Semiparametric SDF Estimators for Pooled, Non-Traded Cash Flows

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Declaration of interest

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Abstract

This paper analyzes stochastic discount factor estimation methodologies suited for pooled, non-traded cash flow streams such as the fund-level cash flows of private equity funds. The asymptotic inference framework for our semiparametric nonlinear least squares estimator draws on a spatial notion, i.e., the idea that the economic distance between distinct private equity funds can be measured. The empirical and Monte Carlo simulation results indicate (i) that our method can improve the popular Generalized Method of Moments approach of Driessen et al. (2012), but (ii) that estimator variance for typical data sizes is still high. Thus, we conjecture that traditional semiparametric extremum estimators like the one described by us shall be exclusively used for single-factor models until considerably more vintage year information for private equity funds is available.

1 Introduction

Private equity has outgrown its niche, sitting today on more than $9 trillion in assets under management, yet rigorous asset-pricing tools have not kept pace with this ascent. The empirical analysis and risk assessment of private equity and other non-traded cash flows remain fundamentally challenging due to the absence of market-based valuations and the inherent frictions of private markets (i.e., under incomplete information). Unlike public assets with trusted and tradeable valuations (in liquid secondary markets), private equity investments generate irregular, infrequently observed cash flows for which standard return-based asset pricing techniques are unsuitable.

We address this gap by proposing a semiparametric stochastic discount factor (SDF) estimator tailored to fund-level cash flows that refines the SDF estimators of Driessen et al. (2012) and Korteweg and Nagel (2016). Our nonlinear least squares estimator stems from the class of Least-Mean-Distance (LMD) estimators, which are easier to handle than traditional Generalized Method of Moments (GMM) approaches (Pötscher and Prucha, 1997). Our LMD estimator arguably both simplifies and generalizes the GMM methodology of Driessen et al. (2012), where we provide the asymptotic inference framework that was missing in the original paper. The asymptotic inference formulations rely on the concept of spatial (near-epoch) dependence between funds as proposed by Korteweg and Nagel (2016). In this context, it is paramount to quantify the economic distance between funds by a measure like absolute vintage year difference or cash flow overlap[1].

For fund-level cash flow data of private equity funds, we document an asymptotic bias term for cash-flow-based SDF estimators like Driessen et al. (2012) and Korteweg and Nagel (2016) that persists also in large samples. The bias term arises due to the pooled nature of fund-level cash flows and more specifically because of

[1] However, this economic inter-fund *distance* refers **not** to the term Least-Mean-*Distance* estimator.

the different starting dates of the underlying deals (in the fund investment period). Our estimator offers simple averaging over multiple discounting dates as one option to control (but not eliminate) the bias term.

In the empirical application of our new estimator, we test simple linear and exponentially affine SDF models that can draw on the five return factors associated with the q^5 investment factor model recently proposed by Hou et al. (2020). Based on a Spatial Heteroskedasticity and Autocorrelation Consistent (SHAC) covariance matrix estimator, we calculate asymptotic standard errors for the model coefficients. Moreover, we assess the small-sample variance of coefficient estimates and the out-of-sample performance of the different SDF models by hv-block cross-validation, which accounts for the inter-vintage-year dependence of private equity funds (Racine, 2000). We test one- and two-factor models for the most prominent private capital fund types Private Equity (PE) and Venture Capital (VC). The empirical two-factor model results are rather disappointing; not more than the single-market-factor model results seem reasonable given the high estimator variance.

The paper is structured as follows. Section 2 introduces our semiparametric LMD estimator and its corresponding asymptotic inference framework. Section 3 applies the method to estimate q^5-investment-factor SDFs for various private equity fund types using simulated and real-world cash flows. Section 4 concludes.

2 Methodology

Our general SDF estimation framework is similar to that of Driessen et al. (2012) and Korteweg and Nagel (2016); the subtle but important differences are discussed in Section 2.5.

In a nutshell, the Driessen et al. (2012) estimator determines the estimated alpha and beta parameters of a linear factor model by minimizing the squared net present value (NPV) of all n fund cash flows in a least-squares optimization problem.

$$\left(\hat{\alpha}, \hat{\beta}\right) = \arg\min_{\alpha,\beta} \sum_{i=1}^{n} \left[NPV_i(\alpha, \beta)\right]^2 \tag{1}$$

with NPV for the ith fund

$$NPV_i(\alpha, \beta) = \sum_{t \in \{t_{0,i}, t_{0,i}+1, \dots\}} \frac{\text{CashFlow}_{i,t}}{\prod_{s=t_{0,i}+1}^{t} \left(1 + r_{\text{free},s} + \alpha + \beta r_{\text{market},s}\right)}$$

Here, the Driessen et al. (2012) approach relies on the identifying assumption that the model is correctly specified, so that $\mathbb{E}[NPV_i(\theta_0)] = 0$.

In this Methodology section, we will show:

1. Why this estimator is unconditionally unbiased for pooled cash flows when we use this Net Present Value approach and analyze the bias term introduced by the Net Future Value approach (Section 2.1).

2. How to modify the estimator to be able to control the bias term (Section 2.2).

3. How we can estimate asymptotic standard errors for the parameters alpha and beta (Sections 2.3 and 2.4).
4. How our estimator compares to similar approaches (Section 2.5.)

Symbol	Description
n	Number of cross-sectional units (funds or portfolios)
V	Number of distinct vintage years
n_v	Number of portfolios in vintage year v
v_i	Vintage year of fund/portfolio i
J	Number of deals per fund
$K_{i,j}$	Number of divestment cash flows for deal j of fund i
$CF_{i,t}$	Net cash flow of fund i at time t
Ψ_t , M_t	Multi-period and single-period SDF
$\Psi_{\tau,t}(\theta)$	SDF ratio Ψ_t/Ψ_τ, parameterized by θ
$\theta = (\alpha, \beta)$	SDF parameter vector (intercept and factor loadings)
θ_0	True parameter vector
$\delta_{i,j}$	Deal-level pricing error
$\epsilon_{\tau,i}$	Fund pricing error at discounting date τ
$\bar{\epsilon}_i$	Weighted average pricing error of fund i
a_i	Fund-level alpha (abnormal return; distinct from SDF α)
$\mathcal{T}_i$	Set of discounting dates for fund i
w_i	Weighting factor for fund i
$L(\cdot)$	Loss function (e.g., $L(x) = x^2$)
$Q_n(\theta)$	Sample objective function
$d_{i,j}$	Economic distance between units i and j
D $(= b_n)$	SHAC bandwidth parameter
H	Hessian of population objective at θ_0
Λ	Long-run covariance matrix of the score
Σ	Asymptotic covariance matrix of $\hat{\theta}$

Table 1: Key notation for the Methodology section.

2.1 Asset Pricing for Pooled Cash Flows

Let private equity fund $i = 1, 2, \ldots, n$ be characterized by its net cash flows $CF_{i,t}$ (i.e., distributions minus contributions) and its net asset values $NAV_{i,t}$ with discrete time index $t = 0, 1, 2, \ldots, T$. To increase the mathematical tractability of the problem, we assume a deal-by-deal data generating process (DGP) for CF where each fund deal consists exactly of one investment and one divestment cash flow in combination with a simple return model for the multi-period deal returns. This means the fund-level cash flow process $(CF_{i,t})_{t=0,1,\ldots,T}$ is an aggregation of deal-level cash flow pairs consisting of one negative at deal inception and at least one positive cash flow later $CF_{i,t} = \sum_j^J cf_{j,i,t}$.

Assumption 1. *Deal-level data generation process:*
1. Each fund i consists of J underlying deals.
2. Each deal is characterized by exactly one, negative investment cash flow, denoted by $\mathrm{Inv}_{i,j}$, which occurs at time $t^{\mathrm{Inv}}_{i,j} \in \{0, 1, \ldots, T-1\}$. It holds $\mathrm{Inv}_{i,j} < 0$.
3. Each deal is characterized by a positive divestment cash flow stream, denoted by $(\mathrm{Div}_{i,j,k})_{k=1,\ldots,K_{i,j}}$, which occur after the investment cash flow $t^{\mathrm{Div}}_{i,j,k} > t^{\mathrm{Inv}}_{i,j}$ for all k. It holds $\mathrm{Div}_{i,j,k} > 0$.
4. The cumulative fund cash flows are generated by summarizing over all deal-level cash flows, i.e., $\sum_{t=0}^{T} CF_{i,t} = \sum_{j=1}^{J} \left(\mathrm{Inv}_{i,j} + \sum_{k=1}^{K_{i,j}} \mathrm{Div}_{i,j,k} \right)$ for all i.

From asset pricing theory, we know that we can use a stochastic discount factor Ψ_t to price each underlying deal

$$\mathbb{E}\left[\delta_{i,j} \,\middle|\, \mathcal{F}_{t^{\mathrm{Inv}}_{i,j}}\right] = 0 \quad \forall \; i, j \tag{2}$$

where we denote the deal-level pricing error by

$$\delta_{i,j} := \mathrm{Inv}_{i,j} + \sum_{k=1}^{K_{i,j}} \frac{\Psi_{t^{\mathrm{Div}}_{i,j,k}}}{\Psi_{t^{\mathrm{Inv}}_{i,j}}} \mathrm{Div}_{i,j,k} \tag{3}$$

Before we can derive the expected value for the fund-level pricing errors, let us analyze the deal-level counterparts.

Case A: Discounting deals to a date before investment ($\tau < t^{\mathrm{Inv}}_{i,j}$) First we apply the Law of Iterated Expectations (the so-called "tower property") to determine the fair value of individual deal cash flows at time $\tau < t^{\mathrm{Inv}}_{i,j}$.

$$\mathbb{E}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau} \delta_{i,j} \mid \mathcal{F}_\tau\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau} \delta_{i,j} \mid \mathcal{F}_{t^{\mathrm{Inv}}_{i,j}}\right] \mid \mathcal{F}_\tau\right] \tag{4}$$

Since the term $\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau}$ is fully known at time $t^{\mathrm{Inv}}_{i,j}$, it can be pulled out of this inner expectation to receive

$$\mathbb{E}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau} \delta_{i,j} \mid \mathcal{F}_\tau\right] = \mathbb{E}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau} \underbrace{\mathbb{E}\left[\delta_{i,j} \mid \mathcal{F}_{t^{\mathrm{Inv}}_{i,j}}\right]}_{=0 \text{ by Eq. 2}} \mid \mathcal{F}_\tau\right] = \mathbb{E}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau} \cdot 0 \mid \mathcal{F}_\tau\right] = 0 \tag{5}$$

Case B: Discounting deals to a date after investment ($\tau > t^{\mathrm{Inv}}_{i,j}$) When the discounting/compounding date τ is after the investment date, the term $\Psi_{t^{\mathrm{Inv}}_{i,j}}/\Psi_\tau$ is not known at $t^{\mathrm{Inv}}_{i,j}$ (it depends on future SDF realizations). Thus, the previous logic does not hold. Instead, we have a non-zero term that represents the accumulated realized pricing error.

Let $\tau > t^{\mathrm{Inv}}_{i,j}$. The expectation becomes:

$$\mathbb{E}\left[\frac{\Psi_{t^{\text{Inv}}_{i,j}}}{\Psi_\tau}\delta_{i,j} \mid \mathcal{F}_\tau\right] = \frac{\Psi_{t^{\text{Inv}}_{i,j}}}{\Psi_\tau}\mathbb{E}\left[\delta_{i,j} \mid \mathcal{F}_\tau\right] \tag{6}$$

Since we condition on $\mathcal{F}_\tau$ and $t^{\text{Inv}}_{i,j} < \tau$, the SDF ratio is known and can be pulled out. Then the term $\mathbb{E}\left[\delta_{i,j} \mid \mathcal{F}_\tau\right]$ is the conditional expectation of the pricing error given information available at time τ. It decomposes as:

$$\mathbb{E}\left[\delta_{i,j} \mid \mathcal{F}_\tau\right] = \underbrace{\text{Inv}_{i,j} + \sum_{t^{\text{Div}}_{i,j,k} \leq \tau} \frac{\Psi_{t^{\text{Div}}_{i,j,k}}}{\Psi_{t^{\text{Inv}}_{i,j}}}\text{Div}_{i,j,k}}_{\text{Realized Cash Flows}} + \underbrace{\mathbb{E}\left[\sum_{t^{\text{Div}}_{i,j,k} > \tau} \frac{\Psi_{t^{\text{Div}}_{i,j,k}}}{\Psi_{t^{\text{Inv}}_{i,j}}}\text{Div}_{i,j,k} \mid \mathcal{F}_\tau\right]}_{\text{Expected Future Cash Flows}} \tag{7}$$

Generally, this sum can become (is usually) non-zero because it includes realized shocks (abnormal returns) that occurred between $t^{\text{Inv}}_{i,j}$ and τ.

Case C: Compounding deals to a future date Let us analyze the following Net Future Value (NFV) case, where $t^{\text{Future}}_{i,j}$ denotes a compounding target date after the deal investment date (i.e., $\tau < t^{\text{Inv}}_{i,j} < t^{\text{Future}}_{i,j}$).

$$\begin{aligned}
\mathbb{E}\left[\frac{\Psi_{t^{\text{Future}}_{i,j}}}{\Psi_\tau}\delta_{i,j} \mid \mathcal{F}_\tau\right] &= \mathbb{E}\left[\frac{\Psi_{t^{\text{Inv}}_{i,j}}}{\Psi_\tau}\frac{\Psi_{t^{\text{Future}}_{i,j}}}{\Psi_{t^{\text{Inv}}_{i,j}}}\delta_{i,j} \mid \mathcal{F}_\tau\right] \\
&= \mathbb{E}\left[\frac{\Psi_{t^{\text{Inv}}_{i,j}}}{\Psi_\tau}\mathbb{E}\left[\frac{\Psi_{t^{\text{Future}}_{i,j}}}{\Psi_{t^{\text{Inv}}_{i,j}}}\delta_{i,j} \mid \mathcal{F}_{t^{\text{Inv}}_{i,j}}\right] \mid \mathcal{F}_\tau\right] \\
&= \mathbb{E}\left[\frac{\Psi_{t^{\text{Inv}}_{i,j}}}{\Psi_\tau}\left(\text{Cov}\left(\frac{\Psi_{t^{\text{Future}}_{i,j}}}{\Psi_{t^{\text{Inv}}_{i,j}}}, \delta_{i,j} \mid \mathcal{F}_{t^{\text{Inv}}_{i,j}}\right) + \mathbb{E}\left[\frac{\Psi_{t^{\text{Future}}_{i,j}}}{\Psi_{t^{\text{Inv}}_{i,j}}} \mid \mathcal{F}_{t^{\text{Inv}}_{i,j}}\right]\underbrace{\mathbb{E}\left[\delta_{i,j} \mid \mathcal{F}_{t^{\text{Inv}}_{i,j}}\right]}_{=0 \text{ by Eq. 2}}\right) \mid \mathcal{F}_\tau\right] \\
&= \mathbb{E}\left[\frac{\Psi_{t^{\text{Inv}}_{i,j}}}{\Psi_\tau}\text{Cov}\left(\frac{\Psi_{t^{\text{Future}}_{i,j}}}{\Psi_{t^{\text{Inv}}_{i,j}}}, \delta_{i,j} \mid \mathcal{F}_{t^{\text{Inv}}_{i,j}}\right) \mid \mathcal{F}_\tau\right]
\end{aligned} \tag{8}$$

The term can generally become non-zero, representing the covariance between the deal's pricing error and the future level of the SDF, effectively a risk premium for the timing mismatch.

