

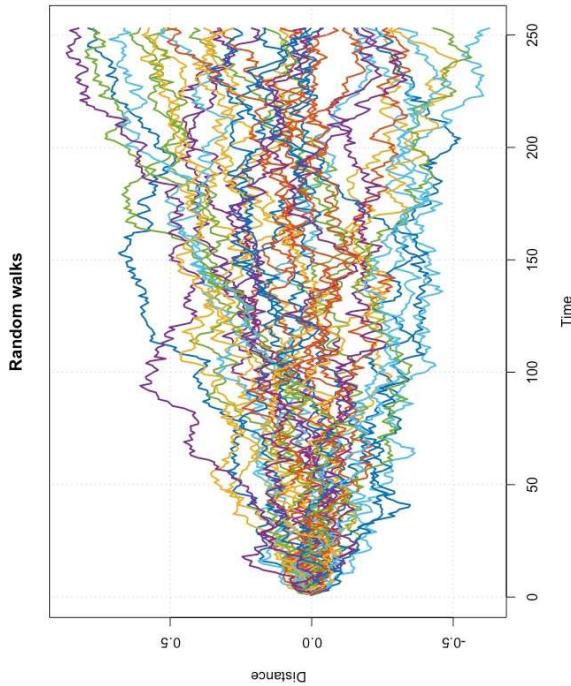


15.455x Mathematical Methods of Quantitative Finance

Week 6: Continuous-Time Finance (continued)

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Finance at MIT
Where ingenuity drives results



Probability density for random walks

Probabilities for random walks

- Since a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ has probability density

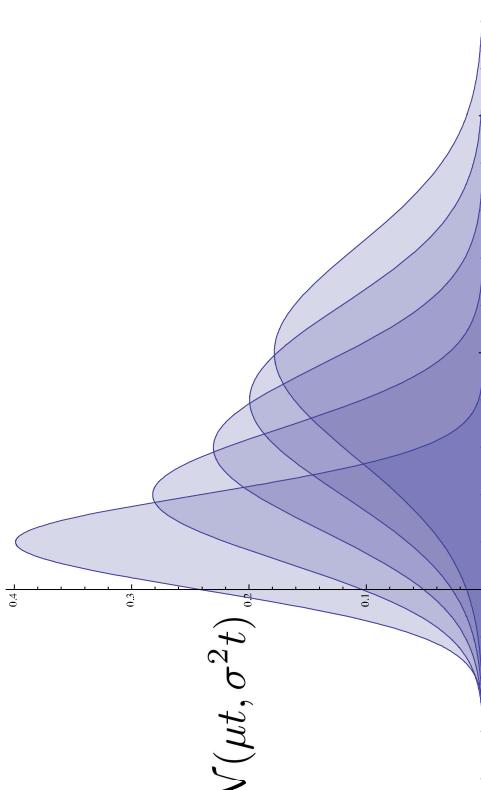
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)},$$

a time-dependent stochastic process where $X_t \sim \mathcal{N}(\mu t, \sigma^2 t)$ has probability density

$$p(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(x-\mu t)^2/(2\sigma^2 t)}$$

- This function satisfies the partial differential equation

$$\frac{\partial p}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial p}{\partial x} = 0$$



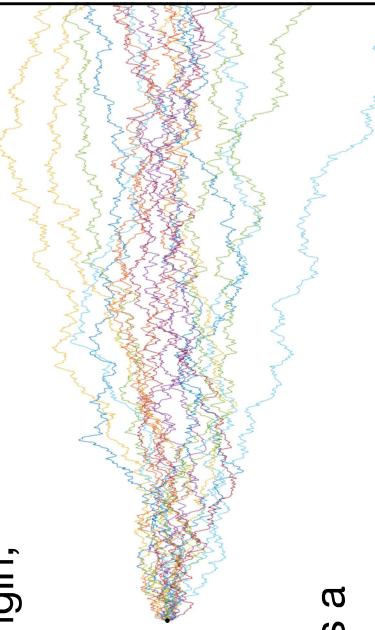
Probabilities for random walks

- More generally, for a random walk that begins elsewhere than the origin,

$$p(x_T, T; x_0, t_0) = \frac{1}{\sqrt{2\pi\sigma^2(T-t_0)}} \exp \left[-\frac{[(x_T - x_0) - \mu(T - t_0)]^2}{2\sigma^2(T - t_0)} \right]$$

- Even though the starting point isn't random, this can be analyzed as a function of its initial coordinates.
 - Notice that it depends only on coordinate differences.
 - It satisfies the "backward" equation

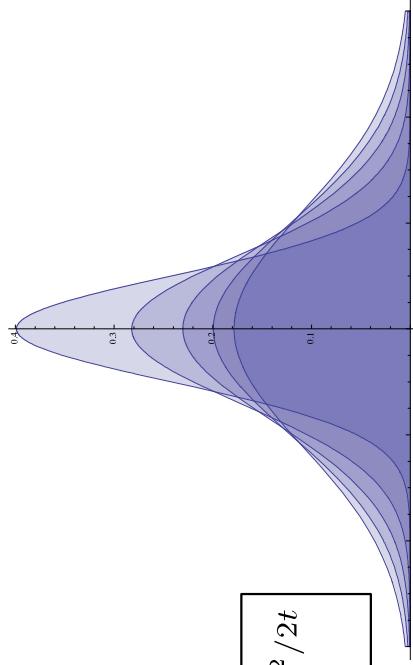
$$\frac{\partial p}{\partial t_0} + \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x_0^2} + \mu \frac{\partial p}{\partial x_0} = 0$$



