23} c_4\dot{\theta}_1 - \lambda_2\, s_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_3 \\ \lambda_2 c_3\dot{\theta}_2 \end{bmatrix}\right]$$

$$\begin{aligned}\mathrm{T}_L = {}& \frac{1}{2}\, m_L(-\lambda_2 c_2 s_4\dot{\theta}_1 + \lambda_3 s_{23} s_4\dot{\theta}_1 + \lambda_2\, s_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_2 - \lambda_3 c_4\dot{\theta}_3)^2 \\ & + \frac{1}{2}\, m_L(-\lambda_2 c_2 c_4\dot{\theta}_1 + \lambda_3 s_{23} c_4\dot{\theta}_1 - \lambda_2\, s_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_2 + \lambda_3 s_4\dot{\theta}_3)^2 + \frac{1}{2}\, m_L(\lambda_2 c_3\dot{\theta}_2)^2\end{aligned} \tag{15}$$

$$\mathrm{T}_L = \frac{1}{2}\, m_L(\lambda_3^2 s_{23}^2 + \lambda_2^2 c_2^2 - 2\lambda_2\lambda_3 c_2 s_{23})\dot{\theta}_1^2 + \frac{1}{2}\, m_L(\lambda_2^2 + \lambda_3^2 - 2\lambda_2\lambda_3 s_3)\dot{\theta}_2^2 + \frac{1}{2}\, m_L\lambda_3^2\dot{\theta}_3^2 - m_L(\lambda_3^2 - \lambda_2\lambda_3 s_3)\dot{\theta}_2\dot{\theta}_3$$

$$\mathrm{P}_L = -m_L({}^0g)^T\,{}^0p_4 = -m_L\begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}^T \begin{bmatrix} \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} \\ \lambda_2 s_1 c_2 - \lambda_3 s_1 s_{23} \\ \lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23} \end{bmatrix} = m_L g\,(\lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23})$$

# 4 LAGRANGIAN

Finally, the Lagrangian $\mathcal{L}$ can be calculated as,

$$\mathcal{L} = \textstyle\sum_1^n \mathrm{T}_i - \sum_1^n \mathrm{P}_i\,, \quad n = 4 \tag{16}$$

Using equation 16, in addition to the kinetic energies $\mathrm{T}_i$ and potential energies $\mathrm{P}_i$ from equations 11 to 14, as well as taking the point load into account from equation 15, followed by rearranging, yields the Lagrangian as, rearranging and taking the point load energy into account as well yields,

$$\begin{aligned}\mathcal{L} = {}& \Big(\frac{1}{2}\, I_{1A} + \frac{1}{2}\, m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + \frac{1}{2}\, I_{2A} s_2^2 + \frac{1}{2}\, I_{2L} c_2^2 + \frac{1}{2}\, m_3\lambda_2^2 c_2^2 + \frac{1}{2}\, m_3\lambda_{c3}^2 s_{23}^2 - m_3\lambda_2\lambda_{c3} c_2 s_{23} + \frac{1}{2}\, I_{3L} s_{23}^2 + \frac{1}{2}\, I_{3A} c_{23}^2 \\ & + \frac{1}{2} m_4(\lambda_3 - \lambda_{c4})^2 s_{23}^2 + \frac{1}{2} m_4\lambda_2^2 c_2^2 - m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 s_{23} + \frac{1}{2} I_{4L} s_{23}^2 + \frac{1}{2} I_{4A} c_{23}^2 + \frac{1}{2} m_L\lambda_3^2 s_{23}^2 \\ & + \frac{1}{2} m_L\lambda_2^2 c_2^2 - m_L\lambda_2\lambda_3 c_2 s_{23}\Big)\dot{\theta}_1^2 \\ & + \Big(\frac{1}{2}\, m_2(\lambda_2 - \lambda_{c2})^2 + \frac{1}{2}\, I_{2L} + \frac{1}{2}\, m_3\lambda_2^2 + \frac{1}{2}\, m_3\lambda_{c3}^2 - m_3\lambda_2\lambda_{c3} s_3 + \frac{1}{2}\, I_{3L} + \frac{1}{2} m_4(\lambda_2 + \lambda_3 - \lambda_{c4})^2 \\ & + \frac{1}{2} I_{4L} + \frac{1}{2} m_L\lambda_2^2 + \frac{1}{2} m_L\lambda_3^2 - m_L\lambda_2\lambda_3 s_3\Big)\dot{\theta}_2^2 \\ & + \Big(\frac{1}{2} m_3\lambda_{c3}^2 + \frac{1}{2}\, I_{3L} + \frac{1}{2} m_4(\lambda_{c4} - \lambda_3)^2 + \frac{1}{2} I_{4L} + \frac{1}{2} m_L\lambda_3^2\Big)\dot{\theta}_3^2 + \Big(\frac{1}{2} I_{4A}\Big)\dot{\theta}_4^2 - (I_{4A} c23)\,\dot{\theta}_1\dot{\theta}_4 \\ & + \Big(m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2 s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2 s_3)\,(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L\big(\lambda_3^2 - \lambda_2\lambda_3 s_3\big)\Big)\dot{\theta}_2\dot{\theta}_3 \\ & - m_1 g\,(\lambda_1 - \lambda_{c1}) - m_2 g\,(\lambda_1 - (\lambda_2 - \lambda_{c2}) s_2) - m_3 g\,(\lambda_1 - \lambda_2 s_2 - \lambda_{c3} c_{23}) \\ & - m_4 g\,(\lambda_1 - \lambda_2 s_2 - (\lambda_3 - \lambda_{c4}) c_{23}) - m_L g\,(\lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23})\end{aligned} \tag{17}$$