This result highlights a fundamental dichotomy. For *ex-ante* asset pricing tests, the Net Future Value approach is misspecified: the non-zero conditional covariance implies that the conditional moment condition $\mathbb{E}[m^{\text{NFV}}_{i,j}(\theta_0) \mid \mathcal{F}_\tau] = 0$ is violated, leading to biased parameter estimates if not explicitly corrected for this risk premium. Conversely, for *ex-post* performance benchmarking, this compounding approach remains valuable. By compounding realized cash flows to a future date T, we facilitate an intuitive comparison of the fund's generated terminal wealth against a public benchmark, provided one acknowledges that the resulting *alpha* implicitly includes the compensation for the timing risk identified above. This entails that any *alpha* measured via Net Future Values is mechanically a composite metric. Specifically, we can mathematically decompose the expected NFV pricing error into two distinct components:

$$\underbrace{\mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\text{Future}}}}{\Psi_\tau}\delta_{i,j} \mid \mathcal{F}_\tau\right]}_{\text{Measured Alpha}} = \underbrace{\mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau}\frac{a_{i,j}^{\text{True}}}{R_{f,t_{i,j}^{\text{Inv}}\to t_{i,j}^{\text{Future}}}} \mid \mathcal{F}_\tau\right]}_{\text{True Skill}} \tag{9}$$
$$+\underbrace{\mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau}\text{Cov}\left(\frac{\Psi_{t_{i,j}^{\text{Future}}}}{\Psi_{t_{i,j}^{\text{Inv}}}}, \delta_{i,j} \mid \mathcal{F}_{t_{i,j}^{\text{Inv}}}\right) \mid \mathcal{F}_\tau\right]}_{\text{Timing Risk Premium}}$$

where R_f is the gross risk-free return over the holding period and a^{True} is the deal's expected pricing error at inception (fund-level abnormal return, distinct from the SDF intercept α). Even if a manager has no skill ($a^{\text{True}} = 0$), the measured alpha will include the *Timing Risk Premium* which is the (discounted) covariance between the future SDF realization and the deal valuation. This is not a flaw but a feature of ex-post benchmarking: it correctly reflects that the investor was exposed to this specific path of SDF realizations. However, for ex-ante pricing tests, the presence of this non-zero *Timing Risk Premium* is problematic. Standard econometric estimators are designed to find parameters that set the average pricing error to zero. If the theoretical moment contains this positive premium, the estimator will be forced to compensate by biasing the estimated alpha downwards or distorting the SDF parameters to “explain away” the *Timing Risk Premium.*

Fund-level pricing error For the pooled, fund-level cash flow stream, we assume that the true fund valuation $V_{i,\tau}$ is generally not observable for us

$$V_{i,\tau} := \mathbb{E}\left[\sum_{t=\tau}^{T}\frac{\Psi_t}{\Psi_\tau}CF_{i,t}\,|\mathcal{F}_\tau\right] \quad \forall\ \tau \geq\ \min_j t_{i,j}^{\text{Inv}} \tag{10}$$

For active funds with $\tau > \min_j t_{i,j}^{\text{Inv}}$, we merely can observe reported fund NAVs as proxies for the true fund value, which are known to suffer from stale pricing and appraisal-smoothing problems (Brown et al., 2019; Korteweg, 2019; Jenkinson et al., 2020)[2]. Additionally, we define the vintage year for each fund in a mathematically convenient way as its “starting year” $v_i := \text{YEAR}\left(\min_j t_{i,j}^{\text{Inv}}\right)$.

For any discounting date $\tau \leq \min_j t_{i,j}^{\text{Inv}}$, the expected fund pricing error is equal to zero.

Proposition 1. *(Price of a pooled cash flow stream at fund inception)*

$$\mathbb{E}\left[\sum_{t=\tau}^{T}\frac{\Psi_t}{\Psi_\tau}CF_{i,t}\,|\mathcal{F}_\tau\right] = 0 \tag{11}$$

if $\tau \leq\ \min_j t_{i,j}^{\text{Inv}}$.

[2]In the empirical section, we will treat the most recent NAV as the final distribution cash flow for non-liquidated funds to increase the sample data.

Proof. Let us first analyze the case $\tau \leq \min_j t_{i,j}^{\text{Inv}}$ by expressing the fund cash flows in term of discounted deal pricing errors using Equation 3.

$$\mathbb{E}\left[\sum_{t=\tau}^{T} \frac{\Psi_t}{\Psi_\tau} CF_{i,t} \,|\mathcal{F}_\tau\right] = \mathbb{E}\left[\sum_{j=1}^{J} \frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau} \delta_{i,j} \,|\mathcal{F}_\tau\right] \tag{12}$$

Then we can simply plug in Equation 5 to receive the postulated zero expected fund pricing error.

$$\mathbb{E}\left[\sum_{j=1}^{J} \frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau} \delta_{i,j} \,|\mathcal{F}_\tau\right] = \mathbb{E}\left[\sum_{j=1}^{J} \frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau} \cdot 0 \,|\mathcal{F}_\tau\right] = 0 \tag{13}$$

□

Next, let us analyze the case $\tau > \min_j t_{i,j}^{\text{Inv}}$ for Equation 10. For any deal j with $t_{i,j}^{\text{Inv}} < \tau$, the conditional expectation $\mathbb{E}[\delta_{i,j}|\mathcal{F}_\tau]$ represents the realized abnormal return of the deal up to time τ. For a generic risky asset, this realized return is a random variable with non-zero variance. Therefore, the condition that Equation 10 equals 0 holds *ex-post* only by chance (i.e., if realized shocks cancel out exactly). Ex-ante, without knowledge of realizations, $\tau \leq \min_j t_{i,j}^{\text{Inv}}$ is the necessary condition to guarantee a zero expected pricing error.

2.2 Least-Mean-Distance estimator

In this subsection, we introduce a new SDF estimator designed to analyze the effect of different discounting dates τ on the bias and variance of the estimated SDF parameters. In the previous subsection, we demonstrated that only the fund inception date yields unconditionally unbiased parameter estimates in a NPV-style optimization formulation. However, to be able to empirically test the impact of mixing in biased NFV terms, we choose to generalize our estimation framework so we can handle both cases flexibly. With this respect, our general idea is to rather average over multiple "suitable" discounting date candidates τ than to select only one candidate for the "unconditionally optimal" discounting date.

Henceforth, we assume that the underlying transactions within a private equity fund cannot be distinguished individually, and that only the funds total (pooled) cash flows are observable. The stochastic discount factor $\Psi_{\tau,t}$ is used to calculate the time-τ "price" $P_{\tau,t,i}$ of a **single** time-t cash flow of any given PE fund i

$$P_{\tau,t,i} := \Psi_{\tau,t} \cdot CF_{i,t} = \frac{\Psi_t}{\Psi_\tau} \cdot CF_{i,t} \qquad \forall \quad \tau, t, i \tag{14}$$

with multi-period SDF $\Psi_t = \prod_{k=1}^{t} M_k$ and $M_0 = 1$. As SDFs are commonly parameterized by a vector $\theta \in \mathbb{R}^p$, i.e., $\Psi_{\tau,t} \equiv \Psi_{\tau,t}(\theta)$, our goal is to find an estimation method for the optimal θ. We denote this optimal/best/true parameter vector as θ_0. We call the numerator Ψ_t the discount part of the multi-period SDF $\Psi_{\tau,t}$ (used for present value calculations) and the denominator Ψ_τ the compound part (used for future value calculations). For each fund i and all points τ within a common fund lifetime, the empirical "pricing error" $\epsilon_{\tau,i}$ of **all** fund cash flows is calculated as time-τ "present value"

$$\epsilon_{\tau,i} := \sum_{t=0}^{T} P_{\tau,t,i} \qquad \forall \quad \tau, i \tag{15}$$

We use the terms "price", "pricing error" and "net present value" in quotation marks to acknowledge the theoretical asset pricing problem which can arise for pooled cash flows and has been described in the previous subsection.

To better analyze the impact of different discount date τ on the estimator's bias and variance, we define the (w_i-weighted) average pricing error $\bar{\epsilon}_i$ that averages over the set $\mathcal{T}_i$

$$\bar{\epsilon}_i = w_i \cdot \frac{1}{\operatorname{card}(\mathcal{T}_i)} \sum_{\tau \in \mathcal{T}_i} \epsilon_{\tau,i} \qquad \forall \quad i \tag{16}$$

where $\mathcal{T}_i$ gives the set of discounting dates τ for fund i which is more thoroughly described below. Finally, the scalar weighting factor w_i can be (i) one divided by the fund's invested capital for equal weighting of funds, (ii) one divided by the vintage year sum of invested capital for vintage year weighting, (iii) the scalar one for fund-size weighting, or (iv) some macroeconomic deflator.

The role of $\mathcal{T}_i$ The set $\mathcal{T}_i$ controls which discounting dates enter the averaged pricing error $\bar{\epsilon}_i$ in Equation 16, as visualized in Figure 1. The smallest choice $\mathcal{T}_i = \{\min_j t_{i,j}^{\text{Inv}}\}$ corresponds to a classical NPV calculation and is used by both Driessen et al. (2012) and Korteweg and Nagel (2016). The largest candidate set includes all periods from inception to the present.

Averaging over multiple dates in $\mathcal{T}_i$ was originally motivated by the "exploding alpha" problem first described by Driessen et al. (2012): when the initial cash flow is very small, an arbitrarily large alpha can discount all subsequent cash flows close to zero, rendering the beta factors irrelevant and distorting estimation. Averaging over $\mathcal{T}_i$ mitigates this degeneracy.

However, as shown in Section 2.1, including future value dates in $\mathcal{T}_i$ introduces a timing risk premium bias. The bias–variance tradeoff of different $|\mathcal{T}_i|$ choices is studied empirically in the simulation study (Section 3.3).

Assumption 2. *To guarantee an unconditionally unbiased estimator, we assume to always only discount to the fund start date* $\mathcal{T}_i = \left\{\min_j t_{i,j}^{\text{Inv}}\right\}$.

Remark 1. *Assumption 2 guarantees that* $\mathbb{E}[\bar{\epsilon}_i(\theta_0)] = 0$ *under the true parameter. For the quadratic loss* $L(x) = x^2$*, the population objective* $Q_0(\theta) = -\mathbb{E}[\bar{\epsilon}_i(\theta)^2] = -\operatorname{Var}[\bar{\epsilon}_i(\theta)] - (\mathbb{E}[\bar{\epsilon}_i(\theta)])^2$ *involves both variance and bias terms. Consistency then requires that* θ_0 *uniquely minimizes this composite criterion (see Proposition 2).*

To find the optimal value for θ, we select an estimator from the broad class of extremum estimators.

Definition 1. *Extremum estimator (Newey and McFadden, 1994, Equation 1.1): An estimator* $\hat{\theta}$ *is an extremum estimator if there is an objective function* $Q_n(\theta)$ *such that*

$$\hat{\theta} = \arg\max_{\theta} \; Q_n(\theta)$$

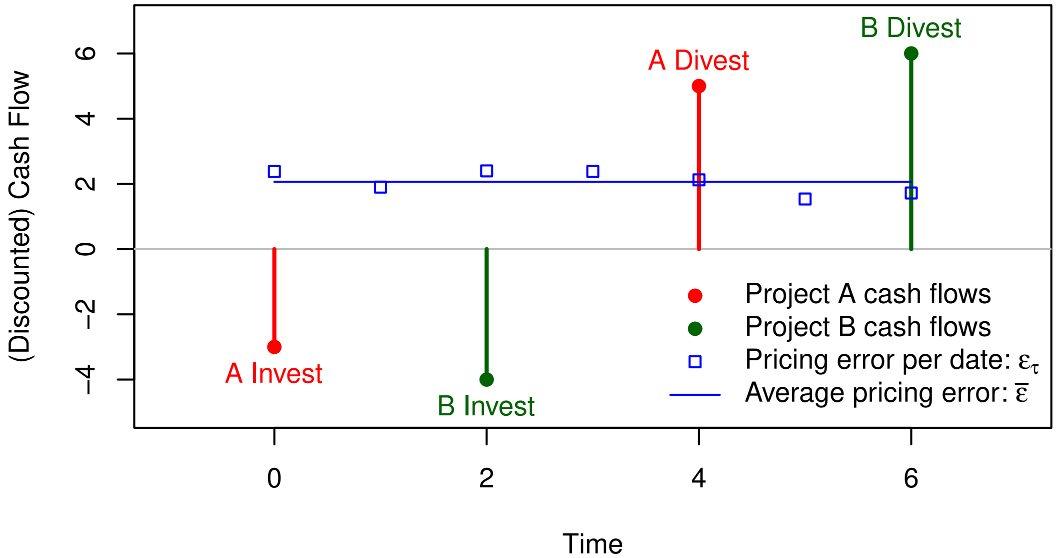

Figure 1: How to calculate and interpret the average pricing error introduced by Equation 16: The time index t is relevant for the net cash flows (black dots). The time index τ is used for the pricing error, i.e., the sum of discounted net cash flows (blue boxes). The weighted average of these "pricing errors" gives the average pricing error $\bar{\epsilon}$ as defined in Equation 16 (solid blue line). In this example, $\text{card}(\mathcal{T}_i) = 7$, i.e., the number of blue boxes.

for $\theta \in \Theta$ where Θ is the set of all possible parameter values.

Concretely, our Least Mean Distance (LMD) estimator (Pötscher and Prucha, 1997, Equation 7.1) minimizes the average loss of $\bar{\epsilon}$. For consistency with Definition 1, we will technically maximize the average negative loss of $\bar{\epsilon}$, which is equivalent.

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \; Q_n(\theta) \quad \text{with} \quad Q_n(\theta) = -\frac{1}{n} \sum_{i=1}^{n} L\left(\bar{\epsilon}_i\right) \tag{17}$$

where $L : \mathbb{R}^1 \to \mathbb{R}^1$ denotes a loss function, e.g., $L(x) = (x - 0)^2$ in the case of nonlinear least squares. Throughout the paper, the weighted average fund pricing error $\bar{\epsilon}_i \equiv \bar{\epsilon}_i(\theta)$ is regarded as nonlinear random function of the SDF parameter θ.

2.3 Asymptotic framework

By the value-additivity principle (Hansen and Richard, 1987), our SDF model holds for any aggregation level of cash flows. In practice, we form vintage-year portfolios as cross-sectional units, which mitigates idiosyncratic noise and GP financial engineering artifacts while preserving sufficient test assets for parameter identification.

Assumption 3. *For each vintage year, we pool fund cash flows to form n_v portfolios. Thus, $n = \sum_{v=1}^{V} n_v$. The boundary cases are (i) single-fund portfolios and (ii)* *one portfolio per vintage year. Both equal-weighed and fund-value-weighted portfolios are considered.*

Without loss of generality, we refer to cross-sectional units as funds. The detailed comparison of aggregation levels is provided in Section 3.3.

The pricing error $\bar{\epsilon}_i$ is naturally indexed by both fund identity and vintage year, yielding a random field $\bar{\epsilon}_i \equiv \bar{\epsilon}_{i,v_i}$ that fits neither standard time-series, cross-sectional, nor panel data frameworks (cf. Figure 2). We address this by adopting a spatial framework similar to Korteweg and Nagel (2016), where they measure the economic distance between funds as a function of the cash flow overlap. Thus, we generally consider $\bar{\epsilon}$ from Equation 16 as a random field (cf. Figure 2). In our case, it is convenient to interpret the fund vintage year v_i as second dimension in our pricing error random field, i.e., $\bar{\epsilon}_i \equiv \bar{\epsilon}_{i,v_i}$.

Yet, in this section, we mainly follow the time-series asymptotic framework of Pötscher and Prucha (1997) since our "spatial" distance measure (between vintage years) is time, and adaptation to our case is thus straightforward. If we observe only one fund per vintage year (or, equivalently, form vintage year portfolios), we will easily see that the framework of Pötscher and Prucha (1997) with time-series indexing can be applied without any major modification.

2.3.1 Vintage year asymptotics

We assume that the "spatial" (i.e., economic) distance between cross-sectional units, i.e., private equity funds/portfolios, can be measured quantitatively[3]. Our "cross-sectional" asymptotic theory lets the number of funds go to infinity $n \to \infty$. To expose our SDF to many distinct covariate realizations (economic conditions), we also want the number of vintages to increase asymptotically.

