15.455x Mathematical Methods of Quantitative Finance

Week 4: Continuous-Time Finance

Paul F. Mende
MIT Sloan School of Management

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Continuous-time finance

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From discrete to continuous time

Discrete-time processes

- Exact
- Complete
- Useful
- Computable
- Extensible

Continuous-time

- Distinguish continuity of time from continuity of processes
- Applicability: pros and cons
- Limit from discrete to continuous is instructive

Scaling the random walk

Basic building block is **elementary random walk** with multiple steps of unit size

$$\mathbb{E}_t[z_s] = 0, \quad t < s$$
$$\mathrm{Var}(z_t) = 1$$
$$\mathrm{Cov}(z_t, z_s) = \delta_{ts}$$

Define basic RW random variable as sum of many steps

- Shift in origin
- Conditional expectation
- Difference of paths

$$B_{1,T} = \sum_{t=t_0+1}^{t_0+T} z_t$$
$$\mathbb{E}_t[B_{1,T}] = 0, \quad t \leq t_0$$
$$\mathrm{Var}_t(B_{1,T}) = T$$
$$\mathrm{Var}_{t_1}(B_{1,T}) = T - (t_1 - t_0), \quad t_0 \leq t_1 \leq T$$

$$B_{1,T'} - B_{1,T} = f(T - T')$$

- Time-translation invariance

Take it to the limit

Whenever constructing any limiting process, always ask these questions:

- Does the limit converge?
- In what sense does the result represent the converging process?
- Is the process unique?
  - Can multiple processes posses the same limit?
  - Can there be different limits by varying the limiting process?

Scaling the random walk

Study the sum of many steps as scale changes for **time step** and **step size**

Can we subdivide interval while preserving distribution of terminal values?

- **Case I**: rescale time step only:

$$\text{Let } \Delta t = T/n,$$

$$B_{\Delta t,T} \equiv \sum_{t=1}^{n} z_t.$$

$$\mathbb{E}[B_{\Delta t,T}] = 0$$

$$\text{Var}(B_{\Delta t,T}) = n\text{Var}(z_t) = n$$

$$\lim_{n\to\infty} \text{Var}(B_{\Delta t,T}) = \infty$$

Scaling the random walk

Study the sum of many steps as scale changes for **time step** and **step size**

Can we subdivide interval while preserving distribution of terminal values?

- **Case II**: rescale time step and step size:

$$\text{Let } \Delta t = T/n, \quad \epsilon_t \equiv \lambda z_t,$$

$$B_{\Delta t,T} = \sum_{t=1}^{n} \epsilon_t = \lambda \sum_{t=1}^{n} z_t$$

$$\mathbb{E}[B_{\Delta t,T}] = 0,$$

$$\mathrm{Var}(B_{\Delta t,T}) = n\mathrm{Var}(\epsilon_t) = n\lambda^2 \mathrm{Var}(z_t) = n\lambda^2$$

$$\text{Suppose } \lambda = 1/n \to 0 \text{ as } n \to \infty.$$

$$\text{Then } \lim_{n\to\infty} \mathrm{Var}(B_{\Delta t,T}) = n\lambda^2 \to 0.$$

Scaling the random walk

Study the sum of many steps as scale changes for **time step** and **step size**

Can we subdivide interval while preserving distribution of terminal values?

- **Case III**: rescale time step and step size **simultaneously** in specific relationship:

$$\text{Let } \Delta t = T/n, \quad \lambda = \sqrt{\Delta t} = \sqrt{T/n}, \quad \epsilon_t \equiv \lambda z_t$$

$$B_{\Delta t, T} = \sum_{t=1}^{n} \epsilon_t = \sqrt{\Delta t} \sum_{t=1}^{n} z_t$$

$$\mathbb{E}[B_{\Delta t, T}] = 0,$$

$$\mathrm{Var}(B_{\Delta t, T}) = n\mathrm{Var}(\epsilon_t) = n\Delta t\, \mathrm{Var}(z_t) = T$$

$$\lim_{\Delta t \to 0} B_{\Delta t, T} \sim \mathcal{N}(0, T)$$

Properties of the limit

Brownian motion paths

- Everywhere continuous, nowhere differentiable
- Convergence in distribution

Why construct as limit of discrete process?

