A spatial stochastic discount factor estimator for private equity funds

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Abstract

This paper proposes an improved stochastic discount factor estimation methodology suited for fund-level cash flows of private equity funds. The asymptotic inference framework for this semiparametric least-mean-distance estimator draws on a spatial notion, i.e., the idea that the economic distance between distinct private equity funds can be measured. The empirical and Monte Carlo simulation results reveal high estimator variance for typical data sizes. Thus, we conjecture that naive semiparametric M-estimators like ours shall be exclusively used for single-factor models until considerably more vintage year information for private equity funds is available.

1 Introduction

Do investments in Private Equity (PE) funds offer abnormal returns to fund investors when risk-adjusted for public market factors? Currently, a popular approach to answer this question is to evaluate private equity fund cash flows by Stochastic Discount Factor (SDF) models that draw on public market return covariates. The basic idea for SDF model estimation is that the sum of all discounted fund net cash flows is expected to be zero when the true SDF is applied. Unfortunately, there is no conclusion about the best methodology to estimate these SDF models, as a variety of proposals coexists in the academic private equity fund literature (Driessen et al., 2012; Buchner, 2014; Korteweg and Nagel, 2016; Ang et al., 2018; Gredil et al., 2019).

This paper aims to revise and enhance existing semiparametric approaches. Especially our conclusions from the insightful Driessen et al. (2012) and Korteweg and Nagel (2016) articles lead us to suggest an improved Least-Mean-Distance (LMD) estimator for SDF models. It can be applied to fund-level cash flow data of private equity funds. On the one hand, we provide asymptotic inference formulations that rely on the concept of spatial (near-epoch) dependence between funds following the pioneering idea in Korteweg and Nagel (2016). In this context, it is paramount to quantify the economic distance between funds by a measure like absolute vintage year difference or cash flow overlap[1]. On the other hand, our LMD estimator arguably generalizes the Driessen et al. (2012) methodology, where we provide the asymptotic inference framework that was missing in the original paper. Additionally, we propose a simple solution to the 'exploding alpha' issue briefly mentioned in their paper. Our Monte Carlo results suggest that the same modification dramatically reduces the inherent small-sample bias associated with the original Driessen et al. (2012) estimator.

In the empirical application of our new estimator, we test simple linear and exponentially affine SDF models that can draw on the five return factors associated with the q^5 investment factor model recently proposed by Hou et al. (2020). Based on a Spatial Heteroskedasticity and Autocorrelation Consistent (SHAC) covariance matrix estimator, we calculate asymptotic standard errors for the model coefficients. Moreover, we assess the small-sample variance of coefficient estimates and the out-of-sample performance of the different SDF models by *hv*-block cross-validation, which accounts for the inter-vintage-year dependence of private equity funds (Racine, 2000). We test one- and two-factor models for the following private equity fund types: Private Equity, Venture Capital, Private Debt, Real Estate, Natural Resources, and Infrastructure. All two-factor model results are rather devastating; not more than the single-market-factor model results seem reasonable given the high estimator variance.

The paper is structured as follows. Section 2 introduces our semiparametric LMD estimator and its corresponding asymptotic inference framework. Section 3 applies the method to estimate q^5-investment-factor SDFs for various private equity fund types using simulated and real-world cash flows. Section 4 concludes.

[1]However, this economic inter-fund *distance* refers **not** to the term Least-Mean-*Distance* estimator.

2 Methodology

2.1 Least-Mean-Distance estimator

Our general SDF setting is similar to that of Driessen et al. (2012) and Korteweg and Nagel (2016); the subtle differences are discussed in section 2.5.

Let fund $i = 1, 2, \ldots, n$ be characterized by its net cash flows $CF_{t,i}$ (i.e., distributions minus contributions) and its net asset values $NAV_{t,i}$ with discrete time index $t = 1, 2, \ldots, T$. The data generating processes (DGPs) for CF and NAV are left unspecified. For a non-liquidated fund we treat the most recent NAV as final distribution cash flow. The stochastic discount factor $\Psi_{\tau,t}$ can be used to calculate the (realized) time-τ price $P_{\tau,t,i}$ of a **single** time-t cash flow of any given PE fund i

$$P_{\tau,t,i} = \Psi_{\tau,t} \cdot CF_{t,i} \qquad \forall \quad \tau, t, i \tag{1}$$

As SDFs are commonly parameterized by a vector $\theta \in \mathbb{R}^p$, i.e., $\Psi_{t,\tau} \equiv \Psi_{t,\tau}(\theta)$, our goal is to find an estimation method for the optimal θ. For each fund i and all points τ within a common fund lifetime, the pricing error $\epsilon_{\tau,i}$ of **all** fund cash flows is calculated as net present value

$$\epsilon_{\tau,i} = \sum_{t=1}^{T} P_{\tau,t,i} \qquad \forall \quad \tau, i \tag{2}$$

We define the w_i-weighted and $\mathcal{T}_i$-averaged fund pricing error as

$$\bar{\epsilon}_i = w_i \cdot \frac{1}{\text{card}(\mathcal{T}_i)} \sum_{\tau \in \mathcal{T}_i} \epsilon_{\tau,i} \qquad \forall \quad i \tag{3}$$

where $\mathcal{T}_i$ gives the set of (relevant) present value times τ for fund i (cf. figure 1). A present value date $\tau \in \mathcal{T}_i$ is a discretionary time point where all fund cash flows are discounted to. The cardinality $\text{card}(\mathcal{T}_i)$ gives the number of present value dates used for the ith fund. The smallest possible set $\mathcal{T}_i$ contains just a fund's starting date; in this case, $\text{card}(\mathcal{T}_i)$ consequently is one. The largest set contains all time periods bigger than the fund's starting date until now. The optimal set size of $\mathcal{T}$ is studied by Monte Carlo simulations in subsection 3.3. There we show that controlling for the optimal size of $\mathcal{T}$ decreases the small-sample bias and variance of the original Driessen et al. (2012) estimator that just discounts all cash flows to the fund inception date. Additionally, each fund i is characterized by its vintage year which can be expressed by $v_i = \min(\mathcal{T}_i) \in 1, 2, \ldots, V$, where V denotes the maximum vintage year used in a given data set. Finally, the scalar weighting factor w_i can be (i) one divided by the fund's invested capital for equal weighting of funds, (ii) one divided by the vintage year sum of invested capital for vintage year weighting, (iii) the scalar one for fund-size weighting, or (iv) some macroeconomic deflator.

To find θ, our LMD estimator minimizes the average loss of $\bar{\epsilon}$

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \; S_n(\theta) \quad \text{with} \quad S_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} L\left(\bar{\epsilon}_i\right) \tag{4}$$

where L denotes a loss function, e.g., $L(x) = (x - 0)^2$. Throughout the paper, the weighted average fund pricing error $\bar{\epsilon} \equiv \bar{\epsilon}(\theta)$ is regarded as nonlinear random function of the SDF parameter θ.

2.2 Cross-sectional unit: Individual fund vs. portfolio of funds

According to the classical value-additivity assumption in Hansen and Richard (1987) SDF models invariably shall hold for all pooled or unpooled assets. So, in theory, it is not important if the test assets for our SDF are portfolio or individual fund cash flows. Practically it makes a difference and there are arguments both for and against portfolio formation.