Assumption 4. *Vintage year asymptotics:*

1. *The number of vintage years $V \to \infty$ as $n \to \infty$.*
2. *The number of funds per vintage year is bounded by some positive constant.*[4]
3. *The maximum fund lifetime is also bounded by a positive constant.*
4. *The economic distance between fund i and j is measured by the vintage year difference $d_{i,j} = |v_i - v_j| + \rho_0 \mathbf{1}_{i \neq j}$ with minimum distance $\rho_0 > 0$. Note that two distinct funds from the same vintage year have distance $d_{i,j} = \rho_0$, reflecting our assumption that within-vintage-year dependence is captured by the SHAC kernel at the smallest lag.*

In terms of the spatial estimation literature, this assumption postulates increasing domain asymptotics and rules out so-called infill asymptotics (cf. Figure 2). The

[3]Generally, the economic distance measure could include multiple dimensions, e.g., temporal, geographic, and industry sector proximity. This could be an interesting topic for future research.

[4]This assumption is naturally satisfied when forming vintage-year portfolios (case (ii) of Assumption 3, with $n_v = 1$). For the individual fund case (i), it ensures that the SHAC estimator remains consistent and that standard errors scale with the number of vintage years rather than the total fund count. Relaxing this assumption would require cluster-robust inference at the vintage-year level.

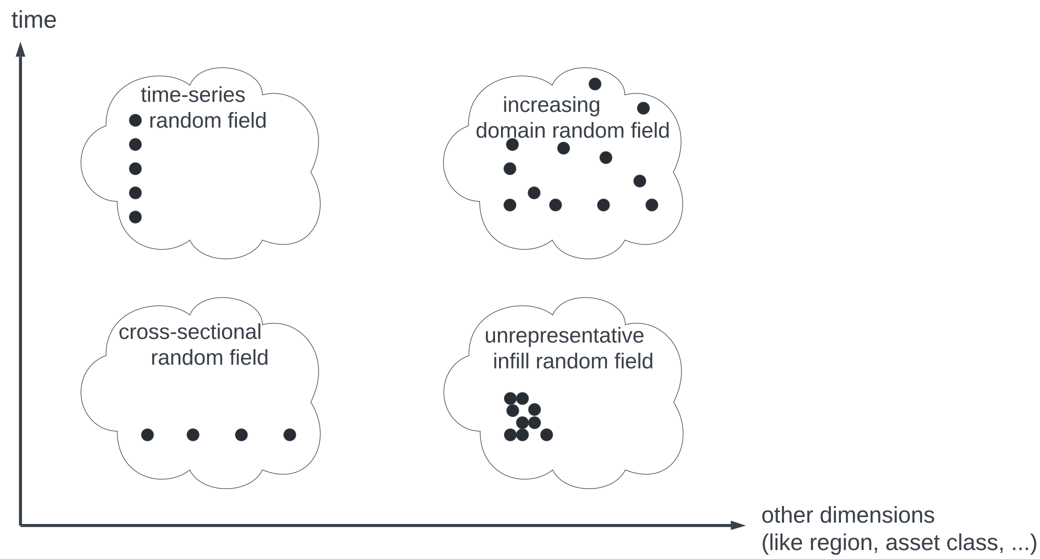

Figure 2: Visualization of generic random field types. Each black dot marks a different observation i of the cash flow data. Importantly, the time axis does **not** correspond to the index t in $CF_{i,t}$ (rather to vintage years v_i). Comparing the four choices, we want to avoid an infill random field but prefer our data to constitute an increasing domain random field. The infill random field is even asymptotically "too clustered" or better "too unrepresentative" to allow for meaningful estimation and inference. The time-series and cross-sectional random fields correspond to the standard cases in the literature but could turn out too restrictive for a general approach. By smart design (like portfolio formation), we often can map an increasing domain random field to simpler time-series or cross-sectional versions.

minimum distance term ρ_0 is a means to ensure these increasing domain asymptotics (Jenish and Prucha, 2012, Assumption 1). Infill asymptotics corresponds to the approach of Driessen et al. (2012), who fix the number of portfolios N and let the number of underlying funds per portfolio $n_i \to \infty$. While this yields consistency for a given set of portfolios, it fundamentally conditions on the observed macro environment: with fixed N, no new vintage yearsand hence no new factor realizationsenter the estimation. In contrast, our increasing domain framework ($V \to \infty$) permits inference that accounts for cross-vintage variation in economic conditions, which is necessary for the unconditional identification of SDF parameters (see Section 2.1). Finally, a further advantage of the *nonlinear least squares* formulation in Equation 17 is that it avoids the well-known complications arising when the number of GMM moment conditions grows with sample size (Han and Phillips, 2006; Newey and Windmeijer, 2009).

2.3.2 Law of large numbers

Under Assumption 2 (NPV-only discounting), Proposition 1 guarantees $\mathbb{E}[\bar{\epsilon}_i(\theta_0)] = 0$. For the general case where $\mathcal{T}_i$ includes future value dates, a residual bias may persist (see Section 2.1). In both cases, the LLN ensures that the sample objective $Q_n(\theta)$ converges uniformly to its population counterpart. To approach this expected value in $\mathbb{E}[\bar{\epsilon}_i(\theta_0)]$, we rely on a spatial law of large numbers instead of applying a time-series law of large numbers. Here, we want to explicitly acknowledge the statistical dependence of pricing errors from adjacent vintage years.

Assumption 5. *Primitive conditions for uniform convergence (cf. Jenish and Prucha, 2009, Theorem 6):*

1. Near-epoch dependence (NED): *For each $\theta \in \Theta$, the random field $\{L(\bar{\epsilon}_i(\theta))\}_{i=1,\dots,n}$ is NED of size $-a$ (with $a > 0$) on an α-mixing random field $\{\eta_v\}_{v\geq 1}$ indexed by vintage years, i.e., $\|L(\bar{\epsilon}_i(\theta)) - \mathbb{E}[L(\bar{\epsilon}_i(\theta)) \mid \sigma(\eta_v : |v - v_i| \leq m)]\|_p \leq c_i \cdot \nu(m)$ with $\sup_i c_i < \infty$ and $\nu(m) = O(m^{-a})$.*
2. Mixing decay: *The base random field $\{\eta_v\}$ is α-mixing with mixing coefficients $\alpha(d) = O(d^{-\gamma})$ for some $\gamma > 2(p-1)/(p-2)$ where $p > 2$.*
3. Moment conditions: $\sup_{\theta\in\Theta} \mathbb{E}[|L(\bar{\epsilon}_i(\theta))|^p] < \infty$ *for some $p > 2$, uniformly in i.*
4. Stochastic equicontinuity: *The family $\{L(\bar{\epsilon}_i(\cdot)) : i = 1, \dots, n\}$ is stochastically equicontinuous in θ, which is satisfied if L is Lipschitz and $\bar{\epsilon}_i(\theta)$ is Lipschitz in θ uniformly in i.*

Under Assumptions 3–5, the ULLN $\sup_{\theta\in\Theta} |Q_n(\theta) - \mathbb{E}[Q_n(\theta)]| \xrightarrow{p.} 0$ follows from (Jenish and Prucha, 2009, Theorem 6). Economically, the NED condition means that the pricing error of a fund in vintage v can be well-approximated using only information from nearby vintages $\{v - m, \dots, v + m\}$, with approximation quality improving in m, which is natural since cash flow overlap decays with vintage year distance (cf. Assumption 4). In practice, we treat funds with distance $d_{i,j} > D$ as approximately independent for the purpose of covariance estimation (cf. the bandwidth parameter $b_n = D$ in Section 2.4). Since nominal fund sizes tend to grow over time, unweighted pricing errors $\bar{\epsilon}_i$ may be non-stationary; the weighting factor w_i from Equation 16 (e.g., scaled by invested capital) normalizes the pricing errors so that the moment condition (3) is satisfied.

2.3.3 Consistency

The estimator $\hat{\theta}$ shall converge in probability to the true parameter value θ_0 as the number of distinct vintage years goes to infinity.

Assumption 6. *(Identification) The population objective $Q_0(\theta)$ has a unique maximizer at θ_0, i.e., $Q_0(\theta) < Q_0(\theta_0)$ for all $\theta \neq \theta_0$.*

Remark 2. *For $L(x) = x^2$, this requires that no alternative parameter vector $\theta \neq \theta_0$ can simultaneously achieve zero bias ($\mathbb{E}[\bar{\epsilon}(\theta)] = 0$) and equal variance ($\mathrm{Var}[\bar{\epsilon}(\theta)] =$*

$\mathrm{Var}[\bar{\epsilon}(\theta_0)]$). *This is economically natural: mis-specifying the SDF factor loadings introduces systematic pricing errors whose variance exceeds the irreducible noise at* θ_0. *The condition is analogous to the standard rank condition in linear GMM.*

Proposition 2 (Consistency)**.** *Under Assumptions 3–6,* $\hat{\theta} \xrightarrow{p} \theta_0$ *as* $n \to \infty$.

Proof. We verify the four conditions of (Newey and McFadden, 1994, Theorem 2.1) for the extremum estimator $\hat{\theta} = \arg\max_{\theta \in \Theta} Q_n(\theta)$ with population objective $Q_0(\theta) := \mathbb{E}[Q_n(\theta)] = -\mathbb{E}\big[L\big(\bar{\epsilon}_i(\theta)\big)\big]$.

1. *Identification.* Since $L \geq 0$, we have $Q_0(\theta) \leq 0$ for all θ. For the case $L(x) = x^2$,
$$Q_0(\theta) = -\mathrm{Var}\big[\bar{\epsilon}(\theta)\big] - \big(\mathbb{E}[\bar{\epsilon}(\theta)]\big)^2.$$
Under Assumption 2, Proposition 1 yields $\mathbb{E}[\bar{\epsilon}(\theta_0)] = 0$, so the bias term vanishes at θ_0. For any $\theta \neq \theta_0$, at least one of (a) the bias term $(\mathbb{E}[\bar{\epsilon}(\theta)])^2 > 0$ or (b) the variance term $\mathrm{Var}[\bar{\epsilon}(\theta)] > \mathrm{Var}[\bar{\epsilon}(\theta_0)]$ must hold, ensuring $Q_0(\theta) < Q_0(\theta_0)$. This is the standard identification condition requiring that the true parameter is the unique minimizer of the expected loss.

2. *Compactness.* The parameter space Θ is compact by economic reasoning: factor loadings and intercepts are bounded by plausible ranges implied by the risk–return tradeoff.

3. *Continuity.* Continuity of $Q_0(\theta)$ follows from the continuous dependence of $\Psi_{\tau,t}(\theta)$ on θ and the dominated convergence theorem, which is a standard regularity condition.

4. *Uniform convergence.* By Assumption 5, $\sup_{\theta \in \Theta} |Q_n(\theta) - Q_0(\theta)| \xrightarrow{p} 0$, which directly satisfies the ULLN requirement of (Newey and McFadden, 1994, Section 2.1).

□

2.3.4 Central limit theorem

To assess the large-sample significance of our parameter estimates in Subsection 2.4, we establish the asymptotic distribution of $\hat{\theta}$.

Proposition 3 (Asymptotic Normality)**.** *Under the conditions of Proposition 2, suppose additionally that (i)* θ_0 *is interior to* Θ*, (ii)* $Q_0(\theta)$ *is twice continuously differentiable at* θ_0 *with nonsingular Hessian* $H := \nabla_{\theta\theta} Q_0(\theta_0)$*, and (iii) a spatial CLT holds for the score. Then*

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma) \quad \text{as } n \to \infty,$$

with sandwich covariance matrix $\Sigma = H^{-1} \Lambda \, H^{-1}$*, where* $\Lambda := \lim_{n\to\infty} n \cdot \mathrm{Var}\big[\nabla_\theta Q_n(\theta_0)\big]$ *is the long-run covariance matrix of the score.*

Proof. The result follows from the general asymptotic normality theorem for extremum estimators (Newey and McFadden, 1994, Theorem 3.1). The key nonstandard ingredient is condition (iii): a CLT for the spatially dependent score

$\nabla_\theta Q_n(\theta_0)$. Under the spatial near-epoch dependence structure posited in Assumption 5 and the vintage year asymptotics of Assumption 4, this follows from (Jenish and Prucha, 2012, Theorem 4), which extends the Pötscher and Prucha (1997) framework to spatial random fields. The remaining conditions of (Newey and McFadden, 1994, Theorem 3.1), i.e., interior θ_0, smooth Q_0, and stochastic equicontinuity of the score, are standard regularity conditions satisfied by the continuous dependence of $\Psi_{\tau,t}(\theta)$ on θ. $\square$

Remark 3. *The spatial CLT of (Jenish and Prucha, 2012, Theorem 4) requires the same NED and mixing conditions as Assumption 5, supplemented by a Lindeberg-type condition on the score contributions $\nabla_\theta L(\bar{\epsilon}_i)$. The Lindeberg condition is implied by the uniform moment bound in Assumption 5(3) applied to the score.*

2.4 Large sample inference

From Proposition 3, the asymptotic covariance matrix takes the sandwich form $\Sigma = H^{-1}\Lambda\, H^{-1}$[5] with H and Λ as defined therein. We now describe how to construct consistent estimators $\hat{H}$ and $\hat{\Lambda}$.

We define the corresponding finite sample estimators analogously to Pötscher and Prucha (1997, Chapters 12, 13.1), and numerically approximate the first and second partial derivatives by finite differences[6]. Specifically, we use the following central difference approximations (with "small" δ) (Eu, 2017, Algorithm 2):

$$f_x(x,y) \approx \frac{f(x+\delta,y)-f(x-\delta,y)}{2\delta}$$

$$f_{xx}(x,y) \approx \frac{f(x+\delta,y)+f(x-\delta,y)-2f(x,y)}{\delta^2}$$

$$f_{xy}(x,y) \approx \frac{f(x+\delta,y+\delta)+f(x-\delta,y-\delta)-f(x+\delta,y-\delta)-f(x-\delta,y+\delta)}{4\delta^2}$$

The Hessian term $\hat{H}$ is relatively straightforward

$$\hat{H} = -\frac{1}{n}\sum_{i=1}^{n} \nabla_{\theta\theta} L\left(\bar{\epsilon}_i\right)$$

The consistency $\hat{H} \xrightarrow{p} H$ is implied by the ULLN (Assumption 5) applied to the second derivatives. Due to the spatial near-epoch dependence, the involved and computationally expensive part is to consistently estimate $\hat{\Lambda}$ by a Spatial Heteroskedasticity and Autocorrelation Consistent (SHAC) covariance matrix estimator (Kim and Sun, 2011, Equation 2)

$$\hat{\Lambda} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{i,j}\left[\nabla_\theta L\left(\bar{\epsilon}_i\right)\left(\nabla_\theta L\left(\bar{\epsilon}_j\right)\right)^\top\right] \tag{18}$$

[5]Since H is the Hessian of a scalar objective, it is symmetric, so $H^{-1} = (H^{-1})^\top$.

[6]As an alternative to finite differences the widespread Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm can be applied to approximate the Hessian (Nocedal and Wright, 2006, Section 6.1).

Since $\nabla_\theta Q_n(\theta) = -\frac{1}{n}\sum_i \nabla_\theta L(\bar{\epsilon}_i)$, the score contributions are $s_i(\theta) := \nabla_\theta L(\bar{\epsilon}_i)$, and $\hat{\Lambda}$ equivalently estimates $n \cdot \mathrm{Var}[\nabla_\theta Q_n]$.

We define the kernel weight k as

$$k_{i,j} \equiv K\left(\frac{d_{i,j}}{b_n}\right)$$

with kernel function $K : \mathbb{R} \to [0,1]$ that satisfies $K(0) = 1$, $K(x) = K(-x)$, $\int_{-\infty}^{\infty} K^2(x)dx < \infty$, and $K(\cdot)$ continuous at zero and at all but a finite number of other points. A common choice is the Bartlett kernel $K_{BT}(x) = \max(0, 1 - |x|)$; see equation 2.7 in Andrews (1991) for other popular kernel choices. This means absolute vintage year differences larger than the bandwidth (or truncation) parameter $b_n = D$ are considered independent and are thus excluded from the $\hat{\Lambda}$ estimation formula.

