15.455x Mathematics Methods of Quantitative Finance

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Review of Linear Algebra

About the word "solution"... *a warm-up exercise*

- **How many** solutions to the following? Write down your answers **now**.

$$3x = 6 \tag{1}$$

$$3x = y \tag{2}$$

$$x^3 = 1 \tag{3}$$

$$x^2 + y^2 = 1 \tag{4}$$

$$\begin{cases} x + 2y & = 8 \\ 3x + 4y & = 6 \end{cases} \tag{5}$$

$$\begin{cases} x + 2y + 3z + 4t & = 0 \\ 5x + 6y + 7z + 8t & = 0 \\ x + 2y + 4z + 8t & = 2 \end{cases} \tag{6}$$

Vectors and vector spaces

- A **vector space** consists of a set of elements, called **vectors**, that is **closed** under the operations of **vector addition** and **scalar multiplication**.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Properties of vector addition

- Commutative

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

- Associative

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

- Identity

$$\mathbf{v} + \mathbf{0} = \mathbf{v}$$

- Inverse

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

Properties of scalar multiplication

- Scalars form a field (here, the real numbers)
- Associative $$a(b\mathbf{v}) = (ab)\mathbf{v}$$
- Distributive $$a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$$ $$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$
- Identity $$1 \cdot \mathbf{v} = \mathbf{v}$$

Buzzword bingo

- Definitions
  - Vectors
  - Vector space
  - Addition
  - Scalar multiplication
  - Closure
  - Subspace
  - Dimension
  - Span
  - Basis
  - Kernel
  - Image
  - Rank
  - Nullity
  - Null space
  - Singular matrix
  - Linear transformation
  - Linear operator
  - Dual space
  - Eigenvalue
  - Eigenvector
  - Diagonalization
  - Change of basis
  - Adjoint transformation
  - Inner product
  - Orthogonal
  - Quadratic form
  - Gram-Schmidt process
  - ....

Linear dependence, basis, and dimension

Linear dependence

- A **linear combination** is a sum of vectors multiplied by arbitrary scalars

$$\mathbf{w} = \sum_i a_i \mathbf{v_i}$$

- A set of vectors is **linearly dependent** if there are constants, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0$$

- If no such set of constants exist, the set of vectors is **linearly independent.**

Linear dependence

- Linear dependence means we can write at least one of the vectors in terms of the others.

$$\mathbf{v}_1 = -\frac{1}{a_1}(a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n)$$

(Here is where we use that the scalars must be a ***field*** *so that there is an inverse for non-zero scalars.)*

- So is there a **finite set** of vectors that be used to express all the others as linear combinations?

Spanning sets

- One way is to **define** a vector space so that it's true. We call the set of all linear combinations of a given set of vectors the **span** of that set.

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \left\{ \sum_{i=1}^{k} a_i \mathbf{v_i}, \quad \forall a_i \in \mathbb{R} \right\}$$

- Because it is to designed to be closed under addition and scalar multiplication, it forms a vector space.

- If every element of a vector space *V* can be expressed as a linear combination of a given set, then that set is said to **span the vector space**.

Basis

- If, in addition to spanning *V*, the vectors in the spanning set are linearly independent, then they form a **basis** for *V.*
  - A basis is a minimal, independent set of vectors that spans the space.
  - The number of vectors in the basis set is called the **dimension** of the vector space.
  - The choice of basis vectors is **not unique**.
  - Changing the basis, however, does not change the dimension.

Coordinates and notation

- Given a basis, the expression of any vector as a linear combination in terms of the basis vectors is **unique**.

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n}$$

- The coefficients are called the **coordinates with respect to the basis**.

- We use vector notation to denote this linear combination compactly as

$$\mathbf{v} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots c_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Subspaces

- A **subspace** S of a vector space V is a vector space which is a subset of V.
- The span of a **subset** of basis vectors of V defines a subspace of V.
- Any linearly independent set of $k < n$ vectors of V defines a subspace.
  - The set can constitute a basis of the subspace.
  - The dimension of the subspace is k.
  - Example: polynomials of degree 3 or less vanishing at $x=1$.

