Private Capital Fund Risk Modeling

A simulation approach customized for the Solvency II framework

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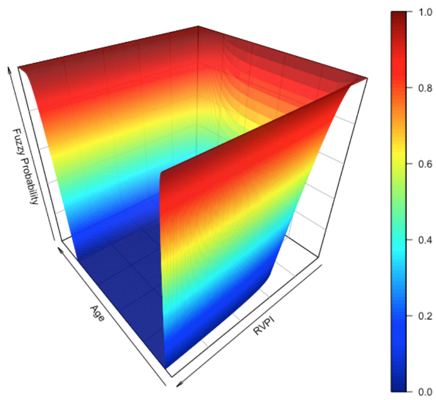

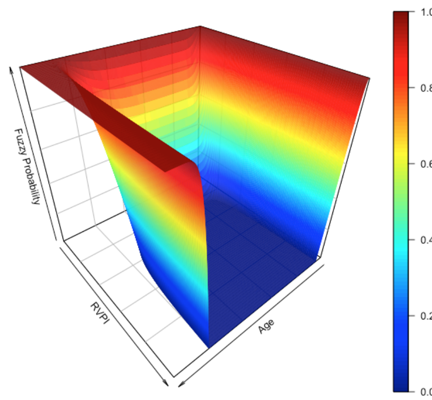

Abstract

In this paper, we develop statistical risk model tools for a portfolio of private capital funds, which enables us to estimate one-year Value-at-Risks for these portfolios. More specifically, we offer two complementary approaches (historical and Monte-Carlo) to simulate probability distributions of returns for various, heterogeneous alternative asset portfolios. These methods incorporate both macroeconomic (respectively public market data) and fund specific input parameters (risk factors) in order to come up with a sound risk estimate for an investor's "exposure to private capital". However, much effort has to be expended beforehand to define a proper risk and performance measure for an illiquid asset class like private equity. The comprehensive model is composed of several sub-modules, which draw on various types of regression-, simulation- and statistical learning methods. Throughout the entire paper, we will repeatedly establish links between our findings and the new Solvency II framework.

Keywords: Private Capital Risk Model, Alternative Asset Portfolio Analysis, One-Year Value-at-Risk, Historical Simulation, Monte-Carlo Simulation, Solvency II

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Conventions and abbreviations

AI: Alternative Investment

AMT: Asset Metrix Type

BO: Buy Out

CAPM: Capital Asset Pricing Model

c.d.f. cumulative distribution function

CI/FO: Chief Investment/Financial Officer

CVaR Conditional Value at Risk

DD: Distressed Debt

FOF: Fund of Funds

GP General Partner (of Limited Partnership)

HGB: Handelsgesetzbuch

IFRS: International Financial Reporting Standards

i.i.d.: independent and identically distributed

Infra: Infrastructure

IRR: Internal Rate of Return

LP: Limited Partner (of Limited Partnership)

MCR: Minimum Capital Requirement

MEZZ: Mezzanine

MC: Monte Carlo

MLE: Maximum Likelihood Estimation

NatRes: Natural Resources

NAV: Net Asset Value

NPV: Net Present Value

ORSA: Own-Risk and Solvency Assessment (supervisory report)

PCF: Private Capital Fund

PCC: Pair Copula Construction

p.d.f.: probability density function

P&L: Profit and Loss

R^{NAV}: Net Asset Value Return [see Eq. (4)]

RaR: Return at Risk

RE: Real Estate

SCR: Solvency Capital Requirement

SEC: Secondaries

SGT: Skewed Generalized t (Distribution)

TFTPF: Ten Fund Test Portfolio

US-GAAP: United States Generally Accepted Accounting Principles

VaR Value at Risk

VC: Venture Capital

xR^{NAV}: Excess Net Asset Value Return [see Eq. (15)]

1 Introduction

Private equity/capital has the notorious reputation of being a quasi-mystical asset class offering the compelling combination of high return potentials associated with relatively low (systematic) risk exposures. Naturally, in the financial community, there are manifold views on this bold statement. Some research findings even consider the "alleged superiority of private equity/capital" as a big misunderstanding. So, in this paper, we want to shed light on the general question regarding the "true" return and risk profile of private capital funds (PCFs) with the striking feature of a hypothetical (risk) horizon of just one year. This involves unique analysis, since most commonly private capital research focuses on multi-year horizons to cover the overall (cash-flow based) performance of the private and thus illiquid investment-vehicles. While PCF studies with intermediate horizons (of one year) come with several straits, the biggest upsides are (a) the better comparability to publicly traded assets and (b) the compatibility to the new Solvency II framework. Solvency II is an EU-wide insurer regulation harmonizing and amending the risk management of all insurance undertakings, operating within the European Union. A key ingredient in this framework is the determination of the "Solvency Capital Requirement" (SCR), which insurance companies must hold to alleviate the risk of insolvency. In the course of this treatise, we aim at developing customized statistical risk models to assess the *appropriate* SCR for insurers' interests in PCFs.

Conveniently, the quantitative PCF risk modeling approaches, presented in this paper, are accompanied by some promising scientific findings. Our major academic contributions are

1. the investigation of PCF performance over one-year horizons by means of returns (R^{NAV}) calculated with intermediate fund valuations called Net Asset Values (NAVs),

2. the development of linear multi-factor-models, i.e. implicitly identifying risk factors, to decompose R^{NAV}'s for various PCF-types by drawing on a well-established econometric approach (and adapting it to PCF peculiarities), and

3. the comparison of historical and Monte Carlo simulation procedures to estimate the overall R^{NAV}-distribution for portfolios composed of several PCFs.

Whereas most related articles in the finance literature primarily address the performance of aggregated (private capital) asset classes, we devise methodologies with a rigorous and position-based risk management emphasis tailored for PCF investments (especially when they are recognized in insurers' balance sheets). However, the stochastic models are readily applicable to other settings besides Solvency II, as the algorithms used for simulating return distributions of mixed (alternative) investment portfolios are fairly universal. Once the comprehensive return distribution is known, the full range of risk measures like Value at Risk (VaR), volatility, etc. can be derived easily.

Section 2 provides a brief (qualitative) overview of (2.1) private capital basics (like definitions, characteristics, and terminology), and (2.2) the most relevant paragraphs of the Solvency II legislation. In Section 3 the (3.1) private capital fund and (3.2) public market data, used in analysis throughout the paper, is introduced. In Section 4 models for replicating PCF portfolio returns are developed. After defining an appropriate risk variable and measure in Section 4.1, a feasible historical PCF portfolio simulation approach is discussed in Section 4.2. Next, regression-based factor-models for various PCF-types are specified in Section 4.3, which serve as building blocks for the Monte Carlo simulation procedure presented in Section 4.4. Section 5 features mainly Monte Carlo model results, application, and examples. Section 6 finally concludes.

All data manipulations and statistical computations used for model development, analysis or simulation are implemented in the popular programming language R.

2 Private Capital in the Solvency II Framework

2.1 Alternative Investments and Private Capital

2.1.1 Taxonomy and Definitions

This thesis studies the risk of private capital funds. Unfortunately, there are no universal definitions for terms like alternative assets or private capital. It is rather common that various financial practitioners, advisors, and scientists establish their own taxonomy to categorize securities in the alternative investment (AI) universe in alignment with their specific needs and objectives. A good starting point for delineating AIs is [CABK15] Chapter 1 "What is an Alternative Investment", where several methods for identifying alternative investments are suggested. On the one hand, AIs can be defined *by exclusion*, i.e. considering "any investment that is not simply a long position in traditional investments" alternative (see [CABK15], p. 3). BlackRock, the world's largest asset manager, partly shares this (rather vague) conception, stating, that alternative investments may not be regarded as an own unique asset class; rather "alternatives represent different approaches to investing across a variety of markets and asset classes."[1] On the other hand, AI definitions *by inclusion* can be used, where all perceived AI-types have to be listed explicitly; e.g. [CABK15] determine four distinct categories

1. Real assets (including natural resources, commodities, real estate, infrastructure, and intellectual property)
2. Hedge funds (including managed futures)
3. Private equity (including mezzanine and distressed debt)
4. Structured products (including credit derivatives)

for AIs (see [CABK15], p. 4). Categorizing (alternative) investments in rigid schemes may in many cases be problematic, as every classification proposal probably lacks a certain sub-category. With the general ambiguities in mind, we define the term "Private Capital Fund" (PCF) straightforward by focusing rather on the fund investment style than the fund's underlying assets.

Private Capital Fund The term "Private Capital Fund" is used throughout this paper for funds, i.e. investment vehicles, pursuing a particular - *private equity like* - investment style, which is briefly characterized by the following points[2]:

1. LIMITED PARTNERSHIPS: Private capital funds are established as financial intermediaries between investor and investment and are commonly structured as limited partnerships. The *general partner* (GP), in most cases an asset management firm, serves as investment-fund-manager and the *limited partners* (LPs) are several (institutional) investors, who provide the bulk of the capital for the partnership's undertakings.
2. COMMITMENT: Limited partners do not deposit their share in the partnership at inception. Rather they *commit* a certain investment

[1] See BlackRock's market commentary "10 myths surrounding alternative investments" (URL: https://www.blackrock.com/au/individual/literature/market-commentary/10-myths-about-alternative-investments-2015-au.pdf).

[2] These points are depicted in [MM05] Chapter 2.1 in greater detail, where the main features of private equity funds are described nicely.

amount up front. Then each time a new investment is transacted or costs, fees, etc. have to be paid, only the required proportion of the committed capital is called to finance the expenses.

3. Contributions: These capital draw-downs are called *contributions* (to the private capital fund). As most of the investments, which are typically acquisitions of controlling stakes in private companies, are conducted in the first 4-5 years of a fund's lifetime, the main part of the contributions should occur in this period.

4. Distributions: At later times the investments are harvested. The proceeds in form of dividends, interest payments or cash from divestments are *distributed* to the limited partners as soon as practical. All investments have to be realized within the contractually limited lifetime of a private capital fund; that are typically 10 years, sometimes with a provision for extension of 2-3 years.

5. Closed-end funds: Committing capital to a private capital fund is therefore clearly a long-term investment with a self-liquidating feature.

6. Illiquidity: As there is little, if any, opportunity for redemption of contributed or even yet undrawn committed capital, the *illiquidity* of both the share in the limited partnership and the underlying "private" assets is a crucial characteristic of every private capital fund investment.

7. Net Asset Value: Assets held by a private capital fund are not registered or publicly traded on an exchange. Thus, no market value exists for those per definition. Therefore, the general partner provides subjective (maybe stale) valuations (called NAVs) of the fund's underlying securities on a quarterly basis, to inform his limited partners about the current status of fund investments.

8. Cost Structure: General partners may enjoy a lavish compensation for their investment services. Management fees often range from 1.5% to 2.5% per annum for current assets under management. Additionally, "carried interest" of typically 20% of the profits, exceeding a pre-agreed "hurdle rate", serves as an incentive for the general partner.

Hence, the fund investment style, characterized by the eight points above (and thus not by the underlying fund assets), is the acceptance criteria for determining PCFs. Concretely, there are PCFs investing in real assets, private equity, and private debt[3].

2.1.2 Valuation and Recognition

The valuation and financial recognition of (current) private capital investments may be a demanding practice for general and especially for limited partners. This problem is recognized both in the academic literature and in the relevant accounting laws. Moreover, there are several practical valuation guidelines for private equity/capital issued by different alternative investment associations.[4] They all turn on the two main questions:

1. How should the general partner determine net asset values?
2. How should the limited partner recognize his private capital investments on his balance sheet?

These two points collapse for the private capital investor to one issue: to what extent can he trust the NAV reported by the general partner. On that score [CGW10] claim in their introduction:

> "Disclosure of performance to the investor is burdened by two main difficulties. On one hand, valuation requires sufficient information on the performance of the firm, whereas on the other hand, even if sufficient information is available, PE firms may choose to disclose information strategically." (see [CGW10] p. 335)

This statement gives rise to doubts, that NAV figures may constitute a *fair value* representation of private capital investments, given the latitude the GP usually has in reckoning NAV figures. The validity concerns remain after looking into the world's two most commonly used general accounting standards, which are US-GAAP and IFRS. Alas, they do not share the same notion about using net asset values as "expedient" for accounting purposes. This is addressed explicitly in the "Basis for conclusions on IFRS 13 *Fair Value Measurement*":

[3] Further readings conveying a detailed overview of AI and PCFs are [MM05, CGW10, CABK15, BCK12].

[4] See e.g. International Private Equity and Venture Capital Valuation Guidelines (Edition December 2015) for a comprehensive treatment of "current best practice, on the valuation of private equity Investments." (URL: http://www.privateequityvaluation.com/download/i/mark_dl/u/4012990401/4625734325/151222%20IPEV%20Valuation%20Guidelines%20December%202015%20Final.pdf)

> "There are different accounting requirements in IFRSs and US-GAAP for measuring the fair value of investments in investment companies. Topic 946 Financial Services—Investment Companies in US-GAAP requires an investment company to recognize its underlying investments at fair value at each reporting period. Topic 820 provides a practical expedient that permits an entity with an investment in an investment company to use as a measure of fair value in specific circumstances the reported net asset value without adjustment. IFRS 10 Consolidated Financial Statements requires an investment company to consolidate its controlled underlying investments. Because IFRSs do not have accounting requirements that are specific to investment companies, the IASB decided that it would be difficult to identify when such a practical expedient could be applied given the different practices for calculating net asset values in jurisdictions around the world."[5]

So, IFRS regulations do not ultimately dictate under which circumstances the net asset value may be used for financial accounting, as these more or less leave the issue intentionally open. But, by implication, the IFRS accounting standard setter can imagine situations, where *at cost* valuation methodologies for private capital investments are more appropriate (than using NAVs). This *at cost* recognition would obviously be more compatible to the caution principle[6] anchored in german accounting laws. The opposite attitude towards recognition *at net asset value* is displayed in US-GAAP, as it allows this proceeding as a "practical expedient" (see: US-GAAP ASC 820). In summary, HGB, IFRS, and even US-GAAP can hardly disguise their suspicion towards reported net asset values and, unsurprisingly, this view is affirmed by academic and practical discussions.

2.2 Structure of Solvency II Risk Models

2.2.1 Solvency II Legislation

Solvency II is a project of the European Commission for a fundamental reform of insurance supervision in Europe, especially the solvency rules con-

[5] See IFRS 13 BC 238(a) (URL: https://www.casrilanka.com/casl/images/ stories/content/publications/publications/accounting_ standards/ifrs/40._ifrs_13_-_fair_value_measurement. pdf).

[6] The german *Vorsichtsprinzip* is an essential principle in the *Handelsgesetzbuch* (see: §252 1(4) HGB).

cerning equity capital adequacy of insurance and reinsurance companies. Therefore it aims for a prospective and risk-based supervisory approach. The essential Solvency II legislation consists of Directive 2009/138/EC and Commission Delegated Regulation (EU) 2015/35. These two legislative texts have to be incorporated into national laws by European Union member states by January 2016.[7]

Three Pillar Structure Similar to the Basel II banking regulations the Solvency II framework consists of three main pillars:

- Pillar 1 gives (rather quantitative) provisions for calculating risk measures for re/insurance undertakings, like the Minimum Capital Requirement (MCR) and Solvency Capital Requirement (SCR). Thereby MCR and SCR can be regarded as soft and hard floors respectively because the MCR is less demanding than the SCR as "the Minimum Capital Requirement shall neither fall below 25% nor exceed 45% of the undertaking's Solvency Capital Requirement" (see Article 129 (3) of Directive 2009/138/EC). Here, clearly, the SCR is of higher importance, as it deals with the more adverse situations. For SCR calculation each insurance company may choose between adopting the "standard formula" (see Section 2.2.2) or developing a full or partial "internal model" (see Section 2.2.3).

- Pillar 2 addresses (rather qualitative) governance and risk management standards for re/insurers. Here, the regulator requires a "regular supervisory reporting", which consist of an "Own-risk and solvency assessment supervisory report" (ORSA) among others. This ORSA report aims at ensuring a complete and holistic risk perception, exhibited from management or supervisory bodies, and therefore shall present "the qualitative and quantitative results of the own risk and solvency assessment and the conclusions drawn by the insurance or reinsurance undertaking from those results" (see Article 306 (a) of Commission Delegated Regulation (EU) 2015/35).

- Pillar 3 sets out some disclosure and transparency requirements for re/insurance undertakings.

Asset Valuation Unfortunately, the Solvency II legislation does not remedy the difficulties associated with private capital asset valuation and recognition

[7] E.g., the german "Gesetz zur Modernisierung der Finanzaufsicht über Versicherungen" entered into force as from 1. January 2016.

discussed in Section 2.1.2. Generally, the valuation standards in the Solvency II framework are similar to IFRS (see Article 9 of Commission Delegated Regulation (EU) 2015/35). This becomes exemplarily apparent in Article 75 1(a) of Directive 2009/138/EC, which specifies the fundamental asset valuation principle in the Solvency II framework, thus "assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction".

The Commission Delegated Regulation (EU) 2015/35 defines the distinct valuation methodologies more precisely. In Article 10 (7) three approaches for "alternative valuation methods" are listed, which shall be applied, if quoted market prices for a given asset are not observable:

- market approach
- income approach
- cost approach

Undeniable, this requirement is rather vague and abstract, as we can argue, that the general partner will probably use one (or more) of these methods to determine the limited partnership's net asset value. Consequently, this provision may suggest adopting private capital fund NAVs as economically reasonable expedients, i.e. value proxies, in the Solvency II framework.

Solvency Capital Requirement The calculation of the SCR can be regarded as the heart of Pillar 1, as the regulator demands in Article 100 of Directive 2009/138/EC, that the "Member States shall require that insurance and reinsurance undertakings hold eligible own funds covering the Solvency Capital Requirement." In order to ensure this precept, the "Solvency Capital Requirement shall be calibrated so as to ensure that all quantifiable risks to which an insurance or reinsurance undertaking is exposed are taken into account. It shall cover existing business, as well as the new business expected to be written over the following 12 months. With respect to existing business, it shall cover only unexpected losses. It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5% over a one-year period" (see Article 101 (3) of Directive 2009/138/EC). As mentioned above, the regulator grants insurers two options to calculate this 99,5% Value-at-Risk; either the standard formula can be applied or individual internal modeling approaches may be expended to come up with an own risk estimate.

<table>
<tr><th colspan="8">Solvency Capital Requirement (SCR)</th></tr>
<tr><td colspan="6">Basis Solvency Capital Requirement (BSCR)</td><td>Adj.</td><td>Op.</td></tr>
<tr><th>Market</th><th>Health</th><th>Default</th><th>Life</th><th>Non-Life</th><th>Intangible</th><td rowspan="7"></td><td rowspan="7"></td></tr>
<tr><td>Interest Rate</td><td rowspan="6">...</td><td rowspan="6">...</td><td rowspan="6">...</td><td rowspan="6">...</td><td rowspan="6">...</td></tr>
<tr><td>Equity</td></tr>
<tr><td>Property</td></tr>
<tr><td>Spread</td></tr>
<tr><td>Currency</td></tr>
<tr><td>Concentration</td></tr>
</table>

Tab. 1: The overall structure of the SCR standard formula.

2.2.2 Standard Formula

The essential structure of the SCR standard formula is specified in Article 103 and 104 of Directive 2009/138/EC in combination with Article 87 of Commission Delegated Regulation (EU) 2015/35. The relevant parts of the overall model structure are outlined in Table 1, to illustrate the modular approach underlying the SCR calculation. All modules are then aggregated in a next step by using predefined correlation matrices to capture diversification effects (see Annex IV of Directive 2009/138/EC).

Naturally, as we are concerned with private capital we have to take a closer look at the Market Risk module (see Article 168 of Commission Delegated Regulation (EU) 2015/35), and there especially into the Equity sub-modules, which is highlighted in Table 2. Here a distinction between Type 1 and 2 Equities becomes evident, as these two classes are exposed to different "instantaneous decreases" in the standard formula (see Article 169 of Commission Delegated Regulation (EU) 2015/35).[8] Furthermore, the residual/remainder-character of Type 2 Equities should be emphasized, as they "shall also comprise all assets other than those covered in the interest rate risk sub-module, the property risk sub-module or the spread risk sub-module, including the assets and indirect exposures referred to in Article 84(1) and (2) where a look-through approach is not possible" (see Article 168 (3) of Commission Delegated Regulation (EU) 2015/35). This Look-through approach, which rules that "the Solvency Capital Requirement shall be calculated on the basis of each of the underlying assets of collective investment undertakings and other investments packaged as funds", is one of the

[8] The instantaneous decrease shall be 39% (+ symmetric adjustment) for Type 1 Equities and 49% (+ symmetric adjustment) for Type 2 Equities.

Equity						
Type 1 Equities		*Type 2 Equities*				
Public Equity (listed in OECD or EEA)	Alternative Investment Funds	Public Equity (listed in non-OECD or non-EEA)	non-listed Equity	Commodities	other alternative Assets	all residual Assets

Tab. 2: Asset classification in the equity sub-module.

general provisions for the standard formula (see Article 84 of Commission Delegated Regulation (EU) 2015/35).

The most important insight with respect to private capital funds is their explicit subsumption under Type 1 Equities if one of the - non-restrictive - conditions of Article 168 (6) of Delegated Regulation (EU) 2015/35 is met. This classification comes with the advantage of applying the more pleasant instantaneous decreases factor of 39%. However, in some cases, the subsumption under Type II Equities (with its instantaneous decreases factor of 49%) may be more favorable for the insurer as diversification effects between Type I and II Equities are considered in the standard formula (via a Type I & II Equity correlation matrix).

2.2.3 Internal Model

Instead of insisting on the obligatory implementation of the standard formula, the regulator grants insurers the option to use their own full or partial internal models to calculate the SCR. Whereby, partial internal models can cover one or more risk modules, or sub-modules, of the Basic Solvency Capital Requirement. Moreover, the directives are quite generous with regard to the calibration of partial models:

> "Insurance and reinsurance undertakings may use a different time period or risk measure than that set out in Article 101(3) for internal modeling purposes as long as the outputs of the internal model can be used by those undertakings to calculate the Solvency Capital Requirement in a manner that provides policy holders and beneficiaries with a level of protection equivalent to that set out in Article 101." (see Article 122 (1) of Directive

2009/138/EC).

Clearly, determining an equivalent level of protection with some degree of certainty may be a highly dodgy endeavor. It remains to be seen, what legal opinion about this specific issue will prevail in the future.

