On Semiparametric SDF Estimators for Pooled, Non-Traded Cash Flows

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August 17, 2025

Keywords

Stochastic Discount Factor, Semiparametric Estimation, Spatial Dependence, Generalized Method of Moments (GMM), Private Equity, Illiquid Assets

Acknowledgements

We thank Nicolas Dütsch, Philipp Abel, and Kai Urban for valuable feedback and helpful comments that greatly improved the paper.

Declaration of interest

The authors report no conflict of interest. The authors alone are responsible for the content and writing of the paper.

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Abstract

This paper proposes an improved stochastic discount factor estimation methodology suited for fund-level cash flows of private equity funds and other pooled, non-traded cash flow streams. The asymptotic inference framework for our semiparametric nonlinear least squares estimator draws on a spatial notion, i.e., the idea that the economic distance between distinct private equity funds can be measured. The empirical and Monte Carlo simulation results indicate (i) that our method can improve the popular Generalized Method of Moments approach of Driessen et al. (2012), but (ii) that estimator variance for typical data sizes is still high. Thus, we conjecture that traditional semiparametric extremum estimators like the one described by us shall be exclusively used for single-factor models until considerably more vintage year information for private equity funds is available.

1 Introduction

Private equity has outgrown its niche, sitting today on more than $9 trillion in assets under management, yet rigorous asset-pricing tools have not kept pace with this ascent. The empirical analysis and risk assessment of private equity and other non-traded cash flows remain fundamentally challenging due to the absence of market-based valuations and the inherent frictions of private markets (i.e., under incomplete information). Unlike public assets with trusted and tradeable valuations (in liquid secondary markets), private equity investments generate irregular, infrequently observed cash flows for which standard return-based asset pricing techniques are unsuitable.

Describe complexity of pooled cash flows.

We address this gap by proposing a semiparametric stochastic discount factor (SDF) estimator tailored to fund-level cash flows that refines the SDF estimators of Driessen et al. (2012) and Korteweg and Nagel (2016). Our nonlinear least squares estimator stems from the class of Least-Mean-Distance (LMD) estimators, which are easier to handle than traditional Generalized Method of Moments (GMM) approaches (Pötscher and Prucha, 1997). It can be readily applied to fund-level cash flow data of private equity funds. Our LMD estimator arguably both simplifies and generalizes the GMM methodology of Driessen et al. (2012), where we provide the asymptotic inference framework that was missing in the original paper. The asymptotic inference formulations rely on the concept of spatial (near-epoch) dependence between funds as proposed by Korteweg and Nagel (2016). In this context, it is paramount to quantify the economic distance between funds by a measure like absolute vintage year difference or cash flow overlap[1].‘ Additionally, we propose a simple solution to the “exploding alpha” issue briefly mentioned in the Driessen et al. (2012) paper. Our Monte Carlo results suggest that the same modification (that solves the “exploding alpha” issue) dramatically reduces the inherent small-sample bias associated with the original Driessen et al. (2012) estimator.

[1] However, this economic inter-fund *distance* refers **not** to the term Least-Mean-*Distance* estimator.

In the empirical application of our new estimator, we test simple linear and exponentially affine SDF models that can draw on the five return factors associated with the q^5 investment factor model recently proposed by Hou et al. (2020). Based on a Spatial Heteroskedasticity and Autocorrelation Consistent (SHAC) covariance matrix estimator, we calculate asymptotic standard errors for the model coefficients. Moreover, we assess the small-sample variance of coefficient estimates and the out-of-sample performance of the different SDF models by *hv*-block cross-validation, which accounts for the inter-vintage-year dependence of private equity funds (Racine, 2000). We test one- and two-factor models for private equity fund types: Private Equity, Venture Capital, Private Debt, Real Estate, Natural Resources, and Infrastructure. All two-factor model results are rather devastating; not more than the single-market-factor model results seem reasonable given the high estimator variance.

The paper is structured as follows. Section 2 reviews the related literature. Section 3 introduces our semiparametric LMD estimator and its corresponding asymptotic inference framework. Section 4 applies the method to estimate q^5-investment-factor SDFs for various private equity fund types using simulated and real-world cash flows. Section 5 concludes.

2 Literature

For private equity funds, performance evaluation under the SDF framework is currently an active research topic (Driessen et al., 2012; Franzoni et al., 2012; Buchner, 2014, 2016a,b; Korteweg and Nagel, 2016; Ang et al., 2018; Gredil et al., 2020). Previously, SDF performance evaluation was already applied for public stock markets by, e.g., Farnsworth et al. (2002) and for hedge fund returns by Li et al. (2016).

In our literature review, we focus on the most relevant parametric and semiparametric estimation methods for private capital fund cash flows.

2.1 Parametric approach

Ang et al. (2018) contribute a parametric approach using Bayesian Markov Chain Monte Carlo methods to estimate latent return factors from present value ratios. Ang et al. (2018) use fund-level data provided by Preqin to estimate a "PE return index based on historical fund cash flows." As Data Generating Process (DGP), they assume a log-linear factor model for each fund distribution

$$D_T = C_t \prod_{\tau=t+1}^{T} g_\tau \tag{1}$$

where D_T is a given fund distribution and C_t a contribution with $t < T$. The return dynamics $\ln(g_t) = \ln(g_t^M) + \epsilon_t$ where g_t^M is the market component of the fund return and ϵ_t its idiosyncratic part that is normal and i.i.d. The state space dynamics of the filtering problem describe the factor model structure of g as

$$g_t^M = \alpha + r_t^{\text{free}} + \beta^\top F_t + f_t^{\text{PE}}$$

where α is the average excess return, r_t^{free} is the risk-free rate, F_t is a vector of public market (long-short) factor returns, β the corresponding factor loadings, and f_t^{PE} is "an asset-class-specific latent factor with mean zero that is orthogonal to the traded factors, F_t." They assume each fund holds N investments $i = 1, 2, \ldots, N$; t_i notes the investment date of the first deal and T_i is the corresponding exit date. For each fund, they then postulate the following Present Value Ratio (PVR) to hold

$$PVR = \ln\left(\frac{PV_D}{PV_C}\right) \cong \ln(u) \qquad \text{where} \quad \ln(u) \sim N\left(-0.5\sigma^2, \sigma^2\right) \tag{2}$$

with present value of distributions

$$PV_D = \sum_{i=1}^{N} \frac{D_{T_i}}{g_{t_1} \cdots g_{T_i}}$$

and present value of contributions

$$PV_C = \sum_{i=1}^{N} \frac{C_i}{g_{t_1} \cdots g_{T_i}} U_{t_i,T_i}$$

where U_{t_i,T_j} contains the idiosyncratic terms

$$U_{t_i,T_j}^i = \exp\left(\epsilon_{t_i+1} + \cdots + \epsilon_{T_i}\right)$$

Here, the approximation ($\cong$) of the PVR by $\ln(u)$ relies on a lognormal central limit theorem (CLT) obtained from the textbook of Baker and Trietsch (2013). For this CLT to hold, they need the two **regularity conditions** that there is no dominantly (i) large or (ii) long investment as N goes to infinity. They estimate all relevant model parameters, i.e., α, β and the latent residual return time-series f^{PE}, by a Bayesian Markov Chain Monte Carlo method that minimizes "one large error term per fund in the estimation process". Thus Ang et al. (2018) provide the first SDF estimation method in the literature that not only estimates the public market factors but additionally an **asset-class-specific latent factor**. Because of the high dimensionality of f^{PE} (i.e., a return time-series), they need to employ an estimation method like Gibbs sampling that can handle more parameters to estimate than actual data points. Inspired by these "latent factors", Tausch and Pietz (2024) propose a machine-learning approach to estimate the error-term time series of public factor models for private capital funds.

2.2 Semiparametric approaches

Semiparametric approaches omit (i) parametric specifications for the idiosyncratic error term and (ii) often also formulations of the concrete Data Generating Process (DGP).

Driessen et al. (2012) develop the first SDF estimator tailored for typical PE fund-level cash flows. Their goal is to estimate a linear factor model that minimizes the squared $DNCF$ of PEFs. Generically, they solve the following minimization problem

$$\theta^{\text{SDF}} = \arg\min_{\theta^{\text{SDF}}} \sum_{i=1}^{N} \left[\sum_{t=1}^{T_i} \left(\frac{D_{it} - C_{it}}{\Psi_t^{-1}} \right) \right]^2 \tag{3}$$

where N is the number of PEF portfolios in the dataset. $\theta^{\text{SDF}} = (\alpha, \beta)$ are the coefficients of a linear factor model SDF with $\Psi_t = (\prod_{\tau=0}^{t} R_\tau)^{-1}$ where R_τ follows a **linear return factor model**

$$R_\tau = 1 + \alpha_\tau + \beta^\top F_\tau + \epsilon_\tau \tag{4}$$

where the risk-free rate is the first element of the vector F_τ. Plugging these terms into Equation 3 yields

$$(\alpha, \beta) = \arg\min_{\alpha,\beta} \sum_{i=1}^{N} \left(\sum_{t=1}^{T_i} \frac{D_{i,t} - C_{i,t}}{\prod_{\tau=0}^{t}(1 + \alpha_\tau + \beta^\top F_\tau + \epsilon_\tau)} \right)^2$$

which constitutes a nonlinear least squares optimization problem. However, Driessen et al. (2012) **interpret their approach as Generalized Method of Moments (GMM) estimator** with N moment conditions and identity weighting matrix where asymptotically the number of underlying investments per portfolio i tends to infinity. In our view, this cross-sectional GMM interpretation is both unnecessarily (i) complex and (ii) unrealistic. Especially, their GMM derivation (in the internet appendix) takes as starting point a DGP similar to Equation 1 but does not show why their nontraditional GMM interpretation is favorable to traditional (and simpler) non-linear least squares as defined, e.g., by Pötscher and Prucha (1997). Instead of using individual funds, they form N **vintage year portfolios** where they pool all fund cash flows from a given vintage "to lower the effect of idiosyncratic shocks," similar to Franzoni et al. (2012). To obtain standard errors for their coefficient estimates, the authors rely on cross-sectional bootstrapping **instead of providing** an asymptotic inference formula. Empirically, the paper draws on the problematic Thomson Venture Economics (TVE) fund-level dataset that exhibits some now well-known data errors as discussed by Harris et al. (2014). For VC, they estimate a one-factor model with a market beta factor of 2.73 and an annualized negative alpha of -12.3%. For BO, one-factor model coefficients are 1.31 for the market factor, and the annualized alpha is again negative with -4.8%. These highly negative alpha terms can be at least partially attributed to the aforementioned data errors in the TVE dataset. However, surprisingly from a methodological viewpoint, both alpha terms must be considered insignificantly different from zero due to high bootstrap standard errors. In summary, we still consider the Driessen et al. (2012) paper a seminal contribution to the PE literature as it proposes the first semiparametric SDF estimator for typical PEF (fund-level) datasets.

Korteweg and Nagel (2016) introduce the Generalized Public Market Equivalent (GPME) framework, which is a new SDF-based performance evaluation methodology for PE fund-level cash flows. The general idea is to first estimate a given SDF just on a public market dataset and then, in the second step, evaluate PEF cash flows by a traditional NPV approach

$$GPME = \sum_t \Psi_t^{\text{public}} CF_t^{\text{PEF}}$$

where Ψ_t^{public} is a generic SDF estimated on public data and CF_t^{PEF} are all cash flows of a (liquidated) PEF. Here, GPME is simply our DNCF from Equation **??**. However, their paper also offers one new and important methodical insight for SDF estimation using PE cash flow data. They are the first to realize that for asymptotic statistical inference, a **spatial heteroskedasticity and autocorrelation consistent (SHAC)**[2] covariance matrix estimator that incorporates the economic distance between PE fund pairs proves very useful. To estimate their exponentially affine SDF, they apply a time-series Generalized Method of Moments (GMM) approach where they, for simplicity, assume an identity weighting matrix. This corresponds to a very similar minimization problem as the one from Equation 3, which is used by Driessen et al. (2012). They form public replication portfolios that mimic PEF contributions and distributions patterns and use these synthetic cash flows for SDF estimation. Interestingly, they claim that an exponentially affine SDF is especially suited for "irregularly spaced, skewed, and endogenously timed payoffs" of VC investments. Moreover, they state: "With irregularly spaced and skewed VC cash flow data, **linear factor models are not readily applicable** without strong distributional assumptions." Empirically, the authors calculate their GPME metric for VC funds from the Preqin fund-level dataset and start-ups from Sand Hill Econometrics' deal-level data. Unfortunately, they relinquish to also test a linear factor model as an alternative to their exponential affine SDF to underpin their strong allegation about the correct SDF choice. It is also not clear why they cannot use a public SDF that has been estimated on a traditional public stock return (not cash flow) dataset (comparable to the Gredil et al. (2020) approach, which is discussed in the next paragraph). From this perspective, their non-traditional GMM estimator and the construction of the public benchmark portfolio cash flows seem unnecessarily complicated and artificial.