- Historical
- Conceptual
- Computational
- LLN, CLT, universality

Issues of uniqueness, completeness more subtle

Alternative limiting processes

- Causal structure
- Non-anticipating
- Jumps
- Cadlag, caglad

Full circle: limit of the limit

Consider behavior of Brownian paths that are separated by finite vs. infinitesimal times

- Finite: terminal values normally distributed with variance proportional to time

$$X(t_1, t_2) = B(t_2) - B(t_1),$$
$$X \sim \mathcal{N}(0, t_2 - t_1), \quad t < t_1 \leq t_2$$

- Infinitesimal: use process as fundamental building block, analogous to unit step RW

$$\mathrm{d}B_t \sim \mathcal{N}(0, \mathrm{d}t)$$

$$\mathrm{Cov}(\mathrm{d}B_t, \mathrm{d}B_{t'}) = \begin{cases} 0, & t \neq t' \\ \mathrm{d}t, & t = t' \end{cases}$$

$$B(T) = B(0) + \int_0^T \mathrm{d}B_t$$

Stochastic integrals and SDE's

Differential form useful in developing closed-form analytical models

- Stochastic differential equations (SDE) reduce to partial differential equations (PDE)

Integral form useful in Monte Carlo simulations

- Example: generate ensemble of time-dependent price paths, compute solutions as risk-neutral expectations of discounted payoffs

$$V(0) = e^{-rT}\mathbb{E}^Q[V(T)]$$

Scales for drift and volatility

- Use elementary Brownian motion to build prices processes. Recall that for Δt=1,

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \mu_0 + \sigma_0 z_t \sim \mathcal{N}(\mu_0, \sigma_0^2)$$

$$\log\left(\frac{S_T}{S_0}\right) = \log\left(\frac{S_T}{S_{T-1}}\frac{S_{T-1}}{S_{T-2}}\cdots\frac{S_1}{S_0}\right) \sim \mathcal{N}(\mu_0 T, \sigma_0^2 T)$$

- So take limit as time step shrinks, holding scaling parameters fixed:

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \lim_{\Delta t\to 0}\left[\sum_{t=1}^{T/\Delta t}(\mu\Delta t) + \sum_{t=1}^{T/\Delta t}(\sigma z_t\sqrt{\Delta t})\right] \\ &= \mu T + \sigma\int \mathrm{d}B_t \sim \mathcal{N}(\mu T, \sigma^2 T)\end{aligned}$$

Scales for drift and volatility

More generally, if drift and volatility depend on time deterministically,

$$\log\left(\frac{S_{t_2}}{S_{t_1}}\right) = \int_{t_1}^{t_2} \mu(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \sigma(t)\,\mathrm{d}B_t$$

Itô processes and Itô's lemma

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Itô process

Define an **Itô process** as stochastic process of the form

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t$$

How do functions behave?

$$\text{If } \mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t, \text{ then what is } \mathrm{d}\left(F(X)\right)?$$

- The usual chain rule would say

$$\mathrm{d}F(t, X) = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X$$

- However since *X* is **nowhere differentiable**, this **does not** hold.

Itô's lemma

Ideas behind proof:

- Although *X* is not differentiable, the function *F* is, so use Taylor's theorem to expand it
- Identify leading order terms in d*t*
- Look for **convergence in probability**, and identify terms with vanishing variance as non-stochastic

$$F'(x) = \lim_{\Delta x \to 0} \left(\frac{F(x+\Delta x) - F(x)}{\Delta x} \right)$$

Replace standard limit with distributional one

$$\lim_{\Delta x \to 0} \text{Prob} \left[\left(\frac{F(x+\Delta x) - F(x)}{\Delta x} - F'(x) \right)^2 > 0 \right] = 0,$$

$$\lim_{\Delta x \to 0} \mathbb{E} \left[\left(\frac{F(x+\Delta x) - F(x)}{\Delta x} - F'(x) \right)^2 \right] = 0$$

Itô's lemma

Expanding,

$$
\begin{aligned}
&\text{If } \mathrm{d}X_t = a \ \mathrm{d}t + b \ \mathrm{d}B_t, \ \text{then} \\
&\quad \mathrm{d}F = \Bigg(\frac{\partial F}{\partial t} \mathrm{d}t + \frac{\partial F}{\partial X} \mathrm{d}X + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (\mathrm{d}X)^2 \\
&\qquad\qquad + \frac{\partial^2 F}{\partial t \partial X} \mathrm{d}t \, \mathrm{d}X + \frac{1}{2} \frac{\partial^2 F}{\partial t^2} (\mathrm{d}t)^2 + \mathcal{O}\left((\mathrm{d}t)^3, (\mathrm{d}X)^3, \cdots \right) \Bigg) \\
&\qquad = \left(\frac{\partial F}{\partial t} \mathrm{d}t + \frac{\partial F}{\partial X} [a \ \mathrm{d}t + b \ \mathrm{d}B_t] + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} [a \ \mathrm{d}t + b \ \mathrm{d}B_t]^2 + \cdots \right)
\end{aligned}
$$