In the risk premium literature, portfolio formation mainly helps to attenuate the errors-in-variables bias connected to two-pass asset pricing methods (Jegadeesh et al., 2019; Pukthuanthong et al., 2019). As this is no issue in our case, we could use individual funds. Cochrane (2011) argues that portfolio sorting (seen as an auxiliary nonparametric regres-

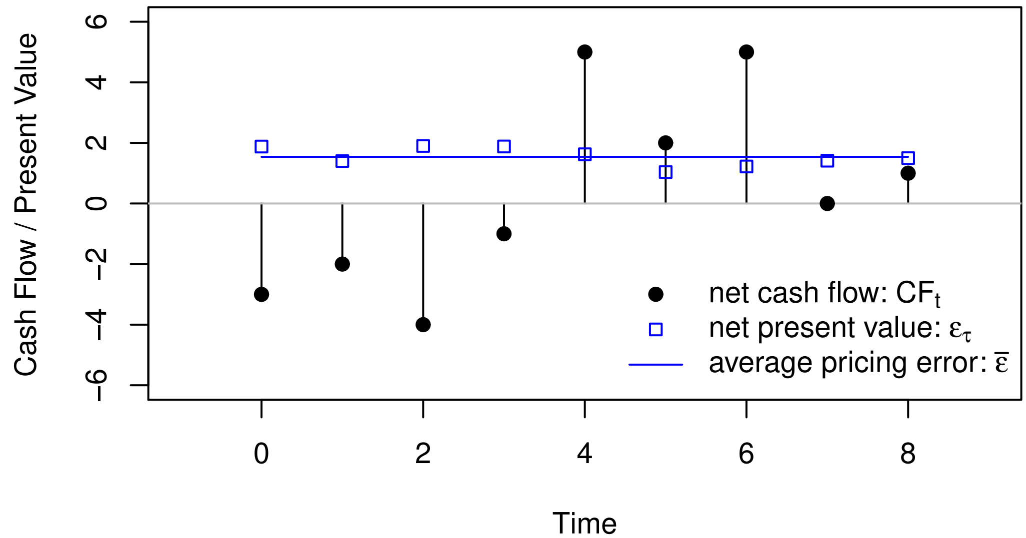

Figure 1: How to calculate and interpret the average pricing error? The time index t is relevant for the net cash flows (black dots). The time index τ is used for the net present values of this net cash flow stream (blue boxes). The weighted average of these net present values gives the average pricing error $\bar{\epsilon}$ as defined in equation 3 (solid blue line).

sion that imposes linearity on the relationship between returns and characteristics) shall be replaced by multivariate panel models due to the curse of dimensionality. Following the same nonparametric regression viewpoint, Cattaneo et al. (2019) derive a nonparametric framework where the optimal number of portfolios sorts acts as a data-dependent tuning parameter that grows with sample size. Generally, the larger the portfolios, the easier any given SDF can price their cash flows since fewer test assets remain.

In the case of private equity funds, the pooling of fund cash flows helps to counter GP financial engineering[2], which might both change and mask the true risk profile of observed LP cash flows. Especially for private equity funds, portfolio formation based on vintage year is compelling due to its time-series-like indexing as done by Driessen et al. (2012). This procedure also offers substantial computational benefits as it drastically decreases the number of cross-sectional units. Further, as stated in Ang et al. (2020), portfolio formation

[2]GPs may use bridge credit facilities below the hurdle rate to boost the fund's internal rate of return. This increases the probability of observing funds with only positive or only negative cash flows. Yet, we want to avoid (the possibility of) cross-sectional units that exhibit just cash flows with the same algebraic sign. Realistic SDFs never can price these cash flow streams.

allows more precise factor loading estimates due to decreasing idiosyncratic risk, but at the expense of sacrificing cross-sectional information. Finally, small (or fixed) T and large N set-ups may face finite sample problems (Raponi et al., 2020).

Assumption 1. *For each vintage year, we pool fund cash flows to form n_v portfolios that serve as cross-sectional units. The two boundary cases are (i) single fund portfolios and (ii) just one portfolio per vintage year.*

Without loss of generality, we refer to our cross-sectional units as funds, although this is just a special case of a size-n_v-portfolio. In the simulation study in subsection 3.3, we compare both boundary cases (i) individual funds and (ii) vintage year portfolios.

2.3 Asymptotic framework

To allow for multiple funds from the same vintage year in assumption 1, we employ an auxiliary 'spatial' notion as originally proposed by Korteweg and Nagel (2016). The spatial viewpoint is just a technical means to switch from time-series-like to more panel-data-like indexing. Unlike typical panel data, we do not follow multiple subjects over time, but for each point in time, we exclusively observe multiple new cross-sectional units (i.e., funds from that vintage year). This unusual two-dimensional indexing causes problems in the PE literature as it neatly fits neither in the (i) time-series, (ii) cross-sectional, nor (iii) panel data literature.

However, in this section, we mainly follow the time-series asymptotic framework of Pötscher and Prucha (1997) since our 'spatial' distance measure is time and adaption to our case is thus straightforward. If we observe just one fund per vintage year (or, equivalently, form vintage year portfolios), we can easily see that the framework of Pötscher and Prucha (1997) with time-series indexing can be directly applied (without any major modification).

2.3.1 Vintage year asymptotics

We assume that the 'spatial' (i.e., economic) distance between cross-sectional units, i.e., private equity funds/portfolios, can be measured quantitatively[3]. Here our asymptotic theory lets the number of funds go to infinity $n \to \infty$. However, to expose our SDF to enough distinct covariate realizations (economic conditions), identification of model parameters requires a sufficient number of funds from different vintage years in the fund-level data set used for model estimation as emphasized by Driessen et al. (2012) and Korteweg and Nagel (2016).

Assumption 2. *(i) The number of vintage years $V \to \infty$ as $n \to \infty$. (ii) The number of funds per vintage year is bounded by some positive constant. (iii) The maximal fund lifetime is also bounded by a positive constant. (iv) The economic distance between fund i and j is measured by the vintage year difference $d_{i,j} = v_i - v_j$.*

In terms of the spatial estimation literature, this assumption postulates increasing domain asymptotics and rules out so-called infill asymptotics. Infill asymptotics corresponds to the assumption of Driessen et al. (2012) that the number of funds per vintage tends to infinity.

2.3.2 Law of large numbers

The global moment condition underlying our estimation approach is that the expected value of $\bar{\epsilon}$ shall be zero if we use the optimal SDF parameter θ_0. This technically means, instead of applying a time-series law of large numbers, we rely on a spatial (cross-sectional) law of large numbers, but acknowledge the statistical dependence of pricing errors from adjacent vintage years.

Assumption 3. *The (i) time-trend and (ii) dependence structure of $\bar{\epsilon}$ shall allow*

$$n^{-1} \sum_{i=1}^{n} \bar{\epsilon}_i \overset{a.s.}{\to} E[\bar{\epsilon}] \quad as \quad V, n \to \infty$$

[3]Generally, the economic distance measure could include multiple dimensions, e.g., temporal, geographic, and industry sector proximity.

Specifically, we assume the process $\bar{\epsilon}$ to be is spatial near-epoch dependent with respect to fund vintage years (Jenish and Prucha, 2012), i.e., two funds with distance $d_{i,j} > D$ are assumed to be independent.

To satisfy the time trend part (i) of this law of large number assumption, the weighting factor w, introduced in equation 3, can be used to make $\bar{\epsilon}$ stationary. Spatial near-epoch dependence with respect to fund vintage years formalizes the simple idea that two fund pricing errors $\bar{\epsilon}$ with a small absolute vintage year difference are supposed to be dependent sine they are exposed to the same macroeconomic conditions. In contrast, two funds with a large absolute vintage year difference can be assumed independent.

2.3.3 Consistency

The estimator $\hat{\theta}$ shall converge in probability to the true parameter value θ_0 as the number of distinct vintage years in our data set goes to infinity. Multiple funds for a specific vintage year are not necessarily required but provide additional information that we want to exploit if available.

Assumption 4. *Consistency of $\hat{\theta}$ requires $\hat{\theta} \xrightarrow{p} \theta_0$ as $V, n \to \infty$. Thus $E[\bar{\epsilon}] = 0$ if and only if $\theta = \theta_0$. The parameter space is compact $\theta \in \Theta$.*

Compactness of Θ can be assured by lower and upper bounds for all parameters that can be justified by economic reasoning. In our case, e.g., a market beta factor of ten seems implausible for PE funds because of the implied risk and return expectations.

2.3.4 Central limit theorem

To assess the large-sample significance of our parameter estimates (in the following subsection 2.4), we want to describe the asymptotic distribution of the parameter vector as a normal distribution.

Assumption 5. *(i) $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \mathbf{\Sigma})$ as $V, n \to \infty$ with covariance matrix $\mathbf{\Sigma}$.*

(ii) The covariance matrix $\mathbf{\Sigma}$ can be characterized by Pötscher and Prucha (1997, Theorem 11.2.b, Theorem H.1).