Diffusion equation, random walks, and probability

- In the special case of pure Brownian motion, $\mu = 0, \sigma = 1$ the probability density obeys the diffusion equation

$$\frac{\partial p_0}{\partial t} = \frac{1}{2} \frac{\partial^2 p_0}{\partial z^2}$$



- The PDE has many solutions

$$p_0(z, t) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}$$

- The Gaussian solution
- describes probability **concentrated at the origin initially that diffuses** over time.
- Increasing likelihood that the endpoint for the walk will be found far from its starting point.
 - Only defined for $t > 0$ due to the square root.

Diffusion equation, random walks, and probability

- This **special** solution can be used to obtain the **general** solution:

For initial conditions $p(z, t = 0) = f(z)$ the general solution is given by

$$p(z, t) = \int p_0(z - w, t) f(w) \, dw = \frac{1}{\sqrt{2\pi t}} \int e^{-(z-w)^2/2t} f(w) \, dw$$

- Examples:

$$f(z) = z^2$$

$$f(z) = e^{az}$$

$$f(z) = \cos(\lambda z)$$

$$f(z) = \theta(z - \kappa) = \begin{cases} 1, & z > \kappa \\ 0, & z < \kappa \end{cases}$$

Diffusion equation, random walks, and probability

- This **special** solution can be used to obtain the **general** solution:

For initial conditions $p(z, t = 0) = f(z)$ the general solution is given by

$$p(z, t) = \int p_0(z - w, t) f(w) \, dw = \frac{1}{\sqrt{2\pi t}} \int e^{-(z-w)^2/2t} f(w) \, dw$$

- Verify solution and initial conditions:

$$\bullet \left[\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial z^2} \right] p(z, t) = \int \left(\left[\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial z^2} \right] p_0(z - w, t) \right) f(w) \, dw = 0;$$

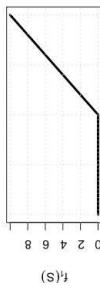
$$\bullet \lim_{t \rightarrow 0} p(z, t) = \lim_{t \rightarrow 0} \frac{1}{\sqrt{2\pi}} \int e^{-u^2/2} f(z + u\sqrt{t}) \, du, \quad \text{using } u = (w - z)/\sqrt{t} \\ = f(z)$$

Special functions

A few special functions

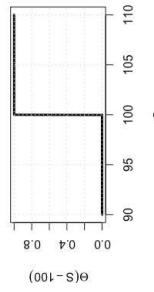
Let's pause to define a few convenient functions, starting by re-writing the familiar payoff function for a call option using absolute value.

$$f_1(S) = \max(S - K, 0) = \frac{1}{2} (|S - K| + S - K)$$



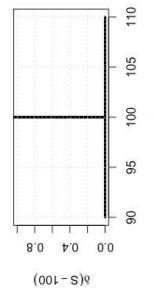
- The slope of the payoff function is the **step function**, which takes values either zero or one.

$$\frac{d}{dS} f_1(S) \equiv \theta(S - K) = \begin{cases} 1 & \text{if } S > K, \\ 0 & \text{otherwise} \end{cases}$$



- The derivative of the step function is the **Dirac delta function**, which is zero almost everywhere – and also has unit area "under the curve"!

$$\frac{d^2}{dS^2} f_1(S) \equiv \delta(S - K) = \begin{cases} 0 & \text{if } S \neq K, \\ \infty & \text{otherwise} \end{cases}$$

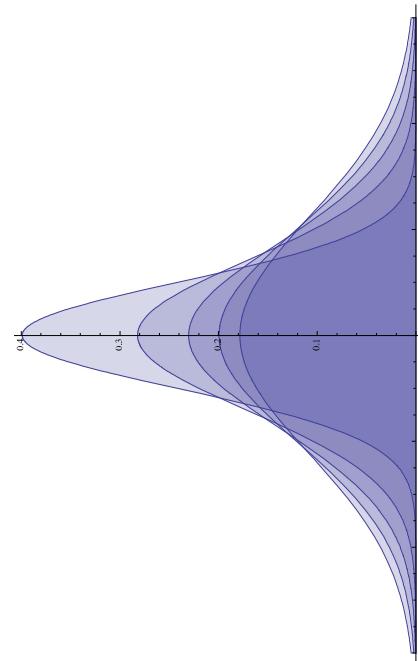


Dirac delta function

- Limit of Gaussian as width goes to zero
 - Singular at zero
 - Integral for area under the curve is one

$$\delta(x) = \lim_{t \rightarrow 0} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & x = 0 \end{cases}$$
$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