3 Private and Public Data

Before preparing a Solvency II risk model, a survey of insurers' financial asset allocations should shed light on some general features and challenges disclosed in insurance companies' balance sheets. The GSAM Insurance Survey (from April 2016)[9] reveals, that Chief Investment/Financial Officers of responding insurers have the highest return expectations for private equity in 2016 and therefore many CI/FOs plan to increase investments to private equity in the next 12 months. To be more precise 26% of respondents plan to increase, 26% plan to maintain, 3% plan to decrease, and 44% do not invest at all into private equity. Thus, private equity/capital expansion is certainly manageable, since insurers' allocation to alternative investments is generally low. The National Association of Insurance Commissioners' Capital Market Special Report on the subject "U.S. Insurer Exposure to Schedule BA (Other Long-Term Invested Assets): Focus on Private Equity, Hedge Funds and Real Estate" gives an answer to the question of how much capital the average (US) insurer allocates to private capital:

> "Insurer exposure to unaffiliated investments that comprised PE [Private Equity], HFs [Hedge Funds] and RE [Real Estate] were $70 billion, $16 billion and $17 billion, respectively, at year-end 2014 (totaling $103 billion, or 1.8% of total cash and invested assets)."[10]

A look into Allianz Group's annual report for the financial year 2015 reveals investments to unlisted, i.e. private, investment funds of 9.2 billion Euro and an open commitment of 5.46 billion Euro, whereas total assets are 849 billion Euro and cash and cash equivalents totaling to 14.842 billion Euro[11]. The great majority of Allianz Group's investment assets are currently government

[9] See GSAM Insurance Survey (April 2016) (URL:https://www.gsam.com/content/dam/gsam/direct-links/us/en/institutions/2016-gsam-insurance-survey.pdf?sa=n&rd=n)

[10] See National Association of Insurance Commissioners' "Capital Market Special Report" from 4. March 2016 (URL: http://www.naic.org/capital_markets_archive/160304.htm).

[11] See (english) Annual Report 2015 - Allianz Group, pages 135 and 219. (URL: https://www.allianz.com/en/investor_relations/results-reports/annual-reports)

and corporate bonds. All in all, PCFs are yet appreciated with a minority status within insurance undertakings, by now; but the importance appears to be rising.

3.1 Prequin Cash Flow Data on Fund Level

We use Prequin (US-Dollar cash flow) data on fund level as of October 2015 for all subsequent PCF analysis, which contain the following information about private capital funds:

- Fund ID
- Firm ID
- Vintage
- Category Type (i.e. investment target/style)
- Fund Status (i.e. still active or yet liquidated)
- Fund Size (in fund currency and in USD)
- Fund Focus (i.e. geographic/regional focus with levels: US, Europe, and Rest of World)
- Transaction Date
- Transaction Category (with levels: Call, Distribution, and [Net Asset] Value)
- Transaction Amount
- Cumulative Contribution
- Cumulative Distribution
- Net Cash Flow
- Fund Name
- Firm Name

Data Cleaning and Editing

In the first data processing step, we aim to detect and remove funds with large “data gaps”, i.e. long time periods without data updates. Especially with old vintages, there are quite some funds without NAV data for several years. These funds are obviously not suitable for further analysis, as ideally NAVs should be reported continuously in a quarterly interval throughout a fund’s lifetime. Moreover, we extract the first reported transaction dates (of cash flow and NAV) for each fund and check, if first reported NAV date, first reported cash flow date, and reported vintage year exhibit irregularities or inconsistencies. Another task is the determination of each fund’s “death date”, provided that the fund in question is “economically dead”. The definition of living vs. dead funds is anything but straightforward and it is considerable that this distinction is only possible through the application of subjective - more or less restrictive - decision rules. An obvious candidate for a fund’s death date is the point in time where the NAV (and the callable open commitment) falls below a specified threshold. But unfortunately, some dubious (or even flawed) entries in the data set can be observed, which prevent easy solutions for the death date problem from delivering the desired “correct” classification in all cases.

After cleaning the data set, we map the Prequin Category Types to the Asset Metrix Type (AMT) taxonomy, which aggregates Prequin’s 28 levels to 10 levels of AMTs[12].

Summary of Data Set after Cleaning and Editing

The final data set used for further analysis and modeling contains 2,977 different funds. Thereof 419 funds may be regarded as economically dead[13], although just 398 funds are classified as officially “liquidated” in the Fund Status information. The geographical decomposition of funds, indicated by the Fund Focus variable, overweights US-based funds with 2,270 entries; by contrast, Europe exhibits 424 and Rest of World just 283 funds. The distribution of vintages clearly reflects the growing number of private capital funds over time, as the data set contains 41 funds with vintages between 1984 and 1989, 429 funds with vintages in the 1990’s, 1411 funds with vintages in the 2000’s, and 896 funds with vintages between 2010 and 2015.

[12] AMT abbreviations: BO = Buy Out, VC = Venture Capital, RE = Real Estate, FOF = Fund of Funds, DD = Distressed Debt, NatRes = Natural Resources, MEZZ = Mezzanine, SEC = Secondaries, Infra = Infrastructure.

[13] Decision rule, e.g. NAV/Commitment-ratio below 5% threshold for funds with age > 10 year.

The predominant Asset Metrix Type levels are by far Buy Out and Venture Capital, as this two types represent more than half of all funds in the data set: Buy Out (994 funds), Venture Capital (723), Real Estate (374), Fund of Funds (337), Distressed Debt (141), Natural Resources (100), Mezzanine (99), Secondaries (80), Infrastructure (70), and Other (59). Table 3 ultimately provides an aggregated overview of the overall distribution of private capital funds in the data set.

3.2 Public Market Data

The public market data required for eventual PCF risk models are summarized in the table below. Alas, no suitable data for commodities nor infrastructure investments could be detected.

Asset Class	Region	Factor	Data	Currency	Source
Public Equity	World	Return	[FF93]: Rm-Rf	USD	mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
	US	Return	NASDAQ 100	USD	www.quandl.com/data/BCB/7847
	US	Return	[PS03]: Traded 10-1 liquidity portfolio	USD	faculty.chicagobooth.edu/lubos.pastor/research/
Fixed Income	US	Spread	OA High-Yield Spreads (corporate)	USD	www.quandl.com/data/ML/HYOAS
	US	Yield	Treasury Yield Curve Rate (one-year)	USD	www.quandl.com/data/USTREASURY/YIELD
	World	Return	Vanguard Total Bond Market Index	USD	www.quandl.com/data/YAHOO/FUND_VBMFX
Real Estate	US	Return	Real Estate Price Index (commercial)	USD	www.quandl.com/data/FED/FL075035503_Q
Commodities	na	na	na	na	na
Infrastructure	na	na	na	na	na

Generally, public market data availability (especially for high-yield spreads and real estate indices) is considerably better for the United States than for Europe or Rest of World. Therefore, all analysis (throughout this paper) are conducted from a US-Dollar PCF investor's perspective. This also allows us to ignore tedious currency issues since the Prequin data set does not contain information on the transaction currency of individual PCF underlying investments anyway. Guessing the "fund currency", which has to be a

Vintage	Total	by AMT			by AMT & Region			
		BO	VC	all other	BO.US	BO.EU	VC.US	VC.EU
1984	1	1	0	0	1	0	0	0
1985	7	2	5	0	2	0	4	1
1986	9	1	6	2	1	0	6	0
1987	8	5	2	1	5	0	2	0
1988	10	5	4	1	5	0	4	0
1989	6	2	4	0	2	0	4	0
1990	14	4	8	2	3	1	7	1
1991	7	1	4	2	1	0	3	0
1992	23	9	9	5	9	0	8	1
1993	22	10	9	3	8	2	8	0
1994	35	18	11	6	16	2	11	0
1995	34	14	15	5	11	2	14	1
1996	49	21	16	12	18	1	15	1
1997	66	26	23	17	20	4	20	2
1998	92	43	33	16	32	11	31	0
1999	87	31	40	16	25	5	38	1
2000	145	43	85	17	38	5	75	4
2001	102	25	46	31	19	5	41	5
2002	75	24	28	23	19	5	23	4
2003	61	18	19	24	15	3	17	1
2004	93	28	30	35	24	3	29	0
2005	157	59	34	64	38	13	28	4
2006	209	75	42	92	55	16	41	0
2007	238	82	56	100	46	20	43	4
2008	223	77	40	106	45	21	33	0
2009	108	35	16	57	18	12	13	2
2010	155	46	20	89	30	7	20	0
2011	218	61	25	132	25	17	19	3
2012	201	64	25	112	43	12	19	4
2013	246	73	28	145	42	13	23	4
2014	222	72	32	118	46	17	24	5
2015	54	19	8	27	9	6	4	3

Tab. 3: Summary of final private capital fund data set

mixture of underlying fund investment currencies, on the basis of Prequin's Fund Focus information may do more harm than, for the sake of simplicity, assuming US-Dollar as the worldwide basis currency[14]. In the Solvency II framework, there is, fortunately, an appropriate currency risk sub-module considering foreign exchange on both the asset and the liability side of an insurance company's balance sheet, which confirms our approach of neglecting currency effects. So all subsequent analysis/models exhibit a global orientation towards a US-Dollar investor, although Solvency II is an exclusive project of the European Union. As 2,270 out of 2,977 funds in the Prequin data set display a US region label, the US-Dollar focus still may be judicious for public market data.

4 Replicating the Portfolio Risk

Risk models for private capital funds have to capture all asset class peculiarities to guarantee sound risk estimates. Consequently, standard textbook solutions for public equity/debt are not suitable without major adjustments. To get a good first impression of the challenges associated with the risk measurement of private capital funds, industry associations (may) offer valuable insights in their publications. The European Private Equity and Venture Capital Association (EVCA) defines in its "Risk Measurement Guidelines" from January 2013[15] six criteria for a risk model for portfolios of private equity funds. According to that, the model should be (1) complete, (2) unbiased, (3) monotonic, (4) observable, (5) reconcilable, and (6) interrelated to ensure a reasonable assessment of the degree of uncertainty inherent in future cash flows and returns (of private equity). Further, four distinct types of risks are specified in these guidelines; correspondingly limited partners have to consider (a) funding, (b) liquidity, (c) market and (d) capital risk for their private capital investments. With this in mind, a scientific risk engineer needs to select the appropriate methodology in the planning stage of her risk model, which is best aligned with the *relevant* points mentioned above. In the Solvency II context, e.g. funding and illiquidity risk may not be that important, as big insurance undertakings ideally have more than sufficient liquidity to meet all capital calls and, secondly, have long enough investment horizons (and relatively small private capital exposure) to avoid forced secondary sales of private capital interests.

[14] If we assume $\mathbb{E}\left(\triangle \mathsf{FX.rate}\right) = 0$ for each analysis date of e.g. regression models, a potential FX-bias averages out over sufficiently long analysis horizons.

[15] URL: http://www.investeurope.eu/media/10083/evca-Risk-Measurement-Guidelines-January-2013.pdf

Slightly distinct conceptions of private equity risk exist in the academic literature. The main focus there is clearly on the evaluation of the attractive of the risk/return profiles of Buy Out or Venture Capital fund investments to finally appraise the entire asset class. This results in most cases in an alpha/beta-decomposition of private equity returns on the basis of cash flow data. Therefore, customized generalized method of moments (GMM) or maximum likelihood estimation (MLE) methods are developed (and applied) in these papers, as the classical time-series OLS regression approach used in public equity is not feasible (see e.g. [DLP12] for a GMM- and [Coc05] for a MLE example).

In the Solvency II context, the focus is on market risk, which is measured by means of 99.5% Value-at-Risks (VaRs) with a one-year horizon. In order to determine VaRs with our own unique approaches, we define appropriate performance and risk measures in Section 4.1. In Section 4.2 a feasible historical PCF simulation method is introduced and critically analyzed. Section 4.3 develops regression-based linear multi-factor models on portfolio and single fund level; many challenges and remedies have to be discussed throughout the paragraph. Ultimately, the single fund level models of Section 4.3.2 serve as fundamental building blocks for the construction of Monte Carlo simulation prototypes in Section 4.4.

4.1 Defining an Appropriate Risk Variable and Measure

The first step in the risk modeling of private capital funds is the determination of the dependent variable whose risk should be assessed. This involves answering the following three questions:

1. Question: Which horizon?
2. Question: Which performance measure?
3. Question: Which risk measure?

In the private capital (fund) context, this task is not that unsophisticated as it might seem. Particularly, certain (private capital specific) assumptions need to be established in advance, in order to attain suitable conclusions. The most substantial one affects the notion of reported net asset values. Here we suppose that private capital investors - and especially insurers - use these NAVs for fair-value accounting purposes. This view implicitly vindicates a *market-value like* treatment of NAVs (in e.g. return calculation formulas). As mentioned in Section 2.1.2 this perception is controversial but reported net asset values might nevertheless be the most expedient estimator of the present value of a private capital fund's future cash flows. Clearly, this estimator exhibits large-scale staleness[16] as general partners update NAVs only on a quarterly basis.

Another field of conjectures is the re-investment and finance assumptions inherent in various performance calculation approaches, which become essential and important particularly in multi-year horizons. These assumptions simply attempt to incorporate *dynamic* elements into *static* performance measurement schemes. Apparently, such easy solutions are limited in their ability to model more complex dynamics. However, if the (risk or performance measurement) horizon is short enough, active re-investment/finance assumptions become negligible.

Horizon

As outlined in Section 2.2.1, the Solvency Capital Requirement "shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5% over a one-year period" (see Article 101 (3) of Directive 2009/138/EC). So we outright constitute a risk measurement horizon of one year or respectively four quarters, i.e. $h = 4q$, for the remainder of this paper[17].

Performance Measure

In this section, we establish a link between the two most commonly used performance measures in the private capital universe; these are the Total Value to Paid In ratio (TVPI) and the Internal Rate of Return (IRR). Hence, the concept of "NAV-Return" (R^{NAV}) is introduced and the adaption of all performance measures to a one-year horizon is outlined concisely. In Appendix A additional PCF performance measures are discussed.

At first, we define the vectors of contribution cash flows

$$C = \left[C_{t_0}, C_{t_1}, C_{t_2}, ..., C_{t_n}, ..., C_{t_{N-1}}, C_T\right]$$

[16] The stale valuation problem in the private capital context is addressed explicitly by Susan E. Woodward from Sand Hill Econometrics in [Woo09] (URL: http://www.sandhillecon.com/pdf/MeasuringRiskForVentureAndBuyouts.pdf).

[17] In general, there are several considerations regarding the choice of suitable Value at Risk parameters (horizon and confidence level), which are summarized in [MFE05] on page 42. With respect to horizon, they remark that "the risk-management horizon $\triangle$ should reflect the time period over which a financial institution is committed to hold its portfolio."

and distribution cash flows

$$D = \left[D_{t_0}, D_{t_1}, D_{t_2}, ..., D_{t_n}, ..., D_{t_{N-1}}, D_T\right].$$

Thus all cash flows occur at discrete points in time and

$$t \in [t_0, t_1, t_2, \ldots, t_n, \ldots, t_{N-1}, T]$$

can be interpreted as fund age, whereby variable N represents the amount of equidistant periods between fund start at t_0 and fund end at $t_N = T$. Moreover, the fund's net asset value can be assigned for each age t. To ease the time notation we allow to define dates with e.g. $t_n - 4q + 1d$, which means the date four quarters before t_n plus one day.

Total Value to Paid In Ratio (TVPI) The TVPI at time t_n is calculated as the ratio between the sum of current net asset value and cumulative distributions in the numerator and the cumulative contributions in the denominator:

$$TVPI_{t_n} = \frac{NAV_{t_n} + CD_{t_n}}{CC_{t_n}} = RVPI_{t_n} + DPI_{t_n} = \frac{NAV_{t_n}}{CC_{t_n}} + \frac{CD_{t_n}}{CC_{t_n}} \quad (1)$$

Obviously, the TVPI (formula) can also be split into its two components Residual Value to Paid In (RVPI) and Distributions to Paid In (DPI). Cumulative quantities are computed straightforward as the sum of all distribution- or respectively contribution- cash flows up to time t_n, i.e. $CD_{t_n} = \sum_{i=0}^{n} D_{t_i}$ or respectively $CC_{t_n} = \sum_{i=0}^{n} C_{t_i}$. Next, a decomposition of the TVPI formula may offer some insights with respect to the ratio's interpretation:

$$\begin{aligned} TVPI_{t_n} &= \frac{NAV_{t_n} + \sum_{i=0}^{n} D_{t_i}}{\sum_{i=0}^{n} C_{t_i}} \\ &= \frac{\sum_{i=0}^{n} C_{t_i} + \sum_{i=1}^{n} (\triangle NAV_{t_i,1} + \triangle NCF_{t_i,1})}{\sum_{i=0}^{n} C_{t_i}} \\ &= 1 + \frac{\sum_{i=1}^{n-1} (\triangle NAV_{t_i,1} + \triangle NCF_{t_i,1}) + (\triangle NAV_{t_n,1} + \triangle NCF_{t_n,1})}{\sum_{i=0}^{n-1} C_{t_i} + C_{t_n}} \end{aligned} \quad (2)$$

where the change of net asset value $\triangle NAV$ and the change of net cash flow $\triangle NCF$ are calculated as

$$\triangle NAV_{t_n,x} = NAV_{t_n} - NAV_{t_{n-x}}$$

$$\triangle NCF_{t_n,x} = \sum_{i=n-x}^{n} (D_{t_i} - C_{t_i})$$

with x being e.g. the performance/risk measurement horizon.

In the last expression of Eq. (2) only the new period "value-creation term" $(\triangle NAV_{t_n,1} + \triangle NCF_{t_n,1})$ and the new contribution term C_{t_n} for the period between t_{n-1} and t_n are unknown a priori at time t_{n-1}. This illustrates the fact, that the TVPI ratio at time t_n reflects the comprehensive fund performance since inception and thus incorporates much old, i.e. already known at time t_{n-1}, information. TVPIs are usually calculated as well for single private capital funds as for portfolios of private capital funds, but in both cases rather for benchmarking than for risk measurement/management purposes.

Internal Rate of Return (IRR) The second standard metric to evaluate private capital fund performance is the IRR, initially used in capital budgeting to measure and compare the profitability of investments. Computationally this concept is more complicated, as an internal rate of return is a discount rate that makes the net present value (NPV) of all net cash flows from a particular fund (or a portfolio of funds) equal to zero. Therefore IRR calculation is equivalent to the following root finding problem:

$$NPV_{t_N} := \sum_{i=0}^{N} \frac{D_{t_i} - C_{t_i}}{(1 + IRR_{t_N})^{t_i}} \stackrel{!}{=} 0 \tag{3}$$

Generally, numerical methods have to be applied to find solutions to Eq. (3). However, there are situations where no IRR at all or, vice versa, multiple IRRs fulfill Eq. (3). Furthermore, as the summation in Eq. (3) involves all N time-steps, the conclusive and terminal IRR can only be determined ex-post. And there is yet another critique or drawback of the internal rate of return since the IRR's economical interpretation is not always straightforward - especially in the context of mutually exclusive projects - and may cause confusing and fallacious conclusions on a portfolio level.

NAV Return Both performance measures, introduced so far, describe the overall or cumulative performance from fund inception t_0 to date t_n or t_N. Though, the risk measurement - and therefore the performance measurement - horizon was set to one year (or respectively four quarters) beforehand. The derivation of intermediate performance metrics requires the incorporation of the net asset values at period, i.e. horizon, start and end. One possible, obvious formula for a one-year "NAV Return" from time t_{n-4q} to time t_n is:

$$\begin{aligned} 1 + R_{t_n}^{NAV} &= \frac{NAV_{t_n} + \sum_{i=n-4q+1d}^{n}(D_{t_i} - C_{t_i})}{NAV_{t_{n-4q}}} \\ &= \frac{NAV_{t_n} + \triangle NCF_{t_n,4q}}{NAV_{t_n} - \triangle NAV_{t_n,4q}} \\ &= \frac{NAV_{t_{n-4q}} + (\triangle NAV_{t_n,4q} + \triangle NCF_{t_n,4q})}{NAV_{t_{n-4q}}} \end{aligned} \tag{4}$$

In the last expression of Eq. (4) only the value-creation term for the new year is unknown a priori. This R^{NAV} formula has the (remarkable) property that it can generate returns of less than -100% in situations with e.g. large contributions in the respective period. On the other hand, $R_{t_n}^{NAV}$ can get very big in cases with large distributions in a given period. Despite these peculiarities, this specific NAV return construct[18] is the most natural PCF performance measure (for intermediate, one-year horizons) in the Solvency II context, as it exhibits the right denominator and the most reasonable numerator for our risk management purposes (see Eq. (6)).

All approaches mentioned above (and in Appendix A) can be extended straightforward onto (total) portfolio level, where other non-PCF "sources of cash flows" may be incorporated to calculate portfolio returns. However, the multiplicative conversion between different period lengths

$$(1 + R_{yearly}) \neq (1 + R_{quarter1})\cdot(1 + R_{quarter2})\cdot(1 + R_{quarter3})\cdot(1 + R_{quarter4})$$

generally does not hold for performance measures like R^{NAV}.

Risk Measure

For a third and final time, we cite that the Solvency Capital Requirement "shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5% over a one-year period" (see Article 101 (3) of Directive 2009/138/EC).

[18] In Section "Other Approaches to Estimate Risk and Abnormal Return - Using Intermediate NAVs" [Pha10] uses the equivalent of Eq. (4) to derive quarterly excess returns for his NAV-based regression approach.

Value at Risk Value at Risk (VaR) is the most common statistical approach to quantify the current downside risk of a financial position or portfolio (as a capital adequacy requirement). [Jor06] defines Value at Risk on page 106 as "the worst loss over a target horizon such that there is a low, pre-specified probability that the actual loss will be larger." To be more precise, one can mathematically define VaR at confidence level $\alpha \in [0,1]$ (in our case $\alpha = 0.995$) for a given horizon h as

$$VaR_\alpha (L_h) = \inf \{l \in \mathbb{R} : P(L_h > l) \leq 1 - \alpha\} \tag{5}$$

or alternatively, using the generalized inverse distribution function $F_{L_h}^{-1}(\alpha)$, as

$$VaR_\alpha (L_h) = F_{L_h}^{-1}(\alpha) := \inf \{l \in \mathbb{R} : F_{L_h}(l) \geq \alpha\}$$

with $F_{L_h}(l) = P(F_{L_h} \leq l)$ (see [MFE05], p. 38). Obviously, our relevant loss distribution has to be derived with the return formulas introduced in the previous section. A loss for a given horizon h is simply

$$L_h := V_{t_n - h} - V_{t_n} = -V_{t_n - h} \cdot R_h$$

the difference of the fund/portfolio values V at time t_n and t_{n+h} or - more conveniently for our case - the negative of the product of V_{t_n} and the fund (or portfolio) return over the predefined horizon[19]. Here $V_{t_{n+h}}$ and R_h are unknown quantities at time t_n. Thus they are perceived as random variables in the VaR framework, which makes L_h a random variable, too. The determination or derivation of L_h's cumulative distribution function (c.d.f.) is eventually the core exercise in VaR calculation (or estimation).