In a more recent paper Korteweg and Nagel (2024) extend their GPME methodology to obtain risk-adjusted benchmarks for single PE funds. Hüther et al. (2022) adapt the GPME method to estimate their so-called Credit Market Equivalent which relies on an SDF that prices the bonds issued by buyout-held companies. Interestingly, Hüther et al. (2022) find that public credit market factors can better price PE cash flows than the traditional public stock market factors. Similarly, Giommetti and Jørgensen (2021) test the GPME approach using a CAPM SDF and a "Discount Rate News" SDF. Although we think pricing PE cash flows instead of public market cash flows is the more natural approach (as Giommetti and Jørgensen (2021) and Hüther et al. (2022) did), Korteweg and Nagel (2016, 2024) still mark highly significant contributions to the PE-related SDF literature.

The unpublished working paper of Gredil et al. (2020) evaluates PE fund-level cash flows by means of asset pricing tests for "off-the-shelf" SDFs. They examine SDFs that shall be presumably better aligned with typical PE investor preferences than traditional factor models like Fama and French (2015). Specifically, they test

[2]A SHAC estimator will be later applied and defined by Equation 18.

two leading consumption-based asset pricing models: the long-run risk model of Bansal and Yaron (2004) and the external habit formation model of Campbell and Cochrane (1999) among other simpler SDF alternatives. Notably, in the first place, the authors take SDF parameters from the existing literature rather than estimating their own SDF coefficients. This is comparable to the Korteweg and Nagel (2016) generalized PME framework, which also does not uses any PE cash flows for SDF estimation. Their SDF can thus be considered as universal SDFs that can price all cash flows in a given economy (not just PE cash flows specifically). Just in their additional results section Gredil et al. (2020) "are trying to 'fine-tune' the off-the-shelf SDFs to reduce the benchmark pricing errors". Empirically, they obtain unrealistically large negative risk-free rate parameters when fitting SDFs to fund cash flows from the Burgiss database. Most interestingly, they claim to "show that cash flow NPV-based measures of performance for long-duration investment vehicles like PE funds are biased relative to per-period abnormal return estimates." In other words, they criticize asset pricing tests that are based on a general NPV-equal-to-zero condition (cf. Equation **??**). Their derived bias term is non-zero if there exits any auto-correlation in the time-series of (unobserved) pricing errors $e_t = \Psi_t R_t - 1$ were Ψ_t is a SDF and R_t is the latent periodic PE return. Here they assume a data generating process similar to Equation 1, which is used by Ang et al. (2018). To better understand their "compounding error" **small-sample bias**, it is important to mention that in small samples, we may measure a nonzero autocorrelation for some error terms even if this effect vanishes asymptotically and the true expected autocorrelation is zero. So, we generally agree that asset pricing tests for PE cash flow datasets are always weaker than standard time-series asset pricing tests that could draw on the latent true return time-series of PE returns. Moreover, their idea to **price the difference between PE cash flows and a suitable public replication strategy** instead of just PE cash flows (as a resolution of their bias issue) is picked up by the quadratic hedging strategies derived by Tausch (2019).

3 Methodology

Our general SDF estimation framework is similar to that of Driessen et al. (2012) and Korteweg and Nagel (2016); the subtle but important differences are discussed in Section 3.7.

3.1 Asset Pricing for Pooled Cash Flows

Let fund $i = 1, 2, \ldots, n$ be characterized by its net cash flows $CF_{t,i}$ (i.e., distributions minus contributions) and its net asset values $NAV_{t,i}$ with discrete time index $t = 0, 1, 2, \ldots, T$. To increase the mathematical tractability of the problem, we assume a deal-by-deal data generating processes (DGP) for CF where each fund deal consists exactly of one investment and one divestment cash flow in combination with a simple return model for the multi-period deal returns. This means the fund-level cash flow process $(CF_{i,t})_{t=0,1,\ldots,T}$ is an aggregation of deal-level cash flow pairs consisting of one negative at deal inception and at least one positive cash flow later $CF_{t,i} = \sum_j^J cf_{j,i,t}$.

Assumption 1. *Deal-level data generation process:*
1. Each fund i consists of J underlying deals.
2. Each deal is characterized by exactly one, negative investment cash flow, denoted by $\text{Inv}_{i,j}$, which occurs at time $t_{i,j}^{\text{Inv}} \in \{0, 1, \ldots, T-1\}$. It holds $\text{Inv}_{i,j} < 0$.
3. Each deal is characterized by a positive divestment cash flow stream, denoted by $(\text{Div}_{i,j,k})_{k=1,\ldots,K}$, which occur at after the investment cash flow $t_{i,j,k}^{\text{Div}} > t_{i,j}^{\text{Inv}}$ for all k. It holds $\text{Div}_{i,j,k} > 0$.
4. The cumulative fund cash flows are generated by summarizing over all deal-level cash flows, i.e., $\sum_{t=0}^{T} CF_{i,t} = \sum_{j=1}^{J} \left(\text{Inv}_{i,j} + \sum_{k}^{K} \text{Div}_{i,j,k} \right)$ for all i.

From asset pricing theory, we know that we can use a stochastic discount factor Ψ_t to price each underlying deal

$$\mathbb{E}\left[\delta_{i,j} \middle| \mathcal{F}_{t_{i,j}^{\text{Inv}}}\right] = 0 \quad \forall\, i, j \tag{5}$$

where we denote the deal-level pricing error by

$$\delta_{i,j} := \text{Inv}_{i,j} + \sum_{k}^{K} \frac{\Psi_{t_{i,j,k}^{\text{Div}}}}{\Psi_{t_{i,j}^{\text{Inv}}}} \text{Div}_{i,j,k} \tag{6}$$

For the pooled, fund-level cash flow stream, we assume that the true fund valuation $V_{i,\tau}$ is not observable for us

$$V_{i,\tau} := \mathbb{E}\left[\sum_{t=\tau}^{T} \frac{\Psi_t}{\Psi_\tau} CF_{i,t} \,|\mathcal{F}_\tau\right] \quad \forall\, \tau \geq \min_j \text{Inv}_{i,j} \tag{7}$$

to acknowledge the stale-pricing problem inherent to private capital fund net asset values (NAVs). In other words, we only trust fund cash flows but not fund NAVs in private markets[3].

In this realistic setting, the absence of observable market valuations considerably contributes to the difficulty of our pricing problem. However, also the existence of only pooled cash flows, instead of granular deal-by-deal cash flows, introduces issues for pricing approaches like Driessen et al. (2012) that discount to the fund inception date (see Equation 3). In the following, we will demonstrate why we cannot easily price pooled fund-level cash flow streams without introducing **inevitable bias terms** (even for fund inception date $\tau = \min_j t_{i,j}^{\text{Inv}}$). Generally, all deal-level cash flows will produce the following "bias term"

$$\mathbb{E}\left[\frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau} \delta_{i,j} \,|\mathcal{F}_\tau\right] = \text{Cov}\left[\frac{\Psi_{t_{i,j}^{\text{Inv}}}}{\Psi_\tau},\ \delta_{i,j}\right] \quad \forall\, \tau < t_{i,j}^{\text{Inv}} \tag{8}$$

Only for the trivial case of Equation 5, where the investment date coincides with the discounting date $\tau = t_{i,j}^{\text{Inv}}$, the covariance term necessarily equals zero.

For the fund-level cash flows, we therefore introduce the following proposition.

[3] In the empirical section, we empirically treat the most recent NAV as the final distribution cash flow for non-liquidated funds.

Proposition 1. *Price of a pooled cash flow stream at fund inception:*

$$\mathbb{E}\left[\sum_{t=\tau}^{T}\frac{\Psi_t}{\Psi_\tau}CF_{i,t}\,|\mathcal{F}_\tau\right]=\sum_{j=1}^{J}\mathrm{Cov}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau},\ \delta_{i,j}\right]\ \forall\ i \tag{9}$$

with fund inception date $\tau = \ \min_j \mathrm{Inv}_{i,j}$.

We can simply proof the proposition as follows.

Proof. We start with using Point 4 of Assumption 1 which states that fund-level cash flows are the sum of deal-level cash flows. Further, we stipulate in the proposition that no deal-level cash flow occurs before τ. Thus, the expected value of discounted fund-level cash flows needs to equal the expected value of discounted deal-level cash flows.

$$\mathbb{E}\left[\sum_{t=\tau}^{T}\frac{\Psi_t}{\Psi_\tau}CF_{i,t}\,|\mathcal{F}_\tau\right]=\mathbb{E}\left[\sum_{j=1}^{J}\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau}\delta_{i,j}\,|\mathcal{F}_\tau\right] \tag{10}$$

Using Equation 8, we can rewrite

$$\mathbb{E}\left[\sum_{j=1}^{J}\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau}\delta_{i,j}\,|\mathcal{F}_\tau\right]=\mathbb{E}\left[\sum_{j=1}^{J}\mathrm{Cov}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau},\ \delta_{i,j}\right]|\mathcal{F}_\tau\right] \tag{11}$$

Linearity of expectations then yields the result we want to proof.

$$\mathbb{E}\left[\sum_{j=1}^{J}\mathrm{Cov}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau},\ \delta_{i,j}\right]|\mathcal{F}_\tau\right]=\sum_{j=1}^{J}\mathrm{Cov}\left[\frac{\Psi_{t^{\mathrm{Inv}}_{i,j}}}{\Psi_\tau},\ \delta_{i,j}\right] \tag{12}$$

□

Corollary 1. *If and only if all deal investment dates coincide with the fund inception date, i.e.,* $\mathrm{Inv}_{i,j} = \min_j \mathrm{Inv}_{i,j}\forall j$*, we have*

$$\mathbb{E}\left[\sum_{t=\tau}^{T}\frac{\Psi_t}{\Psi_\tau}CF_{i,t}\,|\mathcal{F}_\tau\right]=0 \tag{13}$$

for the standard case where we do not assume independence between Ψ *and* $\delta_{i,j}$.

3.2 Least-Mean-Distance estimator

In this subsection, we introduce a new SDF estimator designed to analyzed the effect of different discounting dates τ on the bias and variance of the estimated SDF parameters. In the previous subsection, we demonstrated that for a pooled cash flow stream, consisting of at least two deals with different investment start dates, **no correct discounting date exists**. Thus our general idea is to rather average over multiple suitable discounting date candidates τ than to try to select only one candidate for the "best" discounting date (which is empirically unknown).