Linear transformations

- Functions and mappings
- Linear functions
- A **linear transformation** on a vector space $T : V \rightarrow W$ obeys

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T\mathbf{v}_1 + T\mathbf{v}_2,$$
$$T(c\mathbf{v}) = cT\mathbf{v}.$$

- This simple property of **linearity** means that any linear transformation is completely described by its action on the basis vectors of a space.

Linear transformations

- Consider the transformation *T* acting on an arbitrary vector, which is expressed as a linear combination of basis vectors. Then by linearity,

$$
\begin{aligned}
T\mathbf{v} &= T\left(c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n}\right) \\
&= c_1(T\mathbf{u_1}) + c_2(T\mathbf{u_2}) + \cdots + c_n(T\mathbf{u_n})
\end{aligned}
$$

- Therefore if we know how *T* acts on **each** basis vector in the vector space *V*, we can express the action of *T* on **any** vector by taking linear combinations of these *n* results.

Matrix of a linear transformation

- Let's use column notation for vectors in the target space *W*.

- *T*'s action on each basis vector of *V* gives some vector in *W*, so let's write them in general form as

$$T\mathbf{u_1} = \begin{pmatrix} m_{11} \\ m_{21} \\ \vdots \\ m_{s1} \end{pmatrix}, \quad T\mathbf{u_2} = \begin{pmatrix} m_{12} \\ m_{22} \\ \vdots \\ m_{s2} \end{pmatrix}, \cdots \quad T\mathbf{u_n} = \begin{pmatrix} m_{1n} \\ m_{2n} \\ \vdots \\ m_{sn} \end{pmatrix}$$

- *T* is then characterized by *n* column vectors (the dimension of *V*), each of length *s* (the dimension of *W*).

Matrix of a linear transformation

- Combine these *n* columns to form the matrix *M* corresponding to the linear transformation.

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{s1} & m_{s2} & \cdots & m_{sn} \end{pmatrix}$$

- The matrix *M* depends on the choice of bases in *V* and *W.*

- When *M* acts on a column vector of *V*, the result will be a linear combination of the columns of *M*.

Matrix of a linear transformation

- In column notation,

$$M\mathbf{x} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{s1} & m_{s2} & \cdots & m_{sn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} m_{11}x_1 + m_{12}x_2 + \cdots + m_{1n}x_n \\ m_{21}x_1 + m_{22}x_2 + \cdots + m_{2n}x_n \\ \vdots \\ m_{s1}x_1 + m_{s2}x_2 + \cdots + m_{sn}x_n \end{pmatrix}$$

- In components, this transformation rule reads

$$(M\mathbf{x})_i = \sum_{j=1}^{n} m_{ij}x_j, \qquad i = 1, 2, \ldots, s$$

Linear transformations of the plane

- In two dimensions, let's write this as

$$M\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

- Acting on this equation from the left with a new linear transformation gives a rule for multiplication of matrices

$$M'M = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}$$

- Each column of the product is the result of acting with *M*' on the corresponding column of *M*.

Matrix multiplication

- Matrix multiplication defined by **composition** of linear transformations

$$(M'M)_{ij} = \sum_{k=1}^{s} M'_{ik} M_{kj}$$

- Properties:
  - Associative
  - Distributive
  - NOT commutative
  - Identity

$$M_1(M_2 M_3) = (M_1 M_2) M_3$$
$$M_1(M_2 + M_3) = M_1 M_2 + M_1 M_3$$
$$M_1 M_2 \neq M_2 M_1 \text{ (in general)}$$
$$MI = IM = M$$

- (Bonus fact: Matrices also form a vector space of their own.)

Image and kernel

- Two important subspaces can be defined with respect to a linear operator

$$T : V \to W$$

- The **image** of *T* is the set of all vectors in *W* that can be reached from *V*

$$\operatorname{Im} T = \{\mathbf{w} | \exists \mathbf{v} \in V, \quad T\mathbf{v} = \mathbf{w}\} \subset W$$

- The **kernel** of *T* is the set of all vectors in *V* "annihilated" by *T*

$$\operatorname{Ker} T = \{\mathbf{v} \in V | T\mathbf{v} = 0\} \subset V$$

Image and kernel

- The **rank** of a linear transformation is the dimension of the image. It is the number of linearly independent columns of a matrix.