In large samples, the vector of parameter standard errors can thus be estimated by

$$\mathrm{SE}(\hat{\theta}) = \sqrt{\mathrm{diag}\left[n^{-\frac{1}{2}}\hat{H}^{-1}\hat{\Lambda}\,\hat{H}^{-1}(n^{-\frac{1}{2}})^\top\right]} = \sqrt{\mathrm{diag}\left[\hat{H}^{-1}\hat{\Lambda}\,\hat{H}^{-1}\right]\cdot\frac{1}{n}} \tag{19}$$

The Wald test statistic for linear hypotheses $\mathcal{H}_0 : R\theta = r$ and $\mathcal{H}_1 : R\theta \neq r$ is constructed as (where $\mathcal{H}_0$ and $\mathcal{H}_1$ are hypotheses and not Hessian terms H)

$$W = (R\hat{\theta} - r)^\top\left[R\frac{\hat{H}^{-1}\hat{\Lambda}\,\hat{H}^{-1}}{n}R^\top\right]^{-1}(R\hat{\theta} - r) \overset{H_0}{\sim} \chi^2_q$$

where $\hat{\theta}$ is the $p \times 1$ parameter vector, R is a $q \times p$ matrix, and r is a $q \times 1$ vector. Usually, we select R as $p \times p$ identity matrix, and r as $p \times 1$ vector (e.g., of zeros). Under the null hypothesis, W is chi-squared distributed with q degrees of freedom. As large values of W indicate the rejection of H_0, the corresponding p-value is calculated as $1 - F_{\chi^2_q}(W)$ where $F_{\chi^2_q}$ is the cumulative distribution function of a chi-squared random variable with q degrees of freedom.

However, given the limited amount of available private equity data (typically the oldest vintages start in the 1980s), asymptotic characterizations of Σ and $\mathrm{SE}(\hat{\theta})$ are of limited importance. In empirical applications, the small sample behavior of an estimation method for private equity data is more relevant than its asymptotic theory. Moreover, the standard asymptotic distribution associated with an estimator is generally not valid for post-model-selection inference, i.e., if a model selection procedure is applied to find the best model from a collection of competitors (Leeb and Pötscher, 2005).

2.5 Comparison to similar estimators

Our Least-Mean-Distance (LMD) estimator introduced in Section 2.2 belongs to the class of semiparametric nonlinear M-estimators as defined in Pötscher and Prucha (1997) which are extremum estimators. To gain more flexibility and avoid unneeded complexity, we intentionally opt against the most prominent semiparametric nonlinear M-estimator framework, i.e., classical time-series Generalized Method of Moments (GMM) (Hansen, 1982, 2012). A classical GMM approach requires the construction of stationary, ergodic time-series of moment conditions that are used to empirically estimate the expected value of pricing errors in Equation 15. The stationarity requirement of classical time-series GMM limits (i) more elaborate weighting schemes for w, like fund-size weighting, and (ii) the usage of fund cash flows from non-realized vintages.

2.5.1 Comparison to Driessen et al. (2012)

The Driessen et al. (2012) approach is most closely related to our methodology. One important difference is that we select a simpler and more flexible LMD estimator instead of a cross-sectional GMM approach. In our view, the choice of the more complex cross-sectional GMM just introduces conceptual complexity without changing the estimating equations since the underlying formulas are basically the same as for our LMD estimator[7].

As a first limitation, they have to regard vintage-year portfolios as their cross-sectional units; we can also use individual funds. In this context, Driessen et al. (2012) state that "to identify β, it is essential that the different FoFs are exposed to different market returns." We note that their estimator is valid as a cross-sectional approach: identification requires only that the cross-section of funds provides sufficient variation in factor exposure at a given market return realization[8]. That said, having FoFs exposed to different market environments (i.e., different vintage years) does improve statistical power and is the basis for our increasing domain asymptotics.

Second, the Driessen et al. (2012) asymptotic theory is based on infill asymptotics: they fix the number of vintage year portfolios n and let the number of underlying funds per portfolio $n_i \to \infty$. As discussed in Section 2.3, this conditions inference on the observed macro environment since no new vintage years enter the estimation. Moreover, since each portfolio constitutes a moment condition from the GMM perspective, standard GMM theory requires n to remain fixed (Han and Phillips, 2006; Newey and Windmeijer, 2009). In contrast, our increasing domain framework lets the number of vintage years $V \to \infty$ while bounding the number of funds per vintage, enabling inference that reflects cross-vintage variation in economic conditions.

Further, Driessen et al. (2012) discount all fund cash flows exclusively to the fund inception date, corresponding to $\mathcal{T}_i = \{\min_j t^{\text{Inv}}_{i,j}\}$ in our notation. Under Assumption 2, this is also the only choice that yields unconditionally unbiased estimates. Our framework generalizes this by allowing $\mathcal{T}_i$ to contain additional discounting dates, which we use to empirically study the bias–variance tradeoff analyzed in Section 2.1. Although Driessen et al. (2012) describe their estimator as a one-step GMM approach, it can be viewed as a special case of our LMD estimator. Specifically, Equation 17 from our methodology is a generalization of equation 3 from their paper. Formally, setting $\mathcal{T}_i = \{\min_j t^{\text{Inv}}_{i,j}\}$, $w_i = 1$, and $L(x) = x^2$ in

[7] The formulas are this similar because Driessen et al. (2012) use the identity matrix as GMM weighting matrix and skip the second GMM step.

[8] In a classic cross-sectional regression, we only have one market return realization.

Equation 17 recovers their objective (Equation 1). Consequently, if someone accepts the assumptions from Subsection 2.3, our large sample inference framework from Subsection 2.4 applies to their case without any significant modification. Finally, Driessen et al. (2012) apply simple cross-sectional bootstrapping to obtain standard errors; in contrast, in Subsection 3.2, we use a cross-validation technique that is adapted to the near-epoch dependence of the PE fund data.

2.5.2 Comparison to Korteweg and Nagel (2016)

Korteweg and Nagel (2016) were the first to employ a spatial framework for asymptotic inference on private equity fund data. They measure the economic distance between funds by the degree of cash flow overlap and use the spatial HAC estimator of Conley (1999) to construct robust standard errors. Our spatial HAC framework similarly accounts for cross-vintage dependence but draws on the near-epoch dependent theory of Pötscher and Prucha (1997); Kim and Sun (2011); Jenish and Prucha (2012). A key methodological difference is that Korteweg and Nagel (2016) estimate the SDF on synthetic public market portfolios using a time-series GMM estimator and then apply the resulting model to evaluate PE cash flows, whereas we estimate the SDF directly from PE fund cash flows. Korteweg and Nagel (2016) choose this procedure since "tests that use the ex post equity premium as a performance benchmark should be more powerful than tests that use the ex ante equity premium." Specifically, we obtain the estimator of (Korteweg and Nagel, 2016, Equation 18) in our framework if we replace $Q_n(\theta)$ in Equation 17 by Equation 20.

$$Q_n(\theta) = L\left(\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i\right) \quad \text{with} \quad L(x) = x^\top W x = x^\top I x \tag{20}$$

with identity matrix I as weighting matrix W. Note that since $L(x) = x^\top W x \geq 0$, maximizing $-L(\cdot)$ is equivalent to minimizing $L(\cdot)$; we omit the negative sign in Equation 20 for notational simplicity; accordingly, Equation 20 is to be minimized with respect to θ. In accordance with classical GMM, the function $\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i : \mathbb{R}^{n\times T} \times \Theta \to \mathbb{R}^m$ should be perceived as multidimensional where the dimensionality of the function output corresponds to the number of moment conditions. In contrast to our scalar LMD setting ($\bar{\epsilon}_i \in \mathbb{R}$), here $\frac{1}{n}\sum_i \bar{\epsilon}_i \in \mathbb{R}^m$ may be multidimensional and $L(x) = x^\top W x$ is the standard quadratic GMM objective.

Time-series GMM estimators inherently bear the risk of under-identification if the corresponding time-series is constructed by pooling all fund cash flows from a given fund type. When PE fund cash flows are pooled into a single moment condition as in Equation 20 with $m = 1$, the resulting estimator has just one moment condition, which is insufficient to identify multi-parameter models[9]. Note that Korteweg and Nagel (2016) avoid this issue by estimating the SDF from public market data rather than PE cash flows; the under-identification concern arises only if one applies their framework directly to PE data, as we do in our estimator. To counter under-identification, additional characteristic-based fund portfolios could be formed

[9]In contrast, our estimator corresponds to the opposite edge case with asymptotically an infinite number of LMD "moment conditions" (units to price) as we let $n \to \infty$.

to increase the number of moment conditions per fund type; also, random portfolios combined with bootstrapping could make sense. Yet, Korteweg and Nagel (2016) take another approach and introduce the concept of Generalized Public Market Equivalent (GPME), which elegantly avoids the under-identification issue. Firstly, a public market SDF model is estimated by pricing public trading strategies that shall replicate PE funds instead of directly pricing the observed PE fund cash flows. Only in a second step are these models applied to evaluate private equity fund cash flows.

In summary, our LMD estimator generalizes Driessen et al. (2012) by (a) permitting flexible discounting dates via $\mathcal{T}_i$, (b) employing increasing domain rather than infill asymptotics, and (c) providing SHAC-based inference. Relative to Korteweg and Nagel (2016), we directly price PE cash flows rather than pricing public replicating strategies, which avoids potential misspecification of the public SDF but requires stronger identification assumptions from the PE data alone. Table 2 summarizes the most prominent distinctions between the three approaches.

	Driessen et al. (2012)	Korteweg and Nagel (2016)	Our approach
Estimator type	Cross-sectional NLS (one-step GMM)	Time-series GMM (public SDF)	Nonlinear Least-Mean-Distance
Cash flows priced	PE fund cash flows	Public replicating strategies	PE fund cash flows
Discounting dates	Fund inception only	Fund inception only	Flexible via $\mathcal{T}_i$
Asymptotics	Infill ($n_i \to \infty$, N fixed)	$V \to \infty$	Increasing domain ($V \to \infty$, n_v bounded)
Inference	Bootstrap	Spatial HAC	Spatial HAC
Cross-sectional unit	Vintage year portfolio	Single fund	Both (tested empirically)
SDF specification	Linear factor model	Exponentially affine	Both (tested empirically)

Table 2: Comparison of SDF estimation frameworks for private equity.

3 Empirical application

3.1 Data

We use the Preqin cash flow data set as of 14th April 2022 that is well known in the academic private equity literature (Harris et al., 2014; Korteweg and Nagel, 2016; Ang et al., 2018). To keep the empirical analysis clear and concise, we filter the dataset for Private Equity (PE) funds.

The Private Equity sample contains 2745 distinct funds spreading over 35 vintage years. "The vintage year distribution is as follows: 1983 (1), 1985 (2), 1986 (4), 1987

(5), 1988 (7), 1990 (2), 1991 (4), 1992 (9), 1993 (13), 1994 (20), 1995 (16), 1996 (20), 1997 (24), 1998 (59), 1999 (46), 2000 (72), 2001 (51), 2002 (41), 2003 (45), 2004 (64), 2005 (117), 2006 (154), 2007 (176), 2008 (183), 2009 (83), 2010 (92), 2011 (146), 2012 (136), 2013 (142), 2014 (182), 2015 (152), 2016 (186), 2017 (143), 2018 (157), 2019 (190). The region distribution is as follows: Africa (20), Asia (150), Australasia (39), Europe (681), Latin America & Caribbean (33), Middle East (14), North America (1807). The strategy distribution is as follows: Balanced (69), Buyout (1354), Co-Investment (87), Co-Investment Multi-Manager (59), Direct Secondaries (30), Fund of Funds (643), Growth (331), Secondaries (152), Turnaround (19).

For this subset of Private Equity funds, we extract all (i) equal-weighted and (ii) fund-size-weighted cash flow series. For non-liquidated funds, we treat the latest net asset value as final cash flow. We explicitly refrain from excluding the most recent vintage years. Thus, the minimum vintage year is 1983 (just for PE), and the maximum is 2019 since we filtered out funds with younger vintage.

The public market factors that enter our SDF draw on the US data set of the recently popularized q^5 investment factor model sourced from `http://global-q.org/factors.html` (Hou et al., 2015, 2020). Their five-factor model includes the market excess return (MKT), a size factor (ME), an investment factor (IA), a return on equity factor (ROE), and an expected growth factor (EG).

3.2 Model and estimator specifications

We test a simple linear SDF model similar to Driessen et al. (2012)

$$\Psi_{\tau,t}^{\mathrm{SL}}(\theta) = \frac{\prod_{h=0}^{\tau}\left(1+\alpha+r_h+\sum_j\ \beta_j\ F_{j,h}\right)}{\prod_{h=0}^{t}\left(1+\alpha+r_h+\sum_j\ \beta_j\ F_{j,h}\right)} \tag{21}$$

and an exponential affine SDF model adapted from Korteweg and Nagel (2016)

$$\Psi_{\tau,t}^{\mathrm{EA}}(\theta) = \exp\left[\sum_{h=0}^{\tau} X_h \sum_{h=0}^{t} -X_h\right] \tag{22}$$

with

$$X_h = \alpha + \log(1+r_h) + \sum_{j\in J} \beta_j \cdot \log(1+F_{j,h})$$

with (arithmetic) risk-free return $r = R_{rf} - 1$, (arithmetic) zero-net-investment portfolio returns F_j, and parameter vector $\theta = (\alpha, \beta)$. To avoid overfitting, we just test six simple SDF models that contain {MKT} alone or {MKT} plus {ME or IA or ROE or EG or Alpha}. In Equation 17, we use the quadratic loss function $L(x) = x^2$.

To assess the parameter significance, we compute the asymptotic standard errors as outlined in Subsection 2.4. For the Bartlett kernel's bandwidth $b_n = D$ we select 12 years, i.e., funds with absolute vintage year differences larger than 12 years are assumed to be independent.

Additionally, we want to test the finite - or, more honestly, small - sample parameter significance and the out-of-sample performance of our SDF models. To account for the dependency between funds from adjacent vintage years caused by overlapping fund cash flows, we draw on hv-block cross-validation (Racine, 2000). Therefore, we form three partitions for several vintage year groups. As larger validation sets are preferred for model selection, the validation set (v-block) always contains funds of three neighboring vintage years (e.g., 2000, 2001, 2002). To reduce the dependency between training and validation set, we remove all funds from three-year-adjacent vintage years, i.e., the h-block (e.g., 1997, 1998, 1999, 2003, 2004, 2005). Funds from the remaining vintage years enter the training set and are thus used for model estimation (e.g., 1985-1996, 2006-2019). We apply ten-fold cross validation using the ten validation sets described in Table 3. This means we replace the bootstrap standard error calculation of Driessen et al. (2012) by hv-block cross-validation since the new method (i) accounts for near-epoch-dependence, (ii) focuses directly on the out-of-sample performance of the SDF models, and (iii) is computationally cheaper.

Remark 4 (Definition of Horizon)**.** *Throughout the empirical section, we use the term* Horizon H *to denote the maximum compounding duration in years used in the set of discounting dates $\mathcal{T}$. Formally, assuming a contiguous set of monthly discounting dates $\mathcal{T} = \{1, 2, \dots, m_{\max}\}$ (where $m_{\max}$ is the maximum month), the Horizon is defined as $H = m_{\max}/12$. Consequently, estimating a model with a Horizon H implies that the estimator uses the set of discounting dates $\mathcal{T}_H = \{1, \dots, 12 \cdot H\}$.*

Remark 5 (Alpha Bounds)**.** *In our optimization procedure, we limit the Alpha factor α to the range of $[-1\%, +1\%]$ per month to avoid the "exploding alpha" issue documented in (Driessen et al., 2012, Footnote 8): "Note that one needs to put an upper bound on α when performing the optimization in equation (3). This is to exclude the possibility that α tends to infinity." In this degenerate scenario, an infinite alpha can almost perfectly price any cash flow stream, rendering the beta risk factors irrelevant.*

training.before	h-block.before	v-block	h-block.after	training.after
estimation	remove	validation	remove	estimation
start-1984	1985,1986,1987	1988,1989,1990	1991,1992,1993	1994-end
start-1987	1988,1989,1990	1991,1992,1993	1994,1995,1996	1997-end
start-1990	1991,1992,1993	1994,1995,1996	1997,1998,1999	2000-end
start-1993	1994,1995,1996	1997,1998,1999	2000,2001,2002	2003-end
start-1996	1997,1998,1999	2000,2001,2002	2003,2004,2005	2006-end
start-1999	2000,2001,2002	2003,2004,2005	2006,2007,2008	2009-end
start-2002	2003,2004,2005	2006,2007,2008	2009,2010,2011	2012-end
start-2005	2006,2007,2008	2009,2010,2011	2012,2013,2014	2015-end
start-2008	2009,2010,2011	2012,2013,2014	2015,2016,2017	2018-end
start-2011	2012,2013,2014	2015,2016,2017	2018,2019,2020	2021-end

Table 3: Partitions used for hv-block cross-validation.