- Assigns to any function it is integrated against its value at zero
- Properly speaking, a "generalized function" or functional

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$
$$\int_{-\infty}^{\infty} \delta(x - y) f(x) dx = f(y)$$


Green's functions

Modifying the special solution slightly gives the **Green's function** that can be used to construct solutions to an **inhomogeneous equation**. Define

$$G(z, t) = p_0(z, t)\theta(t) = \frac{\theta(t)}{\sqrt{2\pi t}}e^{-z^2/2t},$$

$$\mathcal{D}G(z, t) = p_0(z, t)\delta(t) = \delta(z)\delta(t).$$

Then if there is a fixed function $h(z, t)$ on the right hand side, G gives a solution:

$$p(z, t) = \int G(z - z', t - t')h(z', t')dz' dt' = \int_0^\infty \int_{-\infty}^\infty \frac{e^{-(z-z')^2/(2(t-t'))}}{\sqrt{2\pi(t-t')}} h(z', t')dz' dt'$$

$$\mathcal{D}p(z, t) = \int \delta(z - z')\delta(t - t')h(z', t')dz' dt' = h(z, t),$$

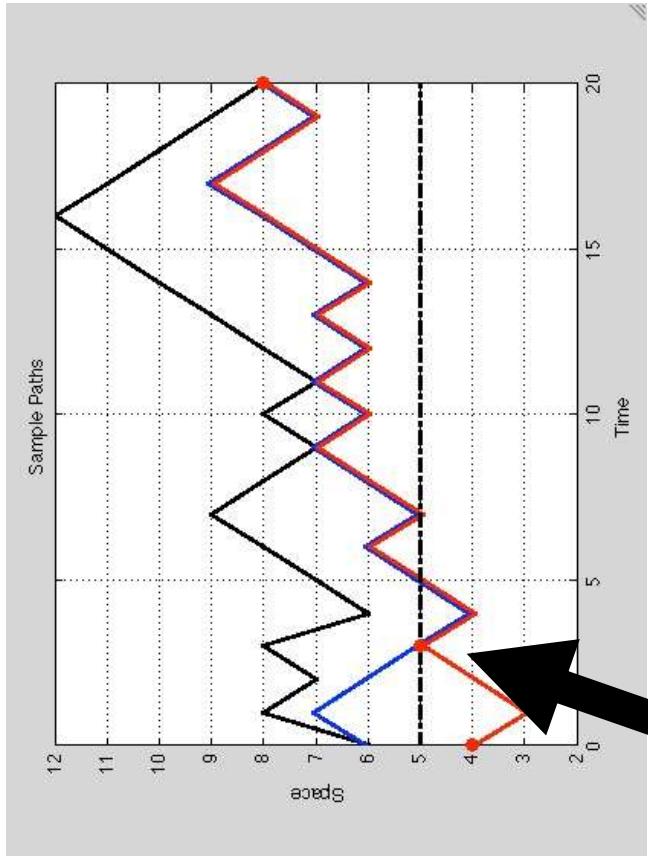
$$\frac{\partial p}{\partial t} - \frac{1}{2} \frac{\partial^2 p}{\partial z^2} = h(z, t)$$

Reflections, barriers, and survival probabilities

Survival probabilities

What is probability to get from point A to point B... without ever hitting point C?

- "Absorbing barrier" to represent events such as default
 - Mean time to hit barrier?
 - Probability to not have hit through time t ?
- Method of images
 - Compute unrestricted probability to go from A to B
 - **Subtract** unrestricted probability to go from A^* to B, where A^* is the image point, i.e., reflection below the barrier of the point A.



Reflect portion of blue path at **first passage** through barrier to get red path

Survival probabilities

Probability to arrive without crossing barrier at z^* , without drift:

$$\begin{aligned} p_s(z, t) &= p_0(z - z_0, t) - p_0(z - [2z^* - z_0], t) \\ &= \frac{1}{\sqrt{2\pi t}} \left(e^{-(z-z_0)^2/2t} - e^{-(z-[2z^*-z_0])^2/2t} \right) \end{aligned}$$

- The survival probability density obeys boundary condition

$$p_s(z^*, t) = 0$$

- Therefore the complete solution for $t > 0$ is

$$p_s(z, t) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \left(e^{-(z-z_0)^2/2t} - e^{-(z+z_0-2z^*)^2/2t} \right) & z > z^*, \\ 0 & z \leq z^*. \end{cases}$$

Survival probabilities

- Probability to arrive, **including drift** term, breaks symmetry.
- Use boundary condition $p_s(z^*, t) = 0$ to determine constant prefactor in "image" term

$$\begin{aligned} p_s(z, t) &= p(z - z_0, t) - C p(z - [2z^* - z_0], t) \\ &= \frac{1}{\sqrt{2\pi\sigma^2 t}} \left(e^{-(z - \mu t - z_0)^2 / 2\sigma^2 t} - C e^{-(z - \mu t + z_0 - 2z^*)^2 / 2\sigma^2 t} \right), \quad C = e^{-2\mu(z_0 - z^*)/\sigma^2} \end{aligned}$$

