Forward Kinematics

1 Transformation Matrices

The DH table for the QArm manipulator is presented in Table 1 (See 1. Frame Assignments for more information).

$\boldsymbol{i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$	$\boldsymbol{d_i}$	$\boldsymbol{\theta_i}$
1	0	$-\pi/2$	λ_1	θ_1
2	λ_2	0	0	θ_2
3	0	$-\pi/2$	0	θ_3
4	0	0	λ_3	θ_4

Table 1. DH Table

Begin with the general form of the transformation matrix,

$$^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\ \cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\ \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Using Equation 1 and Table 1, we can find all the transformation matrices. For brevity, we can abbreviate cosines as c and sines as s. In addition, the revolute joint variables θ will be dropped. Thus, the term $\cos\theta_1$ can be abbreviated as c_1. Similarly, a term such as $\sin(\theta_1+\theta_2)$ can be abbreviated as s_{12}.

$$^0T_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & \lambda_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^1T_2 = \begin{bmatrix} c_2 & -s_2 & 0 & \lambda_2\ c_2 \\ s_2 & c_2 & 0 & \lambda_2\ s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$^2T_3 = \begin{bmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^3T_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & \lambda_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

2 Forward Kinematics Formulation

The joint vector $\vec{\theta}$ represents the joint states of the manipulator,

$$\vec{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \tag{3}$$

and the position $\vec{p}$ of the end-effector {4} (expressed in base frame {0})

$$\overrightarrow{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^0p_4 \tag{4}$$

The goal of the forward kinematics formulation is to provide a mapping from the joint space to the position space

$$\overrightarrow{p} = f_{FPK}\left(\overrightarrow{\theta}\right) \tag{5}$$

Using the matrices in equation 2, we can proceed to derive the matrices ${}^0T_2$, ${}^0T_3$ and ${}^0T_4$.

$${}^0T_2 = {}^0T_1{}^1T_2 = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & \lambda_2c_1c_2 \\ s_1c_2 & -s_1s_2 & c_1 & \lambda_2s_1c_2 \\ -s_2 & -c_2 & 0 & \lambda_1 - \lambda_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$${}^0T_3 = {}^0T_2{}^2T_3 = \begin{bmatrix} c_1c_{23} & s_1 & -c_1s_{23} & \lambda_2c_1c_2 \\ s_1c_{23} & -c_1 & -s_1s_{23} & \lambda_2s_1c_2 \\ -s_{23} & 0 & -c_{23} & \lambda_1 - \lambda_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$${}^0T_4 = {}^0T_3{}^3T_4 = \begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & -c_1c_{23}s_4 + s_1c_4 & -c_1s_{23} & \lambda_2c_1c_2 - \lambda_3c_1s_{23} \\ s_1c_{23}c_4 - c_1s_4 & -s_1c_{23}s_4 - c_1c_4 & -s_1s_{23} & \lambda_2s_1c_2 - \lambda_3s_1s_{23} \\ -s_{23}c_4 & s_{23}s_4 & -c_{23} & \lambda_1 - \lambda_2s_2 - \lambda_3c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Consider the vector representing the position of the end-effector frame {4} $\overrightarrow{p}$. From the matrix ${}^0T_4$ in equation 8, we can extract this vector expressed in frame {0} as ${}^0p_4$, and can also extract the orientation (via rotation matrix) ${}^0R_4$ of the end-effector frame {4} with respect to the base frame {0}.

$$\begin{aligned} p_x &= \lambda_2c_1c_2 - \lambda_3c_1s_{23} \\ p_y &= \lambda_2s_1c_2 - \lambda_3s_1s_{23} \\ p_z &= \lambda_1 - \lambda_2s_2 - \lambda_3c_{23} \end{aligned} \tag{9}$$

Equation 9 represents the forward kinematics formulation f_{FPK}. Note that the wrist rotation angle θ_4 does not play a part in the position of the end-effector, and correspondingly does not appear in f_{FPK}. However, it does play a part in the end-effector orientation,

$${}^0R_4 = \begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & -c_1c_{23}s_4 + s_1c_4 & -c_1s_{23} \\ s_1c_{23}c_4 - c_1s_4 & -s_1c_{23}s_4 - c_1c_4 & -s_1s_{23} \\ -s_{23}c_4 & s_{23}s_4 & -c_{23} \end{bmatrix} \tag{10}$$