15.455x – Mathematical Methods for Quantitative Finance

Recitation Notes #2

Exercise: Let z and z' be two independent, normalized random variables. Define the stochastic process

$$X_t \equiv z\cos(\omega t) + z'\sin(\omega t),$$

where ω is a constant. Show that X_t is weakly stationary.

Solution: All we have to do is compute and apply linearity (and a pinch of trigonometry). $\mathrm{E}\left[X_t\right] = 0$ since z, z' have zero mean. For the variance,

$$\begin{aligned}
\mathrm{Var}(X_t) = \mathrm{E}\left[X_t^2\right] &= \mathrm{E}\left[z^2\cos^2(\omega t) + 2zz'\cos(\omega t)\sin(\omega t) + z'^2\sin^2(\omega t)\right] \\
&= \cos^2(\omega t)\,\mathrm{E}\left[z^2\right] + 2\cos(\omega t)\sin(\omega t)\,\mathrm{E}\left[zz'\right] + \sin^2(\omega t)\,\mathrm{E}\left[z'^2\right] \\
&= \cos^2(\omega t) + \sin^2(\omega t) = 1.
\end{aligned}$$

Now for the autocovariance, or two-point function, observe that for any $t \neq s$

$$\begin{aligned}
\mathrm{E}\left[X_t X_s\right] &= \mathrm{E}\left[\left(z\cos(\omega t) + z'\sin(\omega t)\right)\left(z\cos(\omega s) + z'\sin(\omega s)\right)\right] \\
&= \cos(\omega t)\cos(\omega s) + \sin(\omega t)\sin(\omega s) \\
&= \cos\left(\omega(t-s)\right),
\end{aligned}$$

which is a function of the time difference $t - s$ only.

Exercise:

Let a stochastic process be defined by

$$X_t = z_t + \theta z_{t-2},$$

where z_t are independent, normalized random variables. Is X_t stationary? Find the mean, variance, and autocovariance function.

Solution:

The process is stationary because the defining equation has the same form if all the t are shifted by a constant. The first two moments can be computed by using linearity.

$$\begin{aligned} \mathrm{E}\left[X_t\right] &= \mathrm{E}\left[z_t\right] + \theta\ \mathrm{E}\left[z_{t-2}\right] = 0, \\ \mathrm{E}\left[X_t^2\right] &= \mathrm{E}\left[\left(z_t^2 + 2\theta z_t z_{t-2} + \theta^2 z_{t-2}^2\right)\right] = 1 + \theta^2. \end{aligned}$$

What about the autocovariance function (ACF)? From its definition,

$$\begin{aligned} \gamma_k = \mathrm{E}\left[X_t X_{t-k}\right] &= \mathrm{E}\left[\left(z_t + \theta z_{t-2}\right)\left(z_{t-k} + \theta z_{t-k-2}\right)\right] \\ &= \theta\,\mathrm{E}\left[z_{t-2} z_{t-k}\right] + \theta\,\mathrm{E}\left[z_t z_{t-k-2}\right] \\ &= \begin{cases} \theta, & k = \pm 2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Exercise:

Define a random variable A as the average of the next four observations of the X_t defined above,

$$A = \frac{1}{4}\left(X_1 + X_2 + X_3 + X_4\right).$$

What are the mean and variance of A?

Solution:

The mean is zero since each X_t is mean zero. In computing the variance, we can do the algebra to express results in terms of the expectations we just computed. It is non-zero only for expectations of two X_t that are either zero or two time steps apart.

$$\begin{aligned} \mathrm{Var}(A) = \mathrm{E}\left[A^2\right] &= \frac{1}{16}\,\mathrm{E}\left[\Big(X_1 + X_2 + X_3 + X_4\Big)^2\right] \\ &= \frac{1}{16}\Big(4\,\mathrm{E}\left[X_1^2\right] + 2\,\mathrm{E}\left[X_1 X_3\right] + 2\,\mathrm{E}\left[X_2 X_4\right]\Big) \\ &= \frac{1}{16}\Big[4(1+\theta^2) + 4\theta\Big] = \frac{1}{4}(1 + \theta + \theta^2). \end{aligned}$$

Exercise:

Use recursion to show that the AR(1) process can be expressed as an MA process...of infinite order.

Solution:

Let's simplify the defining equation

$$R_t - \mu = -\lambda(R_{t-1} - \mu) + \sigma z_t$$

by introducing $Y_t = (R_t - \mu)/\sigma$, in terms of which we can continue to substitute:

$$\begin{aligned}
Y_t &= z_t - \lambda Y_{t-1} & &= z_t - \lambda\big[z_{t-1} - \lambda Y_{t-2}\big] \\
&= z_t - \lambda z_{t-1} + \lambda^2 Y_{t-2} & &= z_t - \lambda z_{t-1} + \lambda^2\big[z_{t-2} - \lambda Y_{t-3}\big] \\
&= z_t - \lambda z_{t-1} + \lambda^2 z_{t-2} - \lambda^3 Y_{t-3} & &= z_t - \lambda z_{t-1} + \lambda^2 z_{t-2} - \lambda^3\big[z_{t-3} - \lambda Y_{t-4}\big] \\
&= \cdots
\end{aligned}$$

If we were to continue the substitutions indefinitely (and recalling that $|\lambda| < 1$), we would obtain

$$Y_t = \sum_{k=0}^{\infty} (-\lambda)^k z_{t-k}.$$

Because this is a semi-infinite sum of z's extending into the past, we see immediately that $\mathrm{E}\left[z_s Y_t\right] = 0$ for every $t < s$. In particular, it means that

$$\mathrm{E}\left[z_t \left(R_{t-1} - \mu\right)\right] = 0.$$