The formal proof of assumption 5 may be derived in analogy to the GMM case in (Jenish and Prucha, 2012, Theorem 4) that shows that the general structure of the Pötscher and Prucha (1997) framework also applies to the spatial near-epoch dependent case.

2.4 Large sample inference

In the time-series near-epoch-dependent LMD literature, the covariance matrix $\mathbf{\Sigma}$ can be characterized according to Pötscher and Prucha (1997, Theorem 11.2.b, Theorem H.1):

$$\mathbf{\Sigma} = C^{-1}\Lambda(C^{-1})^{\top}$$

with expected Hessian matrix converging to C as $V, n \to \infty$

$$E\left(\nabla_{\theta\theta} S_n\right) \to C$$

and the expected covariance matrix of gradients converging to Λ as $V, n \to \infty$

$$nE\left[\nabla_{\theta} S_n (\nabla_{\theta} S_n)^{\top}\right] \to \Lambda$$

Here, the gradient vector $\nabla_{\theta} S_n$ is denoted as column vector. We define the corresponding finite sample estimators analogously to Pötscher and Prucha (1997, Chapters 12, 13.1), and numerically approximate the first and second partial derivatives by finite differences ($\delta \to 0$):

$$f_x(x, y) \approx \frac{f(x+\delta, y) - f(x-\delta, y)}{2\delta}$$

$$f_{xx}(x, y) \approx \frac{f(x+\delta, y) + f(x-\delta, y) - 2f(x, y)}{\delta^2}$$

$$f_{xy}(x,y) \approx \frac{f(x+\delta,y+\delta)+f(x-\delta,y-\delta)-f(x+\delta,y-\delta)-f(x-\delta,y+\delta)}{4\delta^2}$$

$\hat{C}$ is relatively straightforward

$$\hat{C} = \frac{1}{n}\sum_{i=1}^{n} \nabla_{\theta\theta} L\left(\epsilon_i\right)$$

Due to the spatial near-epoch dependence, the involved and computationally expensive part is to consistently estimate $\hat{\Lambda}$ by a Spatial Heteroskedasticity and Autocorrelation Consistent (SHAC) covariance matrix estimator (Kim and Sun, 2011, equation 2)

$$\hat{\Lambda} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{i,j}\left[\nabla_{\theta} L\left(\epsilon_i\right)\left(\nabla_{\theta} L\left(\epsilon_j\right)\right)^{\top}\right] \tag{5}$$

We define the kernel weight k as

$$k_{i,j} \equiv K\left(\frac{d_{i,j}}{b_n}\right)$$

with kernel function $K : \mathbb{R} \to [0,1]$ satisfies $K(0) = 1$, $K(x) = K(-x)$, $\int_{-\infty}^{\infty} K^2(x)dx < \infty$, and $K(\cdot)$ continuous at zero and at all but a finite number of other points. A common choice is the Bartlett kernel $K_{BT}(x) = \max(0, 1-|x|)$; see equation 2.7 in Andrews (1991) for other popular kernel choices. This means absolute vintage year differences larger than the bandwidth (or truncation) parameter $b_n = D$ are considered independent and are thus excluded from the $\hat{\Lambda}$ estimation formula.

In large samples, the vector of parameter standard errors can thus be estimated by

$$\text{SE}(\hat{\theta}) = \sqrt{\text{diag}\left[n^{-\frac{1}{2}}\hat{C}^{-1}\hat{\Lambda}(\hat{C}^{-1})^{\top}(n^{-\frac{1}{2}})^{\top}\right]} = \sqrt{\text{diag}\left[\hat{C}^{-1}\hat{\Lambda}(\hat{C}^{-1})^{\top}\right]\cdot\frac{1}{n}}$$

However, given the limited amount of available private equity data (typically the oldest vintages start in the 1980s), asymptotic characterizations of $\mathbf{\Sigma}$ and $\text{SE}(\hat{\theta})$ are of limited importance. In empirical applications, the small sample behavior of an estimation method for private equity data is more relevant than its asymptotic theory. Moreover, the standard asymptotic distribution associated with an estimator is generally not valid for post-model-selection inference, i.e., if a model selection procedure is applied to find the best model from a collection of competitors (Leeb and Pötscher, 2005).

2.5 Comparison to similar estimators

Our Least-Mean-Distance (LMD) estimator developed in section 2.1 belongs to the class of semiparametric nonlinear M-estimators as defined in Pötscher and Prucha (1997). We intentionally opt against the most prominent semiparametric nonlinear M-estimator framework, i.e., classical time-series Generalized Method of Moments (GMM) (Hansen, 1982, 2012). A classical GMM approach requires the construction of stationary, ergodic time-series of moment conditions that are used to empirically estimate the expected value of pricing errors in equation 2. The stationarity requirement of classical time-series GMM limits (i) more elaborate weighting-schemes for w, like fund-size weighting, and (ii) the usage of fund cash flows from non-realized vintages.

2.5.1 Driessen et al. (2012)

The Driessen et al. (2012) approach is most closely related to our methodology. However, they regard vintage year portfolios as their cross-sectional units; we can also use individual funds. The Driessen et al. (2012) asymptotic theory assumes the number of funds (or deals) per vintage year portfolio to go to infinity. Our asymptotic theory lets both (i) the number of vintage years and (ii) the number of funds go to infinity, but bounds the number of funds per vintage year. Further, Driessen et al. (2012) discount all fund cash flows just to the first cash flow date (like in a classical net present value calculation). In contrast, we additionally average over all dates within $\mathcal{T}_i$ to alleviate the exploding alpha issue briefly mentioned in their paper (and more thoroughly so in an earlier working paper version). Although Driessen et al. (2012) describe their estimator as a one-step GMM approach, we consider it a special case of our LMD estimator. Specifically, equation 4 from our paper is a generalization of equation 3 from their paper. Consequently, if someone accepts the assumptions from subsection 2.3, our large sample inference framework from subsection 2.4 applies to their case without any significant modification. Finally, Driessen et al. (2012) apply simple cross-sectional bootstrapping to obtain standard errors; in contrast, in subsection 3.2 we use a cross-validation technique that is adapted to the near-epoch dependence of the PE fund data.

2.5.2 Korteweg and Nagel (2016)

Korteweg and Nagel (2016), first of all, realized the usefulness of employing an auxiliary spatial framework to establish asymptotic inference results for a fund-level panel dataset of private equity funds. They measure the economic distance between two private equity funds (by the degree of cash flow overlap) to account for the cross-sectional dependence between funds. Concretely, their asymptotic inference framework draws on the spatial HAC estimator of Conley (1999); our spatial HAC framework uses Pötscher and Prucha (1997); Kim and Sun (2011); Jenish and Prucha (2012). However, they ultimately utilize a classical GMM estimator, thus a time-series law of large numbers. Specifically, we obtain the estimator of (Korteweg and Nagel, 2016, equation 18) in our framework if we replace $S_n(\theta)$ in equation 4 by equation 6.

$$S_n(\theta) = L\left(\frac{1}{n}\sum_{i=1}^{n}\bar{\epsilon}_i\right) \quad \text{with} \quad L(x) = x^2 \tag{6}$$

Time-series GMM estimators inherently bear the risk of under-identification, if the corresponding time-series is constructed by pooling all fund cash flows from a given fund type. Exactly this happens in equation 6 where we consequentially obtain a GMM estimator with just one moment condition. To counter under-identification, additional characteristic-based fund portfolios could be formed to increase the number of moment conditions per fund type; also, random portfolios combined with bootstrapping make sense. Yet, Korteweg and Nagel (2016) take another approach and introduce the concept of Generalized Public Market Equivalent (GPME), which elegantly avoids the under-identification issue. Firstly, a public market SDF model is estimated by pricing public trading strategies that shall replicate PE funds instead of directly pricing the observed PE fund cash flows. Only in a second step, these public market SDF models are applied to evaluate private equity fund cash flows.

Given these differences, our approach may not be perceived as straightforward generalization of the Korteweg and Nagel (2016) framework. In contrast, our LMD estimator generalizes the Driessen et al. (2012) method. Table 1 summarizes the most prominent distinctions between the three approaches.