Henceforth, we assume that the underlying transactions within a private equity fund cannot be distinguished individually, and that only the funds total (pooled) cash flows are observable. The stochastic discount factor $\Psi_{\tau,t}$ is used to calculate the time-τ "price" $P_{\tau,t,i}$ of a **single** time-t cash flow of any given PE fund i

$$P_{\tau,t,i} := \Psi_{\tau,t} \cdot CF_{t,i} = \frac{\Psi_t}{\Psi_\tau} \cdot CF_{t,i} \qquad \forall \quad \tau, t, i \tag{14}$$

with multi-period SDF $\Psi_t = \prod_{k=0}^{t} M_k$. As SDFs are commonly parameterized by a vector $\theta \in \mathbb{R}^p$, i.e., $\Psi_{t,\tau} \equiv \Psi_{t,\tau}(\theta)$, our goal is to find an estimation method for the optimal θ. We denote this optimal/best/true parameter vector as θ_0. We call the numerator Ψ_t the discount part of the multi-period SDF $\Psi_{\tau,t}$ (used for present value calculations) and the denominator Ψ_τ the compound part (used for future value calculations). For each fund i and all points τ within a common fund lifetime, the empirical "pricing error" $\epsilon_{\tau,i}$ of **all** fund cash flows is calculated as time-τ "present value"

$$\epsilon_{\tau,i} := \sum_{t=1}^{T} P_{\tau,t,i} \qquad \forall \quad \tau, i \tag{15}$$

We use the terms "price", "pricing error‘" and "net present value" in quotation marks to acknowledge the theoretical asset pricing problem which can arise for pooled cash flows and has been described in the previous subsection.

To better analyze the impact of different discount date τ on the estimator's bias and variance, we define the (w_i-weighted) average pricing error $\bar{\epsilon}_i$ that averages over the set $\mathcal{T}_i$

$$\bar{\epsilon}_i = w_i \cdot \frac{1}{\mathrm{card}(\mathcal{T}_i)} \sum_{\tau \in \mathcal{T}_i} \epsilon_{\tau,i} \qquad \forall \quad i \tag{16}$$

where $\mathcal{T}_i$ gives the set of discounting dates τ for fund i which is more thoroughly described in the next Subsection 3.3. Additionally, each fund i is characterized by its vintage year which can be expressed by $v_i = \min(\mathcal{T}_i) \in \{1, 2, \ldots, V\}$, where V denotes the maximum vintage year used in a given data set. Finally, the scalar weighting factor w_i can be (i) one divided by the fund's invested capital for equal weighting of funds, (ii) one divided by the vintage year sum of invested capital for vintage year weighting, (iii) the scalar one for fund-size weighting, or (iv) some macroeconomic deflator.

To find the optimal value for θ, we select an estimator from the broad class of extremum estimators.

Definition 1. *Extremum estimator (Newey and McFadden, 1994, Equation 1.1): An estimator $\hat{\theta}$ is an extremum estimator if there is an objective function $Q_n(\theta)$ such that*

$$\hat{\theta} = \max_{\theta} \; Q_n(\theta)$$

for $\theta \in \Theta$ where Θ is the set of all possible parameter values.

Concretely, our LMD estimator (Pötscher and Prucha, 1997, Equation 7.1) minimizes the average loss of $\bar{\epsilon}$

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \; Q_n(\theta) \quad \text{with} \quad Q_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} L\left(\bar{\epsilon}_i\right) \tag{17}$$

where $L : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ denotes a loss function, e.g., $L(x) = (x - 0)^2$ in the case of nonlinear least squares. Throughout the paper, the weighted average fund pricing error $\bar{\epsilon}_i \equiv \bar{\epsilon}_i(\theta)$ is regarded as nonlinear random function of the SDF parameter θ.

3.3 Future and present value dates: the set $\mathcal{T}$

This subsection helps to explain the importance of the set $\mathcal{T}$ from Equation 16. Initially, we introduced averaging over $\mathcal{T}$ to counter the "exploding alpha" issued described by Driessen et al. (2012): In a net present value formula, an discount factor with a very large alpha term discounts all cash flows close to zero. Thus, in this degenerate situation, the beta factors become irrelevant – an infinite alpha perfectly prices all possible cash flow streams. Even more importantly, our simulation study in Section 4.3 indicates that averaging over $\mathcal{T}$ seems to decrease the small-sample bias of the estimator empirically.

A discounting date $\tau \in \mathcal{T}_i$ is a discretionary point in time where all fund cash flows are discounted to. The cardinality $\text{card}(\mathcal{T}_i) = |\mathcal{T}_i|$ gives the number of discounting dates used for the ith fund. The smallest possible set $\mathcal{T}_i$ contains just the fund's starting date; in this case, $\text{card}(\mathcal{T}_i)$ consequently is one. This corresponds to a typical NPV calculation in finance and is also used by Driessen et al. (2012) and Korteweg and Nagel (2016). In contrast, the largest candidate set for $\mathcal{T}$ contains all time periods bigger than the fund's starting date until now. In Subsection 4.3, we study the optimal set size of $\mathcal{T}$ by Monte Carlo simulations. There we show in our example that controlling for the optimal size of $\mathcal{T}$ can decrease the small-sample bias and variance of the original Driessen et al. (2012) estimator that just discounts all cash flows to the fund inception date. As we average over $\mathcal{T}_i$ in Equation 16 we call $\bar{\epsilon}_i$ the $\mathcal{T}_i$-averaged pricing error, as visualized in Figure 1.

3.4 Cross-sectional unit: individual fund vs. portfolio of funds

According to the classical value-additivity assumption in Hansen and Richard (1987), SDF models invariably shall hold for all pooled or unpooled assets (compare to Assumption **??**). So, in theory, it is not important if the test assets for our SDF are portfolio or individual fund cash flows when the investment dates are the same (see Corollary 1). Empirically it makes a difference, and there are arguments both for and against portfolio formation.

In the risk premium literature, portfolio formation mainly helps to attenuate the errors-in-variables bias connected to two-pass asset pricing methods (Jegadeesh et al., 2019; Pukthuanthong et al., 2019). As this is no issue in our case, we could use individual funds. Cochrane (2011) argues that portfolio sorting (seen as an auxiliary nonparametric regression that imposes linearity on the relationship between returns and characteristics) shall be replaced by multivariate panel models due to the curse of dimensionality. Following the same nonparametric regression viewpoint, Cattaneo et al. (2019) derive a nonparametric framework where the optimal number of portfolio sorts acts as a data-dependent tuning parameter that grows with sample size. Generally, the larger the portfolios, the easier any given SDF can price their

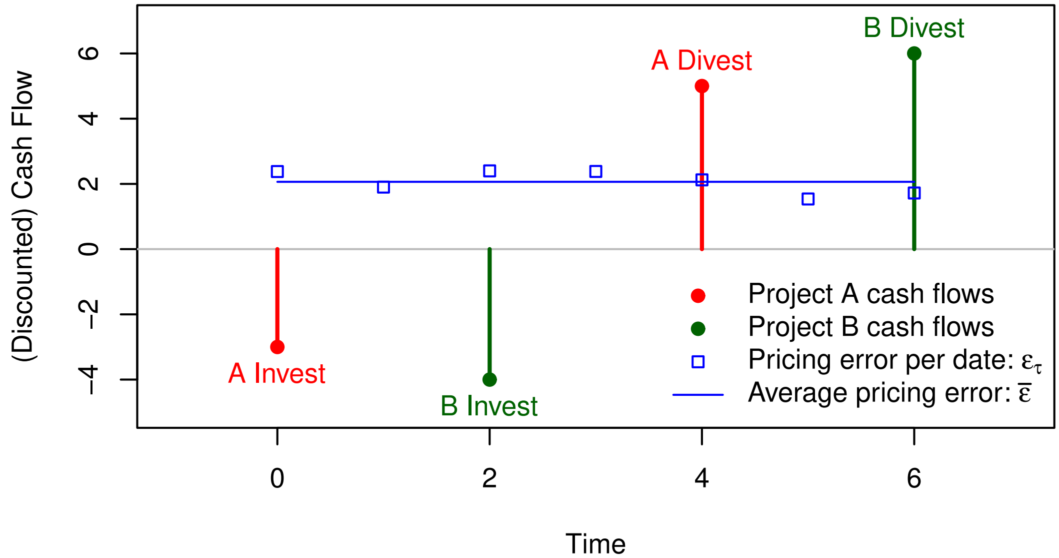

Figure 1: How to calculate and interpret the average pricing error? The time index t is relevant for the net cash flows (black dots). The time index τ is used for the pricing error, i.e., the sum of discounted net cash flows (blue boxes). The weighted average of these "pricing errors" gives the average pricing error $\bar{\epsilon}$ as defined in Equation 16 (solid blue line). In this example, card$(\mathcal{T}_i) = 7$, i.e., the number of blue boxes.

cash flows since fewer test assets remain.

In the case of private equity funds, the pooling of fund cash flows helps to counter GP financial engineering[4], which might both change and mask the true risk profile of observed LP cash flows. Especially for private equity funds, portfolio formation based on vintage years is compelling due to its time-series-like indexing as done by Driessen et al. (2012). This procedure also offers substantial computational benefits as it drastically decreases the number of cross-sectional units. Further, as stated in Ang et al. (2020), portfolio formation allows more precise factor loading estimates due to decreasing idiosyncratic risk, but at the expense of sacrificing cross-sectional information. Finally, small (or fixed) T and large N set-ups may face finite sample problems (Raponi et al., 2020).

Assumption 2. *For each vintage year, we pool fund cash flows to form n_v portfolios that serve as cross-sectional units. Thus, $n = \sum_{v=1}^{V} n_v$. The two boundary cases are (i) single fund portfolios and (ii) just one portfolio per vintage year.*

Without loss of generality, we refer to our cross-sectional units as funds, although this corresponds to a special case of our portfolio concept. In the simulation study

[4]GPs may use bridge credit facilities below the hurdle rate to boost the fund's internal rate of return. This increases the probability of observing funds with only positive or only negative cash flows. However, we want to avoid (the possibility of) cross-sectional units that exhibit just cash flows with the same algebraic sign. Realistic SDFs never can price these cash flow streams.

in Subsection 4.3, we compare both boundary cases (i) individual funds and (ii) vintage year portfolios.

Thinking more broadly, we could even imagine more extreme boundary cases: (iii) on the one hand, we could pool *all* fund cash flows to form just *one* global moment condition for private equity similar to Korteweg and Nagel (2016) and accept potential under-identification; (iv) on the other hand, we could operate on underlying deal level like Buchner (2014, 2016a,b) and use gross-of-fee cash flows.

3.5 Asymptotic framework

To allow for multiple funds from the same vintage year in Assumption 2, we employ an auxiliary "spatial" notion as originally proposed by Korteweg and Nagel (2016). The spatial viewpoint is just a technical means to switch from time-series-like to more panel-data-like indexing. Unlike typical panel data, we do not follow multiple subjects over time, but for each point in time, we exclusively observe multiple new cross-sectional units (i.e., funds from that vintage year). This unusual two-dimensional indexing causes problems in the PE literature as it neatly fits neither in the (i) time-series, (ii) cross-sectional, nor (iii) panel data literature. Thus, we generally consider $\bar{\epsilon}$ from Equation 16 as random field (cf. Figure 2). In our case, it is convenient to interpret the fund vintage year v_i as second dimension in our pricing error random field, i.e., $\bar{\epsilon}_i \equiv \bar{\epsilon}_{i,v_i}$.

Yet, in this section, we mainly follow the time-series asymptotic framework of Pötscher and Prucha (1997) since our "spatial" distance measure (between vintage years) is time, and adaption to our case is thus straightforward. If we observe just one fund per vintage year (or, equivalently, form vintage year portfolios), we will easily see that the framework of Pötscher and Prucha (1997) with time-series indexing can be applied without any major modification.

3.5.1 Vintage year asymptotics

We assume that the "spatial" (i.e., economic) distance between cross-sectional units, i.e., private equity funds/portfolios, can be measured quantitatively[5]. Our "cross-sectional" asymptotic theory lets the number of funds go to infinity $n \to \infty$. To expose our SDF to many distinct covariate realizations (economic conditions), we also want the number of vintages to increase asymptotically.

Assumption 3. *Vintage year asymptotics:*

1. *The number of vintage years $V \to \infty$ as $n \to \infty$.*
2. *The number of funds per vintage year is bounded by some positive constant.*
3. *The maximal fund lifetime is also bounded by a positive constant.*
4. *The economic distance between fund i and j is measured by the vintage year difference $d_{i,j} = |v_i - v_j| + \rho_0 1_{i \neq j}$ with minimum distance $\rho_0 > 0$.*

[5] Generally, the economic distance measure could include multiple dimensions, e.g., temporal, geographic, and industry sector proximity. This could be an interesting topic for future research.