- Fundamental Theorem of Linear Transformations:

$$\dim V = \dim(\operatorname{Im} T) + \dim(\operatorname{Ker} T)$$

- If the kernel is empty, i.e., has dimension=0, then the rank of the transformation is the dimension of *V.*

- If in addition, *V* and *W* have the same dimension, then *T* is **invertible**.

Some properties of determinants

- A square matrix has an **inverse** if and only if Det *M* is non-zero

- For a 2x2 matrix, $$\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

- Product rule: $$\mathrm{Det}(MM') = (\mathrm{Det}\, M)(\mathrm{Det}\, M')$$

- Scalar multiplication: $$\mathrm{Det}(cM) = c^n\, \mathrm{Det}\, M, \quad M : \mathbb{R}^n \to \mathbb{R}^n$$

- A **singular** matrix, where Det *M=0,* has a non-trivial **kernel**

Some properties of determinants

- Det M is **linear** as function of its individual rows or columns
- Det M is **antisymmetric** under interchange of adjacent rows or columns
- If any rows or columns are linearly dependent, Det $M = 0.$
- The determinant is the (oriented) **volume** of the image of the unit (hyper)cube

Some properties of the trace

- The **trace** is defined as the sum of the **diagonal** elements of a square matrix.
- The trace is **invariant** under a change of basis.
- The trace of the **identity** matrix in an n-dimensional space is n.
- The trace of a product is invariant under **cyclic** changes in the order; e.g.,

$$\begin{aligned} \operatorname{Tr} AB &= \operatorname{Tr} BA \\ \operatorname{Tr} ABC &= \operatorname{Tr} BCA = \operatorname{Tr} CAB \end{aligned}$$

Matrix inverse

- For 2x2 matrices, the inverse when Det *M* is non-zero is given by

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- The inverse acts on the left or the right to give the identity matrix

$$M(M^{-1}) = (M^{-1})M = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrices and inverses

- Example: Rotation matrices
  - Columns are **image** of basis vectors
  - Single parameter for angle
  - No fixed points
  - Determinant = 1
  - Inverse matrix = reverse rotation

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad M_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

R Basics

- For matrices larger than 2X2, use a computer
- Distinguish functions of linear algebra from other operators
  - Examples (RTM!)
    - ✦A %*% B – Matrix multiplication of A and B
    - ✦A %*% A – Square of a matrix
    - ✦A * B – Element-by-element multiplication of components
    - ✦exp(M) – Exponential of elements, **not** exponential of the matrix
- Beware of numerical issues and instabilities
  - Reals, rounding
  - Zero

R Basics

- Example: solve

$$
\begin{aligned}
x + 2y + 3z &= 3 \\
4x + 5y + 6z &= 6 \\
7x + 8y + 9z &= 9
\end{aligned}
$$

- Write in matrix form $M\mathbf{v} = \mathbf{b}$ and consider solution of form

$$\mathbf{v} = M^{-1}\mathbf{b}$$

$$
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
=
\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}
$$

R Basics

```
> M <- matrix(c(1,2,3,4,5,6,7,8,9),byrow=TRUE,nrow=3)

> M
     [,1] [,2] [,3]
[1,]    1    2    3
[2,]    4    5    6
[3,]    7    8    9

> solve(M)
Error in solve.default(M) :
  system is computationally singular: reciprocal condition number = 1.54198e-18


> det(M)
[1] 6.661338e-16


> sum(diag(M))
[1] 15
```

R Basics

- Example: solve

$$\begin{aligned} x + 2y + 3z &= 3 \\ 4x + 5y + 6z &= 6 \\ 7x + 8y + 10z &= 9 \end{aligned}$$

- Write in matrix form $M\mathbf{v} = \mathbf{b}$ and consider solution of form

$$\mathbf{v} = M^{-1}\mathbf{b}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

R Basics

```
> M <- matrix(c(1,2,3,4,5,6,7,8,10),byrow=TRUE,nrow=3)
> b <- c(3,6,9)

> det(M)
[1] -3

> Minv <- solve(M) ; Minv
           [,1]      [,2] [,3]
[1,] -0.6666667 -1.333333    1
[2,] -0.6666667  3.666667   -2
[3,]  1.0000000 -2.000000    1

> v <- Minv %*% b ; v
              [,1]
[1,] -1.000000e+00
[2,]  2.000000e+00
[3,]  1.776357e-15

> M %*% v
     [,1]
[1,]    3
[2,]    6
[3,]    9
```

