15.455x – Mathematical Methods for Quantitative Finance

Recitation Notes #6

1 On solving PDEs

One common approach to solving PDEs is to make an educated guess as to the form of the correct answer and see where it leads. If we're lucky, we can quickly narrow in on a result and then apply boundary conditions to fix it uniquely. If we're unlucky, we'll hit a dead end and need to start over. Such a guess, also known as an "ansatz," might be inspired by the form of the equation.

For example, in the case of the diffusion equation, we might have considered trying a function of product form, $p(z,t) = f(z)g(t)$, since the equation has a simple structure with constant coefficients. If we plug it in, we find

$$\frac{\partial p}{\partial t} = f(z)g'(t), \quad \frac{\partial^2 p}{\partial z^2} = f''(z)g(t), \tag{1}$$

where the primes denote first and second derivatives of the single-variable functions f and g. Substitute these into the PDE and then divide by p to get the relationship

$$\frac{g'(t)}{g(t)} = \frac{1}{2}\frac{f''(z)}{f(z)}. \tag{2}$$

Since the left-hand size depends on t only, while the right-hand size depends on z only, it must be the case that both sides are constant, independent of either z or t. Calling this constant Λ, we can now replace the original PDE with a pair of ordinary differential equations, each with a single variable:

$$g'(t) = \Lambda g(t) \implies g(t) = c_0 e^{\Lambda t}, \tag{3}$$

$$f''(z) = 2\Lambda f(z) \implies f(z) = c_1 e^{\sqrt{2\Lambda}z} + c_2 e^{-\sqrt{2\Lambda}z}. \tag{4}$$

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The coefficients c_0, c_1, c_2 are constants of integration. Any linear combination of terms of the form $f(z)g(t)$ will satisfy the PDE. Boundary conditions can be used to fix the exact form of the solution in each case.

Exercise: Consider the equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial y^2} - (\alpha y)\frac{\partial V}{\partial y} = 0, \tag{5}$$

subject to the initial condition $V(t = 0, y) = e^{-y}$.

Answer: Let's try a solution V of the form

$$V(y, t) = e^{f(t) - yg(t)}, \tag{6}$$

where f and g are unknown functions. Substituting this form into the equation, you can find – and solve – ordinary differential equations for $f(t)$ and $g(t)$. The boundary condition on V implies boundary condtions for f and g.

Using the ansatz $V = e^{f(t)-yg(t)}$ and taking derivatives,

$$\frac{\partial V}{\partial t} = \left(\frac{\mathrm{d}f}{\mathrm{d}t} - y\frac{\mathrm{d}g}{\mathrm{d}t}\right) V, \tag{7}$$

$$\frac{\partial V}{\partial y} = -gV, \tag{8}$$

$$\frac{\partial^2 V}{\partial y^2} = g^2 V, \tag{9}$$

so that

$$\left(\frac{\mathrm{d}f}{\mathrm{d}t} - y\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{1}{2}\sigma^2 g^2 + (\alpha y)g\right) V = 0. \tag{10}$$

The coefficient of V must vanish. It consists of terms that are linear in y or constant, and they must separately vanish in order for the equation to hold in general. That is,

$$\left[\left(\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{1}{2}\sigma^2 g^2\right) + y\left(\alpha g - \frac{\mathrm{d}g}{\mathrm{d}t}\right)\right] V = 0. \tag{11}$$

Setting the coefficient of y to zero gives an equation for $\mathrm{d}g/\,\mathrm{d}t = \alpha g$, which has the solution

$$g(t) = ce^{\alpha t}. \tag{12}$$

The boundary conditions $V(0, y) = e^{-y}$ implies boundary conditions $f(0) = 0, g(0) = 1$. That fixes the integration constant $c = 1$, so that $g(t) = e^{\alpha t}$. The equation for $\mathrm{d}f/\mathrm{d}t$ gives

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{\sigma^2}{2} g^2 = -\frac{\sigma^2}{2} e^{2\alpha t}. \tag{13}$$

Integrating this equation and imposing $f(0) = 0$, we find

$$f(t) = \frac{\sigma^2}{4\alpha} \left(1 - e^{2\alpha t}\right). \tag{14}$$

Therefore the solution is

$$V(y, t) = \exp\Big(f(t) - yg(t)\Big) \tag{15}$$

$$= \exp\Big(\frac{\sigma^2}{4\alpha} \left(1 - e^{2\alpha t}\right) - ye^{\alpha t}\Big). \tag{16}$$

2 Boundaries and barriers

Exercise: A trigger call option gives its own the right, but not the obligation, to buy underlying stock at price K if the stock value S_T exceeds X at expiration. The underlying follows the Itô process $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$. What PDE does the option satisfy? What is its price?