In terms of the spatial estimation literature, this assumption postulates increasing domain asymptotics and rules out so-called infill asymptotics (cf. Figure 2). The minimum distance term ρ_0 is a means to ensure these increasing domain asymptotics (Jenish and Prucha, 2012, Assumption 1). Infill asymptotics corresponds to the assumption of Driessen et al. (2012) that the number of funds per vintage tends to infinity.

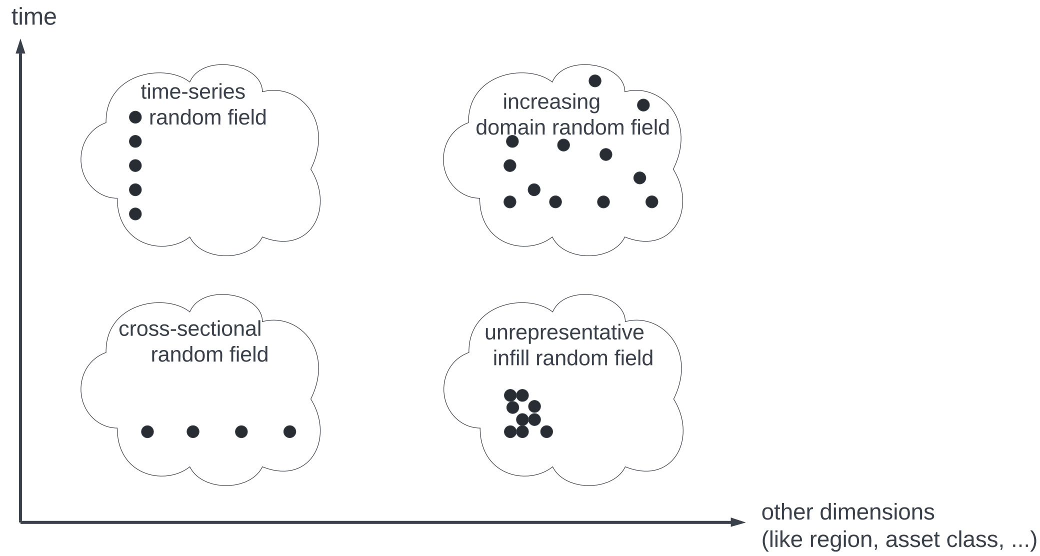

Figure 2: Visualization of generic random field types. Each black dot marks a different observation i of the cash flow data. Importantly, the time axis does **not** correspond to the index t in $CF_{t,i}$ (rather to vintage years v_i). Comparing the four choices, we want to avoid an infill random field but prefer our data to constitute an increasing domain random field. The infill random field is also asymptotically "too clustered" or better "too unrepresentative" to allow for meaningful estimation and inference. The time-series and cross-sectional random fields correspond to the standard cases in the literature but could turn out too restrictive for a general approach. By smart design (like portfolio formation), we often can map an increasing domain random field to simpler time-series or cross-sectional versions.

GMM estimators typically have a fixed number of moment conditions. Thus, GMM estimators, where the number of moment conditions is allowed to grow with sample size, require special attention (Han and Phillips, 2006; Newey and Windmeijer, 2009). In many cases, it is probably most convenient to limit the maximum to a finite number of moment conditions (i.e., not each vintage year should form a moment condition). In this paper, we employ a nonlinear least squares estimators since they do not suffer from this "number of moment condition" limitation.

3.5.2 Law of large numbers

The global moment condition underlying our estimation approach is that the expected value of $\bar{\epsilon}$ shall be zero if we use the optimal SDF parameter θ_0. To approach this expectation, we rely on a spatial (cross-sectional) law of large numbers instead of applying a time-series law of large numbers. Here, we want to explicitly acknowledge the statistical dependence of pricing errors from adjacent vintage years.

Assumption 4. *Uniform Law of Large Numbers (ULLN) for random fields (Jenish and Prucha, 2009, Equation 6):*
The (i) time-trend and (ii) dependence structure of $\bar{\epsilon}$ shall allow

$$\sup_{\theta \in \Theta} |Q_n(\theta) - \mathbb{E}[Q_n(\theta)]| \xrightarrow{p.} 0 \quad as \quad n \to \infty$$

where $Q_n(\theta)$ is given by Equation 17.

Specifically, we could assume (as a so-called primitive condition) the random field $\bar{\epsilon}$ to be spatial near-epoch dependent with respect to fund vintage years (Jenish and Prucha, 2009, 2012), i.e., two funds with distance $d_{i,j} > D$ are assumed to be independent.

To satisfy the time trend part (i) of this law of large number assumption, the weighting factor w, introduced in Equation 16, can be used to make $\bar{\epsilon}$ stationary. Spatial near-epoch dependence with respect to fund vintage years formalizes the simple idea that two fund pricing errors $\bar{\epsilon}$ with a small absolute vintage year difference are supposed to be dependent since they are exposed to the same macroeconomic conditions. In contrast, two funds with a large absolute vintage year difference can be assumed independent.

Generally, a law of large numbers only applies if the random variables under consideration are not "too fat-tailed" and the stochastic process is ergodic (Taleb, 2020).

3.5.3 Consistency

The estimator $\hat{\theta}$ shall converge in probability to the true parameter value θ_0 as the number of distinct vintage years in our data set goes to infinity. Multiple funds for a specific vintage year are not necessarily required but provide additional information that we want to exploit if available.

Lemma 1. *A modified version of (Newey and McFadden, 1994, Theorem 2.1) holds, i.e., if there is a function $Q_0(\theta)$ such that*

1. *Identification: $Q_0(\theta)$ is uniquely minimized at θ_0,*
2. *Boundedness: Θ is compact,*
3. *Continuity: $Q_0(\theta)$ is continuous,*
4. *Uniform convergence: $\hat{Q}_n(\theta)$ converges uniformly in probability to $Q_0(\theta)$,*

then $\hat{\theta} \xrightarrow{p} \theta_0$ as $n \to \infty$.

Proof. The general proof is given in (Newey and McFadden, 1994, Chapter 2) for a max instead of min extremum estimator. Thus, we only recapitulate the four conditions required by the lemma in our specific context.

1. Obviously, we have to replace "maximized at θ_0" by "minimized at θ_0" compared to the exposition of (Newey and McFadden, 1994, Chapter 2). Then, similarly to Equation **??**, we need to first show that $\mathbb{E}[\bar{\epsilon}(\theta_0)] = 0$. Secondly, we know $Q_0(\theta_0) = \mathbb{E}[L(\bar{\epsilon}(\theta_0))] \geq 0$, e.g., for $L(x) = x^2$ we have $Q_0(\theta) = \mathbb{E}\left[(\bar{\epsilon}(\theta))^2\right] = \mathrm{Var}[\bar{\epsilon}(\theta)] + (\mathbb{E}[\bar{\epsilon}(\theta)])^2$ where the second summand can be perceived as bias term that is zero for θ_0. The variance term $\mathrm{Var}[\bar{\epsilon}(\theta)]$ for a simplified DGP is analyzed by Corollary **??**.

2. Compactness of Θ can be assured by lower and upper bounds for all parameters that can be justified by economic reasoning. In our case, e.g., a market beta factor of ten seems implausible for PE funds because of the implied risk and return expectations.

3. Continuity of the limit is a quiet weak and thus a standard regularity condition.

4. The second standard regularity condition is given by Assumption 4 which satisfies the definition of uniform convergence in probability (Newey and McFadden, 1994, Section 2.1) . To make this obvious, we can write $\hat{Q}_n(\theta) = Q_n(\theta) = n^{-1}\sum_{i=1}^{n} L(\bar{\epsilon}_i)$ and $Q_0(\theta) = \mathbb{E}[Q_n(\theta)]$ and compare it to Assumption 4.

□ □

3.5.4 Central limit theorem

To assess the large-sample significance of our parameter estimates (as done in the following Subsection 3.6), we want to describe the asymptotic distribution of the parameter vector as a normal distribution.

Proposition 2. *With estimator consistency established in Lemma 1, and the five (technical) conditions from (Newey and McFadden, 1994, Theorem 3.1) satisfied, it holds*

1. $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma)$ *as* $V, n \to \infty$ *with covariance matrix* Σ*, and*
2. *The covariance matrix* Σ *can be characterized by Pötscher and Prucha (1997, Theorem 11.2.b, Theorem H.1) (as outlined in the next Section 3.6).*

Proof. The extended proof of Proposition 2 may be derived in analogy to the GMM case in (Jenish and Prucha, 2012, Theorem 4) that shows that the general structure of the Pötscher and Prucha (1997) framework also applies to the spatial near-epoch dependent case. Alternatively (and easier), the estimator from Equation 17 can be clearly formulated as extremum estimator in alignment with our Definition 1. In consequence, (Newey and McFadden, 1994, Theorem 3.1), which generally describes the asymptotic normality of extremum estimators, is directly applicable to obtain the stated result. Thus, all details of the proof can be found in the original reference (Newey and McFadden, 1994, Chapter 3). □ □

3.6 Large sample inference

In this subsection, we demonstrate how to empirically apply Proposition 2 to obtain the asymptotic standard errors for our estimator from Equation 17. In the time-series, near-epoch-dependent LMD literature, the covariance matrix Σ can be characterized according to Pötscher and Prucha (1997, Theorem 11.2.b, Theorem H.1):

$$\Sigma = H^{-1}\Lambda(H^{-1})^{\top}$$

with expected Hessian matrix converging to H as $n \to \infty$

$$\mathbb{E}\left[\nabla_{\theta\theta} Q_n\right] \xrightarrow{p} H$$

and the expected covariance matrix of gradients converging to Λ as $n \to \infty$

$$n \cdot \mathbb{E}\left[\nabla_{\theta} Q_n (\nabla_{\theta} Q_n)^{\top}\right] \xrightarrow{p} \Lambda$$

Here, the gradient vector $\nabla_{\theta} Q_n$ is denoted as column vector. We define the corresponding finite sample estimators analogously to Pötscher and Prucha (1997, Chapters 12, 13.1), and numerically approximate the first and second partial derivatives by finite differences[6]. Specifically, we use the following central difference approximations (with "small" δ) (Eu, 2017, Algorithm 2):

$$f_x(x, y) \approx \frac{f(x+\delta, y) - f(x-\delta, y)}{2\delta}$$

$$f_{xx}(x, y) \approx \frac{f(x+\delta, y) + f(x-\delta, y) - 2f(x, y)}{\delta^2}$$

$$f_{xy}(x, y) \approx \frac{f(x+\delta, y+\delta) + f(x-\delta, y-\delta) - f(x+\delta, y-\delta) - f(x-\delta, y+\delta)}{4\delta^2}$$

The Hessian term $\hat{H}$ is relatively straightforward

$$\hat{H} = \frac{1}{n} \sum_{i=1}^{n} \nabla_{\theta\theta} L\left(\epsilon_i\right)$$

Due to the spatial near-epoch dependence, the involved and computationally expensive part is to consistently estimate $\hat{\Lambda}$ by a Spatial Heteroskedasticity and Autocorrelation Consistent (SHAC) covariance matrix estimator (Kim and Sun, 2011, Equation 2)

$$\hat{\Lambda} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{i,j} \left[\nabla_{\theta} L\left(\epsilon_i\right) \left(\nabla_{\theta} L\left(\epsilon_j\right)\right)^{\top}\right] \tag{18}$$

We define the kernel weight k as

$$k_{i,j} \equiv K\left(\frac{d_{i,j}}{b_n}\right)$$

[6] As an alternative to finite differences the widespread Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm can be applied to approximate the Hessian (Nocedal and Wright, 2006, Section 6.1).

with kernel function $K : \mathbb{R} \to [0,1]$ satisfies $K(0) = 1$, $K(x) = K(-x)$, $\int_{-\infty}^{\infty} K^2(x)dx < \infty$, and $K(\cdot)$ continuous at zero and at all but a finite number of other points. A common choice is the Bartlett kernel $K_{BT}(x) = \max(0, 1 - |x|)$; see equation 2.7 in Andrews (1991) for other popular kernel choices. This means absolute vintage year differences larger than the bandwidth (or truncation) parameter $b_n = D$ are considered independent and are thus excluded from the $\hat{\Lambda}$ estimation formula.