Systems of linear equations

Systems of linear equations

- Let's consider two kinds of systems of linear equations where in general we have ***s* equations** with ***n* unknowns**.

$$\begin{cases} M\mathbf{v} = \mathbf{b} & \text{inhomogeneous, or} \\ M\mathbf{v} = 0 & \text{homogeneous,} \end{cases}$$

$$\text{where } \mathbf{v} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^s,\ M : \mathbb{R}^n \to \mathbb{R}^s$$

- Expect to find, roughly, that the "inhomogeneous" equation has
  - **One** solution if $s = n$
  - **No** solutions if $s > n$
  - **Infinitely many** solutions if $s < n$

- The exact situation depends on the dimension of the image and kernel...
  - If $r = \mathrm{rank}(M)$ is smaller than it could be, nature of solutions changes.

Case 1: $s = n$

- When s=n, there are **the same number** of equations as unknowns, and the matrix M is square. As we have seen, there are two sub-cases:

- If det $M \neq 0$, then M is invertible and there is a **unique solution**,

$$M\mathbf{v} = \mathbf{b} \implies \mathbf{v} = M^{-1}\mathbf{b}$$

- If det M=0, then the m equations are **not independent**. M has a **non-zero kernel**, and the rank is less than the dimension of the target space.

$$\dim(\mathrm{Im}\, M) = r = n - \dim(\ker M) < n = s$$

- So there will be some vectors **b** for which there is no solution.

Case 1: *s* = *n* and non-singular

- R example:

$$\boxed{\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 2 \\ 7x + 8y + 10z &= 3 \end{aligned}}$$

  - **solve(M)** inverts matrix
  - **solve(M,b)** gives **unique solution**

Solution:

$$\mathbf{v} = \begin{pmatrix} -1/3 \\ 2/3 \\ 0 \end{pmatrix}$$

```
> M <- matrix(c(1,2,3,4,5,6,7,8,10),byrow=T,nrow=3)
> b <- matrix(c(1,2,3),ncol=1)
> v <- solve(M,b)

> M
     [,1] [,2] [,3]
[1,]    1    2    3
[2,]    4    5    6
[3,]    7    8   10

> solve(M)
           [,1]      [,2] [,3]
[1,] -0.6666667 -1.333333    1
[2,] -0.6666667  3.666667   -2
[3,]  1.0000000 -2.000000    1

> v
           [,1]
[1,] -0.3333333
[2,]  0.6666667
[3,]  0.0000000


> M %*% v
     [,1]
[1,]    1
[2,]    2
[3,]    3
```

Case 1b: *s* = *n* and singular

- R example:

$$
\begin{aligned}
x + 2y + 3z &= 1 \\
4x + 5y + 6z &= 2 \\
7x + 8y + 9z &= 3
\end{aligned}
$$

  - **solve** using **qr** gives a **particular solution** if it exists
  - Add any multiple of kernel for general solution

$$\text{Solution: } \mathbf{v} = \mathbf{v}_p + c\mathbf{z}$$

$$\mathbf{v}_p = \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix}, \ \mathbf{z} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

```
> M1 <- matrix(1:9,byrow=T,nrow=3)
> b1 <- matrix(c(1,2,3),ncol=1)
> v1 <- solve(M1,b1)
Error in solve.default(M1, b1) :
  system is computationally singular: reciprocal condition number =
1.54198e-18
> v1_p <- qr.solve(M1,b1)
Error in qr.solve(M1, b1) : singular matrix 'a' in solve

> M1
     [,1] [,2] [,3]
[1,]    1    2    3
[2,]    4    5    6
[3,]    7    8    9

> det(M1)
[1] 6.661338e-16

> qr(M1)$rank
[1] 2

> v1_p <- solve(qr(M1,LAPACK=TRUE),b1)
> v1_p
           [,1]
[1,] -0.1147976
[2,]  0.2295953
[3,]  0.2185357

> M1 %*% v1_p
     [,1]
[1,]    1
[2,]    2
```

Case 2: $s > n$

- When there are **more equations than unknowns**, then there are **no solutions** for at least some values of **b**.