Moments of truth

Compute moments up to fourth order:

$$\begin{aligned}\mathbb{E}[\,\mathrm{d}B_t] = 0, \quad \mathbb{E}[(\,\mathrm{d}B_t)^2] &= \,\mathrm{d}t \\ \mathbb{E}[(\,\mathrm{d}B_t)^3] = 0, \quad \mathbb{E}[(\,\mathrm{d}B_t)^4] &= 3(\,\mathrm{d}t)^2\end{aligned}$$

So for the Itô process,

$$\begin{aligned}\mathbb{E}[\,\mathrm{d}X_t] &= \mathbb{E}[a\ \mathrm{d}t + b\ \mathrm{d}B_t] = a\ \mathrm{d}t, \\ \mathbb{E}[(\,\mathrm{d}X_t)^2] &= \mathbb{E}[(a\ \mathrm{d}t + b\ \mathrm{d}B_t)^2] = a^2\,(\,\mathrm{d}t)^2 + b^2\ \mathrm{d}t, \\ \mathrm{Var}(\,\mathrm{d}X_t) &= b^2\ \mathrm{d}t, \\ \mathrm{Var}\left((\,\mathrm{d}X_t)^2\right) &= \mathbb{E}[(\,\mathrm{d}X_t)^4] - \mathbb{E}[(\,\mathrm{d}X_t)^2]^2 = 2b^4(\,\mathrm{d}t)^2 + \mathcal{O}((\,\mathrm{d}t)^3)\end{aligned}$$

Itô's lemma

Since variance of higher terms vanish to order d*t,* treat them as non-stochastic.

$$
\begin{aligned}
\mathrm{d}F &= \left(\frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial X}\,\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\,\mathrm{d}X)^2 + \cdots \right) \\
&= \left(\frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial X}\left[a\ \mathrm{d}t + b\ \mathrm{d}B_t\right] + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\left[b^2\ \mathrm{d}t\right] \right) \\
&= \left[\frac{\partial F}{\partial t} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2} \right] \mathrm{d}t + \left[\frac{\partial F}{\partial X} \right] \mathrm{d}X \\
&= \left[\frac{\partial F}{\partial t} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2} + a\frac{\partial F}{\partial X} \right] \mathrm{d}t + \left[b\frac{\partial F}{\partial X} \right] \mathrm{d}B
\end{aligned}
$$

This is the desired result for d*F*, which is therefore also an Itô process.

Itô's lemma

Heuristic: expand and replace
$$(\mathrm{d}B_t)^2 \to \mathrm{d}t,$$
$$(\mathrm{d}X_t)^2 \to b^2\ \mathrm{d}t$$

Then the differential has one additional term beyond the usual chain rule,

$$\boxed{\mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\mathrm{d}t}$$

The **differential** of a function of an Itô process **is itself an Itô process**.

Itô's lemma

Examples:

$$\text{Let } \frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}B.$$

$$\text{Then } \mathrm{d}S = (\mu S)\mathrm{d}t + (\sigma S)\mathrm{d}B, \text{ and}$$

$$\mathrm{d}F = \mathrm{d}(\log S) = \left[\mu - \frac{\sigma^2}{2}\right]\mathrm{d}t + \sigma \mathrm{d}B$$

- Lognormal variable
- Geometric Brownian motion
- Same volatility, lower drift

Itô process dynamics

Generalized random walk: variable scale and variable drift

- General process with coefficients that are integrable functions depending on *X* and *t*.

$$\mathrm{d}X = a(X,t)\mathrm{d}t + b(X,t)\mathrm{d}B$$

- Integrating the differential form gives the distribution from which the path segment is drawn. Constant or time-varying coefficients give **normally distributed** paths

$$\mathrm{d}X = \mu \mathrm{d}t + \sigma \mathrm{d}B, \quad X_{t_2} - X_{t_1} \sim \mathcal{N}\left((\mu(t_2 - t_1), \sigma^2(t_2 - t_1)\right)$$

$$\mathrm{d}X = a(t)\mathrm{d}t + b(t)\mathrm{d}B, \quad X_{t_2} - X_{t_1} \sim \mathcal{N}\left(\int_{t_1}^{t_2} a(t)\mathrm{d}t, \int_{t_1}^{t_2} b(t)^2 \mathrm{d}t\right)$$