	Driessen et al. (2012)	Korteweg and Nagel (2016)	Our approach
M-estimator	Least-Mean-Distance	Generalized Method of Moments	Least-Mean-Distance
Pricing error averaging	No	No	Yes
Cash flows priced	PE cash flows	public cash flows	PE cash flows
Asymptotics	cross-sectional #funds $\to \infty$	time-series #vintages $\to \infty$	spatial # of both $\to \infty$
Inference	bootstrap	spatial HAC	cross-validation & spatial HAC
Cross-sectional unit	vintage year portfolio	single fund	testing both
SDF	simple linear	exponentially affine	testing both

Table 1: Comparison to similar estimation frameworks.

3 Empirical application

3.1 Data

We use the Preqin cash flow data set as of 26th February 2020. We pool all regions and analyze the following fund types (using the Preqin asset class classification): PE ("Private Equity"; 2248 distinct funds in data set; 36 vintage years), VC ("Venture Capital"; 871; 36), RE ("Real Estate"; 742; 27), PD ("Private Debt"; 441; 31), INF ("Infrastructure", 144; 17), NR ("Natural Resources", 138; 26). For these fund types, we extract all (i) equal-weighted and (ii) fund-size-weighted cash flow series. For non-liquidated funds, we treat the latest net asset value as final cash flow. We explicitly refrain from excluding the most recent vintage years. Thus, the minimum vintage year is 1983 (just for PE) and the maximum is 2019.

The public market factors that enter our SDF draw on the US data set of the recently popularized q^5 investment factor model sourced from `http://global-q.org/factors.html` (Hou et al., 2015, 2020). Their five-factor model includes the market excess return (MKT), a size factor (ME), an investment factor (IA), a return on equity factor (ROE), and an expected growth factor (EG).

3.2 Model and estimator specifications

We test a simple linear SDF model as in Driessen et al. (2012)

$$\Psi^{\mathrm{SL}}_{\tau,t}(\theta) = \prod_{h=1}^{t} \left(1 + \alpha + r_h + \sum_{j} \beta_j \; F_{j,h}\right)^{-1} \prod_{h=1}^{\tau} \left(1 + \alpha + r_h + \sum_{j} \beta_j \; F_{j,h}\right) \tag{7}$$

and an exponential affine SDF model adapted from Korteweg and Nagel (2016)

$$\Psi^{\mathrm{EA}}_{\tau,t}(\theta) = \exp\left[-\sum_{h=\tau}^{t}\left(\alpha + \log(1+r_h) + \sum_{j\in J} \beta_j \cdot \log(1+F_{j,h})\right)\right] \tag{8}$$

with (arithmetic) risk-free return r, (arithmetic) zero-net-investment portfolio returns F_j, and parameter vector $\theta = (\alpha, \beta)$. To avoid overfitting, we just test six simple SDF models that contain $\{$MKT$\}$ alone or $\{$MKT$\}$ plus $\{$ME or IA or ROE or EG or Alpha$\}$. In equation 4, we use the quadratic loss function $L(x) = x^2$.

To assess the parameter significance, we compute the asymptotic standard errors as outlined in subsection 2.4. For the Bartlett kernel's bandwidth $b_n = D$ we select 12 years, i.e., funds with absolute vintage year differences larger than 12 years are assumed to be independent.

Additionally, we want to test the finite - or more honestly small - sample parameter significance and the out-of-sample performance of our SDF models. To account for the dependency between funds from adjacent vintage years caused by overlapping fund cash flows, we draw on *hv*-block cross-validation (Racine, 2000). Therefore, we form three partitions for several vintage year groups. As larger validation sets are preferred for model selection, the validation set (v-block) always contains funds of three neighboring vintage years (e.g. 2000, 2001, 2002). To reduce the dependency between training and validation set, we remove all funds from three-year-adjacent vintage years, i.e., the h-block (e.g. 1997, 1998, 1999, 2003, 2004, 2005). Funds from the remaining vintage years enter the training set and are thus used for model estimation (e.g. 1985-1996, 2006-2019). We apply ten-fold cross validation using the ten validation sets described in table 2. This means, we replace the bootstrap standard error calculation of Driessen et al. (2012) by hv-block cross-validation since the new method (i) accounts for near-epoch-dependence, (ii) focuses directly on the out-of-sample performance of the SDF models, and (iii) is computationally cheaper.

training.before	h-block.before	v-block	h-block.after	training.after
estimation	remove	validation	remove	estimation
start-1984	1985,1986,1987	1988,1989,1990	1991,1992,1993	1994-end
start-1987	1988,1989,1990	1991,1992,1993	1994,1995,1996	1997-end
start-1990	1991,1992,1993	1994,1995,1996	1997,1998,1999	2000-end
start-1993	1994,1995,1996	1997,1998,1999	2000,2001,2002	2003-end
start-1996	1997,1998,1999	2000,2001,2002	2003,2004,2005	2006-end
start-1999	2000,2001,2002	2003,2004,2005	2006,2007,2008	2009-end
start-2002	2003,2004,2005	2006,2007,2008	2009,2010,2011	2012-end
start-2005	2006,2007,2008	2009,2010,2011	2012,2013,2014	2015-end
start-2008	2009,2010,2011	2012,2013,2014	2015,2016,2017	2018-end
start-2011	2012,2013,2014	2015,2016,2017	2018,2019,2020	2021-end

Table 2: Partitions used for hv-block cross-validation.

3.3 Simulation study

Our Monte Carlo experiments examine the following questions related to the bias and variance of our estimation methodology in finite samples. Is it beneficial to use vintage year portfolios instead of individual funds? Which SDF model performs better when we also use the corresponding data generating process (i.e., assume correct model specification)? How is estimator precision affected by varying numbers of vintage years and cross-sectional units? Which is the optimal set of present value times $\mathcal{T}$?

We use historical q-investment factors from 1986 to 2005 and simulate 20 funds for each of these 20 vintage years. Each fund contains 15 deals with equal investment amounts and exactly one divestment cash flow. Deals are entered within the first five years of fund lifetime following a discrete uniform distribution and afterward held between one to ten years again uniformly distributed. The deal returns are generated by the simple linear or exponential affine SDF models described in equations 7 and 8. In the base case, we just use the MKT factor with $\beta_{\text{MKT}} = 1$ and in each month add a normal i.i.d. error term with standard deviation $\sigma = 0.2$ and zero mean. Additionally, we test an intercept term α of -0.25% per month and a high β_{MKT} of 2.5. In the exponential affine case, we adjust the log-normally distributed error mean to zero by subtracting $0.5\sigma^2$. If a negative return exceeds -100%, the company defaults with a zero exit cash flow. In contrast, the error term in the simulations of Driessen et al. (2012) is more well-behaved as it follows a shifted lognormal distribution that, even with arbitrarily high error term variance, just allows for returns below say -99%, if the market return is close to its lower bound (see equation 9 in their online appendix). In our base case, the set of present value dates $\mathcal{T}$ contains all months from the first cash flow to maximum month 180. To assess our estimator's bias and variance, we simulate 1000 test scenarios for vintage year portfolios and just 200 test cases when using individual funds due to memory restrictions.

Cross-sectional unit i: As presumed in subsection 2.2, vintage year portfolio results appear to have lower bias and variance when compared to individual funds. For the simple linear SDF and maximum month 180, the mean and standard deviation of the coefficient estimate $\hat{\beta}_{\text{MKT}}$ is 1.016 (0.2) for the vintage year portfolio and 1.096 (0.376) for individual funds. More results are depicted in figure 2. However, for individual funds, we just simulate 200 iterations due to the high computational cost.