3.3 Simulation study

Our Monte Carlo experiments examine the following questions related to the bias and variance of our estimation methodology in finite samples.[10] Is it beneficial to use vintage-year portfolios instead of individual funds? Which SDF model performs better when we also use the corresponding data generating process (i.e., assume correct model specification)? How is estimator precision affected by varying numbers of vintage years and cross-sectional units? Which is the optimal set of discounting dates $\mathcal{T}$?

We use historical q-investment factors from 1986 to 2005 and simulate 20 funds for each of these 20 vintage years. This means we condition our simulation analysis on realized q-investment factors paths (see the Macro-Conditioning Kernel discussion in Section 2.1). Each fund contains 15 deals with equal investment amounts and exactly one divestment cash flow. Deals are entered within the first five years of the fund lifetime following a discrete uniform distribution and afterward held between one to ten years again uniformly distributed. The deal returns are generated by the simple linear or exponential affine SDF models described in Equations 21 and 22. In the base case, we just use the MKT factor with $\beta_{\text{MKT}} = 1$ and in each month, add a normal i.i.d. error term with standard deviation $\sigma = 0.2$ and zero mean. Additionally, we test an intercept term α of -0.25% per month and a high β_{MKT} of 2.5. In the exponential affine case, we adjust the lognormally distributed error mean to zero by subtracting $0.5\sigma^2$. If a negative return exceeds -100%, the company defaults with a zero exit cash flow. In contrast, the error term in the simulations of Driessen et al. (2012) is more well-behaved as it follows a shifted lognormal distribution that, even with arbitrarily high error term variance, just allows for returns below say -99%, if the market return is close to its lower bound (see equation 9 in their online appendix). In our base case, the set of discounting dates $\mathcal{T}$ contains all months from the first cash flow to the maximum month 180. To assess our estimator's bias and variance, we simulate 1000 test scenarios for both vintage-year and single-fund portfolios.

Cross-sectional unit i: As presumed in Subsection 2.3, vintage year portfolio results appear to have lower bias and variance when compared to individual funds. For the simple linear SDF and maximum month 180, the mean and standard deviation of the coefficient estimate $\hat{\beta}_{\text{MKT}}$ is 1.016 (0.2) for the vintage year portfolio and 1.096 (0.376) for individual funds. More results are depicted in Figure 3. However, for individual funds, we only simulate 200 iterations due to the high computational cost.

This finding has two important implications: On the one hand, vintage year portfolio formation can substantially decrease our estimator's bias and variance. On the other hand, it also dramatically reduces the number of cross-sectional units and consequentially impairs the importance of asymptotic results. These considerations may explain the choice of Korteweg and Nagel (2016) to use individual funds as cross-sectional units in their asymptotic SHAC framework to obtain smaller stan-

[10] As with each simulation study, it investigates the ability to identify the assumed data generating process rather than the corresponding SDF model.

dard error estimates.

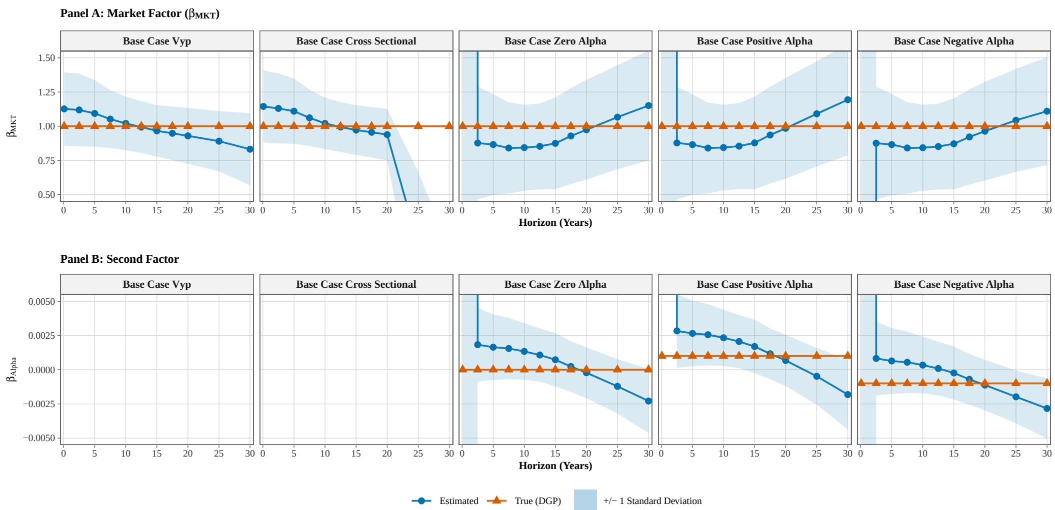

Figure 3: **Simulation ($\beta_{\mathbf{MKT}}$, α):** Single-fund vs. vintage-year portfolios.

Varying vintages V and portfolio sizes n/V: To test the effect of varying data sizes available for MKT factor estimation, we in/decrease the (i) number of vintage years and (ii) the number of funds per vintage year. Here we use vintage year portfolios and the simple linear SDF. We test five different scenarios (from left to right in Figures 5 and 4): (1) Years 1986-2005 with 40 funds per vintage (i.e., the base case with 40 instead of 20 funds per vintage). (2) Years 1967-2005 with 10 funds per vintage. (3) Years 1967-2005 with 20 funds per vintage. (4) Years 1986-1995 with 20 funds per vintage. (5) Years 1996-2005 with 20 funds per vintage.

Interpretation: For our simple data generating process, increasing the number of deals/funds per vintage year portfolio appears to decrease the estimator's variance more effectively than adding more vintage years.

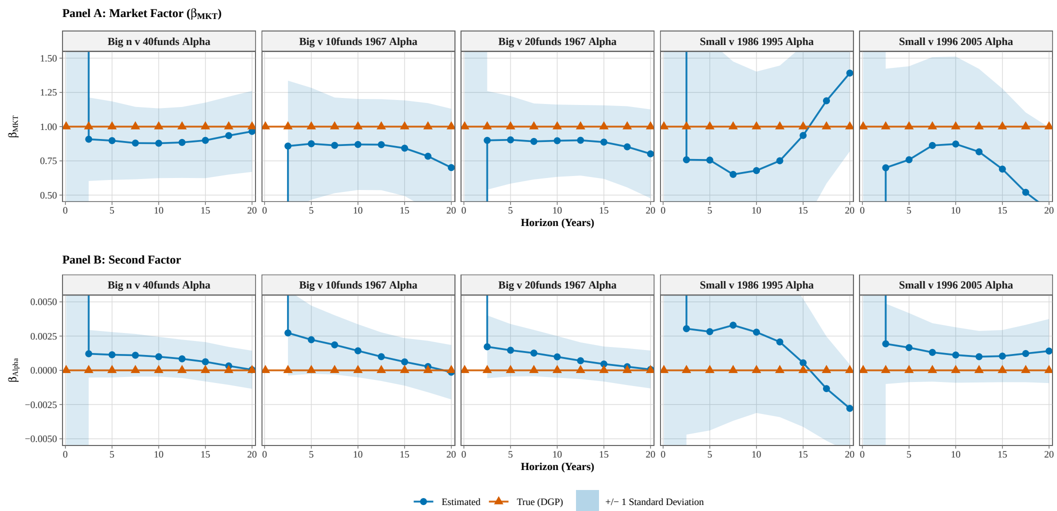

Figure 4: **Simulation ($\beta_{\mathbf{MKT}}$, α):** Varying vintages V and portfolio sizes n/V.

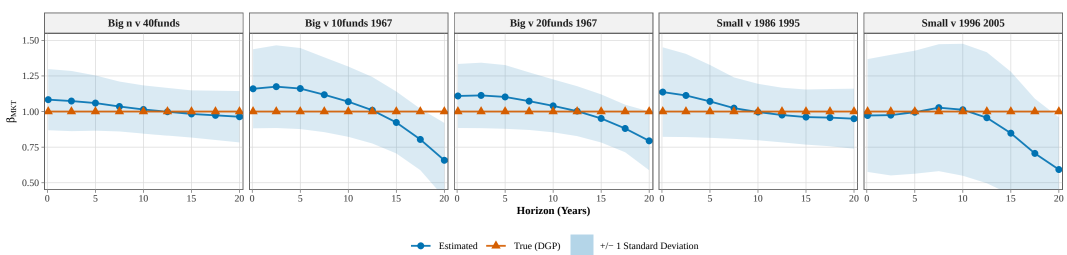

Figure 5: **Simulation ($\beta_{\mathbf{MKT}}$):** Varying vintages V and portfolio sizes n/V.

Two-factor models (q^5 factors): Figure 6 reports simulation results for two-factor models that pair MKT with each of the four non-market q^5 factors (ME, IA, ROE, EG), with true loadings $\beta_{\text{MKT}} = 1$ and $\beta_{\text{2nd}} = 0.5$. A common pattern emerges across all four specifications: the second-factor loading is substantially overestimated at short horizons and attenuated toward — or beyond — the true value at long horizons, while the market factor absorbs a compensating bias in the opposite direction. Estimation variance is markedly larger than in the single-factor case, reflecting the collinearity between correlated factors in a limited cross-section of vintage year portfolios. Table 4 summarizes the key estimates at the shortest horizon and at 15 years, which is approximately the bias-minimizing horizon for most models; we highlight the most notable factor-specific features below.

The ME model is the only specification in which $\hat{\beta}_{\text{2nd}}$ undergoes a sign change (at roughly 17.5 years), declining from 0.86 to -0.02 over the full horizon range — a severe attenuation that would qualitatively alter economic conclusions. For the IA model, the second-factor bias is large at short horizons ($\hat{\beta}_{\text{IA}} \approx 1.06$ vs. truth 0.5) but both loadings converge close to their true values near the 17.5-year horizon ($\hat{\beta}_{\text{MKT}} = 0.97$, $\hat{\beta}_{\text{IA}} = 0.52$). The EG model exhibits the largest overall estimation variance ($\sigma(\hat{\beta}_{\text{EG}}) \approx 2.0$ at the shortest horizon) but achieves a reasonably accurate joint estimate around 15–17.5 years ($\hat{\beta}_{\text{MKT}} = 0.91$–$1.02$, $\hat{\beta}_{\text{EG}} = 0.47$–$0.56$). The ROE model is the exception: both loadings remain persistently biased at all horizons, with $\hat{\beta}_{\text{MKT}}$ never exceeding 0.91 and $\hat{\beta}_{\text{ROE}}$ never falling below 0.62. These results suggest that empirical researchers should exercise caution when interpreting two-factor SDF estimates for private equity, and that the bias-minimizing horizon is model-dependent, typically falling in the 12.5–17.5-year range.

	$\hat{\beta}_{\text{MKT}}$		$\hat{\beta}_{\text{2nd}}$		$\sigma(\hat{\beta}_{\text{MKT}})$		$\sigma(\hat{\beta}_{\text{2nd}})$	
Model	0 yr	15 yr	0 yr	15 yr	0 yr	15 yr	0 yr	15 yr
ME + MKT	1.15	0.97	0.86	0.59	0.28	0.21	0.46	0.40
IA + MKT	0.97	0.92	1.06	0.64	0.34	0.30	0.80	0.53
ROE + MKT	0.84	0.83	1.04	0.69	0.43	0.49	0.72	0.49
EG + MKT	0.87	0.91	1.04	0.56	0.65	0.45	2.00	0.31

True values: $\beta_{\text{MKT}} = 1$, $\beta_{\text{2nd}} = 0.5$.

Table 4: Two-factor simulation summary: mean estimates and standard deviations at the shortest (0 yr) and the 15-year horizon.

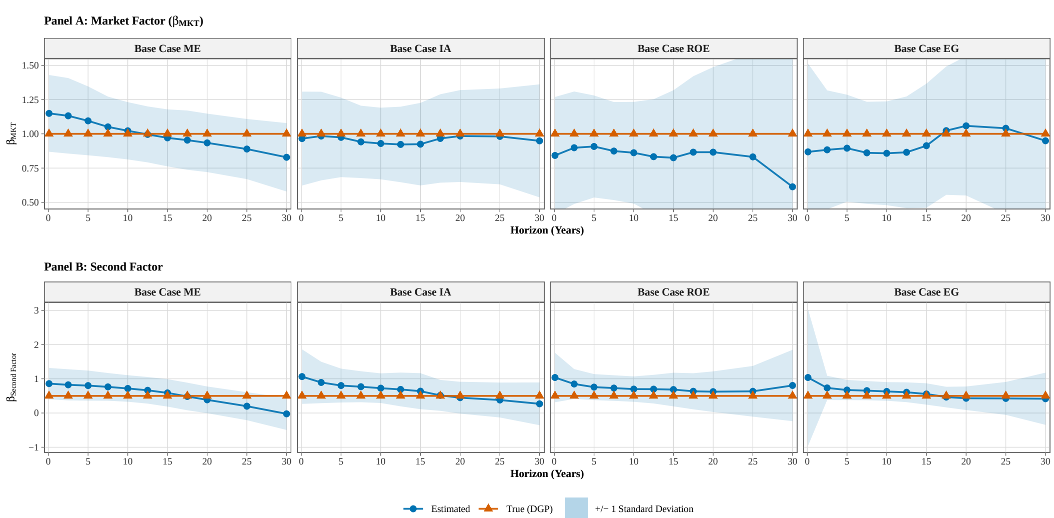

Figure 6: **Simulation (q^5 factors):** Two-factor models using VYP.

SDF model Ψ: In our base case with vintage year portfolios, the exponential affine SDF shows a mean and standard deviation of 1.011 (0.175) compared to the 1.016 (0.2) achieved by the simple linear SDF. Generally, the exponential affine SDF model and the simple linear SDF model exhibit similar bias and variance, cf. panels A and B in Table **??**. Figure **??** visualizes the true $\beta = 1$ case, which shows that the estimation results are not overly sensitive to the choice of the SDF model.

Moreover, the perceived superiority of exponential affine SDFs is probably rather theoretical than practical as other proponents also emphasize their universality mainly from a mathematical perspective without providing supportive empirical or simulation results (Gourieroux and Monfort, 2007; Bertholon et al., 2008).

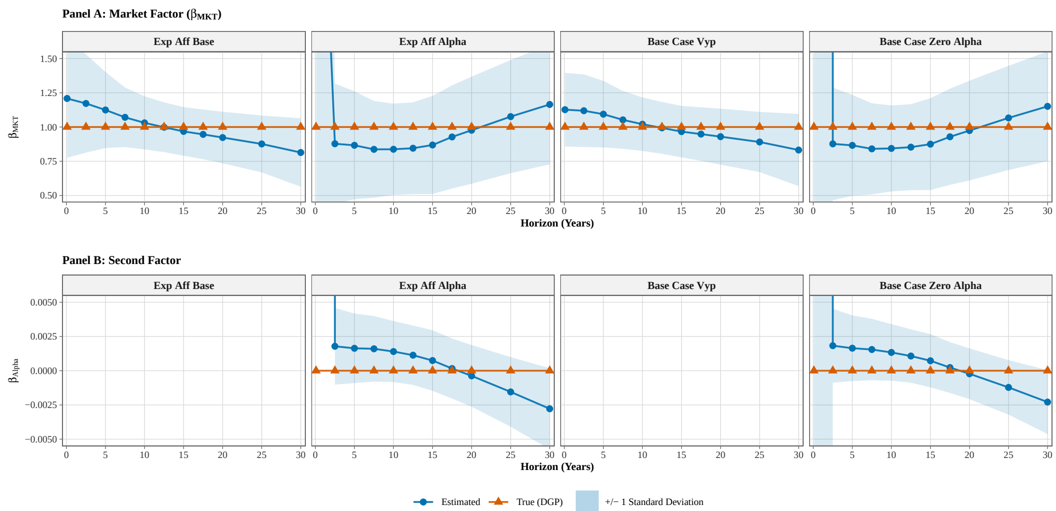

Figure 7: **Simulation ($\beta_{\mathbf{MKT}}$, α):** Exponential affine vs. linear models.