- Integrate more general differentials, reversing Itô formula

Stochastic differential equations

Insights from **form** of equations (without solving), from **solutions** to equations, or from **transformation** into new equations (e.g., PDE)

Itô processes

Brownian motion with drift

$$\mathrm{d}S_t = \mu\mathrm{d}t + \sigma\mathrm{d}B_t$$
$$S_T = S_0 + \mu T + \sigma(B_T - B_0)$$

- Allows possible negative prices.
- Is this a problem in practice (i.e., if probability is sufficiently low)?
- Could a large enough drift drift term and sufficiently positive initial value prevent negative prices?

Itô processes

Geometric Brownian motion with drift

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \\ \mathrm{d}(\log S_t) &= \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}B_t \\ S_T &= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma(B_T - B_0)} \end{aligned}$$

- Excludes negative prices.
- Drift and variance go to zero as *S* approaches zero.
- As a model for asset prices, does not allow for bankruptcy, credit defaults,...
- Is it empirically a good fit for data?

Itô processes

Ornstein–Uhlenbeck process

$$\mathrm{d}S_t = \lambda(\bar{S} - S_t)\mathrm{d}t + \sigma \mathrm{d}B_t$$

- Unbounded process
- Sign of lambda
- **Mean-reversion** dynamics
- Drift term moves *S* toward mean value
- Symmetric around mean value
- Restoring force **proportional** to distance from mean value
- Random shocks unbiased
- Constructed out of simple Brownian plus simple deterministic piece
- Can generalize to let the mean value itself be slowly varying

Itô processes

Cox–Ingersoll–Ross process

$$\mathrm{d}\rho_t = \lambda(\bar{\rho} - \rho_t)\mathrm{d}t + \sigma\sqrt{\rho_t}\mathrm{d}B_t$$

$$\text{Let } F = \sqrt{\rho}, \quad \frac{\partial F}{\partial \rho} = \frac{1}{2\sqrt{\rho}}, \quad \frac{\partial^2 F}{\partial \rho^2} = -\frac{1}{4}\rho^{-3/2}$$

$$\mathrm{d}F = \left(\frac{4\lambda\bar{\rho} - \sigma^2}{8F} - \frac{1}{2}\lambda F\right)\mathrm{d}t + \frac{1}{2}\sigma\mathrm{d}B_t$$

- **Mean-reversion** dynamics
- **Avoids origin** for $2\lambda\bar{\rho} > \sigma^2$
- Interest rates and term structure

From SDE to PDE: The Black-Scholes equation

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Itô's lemma applied to a special portfolio

Consider: Let $V = V(t, S)$ and $\mathrm{d}S = (\mu S)\mathrm{d}t + (\sigma S)\mathrm{d}B$.

$$\mathrm{d}V = \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 V}{\partial S^2} \right) \mathrm{d}t + \left(\frac{\partial V}{\partial S} \right) \mathrm{d}S$$

Now combine the two previous in a portfolio:

$$\begin{aligned} \pi &\equiv V - \Delta S, \quad \Delta \text{ constant.} \\ \mathrm{d}\pi &= \mathrm{d}V - \Delta \mathrm{d}S \\ &= \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 V}{\partial S^2} \right) \mathrm{d}t + \left(\frac{\partial V}{\partial S} - \Delta \right) \mathrm{d}S \end{aligned}$$

Only last term is stochastic...and its **coefficient vanishes** for special choice of delta.

Black-Scholes equation

- The **stochastic term vanishes** for the evolution of this portfolio if $\Delta = \dfrac{\partial V}{\partial S}$

- Since the right-hand side is in general **time-varying**, so is quantity of shares held in hedging portfolio.

- Because there is **no risk** remaining, the portfolio growth rate is risk-free

$$\mathrm{d}\pi = (r\pi)\mathrm{d}t = r(V - \Delta S)\mathrm{d}t = \left(rV - rS\frac{\partial V}{\partial S}\right)\mathrm{d}t$$

- Equating coefficients of d*t*

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Black-Scholes equation

This gives the non-stochastic partial differential equation

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0}$$

Black-Scholes equation

What do we know about the solution from the equation alone?

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0}$$

- Explicit parameters:
  - Volatility
  - Risk-free rate
  - Independent of drift rate. (Where did μ go?)

- Implicit parameters: strike price, expiration date, type (call/put/exotic) will be set by boundary conditions