This finding has two important implications: On the one hand, vintage year portfolio formation can substantially decrease our estimator's bias and variance. On the other hand, it

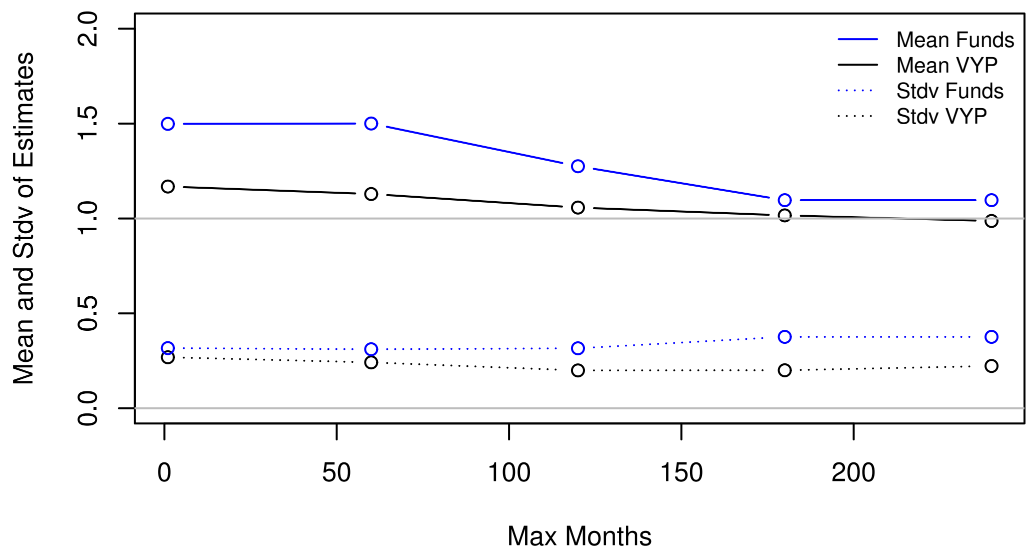

Figure 2: Simulation results comparing individual funds vs. vintage year portfolios (VYPs) with true $\beta = 1$ and simple linear SDF.

also dramatically reduces the number of cross-sectional units and consequentially impairs the importance of asymptotic results. This considerations may explain the choice of Korteweg and Nagel (2016) to use individual funds as cross-sectional units in their asymptotic SHAC framework to obtain smaller standard error estimates.

SDF model Ψ: In our base case with vintage year portfolios, the exponential affine SDF shows a mean and standard deviation of 1.011 (0.175) compared to the 1.016 (0.2) achieved by the simple linear SDF. Generally, the exponential affine SDF model and the simple linear SDF model exhibit similar bias and variance when comparing panels A and B in table 5. Figure 3 visualizes the true $\beta = 1$ case which shows that the estimation results are not overly sensitive to the choice of the SDF model.

Moreover, the perceived superiority of exponential affine SDFs is probably rather theoretical than practical as other proponents also emphasize their universality mainly from a mathematical perspective without providing supportive empirical or simulation results (Gourieroux and Monfort, 2007; Bertholon et al., 2008).

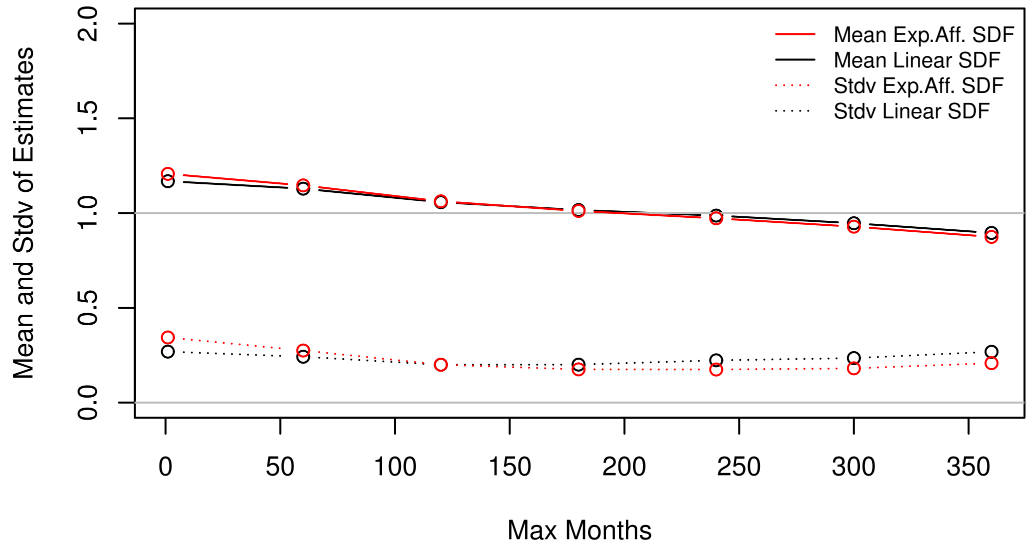

Figure 3: Simulation results comparing exponentially affine and simple linear SDF with true $\beta = 1$ and vintage year portfolios.

Varying vintages V and portfolio sizes n/V: To test the effect of varying data sizes available for MKT factor estimation, we in/decrease the (i) number of vintage years and (ii) the number of funds per vintage year (cf. table 3). Here we use vintage year portfolios and the simple linear SDF. For our simple data generating process, increasing the number of deals/funds per vintage year portfolio appears to decrease the estimator's variance more effectively than adding more vintage years. However, the bias is almost the same for all tested specifications. Generally, we seem to need many new data points to ensure a reasonable variance of our estimator.

	Base	Big n/V	Big V	Big V	Small V	Small V
Start vintage	1986	1986	1967	1967	1986	1996
End vintage	2005	2005	2005	2005	1995	2005
#Funds per vintage	20	40	10	20	20	20
Mean β_{MKT}	1.011	1.020	0.993	1.015	1.027	0.934
Stdv β_{MKT}	0.187	0.133	0.263	0.227	0.232	0.418

Table 3: Simulation study for varying number of vintages and number of funds per vintage. We use vintage year portfolios, the simple linear SDF with true $\beta_{\text{MKT}} = 1$, maximum month 180, and 500 simulation iterations.

Size of set $\mathcal{T}$: The results in table 5 indicate that we can control the bias by an appropriate choice of the set $\mathcal{T}$. The bias almost vanishes when we average over all present value dates in the maximal fund lifetime of 180 months. For smaller or larger sets for $\mathcal{T}$, we find increasing small-sample bias[4].

The same finding also holds when we limit the maximal fund lifetime to ten years by reducing the maximum deal holding period from ten to five years. Here, under correct model specification with $\beta_{\text{MKT}} = 1$, the smallest bias is obtained for maximum month 120: for max. month 60 we get 1.028 (0.116), for max. month 120 we get 1.005 (0.116), and for max. month 180 we get 0.969 (0.13).

In table 5 for both true and false model specifications, the α standard deviation is very high compared to its mean value. This may indicate it is rather delicate to empirically determine private equity's historical outperformance by our semiparametric estimator.

To conclude, our simulations study rationalizes two key practices from the Driessen et al. (2012) paper: (i) vintage year portfolio formation helps to improve estimator precision and (ii) increasing the number of funds per vintage seems to be more effective in controlling estimator variance that increasing the number of vintages[5]. However, our examples with correct specification cannot support the assumption of Korteweg and Nagel (2016) that (iii) the exponential affine SDF is (clearly) superior to the simple liner SDF in a multi-period framework; actually, their bias and variances are quite equal. Moreover, our simulation study suggests that (iv) averaging pricing errors over multiple dates strikingly reduces the bias inherent to the original procedure of Driessen et al. (2012) that just discounts all cash flows to the fund inception date. Actually, choosing the set $\mathcal{T}$ according to the fund lifetime seems to decrease the bias (and to a lesser extend also the variance) more effectively than

[4]Recall that using the minimal set for $\mathcal{T}$, i.e., discounting all cash flows just to the fund inception date, corresponds exactly to the Driessen et al. (2012) approach. Thus, the original Driessen et al. (2012) methodology achieves a suboptimal small-sample bias since it does not average pricing errors over multiple present value dates.

[5]Finding (ii) may explain the choice of Driessen et al. (2012) to employ an asymptotic law that lets the number of deals/funds per vintage tend to infinity.

all other measures combined.

3.4 Empirical results

Following the conclusions from the previous subsection, we use vintage year portfolios to estimate simple linear SDF models with maximum month 180. Asymptotic inference results for the full dataset are exhibited in table 6 for fund-size weighting and in table 8 for equal weighting. The results for hv-block cross-validation are displayed in table 7 for fund-size weighting and in table 9 for equal weighting. We generally analyze the results in a two-step procedure: For a given model specification, we use the cross-validation error (i.e., the average out-of-sample error) to select the best model for each fund type, but analyze the corresponding coefficient estimates from the asymptotic inference tables (estimated on the entire data set). Therefore, for each fund type the SDF models in the asymptotic inference tables 6 and 8 are sorted by the corresponding cross-validation error. Throughout this subsection, we define the statistical significance of coefficient estimates in terms of a t-ratio $\hat{\theta}[SE(\hat{\theta})]^{-1}$ greater than 1.96.