In the PCF context, we derive a "private capital loss function" by setting $R_h = R^{NAV}$ and $V = NAV$

$$L_h = -NAV_{t_n - h} \cdot R^{NAV} = -\left(NAV_{t_n} + \triangle NCF_{t_n,h} - NAV_{t_n - h}\right) \tag{6}$$

Otherwise, we can explicitly modify the VaR formula for our specific risk measure variable, which is one-year R^{NAV}, and the prescribed $\alpha = 99.5\%$ and $h = 4q$ parameters. This yields a newly established Return at Risk (RaR) measure

$$RaR_{t_n} = \inf \left\{r \in \mathbb{R} : P\left(R_{t_n}^{NAV} \leq r\right) \geq 0.5\%\right\} \tag{7}$$

[19] According to this definition, losses are positive and gains are negative.

which can be used to derive the VaR in a subsequent step

$$\begin{aligned} VaR &= -RaR_{t_n} \cdot V_{t_n-h} \\ &= -F^{-1}_{R^{NAV}_{t_n}}(0.5\%) \cdot V_{t_n-h} \\ &= -\inf\left\{r \in \mathbb{R} : P\left(R^{NAV}_{t_n} \leq r\right) \geq 0.5\%\right\} \cdot V_{t_n-h} \end{aligned}$$

where $F^{-1}_{R^{NAV}_{t_n}}$ is the generalized inverse distribution function. Note that Eq. (7) calculates the Return at Risk for time t_{n-4q}, i.e. four quarters before date t_n, or respectively makes a "quantile forecast" for date t_n, which is due to the "backward" definition of $R^{NAV}_{t_n}$ in Eq. (4). Furthermore the one-year horizon ($h = 4q$) is already directly incorporated in Eq. (4) and therefore needs no extra consideration in Eq. (7).

Parametric vs. Non-parametric VaR Estimation There are two possible ways to estimate VaRs or - more generally - quantiles of distributions. In a parametric approach the distribution of the random variable of interest, i.e. R^{NAV} in our case, is assumed to belong to a parametric family. Thus once the most appropriate family is identified, the parameters associated with that distribution have to be estimated (incorporating methods like e.g. method of moments, maximum likelihood, etc.).

Conversely, a non-parametric approach is more general as it does not make any assumption about the shape of the random variable's distribution. Rather the respective historical/empirical distribution is used directly to estimate quantile or VaR figures or for more complex cases simulation approaches are applied to generate an empirical cumulative distribution function. Anyway, both approaches implicitly follow the notion that the past is predictive about the future and, as stated in [Jor06] on page 105, "that the current portfolio is "frozen" over the horizon, like all traditional risk measures, and combines current positions with the uncertainty in the risk factors at the end of the horizon."

Non-subadditivity Property of VaR Despite the high popularity of VaR among risk managers and regulators, there are some fundamental critiques concerning the approach. The most severe (critique) is the fact that VaR is not a coherent risk measure in the sense of [ADEH99]. In their seminal paper, they define a *coherent risk measure* as a function that satisfies the four axiomatic properties of (a) translational invariance , (b) sub-additivity, (c) positive homogeneity, and (d) monotonicity. Unfortunately, VaR does (in general) not meet the desirable feature of sub-additivity, which aims at the risk-reducing nature of diversification, i.e. the idea that the aggregation of financial positions to a portfolio does not increase risk or as outlined in [ADEH99] that “a merger does not create extra risk”.

Conditional Value at Risk Another critique is that VaR does not say anything about the shape of losses in excess of the “VaR threshold”; albeit these regions are naturally most critical in the risk management context. So in order to shed light on the distributional tail area beyond the VaR, the concept of Conditional Value at Risk[20] (CVaR) is introduced

$$CVaR_\alpha(L_h) = E(L_h | L_h \geq VaR_\alpha(L_h)) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L_h)\,du$$

which is the expected value of losses that exceed VaR or the average VaR of all levels $u \in [\alpha, 1]$ (see [MFE05], p. 44).

Conveniently CVaR resolves the non-subadditivity critique of VaR, too, as it satisfies all four coherence axioms. So reporting CVaR in addition to VaR seems generally a good idea to convey a more comprehensive risk assessment.

Value at Risk Decomposition Further portfolio managers may be especially interested in the (value at) risk contribution of particular portfolio components to the total portfolio (value at) risk. Therefore [Jor06] presents in Chapter 7.2 three Value at Risk tools:

- Marginal VaR: “the partial (or linear) derivative with respect to the component position” (beta factors from risk-factor regression),
- Incremental VaR: “evaluate the total impact of a proposed trade on the portfolio”, and
- Component VaR: “additive risk decomposition of portfolio that recognizes the power of diversification”.

4.2 Historical Portfolio Simulation

In this section, we describe a way to simulate the return distribution for a prespecified portfolio consisting of several private capital funds and optionally

[20] Also known as Expected Shortfall, Expected Tail Loss, Tail Conditional Expectation, Tail Value at Risk or Average Value at Risk. See e.g. [ADEH99], [FW15], or [MFE05] Chapter 2.2 and 6.1 for a more detailed consideration of (coherent) risk measures.

public (equity, fixed income, ...) indices *directly from historical data*[21]. Much effort has to be expended to demonstrate a variety of difficulties associated with historical simulations in the PCF context, which turn out to be virtually intractable. Nevertheless, this section features certainly vital lessons learned for further PCF risk-modeling approaches.

The most basic one-year portfolio return formula (for a mixed portfolio of K components) incorporates the R^{NAV} concept for private capital funds and standard percentage changes for the public indices

$$R_{PF} = \mathbf{w}^T\mathbf{R} = \sum_{k=1}^{K} w_k \cdot R_k \tag{8}$$

where the (return-period start-NAV-scaled) weight of the k-th position is given by

$$w_{k,nav} = \frac{NAV_{k,t_{n-4q}}}{\sum_{j=1}^{K} NAV_{j,t_{n-4q}}} \tag{9}$$

and the period t_{n-4q}, t_n return R_k of the k-th element is calculated with Eq. (4) for private capital funds and with

$$R_k = \frac{NAV_{k,t_n}}{NAV_{k,t_{n-4q}}} - 1$$

for public (performance) indices. In the case of publicly traded assets the NAV equals simply public market values. This method for calculating portfolio returns is equal to

$$1 + R_{PF} = \frac{\sum_{k=1}^{K} (NAV_{k,t_n} + \triangle NCF_{k,t_n,4q})}{\sum_{k=1}^{K} NAV_{k,t_n-4q}} \tag{10}$$

calculating the portfolio return directly with aggregated quantities in Eq. (4).

Historical Simulation Idea Estimating the return distribution of a given portfolio via a historical simulation method corresponds basically to randomly sample many returns out of the pool of all (historically) feasible returns of *similar* portfolios, which should be uniformly distributed over time.

[21] See [Jor06] Chapter 10.4 “Historical Simulation Method”, [MFE05] Chapter 2.3.2 “Historical Simulation”, [Ale08] Chapter VI.3 “Historical Simulation”, or [Dow02] Chapter 4 “Non-parametric VaR and ETL” for textbook literature on (public market) historical Value at Risk simulation methods.

4.2.1 Portfolio Similarity and Optimal Scaling

However, it is quite challenging, or virtually intractable, to construct or find just one *similar* "historically feasible" portfolio consisting of private capital funds, regardless of how similarity is defined in the specific case. Someone could e.g. demand matching private capital fund attributes with respect to

- Fund Age,
- Fund Type,
- Fund Region,
- Current NAV (in % of Commitment),
- Current Cumulative Distributions (in % of Commitment), and
- Current Cumulative Contributions (in % of Commitment)

to consider two funds (as) similar. This reasonable, six-dimensional criterion leads directly to a paragon for a problem called the *curse of dimensionality.* Our data set, summarized in Table 3, is by far too meager to make this similar portfolio sampling approach viable.

So, as an expedient, we reduce the criterion's dimension to one and demand only Fund Age to be equal; i.e. just the age structure of the actual and the sampled PCF portfolio have to be (approximately) the same. But as a consequence of this rather loose similarity condition, another problem regarding the weights in the portfolio return formula arises. Since the return period start NAVs of our sampled funds may be totally different than the actual funds' NAVs, the portfolio weights (calculated with Eq. (9)) of the actual and the historically sampled funds may deviate substantially and in a random manner.

Clearly, the "right", i.e. actual, weights can be computed with Eq. (9) using actual portfolio NAVs. Likewise, the historical NAV return can be calculated easily. Though combining these two quantities may yield inconsistent and unreasonable results as

$$R_{PF.sim} \coloneqq (\mathbf{w}_{actual})^T \, \mathbf{R}_{sampled} \neq (\mathbf{w}_{sampled})^T \, \mathbf{R}_{sampled}$$

Particularly, situations with a relatively small, sampled return period start NAV seem problematic as a smaller denominator in Eq. (4) makes the NAV Return more volatile. In a realistic portfolio, volatile NAV Returns are weighted with small weights per definition of Eq. (9); but if we combine actual weights with sampled returns this mechanism is not in charge. As a result, simulated PCF portfolio returns are not/hardly "historically feasible", if we use actual weights.

A natural, but precarious workaround is modification or scaling of weights; e.g. commitment scaling for PCFs

$$w_{k,com} = \frac{\text{Commitment}_{k,t_{n-4q}}}{\sum_{j=1}^{K} \text{Commitment}_{j,t_{n-4q}}} \tag{11}$$

albeit this method is only directly applicable, if there are only PCFs in the given portfolio; the integration of public assets can only be achieved in a subsequent step

$$\begin{aligned} R_{PF} &= x_{private} \cdot R_{private} + x_{public} \cdot R_{public} \\ &= x_{private} \cdot \left(\mathbf{w}_{com}^{T} \mathbf{R}\right) + x_{public} \cdot \left(\mathbf{w}_{nav}^{T} \mathbf{R}\right) \end{aligned} \tag{12}$$

whereby the commitment-weighted private (capital funds) portfolio return $R_{private}$ can be equivalently calculated via a scaled version of Eq. (10)

$$1 + R_{private} = \frac{\sum_{k=1}^{K} \left[s_k \cdot (NAV_{k,t_n} + \triangle NCF_{k,t_n,4q})\right]}{\sum_{k=1}^{K} (s_k \cdot NAV_{k,t_n-4q})} \tag{13}$$

with commitment-scaling parameter

$$s_k = \frac{\text{Commitment}_{k,actual}}{\text{Commitment}_{k,sampled}} \tag{14}$$

In summary, we recognize, that the exact replication of a given actual portfolio is not viable with historical simulation methods. Nevertheless, the approximative, i.e. un-replicative, simulation of historically feasible portfolios (via commitment-scaling) might offer valuable insights; even if, historical simulations are usually unconditional as they do not incorporate conditional (e.g. public market) information in the base case.

4.2.2 Simulation Algorithm

If we accept that a historical simulation technique is merely capable of estimating approximative and unconditional distributions, then the following procedure can be applied to generate a distribution for $R_{private}$, which is the R^{NAV} of a portfolio consisting of K private capital funds:

Iterate N-times

1. Sample Analysis Date, i.e. period start date $t_n - 4q$

2. Sample K different Fund IDs which satisfy the Fund Age similarity criterion[22]

3. Determine weights by Eq. (11) or scaling parameter by Eq. (14)

4. Calculate $R_{private}$ via Eq. (12) or Eq. (13)

to generate N historically feasible PCF portfolio returns.

This algorithm yields a vector $\mathbf{R}_{private}^{Hist.Sim.} = [R_1, R_2, \ldots, R_N]$ of N simulated PCF portfolio returns, which constitutes the empirical cumulative distribution function for a given simulation run. Given the fund data from the Prequin set, we allow the period start date to be in the interval

$$t_n - 4q \in [1995Q1, 1995Q2, \ldots, 2014Q3]$$

and the period end date in

$$t_n \in [1996Q1, 1996Q2, \ldots, 2015Q3]$$

Further, we assume the Analysis Dates to be uniformly distributed in our sampling scheme, which makes our historical simulation equal-analysis-date-weighted and therefore unconditional[23]. Our similarity criterion demands exact matching of Fund Ages (in years) for ages below 10 and for ages of 10 and above we just demand Fund Ages to exceed that threshold. The determination and calculation of weights, scaling parameters and returns is then straightforward.

In a subsequent step, we might be interested in the determination of excess returns, i.e. the difference between a given return and a benchmark return, often used in econometric models. The excess NAV return (exceeding one-year risk-free rate $r^{riskfree.1year}$) on portfolio level can be estimated in our case by two methods

$$\mathbf{w}_{com}^T \left(\mathbf{R} - \mathbf{r}_{currency}^{riskfree.1year} \right) \approx R_{private} - r_{average}^{riskfree.1year}$$

[22] Note, that the "pool sizes of similar funds" are quite heterogenous for different analysis dates and for old fund ages we may sample dead funds!

[23] Our procedure allows "overlapping periods", as all 4 quarters may be sampled in one simulation run. In public market applications, this specification would cause an autocorrelation problem, which leads to an underestimation of risk. However, due to the high idiosyncratic risk in our PCF simulation context, this feature is hardly detectable/apparent in our setting.

Type	Focus	Age.year	Age.week	CC	CD	NAV	Comm
BO	US	1	23	1.4	0	1	10
FOF	ROW	2	75	10	2	8	20
NatRes	EU	3	129	5.5	5	7	5
VC	US	4	181	14.7	3	14	15
BO	ROW	5	233	12	13	2	12.5
RE	EU	6	285	8	2	5	15
DD	US	7	337	1	0	1	10
Infra	ROW	8	389	9.5	3	11	10
VC	EU	9	441	10.5	9	4	10
MEZZ	US	10	493	10	4	5	10

Tab. 4: Ten Fund Test Portfolio

namely, on the one hand the commitment-weighted sum of excess returns on PCF level or on the other hand the difference between the PCF portfolio return and an average (currency weighted) one-year (government bond) yield curve rate. For the rest of the paper, we generally abbreviate excess NAV returns with xR^{NAV} in the text, but denote it with Y in mathematical (regression/simulation linked) formulas, to emphasize the dependent variable character in the regression analysis context:

$$R^{NAV} - r^{riskfree} := xR^{NAV} = Y \tag{15}$$

4.2.3 Ten Fund Test Portfolio

In this section, we apply the historical simulation procedure described above to a test portfolio of ten PCFs (the so-called Ten Fund Test Portfolio, abbreviated TFTPF), which composition is outlined in Table 4. We have to sample 10,000 + X times from the Prequin data set to obtain 10,000 valid NAV returns of PCF portfolios with a similar age structure like the ten fund test portfolio[24]. So not every draw, i.e. simulation iteration, results in a (valid) NAV return figure; this is due to situations, when we sample one or more "dead" Fund IDs. If we simply weight dead funds with weight zero, then again portfolio similarity is not fulfilled. Hence the X void draws are a necessary evil with our straightforward sampling algorithm.

[24] A conservative estimate for the total number of (distinct) historical feasible PCF returns obtainable for the ten fund test portfolio with our historical simulation approach might be about 250 million (78 analysis dates x 10! possible portfolios for each analysis date).

After 18,234 seconds or a little bit more than 5 hours, the code, implemented in R, returned a vector with 10,000 NAV returns. The consequential xR^{NAV} distribution is visualized in Figure 1 and summarized in the following table:

mean	median	st.dev.	skew	kurt	min	max
0.134	0.107	0.322	-	-	-0.599	4.694
Q0.1%	Q0.5%	Q1%	Q2%	Q5%	Q10%	Q25%
-0.531	-0.419	-0.364	-0.310	-0.237	-0.158	0.001
Q75%	Q90%	Q95%	Q98%	Q99%	Q99.5%	Q99.9%
0.214	0.365	0.522	0.870	1.429	2.011	3.429

Briefly worded, xR^{NAV}'s are positively skewed and leptokurtic. Mean (13.4%), median (10.7%), standard deviation (32.2%), 25% quantile (0.1%), and 75% quantile (21.4%) all lie within expectations and seem to confirm private capital's high-risk/high-return reputation. Naturally, the interesting regions are the right and especially - for risk managers - the left tail of the distribution. The rugs, i.e. the dark-green dashes, below the histogram in Figure 1 display single observations and thus convey good "tail density impressions". The 99.5% Value at Risk of xR^{NAV}'s, which corresponds to the 0.5% quantile in the above table, is $-41.9\% \cdot (-\sum NAV)$ and the 99.5% Conditional VaR is $-48.1\% \cdot (-\sum NAV)$.

Figure 2 exposes an apparent time pattern in historically simulated PCF portfolio returns. Neither the mean return nor the volatility seems to be constant over time.

Skewed Generalized t Distribution Fitting [The98] introduced the Skewed Generalized t Distribution (SGT) as a "flexible distribution accommodating the skewness and excess kurtosis often present in financial data". This flexibility manifest itself in the fact, that the SGT distribution can be converted to various established distributions with appropriate parameterization. So we fit a SGT distribution to the empirical xR^{NAV} data from the historical simulation example using maximum likelihood estimation (MLE), to examine, if the SGT distribution can cope with all important features of the empirical distribution. MLE is probably the most applied method of estimating the parameters of a pre-specified statistical model given sample data (i.e. parametric inference). The general MLE procedure is

1. specifying the joint probability density function (p.d.f.) for all obser-

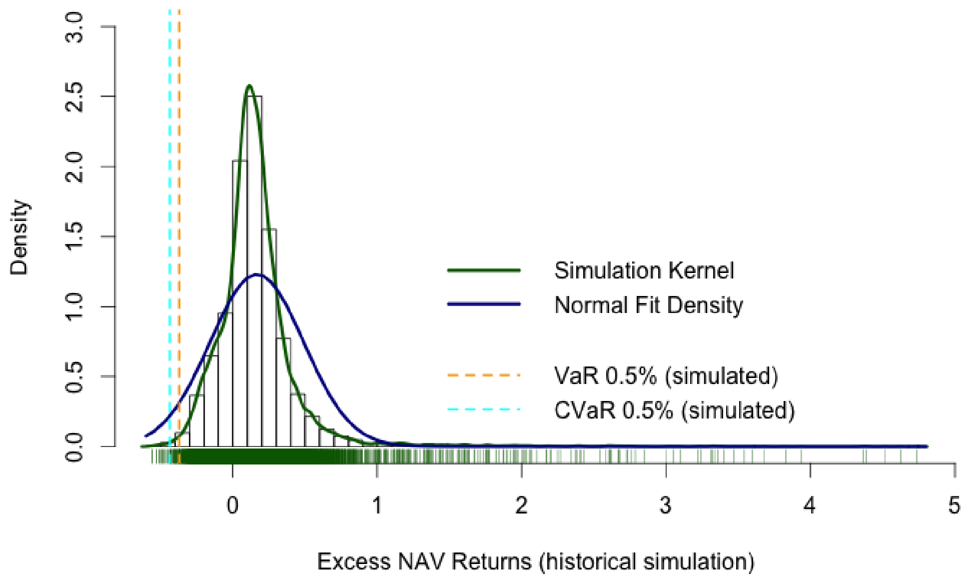

Fig. 1: Historically simulated xR^{NAV} distribution of TFTPF

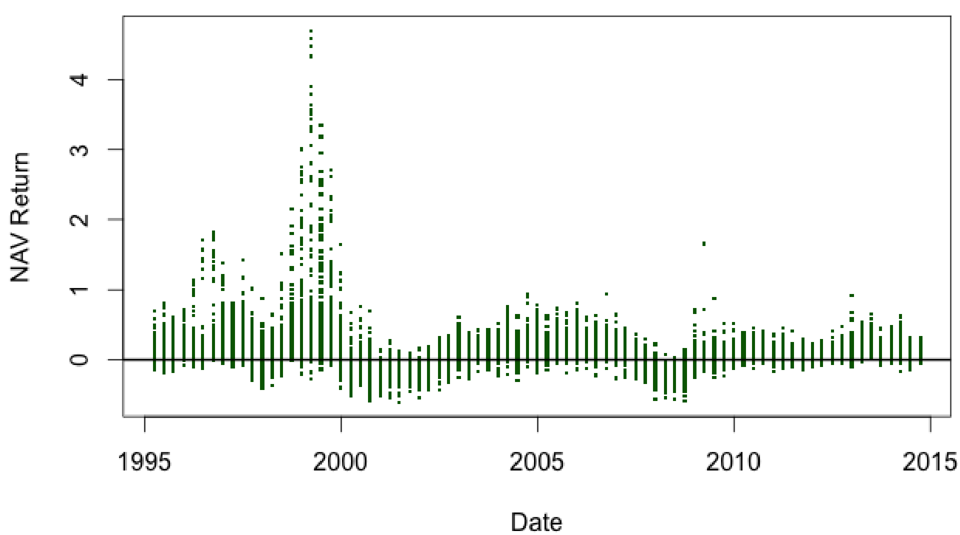

Fig. 2: Time evolution of historically simulated TFTPF returns

vations. For i.i.d. samples the joint p.d.f. is

$$f(\mathbf{x}\,|\theta) = f(x_1\,|\theta) \times f(x_1\,|\theta) \times \ldots \times f(x_n\,|\theta)$$

2. defining the likelihood function $\mathcal{L}$ (by using the joint p.d.f., but considering the observation vector $\mathbf{x}$ to be a fixed parameter and the model parameter vector θ as "flexible" variable)

$$\mathcal{L}(\theta; \mathbf{x}) = f(\mathbf{x}\,|\theta) = \prod_{i=1}^{n} f(x_i\,|\theta)$$

3. taking the natural logarithm of the likelihood function (if convenient)

$$\log \mathcal{L}(\theta; \mathbf{x}) = \sum_{i=1}^{n} f(x_i\,|\theta)$$

4. finding a value of θ, that maximizes the (log-)likelihood function

$$\left\{\hat{\theta}_{mle}\right\} \subseteq \left\{\arg\max_{\theta\in\Theta} \mathcal{L}(\theta; \mathbf{x})\right\}$$

In our case, the (ML) estimation of the five SGT distribution parameters $\mu, \sigma, \lambda, p, q$ is comfortably done in R using the *sgt.mle()* function from the "sgt" package.