Hint: Sketch the option payoff. Can it be replicated by a portfolio of simpler contracts?

Answer: The PDE is the Black-Scholes equation since the stock follows the same Itô process. The payoff for this option if $S_T > X$ is $(S_T - X) + (X - K)$, which can be replicated with a call option of strike X plus $(X - K)$ binary options at the same strike. Therefore

$$\begin{aligned} V &= \left[S\Phi(d_+) - Xe^{-rT}\Phi(d_-)\right] + (X - K)e^{-rT}\Phi(d_-) \\ &= S\Phi(d_+) - Ke^{-rT}\Phi(d_-), \end{aligned}$$

$$\text{where } d_\pm \equiv \frac{\log(S/Xe^{-rT})}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

Exercise: A barrier option is a path-dependent derivative whose payoff depends on the prices realized by the underlying prior to expiration. Knock-out options begin like regular options but they lose all their value if and when the underlying price hits a predetermined level. A knock-in option begins worthless but becomes active if and when the underlying price hits a pre-specified level. The options can be puts or calls, vanilla or binary, etc. For a single barrier, the level can be set either above or below the underlying's current price.

Find the value $C_{\rm do}(S,t)$ of a European down-and-out call with strike price K, expiring at T where the barrier is below the strike price, $X < K$. The underlying is a non-dividend-paying stock that has drift and volatility μ, σ. The risk-free rate $r = 0$.

Answer: This option satisfies the Black-Scholes PDE, subject to boundary conditions that correspond to the terms of the contract. (This example comes from Robert Merton's original 1973 paper on option pricing. We take $r = 0$ to simplify the math.)

Motivated by our study of Brownian motion and survival probabilities in the presence of a default boundary, it is natural to look for a solution that is the difference of the normal vanilla option formula and an "image" option that starts with a value below the barrier. Since the Black-Scholes equation is related to the diffusion equation in transformed variables, we would expect to find the "image price" at $\log \tilde{S} = 2(\log X) - \log S$, or equivalently in terms of $C(S,t) - C(X^2/S,t)$, where C is the value of the vanilla call option with the same strike and expiration. Although it satisfies the boundary condition, the second term doesn't quite satisfy the PDE. Let's look for a solution of the form $f = (S/X)^\alpha C(X^2/S,t)$. Since

$$\frac{\partial f}{\partial t} = \left(\frac{S}{X}\right)^\alpha \frac{\partial C}{\partial t}, \tag{17}$$

$$\frac{\partial f}{\partial S} = \left(\frac{S}{X}\right)^\alpha \left[\frac{\alpha}{S} C + \left(-\frac{X^2}{S^2}\right) C'\right] \tag{18}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{S}{X}\right)^\alpha \left[\frac{\alpha(\alpha-1)}{S^2} C + (2\alpha - 2)\left(-\frac{X^2}{S^2}\right) C' + \left(-\frac{X^2}{S^2}\right)^2 C''\right], \tag{19}$$

setting $\alpha = 1$ eliminates the first two terms and provides the desired second solution:

$$\frac{\partial f}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 f}{\partial S^2} = \left(\frac{S}{X}\right)^\alpha \left[\frac{\partial C}{\partial t} + \frac{(\sigma \tilde{S})^2}{2}\frac{\partial^2 C}{\partial \tilde{S}^2}\right] = 0. \tag{20}$$

Therefore the value for the down-and-out call option when $X < K$ is given by

$$C_{\mathrm{do}}(S,t) = \begin{cases} C(S,t) - \left(\frac{S}{X}\right) C(X^2/S,t), & S \geq X, \\ 0, & S \leq X. \end{cases} \tag{21}$$