In large samples, the vector of parameter standard errors can thus be estimated by

$$\mathrm{SE}(\hat{\theta}) = \sqrt{\operatorname{diag}\left[n^{-\frac{1}{2}}\hat{H}^{-1}\hat{\Lambda}(\hat{H}^{-1})^\top(n^{-\frac{1}{2}})^\top\right]} = \sqrt{\operatorname{diag}\left[\hat{H}^{-1}\hat{\Lambda}(\hat{H}^{-1})^\top\right] \cdot \frac{1}{n}}$$

The Wald test statistic for linear hypotheses $H_0 : R\theta = r$ and $H_1 : R\theta \neq r$ is constructed as (where, H_0 and H_1 are hypotheses and not Hessian terms H)

$$W = (R\hat{\theta} - r)^\top \left[R\frac{\hat{H}^{-1}\hat{\Lambda}(\hat{H}^{-1})^\top}{n}R^\top\right]^{-1}(R\hat{\theta} - r) \overset{H_0}{\sim} \chi^2_q$$

where $\hat{\theta}$ is the $p \times 1$ parameter vector, R is a $q \times p$ matrix, and r is a $q \times 1$ vector. Usually, we select R as $p \times p$ identity matrix, and r as $p \times 1$ vector (e.g., of zeros). Under the null hypothesis, W is chi-squared distributed with q degrees of freedom. As large values of W indicate the rejection of H_0, the corresponding p-value is calculated as $1 - F_{\chi^2_q}(W)$ where $F_{\chi^2_q}$ is the cumulative distribution function of a chi-squared random variable with q degrees of freedom.

However, given the limited amount of available private equity data (typically the oldest vintages start in the 1980s), asymptotic characterizations of Σ and $\mathrm{SE}(\hat{\theta})$ are of limited importance. In empirical applications, the small sample behavior of an estimation method for private equity data is more relevant than its asymptotic theory. Moreover, the standard asymptotic distribution associated with an estimator is generally not valid for post-model-selection inference, i.e., if a model selection procedure is applied to find the best model from a collection of competitors (Leeb and Pötscher, 2005).

3.7 Comparison to similar estimators

Our Least-Mean-Distance (LMD) estimator introduced in Section 3.2 belongs to the class of semiparametric nonlinear M-estimators as defined in Pötscher and Prucha (1997) which are extremum estimators. To gain more flexibility and avoid unneeded complexity, we intentionally opt against the most prominent semiparametric nonlinear M-estimator framework, i.e., classical time-series Generalized Method of Moments (GMM) (Hansen, 1982, 2012). A classical GMM approach requires the construction of stationary, ergodic time-series of moment conditions that are used to empirically estimate the expected value of pricing errors in Equation 15. The stationarity requirement of classical time-series GMM limits (i) more elaborate weighting schemes for w, like fund-size weighting, and (ii) the usage of fund cash flows from non-realized vintages.

3.7.1 Comparison to Driessen et al. (2012)

The Driessen et al. (2012) approach is most closely related to our methodology.

One important difference is that we select a simpler and more flexible LMD estimator instead of a cross-sectional GMM approach. In our view, the choice of the more complex cross-sectional GMM just causes some conceptional issues, whereas the underlying formulas are basically the same as for our LMD estimator[7]. As a first limitation, they have to regard vintage-year portfolios as their cross-sectional units; we can also use individual funds. In this context, we also question their statement that "to identify β, it is essential that the different FoFs are exposed to different market returns" since it is perfectly fine to view their estimator as cross-sectional approach[8]. Second, the Driessen et al. (2012) asymptotic theory assumes the number of funds (or deals) per vintage year portfolio to go to infinity. To comply with standard GMM assumptions, the number of vintage years, which corresponds to the number of moment conditions in their approach, **must be considered fixed** and thus cannot grow asymptotically (Han and Phillips, 2006; Newey and Windmeijer, 2009). For a typical LMD estimator (e.g., nonlinear least squares), this constraint does not exist. Our asymptotic theory lets both (i) the number of vintage years and (ii) the number of funds go to infinity but bounds the number of funds per vintage year.

Further, Driessen et al. (2012) discount all fund cash flows just to the first cash flow date (like in a classical net present value calculation). In contrast, we additionally average over all dates within $\mathcal{T}_i$ to alleviate the exploding alpha issue briefly mentioned in their paper (and more thoroughly so in an earlier working paper version). Although Driessen et al. (2012) describe their estimator as a one-step GMM approach, we consider it a special case of our LMD estimator. Specifically, Equation 17 from our methodology is a generalization of equation 3 from their paper. Consequently, if someone accepts the assumptions from Subsection 3.5, our large sample inference framework from Subsection 3.6 applies to their case without any significant modification. Finally, Driessen et al. (2012) apply simple cross-sectional bootstrapping to obtain standard errors; in contrast, in Subsection 4.2, we use a cross-validation technique that is adapted to the near-epoch dependence of the PE fund data.

3.7.2 Comparison to Korteweg and Nagel (2016)

Korteweg and Nagel (2016), first of all, realized the usefulness of employing an auxiliary spatial framework to establish asymptotic inference results for a fund-level panel dataset of private equity funds. To account for the cross-sectional dependence between funds, they measure the economic distance between two private equity funds (by the degree of cash flow overlap). Concretely, their asymptotic inference framework draws on the spatial HAC estimator of Conley (1999); our spatial HAC framework uses Pötscher and Prucha (1997); Kim and Sun (2011); Jenish and Prucha (2012). However, they ultimately utilize a classical GMM estimator, thus a

[7]The formulas are this similar because Driessen et al. (2012) use the identity matrix as GMM weighting matrix and skip the second GMM step.

[8]In a classic cross-sectional regression, we only have one market return realization.

time-series law of large numbers. Specifically, we obtain the estimator of (Korteweg and Nagel, 2016, Equation 18) in our framework if we replace $Q_n(\theta)$ in Equation 17 by Equation 19.

$$Q_n(\theta) = L\left(\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i\right) \quad \text{with} \quad L(x) = x^\top W x = x^\top I x \tag{19}$$

with identity matrix I as weighting matrix W. In accordance with classical GMM, the function $\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i : \mathbb{R}^{n\times T} \times \Theta \to \mathbb{R}^m$ should be perceived as multidimensional where the dimensionality of the function output corresponds to the number of moment conditions.

Time-series GMM estimators inherently bear the risk of under-identification if the corresponding time-series is constructed by pooling all fund cash flows from a given fund type. Exactly this happens in Equation 19 with $m = 1$ where we consequentially obtain a GMM estimator with just one moment condition[9]. To counter under-identification, additional characteristic-based fund portfolios could be formed to increase the number of moment conditions per fund type; also, random portfolios combined with bootstrapping make sense. Yet, Korteweg and Nagel (2016) take another approach and introduce the concept of Generalized Public Market Equivalent (GPME), which elegantly avoids the under-identification issue. Firstly, a public market SDF model is estimated by pricing public trading strategies that shall replicate PE funds instead of directly pricing the observed PE fund cash flows. Only in a second step these public market SDF models are applied to evaluate private equity fund cash flows.

Given these differences, our approach may not be perceived as a straightforward generalization of the Korteweg and Nagel (2016) framework. In contrast, our LMD estimator generalizes the Driessen et al. (2012) method. Table 1 summarizes the most prominent distinctions between the three approaches.

	Driessen et al. (2012)	Korteweg and Nagel (2016)	Our approach
M-estimator	Cross-sectional Generalized Method of Moments	Time-series Generalized Method of Moments	Least-Mean-Distance
Pricing error averaging	No	No	Yes
Cash flows priced	PE cash flows	public cash flows	PE cash flows
Asymptotics	cross-sectional #funds $\to \infty$	time-series #vintages $\to \infty$	spatial # of both $\to \infty$
Inference	bootstrap	spatial HAC	cross-validation & spatial HAC
Cross-sectional unit	vintage year portfolio	single fund	testing both
SDF	simple linear	exponentially affine	testing both

Table 1: Comparison to similar estimation frameworks.

[9]In contrast, our estimator corresponds to the opposite edge case with asymptotically an infinite number of LMD "moment conditions" as we let $n \to \infty$.

4 Empirical application

4.1 Data

We use the Preqin cash flow data set as of 26th February 2020 that is well known in the academic private equity literature (Harris et al., 2014; Korteweg and Nagel, 2016; Ang et al., 2018). For an overview of the available asset classes and strategies in the unprepared raw Preqin dataset, see Table 5. After data preparation, we pool all regions and group the remaining funds according to the Preqin asset class classification: PE ("Private Equity"; 2248 distinct funds in data set; 36 vintage years), VC ("Venture Capital"; 871; 36), RE ("Real Estate"; 742; 27), PD ("Private Debt"; 441; 31), INF ("Infrastructure", 144; 17), NR ("Natural Resources", 138; 26). For these fund types, we extract all (i) equal-weighted and (ii) fund-size-weighted cash flow series. For non-liquidated funds, we treat the latest net asset value as final cash flow. We explicitly refrain from excluding the most recent vintage years. Thus, the minimum vintage year is 1983 (just for PE), and the maximum is 2019.

The public market factors that enter our SDF draw on the US data set of the recently popularized q^5 investment factor model sourced from `http://global-q.org/factors.html` (Hou et al., 2015, 2020). Their five-factor model includes the market excess return (MKT), a size factor (ME), an investment factor (IA), a return on equity factor (ROE), and an expected growth factor (EG).

4.2 Model and estimator specifications

We test a simple linear SDF model similar to Driessen et al. (2012)

$$\Psi_{\tau,t}^{\mathrm{SL}}(\theta) = \frac{\prod_{h=0}^{\tau}\ \left(1+\alpha+r_h+\sum_j\ \beta_j\ F_{j,h}\right)}{\prod_{h=0}^{t}\ \left(1+\alpha+r_h+\sum_j\ \beta_j\ F_{j,h}\right)} \tag{20}$$

and an exponential affine SDF model adapted from Korteweg and Nagel (2016)

$$\Psi_{\tau,t}^{\mathrm{EA}}(\theta) = \exp\left[\sum_{h=0}^{\tau} X_h \sum_{h=0}^{t} -X_h\right] \tag{21}$$

with

$$X_h = \alpha + \log(1+r_h) + \sum_{j\in J} \beta_j \cdot \log(1+F_{j,h})$$

with (arithmetic) risk-free return $r = R_{rf} - 1$, (arithmetic) zero-net-investment portfolio returns F_j, and parameter vector $\theta = (\alpha, \beta)$. To avoid overfitting, we just test six simple SDF models that contain {MKT} alone or {MKT} plus {ME or IA or ROE or EG or Alpha}. In Equation 17, we use the quadratic loss function $L(x) = x^2$.

To assess the parameter significance, we compute the asymptotic standard errors as outlined in Subsection 3.6. For the Bartlett kernel's bandwidth $b_n = D$ we select 12 years, i.e., funds with absolute vintage year differences larger than 12 years are assumed to be independent.