		MKT Factor			Second Factor		
Weighting	Inference	Coef	SE	SE.indep	Coef	SE	SE.indep
fund-size	asymptotic	0.75	27.06	19.73	0.80	28.95	20.94
fund-size	cross-validation	0.85	0.38	-	0.59	0.51	-
equal	asymptotic	0.76	26.75	16.16	0.76	11.25	6.69
equal	cross-validation	0.84	0.34	-	0.62	0.50	-

Table 4: Top-level overview over tables 6 to 9: Averages of absolute values of coefficient estimates and standard errors.

Table 4 helps to get a rough overview of tables 6 to 9 as it summarizes their absolute column means. Conspicuously, asymptotic standard errors (SEs) seem enormously high and, moreover, contain colossal outliers. The standard errors implied by $hv-$block cross-validation are considerably smaller than the asymptotic SEs and seem to lie within a plausible range. When just looking at asymptotic standard errors of the second factors, fund-size weighting exhibits substantially larger SEs than fund equal-weighting. Assuming independence between funds from different vintages decreases asymptotic SEs by approximately 30-40% compared to a realistic kernel bandwidth of $D = 12$. But even these independent SEs rarely imply statistical significance coefficient estimates with t-ratios bigger than 1.96. In table 6 with fund-size weighting, just one out of 36 models exhibit asymptotically significant MKT and second-factor estimates. In the case of equal-weighting, table 8 also shows just one asymptotically significant model out of 36.

In summary, the results reveal weak two-factor models with MKT plus a second q-investment factor. Likewise, the simulation results from the previous subsection indicate a rather high variance associated with our semiparametric estimator (given the amount of data typically available). Thus, we recommend focusing on single MKT factor models even when their asymptotic t-ratios are below 1.96. At least the hv-block cross-validation standard deviations imply significant one-factor MKT models for fund types PE, VC, PD, INF. In contrast, RE is just significant for equal-weighting, and NR is insignificant for both weighting schemes.

Focus on PE and VC estimates Here, we briefly summarize the one-factor MKT and the two-factor Alpha model estimates for fund types PE (i.e., mainly Buyout and Growth) and VC. For PE, all one-factor MKT model β_{MKT} estimates fall in the range from 1.13 to 1.28. If we add an α term, all β_{MKT} estimates decrease to the range 0.61 to 0.77 with annualized α coefficients of approximately positive 4-5% per year. For VC, the one-factor MKT model β_{MKT} estimates are in the range from 0.80 to 1.14. If we add an α term, all β_{MKT} estimates strongly increase to the range 1.81 to 2.06 with annualized α coefficients of approximately negative 6-7% per year. These results at least weakly indicate - given their insignificant asymptotic standard errors - that PE funds outperform public markets with a market beta coefficient of less than one, which suggests low market risk. On the other hand, VC underperforms public markets with market beta coefficients of roughly two, which implies high market risk. So, even Driessen et al. (2012) use the problematic Thomson

Venture Economics (TVE) dataset for their empirical analysis[6], we obtain similar qualitative results using Preqin data: (i) the market beta of VC seems to be higher than that of PE and (ii) VC, in contrast to PE, appears to exhibit a negative abnormal performance α.

As a robustness check, we reestimate all SDF models on a dataset that just contains funds from vintages older or equal than 2011. Interestingly, the PE and VC results regarding β_{MKT} and α can be qualitatively and also quantitatively confirmed on this 'mostly-liquidated' dataset[7].

4 Conclusion

Theoretically, our Least-Mean-Distance estimator can be easily generalized to estimate SDF models for all kinds of non-traded cash flows. Practically, semiparametric estimators commonly exhibit problematic small sample behavior. Given the amount of currently available private equity fund data, our estimator's variance seems quite large, even for simple SDF model specifications. Specifically, our Monte Carlo simulation results prompt us to conclude that the closely related Driessen et al. (2012) estimator may exhibit more bias and variance than originally assumed in their paper. Especially, the variance of α estimates seems to be too high to allow reliable abnormal performance conclusions. Fortunately, we show that at least the bias can be easily reduced by averaging pricing errors over all dates within the fund lifetime.

In the data-sparse private equity domain with only 20-40 cross-sectional units (i.e., vintage year portfolios) currently available for estimation, asymptotic inference seems not to be overly useful. Thus, we strongly advise to always challenge asymptotic inference results by resampling or cross-validation techniques that are adapted to the dependence structure of overlapping fund cash flows. However, even their conclusions should be double-checked, to avoid unreasonable instances, e.g., when hv-block cross-validation chooses dubious models

[6] Harris et al. (2014) discuss the potential downward bias of the TVE dataset.

[7] All R code and data is available in an online repository.

with negative MKT factor estimates. Since, in our empirical analyses, basically all two-factor models' asymptotic standard errors appear statistically insignificant, we conjecture that naive versions of our SDF estimator shall be exclusively used for a single-MKT-factor model until considerably more vintage year information for private equity funds is available.

If someone wants to estimate more complex SDF models that incorporate additional factors, more structure is needed. This can be parametric assumptions for the data generating process (Ang et al., 2018) or to extract additional information from intermediate net asset values (Gredil et al., 2019; Brown et al., 2020). A first 'modern' approach to the same problem is applying machine learning techniques that regularize/shrink all coefficients other than the MKT factor. Secondly, given the high estimator variance revealed in the simulation study, statistical learning methods that create a strong learner by combining multiple weak learners seem also worth considering (boosting, bagging, model averaging).

References

Andrews, D. W. (1991). Heteroskedasticity and autocorrelation consistent covariance matrix estimation. *Econometrica: Journal of the Econometric Society*, pages 817–858.

Ang, A., Chen, B., Goetzmann, W. N., and Phalippou, L. (2018). Estimating private equity returns from limited partner cash flows. *Journal of Finance*, 73(4):1751–1783.

Ang, A., Liu, J., and Schwarz, K. (2020). Using stocks or portfolios in tests of factor models. *Journal of Financial and Quantitative Analysis*, 55(3):709–750.

Bertholon, H., Monfort, A., and Pegoraro, F. (2008). Econometric asset pricing modelling. *Journal of Financial Econometrics*, 6(4):407–458.

Brown, G. W., Ghysels, E., and Gredil, O. (2020). Nowcasting net asset values: The case of private equity. working paper (as of 2020-04-16).

Buchner, A. (2014). The alpha and beta of private equity investments. working paper (as of 2014-10-24).

Cattaneo, M. D., Crump, R. K., and Farrell, M. H. (2019). Characteristic-sorted portfolios: Estimation and inference. *Review of Economics and Statistics*.

Cochrane, J. H. (2011). Presidential address: Discount rates. *The Journal of Finance*, 66(4):1047–1108.

Conley, T. G. (1999). Gmm estimation with cross-sectional dependence. *Journal of Econometrics*, 92(1):1–45.

Driessen, J., Lin, T.-C., and Phalippou, L. (2012). A new method to estimate risk and return of nontraded assets from cash flows: the case of private equity. *Journal of Financial and Quantitative Analysis*, 47(3):511–535.

Gourieroux, C. and Monfort, A. (2007). Econometric specification of stochastic discount factor models. *Journal of Econometrics*, 136(2):509–530.

Gredil, O., Sorensen, M., and Waller, W. (2019). Evaluating private equity performance using stochastic discount factors. working paper (as of 2019-03-15).

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica*, 50(4):1029–1054.

Hansen, L. P. (2012). Proofs for large sample properties of generalized method of moments estimators. *Journal of Econometrics*, 170:325–330.

Hansen, L. P. and Richard, S. F. (1987). The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models. *Econometrica*, pages 587–613.

Harris, R. S., Jenkinson, T., and Kaplan, S. N. (2014). Private equity performance: What do we know? *The Journal of Finance*, 69(5):1851–1882.