- Integrate over all non-defaulting results, above the barrier, at time t

$$\begin{aligned} p_s(t) &= \int_{z^*}^{\infty} p_s(z, t) dz \\ &= \Phi \left(\frac{\mu t + z_0 - z^*}{\sqrt{\sigma^2 t}} \right) - e^{-2\mu(z_0 - z^*)/\sigma^2} \Phi \left(\frac{\mu t - z_0 + z^*}{\sqrt{\sigma^2 t}} \right) \end{aligned}$$

$\Phi(x) \equiv \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$

Survival probabilities

Application: corporation **non-default probability** for corporate bond pricing

$$z = \text{firm value} = D + E$$

$$z^* = \text{firm debt} = D$$

$$z_0 = \text{firm current value}, \quad z_0 > z^*$$

- How important is it to have high growth rate vs. high initial buffer to protect against default?
- What is required buffer, given growth rate, so that 10-year default probability is less than 25%?
- What is optimal capital structure to fund growth and minimize default probability?

Survival probabilities

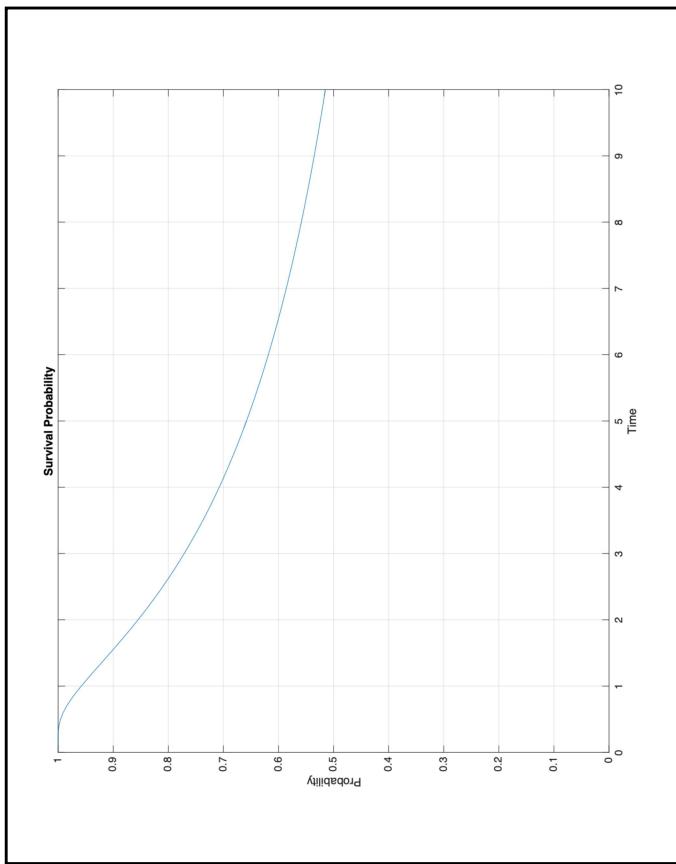
- Sample parameter values (cf. Wise & Bhansali)

$$\mu = 0.01$$

$$\sigma = 0.25$$

$$z_0 - z^* = 0.5$$

- Default entirely due to chance of value diffusion below barrier, absent other sources of business shocks.



Probability densities and expectations

Stock price diffusion

We can also ask about more general future payoffs and expectations.

- The future **expected value** of a function on random paths satisfies the **same differential equation** as the probability density, considered as a function of its initial values.

- Consider the probability density function of the standard stock price path defined by

$$dS = \mu S dt + \sigma S dB$$

The probability $p(S_T, T; S, t)$ satisfies

$$\frac{\partial p}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 p}{\partial S^2} + \mu S \frac{\partial p}{\partial S} = 0$$

What to expect when you're expecting

So consider the future expectation of a function of S_T :

$$\mathbb{E}_t [f(S_T)] = \int p(S_T, T; S, t) f(S_T) dS_T = F(S, t)$$

- The expectation is itself a function of the initial (or current) values of S, t and satisfies the same differential equation, along with the limiting value

$$\lim_{t \rightarrow T} F(S, t) = \int \delta(S_T - S) f(S_T) dS_T = f(S)$$

- For the expectation of a terminal payoff, consider the equation satisfied by its present value

$$V(S, t) = e^{-r(T-t)} F(S, t) = e^{-r(T-t)} \mathbb{E}_t [f(S_T)] = e^{-r(T-t)} \mathbb{E}_t [V(S_T, T)]$$

Drift away

V satisfies a PDE **similar** to Black-Scholes, **except** with a μ -dependent drift

$$\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 V}{\partial S^2} + \mu_S \frac{\partial V}{\partial S} - rV = 0$$

- V would **exactly** satisfy the Black-Scholes PDE if it were instead based on an Itô process where **the drift is replaced by the risk-free rate**

$$dS = rSdt + \sigma SdB$$

- With respect to this evolution equation, the present value of a Black-Scholes contract is given by the expectation of its discounted payoff:

$$e^{-rt}V(S, t) = \mathbb{E}_t [e^{-rT}V(S_T, T)]$$

Black-Scholes solutions

- One method for computing option prices is to evaluate the expectation numerically using Monte Carlo techniques to average over a large number of appropriate paths.
- Another method is to apply the probability density formulas directly. Returning to the original variables for stock price, time, etc.,

$$\begin{aligned} V(s, t) &= \int p(S_T, T; S, t) V(S_T, T) dS_T \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int e^{-(x-x')^2/2\sigma^2(T-t)} f(x') dx', \\ \text{where } f(x') &= g(S') = \max(S' - K, 0) \end{aligned}$$

for a vanilla call option of strike price K expiring at time T .

