15.455x – Mathematical Methods for Quantitative Finance

Recitation Notes #4

Let's look at examples of Itô's lemma in action. The simplest cases are where the stochastic variable X is just pure Brownian motion:

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t = \mathrm{d}B_t.$$

So in terms of the general form of an Itô process, $a = 0, b = 1$.

> **Note on notation:** *With X_t, B_t, S_t, etc. the subscript t is just a reminder that these are time-dependent random variables. We will often drop the subscript to de-clutter the notation. So X and B have <u>exactly the same meaning</u> as X_t, B_t, etc.*

Exercises

The simplest cases are where the stochastic variable X is just pure Brownian motion:

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t = \mathrm{d}B_t.$$

So in terms of the general form of an Itô process, $a = 0, b = 1$. For pure Brownian motion, therefore,

$$\mathrm{d}F = \frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial B}\,\mathrm{d}B + \frac{1}{2}\frac{\partial^2 F}{\partial B^2}\,\mathrm{d}t$$

Exercise: $F(t, B) = t^3 + B^3$. Find $\mathrm{d}F$.

Solution: Taking the required partial derivatives,

$$\begin{aligned}
\mathrm{d}F &= 3t^2\,\mathrm{d}t + 3B^2\,\mathrm{d}B + 3B\,\mathrm{d}t \\
&= 3(t^2 + B)\,\mathrm{d}t + (3B^2)\,\mathrm{d}B.
\end{aligned}$$

Exercise: $F(t, B) = e^{-rt}\sin(\theta B)$. Find $\mathrm{d}F$.

Solution: Taking partial derivatives,

$$dF = e^{-rt}\sin(\theta B)\,dt + \theta e^{-rt}\cos(\theta B)\,dB - e^{-rt}\frac{\theta^2}{2}\sin(\theta B)\,dt$$

$$= e^{-rt}\left[\left(1 - \frac{\theta^2}{2}\right)\sin(\theta B)\right]dt + e^{-rt}\left[\theta\cos(\theta B)\right]dB.$$

If $\theta = \sqrt{2}$, then the first term drops out and $dF = e^{-rt}\left[\sqrt{2}\cos(\sqrt{2}B)\right]dB.$

Exercise: $F(t, B) = \log B$. Find dF.

Solution:

$$dF = \frac{1}{B}\,dB + \frac{1}{2}\left(\frac{-1}{B^2}\right)dt.$$

Exercise: $dX/X = \mu\,dt + \sigma\,dB$. Find a function $F(t, X)$ such that $dF = a\,dt + b\,dB$ with constant coefficients a, b.

Solution: Let's use Itô's lemma in the form where we write the right-hand side in terms of dt and dB (rather than dX):

$$dF = \left[\frac{\partial F}{\partial t} + \frac{(\sigma X)^2}{2}\frac{\partial^2 F}{\partial X^2} + (\mu X)\frac{\partial F}{\partial X}\right]dt + \left[(\sigma X)\frac{\partial F}{\partial X}\right]dB.$$

Each of the expressions in square brackets must be constant, so there are two differential equations for F. Let's start with the second one, since it's much shorter:

$$(\sigma X)\frac{\partial F}{\partial X} = \text{constant} \implies X\frac{\partial F}{\partial X} = \frac{\text{constant}}{\sigma} = C.$$

The left-hand side is a logarithmic derivative, so let's try a very simple form where $F = F(X)$ and does not depend on time. Then

$$X\frac{dF}{dX} = C \implies dF = C\frac{dX}{X}$$
$$F(X) = C\log X.$$

Now observe that this expression automatically makes the coefficient function of dt into a constant as well. In fact, if we choose $C = 1$, we find that

$$dF = \left[\mu - \frac{\sigma^2}{2}\right]dt + \sigma\,dB.$$

We've shown that another route to the differential of $d(\log S)$ seen in modeling stock prices comes from seeking a change of variables that makes the right-hand-side coefficients constant.

Exercise: An asset follow the Ornstein-Uhlenbeck process $dS = \lambda(\bar{S} - S)\,dt + \sigma\,dB$. What PDE is satisfied by derivatives of the asset?

Solution: This case is actually no harder than the derivation of the usual Black-Scholes-Merton equation because the exact form of dS enters through one small feature, the coefficient of dB. Given

$$dS = a(t, S)\,dt + b(t, S)\,dB,$$

$$dV = \left(\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S}\right)dS,$$

$$d(V - \Delta S) = \left(\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS$$

$$= r\,dt(V - \Delta S).$$

Making the same choice of $\Delta = \partial V/\partial S$ in order to zero out the coefficient of dS, we find the generalization

$$\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

which is independent of $a(t, S)$ in the defining process for the underlying. This form gives us a direct route from a defining process to the PDE satisfied by derivatives on the underlying asset.

For the Ornstein-Uhlenbeck process,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$