Hou, K., Xue, C., and Zhang, L. (2015). Digesting anomalies: An investment approach. *Review of Financial Studies*, 28(3):650–705.

Hou, K., Xue, C., and Zhang, L. (2020). An augmented q^5 model with expected growth. *Review of Finance*.

Jegadeesh, N., Noh, J., Pukthuanthong, K., Roll, R., and Wang, J. (2019). Empirical tests of asset pricing models with individual assets: Resolving the errors-in-variables bias in risk premium estimation. *Journal of Financial Economics*, 133(2):273–298.

Jenish, N. and Prucha, I. R. (2012). On spatial processes and aysmptotic inference under near-epoch dependence. *Journal of Econometrics*, 170(1):178–190.

Kim, M. S. and Sun, Y. (2011). Spatial heteroskedasticity and autocorrelation consistent estimation of covariance matrix. *Journal of Econometrics*, 160(2):349–371.

Korteweg, A. and Nagel, S. (2016). Risk-adjusting the returns to venture capital. *Journal of Finance*, 71(3):1437–1470.

Leeb, H. and Pötscher, B. M. (2005). Model selection and inference: Facts and fiction. *Econometric Theory*, 21:21–59.

Pötscher, B. M. and Prucha, I. R. (1997). *Dynamic nonlinear econometric models: Asymptotic theory*. Springer Science & Business Media.

Pukthuanthong, K., Roll, R., and Subrahmanyam, A. (2019). A protocol for factor identification. *The Review of Financial Studies*, 32(4):1573–1607.

Racine, J. (2000). Consistent cross-validatory model-selection for dependent data: hv-block cross-validation. *Journal of Econometrics*, 99:39–61.

Raponi, V., Robotti, C., and Zaffaroni, P. (2020). Testing beta-pricing models using large cross-sections. *The Review of Financial Studies*, 33(6):2796–2842.

Panel A: simple linear SDF

Model==DGP	True	False		False	True	
MaxMonth	$\beta = 1$	$\alpha = 0$	$\beta = 1$	$\beta = 2.5$	$\alpha = -0.25$	$\beta = 2.5$
1 - mean	1.168	1625%	0.003	2.023	5879%	-16.711
1 - stdv	0.269	2792%	9.968	0.342	866%	13.347
60 - mean	1.129	0.138%	0.933	2.103	-0.086%	2.285
60 - stdv	0.242	0.245%	0.363	0.302	0.253%	0.406
120 - mean	1.058	0.112%	0.906	2.063	-0.085%	2.239
120 - stdv	0.200	0.214%	0.313	0.253	0.239%	0.385
180 - mean	1.016	0.041%	0.965	2.052	-0.161%	2.370
180 - stdv	0.200	0.172%	0.334	0.277	0.173%	0.403
240 - mean	0.987	-0.053%	1.077	2.072	-0.277%	2.589
240 - stdv	0.223	0.162%	0.361	0.326	0.118%	0.375
300 - mean	0.946	-0.149%	1.175	2.080	-0.357%	2.714
300 - stdv	0.235	0.174%	0.377	0.398	0.114%	0.366
360 - mean	0.895	-0.245%	1.269	2.048	-0.461%	2.859
360 - stdv	0.268	0.201%	0.399	0.551	0.140%	0.386

Panel B: exponential affine SDF

Model==DGP	True	False		False	True	
MaxMonth	$\beta = 1$	$\alpha = 0$	$\beta = 1$	$\beta = 2.5$	$\alpha = -0.25$	$\beta = 2.5$
1 - mean	1.207	203%	1.276	2.256	692%	1.704
1 - stdv	0.344	314%	0.710	0.290	13%	1.666
60 - mean	1.146	0.126%	0.941	2.264	-0.018%	2.277
60 - stdv	0.275	0.264%	0.386	0.256	0.370%	0.473
120 - mean	1.062	0.107%	0.908	2.221	0.009%	2.205
120 - stdv	0.200	0.237%	0.333	0.187	0.357%	0.448
180 - mean	1.011	0.027%	0.971	2.182	-0.136%	2.358
180 - stdv	0.175	0.211%	0.366	0.168	0.344%	0.505
240 - mean	0.972	-0.088%	1.095	2.144	-0.441%	2.723
240 - stdv	0.174	0.224%	0.406	0.178	0.317%	0.503
300 - mean	0.928	-0.202%	1.203	2.083	-0.717%	2.985
300 - stdv	0.181	0.253%	0.426	0.254	0.340%	0.513
360 - mean	0.874	-0.319%	1.304	1.685	-1.095%	3.272
360 - stdv	0.208	0.291%	0.447	0.772	0.374%	0.586

Table 5: Simulation study to compare the simple linear with the exponential affine SDF and to determine the optimal size of the set $\mathcal{T}$. Here, we always use vintage year portfolios and 1000 simulation iterations. For better readability, $\beta_{\text{MKT}} = \beta$. For the unity and high beta model, we test true and false model specifications (with and without the α term).

	MKT Factor				Second Factor		
Type	Estim.	SE	SE.indep	Factor	Estim.	SE	SE.indep
PE	0.709	2.470	1.153	EG	0.807	4.693	1.960
PE	0.770	7.976	3.348	ROE	1.540	5.140	3.499
PE	1.126	1.003	0.868	MKT	1.126	1.003	0.868
PE	0.644	1.234	0.585	Alpha	0.003	0.036	0.013
PE	1.121	1.023	0.897	ME	0.074	2.021	0.915
PE	1.158	1.125	1.068	IA	-0.338	2.499	1.259
VC	1.053	4.150	2.733	IA	-1.959	2.100	1.767
VC	1.114	3.861	2.894	ME	-1.383	5.102	2.211
VC	1.806	11.391	4.279	Alpha	-0.006	0.124	0.046
VC	0.801	704.455	561.598	MKT	0.801	704.455	561.598
VC	1.429	8.073	3.219	ROE	-1.306	18.055	6.919
VC	1.507	17.322	6.966	EG	-0.904	15.344	5.737
PD	0.885	1.040	1.242	MKT	0.885	1.040	1.242
PD	0.660	0.095	0.039	Alpha	0.002	0.001	0.000
PD	0.826	1.707	1.443	EG	0.143	20.341	7.506
PD	0.849	2.921	2.146	ME	0.301	2.739	1.518
PD	0.887	1.378	1.244	ROE	-0.023	6.925	2.553
PD	0.863	2.942	2.224	IA	0.247	5.306	3.607
RE	0.578	1.827	1.196	MKT	0.578	1.827	1.196
RE	1.303	5.463	2.259	Alpha	-0.006	0.088	0.034
RE	0.200	2.598	1.356	ROE	3.118	2.579	6.629
RE	0.202	3.297	1.965	EG	0.844	2.478	1.828
RE	0.756	3.043	2.192	IA	-1.938	1.879	0.783
RE	0.887	1.167	0.858	ME	-2.059	1.300	0.563
NR	-0.215	2.367	1.976	EG	0.909	14.505	7.475
NR	0.191	3.136	4.242	MKT	0.191	3.136	4.242
NR	-0.674	58.003	24.234	Alpha	0.008	0.230	0.098
NR	-0.020	0.954	2.210	ROE	1.128	4.830	5.066
NR	0.143	3.236	4.116	IA	-0.768	1.808	2.154
NR	0.212	4.209	5.450	ME	-0.575	1.603	1.252
INF	0.824	3.201	2.815	MKT	0.824	3.201	2.815
INF	0.190	7.133	3.288	Alpha	0.005	0.030	0.025
INF	0.317	23.904	10.658	EG	0.848	45.836	20.176
INF	0.470	9.316	4.237	ROE	1.245	5.531	6.134
INF	0.778	5.951	5.424	ME	-0.811	3.712	4.349
INF	0.661	61.329	33.819	IA	-1.108	150.713	85.733

Table 6: Asymptotic inference with fund-size-weighting, max month 180, and $D = 12$.