Black-Scholes solution

$$\text{So } V(S, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{x'=\log K}^{\infty} e^{-(x-x')^2/2\sigma^2(T-t)} (e^{x'} - K) dx'$$

$$= S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-),$$

$$\text{where } d_{\pm} \equiv \frac{\log(S/K)e^{-r(T-t)}}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t} \text{ and } \Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz$$

The "risk-neutral" probability density describes the diffusion of a hypothetical asset with the same volatility as S but with drift rate r :

$$p_{RN}(S_T, T; S, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}S_T} \exp \left[-\frac{\left(\log(S_T/S) - \left(r - \frac{\sigma^2}{2} \right)(T-t) \right)^2}{2\sigma^2(T-t)} \right]$$

Greeks and exotics

The Greeks

It is customary to define various partial derivatives of the solution, including

$$\text{Delta } \Delta \equiv \partial V / \partial S = \begin{cases} \Phi(d_+), & \text{call} \\ \Phi(-d_+) = 1 - \Phi(d_+), & \text{put} \end{cases}$$

$$\text{Gamma } \Gamma \equiv \partial^2 V / \partial S^2 = \frac{\Phi'(d_+)}{\sigma S \sqrt{T-t}},$$

$$\text{Vega } \nu \equiv \partial V / \partial \sigma = \Phi'(d_+) S \sqrt{T-t}$$

- The delta and gamma can be given their own probability/diffusion representation. The vega, which is the derivative with respect to a parameter, cannot.

Black-Scholes solutions: exotic options

Likewise, different payoff functions lead directly to a value formula by plugging into the integral. Example:

For a **binary call option**, with payoff $f(x') = g(S') = \theta(S' - K) = \begin{cases} 1, & S' \geq K, \\ 0, & S' < K \end{cases}$

$$V(S, t) = e^{-r(T-t)} \Phi(d_-)$$

which is directly related to the probability of the stock finishing in the money at time T ...under the risk-neutral measure. This is **not** the real-world probability, which depends on μ

$$p_\mu(S_T, T; S, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}S_T} \exp \left[-\frac{\left(\log \left(S_T / Se^{(\mu-\sigma^2/2)(T-t)} \right) \right)^2}{2\sigma^2(T-t)} \right]$$

Black-Scholes solutions: exotic options

Example: consider a power option whose payoff is a fixed power of S : $X_T = S_T^2$

$$\begin{aligned} X &= S^2, \quad \log X = 2 \log S, \quad d(\log X) = 2 d(\log S) \\ X_t &= S_0^2 e^{2[(\mu - \sigma^2/2)t + \sigma\sqrt{t}Z]}, \\ \mathbb{E}^Q[X_T] &= S_0^2 e^{2(r - \sigma^2/2)T} e^{2\sigma^2 T}, \\ V &= S_0^2 e^{rT + \sigma^2 T}. \end{aligned}$$

where we used the risk-neutral measure and made use of the moment-generating function for Gaussian random variables

$$Y \sim \mathcal{N}(\mu, \sigma^2) \implies f(\lambda) = \mathbb{E}[e^{\lambda Y}] = e^{\lambda\mu + \lambda^2\sigma^2/2}$$

American options

Finance at MIT

Where ingenuity drives results

American exercise

- For American options, there are additional considerations. The owner of the option has the right to exercise at any time, not just at T .
- Should the option be exercised early? If so, when? Since the owner might no longer hold the option at T , we cannot simply apply the earlier formulas.

American perpetual put

Example: consider a put option that **never expires**.

- Its payoff upon exercise, at all times, is $\max(K - S, 0)$, where K is the strike price. The value is time-independent, so it satisfies

$$\frac{(\sigma S)^2}{2} \frac{d^2 V}{dS^2} + rS \frac{dV}{dS} - rV = 0$$

- Let's try a solution of power-law form

$$V(S) = S^\alpha \quad \Rightarrow \quad (\alpha^2 - \alpha) \frac{\sigma^2}{2} + \alpha r - r = 0 \quad \Rightarrow \quad \alpha = 1 \text{ or } -2r/\sigma^2$$

- Since the solution must vanish for increasing S (and assuming $r > 0$),

$$V(S) = cS^{-2r/\sigma^2}$$

American perpetual put

- For $S > K$, don't exercise.
 - However if S is far below K , it could be advantageous to exercise.
 - (Special case: if the stock price S decreases to zero, the option's value can never go higher so there is no point waiting any longer)