Size of set $\mathcal{T}$: This is the big question of the simulations study! Which max compounding horizon is the optimal one? From our diverse simulation results, we can deduce that we cannot determine the optimal horizon from these simulation results.

The results in indicate that we can control the asymptotic bias by an appropriate choice of the set $\mathcal{T}$. Horizon 15 year seems optimal, even if the NPV-only horizon of 0 year is the optimal one in theory (cf. Section 2.1). Recall that using the minimal set for $\mathcal{T}$, i.e., discounting all cash flows just to the fund inception date, corresponds exactly to the Driessen et al. (2012) approach. The Appendix A tries to explain this puzzling result. However, the probability bands (one standard deviation) in the simulation do not indicate decreasing parameter variance with increasing horizon "max compounding month". Thus, how and why does our "variance reduction technique" reduce/affect/control bias rather than variance? To be optimistic, the variance seems slightly smaller near the "optimal" horizon of around 15 years in the simulation and slightly increases for smaller and higher horizons (the effect looks not very large, we need to investigate this thoroughly). We need to solve this puzzle why only the bias term is affected by the additional future compounding dates (= additional Timing Risk Premium terms).

Simulation conclusion: To conclude, our simulations study rationalizes two key practices from the Driessen et al. (2012) paper: (i) vintage year portfolio formation helps to improve estimator precision, and (ii) increasing the number of funds per vintage seems to be more effective in controlling estimator variance than increasing the number of vintages. Thus, finding (ii) may explain the choice of Driessen et al. (2012) to employ an asymptotic law that lets the number of deals/funds per vintage tend to infinity.. However, our examples with correct specification cannot support the assumption of Korteweg and Nagel (2016) that (iii) the exponential affine SDF is (clearly) superior to the simple liner SDF in a multi-period framework; actually, their bias and variances are quite equal. Moreover, our simulation study suggests that (iv) averaging pricing errors over multiple dates strikingly reduces the bias inherent to the original procedure of Driessen et al. (2012) that just discounts all cash flows to the fund inception date. Actually, choosing the set $\mathcal{T}$ close to the fund lifetime seems to decrease the bias (and to a lesser extent also the variance) more effectively than all other measures combined. Appendix A explains why our new *biased* estimator (cf. Eq. 9) can yield better small-sample results due to effects related to the famous bias–variance tradeoff.

3.4 Empirical results

Following the conclusions from the simulation study, we use the Preqin data from Section 3.1 and form vintage-year portfolios (VYPs) to estimate simple linear SDF models with a Horizon of 15 years. For each model specification, we report both asymptotic standard errors (SE_A) and *hv*-block cross-validation standard errors (SE_{CV}). We focus the presentation on PE (i.e., mainly Buyout and Growth) as our primary fund type and report both equal-weighted (EW) and value-weighted (FW) estimates across a grid of compounding horizons. Throughout this subsection, t_A and t_{CV} denote the t-ratios based on asymptotic and cross-validation standard errors, respectively[11].

3.4.1 Single market-factor models

Table 5 and Figure 8 report the single-factor MKT model for PE across horizons from 0.1 to 30 years. The market beta estimate monotonically declines with the compounding horizon: under equal weighting (EW), $\hat{\beta}_{\mathrm{MKT}}$ decreases from 1.63 at 0.1 years to approximately 0.91 at 25–30 years; under value weighting (FW), the corresponding range is 1.78 to 0.97. This hump-shaped pattern at short horizons likely reflects the finite-sample bias documented in the simulation study, whereby the estimator overestimates market exposure when few discounting dates are used. Estimates stabilize beyond approximately 20 years, consistent with the bias-minimizing horizons identified in Section 3.3.

A striking feature of the results is the divergence between asymptotic and cross-validation inference. Asymptotic t-ratios (t_A) are borderline at short horizons (≈ 1.8) and drop well below 1.96 at longer horizons; partly driven by extremely large and erratic asymptotic standard errors (e.g., $\mathrm{SE}_A > 1.7$ for horizons ≥ 15 years). In contrast, cross-validation t-ratios (t_{CV}) are uniformly large, ranging from 5.2 to 10.3 under EW and from 5.99 to 44.5 under FW, clearly indicating significance of the single-factor MKT model. Given the small number of cross-sectional units

[11] All R code and data are available in an online repository: `https://github.com/quant-unit/Fundwise_SDF/tree/master/r_project`.

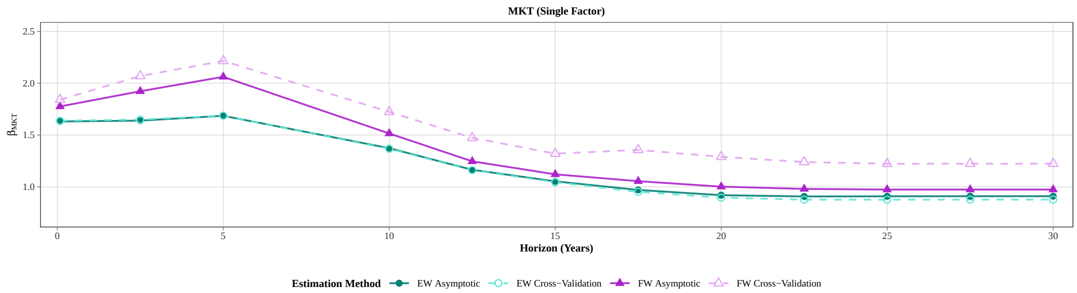

Figure 8: **Empirical (MKT factor):** Single-factor model for PE using VYP.

(vintage-year portfolios) available for estimation, the asymptotic approximation appears unreliable, and we regard the hv-block cross-validation results as more trustworthy. Therefore, drawing on both the simulation evidence and the cross-validation results, we focus on single MKT factor models for the remainder of the empirical analysis, even when their asymptotic t-ratios fall below conventional thresholds.

Table 5: Empirical Estimates: PE — Single-Factor (MKT) Model

	Equal-Weighted (EW)					Value-Weighted (FW)				
Horizon	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}
0.1 yr	1.630	0.897	0.159	1.82	10.33	1.776	1.002	0.041	1.77	44.53
2.5 yr	1.639	0.899	0.185	1.82	8.89	1.922	1.064	0.149	1.81	13.89
5 yr	1.685	0.892	0.188	1.89	8.96	2.061	1.124	0.095	1.83	23.30
10 yr	1.373	0.864	0.172	1.59	7.97	1.515	0.959	0.172	1.58	10.01
12.5 yr	1.164	0.846	0.187	1.38	6.19	1.246	0.925	0.220	1.35	6.68
15 yr	1.053	1.716	0.186	0.61	5.60	1.121	0.908	0.220	1.23	5.99
17.5 yr	0.970	1.588	0.179	0.61	5.31	1.054	1.033	0.197	1.02	6.89
20 yr	0.920	1.856	0.172	0.50	5.22	1.002	1.744	0.176	0.57	7.33
22.5 yr	0.907	1.826	0.162	0.50	5.42	0.980	1.708	0.153	0.57	8.09
25 yr	0.909	1.830	0.158	0.50	5.54	0.974	1.702	0.146	0.57	8.40
27.5 yr	0.909	1.832	0.158	0.50	5.53	0.974	1.702	0.146	0.57	8.39
30 yr	0.909	1.832	0.158	0.50	5.53	0.974	1.702	0.146	0.57	8.39

Notes: $\hat{\beta}$ = parameter estimate; SE_A = asymptotic standard error; SE_{CV} = cross-validation standard error; t_A/t_{CV} = corresponding t-ratios. Significance: $^{*}\,p < 0.05$, $^{**}\,p < 0.01$, $^{***}\,p < 0.005$.

3.4.2 Two-factor models

We now examine two-factor models for PE that pair the MKT factor with either an Alpha intercept or one of the four non-market q^5 factors (Tables 6–10 and Figure 9).

MKT + Alpha (Table 6): Adding an α term increases the MKT loading relative to the single-factor model at most horizons: under EW, $\hat{\beta}_{\text{MKT}}$ ranges from 1.24 to 1.44; under FW, from 1.36 to 1.93. The Alpha coefficient is positive at short horizons ($\hat{\beta}_\alpha \approx 0.003$–$0.005$ per month, corresponding to roughly 3.6–6% annualized) but turns negative at horizons beyond 10 years ($\hat{\beta}_\alpha \approx -0.002$ to -0.004 per month). This sign-reversal across horizons is consistent with the horizon-dependent

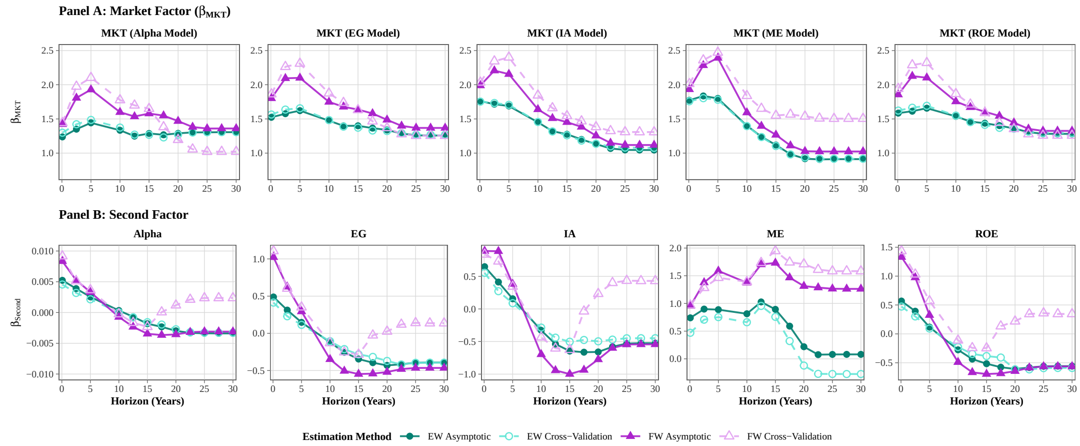

Figure 9: **Empirical (q^5 factors):** Two-factor models for PE using VYP.

bias pattern documented in the simulation study. While a few isolated asymptotic t-ratios are significant (e.g., EW at 5 years: $t_A = 10.8$ for MKT, $t_A = 3.49$ for Alpha), the asymptotic standard errors are highly erratic across horizons (SE_A ranging from 0.13 to over 26), precluding a reliable asymptotic inference for the two-factor model.

Non-market q^5 factors (Tables 7–10): A common pattern emerges across all four specifications (EG, IA, ME, ROE): the second-factor loading is positive at short horizons and declines (often changing sign) as the horizon increases, while the MKT loading absorbs a compensating adjustment. None of the second-factor coefficients are asymptotically significant at any horizon, and the asymptotic standard errors are again extremely large and unstable. Among the q^5 factors, ME stands out with particularly large positive loading estimates under FW at short horizons ($\hat{\beta}_{\mathrm{ME}} \approx$ 1.0–1.6) that remain positive and CV-significant even at long horizons. In contrast, the EG, IA, and ROE loadings uniformly become negative beyond 10 years and remain insignificant under both weighting schemes.

Overall, the two-factor models for PE are fragile: estimates are highly horizon-dependent, asymptotic inference is unreliable, and no second factor consistently adds explanatory power beyond MKT. These findings corroborate the simulation-based recommendation to focus on single-factor MKT models given the limited cross-section of vintage-year portfolios currently available.

3.4.3 Focus on varying vintage cutoffs

Our standard vintage year cutoff in our empirical Preqin dataset is 2019. In this section, we analyze how altering the maximum vintage cutoff from 2019 to older vintages influences the parameter estimates. Concretely, we compare yearly cutoff-vintages from 2021 to 2011 making it 11 distinct cutoff dates. We only consider asymptotic estimates in this section, i.e., we did not perform cross-validation results for the different vintage cutoffs.

Figure 10 shows the market beta estimates for the single-factor model across different vintage cutoffs. We observe a clear upward trend in the market beta as

Table 6: Empirical Estimates: PE — MKT + Alpha Model

	Equal-Weighted (EW)					Value-Weighted (FW)				
Horizon	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}
Panel A: Market Factor (β_{MKT})										
0.1 yr	1.237	1.491	0.323	0.83	4.02	1.423	0.758	0.118	1.88	12.19
2.5 yr	1.350	0.844	0.375	1.60	3.80	1.807	1.744	0.113	1.04	17.50
5 yr	1.442***	0.134	0.376	10.80	3.95	1.929	1.072	0.134	1.80	15.66
10 yr	1.334	2.091	0.373	0.64	3.68	1.599	24.091	0.189	0.07	9.37
12.5 yr	1.249	10.646	0.390	0.12	3.26	1.536	26.945	0.207	0.06	8.19
15 yr	1.287***	0.189	0.388	6.82	3.27	1.579	8.828	0.256	0.18	6.44
17.5 yr	1.268***	0.278	0.372	4.56	3.31	1.549	8.244	0.268	0.19	5.15
20 yr	1.289	3.589	0.387	0.36	3.29	1.468	6.849	0.326	0.21	3.67
22.5 yr	1.303	1.523	0.396	0.86	3.29	1.382	5.413	0.322	0.26	3.28
25 yr	1.304	1.586	0.395	0.82	3.30	1.359	5.457	0.312	0.25	3.27
27.5 yr	1.306	1.571	0.394	0.83	3.32	1.360	5.450	0.311	0.25	3.28
30 yr	1.306	1.571	0.394	0.83	3.32	1.360	5.450	0.311	0.25	3.28
Panel B: Alpha (β_{Alpha})										
0.1 yr	0.005	0.009	0.002	0.61	2.23	0.008	0.005	0.001	1.76	12.48
2.5 yr	0.004*	0.002	0.002	2.12	1.36	0.005	0.010	0.002	0.55	2.16
5 yr	0.003***	0.001	0.002	3.49	1.00	0.003*	0.002	0.002	1.97	1.47
10 yr	0.000	0.004	0.003	0.08	0.01	-0.001	0.156	0.003	-0.00	-0.08
12.5 yr	-0.001	0.184	0.003	-0.00	-0.26	-0.002	0.183	0.003	-0.01	-0.48
15 yr	-0.002*	0.001	0.003	-2.51	-0.50	-0.003	0.070	0.004	-0.05	-0.65
17.5 yr	-0.002*	0.001	0.003	-2.29	-0.59	-0.004	0.067	0.004	-0.05	0.02
20 yr	-0.003	1.841	0.003	-0.00	-0.79	-0.004	0.064	0.005	-0.06	0.26
22.5 yr	-0.003	0.397	0.003	-0.01	-0.92	-0.003	0.059	0.005	-0.05	0.44
25 yr	-0.003	0.324	0.004	-0.01	-0.92	-0.003	0.060	0.005	-0.05	0.50
27.5 yr	-0.003	0.315	0.004	-0.01	-0.93	-0.003	0.060	0.005	-0.05	0.50
30 yr	-0.003	0.315	0.004	-0.01	-0.93	-0.003	0.060	0.005	-0.05	0.50

Notes: $\hat{\beta}$ = parameter estimate; SE_A = asymptotic standard error; SE_{CV} = cross-validation standard error; t_A/t_{CV} = corresponding t-ratios. Significance: * $p < 0.05$, ** $p < 0.01$, *** $p < 0.005$.