The result is visually apparent in Figure 3. Here we can see a good fit in the distribution center, but obvious and serious left and right tail issues. The red (simulated SGT) and green (empirical) rugs nicely indicate, there is by far too much density in the left and too little density in the right tail of the fitted SGT distribution. However, it is noteworthy, that per definition of NAV Returns in Eq. (4) values below -1 are not problematic per se. Nevertheless, we have to conclude, that, unfortunately, the SGT is not "flexible" enough to satisfactorily approximate the ten fund test portfolio returns. Hence even this advanced parametric method can not capture the empirical distribution's complexity.

4.3 Linear Multi-Factor Models

Section 4.2 describes a way to generate (historically feasible) PCF portfolio returns. This section is now interested in developing models to explain these

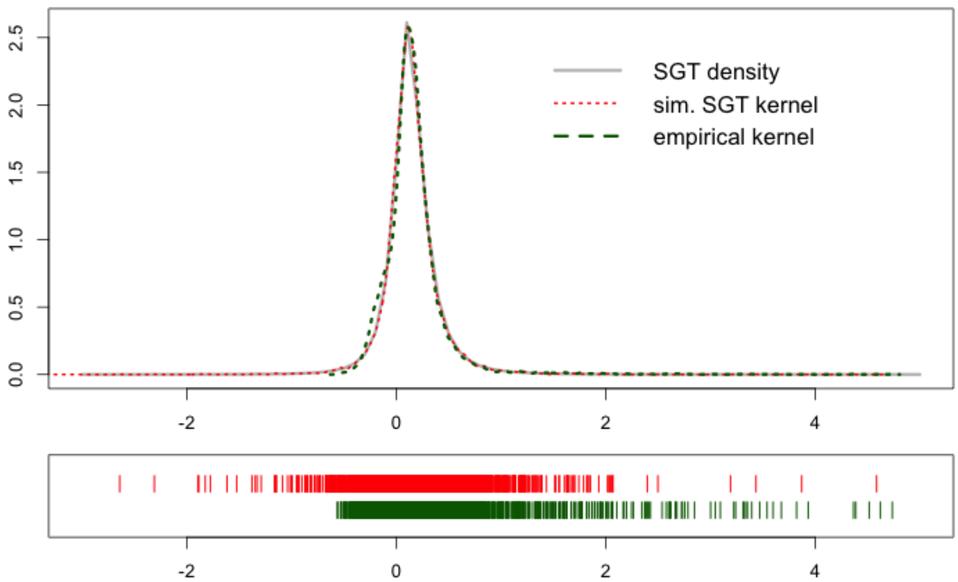

Fig. 3: Fitted skewed generalized t vs. empirical distribution of TFTPF

(and other PCF) returns via dimension reduction techniques[25]. This effort comes with the benefit, that we make the unconditional input variable conditional (some independent variables). More specifically, we seek models which (linearly) decompose PCF returns into multiple systematic risk factors and - very important in our case - residual idiosyncratic risk. Such multi-factor models are categorized by [Con95] into three main types:

- Macroeconomic factor models, that use time-series of observable economic variables like interest rates, inflation, GDP, unemployment rate.
- Fundamental factor models, that use the time-series of returns to mimicking portfolios targeting specific observable asset attributes like e.g. (in public equity) book-to-market ratio, industry classification, or dividend yield.
- Statistical factor models, that treat the factors as unobservable or latent variables which are derived via statistical techniques like principal component analysis.

[25] There is excessive textbook literature of vanilla factor models for (public) asset returns; see e.g. [MFE05] Chapter 3.4 “Dimension Reduction Techniques”, [Tsa10] Chapter 9 “PCA and Factor Models”, or [Ale08] Chapter II.1 “Factor Models”.

The general idea of explaining asset price movements, or deriving (ex-ante) expected asset returns, with a linear multi-factor model was initially popularized by the Arbitrage Pricing Theory (of Capital Asset Pricing) developed by [Ros76].

In this paragraph, we firstly specify a basic linear multi-factor model to explain the excess returns of (historically simulated) PCF portfolios in Section 4.3.1 and then, based on this, develop customized regression-based models on single fund level for different (Asset Metrix) Types in Section 4.3.2.

4.3.1 The Basic Model on Portfolio Level

The basic model on portfolio level is a regression-based, linear multiple factor model with commitment-weighted xR^{NAV}, denoted by Y_i, as dependent variable, which uses *chronological cross-sectional*[26] data stemming from m historically simulated portfolios (therefore $i = 1, 2, \ldots, m$)

$$R_i^{NAV} - r_i^{riskfree} := Y_i = \alpha + \beta_1 X_{i,1} + \ldots + \beta_k X_{i,k} + \epsilon_i$$

with $k \in \mathbb{N}$ possible risk factors X. The intercept is denoted with α, the factor loadings with β and the (uncorrelated) error term with ϵ. In our framework, we explicitly do not introduce any diversification assumptions which eventually allow us to neglect the noise term ϵ. In fact, the error term is essential in the PCF context as it might comprise the private capital specific residual return/risk component, alternative investments are recognized for. Generally, econometricians are eager to find (a model with) as many as possible (significant) explanatory risk factors, as with any additional factor it becomes more reasonable to assume that the noise term is not correlated with the risk factors (as a convenient side effect). Obviously, a model's greater explanatory power is the main effect of more descriptive factors. However, over-fitting is definitely to be avoided.

Factors in the Private Equity Literature There exist several multi-factor models for private equity returns in the academic literature; their focus is

[26] We have multiple observations of PCF xR^{NAV}'s per date, as we - by chance - sample multiple portfolios for at least some dates. Hence our approach implicates overlapping time periods, where each portfolio return is considered as a separate realization of the combined asset class returns plus noise in the given period. So, there exist a natural chronological ordering of our historically simulated data, which makes our data a hybrid (data-type) between time-series and cross-sectional. An accurate or classical time-series regression with one observation per date is hardly viable since we have only 20 years of data and a one-year period.

though the determination of systematic private capital risk, i.e. the risk of the aggregated asset class. Nevertheless, considering four models of the most recent papers gives us a good overview over possible significant factor-candidates. So, we briefly discuss their methodologies and examine their incorporated independent variables.

The [FNP12] model follows [Coc05] by assuming log-normally distributed one-period returns (of single private equity investments) as the dependent variable. Their four independent variables (a) excess equity market return, (b) HML (high minus low book-to-market ratio portfolio return), (c) SML (small minus big market capitalization portfolio return), and (d) IML (illiquid minus liquid portfolio return) originate from [PS03], whereby the first three factors are from the famous [FF93] model. To control for outliers in the return distribution on single investment level[27], they "group individual investments into portfolios" and use the (gross geometric mean of) portfolio log-returns as dependent variable. Factor loadings are then estimated using a cross-sectional OLS regression.

[JKP15] estimate the risk and expected return of listed, i.e. publicly traded, private equity funds (and fund of funds) by generating "value-weighted indices for various categories of listed entities". Ultimately, they apply a time-series OLS regression with dependent variable excess private equity index return and the six independent variables (a) excess equity market return, (b) HML, (c) SMB, (d) MOM ("a momentum factor" introduced by [Car97]), (e) GDP growth (normalized by standard score), and (f) credit spread (normalized by standard score).

[PPH14] propose risk models for various alternative investment types: private equity (particularly: buy out and venture capital), real assets (particularly: real estate, infrastructure, farmland, timberland, and natural resources), and hedge funds (including exotic beta strategies like momentum, carry, value, volatility, etc.). Their dependent variables are (aggregated) alternative asset class returns reported by data providers. Since the authors suppose that "the available asset return series may be smoothed", they "adjust for the smoothing effect," in such a way that their "model assumes that observed index returns represent a 'moving average' of the current and past 'true' investment returns." This method is similar to the [Dim79] approach, as the "model uses transformed risk factor returns that account for the lag structure of the index." Altogether, they consider various independent vari-

[27] [FNP12] use cash flow data of the underlying investments undertaken within PCFs. Thus, they form portfolios of investments, which are comprised in different PCFs, but started at the same (monthly) date.

ables to account for the variety of alternative investment types. The final model for a given (alternative) asset class includes naturally only a significant subset of risk factors. Such factors are e.g. (a) equity market return, (b) HML, (c) SMB, (d) IML, (e) several industry index returns, (f) corporate high-yield spread, and (g) 10-year government bond yield, among others.

[Pha10] describes in his survey, covering different methods of measuring private equity investments' risk and return, an "NAV-Based Regression Approach for Private Equity Funds" using *quarterly* excess NAV returns and a staleness correction in the sense of [Dim79]. "Betas are then estimated via a time-series OLS regression of returns on contemporaneous and lagged risk factors (market excess return)." The dependent variable is exactly the excess NAV return described in Eq. (4) with a quarterly period length and the independent variables are (a) excess equity market return (unlagged, and with lags of 1, 2, 3, and 4 quarters), (b) HML, and (c) SMB.

Variable Selection/ Factor Screening List of possible (risk) factors $X_{1,\dots,k}$ and appearance in above mentioned models:

Variable	Describtion	Used in PCF context by
EMR	Excess market return (also [FF93]-factor)	[FNP12] (a), [JKP15] (a), [PPH14] (a) & (e) {but non-excess in [PPH14]}, [Pha10] (a)
HML	[FF93]-factor	[FNP12] (b), [JKP15] (b), [PPH14] (b), [Pha10] (b)
SMB	[FF93]-factor	[FNP12] (c), [JKP15] (c), [PPH14] (c), [Pha10] (c)
RMW	[FF14]-factor	-
CMA	[FF14]-factor	-
MOM	[Car97]-factor	[JKP15] (d)
IML	[PS03]-factor	[FNP12] (d), [PPH14] (d)
GDP	GDP growth rate	[JKP15] (e)
HYS	(High-)yield spread	[JKP15] (f), [PPH14] (f)
Yield	Yield curve rate	[PPH14] (g)

We now need a method to decide which variables (from the above table) should be included in our regression model. On a naive technical level, we can distinguish between stepwise regression (forward selection/backward elimination) vs. all possible regression selection procedures for finding the *best* subset of explanatory variables; whereas *best* roughly means *as sparse and as uncorrelated as possible.* However, automated stepwise model selec-

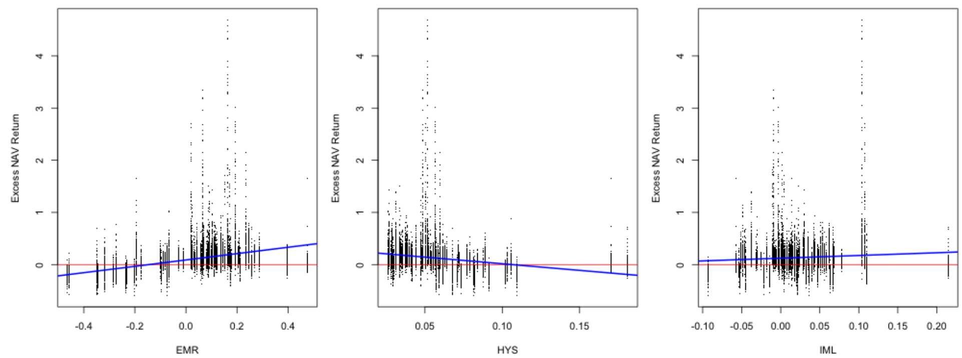

Fig. 4: Independent variables vs. dependent variable of OLS regression

tion procedures may generally be delicate or even in some cases disastrous as it involves repeated hypothesis testing[28].

Basic OLS Regression Results On the bottom line, we decide on a linear model for explaining commitment-weighted xR^{NAV}'s

$$Y_i = \alpha + \beta_{EMR} X_{i,EMR} + \beta_{HYS} X_{i,HYS} + \beta_{IML} X_{i,IML} + \epsilon_i$$

with the three factors excess market return (EMR), high-yield spread (HYS), and illiquid minus liquid portfolio return (IML). The univariate scatter-plots of independent variables vs. dependent variable (see Figure 4) indicate positive linear dependencies between Y and X_{EMR} and X_{IML} and a negative linear dependency between Y and X_{HYS}, which seems reasonable from an economic point of view. With the first sanity check passed, our (basic) linear model can be written in the matrix notation of [RPD98] Chapter 3, as

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon \tag{16}$$

with

Y: the $m \times 1$ column vector of observations on the dependent variable $\left(R^{NAV} - r^{riskfree}\right)$, where m is the sample size (of historically simulated portfolios),

X: the $m \times (1+3)$ matrix consisting of a column of ones, followed by the three column vectors of the observations of independent variables: EMR, HYS, and IML,

[28] See e.g. [KJ13] Chapter 19 "An Introduction to Feature Selection" or [HTF09] Chapter 3.3 "Subset Selection" for general variable selection discussions.

β: the $(1+3) \times 1$ vector of parameters to be estimated (note that the first element of the vector β is the scalar α in our notation), and

ϵ: the $m \times 1$ vector of random errors.

We use the (observed) data $\mathbf{X}$ and $\mathbf{Y}$ to estimate the unknown β and ϵ. The method of least squares allows us to find a closed-form expression[29] for the estimated regression coefficients (i.e. factor loadings)

$$\hat{\beta} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Y}$$

and, moreover, to derive the vector of residuals directly from the sample data

$$\hat{\epsilon} = \mathbf{Y} - \hat{\mathbf{Y}}$$

where the vector of estimated means is given by

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\beta} = \left[\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\right]\mathbf{Y}$$

If we assume that ϵ is an independent and identically distributed (i.i.d.) normal random variable[30] with mean zero and variance σ^2, then $\mathbf{Y}$ is multivariate normally distributed

$$\mathbf{Y} \sim N\left(\mathbf{X}\beta, \sigma^2\mathbf{I}\right)$$

where $\mathbf{I}$ is the identity matrix, since all covariance terms are assumed to be zero. Though [RPD98] notes on page 88 that "this result is based on the assumption that the linear model used is the correct model. If important independent variables have been omitted or if the functional form of the model is not correct, $\mathbf{X}\beta$ will not be the expectation of $\mathbf{Y}$."

The R output of our basic linear model (see Figure 5) might suggest a good and explanatory model at first sight, as all three independent variables

[29] The unique "pseudo-inverse" solution only exists when $\mathbf{X}^T\mathbf{X}$ is non-singular.

[30] "The conventional tests of hypotheses and confidence interval estimates of the parameters are based on the assumption that the estimates are normally distributed. Thus, the assumption of normality of the ϵ_i is critical for these purposes. However, normality is not required for least squares estimation. Even in the absence of normality, the least squares estimates are the best linear unbiased estimates (b.l.u.e.). They are best in the sense of having minimum variance among all linear unbiased estimators" (see [RPD98] p. 77). This is the result of the famous Gauss–Markov theorem, which states, that the residuals do not need to be normally nor i.i.d. distributed, just uncorrelated with expectation zero and homoscedastic with finite variance, to ensure, that the OLS regression coefficients are the "best linear unbiased estimates".

```
Residuals:
    Min      1Q  Median      3Q     Max
-0.7219 -0.1150 -0.0320  0.0577  4.3989

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.246719   0.006408   38.50   <2e-16 ***
EMR          0.618631   0.015308   40.41   <2e-16 ***
HYS         -2.949047   0.099777  -29.56   <2e-16 ***
IML          0.721669   0.060942   11.84   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.2839 on 9853 degrees of freedom
  (2253 observations deleted due to missingness)
Multiple R-squared:  0.2024,    Adjusted R-squared:  0.2022
F-statistic: 833.6 on 3 and 9853 DF,  p-value: < 2.2e-16
```

Fig. 5: R summary output: OLS regression of basic linear model

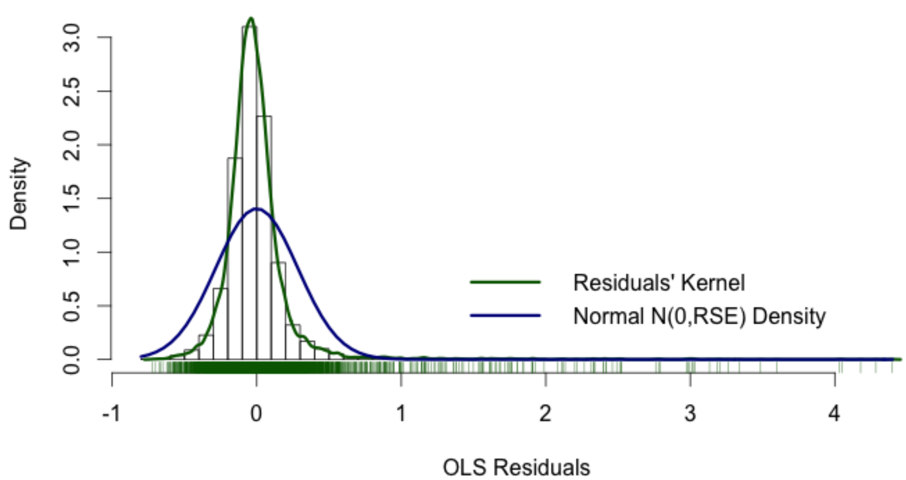

Fig. 6: OLS residuals and normal distribution fit

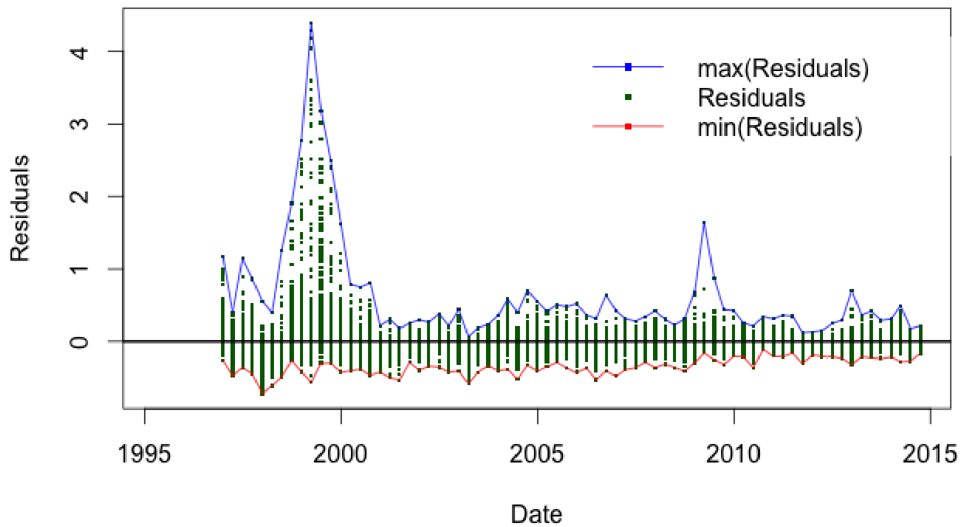

Fig. 7: Time evolution of residuals from OLS regression

seem highly significant with all p-values $< 2e^{-16}$. However, we know that our specific ten fund test portfolio xR^{NAV} data $\mathbf{Y}$ from Section 4.2.3 is positively skewed and leptokurtic. This might indicate, that we generally can not expect normality of idiosyncratic returns, i.e. residuals, in the PCF context. Therefore, the non-normal distribution of residuals from our basic linear model, visualized in Figure 6, is no big surprise; here, the residuals and the dependent variable exhibit quite similar features with respect to their distribution (recall: Figure 1).

As yet, the chronological character of the data is neglected in our regression analysis. Though the historically simulated data inheres a natural sequential order *by groups* of quarterly observations. Standard tests of autocorrelation in regression analysis (like the Durbin-Watson statistic) are therefore not meaningful and even misleading, as there exists no strict order of *each single element* in our data. So the clear time-dependency, i.e. time-pattern of similar adjacent residuals, in Figure 7 should not be labeled "serial correlation problem" to avoid confusion; we can use the terms "pseudo-autocorrelation" or conspicuous "time-pattern" instead. However, the final conclusion, that the i.i.d. assumption of residuals does not hold in our regression, is the same. There seems to be a regime-change in the year 2001 or, otherwise stated, the conditions in the period from 1998 until 2001 cause the right tail in Figure 6 (in a large part).

Appendix B discusses possible remedies to account for violations of the normality and homoscedasticity assumption of regression residuals.

4.3.2 Factor Models for Various Fund Types

In the previous section (and in Appendix B) likely issues associated with the decomposition of xR^{NAV}'s via linear factor models are introduced and summarized using the educational example of the ten fund test portfolio from Section 4.2.3. With the general PCF specific challenges in mind, we now develop individual AMT-factor-models (on single fund level) based on the three-factor model from Eq. (16), which are incorporated in the subsequent Monte Carlo simulation in Section 4.4.

The little more general model formulation in matrix notation here is

$$Y_{AMT} = f\left(X_{AMT} \left| \beta_{AMT}\right.\right) + \epsilon_{AMT}$$

to explain the xR^{NAV}'s of PCFs belonging to one Asset Metrix Type: $AMT = \{\mathrm{BO}, \mathrm{VC}, \mathrm{FOF}, \mathrm{DD}, \mathrm{RE}, \mathrm{MEZZ}, \mathrm{Infra}, \mathrm{NatRes}\}$. In contrast to many other settings, where just the determination of the dependent variable's conditional mean is of exclusive importance, we may not neglect the residual term in the above equation. Firstly, in some regression models (beyond OLS), where possibly $\mathbb{E}(\epsilon) \neq 0$, even the mean of the dependent variable depends on the error distribution[31]. Secondly, the residual (i.e. unexplainable part of the return = idiosyncratic return) distribution is (a) perchance too complex for simple parametric modeling and (b) *extremely* relevant for Value at Risk applications, as they essentially define the tail behavior of return distributions. Therefore, we follow the approach of *separating beta estimation from residual modeling*. Luckily, the (estimated) error term can be easily calculated once the regression coefficients are estimated, as $\hat{\epsilon}_{AMT} = Y_{AMT} - f\left(X_{AMT} \middle| \hat{\beta}_{AMT}\right)$. This method theoretically comes with the benefit, that, even though the models are designed for single fund returns, we can use aggregated (portfolio/index) data to estimate factor loadings, as long as we calculate the corresponding errors with single fund data.