- Boundary condition: the option's value will equal its exercise value when

$$V(\hat{S}) = K - \hat{S} \implies V(S) = (K - \hat{S}) \left(\frac{S}{\hat{S}} \right)^{-2r/\sigma^2}$$

- The option writer must assume that the buyer will choose to maximize V :

$$\begin{aligned} \frac{\partial V}{\partial S} \Big|_{S=\hat{S}} = 0 &\implies \hat{S} = \frac{K}{1 + \sigma^2/2r}, \\ V(S) &= \frac{K\sigma^2/2r}{1 + \sigma^2/2r} \left(\frac{S}{K} (1 + \sigma^2/2r) \right)^{-2r/\sigma^2} \end{aligned}$$

Measures, martingales, and Monte Carlo

Measures and martingales

- An Itô process is a martingale if and only if it has **zero drift**. Measure for Brownian motion.

$$\begin{aligned}\mathbb{E}_t[X_{t'}] &= X_t, \quad t < t' \implies \mathbb{E}_t[\mathrm{d}X_t] = 0 \\ \mathrm{d}X_t &= a \mathrm{d}t + b \mathrm{d}B_t \implies a = 0\end{aligned}$$

- Now consider a **discounted price process**

$$F = e^{-rt} S \text{ where } \mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}B$$

Then

$$\begin{aligned}\frac{\partial F}{\partial S} &= e^{-rt}, \quad \frac{\partial^2 F}{\partial S^2} = 0, \quad \frac{\partial F}{\partial t} = -r e^{-rt} S, \\ \frac{\mathrm{d}F}{F} &= (\mu - r) \mathrm{d}t + \sigma \mathrm{d}B \text{ is a martingale iff } \mu = r.\end{aligned}$$

Risk-neutral pricing

What is the measure for **risk-neutral pricing**?

- Under measure Q , expected return of risky assets equals risk-free rate,
i.e.,

$$\mathbb{E}_t^Q \left[\frac{dS_t}{S_t} \right] = r dt$$

- How do we find the measure Q ? Let's write

$$\frac{dS_t}{S_t} = r dt + (\mu - r) dt + \sigma dB$$

$$= r dt + \sigma dB^Q, \text{ where } dB^Q \equiv \left(\frac{\mu - r}{\sigma} \right) dt + dB$$

- Then the new differential is a martingale:

$$\mathbb{E}_t^Q [dB^Q] = 0, \\ \text{Var}(dB^Q) = dt$$

Risk-neutral pricing

- Heuristic: replace drift with risk-free rate to get risk-neutral process: $\mu \rightarrow r$

$$\frac{dS_t}{S_t} = r dt + \sigma dB_t^Q$$

$$d(\log S_t) = \left(r - \frac{\sigma^2}{2} \right) dt + \sigma dB_t^Q,$$

$$\log S_T / S_0 \sim \mathcal{N} \left(\left(r - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right)$$

- Analogous to discrete-time binomial model results: use risk-neutral, not objective, probabilities to determine pricing.

Risk-neutral pricing

All (no-arbitrage) traded assets have discounted price process that are martingales

$$e^{-rt} X_t = \mathbb{E}_t^Q [e^{-rT} X_T]$$

- For a call option, when interest rate is constant,

$$C_t = e^{-r(T-t)} \mathbb{E}_t^Q [\max(S_T - K, 0)]$$

- Monte Carlo implementation: generate ensemble of equiprobable price paths using **risk-neutral** drift and volatility parameters, compute terminal payoffs, and take average of their discounted present value.

Monte Carlo pricing

- More generally, price any contract from its terminal values, allowing risk-free rate to vary with time

$$V_t = \mathbb{E}_t^Q \left[\frac{V_T}{\beta_T/\beta_t} \right]$$
$$\begin{cases} \mathbb{E}[\cdot] & \text{Sum over paths, equal weights} \\ Q: & \text{Use } r \text{ in evolution} \\ V_T & \text{Terminal value of paths} \\ \beta_T/\beta_t & \text{Discounting } e_t^{\int_t^T r(s) ds} \end{cases}$$

Monte Carlo pricing

- Generate an ensemble of **risk-neutral paths**
 - Use risk-free rate for drift
 - Use random number generation so that all paths are equally probable under **risk-neutral measure**