Additionally, we want to test the finite - or, more honestly, small - sample parameter significance and the out-of-sample performance of our SDF models. To account for the dependency between funds from adjacent vintage years caused by overlapping fund cash flows, we draw on *hv*-block cross-validation (Racine, 2000). Therefore, we form three partitions for several vintage year groups. As larger validation sets are preferred for model selection, the validation set (*v*-block) always contains funds of three neighboring vintage years (e.g., 2000, 2001, 2002). To reduce the dependency between training and validation set, we remove all funds from three-year-adjacent vintage years, i.e., the *h*-block (e.g., 1997, 1998, 1999, 2003, 2004, 2005). Funds from the remaining vintage years enter the training set and are thus used for model estimation (e.g., 1985-1996, 2006-2019). We apply ten-fold cross validation using the ten validation sets described in Table 2. This means we replace the bootstrap standard error calculation of Driessen et al. (2012) by *hv*-block cross-validation since the new method (i) accounts for near-epoch-dependence, (ii) focuses directly on the out-of-sample performance of the SDF models, and (iii) is computationally cheaper.

training.before	*h*-block.before	*v*-block	*h*-block.after	training.after
estimation	remove	validation	remove	estimation
start-1984	1985,1986,1987	1988,1989,1990	1991,1992,1993	1994-end
start-1987	1988,1989,1990	1991,1992,1993	1994,1995,1996	1997-end
start-1990	1991,1992,1993	1994,1995,1996	1997,1998,1999	2000-end
start-1993	1994,1995,1996	1997,1998,1999	2000,2001,2002	2003-end
start-1996	1997,1998,1999	2000,2001,2002	2003,2004,2005	2006-end
start-1999	2000,2001,2002	2003,2004,2005	2006,2007,2008	2009-end
start-2002	2003,2004,2005	2006,2007,2008	2009,2010,2011	2012-end
start-2005	2006,2007,2008	2009,2010,2011	2012,2013,2014	2015-end
start-2008	2009,2010,2011	2012,2013,2014	2015,2016,2017	2018-end
start-2011	2012,2013,2014	2015,2016,2017	2018,2019,2020	2021-end

Table 2: Partitions used for *hv*-block cross-validation.

4.3 Simulation study

Our Monte Carlo experiments examine the following questions related to the bias and variance of our estimation methodology in finite samples.[10] Is it beneficial to use vintage-year portfolios instead of individual funds? Which SDF model performs better when we also use the corresponding data generating process (i.e., assume correct model specification)? How is estimator precision affected by varying numbers of vintage years and cross-sectional units? Which is the optimal set of discounting dates $\mathcal{T}$?

We use historical q-investment factors from 1986 to 2005 and simulate 20 funds for each of these 20 vintage years. Each fund contains 15 deals with equal investment

[10]As each simulation study it more investigates the ability to identify the assumed data generating process than the corresponding SDF model.

amounts and exactly one divestment cash flow. Deals are entered within the first five years of the fund lifetime following a discrete uniform distribution and afterward held between one to ten years again uniformly distributed. The deal returns are generated by the simple linear or exponential affine SDF models described in Equations 20 and 21. In the base case, we just use the MKT factor with $\beta_{\text{MKT}} = 1$ and in each month, add a normal i.i.d. error term with standard deviation $\sigma = 0.2$ and zero mean. Additionally, we test an intercept term α of -0.25% per month and a high β_{MKT} of 2.5. In the exponential affine case, we adjust the lognormally distributed error mean to zero by subtracting $0.5\sigma^2$. If a negative return exceeds -100%, the company defaults with a zero exit cash flow. In contrast, the error term in the simulations of Driessen et al. (2012) is more well-behaved as it follows a shifted lognormal distribution that, even with arbitrarily high error term variance, just allows for returns below say -99%, if the market return is close to its lower bound (see equation 9 in their online appendix). In our base case, the set of discounting dates $\mathcal{T}$ contains all months from the first cash flow to the maximum month 180. To assess our estimator's bias and variance, we simulate 1000 test scenarios for vintage year portfolios and only 200 test cases when using individual funds due to the higher computational costs of simulating the individual fund cash flows.

Cross-sectional unit i: As presumed in Subsection 3.4, vintage year portfolio results appear to have lower bias and variance when compared to individual funds. For the simple linear SDF and maximum month 180, the mean and standard deviation of the coefficient estimate $\hat{\beta}_{\text{MKT}}$ is 1.016 (0.2) for the vintage year portfolio and 1.096 (0.376) for individual funds. More results are depicted in Figure 3. However, for individual funds, we only simulate 200 iterations due to the high computational cost.

This finding has two important implications: On the one hand, vintage year portfolio formation can substantially decrease our estimator's bias and variance. On the other hand, it also dramatically reduces the number of cross-sectional units and consequentially impairs the importance of asymptotic results. These considerations may explain the choice of Korteweg and Nagel (2016) to use individual funds as cross-sectional units in their asymptotic SHAC framework to obtain smaller standard error estimates.

SDF model Ψ: In our base case with vintage year portfolios, the exponential affine SDF shows a mean and standard deviation of 1.011 (0.175) compared to the 1.016 (0.2) achieved by the simple linear SDF. Generally, the exponential affine SDF model and the simple linear SDF model exhibit similar bias and variance, cf. panels A and B in Table 6. Figure 4 visualizes the true $\beta = 1$ case, which shows that the estimation results are not overly sensitive to the choice of the SDF model.

Moreover, the perceived superiority of exponential affine SDFs is probably rather theoretical than practical as other proponents also emphasize their universality mainly from a mathematical perspective without providing supportive empirical or simulation results (Gourieroux and Monfort, 2007; Bertholon et al., 2008). From Section **??**, we also know that the pricing implications of the true SDF (with unknown functional form) and its linear maximum correlation portfolio shall be the

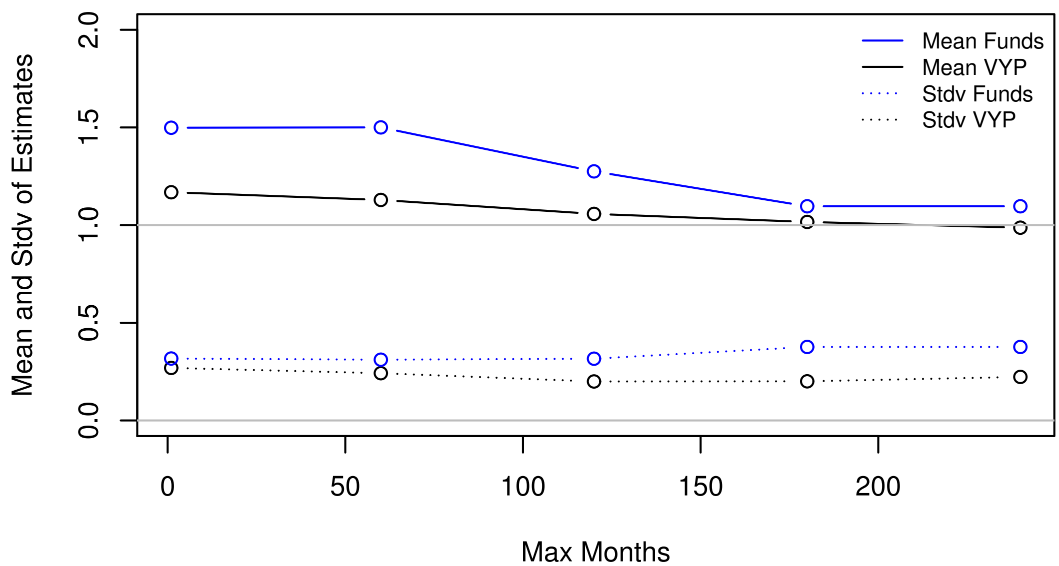

Figure 3: Simulation results comparing individual funds vs. vintage year portfolios (VYPs) with true $\beta = 1$ and simple linear SDF (200 simulation iterations).

same.

Varying vintages V and portfolio sizes n/V: To test the effect of varying data sizes available for MKT factor estimation, we in/decrease the (i) number of vintage years and (ii) the number of funds per vintage year (cf. Table 3). Here we use vintage year portfolios and the simple linear SDF. For our simple data generating process, increasing the number of deals/funds per vintage year portfolio appears to decrease the estimator's variance more effectively than adding more vintage years. However, the bias is almost the same for all tested specifications. Generally, we seem to need many new data points to ensure a reasonable variance of our estimator.

	Base	Big n/V	Big V	Big V	Small V	Small V
Start vintage	1986	1986	1967	1967	1986	1996
End vintage	2005	2005	2005	2005	1995	2005
#Funds per vintage	20	40	10	20	20	20
Mean β_{MKT}	1.011	1.020	0.993	1.015	1.027	0.934
Stdv β_{MKT}	0.187	0.133	0.263	0.227	0.232	0.418

Table 3: Simulation study for varying number of vintages and number of funds per vintage. We use vintage year portfolios, the simple linear SDF with true $\beta_{\text{MKT}} = 1$, maximum month 180, and 500 simulation iterations.

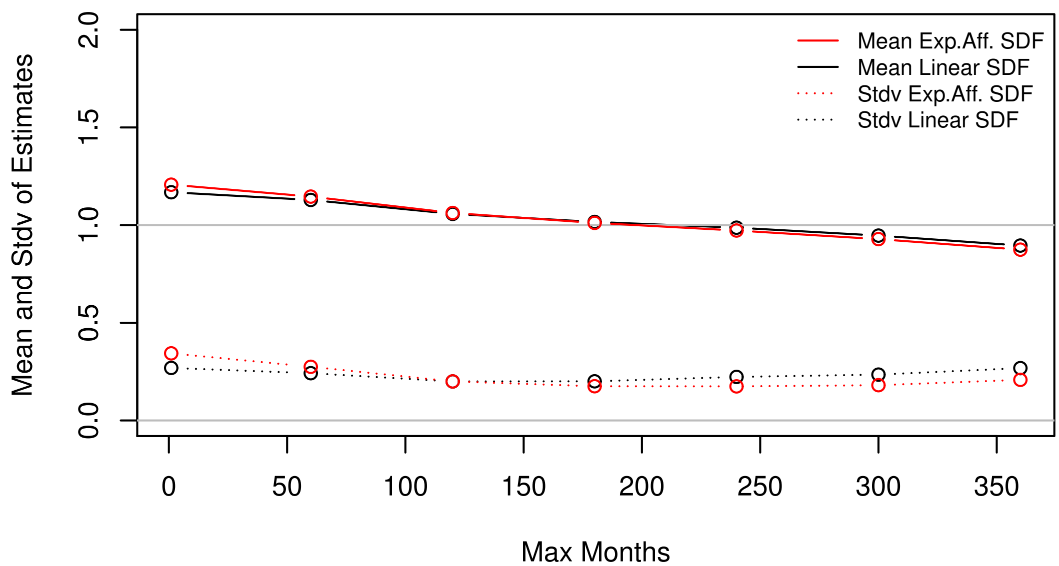

Figure 4: Simulation results comparing exponentially affine and simple linear SDF with true $\beta = 1$ and vintage year portfolios (1000 simulation iterations).

Size of set $\mathcal{T}$: The results in Table 6 indicate that we can control the bias by an appropriate choice of the set $\mathcal{T}$. The bias almost vanishes when we average over all discounting dates in the maximal fund lifetime of 180 months. For smaller or larger sets for $\mathcal{T}$, we find increasing small-sample bias[11].

The same finding also holds when we limit the maximal fund lifetime to ten years by reducing the maximum deal holding period from ten to five years. Here, under correct model specification with $\beta_{\text{MKT}} = 1$, the smallest bias is obtained for maximum month 120, for max. month 60 we get 1.028 (0.116), for max. month 120, we get 1.005 (0.116), and for max. month 180 we get 0.969 (0.13).

In Table 6 for both true and false model specifications, the α standard deviation is very high compared to its mean value. This may indicate it is rather delicate to empirically determine private equity's historical outperformance by our semiparametric estimator.

To conclude, our simulations study rationalizes two key practices from the Driessen et al. (2012) paper: (i) vintage year portfolio formation helps to improve estimator precision, and (ii) increasing the number of funds per vintage seems to be more effective in controlling estimator variance that increasing the number of vintages[12]. However, our examples with correct specification cannot support the assumption of Korteweg and Nagel (2016) that (iii) the exponential affine SDF is

[11] Recall that using the minimal set for $\mathcal{T}$, i.e., discounting all cash flows just to the fund inception date, corresponds exactly to the Driessen et al. (2012) approach. Accordingly, the original Driessen et al. (2012) methodology might achieve a suboptimal small-sample bias since it does not average pricing errors over multiple discounting dates.

[12] Finding (ii) may explain the choice of Driessen et al. (2012) to employ an asymptotic law that lets the number of deals/funds per vintage tend to infinity.