Beta Estimation We can choose between four (valid) regression approaches for estimating the factor loadings of our linear AMT-factor-models. On the one hand, we can use either (minimum diversified) single fund data or (maximum diversified) AMT-index[32] data for the analysis. On the other

[31] The famous bias vs. variance trade-off is related to the issue of biased estimators.
[32] See Appendix B point 6 for a first AMT-index construction proposal.

hand, we can opt between an OLS or an alternative, robust regression technique. Naturally, each procedure comes with its own dis/advantages and neither method guarantees to yield the absolute “true” result, i.e. the “best” linear model. Generally, we expect biased estimators in every approach, as we explicitly separate beta estimation from residuals modeling. This is especially true since analysis with single fund data reveals heteroscedastic errors with respect to the NAV at the analysis date and in some cases time-patterns (or regime changes) in the error distribution. Further analysis shows leptokurtic and (just) slightly positively skewed residuals in the majority of all PCF models. Therefore, *robust* regression procedures (like e.g MM-estimation[33]) seem to be the (more) appropriate choice to obtain regression coefficients explaining the essential behavior of conditional mean xR^{NAV}'s in the single PCF context. The details of robust regression approaches of the M/S/MM-estimation family are best studied in textbooks on robust statistics like [MMY06]. Here MM-estimation is characterized as an advanced robust regression method, combining the advantages of M- and S-estimation, to obtain estimates that have a high breakdown point[34]. This property suggests the use of MM-estimation in our setting with a multitude of xR^{NAV} wild-shots (on single fund level). With diversified index data the OLS method is possibly more justified[35].

In the data issue, we tendentially favor the (equal commitment weighted) index approach, as it promises beta factors for conditional *asset class* or, to be more precisely, mean returns of an (equal commitment weighted) AMT-class-investor. Single fund data probably would have to be time-weighted (and maybe additionally filtered) in a sophisticated manner[36] to attain valid asset class expectations, as single fund observations are highly unbalanced in the time dimension due to PCFs' rising popularity over time. Moreover, classical time-series regression with index data avoids wild-shot problems related

[33] “The asymptotic theory for M-estimates, which includes S- and MM-estimates, has been derived under the assumption that the errors are i.i.d. and hence homoscedastic. These assumptions do not always hold in practice. [...] Actually the assumptions of independent and homoscedastic errors are not necessary for the consistency and asymptotic normality of M-estimates. In fact, it can be shown that these properties hold under much weaker conditions.” [MMY06], p. 153.

[34] “Roughly speaking, the breakdown point (BP) of an estimate $\hat{\theta}$ of the parameter θ is the largest amount of contamination (proportion of atypical points) that the data may contain such that $\hat{\theta}$ still gives some information about θ, i.e., about the distribution of the “typical” points.” [MMY06], p. 58.

[35] Particularly since OLS and MM estimates are quite similar when aggregated with the Index Quagging method (introduced in the next passage); compare Table 5 and 6.

[36] E.g. bootstrapping AMT samples with a desired time-distribution.

AMT	deterministic			stochastic							
	(Int)	Bias Corr	High Yield Spread	Excess Index Returns World Equity	Excess Index Returns NAS DAQ	Excess Index Returns US-RE	Liq 10-1 PF	Q1 adj. R^2	Q2 adj. R^2	Q3 adj. R^2	Q4 adj. R^2
BO	0.188	?	-2.020	0.573	-	-	-	0.60	0.47	0.55	0.66
VC	0.300	?	-4.249	-	0.993	-	-	0.34	0.31	0.28	0.29
FOF	0.058	?	-	0.830	-	-	-	0.14	0.17	0.10	0.11
RE	0.141	?	-1.759	-	-	1.095	-	0.55	0.64	0.74	0.54
DD	-	?	1.036	0.562	-	-	-	0.62	0.63	0.62	0.61

Tab. 5: OLS-estimates of linear model coefficients obtained via Index Quagging explaining one-year xR^{NAV}'s .

AMT	deterministic			stochastic							
	(Int)	Bias Corr	High Yield Spread	Excess Index Returns World Equity	Excess Index Returns NAS DAQ	Excess Index Returns US-RE	Liq 10-1 PF	Q1 adj. R^2	Q2 adj. R^2	Q3 adj. R^2	Q4 adj. R^2
BO	0.191	?	-2.079	0.584	-	-	-	0.78	0.51	0.53	0.69
VC	0.191	?	-3.372	-	0.510	-	-	0.66	0.52	0.66	0.50
FOF	0.084	?	-	0.419	-	-	-	0.04	0.11	0.05	0.40
RE	0.113	?	-1.392	-	-	1.050	-	0.84	na	0.72	0.83
DD	-	?	0.963	0.580	-	-	-	0.64	0.65	0.67	0.78

Tab. 6: Robust MM-estimates of linear model coefficients obtained via Index Quagging explaining one-year xR^{NAV}'s.

AMT	deterministic			stochastic				
	(Int)	Bias Corr	High Yield Spread	Excess Index Returns World Equity	Excess Index Returns NAS DAQ	Excess Index Returns US-RE	Liq 10-1 PF	adj. R^2
MEZZ	0.101	?	-0.922	0.206	-	-	-	0.05
NatRes	0.064	?	-1.042	0.498	-	-	-	0.07
Infra	0.079	?	-0.859	0.261	-	-	-	0.02

Tab. 7: Robust MM-estimates of linear model coefficients obtained from unfiltered single fund data explaining one-year xR^{NAV}'s.

to heteroscedasticity and autocorrelation. Though, the index time-series approach comes with two drawbacks because there exist (a) four equivalently valid time-series with yearly intervals (and non-overlapping periods) starting at different quarters, which are (b) relatively short (20-25 data points, as data starts in the early 1990's). But here we can make a virtue out of necessity. In a subsequent Monte Carlo simulation, we can simply use all four models by sampling with equal probabilities among them, which increases the variability of simulated returns and additionally highlights the stochastic character of every linear model. For communication purposes, we are free to aggregate the four quarterly estimates by averaging out the coefficients. The resulting *quagging*[37] coefficients are finally reported as our best beta factor estimates. The fall-back option for factor loading estimation is a robust regression on the total AMT single fund data set, in cases where the index-quagging approach fails. Clearly, regressions on single fund data come with the advantage of being capable of integrating more fund specific factors (like region, domestic index return, fund age, etc.) into the analysis.

Tables 5 and 6 exhibit the OLS- and MM-estimates of the regression coefficients of the eight linear AMT-factor-models, which are generally similar in sign and magnitude. The factors are subdivided into deterministic, i.e. known at analysis date, and stochastic, i.e. unknown at analysis date, predictors. Additionally, a column for a potential bias correction is inserted in the tables, to put the possibility of a random bias, i.e. non-zero mean residuals, in mind. As the residuals modeling is not tackled yet, no assertions about the sign and magnitude of a potential bias can be made. Thus, in contrast to standard regression model outputs, no (robust) residual standard errors are reported here to avoid confusion. However, we report adjusted R^2 figures for all four quarters (in Tables 5 and 6) to call to mind, that the displayed coefficients are aggregated quantities. In contrast, there is only one coefficient of determination R^2 for the single fund regressions of Table 7, which reveals poor fits for AMTs Mezzanine, Natural Resources, and Infrastructure[38].

Error Modeling The issues of (a) heteroscedastic errors with respect to the NAV at the analysis date and (b) time patterns (or regime changes) in the error distribution were already briefly mentioned in the above paragraph. The sine qua non for modeling the error distribution appropriate for a given

[37] The term *quagging*, short for quarterly aggregating, is inspired by [Bre96]'s famous machine learning technique called bagging.

[38] The main problem with those AMTs is the lack of data and therefore those are admittedly treated as orphans in our analysis. Expending more effort to those AMTs could probably result in enhanced linear models with higher R^2 values.

NAV partition	NAV quantile	Description
bottom 20%	0-20%	discarded portion
top 80%	20%-100%	retained overall set
20-60%	20%-60%	retained wild subset
top 40%	60%-100%	retained gentle subset

Tab. 8: Empirical error NAV partitions

AMT-factor-model is the resolution of these two problems. Only when the empirical error distribution of $\hat{\epsilon}_{AMT}$ is cleaned from unwanted wild-shots, modeling approaches aim at the wanted, reasonable residual distribution.

The severe NAV-heteroscedasticity is caused by the design of R^{NAV} in Eq. (4), which facilitates an increasing variability in R^{NAV} with decreasing NAVs at the analysis date, as the NAV (at analysis date) enters the equation in the denominator. Figure 8 illustrates this phenomenon since there extreme residual wild-shots mainly occur for small NAVs (at analysis date). The simplest remedy of the NAV-heteroscedasticity is the determination of a NAV cut-off-threshold, in such a way that xR^{NAV} observations with period start NAVs below that threshold are eliminated. The dotted blue lines in Figure 8 show e.g. NAV thresholds at the 20% quantile of observed NAVs per AMT. However, the establishment of the most suitable cut-off-value is anything but straightforward, as the VaR in the subsequent Monte Carlo simulation *heavily* depends on the selected threshold figure. Eventually, we decided to reject 20% of the smallest NAVs, i.e. cut-off at the empirical 20% quantile, and further partition the remaining 80% of the data points into two commensurate, complementary NAV-sorted groups, i.e. split at the 60% quantile of the empirical NAV c.d.f. for a given AMT. The NAV-subsets (cf. Table 8) are then used separately to model proper error distributions, which can be compared in the final analysis of the comprehensive model.

Next, peculiar error time-pattern, regime changes or other features (i.e. dependencies) over time have to be addressed, (just) if necessary. For Venture Capital funds we could e.g. separate out the years 1999 until 2001, the time of the so-called “internet bubble”, which was (perhaps) a unique, non-repeating event. Even more important is the assumption of no (temporal) cross-dependencies between the model errors of distinct AMTs to justify an univariate residual modeling (with respect to the subsequent Monte Carlo simulation). If there exist non-negligible inter-AMT-error-dependencies, a single multivariate error distribution for all eight AMTs has to be conceived.

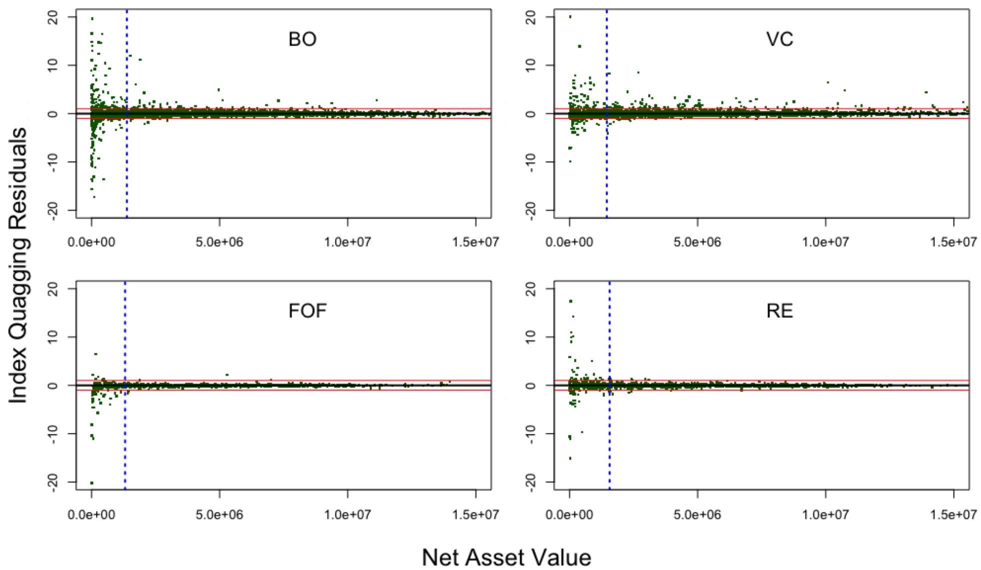

Fig. 8: Plot of Index Quagging residuals vs. NAV

But, if there are no time-dependencies on univariate AMT level, then there can not be multivariate inter-AMT time dependencies.

When all anomalies or sources of wild-shots are filtered out, we can ultimately start modeling the tamed error distributions using stochastic techniques. Preferably, we should first check (i.e. tinker), if the residuals can be decomposed further in a predictive way using e.g. machine learning techniques. If not, errors have to be modeled en bloc; i.e. trying to recreate the empirical distribution of independent residuals arising from linear AMT-factor-models. The most obvious modeling (in the sense of replication) approaches are simple parametric models (like fitting a log-normal distribution) or advanced parametric models (like fitting an SGT distribution or Gaussianize the residuals using the Lambert Way; see Appendix B). Yet, none of this methods yields convincing results. The residual distributions, which are still leptokurtic and positively skewed in most cases (for most AMTs), continue to be too complex, especially at the tails. Therefore, we personally prefer sampling directly from the empirical error distribution as the simplest method, which assures reasonable results with the highest probability, since no model error is introduced by following this (non-parametric) approach. Though, the most promising (parametric) statistical method to tackle the error distributions' complexity is the application of finite mixture distribution models because they nicely decompose the distribution into sub-groups. This is not only pleasant for illustrative purposes, but may be furthermore capable of revealing attractive PCF characteristics latent in the residual distribution. Since the sub-populations are unknown a priori "unsupervised clustering" or "model-based clustering" procedures have to be applied to cluster the distribution into sub-groups. In the R package "mixtools" *EM algorithms* drawing on maximum likelihood estimation in the presence of *incomplete data* accomplish this task[39], as standard MLE approaches are unusable, due to the absence of global likelihood function maxima in the mixture distribution setting. We use this package, and thus follow the derivation and notation of [BCHY09] hereafter, to exemplarily fit two univariate normal mixture distributions with 2 and 3 components, respectively, to the empirical distribution of Index Quagging residuals of VC funds with NAVs above the 20% quantile (see Figure 9). Concretely, we have a random sample of 5,245 error observations of the random variables $X_1, \dots, X_{5245}$, stemming from a mixture of $m = \{2, 3\}$ normal distributions with component densities $\phi\left(\cdot\,|\mu, \sigma\right)$. The density of *each* X_i may then be expressed as

$$g_\theta\left(x_i\right) = \sum_{j=1}^{m} \lambda_j \phi_j\left(x_i\right), \quad x_i \in \mathbb{R},$$

where $\theta = (\lambda, \phi) = \left(\lambda_1, \dots, \lambda_m, \left(\mu_1, \sigma_1\right), \dots, \left(\mu_m, \sigma_m\right)\right)$ denotes the parameter and the weights λ_m are positive and sum to unity. Hence, the empirical residual distribution consists of $n = 5245$ i.i.d. observations $x = (x_1, \dots, x_{5245})$ from the density g_θ. In the missing observations setup of [DLR77] g_θ is called the incomplete-data density, and the associated log-likelihood is $L_x\left(\theta\right) = \sum_{i=1}^{n} \log\, g_\theta\left(x_i\right)$, which is used to determine the maximum likelihood estimator by finding $\hat{\theta}_x = \arg\max_{\theta \in \phi} L_x\left(\theta\right)$. The corresponding complete-data density (for one observation) is

$$h_\theta\left(x_i, z_i\right) = \sum_{j=1}^{m} \mathbb{I}_{z_{ij}} \lambda_j \phi_j\left(x_i\right)$$

where $\mathbb{I}$ is the indicator function, $Z_i = (Z_{ij}, j = 1, \dots, m)$, and $Z_{ij} \in \{0, 1\}$ is a binary Bernoulli-type random variable indicating that individual i comes

[39] The term *incomplete data* is described in [DLR77] on page 1: "Since each iteration of the algorithm consists of an expectation step followed by a maximization step we call it the EM algorithm. [...] The term "incomplete data" in its general form implies the existence of two sample spaces $\mathscr{Y}$ and $\mathscr{X}$ and a many-one mapping from $\mathscr{X}$ to $\mathscr{Y}$. The observed data y are a realization from $\mathscr{Y}$. The corresponding x in $\mathscr{X}$ is not observed directly, but only indirectly through y."

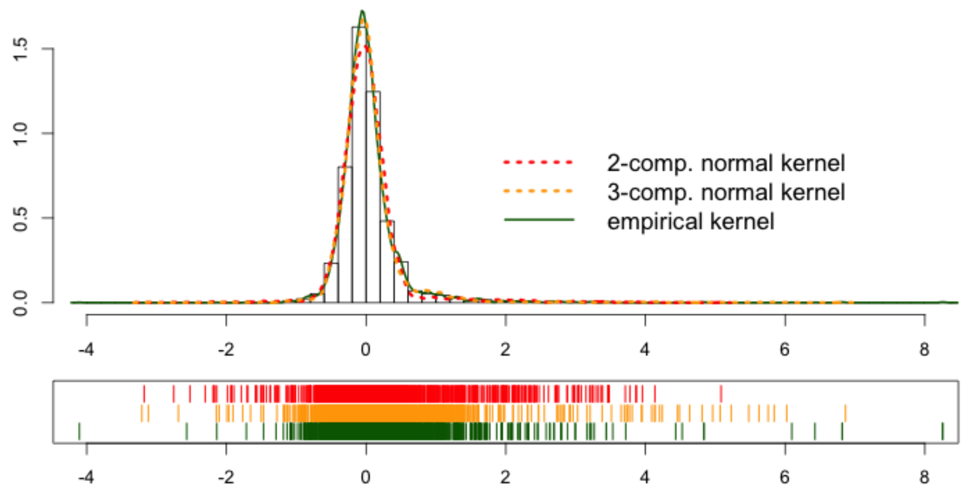

Fig. 9: Simulated 2 & 3-component normal mixtures vs. empirical distribution of VC Index Quagging residuals

form component j, with $\sum_{j=1}^{m} Z_{ij} = 1$. Consequently , $P(Z_{ij} = 1) = \lambda_j$ gives the probability that individual i belongs to the j-th normal distribution and, clearly, $(X_i \,|Z_{ij} = 1) \sim \phi_j(\cdot\,|\mu_j, \sigma_j)$, for $j = 1, \ldots, m$.

The EM algorithm iteratively maximizes the operator

$$Q\left(\theta \,\middle|\theta^{(t)}\right) = \mathbb{E}\left[\log\ h_\theta(X, Z) \,\middle|x, \theta^{(t)}\right]$$

where $\theta^{(t)}$ is the current value at iteration t. The EM-algorithm procedure from iteration t to $t+1$ consists of an E(xpectation) and a M(aximization) step

1. E-step: compute $Q\left(\theta \,\middle|\theta^{(t)}\right)$
2. M-step: set $\theta^{(t+1)} = \arg\max_{\theta\in\Phi} Q\left(\theta \,\middle|\theta^{(t)}\right)$

So in every iteration, we obtain new values for the weights λ and the associated normal density parameters. The algorithm stops if a pre-defined convergence criterion regarding the change in the observed data log-likelihood is met. The procedure clearly depends on the selected starting parameter $\theta^{(1)}$; however we use the package's default setting for obtaining (random) starting values in our specific example.

The fitted 3-component normal-mixture distribution, received after 273 iterations and visualized by the orange lines in Figure 9, is parameterized with

$$\lambda_1 = 0.810 \ : \qquad \mathcal{N}\left(\mu_1 = -0.052,\ \sigma_1 = 0.211\right)$$
$$\lambda_2 = 0.171 \ : \qquad \mathcal{N}\left(\mu_2 = 0.262,\ \sigma_2 = 0.583\right)$$
$$\lambda_3 = 0.019 \ : \qquad \mathcal{N}\left(\mu_3 = 1.939,\ \sigma_3 = 2.170\right)$$

and therefore adds up to a bias correction coefficient of 0.039, which is the mean of this mixture distribution.

The decomposition of xR^{NAV} residuals exhibits again the great importance of the error terms in the context of AMT-factor-models. The general method of separating beta estimation from error modeling allows us to build several linear models for each AMT and then, in a next step, decompose the resulting residual distributions with greater effort than in usual OLS or even robust regression approaches. In Monte Carlo simulation applications, we are free to semi-randomly sample, i.e. proceed in the spirit of bagging, from this repertory of plural β- and ϵ-models[40].

4.4 Fund to Portfolio Aggregation (Monte Carlo)

Monte Carlo (MC) simulation methods, i.e. statistical models or computational algorithms depending on streams of random numbers, are commonly viewed as a powerful and flexible means to generate return or profit and loss distributions for portfolios consisting of several heterogeneous assets with complex dependency structures. Their biggest advantage is the ability to create a great variety of different (factor) scenarios and the corresponding financial outcomes by combining (rather simple) parametric models in a sophisticated way. Random sampling from an MC model thus allows to relatively easily approximate the financial return distribution of a given portfolio, even if analytical solutions to the high-dimensional problem are not available. In our specific case, we attempt to transform and aggregate the individual regression-based AMT-factor-models from Section 4.3.2 to a comprehensive "sample-able" Monte Carlo model tailored for VaR calculation of PCF portfolios. To our knowledge, the MC simulation procedure, adopted in this section, is the first in the academic literature for simulating VaRs with a one-year horizon in the PCF context. [BKSW10] developed the probably most closely related Monte Carlo simulation method to calculate Value-at-Risks for IRRs of venture capital fund investments. Their focus is hence

[40] When we estimate several β-models (a so-called *ensemble of models* in machine learning terminology), feature selection should always be considered as the first step of the model-building process, in theory.

rather "performance projection" than (one-year horizon) risk management. General (public market) textbook literature for VaR simulation by means of MC techniques can be found e.g. in [Jor06] Chapter 12 "Monte Carlo Methods" or in [Ale08] Chapter IV.4 "Monte Carlo VaR".

In Section 4.4.1, we cover the missing part of our MC model, i.e. the construction of multivariate input-factor distributions, which can be considered as the engine of our MC model. In the next Section 4.4.2, the actual MC model is introduced by combining the sub-models for β-estimation, ϵ-modeling, and $\mathbf{X}$-factor construction.

4.4.1 Multivariate Factor Construction

A general (regression-based) linear multi-factor model

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$

consists of three terms on the right-hand side: a factor matrix $\mathbf{X}$, a factor loadings vector β, and a vector of (unexplainable) errors ϵ. In Section 4.3.2 some feasible β-estimation and ϵ-modeling approaches are introduced. Consequently, only the modeling of the independent variables $\mathbf{X}$ is left, to complete a Monte Carlo model of $\mathbf{Y}$. Several eligible approaches for $\mathbf{X}$-generation are therefore presented in this section, i.e. finding one multivariate model for the (public market) factors outlined in Section 3.2.