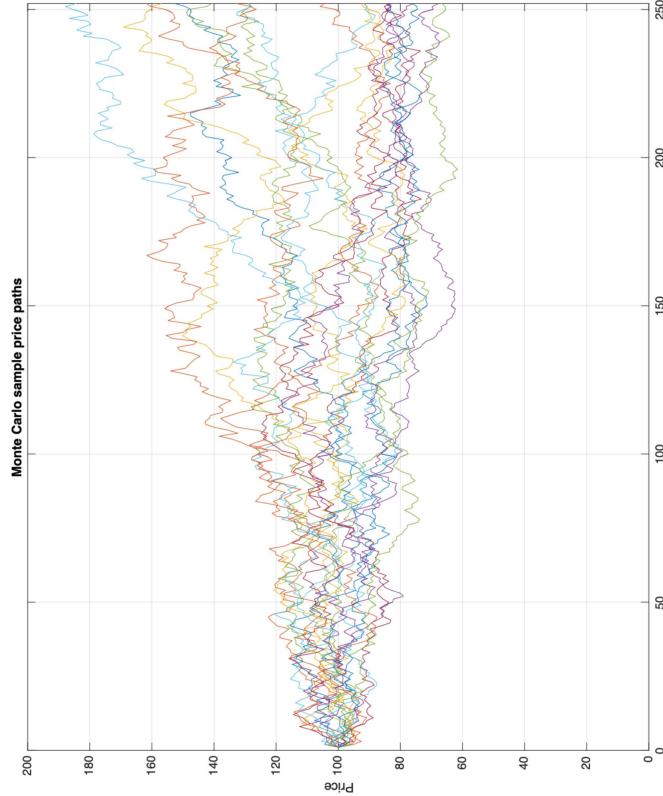
```
MCprice <- function(Price, Strike, Rate, Time, Volatility, Steps, Paths) {  
  # Monte Carlo pricer for vanilla options [8/12/2021 pfm]  
  # Input arguments use consistent units, e.g., annualized  
  # Price: current price of underlying  
  # Strike: strike price of option contract  
  # Rate: risk-free rate  
  # Time: time to expiration  
  # Volatility: number of time steps in discretization  
  # Steps: number of Monte Carlo simulation paths for sampling measure  
  # Paths: number of Monte Carlo simulation paths for sampling measure  
  
  S0 <- Price  
  K <- Strike  
  rf <- Rate  
  T <- Time  
  sigma <- Volatility  
  Nt <- Steps  
  Np <- Paths  
  dt <- T/Nt  
  
  # Select independent, standardized shocks. For example,  
  z <- matrix(sign(rnorm(Nt*Np)),ncol=Np)  
  
  # Define IID returns for each step and path under risk-neutral measure  
  # INSERT CODE HERE  
  
  # Construct stochastic paths and price process  
  # INSERT CODE HERE  
  
  # Define payoff values for derivatives  
  # INSERT CODE HERE  
  
  # Compute call and put values as discounted expected payoffs under RN measure  
  # INSERT CODE HERE  
  
  # Return values  
  return(data.frame(call=Call,put=Put))  
}
```

- Determine terminal payoffs
- Compute discounted present value of average over paths

Monte Carlo pricing

Accuracy, limits, and convergence

- Discrete time steps
- Finite number of sample paths



```
S0 <- 100; K <- 100; T <- 1; rf <- 0.1;
sigma <- 0.3; Nt <- 252; Np <- 1e4;
MCprice(S0,K,rf,T,sigma,Nt,Np)

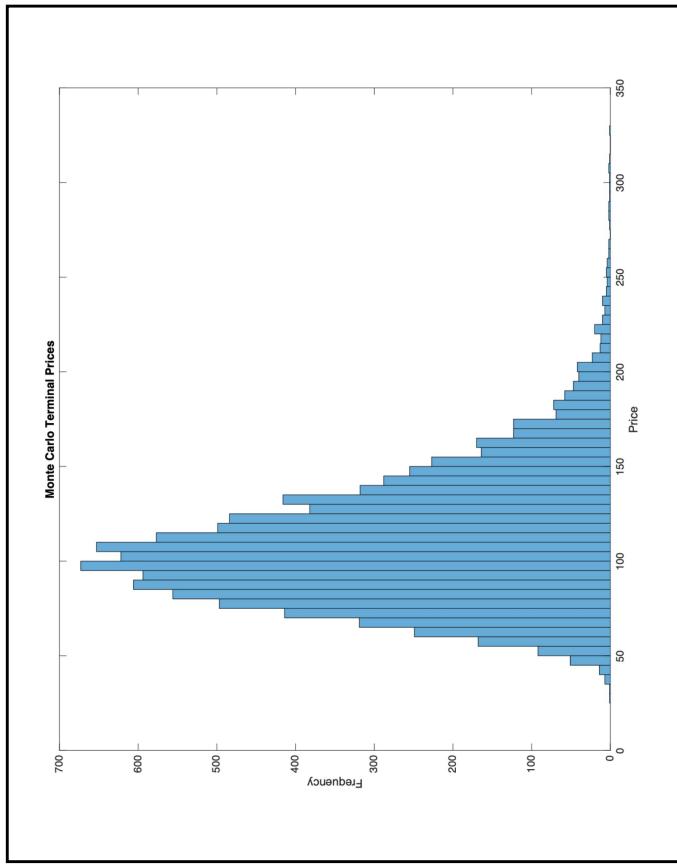
call    put
1 16.93101 7.051155
library(RQuantLib)

EuropeanOption("call", S0, K, theta, rf, T, sigma)$value
[1] 16.73413

EuropeanOption("put", S0, K, theta, rf, T, sigma)$value
[1] 7.217875
```

