Advanced Derivatives Modelling Seminar: Volatility derivatives

There is a market for contracts with direct exposure to volatility. One such contract is the variance swap with a swap payoff where realised variance is exchanged against a fixed strike. The percentage change in value of the stock is is measured each day, squared and averaged over the life of the contract and this is swapped against a fixed strike. The payoff is thus

$$\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2 - K \tag{0.1}$$

For the volatility swap, the payoff is

$$\sqrt{\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2} - K \tag{0.2}$$

For simplicity assume zero interest rates and dividends and stock price follows the process:

$$dS_t = \sigma_t S_t dW_t,$$

with stochastic volatility σ_t. We assume the volatility to be uncorrelated with the brownian driving the stock process.

- Use the independence assumption to write the value of a derivative with payoff $f(S_T)$ as an integral with respect of the density of the integrated variance $\Sigma_{0,T} = \int_0^T \sigma_t^2 dt$.
- Show that if you choose the payoff $f(S_T) = \left(\frac{S_T}{S_0}\right)^p$, the value simplifies to

$$\mathbb{E}\left[e^{\frac{1}{2}p(p-1)\Sigma_{0,T}}\right] \tag{0.3}$$

- We have shown in the lecture that any twice-differentiable payoff at time T may be statically hedged using a portfolio of calls and puts with same maturity T. How does it change/simplify with our assumption of zero rates?

- Write it in the case of the $f(S_T) = \left(\frac{S_T}{S_0}\right)^p$.
- Rewrite the result in terms of log-strike $k = \log \frac{K}{S_0}$.
- In Black-Scholes model, we have Put-Call symmetry: $P(K, T, S, \sigma, r) = C(Se^{rT}, T, Ke^{-rT}, \sigma, r)$. Show that in the case of stochastic volatility, if correlation between stock and volatility is zero, the symmetry relationship still holds.
- Use put-call symmetry to rewrite your static arbitrage formula in terms of calls only.
- Deduce the value V_0 of a contract with payoff $\Sigma_{0,T}$.
- Deduce the value of forward variance $v_t(T) = \mathbb{E}_t\left[\sigma_T^2\right]$ and show that it is a martingale.