Technically, the missing MC component is the vector[41] $\mathbf{X} = (X_1, \ldots, X_d)$ constituting one possible (simulated) factor outcome. Hence, the joint (i.e. multivariate) cumulative distribution function (c.d.f.)

$$F(\mathbf{x}) = F(x_1, \ldots, x_d) = P(X_1 \leq x_1, \ldots, X_d \leq x_d)$$

of the MC model's d input factors is of interest. In this section, several viable $\mathbf{X}$-generation options are introduced. At first, (non-parametric) historical factor outcome sampling is always an option. The most basic (parametric) one is a multivariate normal distribution model for the factors

$$\mathbf{X} \sim \mathcal{N}(\mu, \Sigma)$$

characterized (completely) by the mean vector μ and the covariance matrix Σ. However, in virtue of the normal distribution's parsimony this model is not capable of neither generating leptokurtic or skewed (univariate) factor

[41] In regression models $\mathbf{X}$ is a matrix and β is a vector (see Eq. (16)). In our MC model $\mathbf{X}$ is a vector and β is a matrix (see Eq. (21)).

distributions[42] nor describing complex interdependencies between factors. Therefore, the application of copula methods seems advantageous (and state of the art), as they are a powerful concept to strictly separate dependency modeling from specifying the marginal (cumulative) distribution (function), which is obtained from the joint c.d.f. through

$$F_i(x_i) = P(X_i \leq x_i) = F(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty)$$

The two (bottom-up) steps in copula modeling are

1. univariate (parametric) models F_X for each factor X_i
2. copula model C describing the dependence between factors

whereby this elegant decomposition generally succeeds because of Sklar's theorem.

Sklar's Theorem (1959) *For a d-variate cumulative distribution function $F \in \mathcal{F}(F_1, ..., F_d)$, with j-th univariate margin F_j, the copula associated with F is a distribution function $C: [0,1]^d \to [0,1]$ with $U(0,1)$ margins that satisfies*

$$F(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d)), \ \mathbf{x} \in \mathbb{R}^d \tag{17}$$

and, if F is a continuous d-variate distribution function with univariate margins $F_1, ..., F_d$, and quantile functions $F_1^{-1}, ..., F_d^{-1}$, then

$$C(\mathbf{u}) = F\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right), \ \mathbf{u} \in [0,1]^d$$

is the unique choice (see [Joe15], p. 7).

Thus, a copula C is a joint c.d.f. (of a vector of random variables U) with all univariate margins being standard uniformly distributed $U_i \sim U(0,1)$ and a desired dependency structure inherent. Therefore, inverse transform sampling (also known as inverse probability integral transform) may be applied within Monte Carlo simulations, if all selected marginal distributions have invertible c.d.f.'s (or - to be more realistic - if good (numerical) approximations of the quantile function exist), since

$$P\left(F_{X_i}^{-1}(U_i) \leq x\right) = F_{X_i}(x)$$

[42] The fitting of multivariate generalized hyperbolic distributions (normal mixtures) by applying the EM-algorithm, described in [MFE05] Chapter 3.2.4, may be used to obtain more complex marginals.

or equivalently

$$X_i = F_{X_i}^{-1}(U_i)$$

where $F_i(x_i) \equiv F_{X_i}(x_i)$, $U_i \sim U(0,1)$ is e.g. the i-th component of a copula, $F_{X_i} : \mathbb{R} \to (0,1)$ is the selected marginal c.d.f. of the i-th factor, and $F_{X_i}^{-1} : (0,1) \to \mathbb{R}$ the respective inverse. So in a Monte Carlo implementation, the d simulated draws from dependent uniform distributions are inserted in the corresponding marginal quantile functions. The determination of the univariate marginal (factor) distributions is generally perceived as the easy part of the copula framework. Getting an idea of the (multivariate) dependence structure, i.e. the copula, is considerably harder and [Joe15] notes in the introduction of his extensive copula textbook that "the difficult step in copula construction is the extension from bivariate to multivariate to get flexible dependence" (see [Joe15], p. 2).

Since the building of high-dimensional copulas is regarded very difficult, if heterogeneous and complex dependencies exist between variables, Pair Copula Construction (PCC) techniques are (currently) in the focus of academic research. The idea behind PCC is to create high-dimensional (i.e. multivariate) copulas out of (bivariate) copula pairs and "thus exploiting the richness of the class of bivariate copulas and providing a flexible and convenient way to extend the bivariate theory to arbitrary dimensions" (see [MS12], p. 185). Next, we follow [ACFB09] to outline how to decompose a general multivariate distribution into pair-copulae. First, the joint density function of $\mathbf{X}$ gets factorized as

$$f(\mathbf{x}) = f_d(x_d) \cdot f(x_{d-1} | x_d) \cdot f(x_{d-2} | x_{d-1}, x_d) \cdots f(x_1 | x_2, \ldots, x_d) \quad (18)$$

which is equivalent to

$$f(\mathbf{x}) = c_{1 \cdots d}(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d) \quad (19)$$

(which is obtained by) differentiating Eq. (17) of Sklar's theorem using the chain rule, if F is absolutely continuous with strictly increasing, continuous marginal densities and the (uniquely identified) d-variate copula density $c_{1 \cdots d}$ exists[43]. The general approach of pair-copula decomposition, explained in more detail by [ACFB09] in Section 2, is to express each term in Eq. (18) with appropriate (conditional and unconditional) bivariate pair-copula densities obtained from Eq. (19). For e.g. $d = 3$ the third, i.e. also the last, term of Eq. (18) can be written as

$$f(x_1 | x_2, x_3) = c_{13|2}(F(x_1 | x_2), F(x_3 | x_2)) \cdot c_{12}(F(x_1), F(x_2)) \cdot f_1(x_1) \quad (20)$$

[43] Therefore, the copula has to be differentiable.

with unconditional pair-copula density c_{12} and conditional pair-copula density $c_{13|2}$, applied to the transformed variables $F(x_1 | x_2)$ and $F(x_3 | x_2)$. However, the right hand side of Eq. (20) is not unique, i.e. there exists an alternative pair copula decomposition of $f(x_1 | x_2, x_3)$, since this decomposition is generally just unique for the second term of Eq. (18).

For high-dimensional distributions, there is a considerable number of possible pair-copulae decompositions. This enormous flexibility comes with the price of making PCC confusing and challenging from a conceptional and also from a notational perspective. To stay on top of things, the framework of "regular vines", technically a graphical model generalizing the concept of trees[44], can be used to organize PCC in a more comprehensible way. However, it is just convenient not indispensable to use R-vines (regular), C-vines (canonical), or D-vines in PCC, like [ACFB09] note on page 4, "that the tree structure is not strictly necessary for applying the pair-copula methodology, but it helps identifying the different pair-copula decompositions." Primarily, all copula and marginal c.d.f.'s must have a first derivative, i.e. a density, to employ PCC. Though, this implies that the marginal distributions are or have to be known, which is not realistic in practice[45]. Further [ACFB09] note on page 15, that "full inference for a pair-copula decomposition should in principle consider (a) the selection of a specific factorization, (b) the choice of pair-copula types, and (c) the estimation of the copula parameters." In the R implementation of our MC model, we use the terrifically handy *RVineStructureSelect()* function from the "VineCopula" package to specify and determine a complete, i.e. all three inference components at one go[46], R-vine copula model for our d independent factors. Figure 10 depicts $\sum_{j=1}^{(d-1)} j = \frac{(d-1)^2+(d-1)}{2}$ pairs of 1,000 simulated factor uniforms, generated from the R-vine copula model obtained by the *RVineStructureSelect()* function (with default setting).

In the next step, the marginal factor distributions have to be determined. This is (again) preferably accomplished via MLE for simple parametric models. If (more) appropriate, finite mixture distributions can naturally be es-

[44] The purpose of regular vines (R-vines) is to graphically designate (un/conditional) two-dimensional constraints in multivariate probability distributions.

[45] [ACFB09] state in the section considering inference for a specified pair-copula decomposition on page 12, that "it is important to emphasize that unless the margins are known (which they never are in practice), the estimation method presented below then must rely on the normalized ranks of the data. These are only approximately uniform and independent, meaning that what is being maximized is a pseudo-likelihood."

[46] See [ACFB09] page 14, for a likelihood evaluation algorithm, which numerically optimizes the D-vine log-likelihood.

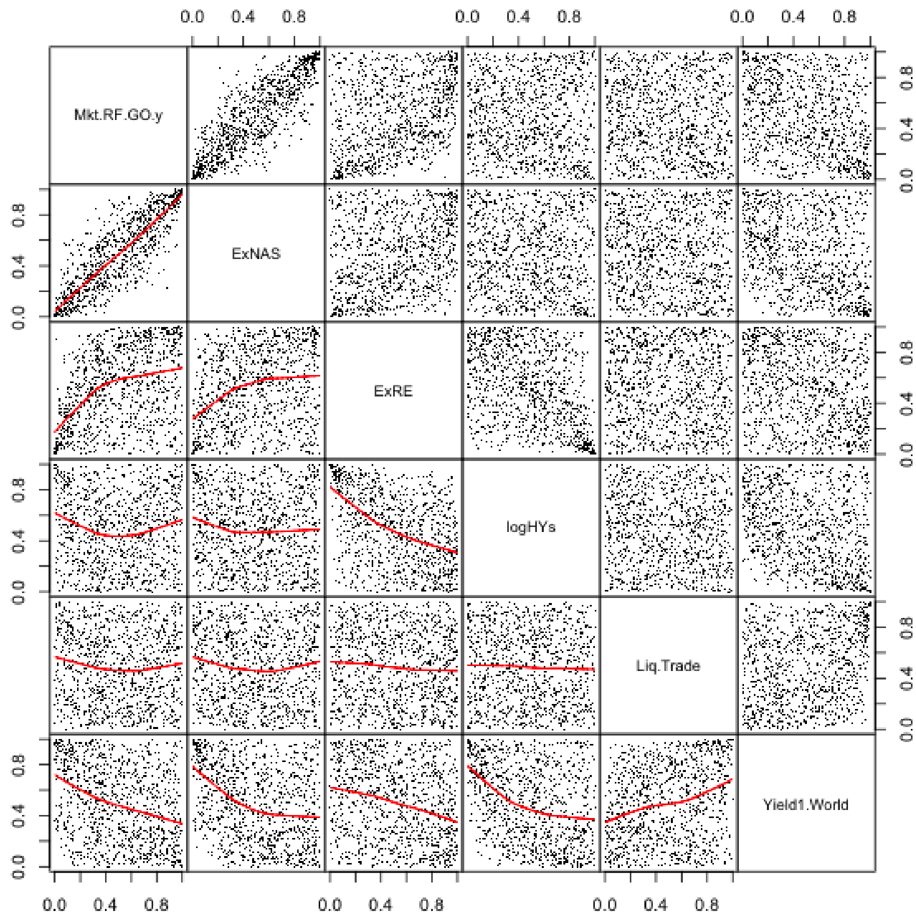

Fig. 10: Pairs of simulated R-vine copula uniforms describing **X**-factor dependencies

timated with the familiar EM-algorithm from Section 4.3.2. Yet, we finally model the marginal distributions of

xR_{WE}: the excess return of a world equity index (generalized) t-distributed with $xR_{WE} = \mu + \sigma T$, where $\mu = 0.066$, $\sigma = 0.185$, and $T \sim t\,(df = 19.39)$,

xR_{NAS}: the excess return of the NASDAQ equity index (generalized) t-distributed with $xR_{NAS} = \mu + \sigma T$, where $\mu = 0.098$, $\sigma = 0.245$, and $T \sim t\,(df = 6.25)$,

xR_{RE}: the excess return of a US commercial real estate index SGT-distributed (Skewed Generalized T) with $xR_{RE} \sim SGT\,(\mu = 0.034,\ \sigma = 0.124,\ \lambda = -0.537,\ p = 1.255,\ q = 3.686)$,

$\log(S_{HY})$: the logarithm of US high-yield spreads log-normally[47] distributed with $\log(S_{HY}) \sim LN\,(\mu = 0.480,\ \sigma = 0.260)\,/100$, and

R_{liq}: the return of the liquidity portfolio factor from [PS03] normally distributed with $R_{liq} \sim N\,(\mu = 0.014,\ \sigma = 0.049)$.

Note, that the high-yield spreads actually need no stochastic modeling at all, since they are know a priori in applications of the MC model. Just, if we want to compare historical with MC simulations, stochastic high-yields spreads are required.

4.4.2 Monte Carlo Model

With the β-estimation and ϵ-modeling approaches from Section 4.3.2 and the **X**-factor construction methods from Section 4.4.1 all building blocks for a Monte Carlo simulation are available. In our Monte Carlo model, we combine multiple AMT-factor-models to build a comprehensive model, which shall be capable of generating PCF portfolio specific xR^{NAV}'s samples. Once again, we have to remark the essential MC model assumptions that there are no inter-AMT-error-dependencies and the NAV-heteroscedasticity is resolved satisfactorily, to justify i.i.d. error sampling/modeling.

Then, we are ready to formulate our Monte Carlo model for the xR^{NAV} vector (of all m single PCF portfolio components) in terms of a parameter

[47] This mean, that $\log(\log(S_{HY})) \sim \mathcal{N}(\mu, \sigma)$, which eventually leads to too much positive skewness. The double logarithmic modeling of high-yield spreads may thus be regarded as a worst case specification; $\log(S_{HY}) \sim \mathcal{N}(\mu, \sigma)$ is probably more appropriate, however perhaps a little bit too optimistic; the alternative parametrization is $\log(S_{HY}) \sim \mathcal{N}(\mu = 1.671\,, \sigma = 0.429)\,/100$.

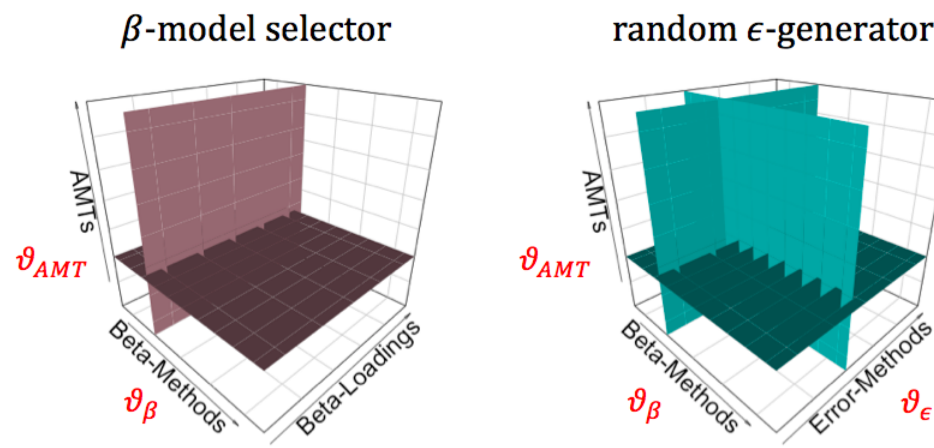

Fig. 11: Illustration of β and ϵ method selection in MC simulation

vector ϑ, which contains the distinct data and methodology selection options on-hand:

$$\mathbf{Y}^{(MC)}(\vartheta) = \beta_{\vartheta_{AMT,\beta}} \mathbf{X}_{\vartheta_{\mathbf{X}}} + \epsilon_{\vartheta_{AMT,\beta,\epsilon}} \tag{21}$$

with

$$\beta_{\vartheta_{AMT,\beta}} = \left(\beta_{i=\left(\vartheta_{AMT,\beta}\right)_i,\, j}\right)$$
$$\mathbf{X}_{\vartheta_{\mathbf{X}}} = \left(X_{j=(\vartheta_X)_j}\right)$$
$$\epsilon_{\vartheta_{AMT,\beta,\epsilon}} = \left(\epsilon_{i=\left(\vartheta_{AMT,\beta,\epsilon}\right)_i}\right)$$

where $i \in [1, 2, ..., m]$, $j \in [1, 2, ..., d, d+1]$, m is the number of PCF portfolio components, and d is the (overall) number of factors in (all) MC β-models. With this specification, we obtain

$\mathbf{Y}^{(MC)}$: the sought $m \times 1$ column vector of (Monte Carlo) simulated single PCF xR^{NAV}'s,

$\beta_{\vartheta_{AMT,\beta}}$: a $m \times (1+d)$ matrix consisting of rows of AMT-appropriate factor loadings,

$X_{\vartheta_{\mathbf{X}}}$: a $(1+d) \times 1$ column vector consisting of a scalar one (for the intercept), followed by all d simulated independent variable outcomes,

$\epsilon_{\vartheta_{AMT,\beta,\epsilon}}$: a $m \times 1$ column vector of randomly sampled AMT-appropriate errors,

which are defined via the parameter vector $\vartheta = (\vartheta_{AMT},\ \vartheta_{\mathbf{X}},\ \vartheta_{\beta},\ \vartheta_{\epsilon})$ with

ϑ_{AMT}: the eight Asset Metrix Types: Buy Out, Venture Capital, Fund of Funds, Real Estate, Distressed Debt, Mezzanine, Infrastructure, and Natural Resources,

$\vartheta_{\mathbf{X}}$: the $\mathbf{X}$-generation options:

$\mathbf{X}$-method: Multivariate normal, historical sampling, and R-vine copulas,

$\mathbf{X}$-data: -

ϑ_{β}: the β-estimation options:

β-method: Ordinary least squares, and robust MM-estimation,

β-data: Quarterly separate AMT-index data, Index Quagging data, and single fund data,

ϑ_{ϵ}: the ϵ-modeling options:

ϵ-method: empirical sampling, and mixture distribution,

ϵ-data: NAV cut-off threshold, further NAV partitions.

Since we have linear portfolios, we can calculate the comprehensive PCF portfolio xR^{NAV}, denoted by $Y^{(PCF)}$, by simply adding the NAV-weighted components of (the column vector) $Y^{(MC)}$ up

$$Y^{(PCF)} = (\mathbf{w}_{NAV})^T\, \mathbf{Y}^{(MC)} \tag{22}$$

where $(\mathbf{w}_{NAV})^T$ is the (transpose of the $m \times 1$ column) vector of corresponding portfolio NAV-weights defined in Eq. (9).

Simulation Algorithm The procedure, symbolized by Eq. (21), has to be iterated k-times to obtain a (Monte Carlo) simulated PCF portfolio xR^{NAV} vector $\mathbf{Y}^{(PF)}$ consisting of k components:

Iterate k-times

1. **Simulate:** the vector $\mathbf{Y}^{(MC)}$

1.a) $\mathbf{X}$-generation: create macro-condition
1.b) β-model-selection: use factor loadings from pre-determined β-model

Option: (randomly) draw a β-model from the repertory of available models[48]

1.c) ϵ-term-sampling: (randomly) draw β-model-appropriate residual terms

Option: sample multiple errors for one factor scenario $\mathbf{X}$

1.d) Calculate Eq. (21) with $\mathbf{X}$, β and ϵ from the previous steps

2. **Calculate:** the scalar $Y^{(PCF)}$ with Eq. (22)

to generate (a sample of) k possible PCF portfolio xR^{NAV}'s.

With the k-dimensional vector $\mathbf{Y}^{(PF)} = \left(Y_1^{(PCF)}, Y_2^{(PCF)}, \ldots, Y_k^{(PCF)}\right) \in \mathbb{R}^k$ of PCF portfolio xR^{NAV}'s, i.e. a length k i.i.d. sequence of MC model outcomes, it is straightforward to calculate the empirical cumulative distribution function

$$\hat{F}_k^{\mathbf{Y}^{(PF)}}(y) := \frac{1}{k}\sum_{i=1}^{k} \mathbb{I}_{Y_i^{(PCF)}<y}$$

and Value at Risks using the *plug-in estimator*, which is the associated empirical generalized inverse function

$$\left(\hat{F}_k^{\mathbf{Y}^{(PF)}}\right)_\alpha^{-1} := \inf\left\{y \in \mathbb{R} : \hat{F}_k^{\mathbf{Y}^{(PF)}}(y) \geq \alpha\right\}$$

in a subsequent step.

Convergence of Empirical Distribution Function By the strong law of large numbers, the estimator $\hat{F}_k^{\mathbf{Y}^{(PF)}}(y)$ converges (point-wise) to the "real, but latent" c.d.f. $F(y)$ as almost surely, for every *fixed* value of y

$$\hat{F}_k^{\mathbf{Y}^{(PF)}}(y) \overset{a.s.}{\to} F(y)$$

[48] The idea of β-model sampling can be compared to the concept of ensemble learning (or bagging) in statistical learning (see [Bre96]). The lecture of machine learning textbooks like [HTF09, KJ13] is advised to get a sense of these concepts. See also point "Bagging" in Appendix B.

Fig. 12: Monte Carlo model concept map

which makes the empirical distribution function an asymptotically consistent estimator. The theorem of Glivenko–Cantelli strengthens this result by proving uniform convergence

$$\left\| \hat{F}_k^{\mathbf{Y}^{(PF)}} - F \right\|_\infty \equiv \sup_{y \in \mathbb{R}} \left| \hat{F}_k^{\mathbf{Y}^{(PF)}}(y) - F(y) \right| \overset{a.s.}{\rightarrow} 0$$

which is stronger than point-wise convergence. However, the *rate of convergence* is yet obtained by the central limit theorem, that uses the notion of convergence in distribution, actually the weakest form of convergence, since it is implied by all other types of convergence.

$$\hat{F}_k^{\mathbf{Y}^{(PF)}}(y) \overset{d}{\rightarrow} F(y) \text{ as } k \rightarrow \infty$$

or

$$\lim_{k \rightarrow \infty} \hat{F}_k^{\mathbf{Y}^{(PF)}}(y) = F(y)$$

Hence, the distribution of $k \cdot \hat{F}_k^{\mathbf{Y}^{(PF)}}(y)$ is binomial for each fixed $y \in]-\infty, \infty[$

$$k \cdot \hat{F}_k^{\mathbf{Y}^{(PF)}}(y) \sim \mathcal{B}(k,\ F(y))$$

and thus $\hat{F}_k^{\mathbf{Y}^{(PF)}}(y)$ is asymptotically normal distributed

$$\hat{F}_k^{\mathbf{Y}^{(PF)}}(y) \sim \mathcal{N}\left(F(y),\ \sqrt{\frac{F(y)\left[1 - F(y)\right]}{k}} \right) \text{ as } k \rightarrow \infty$$

(see [Ser80], p. 57). The main result of this asymptotical property of sample quantiles[49] is that the rate of convergence (for MC VaRs) is of order $k^{-1/2}$, since

$$\hat{F}_k^{\mathbf{Y}^{(PF)}}(y) - F(y) \overset{d}{\rightarrow} k^{-1/2} \cdot \sqrt{F(y)\left[1 - F(y)\right]} \cdot \mathcal{N}(0,\ 1)$$

which means that appending one significant figure of accuracy requires increasing k by a factor of 100. Clearly, we are free to determine confidence intervals via resampling techniques, if we do not want to rely on vague asymptotical properties in practical applications.