Table 7: Empirical Estimates: PE — MKT + EG Model

	Equal-Weighted (EW)					Value-Weighted (FW)				
Horizon	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}
Panel A: Market Factor (β_{MKT})										
0.1 yr	1.524	1.791	0.246	0.85	6.36	1.802**	0.698	0.093	2.58	19.95
2.5 yr	1.576	3.504	0.287	0.45	5.69	2.095	1.725	0.067	1.21	33.91
5 yr	1.621	17.286	0.282	0.09	5.87	2.101	4.304	0.049	0.49	46.93
10 yr	1.474	2.814	0.241	0.52	6.14	1.749	6.056	0.188	0.29	9.97
12.5 yr	1.399	1.379	0.229	1.01	6.06	1.678	4.452	0.184	0.38	9.44
15 yr	1.400	1.279	0.216	1.09	6.34	1.637	2.492	0.191	0.66	8.55
17.5 yr	1.366	1.876	0.225	0.73	5.91	1.580	2.357	0.171	0.67	8.59
20 yr	1.338	1.361	0.244	0.98	5.41	1.487	2.312	0.161	0.64	8.52
22.5 yr	1.287	1.347	0.267	0.96	4.84	1.398	2.280	0.151	0.61	8.45
25 yr	1.259	1.426	0.276	0.88	4.60	1.369	1.832	0.140	0.75	8.95
27.5 yr	1.257	1.439	0.277	0.87	4.59	1.368	1.832	0.140	0.75	8.99
30 yr	1.257	1.439	0.277	0.87	4.59	1.368	1.832	0.140	0.75	8.99
Panel B: EG (β_{EG})										
0.1 yr	0.489	1.081	0.286	0.45	1.44	1.023*	0.444	0.087	2.30	12.68
2.5 yr	0.312	2.137	0.292	0.15	0.78	0.626	1.011	0.493	0.62	1.22
5 yr	0.148	11.338	0.243	0.01	0.47	0.294	2.526	0.525	0.12	0.67
10 yr	-0.120	2.641	0.271	-0.05	-0.39	-0.348	4.373	0.576	-0.08	-0.23
12.5 yr	-0.256	5.720	0.289	-0.04	-0.73	-0.506	3.542	0.548	-0.14	-0.46
15 yr	-0.345	4.565	0.318	-0.08	-0.89	-0.547	2.205	0.523	-0.25	-0.54
17.5 yr	-0.394	10.642	0.380	-0.04	-0.83	-0.542	2.083	0.561	-0.26	-0.05
20 yr	-0.431	7.164	0.383	-0.06	-0.97	-0.520	2.175	0.584	-0.24	0.04
22.5 yr	-0.415	7.434	0.262	-0.06	-1.60	-0.480	2.272	0.646	-0.21	0.18
25 yr	-0.395	8.352	0.288	-0.05	-1.38	-0.465	2.056	0.649	-0.23	0.22
27.5 yr	-0.392	8.473	0.289	-0.05	-1.36	-0.465	2.057	0.643	-0.23	0.21
30 yr	-0.392	8.473	0.289	-0.05	-1.36	-0.465	2.057	0.643	-0.23	0.21

Notes: $\hat{\beta}$ = parameter estimate; SE_A = asymptotic standard error; SE_{CV} = cross-validation standard error; t_A/t_{CV} = corresponding t-ratios. Significance: $^{*}\, p < 0.05$, $^{**}\, p < 0.01$, $^{***}\, p < 0.005$.

Table 8: Empirical Estimates: PE — MKT + IA Model

	Equal-Weighted (EW)					Value-Weighted (FW)				
Horizon	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}
Panel A: Market Factor (β_{MKT})										
0.1 yr	1.760	1.228	0.274	1.43	6.40	1.989	1.846	0.096	1.08	21.16
2.5 yr	1.716	1.479	0.273	1.16	6.35	2.207	1.999	0.138	1.10	17.02
5 yr	1.689	3.710	0.255	0.46	6.70	2.156	5.420	0.097	0.40	24.67
10 yr	1.451	1.126	0.206	1.29	7.06	1.640	5.551	0.059	0.30	31.38
12.5 yr	1.321	1.039	0.199	1.27	6.63	1.510	1.558	0.038	0.97	44.08
15 yr	1.276	1.037	0.191	1.23	6.60	1.453	1.682	0.027	0.86	57.71
17.5 yr	1.200	1.032	0.216	1.16	5.47	1.384	1.691	0.047	0.82	31.31
20 yr	1.132	1.001	0.240	1.13	4.74	1.255	1.776	0.108	0.71	12.83
22.5 yr	1.068	2.835	0.257	0.38	4.28	1.149	1.602	0.169	0.72	7.84
25 yr	1.046	2.831	0.253	0.37	4.27	1.119	1.836	0.184	0.61	7.10
27.5 yr	1.045	2.839	0.249	0.37	4.32	1.118	1.835	0.183	0.61	7.16
30 yr	1.045	2.839	0.249	0.37	4.32	1.118	1.835	0.183	0.61	7.16
Panel B: IA (β_{IA})										
0.1 yr	0.649	2.253	0.531	0.29	1.04	0.889	3.240	0.345	0.27	2.43
2.5 yr	0.413	2.708	0.567	0.15	0.48	0.886	3.182	1.039	0.28	0.70
5 yr	0.156	6.603	0.462	0.02	0.19	0.383	8.311	1.235	0.05	0.28
10 yr	-0.328	4.260	0.471	-0.08	-0.61	-0.696	10.654	0.961	-0.07	-0.45
12.5 yr	-0.538	4.211	0.496	-0.13	-0.89	-0.943	2.463	0.863	-0.38	-0.70
15 yr	-0.646	3.722	0.553	-0.17	-0.91	-0.999	2.453	0.825	-0.41	-0.76
17.5 yr	-0.663	3.640	0.694	-0.18	-0.69	-0.936	2.625	0.995	-0.36	-0.04
20 yr	-0.660	3.449	0.745	-0.19	-0.67	-0.774	2.986	1.090	-0.26	0.21
22.5 yr	-0.580	4.544	0.757	-0.13	-0.62	-0.599	2.783	1.059	-0.22	0.38
25 yr	-0.531	4.858	0.735	-0.11	-0.61	-0.545	2.818	1.029	-0.19	0.42
27.5 yr	-0.525	4.910	0.727	-0.11	-0.62	-0.543	2.820	1.023	-0.19	0.42
30 yr	-0.525	4.910	0.727	-0.11	-0.62	-0.543	2.820	1.023	-0.19	0.42

Notes: $\hat{\beta}$ = parameter estimate; SE_A = asymptotic standard error; SE_{CV} = cross-validation standard error; t_A/t_{CV} = corresponding t-ratios. Significance: * $p < 0.05$, ** $p < 0.01$, *** $p < 0.005$.

Table 9: Empirical Estimates: PE — MKT + ME Model

	Equal-Weighted (EW)					Value-Weighted (FW)				
Horizon	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}
Panel A: Market Factor (β_{MKT})										
0.1 yr	1.773	1.142	0.214	1.55	8.23	1.933*	0.974	0.105	1.98	19.22
2.5 yr	1.836	1.033	0.221	1.78	8.16	2.285**	0.748	0.050	3.05	46.94
5 yr	1.800	1.142	0.218	1.58	8.16	2.392***	0.538	0.041	4.45	59.86
10 yr	1.401	1.008	0.191	1.39	7.31	1.593***	0.464	0.208	3.44	8.83
12.5 yr	1.240	1.426	0.187	0.87	6.61	1.396	1.229	0.149	1.14	11.07
15 yr	1.118	0.939	0.187	1.19	5.90	1.267	1.579	0.111	0.80	13.92
17.5 yr	0.984	1.706	0.187	0.58	5.25	1.110	3.220	0.087	0.34	17.97
20 yr	0.918	1.850	0.184	0.50	5.10	1.028	3.363	0.124	0.31	12.37
22.5 yr	0.908	1.820	0.180	0.50	5.10	1.024	3.135	0.146	0.33	10.31
25 yr	0.911	1.827	0.180	0.50	5.09	1.023	3.129	0.152	0.33	9.90
27.5 yr	0.912	1.830	0.180	0.50	5.09	1.024	3.133	0.153	0.33	9.86
30 yr	0.912	1.830	0.180	0.50	5.09	1.024	3.133	0.153	0.33	9.86
Panel B: ME (β_{ME})										
0.1 yr	0.740	1.980	0.612	0.37	0.77	0.954	1.963	0.226	0.49	4.27
2.5 yr	0.898	2.056	0.391	0.44	1.81	1.379	2.973	0.201	0.46	6.38
5 yr	0.888	2.234	0.296	0.40	2.55	1.587	3.536	0.097	0.45	15.17
10 yr	0.814	12.157	0.417	0.07	1.58	1.373	3.431	0.180	0.40	7.71
12.5 yr	1.027	3.631	0.235	0.28	4.06	1.704	5.756	0.480	0.30	3.61
15 yr	0.893	2.958	0.346	0.30	2.19	1.732	6.762	0.546	0.26	3.56
17.5 yr	0.590	2.507	0.672	0.24	0.48	1.467	14.712	0.298	0.10	5.86
20 yr	0.218	1.749	0.786	0.12	-0.15	1.313	7.376	0.154	0.18	11.11
22.5 yr	0.075	1.734	0.737	0.04	-0.37	1.286	9.366	0.085	0.14	18.87
25 yr	0.080	1.736	0.743	0.05	-0.37	1.265	9.487	0.071	0.13	22.28
27.5 yr	0.080	1.736	0.742	0.05	-0.37	1.263	9.526	0.071	0.13	22.24
30 yr	0.080	1.736	0.742	0.05	-0.37	1.263	9.526	0.071	0.13	22.24

Notes: $\hat{\beta}$ = parameter estimate; SE_A = asymptotic standard error; SE_{CV} = cross-validation standard error; t_A/t_{CV} = corresponding t-ratios. Significance: $^{*}\ p < 0.05$, $^{**}\ p < 0.01$, $^{***}\ p < 0.005$.

Table 10: Empirical Estimates: PE — MKT + ROE Model

	Equal-Weighted (EW)					Value-Weighted (FW)				
Horizon	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}	$\hat{\beta}$	SE_A	SE_{CV}	t_A	t_{CV}
Panel A: Market Factor (β_{MKT})										
0.1 yr	1.585	2.675	0.244	0.59	6.66	1.856*	0.912	0.122	2.04	15.89
2.5 yr	1.613	5.218	0.278	0.31	5.99	2.126	1.530	0.076	1.39	30.31
5 yr	1.657	18.700	0.261	0.09	6.49	2.103	12.191	0.034	0.17	67.65
10 yr	1.538	1.688	0.205	0.91	7.57	1.755	4.738	0.165	0.37	11.31
12.5 yr	1.461	1.249	0.185	1.17	7.87	1.671	2.401	0.165	0.70	10.37
15 yr	1.437	1.213	0.179	1.18	7.90	1.605	2.205	0.166	0.73	9.55
17.5 yr	1.402	1.079	0.196	1.30	6.99	1.544	2.096	0.147	0.74	9.83
20 yr	1.361	1.135	0.208	1.20	6.53	1.445	1.992	0.125	0.73	10.87
22.5 yr	1.307	1.129	0.231	1.16	5.70	1.355	1.919	0.098	0.71	13.07
25 yr	1.283	1.133	0.239	1.13	5.43	1.326	1.843	0.086	0.72	14.64
27.5 yr	1.282	1.134	0.239	1.13	5.41	1.325	1.843	0.086	0.72	14.63
30 yr	1.282	1.134	0.239	1.13	5.41	1.325	1.843	0.086	0.72	14.63
Panel B: ROE (β_{ROE})										
0.1 yr	0.568	2.320	0.477	0.24	0.99	1.330	0.820	0.514	1.62	2.79
2.5 yr	0.393	4.496	0.454	0.09	0.67	0.977	1.259	0.769	0.78	1.34
5 yr	0.117	15.801	0.365	0.01	0.25	0.325	9.526	0.786	0.03	0.73
10 yr	-0.276	2.999	0.395	-0.09	-0.58	-0.489	4.149	0.834	-0.12	-0.13
12.5 yr	-0.437	2.922	0.436	-0.15	-0.79	-0.666	2.513	0.815	-0.26	-0.30
15 yr	-0.516	2.828	0.536	-0.18	-0.72	-0.695	2.280	0.813	-0.30	-0.30
17.5 yr	-0.577	2.667	0.668	-0.22	-0.61	-0.684	2.270	0.928	-0.30	0.14
20 yr	-0.607	3.573	0.227	-0.17	-2.71	-0.645	2.145	0.969	-0.30	0.22
22.5 yr	-0.583	3.639	0.223	-0.16	-2.75	-0.586	2.155	1.052	-0.27	0.33
25 yr	-0.561	3.730	0.240	-0.15	-2.47	-0.564	2.136	1.041	-0.26	0.34
27.5 yr	-0.558	3.742	0.243	-0.15	-2.43	-0.564	2.136	1.020	-0.26	0.34
30 yr	-0.558	3.742	0.243	-0.15	-2.43	-0.564	2.136	1.020	-0.26	0.34

Notes: $\hat{\beta}$ = parameter estimate; SE_A = asymptotic standard error; SE_{CV} = cross-validation standard error; t_A/t_{CV} = corresponding t-ratios. Significance: $^{*}\ p < 0.05$, $^{**}\ p < 0.01$, $^{***}\ p < 0.005$.

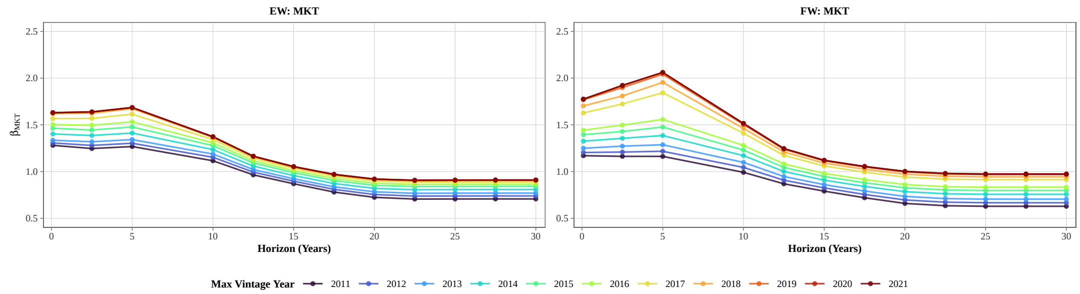

Figure 10: **Sensitivity analysis (MKT factor):** Single-factor models for PE using VYP with different vintage-year cutoffs (asymptotic estimates).

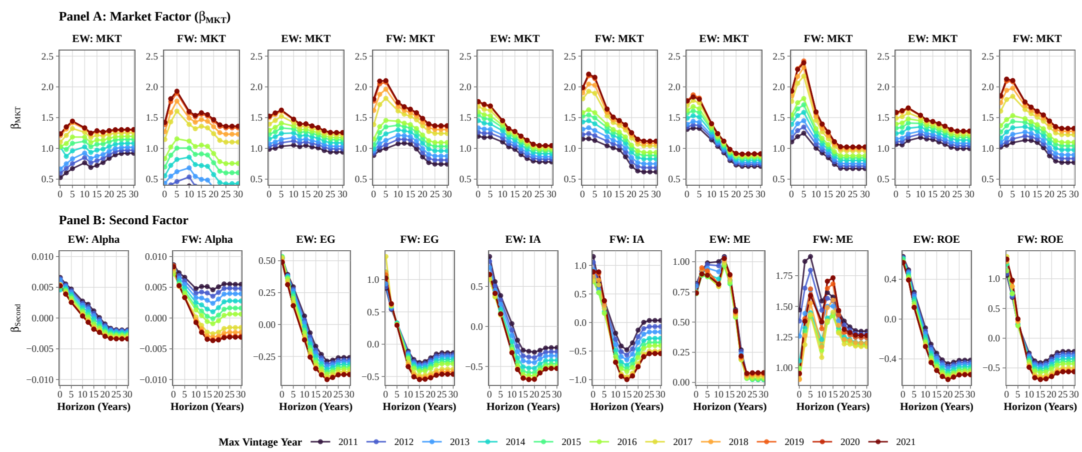

Figure 11: **Sensitivity analysis (q^5 factors):** Two-factor models for PE using VYP with different vintage-year cutoffs (asymptotic estimates).

we include more recent vintages. For equal-weighted (EW) portfolios, $\hat{\beta}_{\text{MKT}}$ rises from approximately 0.87 with a 2011 cutoff to 1.05 with the full sample (2019/2021). Similarly, value-weighted (FW) estimates increase from 0.79 to 1.12. This could suggest that recent vintage years display higher market co-movements or that treating unrealized NAVs as distribution cash flows mechanically increases the β exposures. Interestingly, asymptotic standard errors tend to increase when including the most recent vintages (e.g., jumping from ≈ 0.5 to > 1.0 for FW after 2016), likely because these younger funds contribute fewer informative cash flows given the 15-year horizon.