(clearly) superior to the simple liner SDF in a multi-period framework; actually, their bias and variances are quite equal. Moreover, our simulation study suggests that (iv) averaging pricing errors over multiple dates strikingly reduces the bias inherent to the original procedure of Driessen et al. (2012) that just discounts all cash flows to the fund inception date. Actually, choosing the set $\mathcal{T}$ according to the fund lifetime seems to decrease the bias (and to a lesser extent also the variance) more effectively than all other measures combined.

4.4 Empirical results

Following the conclusions from the previous subsection, we use vintage-year portfolios to estimate simple linear SDF models with maximum month 180. Asymptotic inference results for the full dataset are exhibited in Table 7 for fund-size weighting and in Table 9 for equal weighting. The results for *hv*-block cross-validation are displayed in Table 8 for fund-size weighting and in Table 10 for equal weighting. We generally analyze the results in a two-step procedure: For a given model specification, we use the cross-validation error (i.e., the average out-of-sample error) to select the best model for each fund type but analyze the corresponding coefficient estimates from the asymptotic inference tables (estimated on the entire data set). Therefore, for each fund type the SDF models in the asymptotic inference Tables 7 and 9 are sorted by the corresponding cross-validation error. Throughout this subsection, we define the statistical significance of coefficient estimates in terms of a t-ratio $\hat{\theta}[SE(\hat{\theta})]^{-1}$ greater than 1.96.

		MKT Factor			Second Factor		
Weighting	Inference	Coef	SE	SE.indep	Coef	SE	SE.indep
fund-size	asymptotic	0.75	27.06	19.73	0.80	28.95	20.94
fund-size	cross-validation	0.85	0.38	-	0.59	0.51	-
equal	asymptotic	0.76	26.75	16.16	0.76	11.25	6.69
equal	cross-validation	0.84	0.34	-	0.62	0.50	-

Table 4: Top-level overview over Table 7 to 10: Averages of absolute values of coefficient estimates and standard errors (SEs). We see that asymptotic SEs are much higher than the SEs obtained by cross-validation.

Table 4 helps to get a rough overview of Table 7 to 10 as it summarizes their absolute column means. Conspicuously, asymptotic standard errors (SEs) seem enormously high and, moreover, contain colossal outliers. The standard errors implied by $hv-$block cross-validation are considerably smaller than the asymptotic SEs and seem to lie within a plausible range. When just looking at asymptotic standard errors of the second factors, fund-size weighting exhibits substantially larger SEs than fund equal-weighting. Assuming independence between funds from different vintages decreases asymptotic SEs by approximately 30-40% compared to a realistic kernel bandwidth of $D = 12$. But even these independent SEs rarely imply statistical significance coefficient estimates with t-ratios bigger than 1.96. In Table 7 with fund-size weighting, just one out of 36 models exhibit asymptotically significant MKT and second-factor estimates. In the case of equal-weighting, Table 9 also shows just one asymptotically significant model out of 36.

In summary, the results reveal weak two-factor models with MKT plus a second q-investment factor. Likewise, the simulation results from the previous subsection indicate a rather high variance associated with our semiparametric estimator (given the amount of data typically available). Thus, we recommend focusing on single MKT factor models even when their asymptotic t-ratios are below 1.96. At least the hv-block cross-validation standard deviations imply significant one-factor MKT models for fund types PE, VC, PD, INF. In contrast, RE is just significant for equal weighting, and NR is insignificant for both weighting schemes.

Focus on PE and VC estimates Here, we briefly summarize the one-factor MKT and the two-factor Alpha model estimates for fund types PE (i.e., mainly Buyout and Growth) and VC. For PE, all one-factor MKT model β_{MKT} estimates fall in the range from 1.13 to 1.28. If we add an α term, all β_{MKT} estimates decrease to the range 0.61 to 0.77 with annualized α coefficients of approximately positive 4-5% per year. For VC, the one-factor MKT model β_{MKT} estimates are in the range from 0.80 to 1.14. If we add an α term, all β_{MKT} estimates strongly increase to the range 1.81 to 2.06 with annualized α coefficients of approximately negative 6-7% per year. These results at least weakly indicate - given their insignificant asymptotic standard errors - that PE funds outperform public markets with a market beta coefficient of less than one, which suggests low market risk. On the other hand, VC underperforms public markets with market beta coefficients of roughly two, which implies high market risk. So, even Driessen et al. (2012) use the problematic Thomson Venture Economics (TVE) dataset for their empirical analysis[13], we obtain similar quantitative and qualitative results using Preqin data: (i) the market beta of VC seems to be higher than that of PE, and (ii) VC, in contrast to PE, appears to exhibit a negative abnormal performance α[14].

As a robustness check, we reestimate all SDF models on a dataset that just contains funds from vintages older or equal than 2011. Interestingly, the PE and VC results regarding β_{MKT} and α can be qualitatively and also quantitatively confirmed on this 'mostly-liquidated' dataset[15].

5 Conclusion

Theoretically, our Least-Mean-Distance estimator can be easily generalized to estimate SDF models for all kinds of non-traded cash flows. Practically, semiparametric estimators commonly exhibit problematic small sample behavior. Given the amount of currently available private equity fund data, our estimator's variance seems quite large, even for simple SDF model specifications. Specifically, our Monte Carlo simulation results prompt us to conclude that the closely related Driessen et al. (2012)

[13] Harris et al. (2014) discuss the potential downward bias of the TVE dataset.

[14] Similarly, (Metrick and Yasuda, 2010, Exhibit 4.6) find high beta coefficients (1.63 and 2.04) and small to negative alphas (-2.11% and 0.13%) for VC funds in their lag-return regression.

[15] All R code and data is available in an online repository. `https://github.com/quant-unit/Fundwise_SDF/tree/master/r_project`

estimator may exhibit more bias and variance than originally assumed in their paper. Especially, the variance of α estimates seems to be too high to allow reliable abnormal performance conclusions. Fortunately, we show that at least the bias can be easily reduced by averaging pricing errors over all dates within the fund lifetime.

In the data-sparse private equity domain with only 20-40 cross-sectional units (i.e., vintage year portfolios) currently available for estimation, asymptotic inference seems not to be overly useful. Thus, we strongly advise always challenging asymptotic inference results by resampling or cross-validation techniques adapted to the dependence structure of overlapping fund cash flows. However, even these conclusions should be double-checked to avoid unreasonable instances, e.g., when *hv*-block cross-validation chooses dubious models with negative MKT factor estimates. Unfortunately, using individual funds instead of vintage year portfolios, which yields smaller asymptotic standard errors, constitutes no viable resolution as individual funds show considerably larger small-sample bias and variance in our Monte Carlo example. Since, in our empirical analyses, basically all two-factor models' asymptotic standard errors appear statistically insignificant, we conjecture that naive versions of our SDF estimator shall be exclusively used for a single-MKT-factor model until considerably more vintage year information for private equity funds is available.

If someone wants to estimate more complex SDF models that incorporate additional factors, more structure is needed. These can be parametric assumptions for the data generating process (Ang et al., 2018) or to extract additional information from intermediate net asset values (Gredil et al., 2020; Brown et al., 2021). A first "modern" approach to the same problem is applying machine learning techniques that regularize/shrink all coefficients other than the MKT factor. Secondly, given the high estimator variance revealed in the simulation study, statistical learning methods that create a strong learner by combining multiple weak learners seem also worth considering (boosting, bagging, or model averaging).

Incorporate fees and carry model to DGP for SDF estimation for PE gross-fee performance

Finally, we point to the potentially most interesting topic for future research. Our simulation study indicates that the estimator's bias and variance can be controlled by an appropriate choice for the set $\mathcal{T}$. This set averages the pricing error over multiple discounting dates. In simpler terms, an identification method that utilizes a future value concept instead of net present values obtains more favorable results in our case. The bias in our simulation study is minimal when the set of discounting dates corresponds to the fund lifetime. A parsimonious but general model that allows for misspecification and can explain this $\mathcal{T}$-averaging effect from a mathematical perspective would be highly appreciated.

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A Appendix of tables

Asset Class	Strategy	Fund Count
Infrastructure	Infrastructure Core	32
Infrastructure	Infrastructure Core Plus	58
Infrastructure	Infrastructure Debt	26
Infrastructure	Infrastructure Fund of Funds	6
Infrastructure	Infrastructure Opportunistic	12
Infrastructure	Infrastructure Secondaries	4
Infrastructure	Infrastructure Value Added	36
Natural Resources	Energy	2
Natural Resources	Natural Resources	146
Natural Resources	Timber	19
Private Debt	Direct Lending	3
Private Debt	Direct Lending - Blended / Opportunistic Debt	21
Private Debt	Direct Lending - Junior / Subordinated Debt	9
Private Debt	Direct Lending - Senior Debt	89
Private Debt	Direct Lending - Unitranche Debt	4
Private Debt	Distressed Debt	153
Private Debt	Mezzanine	147
Private Debt	Private Debt Fund of Funds	4
Private Debt	Special Situations	58
Private Equity	Balanced	63
Private Equity	Buyout	1251
Private Equity	Co-Investment	75
Private Equity	Co-Investment Multi-Manager	58
Private Equity	Direct Secondaries	24
Private Equity	Fund of Funds	589
Private Equity	Growth	265
Private Equity	Secondaries	131
Private Equity	Turnaround	18
Real Estate	Real Asset	1
Real Estate	Real Estate Co-Investment	15
Real Estate	Real Estate Core	33
Real Estate	Real Estate Core-Plus	44
Real Estate	Real Estate Debt	106
Real Estate	Real Estate Distressed	27
Real Estate	Real Estate Fund of Funds	29
Real Estate	Real Estate Opportunistic	225
Real Estate	Real Estate Secondaries	15
Real Estate	Real Estate Value Added	315
Venture Capital	Early Stage	262
Venture Capital	Early Stage: Seed	42
Venture Capital	Early Stage: Start-up	37
Venture Capital	Expansion / Late Stage	110
Venture Capital	Venture (General)	513
Venture Capital	Venture Debt	21

Table 5: Summary of asset classes and strategies in the unprepared raw Preqin dataset.

Panel A: simple linear SDF

Model==DGP	True	False		False	True	
MaxMonth	$\beta = 1$	$\alpha = 0$	$\beta = 1$	$\beta = 2.5$	$\alpha = -0.25$	$\beta = 2.5$
1 - mean	1.168	1625%	0.003	2.023	5879%	-16.711
1 - stdv	0.269	2792%	9.968	0.342	866%	13.347
60 - mean	1.129	0.138%	0.933	2.103	-0.086%	2.285
60 - stdv	0.242	0.245%	0.363	0.302	0.253%	0.406
120 - mean	1.058	0.112%	0.906	2.063	-0.085%	2.239
120 - stdv	0.200	0.214%	0.313	0.253	0.239%	0.385
180 - mean	1.016	0.041%	0.965	2.052	-0.161%	2.370
180 - stdv	0.200	0.172%	0.334	0.277	0.173%	0.403
240 - mean	0.987	-0.053%	1.077	2.072	-0.277%	2.589
240 - stdv	0.223	0.162%	0.361	0.326	0.118%	0.375
300 - mean	0.946	-0.149%	1.175	2.080	-0.357%	2.714
300 - stdv	0.235	0.174%	0.377	0.398	0.114%	0.366
360 - mean	0.895	-0.245%	1.269	2.048	-0.461%	2.859
360 - stdv	0.268	0.201%	0.399	0.551	0.140%	0.386

Panel B: exponential affine SDF

Model==DGP	True	False		False	True	
MaxMonth	$\beta = 1$	$\alpha = 0$	$\beta = 1$	$\beta = 2.5$	$\alpha = -0.25$	$\beta = 2.5$
1 - mean	1.207	203%	1.276	2.256	692%	1.704
1 - stdv	0.344	314%	0.710	0.290	13%	1.666
60 - mean	1.146	0.126%	0.941	2.264	-0.018%	2.277
60 - stdv	0.275	0.264%	0.386	0.256	0.370%	0.473
120 - mean	1.062	0.107%	0.908	2.221	0.009%	2.205
120 - stdv	0.200	0.237%	0.333	0.187	0.357%	0.448
180 - mean	1.011	0.027%	0.971	2.182	-0.136%	2.358
180 - stdv	0.175	0.211%	0.366	0.168	0.344%	0.505
240 - mean	0.972	-0.088%	1.095	2.144	-0.441%	2.723
240 - stdv	0.174	0.224%	0.406	0.178	0.317%	0.503
300 - mean	0.928	-0.202%	1.203	2.083	-0.717%	2.985
300 - stdv	0.181	0.253%	0.426	0.254	0.340%	0.513
360 - mean	0.874	-0.319%	1.304	1.685	-1.095%	3.272
360 - stdv	0.208	0.291%	0.447	0.772	0.374%	0.586

Table 6: Simulation study to compare the simple linear with the exponential affine SDF and to determine the optimal size of the set $\mathcal{T}$. Here, we always use vintage year-portfolios and 1000 simulation iterations. For better readability, $\beta_{\text{MKT}} = \beta$. For the unity and high beta model, we test true and false model specifications (with and without the α term).