[49] See e.g. [Bah66, Ser80, HL11] for more details on sample quantile and Monte Carlo Value at Risk properties.

From Excess Return to P&L Distribution Like high-yield spreads, one-year (risk-free, i.e. government bond) yields are known at simulation date. So, we can simply define the vector of simulated losses

$$\mathbf{L} = -\left(\mathbf{Y}^{(PF)} + \mathbf{r}^{riskfree}\right) \cdot \mathbf{NAV}$$

as the sum of portfolio xR^{NAV} and (global average) one-year risk-free rate multiplied by the negative of portfolio NAV at analysis date. Since the calculation is done this easily, we decide to just consider the vector of portfolio xR^{NAV}'s $\mathbf{Y}^{(PF)}$ in the remainder of the paper, which is better suited for comparisons with historical simulation results.

5 Results, Applications, Examples

With complete Monte Carlo and historical simulation models on hand, the time has come to apply both R prototypes. In Section 5.1 the model results of MC and historical simulation approach are contrasted and in Section 5.2 MC model outcomes are confronted with Solvency II standard formula figures. Both comparisons are done on the basis of the ten fund test portfolio introduced in Section 4.2.3. In Section 5.3 diversification effects resulting from increasing PCF portfolio sizes are demonstrated briefly and in Section 5.4 the improved accuracy of MC simulations with increasing iteration counts is evaluated.

5.1 Monte Carlo vs. Historical Simulation

Comparing Monte Carlo and historical simulation results presumes *unconditional* MC sampling *with respect to high-yield spreads*.

Unconditional Ten Fund Test Portfolio Results

In order to replicate the *unconditional* ten fund test portfolio xR^{NAV} distribution, we generate

- 25,000 MC simulated portfolios each for three different residual subsets with exact model specification:
  - **X**-generation: 1,000 macro conditions x 25 error draws per condition, R-vine factor dependencies, log-normally distributed high-yield spreads (–> unconditional simulation),

  - β-estimation: quarterly random AMT-index β-model-sampling, MM-estimated β-coefficients,
  - ϵ-modeling: direct ϵ-sampling from top 80%, top 40% and 20-60% NAV partitions, and

- 12,500 historical (simulation) draws

to obtain four empirical distribution functions of xR^{NAV}'s, which bottom quantiles are summarized in the following table:

Quantile (in %)	min	Q 0.5	Q 1	Q 2	Q 5	Q 10	Q 25	Q 50	mean
Historical	-56%	-37%	-32%	-27%	-20%	-12%	3%	13%	16.2%
MC (top 80)	-126%	-39%	-32%	-26%	-18%	-12%	-2.1%	8.0%	9.2%
MC (top 40)	-93%	-33%	-28%	-23%	-16%	-9.8%	-1.0%	8.4%	9.4%
MC (20-60)	-105%	-39%	-33%	-28%	-19%	-13%	-2.4%	8.6%	10.2%

So, the Solvency II relevant 99.5% xR^{NAV}-at-Risks for the historical, MC with top 80% NAV errors, MC with top 40% NAV errors, and MC with 20-60% quantile NAV errors simulations are -37%, -39%, -33%, and -39% respectively[50]. The lower quantiles generally exhibit similar values for all four approaches, although MC and historical minimum observations differ substantially. The historical median value is 4-5% greater than the MC figures and the mean lies 6-7% higher for the historical simulation. The mean/median differences may be explained with atypically high historical VC observations during the millennium internet boom/bubble and the fact that historical simulation results can differ substantially with respect to the sampled AMT portfolio structure. To be concrete, the historically drawn portfolios (probably) overrepresent BO and VC funds as compared to the ten fund test portfolio. A comprehensive distributional comparison is preferably achieved via density plots and rugs like in Figure 13, where the *unconditional* MC[51] vs. historical simulation of the ten fund test portfolio subset is exemplarily visualized. There the dashed vertical lines (on the left) indicate the corresponding empirical 0.5%-quantiles and the vertical solid lines (in the kernel density centers) display the sample mean values. The gray lines represent the density and 0.5%-quantile obtained from an MC simulation without error sampling, but with bias correction; thus they can be regarded

[50] The corresponding Conditional xR^{NAV}@Risks are -44.0%, -49.4%, -46%, and -49.7%.

[51] Monte Carlo errors are here directly sample from the top 80% NAV subset of linear model residuals.

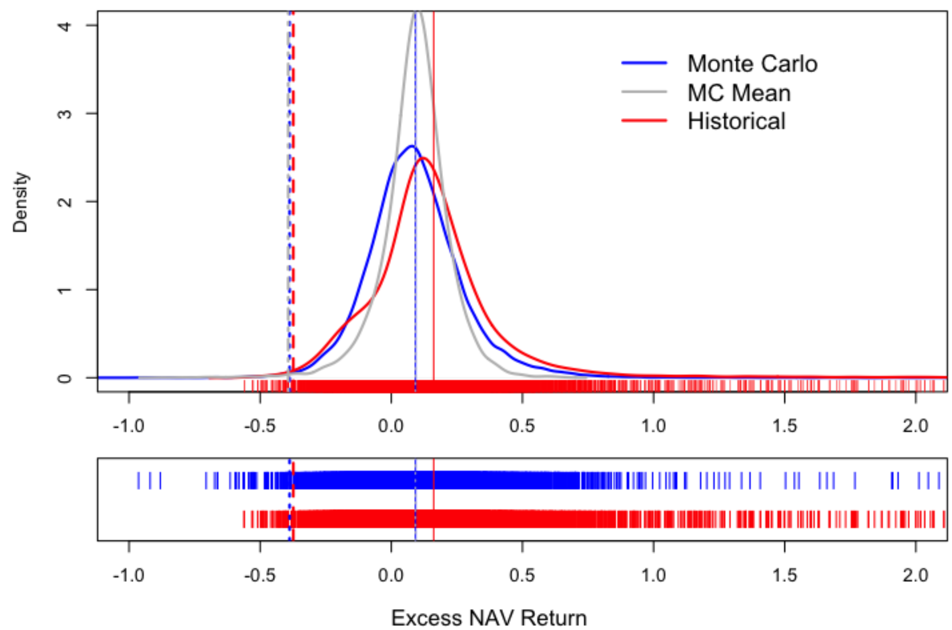

Fig. 13: *Unconditional* MC vs. historical simulation

as constituting the systematic (undiversifiable) risk of the particular PCF portfolio.

Pros and Cons

Despite the (naive) historical simulation model's parsimony and conceptually plainness, its biggest limitation is a fundamental rigidness, which manifests itself in inadequate fund/portfolio similarities and the omitted processing of conditional (public market) information. In contrast, the MC model's flexibility originates from the modular design, which allows incorporating many different methodologies to optimally decompose the return distribution replication problem. However, too much latitude may sometimes tempt MC risk engineers to shoot over the top.

In summary, Table 9 suggests supporting [Ale08]'s notion of preferring MC over historical simulation methods: "Given the substantial limitations, it is difficult to understand why so many banks favor historical VaR over Monte Carlo VaR models. [...] In my view, the great advantage of Monte Carlo simulation is that it uses historical data more intelligently than standard

Simulation	Monte Carlo	Historical
Pro	+ conditional + modular design (enables problem decomposition) + customer can specify certain modules with own expectations + flexible (e.g. for scenario analysis) + NAV weighting	+ parsimonious + conceptually fast & easy implementation + dependencies implicit in historical data
Contra	- more model risk - long(er) R code - conceptually (& computationally) expensive	- more estimation risk - slow(er) R code - unconditional - commitment weighting - no AMT similarity - problematic for very large portfolios
Problems	a) NAV-heteroscedasticity (one-dimensional) b) time-regime-pattern	a) fund-similarity (high-dimensional) b) time-regime-pattern

Tab. 9: MC vs. historical simulation: Pro & Cons

historical simulation does" (see [Ale08], Volume 4, pp. 142-143).

The main advantage of the MC method in our context is that the toughest MC issue comprises of one-dimensional NAV-heteroscedasticity, however, in the historical simulation approach, the intricate fund-similarity problem is of higher dimensionality. As a consequence, we are able to resolve NAV-heteroscedasticity quite satisfactorily, while the question of fund-similarity stays intractable. Just a vast amount of supplementary PCF data may ease the fund-similarity issue in historical PCF simulations since here the historical data is used less intelligently than in Monte Carlo approaches.

5.2 Monte Carlo vs. Standard Formula (Solv. II)

Comparing Monte Carlo VaR and the corresponding Solvency II standard formula figure presumes *conditional high-yield spread* MC sampling.

Conditional Ten Fund Test Portfolio Results

In order to replicate the *conditional* ten fund test portfolio xR^{NAV} distribution, we generate

- 100,000 MC simulated portfolio xR^{NAV}'s each with exact model specification:
  - **X**-generation: 2,000 macro conditions x 50 error draws per condition, R-vine factor dependencies, deterministic high-yield spreads (−> conditional simulation),
  - β-estimation: quarterly random AMT-index β-model-sampling, MM-estimated β-coefficients,
  - ϵ-modeling: direct ϵ-sampling from top 80%, top 40% and 20-60% NAV partitions.

For that reason we use high-yield spreads (6.40%) and (one-year) treasury yield curve rates (0.53%) as of 11th May 2016[52]. The translation of xR^{NAV}'s to NAV-Returns is easily done by adding the risk-free rate, which yields empirical R^{NAV} 0.5%-quantiles of

- $-0.403 + 0.0053 = -39.77\%$ with ϵ-sampling from 20-60%-quantile NAV partition,

[52] Note, that NAVs are stale between end of quarter dates.

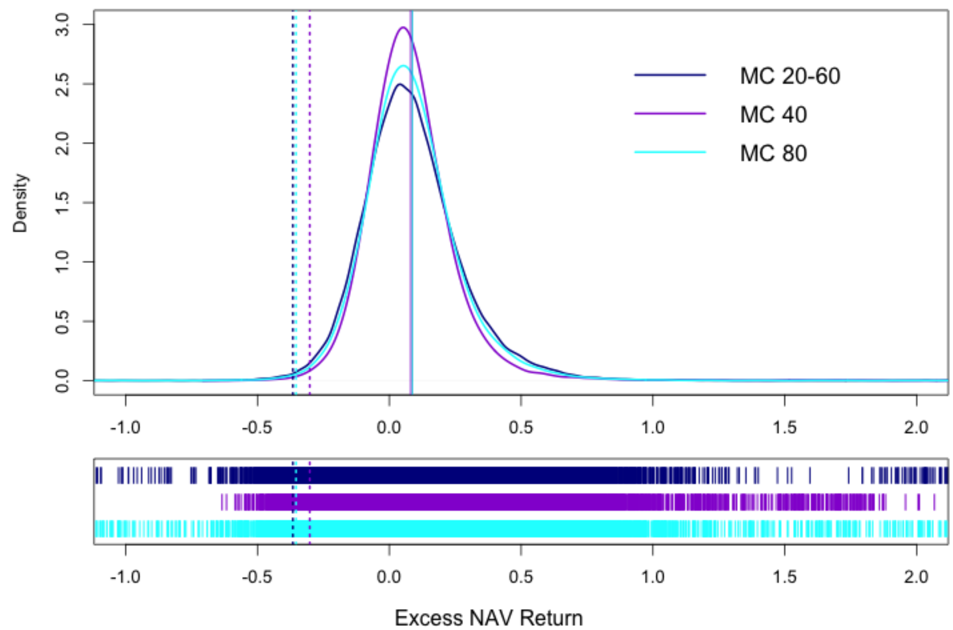

Fig. 14: *Conditional* MC simulation for distinct ϵ-sampling proposals

- $-0.286 + 0.0053 = -28.07\%$ with ϵ-sampling from top 40% NAV partition,
- $-0.380 + 0.0053 = -37.47\%$ with ϵ-sampling from top 80% NAV partition.

The Solvency Capital Requirement is then simply obtained by multiplying with minus 1 (times NAV).

Figure 14 illustrates the distributional differences caused by different error-sampling schemes incorporated in a conditional MC simulation. The dashed vertical lines represent 0.5%-quantiles and the solid lines the corresponding mean values.

Solvency II Standard Formula Results

The application of a look-through approach[53] is definitely one of the guiding principles in the Solvency II framework, whenever feasible. Though, ac-

[53] When deploying the look-through approach, AMT fund types have to be ascribed to distinct market risk sub-models in the Solvency II standard formula:

cording to Commission Delegated Regulation (EU) 2015/35 Article 168 (6) "closed-ended and unleveraged alternative investment funds [...] shall in any case be considered as type 1 [equities]". So the benchmark for our MC model is a Solvency Capital Requirement, corresponding to a 99.5% VaR, of 39% plus a symmetric adjustment[54] of

$$0.5 \cdot \left(\frac{46.12 - 43.85}{43.85} - 0.08 \right) = -0.0141 = -1.41\%$$

which we calculate using an exchange traded fund replicating the MSCI World total return index[55]. Therefore, the Solvency II standard formula quantifies the SCR with 37.59% for PCFs.

So in our case, the most conservative conditional MC SCR figure (39.77%) is slightly above the standard formula SCR benchmark. The MC model with ϵ-sampling from top 80% NAV-subset yields however virtually the same value (37.47%). This proximity might suggest a reasonable MC model calibration with the full top 80% NAV-subset, even though the (nearly) exact match in our example is particular a product of pure chance[56]. Generally, it is not straightforward to predict for which PCF portfolio compositions the SCR obtained by MC simulations will lie above/below the standard formula figure. But, since our estimated β-factor loadings are fairly small, MC model outcomes for highly-diversified PCF portfolios should come below the standard formula SCR. Moreover, do highly-diversified portfolios (particularly in the fund-vintage dimension) suggest direct ϵ-sampling from the "gentle" top 40% NAV subset.

In conclusion, this naive example seems to confirm private equity's subsumption to Type 1 equities in the Solvency II standard formula; in some preliminary Solvency II drafts, private equity was regarded as Type 2 equity with 49% SCR.

- BO, VC, FoF, and Commodities are assigned to the Equity risk sub-module
- DD is assigned to the Spread and Interest risk sub-modules
- RE is assigned to the Property risk sub-module
- Infra is assigned to the Equity risk sub-module (infrastructure carve-out)
- MEZZ is partially assigned to Equity, Spread and Interest risk sub-modules

[54] The formula for the symmetric adjustment of the equity capital charge is outlined in Article 172 of Commission Delegated Regulation (EU) 2015/35. Per construction, the symmetric adjustment works tendentially in the opposite direction than the conditional high-yield spread factor in our factor models.

[55] db X-trackers MSCI World TRN Index UCITS ETF (XMWD) in US-Dollar. (URL: https://www.quandl.com/data/GOOG/LON_XMWD)

[56] Confer MC accuracy results in Section 5.4.

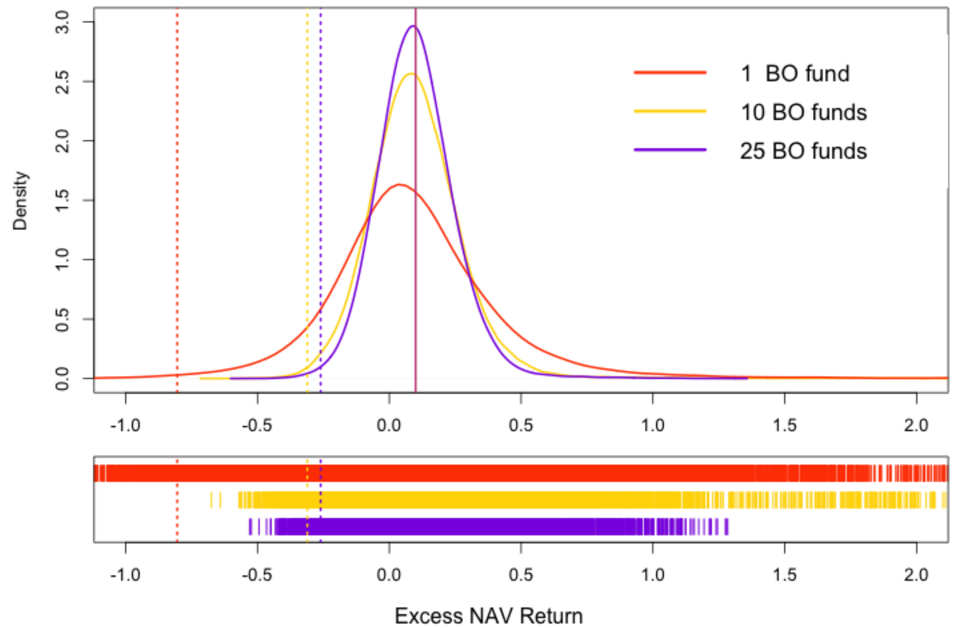

Fig. 15: *Conditional* MC simulation of various-sized BO portfolios

5.3 Diversification Effects

In order to demonstrate MC model diversification effects (by means of homo-AMT portfolios with different component counts to segregate the dispersion impact), we generate

- 100,000 MC simulated portfolio xR^{NAV}'s each with exact model specification:
  - **X**-generation: 2,000 macro conditions x 50 error draws per condition, R-vine factor dependencies, deterministic high-yield spreads (–> conditional simulation),
  - β-estimation: quarterly random AMT-index β-model-sampling, MM-estimated β-coefficients,
  - ϵ-modeling: direct ϵ-sampling from top 80% NAV subset, for portfolios consisting of
    - 1 Buy Out fund,
    - 10 Buy Out funds,

* 25 Buy Out funds.

Here, we again use public market data as of 11th May 2016 to induce conditional MC results. The vertical dashed lines in Figure 15, representing sample 0.5%-quantiles $\left(\hat{F}_k^{\mathbf{Y}^{(PF)}}\right)^{-1}_{\alpha=0.5\%}$, expose the anticipated diversification benefits in a very obvious way. On the one hand, there is a big margin between the one-BO-fund portfolio and the 10/25-BO-funds portfolios and, on the other hand, the gap, i.e. the diversification benefit, between 10 and 25-BO-fund portfolio is relatively narrow.

5.4 Monte Carlo Model Accuracy

Finally, we employ resampling techniques, actually re-run the model several times, to evaluate the precision (convergence properties) of MC simulation outcomes more closely. Once again, the ten fund test portfolio is selected for the analysis. In order to replicate two *conditional* ten fund test portfolio xR^{NAV} distributions, we generate

- 10,000 MC simulated portfolio xR^{NAV}'s (200 macro conditions x 50 error draws per condition), and
- 100,000 MC simulated portfolio xR^{NAV}'s (2,000 macro conditions x 50 error draws per condition); with otherwise equal model specifications:
  - **X**-generation: R-vine factor dependencies, deterministic high-yield spreads –> conditional simulation (as of 11th May 2016),
  - β-estimation: quarterly random AMT-index β-model-sampling, MM-estimated β-coefficients,
  - ϵ-modeling: direct ϵ-sampling from top 80% NAV partition.

For the accuracy assessment of 0.5% quantiles $\left(\hat{F}_k^{\mathbf{Y}^{(PF)}}\right)^{-1}_{\alpha=0.5\%}$ of the resulting two *conditional* ten fund test portfolio xR^{NAV} distributions, we repeatedly run the MC simulations for a 100 times[57]. Consequently, we are able to determine two empirical c.d.f.'s for the quantities of interest $\left(\hat{F}_{k=10,000}^{\mathbf{Y}^{(PF)}}\right)^{-1}_{\alpha=0.5\%}$ and $\left(\hat{F}_{k=100,000}^{\mathbf{Y}^{(PF)}}\right)^{-1}_{\alpha=0.5\%}$. The result of our "naive jackknife" is visualized in Figure 16. Here all mean and medians values (vertical lines) center around -34%, whereas the dispersion for the 10,000 return MC simulation is considerably higher than for the 100,000 example.

[57] The 100 MC simulation re-runs took 36 minutes for the 10,000 and 508 minutes for the 100,000 case.

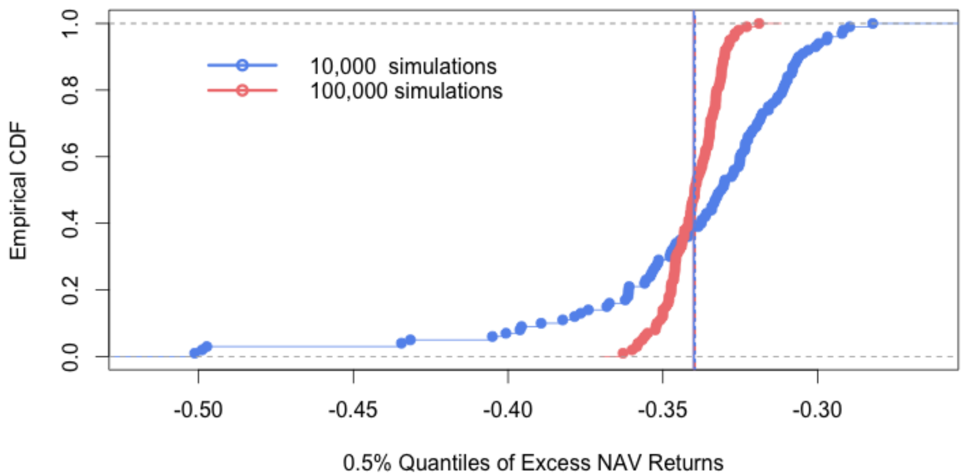

Fig. 16: Empirical c.d.f. of xR^{NAV} 0.5%-quantiles

In real-life cases of application, it is straightforward to determine confidence intervals directly from the appropriate empirical c.d.f. to e.g. communicate MC model accuracy (in the sense of convergence properties) to potential customers. Unfortunately, some time-consuming re-run procedure, which can be regarded as the prerequisite for jackknifing, has to be performed for each new PCF portfolio all over again, since previous accuracy assessments are (maybe not impractical but surely) not reusable for new portfolio compositions. So learning over time, MC model operators should ideally establish a "feel-good" setting of simulation iterations (for a desired accuracy level), in order to make the protracted model re-runs expendable.

The xR^{NAV} 0.5%-quantile evolution over time (actually with increasing MC sample sizes), illustrated in Figure 17, might additionally help to get an impression about the MC model's convergence properties. Here 2,000,000 *conditional* MC portfolio xR^{NAV}'s (20,000 macro conditions x 100 error draws per condition) are simulated for the ten fund test portfolio (with equal model specifications as before) After almost 10 hours of computations in R, this length 2,000,000 vector is used successively to estimate empirical 0.5%-quantiles with gradually extending subsets of the total vector, which displays the evolution to more accurate, i.e. less volatile, estimates the larger the subsets become. The 0.5%-quantile for the total vector is -33.4%, which is the terminal value of the red and green 0.5%-quantile course lines in Figure 17. When applying forward versus backward extension of total vector subsets, slightly different (mirror-symmetric) paths can be observed.