Monte Carlo pricing

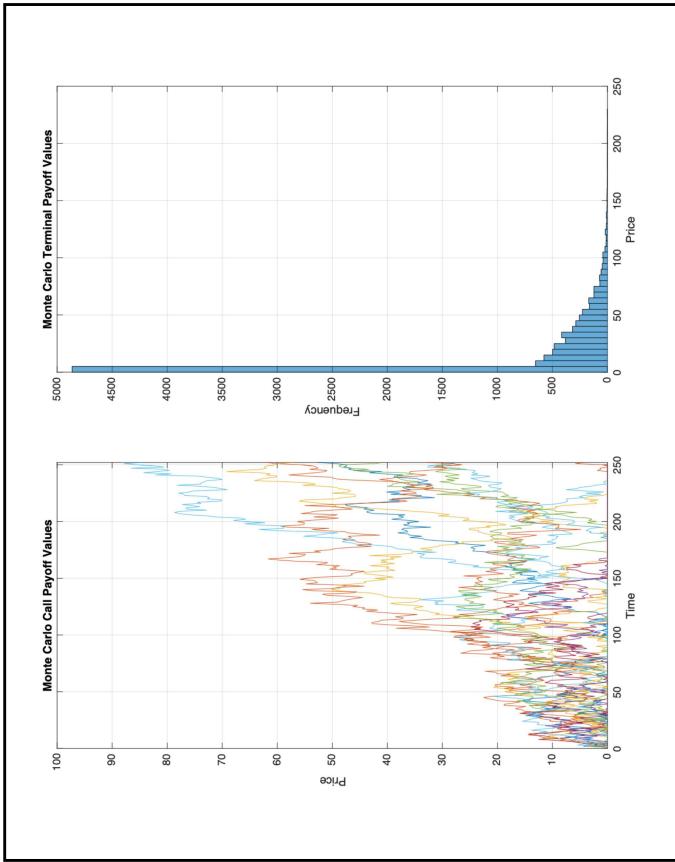
- Price paths lognormally distributed
- Mean value based on risk-neutral, not objective, drift rate
- Volatility identical



Monte Carlo pricing

Implementation of measure:

- Since all paths **equally probable under Q measure**, compute option value using simple arithmetic average of discounted payoffs.



Ito processes in higher dimensions

Itô's lemma: multiple stochastic variables

- For multiple Itô processes, formula generalizes.

$$dX_i = a_i(t, X_1, X_2, \dots) dt + b_i(t, X_1, X_2, \dots) dB_i$$

$$dF = \frac{\partial F}{\partial t} dt + \sum \frac{\partial F}{\partial X_i} dX_i + \frac{1}{2} \sum \rho_{ij} b_i b_j \frac{\partial^2 F}{\partial X_i \partial X_j} dt$$

- Applications
 - Multiple assets, such as a stock index or portfolio
 - Multiple factors, reducing independent sources of correlation
 - Risk models, to determine sources of risk priced in the market
 - Term-structure models for interest rates and derivatives
 - ...

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- Heuristic "rule of thumb" for correlated Brownian motions

$$\begin{aligned} (dB_i)^2 &\rightarrow dt, \\ (dB_i)(dB_j) &\rightarrow \rho_{ij} dt, \\ (dX_i)^2 &\rightarrow b_i^2 dt \\ (dX_i)(dX_j) &\rightarrow \rho_{ij} b_i b_j dt \end{aligned}$$

Itô's lemma

- Example: consider two independent stochastic variables, and

$$F = X_1 X_2 \implies dF = X_1 dX_2 + X_2 dX_1 + (dX_1)(dX_2),$$
$$\frac{dF}{F} = \frac{dX_1}{X_1} + \frac{dX_2}{X_2} + \left(\frac{dX_1}{X_1} \right) \left(\frac{dX_2}{X_2} \right)$$

- Geometric Brownian motions:

$$\frac{dX_i}{X_i} = \mu_i dt + \sigma_i dB_i$$
$$\frac{dF}{F} = \left(\mu_1 + \mu_2 + \frac{\rho_{12}\sigma_1\sigma_2}{F} \right) dt + \sigma_1 dB_1 + \sigma_2 dB_2$$

Itô's lemma

- Example: consider two independent stochastic variables. How does ratio evolve?

$$\begin{aligned} F = \frac{X_2}{X_1} &\implies dF = \frac{dX_2}{X_1} - \frac{X_2 dX_1}{X_1^2} + \frac{1}{2} \left(\frac{2X_2}{X_1^3} \right) (dX_1)^2 - \left(\frac{1}{X_1^2} \right) (dX_1)(dX_2) \\ \frac{dF}{F} &= \frac{dX_2}{X_2} - \frac{dX_1}{X_1} + \left[\frac{\sigma_1^2}{X_1^2} - \frac{\rho_{12}\sigma_1\sigma_2}{X_1 X_2} \right] dt, \\ \text{If } \rho_{12} = 0 &\implies \left(\mu_2 - \mu_1 + \frac{\sigma_1^2}{X_1^2} \right) dt + \sigma_2 dB_2 - \sigma_1 dB_1 \end{aligned}$$

- If drift coefficients are equal, then growth rate of 2 vs. 1 is positive.
However the same is true of the inverse. Contradiction?
- Application: changes of base currency

References

- Books
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