15.455x Sample Exam Questions

These sample exam problems are each worth 24 points. All sub-parts are weighted equally.

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1. Suppose that an asset price S_t follows a lognormal, continuous-time stochastic process,

$$\mathrm{d}S = \mu S \, \mathrm{d}t + \sigma S \, \mathrm{d}B,$$

where μ, σ are constants and B is a standard Brownian motion. Use Itô's lemma to find stochastic differential equations expressing $\mathrm{d}V$ in terms of $\mathrm{d}t$ and $\mathrm{d}B$ for the following functions $V(S,t)$. Are they Itô processes?

(a) $V = \alpha S + \beta$,

(b) $V = S^\gamma$,

(c) $V = e^{r(T-t)}S$,

where α, β, γ, r, and T are constants.

2. Let a stationary discrete-time stochastic process x_t be given by

$$x_t = A + Bx_{t-2} + C_t,$$

where $z_t \sim \mathcal{N}(0, 1)$ is an IID Gaussian white-noise process, and A, B, C are constants.

(a) What is the unconditional mean of the process x_t?

(b) An analyst decides to construct a forecast f_τ for future values of the process by taking its expected value, conditional on information available up through the time t when the forecast is made. That is,

$$f_\tau \equiv \mathrm{E}_t\big[x_\tau \,|\, x_t, x_{t-1}, \ldots\big], \quad \tau > t.$$

Let $A = 0.1$, $B = 0.2$, $C = 0.3$, and suppose that two recent values $x_1 = 0.4$, $x_2 = 0.5$ have just been observed. What are the one-step-ahead and two-step-ahead forecasts? That is, at time $t = 2$, what are the forecasts f_3 and f_4? What is the variance of the forecasts?

3. (a) Consider the quadratic form defined by

$$Q(x, y) = 2x^2 + 12xy - 7y^2.$$

Using Lagrange multipliers, find the location and value of the extrema of Q subject to the constraint $x + 3y = 5$. Determine whether each solution is a maximum, minimum, or neither.

(b) Two assets have correlation ρ, and their volatilities are 2σ and σ respectively. What are the weights of a minimum-variance, fully-invested portfolio of the two assets, and what is its risk? That is, minimize the portfolio variance $\sigma_p^2 = \mathbf{w}^\top C \mathbf{w}$, where C is the covariance matrix and $\mathbf{w}$ is an asset weight vector whose components satisfy the budget constraint $w_1 + w_2 = 1$.

(c) In the problem above, for what values of ρ and σ will the solution also satisfy an inequality constraint $0 \leq w_i \leq 1$? (That is, the optimal portfolio is also unlevered and long-only.)

4. The returns on a set of N assets are believed to follow the mean-reverting process

$$R_{it} - \mu_i = -\lambda(R_{i(t-1)} - \mu_i) + \sigma_i z_{it},$$

where μ_i, σ_i, λ are constants, $i = 1, \ldots N$; $|\lambda| < 1$; and

$$\mathbb{E}[z_{it}] = 0; \quad \mathbb{E}[z_{it} z_{js}] = \begin{cases} 1 & \text{if } t = s \text{ and } i = j, \\ 0 & \text{if } t \neq s \text{ or } i \neq j; \end{cases}$$

A market-neutral long/short trading strategy attempts to profit by investing capital in weights assigned according to

$$w_{it} = -\frac{1}{N}(R_{it} - \overline{R}_t),$$

where the market average return is defined by

$$\overline{R}_t = \frac{1}{N}\sum_{i=1}^{N} R_{it}.$$

Assume there are no transaction costs and the risk-free rate $R_f = 0$. Find the expected portfolio return

$$\mathbb{E}[R_p] = \mathbb{E}\left[\sum_i w_{i(t-1)} R_{it}\right]$$

in terms of the parameters given. Under what conditions is this expected return positive?

5. You are responsible for evaluating alternative investments for the endowment at Martingale Institute of Technology. A promising new market-neutral strategy is under consideration, and the managers from the firm 455 Capital Management LLC have provided backtested results of their investment strategy in the form of three years' worth of monthly simulated excess returns, $\{R_t, t = 1, 2, \ldots, 36\}$. They show a mean annualized excess return of about 12%, a Sharpe ratio of about 1.5, and a beta of 0.0. You need to evaluate the strategy's contribution to the endowment's risk and return profile in order to determine how much capital, if any, should be allocated to this new investment opportunity.

   (a) There are three time-series models in use at the MIT endowment for purposes of evaluating risk profiles: normal random walk, lognormal random walk, or mean-reverting. Which is most appropriate?

   (b) For the model you selected, write down estimators for the strategy's annualized mean, variance, and Sharpe ratio. Are these estimators biased?

   (c) The endowment guidelines call for estimating the mean return to within 1%. How long a time series of returns is required?

   (d) The endowment's chief investment officer is concerned about the effects of possible return smoothing or serial correlation in the data. Which test(s) can you do to identify serial correlation, and what data is required?

6. Two stocks have prices S_1 and S_2 that follow geometric Brownian motion with the same stochastic process $\mathrm{d}B$:

$$\begin{aligned} \mathrm{d}S_1 &= \mu_1 S_1 \; \mathrm{d}t + \sigma_1 S_1 \; \mathrm{d}B, \\ \mathrm{d}S_2 &= \mu_2 S_2 \; \mathrm{d}t + \sigma_2 S_2 \; \mathrm{d}B. \end{aligned}$$

(a) A contract has value $V = S_1 S_2$. You can show that V also follows geometric Brownian motion. What are its drift and volatility parameters?

(b) What is the process followed by $1/V$?

(c) A call option on V has payoff at expiration $\max(S_1 S_2 - K, 0)$. What PDE does the option satisfy?