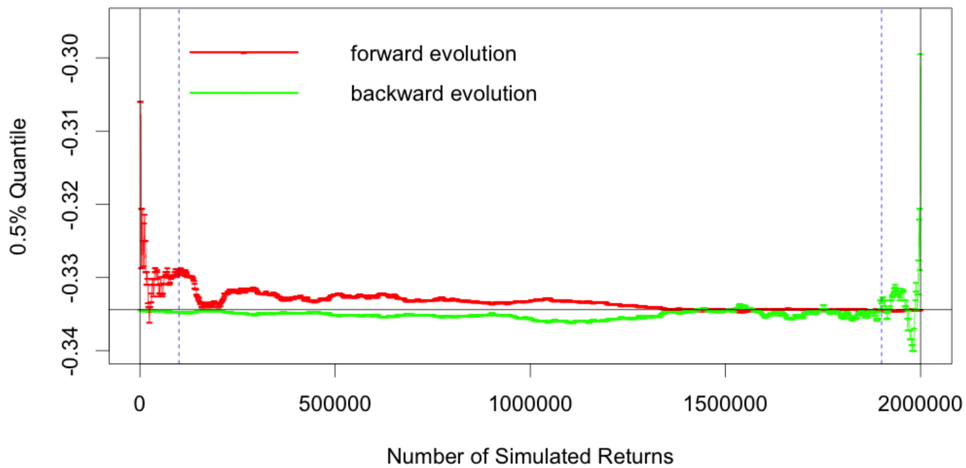

Fig. 17: Empirical 0.5%-quantile evolution of xR^{NAV}'s

A natural accuracy assessment procedure is to repeatedly permute the total vector and analyze the associated volatilities for small sample sizes to (re)use the available data vector more subtly than in the approaches above[58].

Cross-Validation In the AMT-factor-modeling section, only about 2/3 of the available Prequin single fund data is used (as the training set for single fund modeling). The (randomly) omitted 1/3 data subset (the test set) can be utilized to review the linear factor models on single fund level and the error models with respect to out-of-sample errors[59]. Though, for the index factor models the full data set is employed to construct the AMT-indices. Naturally, distinct index-building approaches can be implemented, e.g. fund-size weighting instead of equal commitment-weighting, to revise the AMT-factor-models relying on index data (and to obtain new β-models for bagging as a pleasant side effect). Next, a "leave-X-funds-out" cross-validation procedure of index results may be adopted; the aim of creating "leave-X-funds-out" AMT-indices is again however rather the generation of new, equivalent "sample-able" β-models (and associated residuals) in the Monte Carlo context, than the out-of-bag/sample error assessment, which could (just theoretically) be of interest on β-model level. General, the vali-

[58] In principle, this is jackknifing (deterministic permutations) or bootstrapping (random permutations). Note, that jackknifing in contrast to bootstrapping assures reproducibility.

[59] In the R code, we additionally implement a function to generate bootstrap samples of single fund data.

dation of sub-models is not of primary concern as their stand-alone performance is vacuous for the comprehensive MC model performance. On the other hand, backtests of the (comprehensive) MC model are virtually impossible due to the scarcity of historical PCF data.

6 Conclusion

This paper develops and analyzes private capital fund risk models, which shall be capable of estimating 99.5% Value-at-Risks over a one-year horizon (as regulated in the Solvency II legislation).

Performance Measure The first step towards (strictly position-based) PCF risk models with a one-year horizon is investigating the measurement of intermediate fund performance given the illiquid and stale nature of PCFs. Therefore we (have to) use the concept of net asset value returns (R^{NAV}) that incorporates intermediate NAV figures into the return calculation. Despite the NAV's dubious reputation, the derived xR^{NAV} performance measure can be shown to be the most natural and reasonable dependent variable, especially in the Solvency II context. As the xR^{NAV} measure is comparatively well-behaved for one-year horizons, it could be of scientific interest, if NAV returns continue to be tractable (e.g. in regression analysis) when using just quarterly horizons.

AMT-factor-models Once the dependent variable is defined, individual linear multi-factor-models for decomposing xR^{NAV}'s are developed for eight distinct fund types. Here standard, time-series OLS regression methods are not feasible without major adjustments. In order to facilitate the regression problem, we separate β-factor estimation from ϵ-modeling to achieve an as accurate as possible return decomposition. Several proposals for β-estimation are presented in our paper, which differ with respect to their distinct data editing or estimation methodologies. Our favored β-estimation approach resorts on self-constructed AMT indices starting at four different quarters (for AMTs: BO, VC, FOF, DD, RE); to aggregate the resulting four different factor loadings, the method of Index Quagging (quarterly aggregating) is introduced. As ϵ-modeling of single fund residuals fails with simple parametric approaches, the most promising approaches are (a) direct, empirical ϵ-sampling, and (b) using finite mixture distribution models, which are a convenient means to nicely reveal latent PCF return characteristics. Generalizing multi-factor model results, we can retain, that (a) the public market return β-factors are relatively small, (b) the high-yield spread β-factors are surprisingly big (and highly explanatory), (c) the liquidity factor of [PS03, FNP12] is only significant using single fund data in regressions, and (d) the idiosyncratic error terms are across-the-board quite high and wild. Since most economic publications examining private equity/capital risk and return report remarkably higher public market return β-factors (yet for other performance measures), the topic remains a natural candidate for further research[60]. On the other hand, the much-quoted NAV staleness is also apparent in our analysis as the phenomena induces the significant high-yield spread factor loadings, which is possibly an even better way to tackle the staleness problem than the [Dim79] approach. Concluding the multi-factor modeling section, we have to admit, that we are looking forward to seeing applications of our approach in other settings (e.g. adapted for dynamic models with just quarterly horizons).

Historical and Monte Carlo Simulation To finally replicate PCF portfolio xR^{NAV}'s, we suggest a naive historical simulation method, which unfortunately can not convince due to its unconditional rigidness and its intractable fund similarity problem. Nevertheless, the historical simulation's rough estimates may convey a "good first impression" of PCF portfolio returns, as all sampled returns were actually historically feasible. In contrast, the Monte Carlo simulation approach, which draws on the pre-developed AMT-factor-models, overcomes all conceptional issues (like most prominent NAV-heteroscedasticity) reasonably well and consequently generates quite satisfactory quantile estimates via the generalized inverse of the simulated, empirical c.d.f. of xR^{NAV}'s. As there exist several equivalent β-estimation (e.g. OLS vs. MM-estimation, index data vs. single fund data, quagging coefficients vs. 4 quarterly coefficients) and ϵ-modeling (e.g. empirical sampling, mixture distributions) approaches, the idea of randomly sampling out of the repertory/ensemble of β- and ϵ-models in MC simulations seems natural; the β-model sampling idea corresponds to bagging in an MC simulation context. The missing part of our MC model ($\mathbf{Y} = \beta\mathbf{X} + \epsilon$), i.e. the vector of simulated macro-conditions $\mathbf{X}$, is obtained via pair-copulae construction methods, which is - although conceptually challenging - easily done by R's "VineCopula" package.

Once again, we may highlight, that the general fund similarity problem is the most severe and ambiguous conceptual challenge in either simulation

[60] See [Buc14] for an up-to-date study about "The Alpha and Beta of Private Equity Investments" including further references.

method; it is caused by the static model setups both times. Applications of the MC model indicate with respect to the Solvency II framework, that standard formula and MC model figures of 99.5% Value-at-Risks lie pretty much inside the same range for our ten fund test portfolio. For more diversified portfolios the Solvency Capital Requirement (SCR) figure obtained by MC simulations may come below the standard formula value; whereby extreme macroeconomic conditions may reverse this statement as MC's high-yield spread factor and standard formula's symmetric adjustment go in the opposite direction. Generally, we apply our MC model in this paper just for illustrative purposes; there are manifold model capabilities (by using other model parameterizations) not employed throughout the sample applications, which can be used to study the xR^{NAV} distributions for various PCF portfolio compositions in much greater detail. One obvious, interesting field of further research is the MC model's utilization in order to determine optimal PCF-type portfolio allocations under e.g. Value-at-Risk constraints.

Bottom Line Since private capital funds implement dynamic trading strategies (with mainly untraded assets), the development of a static one-period risk model is naturally associated with some challenges. Dynamic modeling in a multi-period setting seems more appropriate. However, in the Solvency II context, the one-year horizon is mandatory, which makes the static focus on a four-quarter horizon eligible, as here unimportant and unnecessary dimensions are simply dropped to concentrate on the essential. So we can conclude, that this β-bagging-Monte-Carlo-model tailored for the simulation of one-year PCF portfolio returns is truly neat and operational.

Ultra posse nemo obligatur

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A Additional PCF Performance Measures

Modified Internal Rate of Return (M.IRR) To overcome the computational ambiguity and the economical misapplication of IRRs, a modified approach was introduced by [Lin76]:

$$1+M.IRR_{t_N} = \left(\frac{FV_{t_N}\left(D, r_{inv}\right)}{PV_{t_N}\left(C, r_{fin}\right)}\right)^{1/t_N} = \left(\frac{\sum_{i=0}^{N}\left(D_{t_i}\cdot\left(1+r_{inv}\right)^{t_N-t_i}\right)}{\sum_{i=0}^{N}\left(C_{t_i}\cdot\left(1+r_{fin}\right)^{-t_i}\right)}\right)^{1/t_N}$$

Here the future value of all distributions up to time t_N has to be calculated with a pre-determined re-investment rate r_{inv} and the present value of all contributions up to time t_N is computed with a pre-determined finance rate r_{fin}. Clearly, the additional assumptions regarding re-investment and finance rate make the M.IRR performance measure less parsimonious than the primal IRR.

Alternative NAV Returns Some industry practitioners may prefer a more conservative NAV return version called R^{EvB}:

$$1 + R_{t_n}^{EvB} = \frac{NAV_{t_n} + \sum_{i=n-4q+1d}^{n} D_{t_i}}{NAV_{t_n-4q} + \sum_{i=n-4q+1d}^{n} C_{t_i}}$$

More accurate return figures can be obtained with the formulas proposed by [Die66]. The "Modified Dietz" formula is recommended in the EVAC Risk Measurement Guidelines from January 2013[61]:

$$R_{t_n}^{MD} = \frac{NAV_{t_n} - NAV_{t_n-4q} + \sum_{i=n-4q+1d}^{n} (D_{t_i} - C_{t_i})}{NAV_{t_n-4q} + \sum_{i=n-4q+1d}^{n} (C_{t_i} - D_{t_i}) \cdot \frac{t_n - t_i}{t_n - t_{n-4q+1d}}} \tag{23}$$

This method was developed to evaluate portfolio performance in the presence of external in- and out-flows. In some cases a simplified version of Eq. (23) yield similar results. In the "Simple Dietz" method the stringent time-weighting of cash flows is abandoned:

$$R_{t_n}^{SD} = \frac{NAV_{t_n} - NAV_{t_n-4q} + \sum_{i=n-4q+1d}^{n} (D_{t_i} - C_{t_i})}{NAV_{t_n-4q} + 0.5 \cdot \sum_{i=n-4q+1d}^{n} (C_{t_i} - D_{t_i})} \tag{24}$$

The return calculation with Eq. (24) is based on the assumption that all distributions and contributions occur at the half-way point in time within the return period (or are distributed uniformly across the period, and so the cash flows occur on average in the middle of the period).

But in the risk modeling context the more aggressive/volatile "standard" R^{NAV} formula from Eq. (4) may be most appropriate. Another point for that very formula is the nice transition/approximation property between Eq. (4) and the so-called Horizon IRR.

Horizon Internal Rate of Return (H.IRR) The Horizon IRR approaches makes the IRR calculation viable for intermediate horizons - e.g. one year for our purposes - during a fund's lifetime. Consequently Eq. (3) becomes:

$$-NAV_{t_x} + \sum_{i=x+1d}^{x+h-1d} \frac{D_{t_i} - C_{t_i}}{(1 + IRR_{t_x,t_h})^{t_i - t_x}} + \frac{NAV_{t_{x+h}}}{(1 + IRR_{t_x,t_h})^{t_{x+h} - t_h}} = 0 \tag{25}$$

[61] URL: http://www.investeurope.eu/media/10083/ evca-Risk-Measurement-Guidelines-January-2013.pdf

Here the net asset value at period start NAV_{t_x} serves as first out-flow, which may be interpreted as fictional purchasing price of the fund or first investment cash flow, and the net asset value at horizon end $NAV_{t_{x+h}}$ is considered as the final in-flow or divestment cash flow. This method is especially flexible as it allows to compute backward Horizon IRRs for $x = 0$ and forward Horizon IRRs for $x + h = N$.

If we use aggregated cash flow data on a quarterly basis, H.IRR determination corresponds to solving the following equation for r:

$$\begin{aligned} 0 &= -NAV_{t_x} + \frac{\triangle NCF_{t_{x+1q},t_x}}{(1+r)^{1q}} + \frac{\triangle NCF_{t_{x+2q},t_{x+1q}}}{(1+r)^{2q}} + \\ &+ \frac{\triangle NCF_{t_{x+3q},t_{x+2q}}}{(1+r)^{3q}} + \frac{\triangle NCF_{t_{x+4q},t_{x+3q}} + NAV_{t_{x+4q}}}{(1+r)^{4q}} \\ &= -NAV_0 + \frac{\triangle NCF_{Q1}}{(1+r)^{1q}} + \frac{\triangle NCF_{Q2}}{(1+r)^{2q}} + \frac{\triangle NCF_{Q3}}{(1+r)^{3q}} + \\ &+ \frac{\triangle NCF_{Q4} + NAV_{Q4}}{(1+r)^{4q}} \end{aligned} \tag{26}$$

The simplified version of Eq. (26) can be further approximated for yearly cash flows or respectively yearly discounting:

$$0 = \frac{\triangle NCF_{Q1} + \triangle NCF_{Q2} + \triangle NCF_{Q3} + \triangle NCF_{Q4} + NAV_{Q4}}{(1+r)^{4q}} - NAV_0 \tag{27}$$

Now the conjunction of H.IRR and R^{NAV} gets obvious as Eq. (27) is exactly a transformed version of Eq. (4). So for relatively short horizons H.IRR may be approximated by R^{NAV}; but, clearly, the longer the horizons the more the returns obtained by Eq. (25) and Eq. (4) can deviate and the H.IRR is to be preferred, as it submits a more accurate measure of performance.

B Accounting for Non-normality and Heteroscedasticity

In the PCF context, heavily non-normal regression residuals are likely to be constant companions. These non-normal residual distributions may come along with or even directly result from issues related to heteroscedasticity (with respect to NAV, time, etc.). Fortunately, there are several valid remedies to account for (severe) violations of the normality and i.i.d.[62] assumptions of errors ϵ:

1. TRANSFORMATIONS to make $\mathbf{Y}$ more normal distribution like (in a bijective manner):

   (a) LOGARITHMIC transformation using the natural logarithm of the dependent variable. Applying this method can be regarded as affective behavior among econometricians when they are confronted with positively skewed data. In prosaic terminology, the log-transformation is just a special case of a Box-Cox transformation.

   (b) BOX-COX power transformation introduced by [BC64]:

   $$Y_i^{(\lambda)} = \begin{cases} \frac{(Y_i)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \log Y_i & \text{if } \lambda = 0 \end{cases}$$

   The biggest limitation of this popular approach is the non-negativity constraint on the variable to be transformed.

   (c) LAMBERT WAY to Gaussianize data described by [Goe15], on the other hand, has no difficulties with negative values. Thus, it is the most versatile transformation method, mentioned here. Yet, transformations generally contain two major disadvantages, which both concern the back-transformation to the "original, non-normal world/scale". Firstly, the values obtained by applying the back-transformation may be unrealistically extreme (at the tails). This would be a severe problem in Monte Carlo simulations of Conditional Value at Risks. Secondly, the beta factors, i.e. regression coefficients, lose their "straightforward" interpretability, as the economic concepts of idiosyncratic and systematic risk get mixed up or become inseparable in the latent, normally transformed world as a collateral damage.

2. ROBUST REGRESSION analysis is a potent way to reveal difficulties in classical OLS regressions, as *robust* statistical procedures are by design *robust* to violations of its assumptions; i.e. these methods generate reliable estimates even when the assumptions of the statistical model are not *exactly,* but *approximately* true. If there are substantial differences between OLS and robust regression results, then the validity

[62] Respectively: uncorrelated with expectation zero and homoscedastic with finite variance.

of the least squares approach have to be questioned[63]. Since there are no outliers in the normal world of OLS (per definition/assumption) robust methods are a mandatory choice for detecting "wild shots" in regressions with non-normal data. "In all these problems [where fitting is essential in identifying "wild shots"], resistant techniques seem essential, if a good job of identifying "wild shots" is to be possible" (see [Tuk75] p. 18). So robust (regression) approaches are not capable of modeling/explaining all (typical and atypical) observations, but they provide - in contrast to OLS - a comparative good fit to the bulk of the data, if the data contains (a few) atypical observations, which is, as mentioned above, mandatory to recognize outliers as such. Classical OLS regression may be regarded as over-fitting in the presence of non-normal data, as the estimated β-coefficients force the residuals' mean to zero. On the other hand, robust regression approaches do not aim at zero mean residuals and, thus, may generate linear-biased estimators of the conditional mean, i.e. $\mathbb{E}(\mathbf{Y}) \neq \mathbf{X}\hat{\beta}$. This is true for linear models with intercept and a non-symmetric residual distribution, since "then the intercept is asymptotically biased, but the slope estimates are nonetheless consistent." (see [MMY06] p. 100)[64].

3. Tailored Regression for peculiar residual distributions, if the distribution is "too non-normal" for standard robust methods, i.e. the distribution exhibits too many deviations to justify even an approximative normality assumption for the bulk/core/center of the observed residuals (see: contamination breakdown point in the robust regression context, cf. [MMY06] p. 58). Especially in the presence of skewed data, [MMT09] recommend an alternative robust regression approach in the context of Capital-Asset-Pricing-Model (CAPM) applications using the skewed generalized t distribution family (SGT), mentioned in Section 4.2.3. In our case the single factor model of [MMT09] has to be extended to our multiple factor setting.

4. Weighted Regression is applicable, if we possess a reasonable method to detect uncertain/dubious observations, which need not be outliers

[63] Hence, [Tuk75] emphasizes the importance of deploying supplementary robust analysis concurrent to classical methods: "[Good statistical practice] means that any analysis based upon arithmetic means, moments, least squares, to name a few standard cases, needs to be at least accompanied by a resistant/robust analysis if an appropriate one can be found" (see [Tuk75] p. 3).

[64] The R package "robustbase" contains robust regression methods including model selection and multivariate statistics where the authors strive to cover the content of [MMY06].

(or tail observations) in any case. After detection, uncertain observations are, consequently, attached with smaller weights than other more reliable data points[65]. The key to this approach is thus obviously the classification method for and the definition of uncertain observations within $\mathbf{Y}$. It is remarkable that in problems, where highly idiosyncratic observations are characteristic, like in the PCF return context, tail observations are neither outliers nor uncertain observations per se. Thus, both outlier and uncertain observation detection optimally should not be based on putative dependent variable peculiarities, which is however not feasible in most cases (see textbook [Str11]). Robust regression can be regarded as weighted regression, where putative outliers (tail observations) are weighted down to obtain more explanatory beta coefficients for the non-outlier data points.

5. Remove outliers before running an OLS regression. [BKSW10] propose a linear multi-factor model for internal rate of returns (IRRs) of individual venture capital investments. In their setting, they exclude total losses and "outperformers" with IRRs above $+99\%$ from the analysis and, as a result, perform the OLS regression only on the remaining "normal-performer-sample". In the subsequent Monte Carlo simulation of IRRs, the excluded positive and negative outliers are re-injected by directly sampling from both omitted outlier sets. This approach can be regarded as an extreme example of a weighted regression with binary weights, where all outliers, or more specifically tail observations, are classified as uncertain data points. In this way, the procedure, at worst, might have the opposite effect of what is intended by causing additional bias with respect to the regression coefficients.

6. Diversified Portfolios should exhibit "more well-behaved" residuals, i.e. idiosyncratic returns. Therefore, simulating a more diversified portfolio (with e.g. our historical simulation method) could in the best case resolve all error distribution non-normalities. The construction of PCF indices, which are theoretically maximum diversified, is finally the logical continuation of this idea. Our design proposal is building equal-commitment-weighted indices for our eight AMTs (like BO, VC, RE, etc.) and possibly one (private capital) index composed of various AMTs. However, the time-series of yearly index returns will be rather

[65] Weighted least square regression, a special case of a weighted regression, is the extension of an OLS regression for applications with unequal variances of single ϵ_i's (heteroscedasticity). There each observation gets rescaled with the corresponding residual variance to run an OLS regression in the subsequent step (see [RPD98] pp. 413).

short as a consequence of the lack of PCF data before 1990. Next, using quarterly index returns to quadruple the number of data points is not feasible since by construction of NAV returns yearly and quarterly returns are not mutually convertible (in a multiplicative way). Moreover, there is a general critique of the popular portfolio forming approach in [ALS10] in the context of tests of econometric cross-sectional factor models.

7. Non-Paramedic Regressions are designed to perform reasonably well for (many) arbitrary error distributions, as they do not rely on parametric models for the residual distribution. Non-parametric approaches may be the last resort when we finally can not overcome the error distribution's complexity and have to admit that we see no way to capture the error distribution in parametric form. Sampling directly from empirical residuals is thus ultima ratio.

8. Bagging or bootstrap aggregating is a machine learning technique proposed by [Bre96] to improve the stability and accuracy of statistical learning algorithms. Especially for models with high variance and low bias, bagging predictors reduce variance and help to avoid overfitting, as multiple model versions are created by making *bootstrap* samples of the (learning) data set (see e.g. [HTF09] Chapter 8.7). The resulting ensemble/repertory of models is then *aggregated* to one model by averaging their outcomes/coefficients. So in cases, where no persuasive single best model can be obtained, since multiple equivalent models coexist, the application of sampling-based procedures may be a legitimate means. A "real bagging approach" in our context is the implementation of separate regressions on bootstrap samples (of single fund data) and the aggregation of these model results in a subsequent step.