	MKT Factor			Second Factor		
Type	Mean	SD	Factor	Mean	SD	CV-error
PE	0.867	0.276	EG	0.720	0.137	112808.000
PE	0.927	0.305	ROE	1.375	0.420	126801.000
PE	1.276	0.296	MKT	1.276	0.296	151964.000
PE	0.772	0.238	Alpha	0.004	0.002	154805.000
PE	1.317	0.396	ME	0.236	0.664	209319.000
PE	1.311	0.370	IA	0.014	0.703	210650.000
VC	1.045	0.126	IA	-1.890	0.238	11858.000
VC	1.172	0.126	ME	-1.448	0.263	13301.000
VC	1.930	0.356	Alpha	-0.005	0.001	17723.000
VC	0.804	0.363	MKT	0.804	0.363	21852.000
VC	1.527	0.517	ROE	-0.972	0.679	26680.000
VC	1.646	0.678	EG	-0.644	0.556	32730.000
PD	0.887	0.039	MKT	0.887	0.039	7368.000
PD	0.567	0.202	Alpha	0.003	0.001	7917.000
PD	0.763	0.113	EG	0.229	0.141	8758.000
PD	0.862	0.103	ME	0.342	0.256	9834.000
PD	0.812	0.153	ROE	0.258	0.394	11522.000
PD	0.914	0.211	IA	0.472	0.424	18096.000
RE	0.722	0.392	MKT	0.722	0.392	50900.000
RE	1.288	0.345	Alpha	-0.004	0.005	51437.000
RE	0.389	0.446	ROE	2.333	1.507	54689.000
RE	0.465	0.470	EG	0.448	0.629	59316.000
RE	0.847	0.411	IA	-1.262	1.160	65835.000
RE	0.983	0.350	ME	-1.467	1.298	66827.000
NR	-0.047	0.421	EG	0.657	0.557	10559.000
NR	0.318	0.321	MKT	0.318	0.321	11480.000
NR	-0.335	0.763	Alpha	0.006	0.005	11854.000
NR	0.136	0.466	ROE	0.844	1.062	13296.000
NR	0.270	0.508	IA	-0.288	1.124	14479.000
NR	0.416	0.587	ME	0.032	1.079	15789.000
INF	0.862	0.320	MKT	0.862	0.320	14551.000
INF	0.639	0.753	Alpha	0.002	0.004	15069.000
INF	0.766	0.626	EG	0.258	0.495	16004.000
INF	0.837	0.504	ROE	0.090	0.939	18472.000
INF	0.868	0.412	ME	0.078	0.643	18514.000
INF	0.892	0.561	IA	0.081	1.073	23162.000

Table 7: *hv*-block cross-validation with fund-size-weighting and max month 180.

	MKT Factor				Second Factor		
Type	Estim.	SE	SE.indep	Factor	Estim.	SE	SE.indep
PE	0.775	0.638	0.550	EG	0.667	5.558	2.125
PE	0.610	1.064	0.387	Alpha	0.004	0.006	0.002
PE	0.826	20.352	8.308	ROE	1.087	33.514	12.143
PE	1.134	1.050	0.694	MKT	1.134	1.050	0.694
PE	1.146	1.001	0.638	IA	-0.386	1.909	0.813
PE	1.134	1.048	0.702	ME	-0.014	1.797	0.736
VC	1.181	24.418	16.693	ME	-1.277	4.928	4.352
VC	1.137	7.259	6.057	IA	-1.553	3.716	2.139
VC	1.956	4.189	1.520	Alpha	-0.006	0.335	0.117
VC	1.034	2.205	1.758	MKT	1.034	2.205	1.758
VC	1.488	1.801	0.941	ROE	-1.148	4.060	1.424
VC	1.535	2.821	1.336	EG	-0.754	3.626	1.260
PD	0.844	1.245	0.856	MKT	0.844	1.245	0.856
PD	0.502	0.044	0.015	Alpha	0.003	0.000	0.000
PD	0.791	2.478	1.557	ROE	0.222	5.024	1.966
PD	0.736	2.230	1.303	EG	0.213	6.296	2.374
PD	0.844	1.150	0.837	IA	0.076	2.543	1.416
PD	0.833	1.978	1.362	ME	0.323	1.845	0.986
RE	0.743	3.471	2.075	MKT	0.743	3.471	2.075
RE	1.265	7.581	3.331	Alpha	-0.004	0.046	0.017
RE	0.145	3.486	1.614	ROE	3.202	17.447	7.413
RE	0.400	50.493	31.818	EG	0.700	48.813	29.195
RE	0.884	4.928	2.813	ME	-1.782	1.474	0.605
RE	0.795	18.146	10.282	IA	-1.712	13.199	7.657
NR	-0.056	3.693	1.897	ROE	1.934	3.154	2.287
NR	0.000	4.771	2.368	EG	0.814	1.368	1.753
NR	0.425	3.370	5.178	MKT	0.425	3.370	5.178
NR	-0.272	39.463	16.698	Alpha	0.006	0.202	0.081
NR	0.394	0.871	1.401	IA	-0.319	1.734	1.169
NR	0.453	15.719	24.377	ME	0.432	5.574	8.016
INF	0.098	0.055	0.025	Alpha	0.006	0.001	0.000
INF	0.280	20.158	8.983	EG	0.893	21.368	9.700
INF	0.775	19.273	22.509	MKT	0.775	19.273	22.509
INF	0.469	26.226	12.751	ROE	1.030	30.728	16.129
INF	0.758	33.453	36.239	ME	-0.804	16.214	16.161
INF	0.664	630.801	351.730	IA	-0.929	137.937	75.628

Table 8: Asymptotic inference with equal-weighting, max month 180, and $D = 12$.

	MKT Factor			Second Factor		
Type	Mean	SD	Factor	Mean	SD	CV-error
PE	0.886	0.262	EG	0.614	0.217	101444.000
PE	0.719	0.205	Alpha	0.004	0.001	105842.000
PE	0.948	0.267	ROE	0.975	0.407	110926.000
PE	1.250	0.262	MKT	1.250	0.262	127589.000
PE	1.247	0.274	IA	-0.183	0.598	157037.000
PE	1.281	0.323	ME	0.048	0.644	169552.000
VC	1.250	0.153	ME	-1.292	0.234	16305.000
VC	1.183	0.169	IA	-1.507	0.327	16449.000
VC	2.052	0.257	Alpha	-0.006	0.001	18666.000
VC	1.138	0.341	MKT	1.138	0.341	25321.000
VC	1.610	0.431	ROE	-0.946	0.426	26618.000
VC	1.688	0.505	EG	-0.616	0.331	30392.000
PD	0.838	0.029	MKT	0.838	0.029	11290.000
PD	0.458	0.151	Alpha	0.003	0.001	11568.000
PD	0.770	0.058	ROE	0.232	0.111	11572.000
PD	0.707	0.086	EG	0.224	0.091	12194.000
PD	0.825	0.083	IA	0.158	0.305	15071.000
PD	0.837	0.098	ME	0.358	0.328	15441.000
RE	0.803	0.336	MKT	0.803	0.336	43486.000
RE	1.191	0.363	Alpha	-0.003	0.004	45310.000
RE	0.341	0.402	ROE	2.275	1.558	52822.000
RE	0.559	0.379	EG	0.397	0.503	52867.000
RE	0.929	0.339	ME	-1.287	1.174	57341.000
RE	0.852	0.372	IA	-1.129	1.025	57662.000
NR	-0.009	0.203	ROE	1.880	0.562	15631.000
NR	0.200	0.510	EG	0.681	0.372	18981.000
NR	0.572	0.400	MKT	0.572	0.400	20006.000
NR	-0.065	0.729	Alpha	0.006	0.004	20880.000
NR	0.596	0.567	IA	-0.060	0.869	24766.000
NR	0.769	0.771	ME	0.666	1.062	27238.000
INF	0.124	0.608	Alpha	0.007	0.006	14995.000
INF	0.613	0.393	EG	0.402	0.496	15758.000
INF	0.862	0.281	MKT	0.862	0.281	15820.000
INF	0.627	0.360	ROE	0.577	1.374	18297.000
INF	0.810	0.315	ME	-0.129	1.109	19641.000
INF	0.797	0.842	IA	0.051	2.180	33661.000

Table 9: *hv*-block cross-validation with equal-weighting and max month 180.