- Because a "smaller" space is going into a "bigger" one, some vectors **b** in the target space **cannot be reached** from any vector in the "smaller" space. To get technical, from the Fundamental Theorem of Linear Transformations,

$$\dim(\operatorname{Im} M) = r = n - \dim(\ker M) \leq n < s$$

- Although there is no solution in general, some special points (in Im *M)* may yield solutions.

Case 2: *s* > *n*

- R example:

$$
\begin{aligned}
x + 2y &= -1 \\
3x + 4y &= 1 \\
5x + 6y &= 3 \\
7x + 8y &= 5
\end{aligned}
$$

  - **qr.solve(M,b)** gives **particular solution** if it exists
  - **Warning**: need to **check answer** since it also gives values when no solution exists(!)

Solution:

$$\mathbf{v}_p = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

```
> M2 <- matrix(1:8,byrow=T,nrow=4)
> b2 <- matrix(c(-1,1,3,5),nrow=4)
> v2_p <- qr.solve(M2,b2)

> M2
     [,1] [,2]
[1,]    1    2
[2,]    3    4
[3,]    5    6
[4,]    7    8

> v2_p
     [,1]
[1,]    3
[2,]   -2

> b2 <- matrix(c(-1,1,3,6),nrow=4)
> v2_not <- qr.solve(M2,b2)
> v2_not
      [,1]
[1,]  3.50
[2,] -2.35

> M2 %*% v2_not
     [,1]
[1,] -1.2
[2,]  1.1
[3,]  3.4
[4,]  5.7
```

Case 3: $s < n$

- When there are **more unknowns than equations**, then there are **multiple solutions**.

- Because a "bigger" space is going into a "smaller" one, some vectors must be mapped to zero. To get technical, from the Fundamental Theorem of Linear Transformations,

$$\dim(\ker M) = n - \dim(\operatorname{Im} M) \geq n - s > 0$$

- To any "particular solution" can be added any element of the kernel.

Case 3: *s* < *n*

- R example:

$$
\boxed{\begin{aligned} x + 2y + 3z + 4w &= 1 \\ 5x + 6y + 7z + 8w &= 1 \end{aligned}}
$$

  - **qr.solve(M,b)** gives a **particular solution**
  - ker(M) gives (sometimes inconvenient) basis for kernel

Solution: $\mathbf{v} = \mathbf{v}_p + c_1\mathbf{z}_1 + c_2\mathbf{z}_2$, where

$$
\mathbf{v}_p = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\ker M = \operatorname{span}\left\{ \mathbf{z}_1 = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}
$$

```
> M3 <- matrix(1:8,byrow=T,nrow=2)
> b3 <- matrix(c(1,1),ncol=1)
> v3_p <- qr.solve(M3,b3)

> M3
     [,1] [,2] [,3] [,4]
[1,]    1    2    3    4
[2,]    5    6    7    8

> v3_p
     [,1]
[1,]   -1
[2,]    1
[3,]    0
[4,]    0

> ker(M3)
           [,1]       [,2]
[1,]  0.0000000 -0.5477226
[2,]  0.4082483  0.7302967
[3,] -0.8164966  0.1825742
[4,]  0.4082483 -0.3651484
```

Case 3: *s* < *n*

- R example:

$$\begin{aligned} x + 2y + 3z + 4w &= 1 \\ 5x + 6y + 7z + 8w &= 1 \end{aligned}$$

  - **qr.solve(M,b)** gives a **particular solution**
  - ker(M) gives (sometimes inconvenient) basis for kernel

```
ker <- function(M){  # adapted from nullspace
      r <- qr(M)$rank
      cols <- if (r>0) -(1:r) else (1:ncol(M))
      V <- eigen(t(M) %*% M)$vectors[,cols,drop=FALSE]
      if (length(V)==0)
        return(NULL)
      else return(V)
}

> ker(M3)
           [,1]       [,2]
[1,]  0.0000000 -0.5477226
[2,]  0.4082483  0.7302967
[3,] -0.8164966  0.1825742
[4,]  0.4082483 -0.3651484
```