	MKT Factor				Second Factor		
Type	Estim.	SE	SE.indep	Factor	Estim.	SE	SE.indep
PE	0.709	2.470	1.153	EG	0.807	4.693	1.960
PE	0.770	7.976	3.348	ROE	1.540	5.140	3.499
PE	1.126	1.003	0.868	MKT	1.126	1.003	0.868
PE	0.644	1.234	0.585	Alpha	0.003	0.036	0.013
PE	1.121	1.023	0.897	ME	0.074	2.021	0.915
PE	1.158	1.125	1.068	IA	-0.338	2.499	1.259
VC	1.053	4.150	2.733	IA	-1.959	2.100	1.767
VC	1.114	3.861	2.894	ME	-1.383	5.102	2.211
VC	1.806	11.391	4.279	Alpha	-0.006	0.124	0.046
VC	0.801	704.455	561.598	MKT	0.801	704.455	561.598
VC	1.429	8.073	3.219	ROE	-1.306	18.055	6.919
VC	1.507	17.322	6.966	EG	-0.904	15.344	5.737
PD	0.885	1.040	1.242	MKT	0.885	1.040	1.242
PD	0.660	0.095	0.039	Alpha	0.002	0.001	0.000
PD	0.826	1.707	1.443	EG	0.143	20.341	7.506
PD	0.849	2.921	2.146	ME	0.301	2.739	1.518
PD	0.887	1.378	1.244	ROE	-0.023	6.925	2.553
PD	0.863	2.942	2.224	IA	0.247	5.306	3.607
RE	0.578	1.827	1.196	MKT	0.578	1.827	1.196
RE	1.303	5.463	2.259	Alpha	-0.006	0.088	0.034
RE	0.200	2.598	1.356	ROE	3.118	2.579	6.629
RE	0.202	3.297	1.965	EG	0.844	2.478	1.828
RE	0.756	3.043	2.192	IA	-1.938	1.879	0.783
RE	0.887	1.167	0.858	ME	-2.059	1.300	0.563
NR	-0.215	2.367	1.976	EG	0.909	14.505	7.475
NR	0.191	3.136	4.242	MKT	0.191	3.136	4.242
NR	-0.674	58.003	24.234	Alpha	0.008	0.230	0.098
NR	-0.020	0.954	2.210	ROE	1.128	4.830	5.066
NR	0.143	3.236	4.116	IA	-0.768	1.808	2.154
NR	0.212	4.209	5.450	ME	-0.575	1.603	1.252
INF	0.824	3.201	2.815	MKT	0.824	3.201	2.815
INF	0.190	7.133	3.288	Alpha	0.005	0.030	0.025
INF	0.317	23.904	10.658	EG	0.848	45.836	20.176
INF	0.470	9.316	4.237	ROE	1.245	5.531	6.134
INF	0.778	5.951	5.424	ME	-0.811	3.712	4.349
INF	0.661	61.329	33.819	IA	-1.108	150.713	85.733

Table 7: Asymptotic inference with fund-size weighting, max month 180, and $D = 12$.

	MKT Factor			Second Factor		
Type	Mean	SD	Factor	Mean	SD	CV-error
PE	0.867	0.276	EG	0.720	0.137	112808
PE	0.927	0.305	ROE	1.375	0.420	126801
PE	1.276	0.296	MKT	1.276	0.296	151964
PE	0.772	0.238	Alpha	0.004	0.002	154805
PE	1.317	0.396	ME	0.236	0.664	209319
PE	1.311	0.370	IA	0.014	0.703	210650
VC	1.045	0.126	IA	-1.890	0.238	11858
VC	1.172	0.126	ME	-1.448	0.263	13301
VC	1.930	0.356	Alpha	-0.005	0.001	17723
VC	0.804	0.363	MKT	0.804	0.363	21852
VC	1.527	0.517	ROE	-0.972	0.679	26680
VC	1.646	0.678	EG	-0.644	0.556	32730
PD	0.887	0.039	MKT	0.887	0.039	7368
PD	0.567	0.202	Alpha	0.003	0.001	7917
PD	0.763	0.113	EG	0.229	0.141	8758
PD	0.862	0.103	ME	0.342	0.256	9834
PD	0.812	0.153	ROE	0.258	0.394	11522
PD	0.914	0.211	IA	0.472	0.424	18096
RE	0.722	0.392	MKT	0.722	0.392	50900
RE	1.288	0.345	Alpha	-0.004	0.005	51437
RE	0.389	0.446	ROE	2.333	1.507	54689
RE	0.465	0.470	EG	0.448	0.629	59316
RE	0.847	0.411	IA	-1.262	1.160	65835
RE	0.983	0.350	ME	-1.467	1.298	66827
NR	-0.047	0.421	EG	0.657	0.557	10559
NR	0.318	0.321	MKT	0.318	0.321	11480
NR	-0.335	0.763	Alpha	0.006	0.005	11854
NR	0.136	0.466	ROE	0.844	1.062	13296
NR	0.270	0.508	IA	-0.288	1.124	14479
NR	0.416	0.587	ME	0.032	1.079	15789
INF	0.862	0.320	MKT	0.862	0.320	14551
INF	0.639	0.753	Alpha	0.002	0.004	15069
INF	0.766	0.626	EG	0.258	0.495	16004
INF	0.837	0.504	ROE	0.090	0.939	18472
INF	0.868	0.412	ME	0.078	0.643	18514
INF	0.892	0.561	IA	0.081	1.073	23162

Table 8: *hv*-block cross-validation with fund-size weighting and max month 180.

Type	MKT Factor Estim.	SE	SE.indep	Factor	Second Factor Estim.	SE	SE.indep
PE	0.775	0.638	0.550	EG	0.667	5.558	2.125
PE	0.610	1.064	0.387	Alpha	0.004	0.006	0.002
PE	0.826	20.352	8.308	ROE	1.087	33.514	12.143
PE	1.134	1.050	0.694	MKT	1.134	1.050	0.694
PE	1.146	1.001	0.638	IA	-0.386	1.909	0.813
PE	1.134	1.048	0.702	ME	-0.014	1.797	0.736
VC	1.181	24.418	16.693	ME	-1.277	4.928	4.352
VC	1.137	7.259	6.057	IA	-1.553	3.716	2.139
VC	1.956	4.189	1.520	Alpha	-0.006	0.335	0.117
VC	1.034	2.205	1.758	MKT	1.034	2.205	1.758
VC	1.488	1.801	0.941	ROE	-1.148	4.060	1.424
VC	1.535	2.821	1.336	EG	-0.754	3.626	1.260
PD	0.844	1.245	0.856	MKT	0.844	1.245	0.856
PD	0.502	0.044	0.015	Alpha	0.003	0.000	0.000
PD	0.791	2.478	1.557	ROE	0.222	5.024	1.966
PD	0.736	2.230	1.303	EG	0.213	6.296	2.374
PD	0.844	1.150	0.837	IA	0.076	2.543	1.416
PD	0.833	1.978	1.362	ME	0.323	1.845	0.986
RE	0.743	3.471	2.075	MKT	0.743	3.471	2.075
RE	1.265	7.581	3.331	Alpha	-0.004	0.046	0.017
RE	0.145	3.486	1.614	ROE	3.202	17.447	7.413
RE	0.400	50.493	31.818	EG	0.700	48.813	29.195
RE	0.884	4.928	2.813	ME	-1.782	1.474	0.605
RE	0.795	18.146	10.282	IA	-1.712	13.199	7.657
NR	-0.056	3.693	1.897	ROE	1.934	3.154	2.287
NR	0.000	4.771	2.368	EG	0.814	1.368	1.753
NR	0.425	3.370	5.178	MKT	0.425	3.370	5.178
NR	-0.272	39.463	16.698	Alpha	0.006	0.202	0.081
NR	0.394	0.871	1.401	IA	-0.319	1.734	1.169
NR	0.453	15.719	24.377	ME	0.432	5.574	8.016
INF	0.098	0.055	0.025	Alpha	0.006	0.001	0.000
INF	0.280	20.158	8.983	EG	0.893	21.368	9.700
INF	0.775	19.273	22.509	MKT	0.775	19.273	22.509
INF	0.469	26.226	12.751	ROE	1.030	30.728	16.129
INF	0.758	33.453	36.239	ME	-0.804	16.214	16.161
INF	0.664	630.801	351.730	IA	-0.929	137.937	75.628

Table 9: Asymptotic inference with equal weighting, max month 180, and $D = 12$.

	MKT Factor			Second Factor		
Type	Mean	SD	Factor	Mean	SD	CV-error
PE	0.886	0.262	EG	0.614	0.217	101444
PE	0.719	0.205	Alpha	0.004	0.001	105842
PE	0.948	0.267	ROE	0.975	0.407	110926
PE	1.250	0.262	MKT	1.250	0.262	127589
PE	1.247	0.274	IA	-0.183	0.598	157037
PE	1.281	0.323	ME	0.048	0.644	169552
VC	1.250	0.153	ME	-1.292	0.234	16305
VC	1.183	0.169	IA	-1.507	0.327	16449
VC	2.052	0.257	Alpha	-0.006	0.001	18666
VC	1.138	0.341	MKT	1.138	0.341	25321
VC	1.610	0.431	ROE	-0.946	0.426	26618
VC	1.688	0.505	EG	-0.616	0.331	30392
PD	0.838	0.029	MKT	0.838	0.029	11290
PD	0.458	0.151	Alpha	0.003	0.001	11568
PD	0.770	0.058	ROE	0.232	0.111	11572
PD	0.707	0.086	EG	0.224	0.091	12194
PD	0.825	0.083	IA	0.158	0.305	15071
PD	0.837	0.098	ME	0.358	0.328	15441
RE	0.803	0.336	MKT	0.803	0.336	43486
RE	1.191	0.363	Alpha	-0.003	0.004	45310
RE	0.341	0.402	ROE	2.275	1.558	52822
RE	0.559	0.379	EG	0.397	0.503	52867
RE	0.929	0.339	ME	-1.287	1.174	57341
RE	0.852	0.372	IA	-1.129	1.025	57662
NR	-0.009	0.203	ROE	1.880	0.562	15631
NR	0.200	0.510	EG	0.681	0.372	18981
NR	0.572	0.400	MKT	0.572	0.400	20006
NR	-0.065	0.729	Alpha	0.006	0.004	20880
NR	0.596	0.567	IA	-0.060	0.869	24766
NR	0.769	0.771	ME	0.666	1.062	27238
INF	0.124	0.608	Alpha	0.007	0.006	14995
INF	0.613	0.393	EG	0.402	0.496	15758
INF	0.862	0.281	MKT	0.862	0.281	15820
INF	0.627	0.360	ROE	0.577	1.374	18297
INF	0.810	0.315	ME	-0.129	1.109	19641
INF	0.797	0.842	IA	0.051	2.180	33661

Table 10: *hv*-block cross-validation with equal weighting and max month 180.