Figure 11 displays the coefficients for the two-factor models. The second-factor estimates are largely robust to the vintage cutoff, maintaining their general sign and magnitude. For instance, the ME factor loading remains consistently positive (around 0.9 for EW and 1.7 for FW). In contrast, the Alpha coefficient in the MKT+Alpha model exhibits a downward drift, turning from insignificant positive values in the earlier cutoffs to significantly negative values (e.g., -0.18% per month) when including post-2016 vintages.

4 Conclusion

Theoretically, our Least-Mean-Distance estimator can be easily generalized to estimate SDF models for all kinds of non-traded cash flows. Practically, semiparametric estimators commonly exhibit problematic small sample behavior. Given the amount of currently available private equity fund data, our estimator's variance seems quite large, even for simple SDF model specifications. Specifically, our Monte Carlo simulation results prompt us to conclude that the closely related Driessen et al. (2012) estimator may exhibit more bias and variance than originally assumed in their paper. Especially, the variance of α estimates seems to be too high to allow reliable abnormal performance conclusions. Fortunately, we show that at least the bias can be easily reduced by averaging pricing errors over all dates within the fund lifetime.

In the data-sparse private equity domain with only 20-40 cross-sectional units (i.e., vintage year portfolios) currently available for estimation, asymptotic inference seems not to be overly useful. Thus, we strongly advise always challenging asymptotic inference results by resampling or cross-validation techniques adapted to the dependence structure of overlapping fund cash flows. However, even these conclusions should be double-checked to avoid unreasonable instances, e.g., when hv-block cross-validation chooses dubious models with negative MKT factor estimates. Unfortunately, using individual funds instead of vintage year portfolios, which yields smaller asymptotic standard errors, constitutes no viable resolution as individual funds show considerably larger small-sample bias and variance in our Monte Carlo example. Since, in our empirical analyses, basically all two-factor models' asymptotic standard errors appear statistically insignificant, we conjecture that naive versions of our SDF estimator shall be exclusively used for a single-MKT-factor model until considerably more vintage year information for private equity funds is available.

If someone wants to estimate more complex SDF models that incorporate additional factors, more structure is needed. These can be parametric assumptions for the data generating process (Ang et al., 2018) or to extract additional information from intermediate net asset values (Gredil et al., 2020; Brown et al., 2021). A first "modern" approach to the same problem is applying machine learning techniques that regularize/shrink all coefficients other than the MKT factor. Secondly, given the high estimator variance revealed in the simulation study, statistical learning methods that create a strong learner by combining multiple weak learners seem also worth considering (boosting, bagging, or model averaging).

Additional idea for further research is to incorporate fees and carry model to DGP for SDF estimation for PE gross-fee performance or also for better DGP models for fund LPs. With a more realistic DGP we likely become more realistic SDF estimates and more precise terms for the bias and variance terms.

Another idea is to treat NAVs as true value proxies to alleviate the NFV/Timing Risk Premium bias. This would would entail adapting the estimator to neglect realized cash flows and use NAVs instead.

Finally, we point to the potentially most interesting topic for future research. Our simulation study indicates that the estimator's bias and variance can be controlled by an appropriate choice for the set $\mathcal{T}$. This set averages the pricing error over multiple discounting dates. In simpler terms, an identification method that utilizes a future value concept instead of net present values obtains more favorable results in our case. The bias in our simulation study is minimal when the set of discounting dates corresponds to the fund lifetime. A parsimonious but general model that allows for misspecification and can explain this $\mathcal{T}$-averaging effect from a mathematical perspective would be highly appreciated.

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A Bias–Variance Tradeoff of $\mathcal{T}$-Averaging

This appendix shows that $\mathcal{T}$-averaging introduces a small asymptotic bias via the Timing Risk Premium but achieves a net bias reduction in finite samples by lowering the variance of the pricing-error moments. We formalize this claim and reconcile it with the simulation results from Section 3.3.

A.1 Decomposition of $\mathbb{E}[\bar{\epsilon}_i(\theta_0)]$

Starting from the average pricing error (Equation 16) and the deal-level decomposition (Equation 3), we can write

$$\bar{\epsilon}_i(\theta_0) = \frac{1}{|\mathcal{T}_i|} \sum_{\tau \in \mathcal{T}_i} \epsilon_{\tau,i}(\theta_0) = \frac{1}{|\mathcal{T}_i|} \sum_{\tau \in \mathcal{T}_i} \sum_{j=1}^{J} \frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau} \delta_{i,j} \tag{23}$$

Taking the unconditional expectation yields

$$\mathbb{E}[\bar{\epsilon}_i(\theta_0)] = \frac{1}{|\mathcal{T}_i|} \sum_{\tau \in \mathcal{T}_i} \sum_{j=1}^{J} \mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau} \delta_{i,j} \right] \tag{24}$$

For each (τ, j) pair, we decompose via the covariance identity:

$$\mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\mathrm{Inv}}}}{\Psi_\tau}\,\delta_{i,j}\right] = \underbrace{\mathrm{Cov}\left(\frac{\Psi_{t_{i,j}^{\mathrm{Inv}}}}{\Psi_\tau},\,\delta_{i,j}\right)}_{\mathrm{TRP}_{\tau,j}} + \underbrace{\mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\mathrm{Inv}}}}{\Psi_\tau}\right]}_{>0} \cdot \underbrace{\mathbb{E}[\delta_{i,j}]}_{=0} \tag{25}$$

Hence the asymptotic bias of the average pricing error is entirely determined by the averaged Timing Risk Premium:

$$\mathbb{E}[\bar{\epsilon}_i(\theta_0)] = \frac{1}{|\mathcal{T}_i|}\sum_{\tau\in\mathcal{T}_i}\sum_{j=1}^{J}\mathrm{Cov}\left(\frac{\Psi_{t_{i,j}^{\mathrm{Inv}}}}{\Psi_\tau},\,\delta_{i,j}\right) \tag{26}$$

The crucial observation is that this covariance is exactly zero when $\tau \le t_{i,j}^{\mathrm{Inv}}$ (Case A from Section 2.1). When $\tau \le t_{i,j}^{\mathrm{Inv}}$, iterated expectations through $\mathcal{F}_{t_{i,j}^{\mathrm{Inv}}}$ give

$$\mathrm{Cov}\left(\frac{\Psi_{t_{i,j}^{\mathrm{Inv}}}}{\Psi_\tau},\,\delta_{i,j}\right) = \mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\mathrm{Inv}}}}{\Psi_\tau}\,\underbrace{\mathbb{E}\Big[\delta_{i,j}\mid\mathcal{F}_{t_{i,j}^{\mathrm{Inv}}}\Big]}_{=0\text{ by Eq. 2}}\right] = 0 \tag{27}$$

Therefore the bias decomposes as

$$\mathbb{E}[\bar{\epsilon}_i(\theta_0)] = \frac{1}{|\mathcal{T}_i|}\left[\underbrace{\sum_{\substack{\tau\in\mathcal{T}_i,\,j\\ \tau\le t_{i,j}^{\mathrm{Inv}}}} 0}_{\text{NPV terms}} + \underbrace{\sum_{\substack{\tau\in\mathcal{T}_i,\,j\\ \tau> t_{i,j}^{\mathrm{Inv}}}}\mathrm{Cov}\left(\frac{\Psi_{t_{i,j}^{\mathrm{Inv}}}}{\Psi_\tau},\,\delta_{i,j}\right)}_{\text{NFV terms (Timing Risk Premium)}}\right] \tag{28}$$

Remark (misspecification). Under model misspecification, $\mathbb{E}[\delta_{i,j}] \neq 0$ and the second term in Equation 25 no longer vanishes. This introduces an additional bias component proportional to $\mathbb{E}[\Psi_{t_{i,j}^{\mathrm{Inv}}}/\Psi_\tau]\cdot\mathbb{E}[\delta_{i,j}]$ that grows with $|\mathcal{T}_i|$.

A.2 Reconciliation with the simulation study

If this asymptotic bias were the only effect, the NPV-only approach of Driessen et al. (2012) with $\mathcal{T}_i = \{\min_j t_{i,j}^{\mathrm{Inv}}\}$ would exhibit the smallest bias, and adding NFV terms would always increase it. However, the simulation study in Section 3.3 shows the opposite: averaging over larger $\mathcal{T}_i$ *reduces* total bias. The resolution involves two competing effects.

Effect 1: Finite-sample bias from nonlinear estimation For the nonlinear least-squares objective $Q_n(\theta) = -\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i(\theta)^2$, the finite-sample bias of $\hat{\theta}$ is approximately (standard Nagar (1959)-type expansion for M-estimators):

$$\mathbb{E}[\hat{\theta}-\theta_0] \approx -\frac{1}{n}\,H^{-1}\cdot b(\mathrm{Var}[\bar{\epsilon}_i],\,\text{curvature}) \tag{29}$$

where $H = \nabla^2 Q_n(\theta_0)$ denotes the Hessian of the objective function. The term curvature refers to the second-order derivatives $\nabla^2_\theta \bar{\epsilon}_i(\theta_0)$ of the moment conditions with respect to the parameter vector. These derivatives measure how nonlinear the estimating equations are in θ: if the pricing-error moments were perfectly linear in θ, the curvature would vanish and no finite-sample bias would arise. The correction term $b(\cdot)$ is the Nagar-type bias correction from Rilstone et al. (1996) and Newey and Smith (2004); it captures the interaction between the variance of the moment conditions $\mathrm{Var}[\bar{\epsilon}_i(\theta_0)]$ and the curvature of the objective. Specifically, b is proportional to $\mathrm{Var}[\bar{\epsilon}_i(\theta_0)]$: the noisier the moments, the more the nonlinear optimizer is "pulled away" from the true θ_0 in finite samples, because the concavity/convexity of Q_n causes asymmetric responses to positive and negative moment deviations. When each fund provides only one noisy moment (NPV-only), this variance is high, resulting in a substantial finite-sample bias.

Effect 2: Variance reduction through $\mathcal{T}$-averaging When averaging over $|\mathcal{T}_i|$ discounting dates, the variance of the average pricing error is

$$\mathrm{Var}[\bar{\epsilon}_i] = \frac{1}{|\mathcal{T}_i|^2} \sum_{\tau,\tau' \in \mathcal{T}_i} \mathrm{Cov}\big(\epsilon_{\tau,i},\, \epsilon_{\tau',i}\big) \tag{30}$$

The pricing errors $\epsilon_{\tau,i}$ at different τ are positively correlated (they share the same deal cash flows and factor realizations), but they are not perfectly correlated because different τ's weight the SDF path differently, so that exploiting these additional conditional moment restrictions improves efficiency (Newey, 1993). Hence

$$\mathrm{Var}[\bar{\epsilon}_i] < \mathrm{Var}[\epsilon_{\tau_0,i}] \quad \text{for } |\mathcal{T}_i| > 1$$

and this variance reduction directly decreases the finite-sample bias from the nonlinear optimization (cf. Equation 29).

The net effect The total bias in small samples decomposes approximately as

$$\underbrace{\text{Finite-sample bias}}_{\propto \mathrm{Var}[\bar{\epsilon}_i]/n} + \underbrace{\text{Asymptotic TRP bias}}_{\text{Eq. 28}}$$

As $|\mathcal{T}_i|$ increases from one:

- The finite-sample bias *decreases* because $\mathrm{Var}[\bar{\epsilon}_i]$ drops (cf. Equation 29).
- The TRP bias *increases* as more NFV terms enter, but is tempered by two factors:
  - only (τ, j) pairs with $\tau > t^{\mathrm{Inv}}_{i,j}$ contribute non-zero TRP terms,
  - the $1/|\mathcal{T}_i|$ denominator dilutes the accumulating terms.

The simulation study finds the sweet spot where these two effects balance: at $|\mathcal{T}_i|$ corresponding to approximately the fund lifetime (180 months), the variance-reduction benefit dominates the TRP cost.

A.3 Interpretation

The $\mathcal{T}$-averaging mechanism can be understood through three complementary perspectives from the econometrics and asset pricing literature.

Variance reduction via moment combination. In the GMM framework of Hansen (1982), efficiency gains arise from combining multiple moment conditions. Each discounting date $\tau \in \mathcal{T}_i$ contributes a separate pricing-error moment $\epsilon_{\tau,i}(\theta)$, and averaging over $\mathcal{T}_i$ is formally equivalent to combining $|\mathcal{T}_i|$ such moments into a single summary condition. Since these moments share the same underlying deal cash flows and factor paths but weight the SDF trajectory differently, they are positively yet imperfectly correlated. The resulting reduction in $\mathrm{Var}[\bar{\epsilon}_i]$ is the classical Hansen (1982) effect of exploiting additional moment information, which in turn decreases the finite-sample bias of the nonlinear estimator (cf. Equation 29).

The TRP as a bounded risk-premium term. The Timing Risk Premium in Equation 26 is a covariance between a ratio of stochastic discount factors and a deal-level pricing error. Within the Hansen and Jagannathan (1991) framework, the magnitude of such covariances is bounded by the product of the SDF's variability and the payoff's standard deviation. Under correct model specification, the TRP terms are therefore non-zero (i.e., they reflect genuine systematic risk) but their size is disciplined by the Hansen–Jagannathan bound. Consequently, the TRP bias introduced by NFV terms ($\tau > t_{i,j}^{\mathrm{Inv}}$) grows only moderately with $|\mathcal{T}_i|$ and is further diluted by the $1/|\mathcal{T}_i|$ denominator.

Analogy to model averaging. The structure of $\mathcal{T}$-averaging closely mirrors the jackknife model averaging framework of Hansen and Racine (2012), where combining multiple (possibly misspecified) estimators achieves a bias–variance tradeoff: averaging reduces variance at the expense of a small bias from the inclusion of imperfect components. In our setting, the "models" being averaged are the pricing-error moments at different discounting dates τ. Each individual moment is unbiased for the NPV terms but may carry TRP bias for the NFV terms; analogous to the weak misspecification in the Hansen and Racine (2012) setup. Their key insight applies directly: the variance reduction from averaging dominates the bias cost as long as the individual components are not too severely biased, which is precisely what the Hansen and Jagannathan (1991) bound guarantees in our context.

Summary. The $\mathcal{T}$-averaging mechanism does not make the TRP wash out in the sense that each individual TRP term goes to zero. Rather, it achieves a bias–variance tradeoff where the TRP bias introduced by NFV terms is more than compensated by the variance reduction that cures the finite-sample bias from the nonlinear estimation. The fact that the optimal $|\mathcal{T}_i|$ depends on the SDF model specification is consistent with this explanation: a more complex or misspecified model not only increases the TRP covariance terms but also activates the additional bias channel identified in the Remark on misspecification above, shifting the optimal tradeoff point. Finally, the favorable tradeoff observed in our simulation study may partly reflect the conditioning on a single realized factor path, which attenuates the TRP covariance terms relative to their unconditional magnitude. Across different macro regimes, the TRP cost could be larger and the optimal $|\mathcal{T}_i|$ smaller.